Absolute intersection motive

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Abstract. The purpose of this article is to define and study the notion of absolute intersection motive.

Contents

1. Introduction

Let X be a smooth scheme over the field $\mathbb{C}$ of complex numbers. According to Deligne, singular cohomology $H^n(X(\mathbb{C}), \mathbb{Q})$ carries a *mixed Hodge structure* of weights $\geqslant n$, for all $n \in \mathbb{Z}$. Dually, cohomology with compact support $H_c^n(X(\mathbb{C}), \mathbb{Q})$ is equipped with a mixed Hodge structure of weights $\leqslant n$. The canonical morphism

$$H_c^n(X(\mathbb{C}), \mathbb{Q}) \longrightarrow H^n(X(\mathbb{C}), \mathbb{Q})$$

is a morphism of Hodge structures. It therefore factors over a morphism

$$u_n\colon \operatorname{Gr}_n^W H_c^n(X(\mathbb{C}), \mathbb{Q}) \longrightarrow \operatorname{Gr}_n^W H^n(X(\mathbb{C}), \mathbb{Q})$$

between *pure Hodge structures* of weight n, where $\mathrm{Gr}_n^W H_c^n(X(\mathbb{C}),\mathbb{Q})$ denotes the quotient of $H_c^n(X(\mathbb{C}),\mathbb{Q})$ by its part of weights $\leqslant n-1$, and $\mathrm{Gr}_n^W H^n(X(\mathbb{C}),\mathbb{Q})$ denotes the part of $H^n(X(\mathbb{C}),\mathbb{Q})$ of weight n. Therefore, the direct sum

$$\mathcal{H}(u_n) := \ker(u_n) \oplus \mathrm{im}(u_n) \oplus \mathrm{coker}(u_n)$$

is again a pure Hodge structure of weight n.

If X is proper, then $H^n(X(\mathbb{C}),\mathbb{Q}) = H_c^n(X(\mathbb{C}),\mathbb{Q})$ is pure, u_n is an isomorphism, and $\mathcal{H}(u_n) = H^n(X(\mathbb{C}),\mathbb{Q})$. Furthermore, the collection of all $\mathcal{H}(u_n)$, $n \in \mathbb{Z}$ is *of motivic origin*: there is a *Chow motive* $M_{gm}(X)$ over $\mathbb{C}$, the *motive of* X, whose (cohomological) *Hodge theoretic realization* equals $(\mathcal{H}(u_n))_{n\in\mathbb{Z}}$.

The aim of this paper is to provide evidence for the following: whenever X is smooth (but not necessarily proper), the collection $(\mathcal{H}(u_n))_{n\in\mathbb{Z}}$ is of motivic origin: there is a Chow motive $M^{!*}(X)$, the *absolute intersection motive of* X, whose Hodge theoretic realization equals $(\mathcal{H}(u_n))_{n\in\mathbb{Z}}$.

The sequence $(\mathcal{H}(u_n))_{n\in\mathbb{Z}}$ satisfies the following minimality property: the morphism u_n can be represented as the composition of a monomorphism and an epimorphism

$$\mathrm{Gr}_n^W H_c^n(X(\mathbb{C}),\mathbb{Q}) \hookrightarrow \mathcal{H}(u_n) \twoheadrightarrow \mathrm{Gr}_n^W H^n(X(\mathbb{C}),\mathbb{Q}),$$

and whenever u_n is factored through a pure polarizable Hodge structure H of weight n in such a way, then $\mathcal{H}(u_n)$ is a direct factor of H. This property explains the name of our construction: indeed, for any *choice* of compactification $j\colon X \hookrightarrow \widetilde{X}$ of X, *intersection cohomology of* X *with respect to* j, *i.e.*, the collection of pure polarizable Hodge structures $H^n(\widetilde{X}(\mathbb{C}), j_{!*}\mathbb{Q})$ yields factorizations

$$\mathrm{Gr}_n^W H_c^n(X(\mathbb{C}),\mathbb{Q}) \hookrightarrow H^n(\widetilde{X}(\mathbb{C}), j_{!*}\mathbb{Q}) \twoheadrightarrow \mathrm{Gr}_n^W H^n(X(\mathbb{C}),\mathbb{Q}).$$

Thus, $\mathcal{H}(u_n)$ is a direct factor of $H^n(\widetilde{X}(\mathbb{C}), j_{!*}\mathbb{Q})$ for all $n \in \mathbb{Z}$. While the cohomologies $H^n(\widetilde{X}(\mathbb{C}), j_{!*}\mathbb{Q})$ depend on the choice of j—their notion is therefore a *relative* one—the $\mathcal{H}(u_n)$ do not.

The definition of the absolute intersection motive $M^{!*}(X)$ is conditional, but makes sense for any perfect base field k (Definition 3.5). Indeed, it depends on the existence of a *minimal weight filtration* of the *boundary motive* $\partial M_{gm}(X)$. This existence would be ensured if as expected, all Chow motives over k, or at least those necessary to build up $\partial M_{gm}(X)$, were *finite-dimensional* in the sense of Kimura. A priori, given the main result from [20], this means that $M^{!*}(X)$ can be defined as soon as $\partial M_{gm}(X)$ is *of Abelian type.*

Section 2 contains the necessary basics from the theory of minimal weight filtrations. With an eye to the application to the Hodge theoretic context, we take particular care to formulate results for triangulated categories, which are not only equipped with a *weight structure* w à la Bondarko, but equally with a t-structure *transversal* to w. The main result is Theorem 2.9, which gives a characterization of morphisms in the *radical* in terms of their effect on cohomology objects.

Section 3 contains the basic definitions, conditional as we said in the motivic context, but unconditional in the Hodge theoretic one. We thus get the notion of *Hodge theoretic absolute intersection complex* of a smooth $\mathbb{C}$-scheme X (Definition 3.6). Both definitions rely on Construction 3.1, which establishes a bijection between isomorphism classes of *weight filtrations* of some boundary object on the one hand, and isomorphism classes of factorizations on the other.

By definition, the cohomology objects of the Hodge theoretic absolute intersection complex give *absolute intersection cohomology*; its study is the object of Section 4. In particular (Theorem 4.8), we prove, using the theory developed in Section 2, that absolute intersection cohomology of X is isomorphic to $(\mathcal{H}(u_n))_{n\in\mathbb{Z}}$; therefore it satisfies the minimality property mentioned above. In particular (Corollary 4.11), it is a direct factor of intersection cohomology of X with respect to any choice of compactification.

Section 5 contains a great number of examples. We establish the first examples of minimal weight filtrations for motives which are not of Abelian type (Example 5.4, Corollary 5.7). We also see that in general, the absolute intersection motive of a product is unequal to the tensor product of the absolute intersection motives of the factors (Remark 5.11).

Section 6 is concerned with the following problem (Question 6.1): can absolute intersection cohomology be "topologically realized", *i.e.*, does there exist a topological space $X^{!*}$, stratified into topological manifolds, among which $X(\mathbb{C})$, such that absolute intersection cohomology of X equals intersection cohomology relative to $X^{!*}$? None of the examples we found suggests the contrary, and we identify $X^{!*}$ in many cases. Actually, we find that sometimes, but not always, the Alexandrov one-point compactification X^+ can be chosen as $X^{!*}$. Theorem 6.7 gives necessary and sufficient criteria for the equation "$X^{!*} = X^+$" to hold. They basically tell us that whenever Y is a smooth algebraic compactification of X, with complement Z, then "$X^{!*} = X^+$" if and only if "the complement Z has maximal geometric self-interaction". By contrast, "no geometric self-interaction of Z" seems to lead to the solution "$X^{!*} = Y(\mathbb{C})$". We finish the article with an example (Exercice 6.9), where $X^{!*}$ exists, but is neither equal to X^+ nor to $Y(\mathbb{C})$, for any smooth compactification Y of X.

Notation and conventions

For a perfect field k, we denote by Sch/k the category of separated schemes of finite type over k, and by $Sm/k \subset Sch/k$ the full sub-category of objects which are smooth over k. As far as motives are concerned, the notation of this paper is, with a single exception, that of [30] and [31], which in turn follows that of [28]. We refer to [30, Sect. 1] for a concise review of this notation, and of the definition of the triangulated category $DM_{gm}(k)$ of geometrical motives over k.

The exception to the rule concerns the Tate motives, denoted $\mathbb{Z}(n)$ in [28, 30, 31], for which we shall write $\mathbb{1}(n)$.

Let F be a commutative semi-simple Noetherian $\mathbb{Q}$-algebra, in other words, a finite direct product of fields of characteristic zero. The notation $DM_{gm}(k)_F$ stands for the F-linear analogue of $DM_{gm}(k)$ defined in [1, Sect. 16.2.4 and Sect. 17.1.3]. Similarly, we denote by $CHM(k)$ the category opposite to the category of Chow motives, and by $CHM(k)_F$ the pseudo-Abelian completion of the category $CHM(k) \otimes_{\mathbb{Z}} F$. Using [27, Cor. 2], we canonically identify $CHM(k)_F$ with a full additive sub-category of $DM_{gm}(k)_F$.

When we assume a field k to *admit resolution of singularities*, then it will be in the sense of [15, Def. 3.4]: (i) for any separated k-scheme X of finite type, there exists an abstract blow-up $Y \to X$ [15, Def. 3.1] whose source Y is smooth over k, (ii) for any pair of smooth, separated k-schemes X, Y of finite type, and any abstract blow-up $q\colon Y \to X$, there exists a sequence of blow-ups $p\colon X_n \to \ldots \to X_1 = X$ with smooth centers, such that p factors through q. We say that k *admits strict resolution of singularities* if, in (i), for any given dense open subset U of the smooth locus of X, the blow-up $q\colon Y \to X$ can be chosen to be an isomorphism above U, and such that arbitrary intersections of the irreducible components of the complement Z of U in Y are smooth (*e.g.*, $Z \subset Y$ is a normal crossing divisor with smooth irreducible components).

2. Weight structures and t-structures

We make free use of the terminology of and basic results on weight structures [6, Sect. 1.3]. Let us fix an F-linear triangulated category $\mathcal{C}$, which is equipped with a weight structure $w = (\mathcal{C}_{w\leqslant 0}, \mathcal{C}_{w\geqslant 0})$. For simplicity, and in order to be able to apply the results from [7], we assume w to be bounded. The following notion is the key for everything to follow.

Definition 2.1 ([38, Def. 1.3]). Let $M \in \mathcal{C}$, and $n \in \mathbb{Z}$. A *minimal weight filtration concentrated at* n of M is a weight filtration

$$M_{\leqslant n-1} \longrightarrow M \longrightarrow M_{\geqslant n} \stackrel{\delta}{\longrightarrow} M_{\leqslant n-1}[1]$$

($M_{\leqslant n-1} \in \mathcal{C}_{w\leqslant n-1}$, $M_{\geqslant n} \in \mathcal{C}_{w\geqslant n}$) such that the morphism δ belongs to the *radical* of $\mathcal{C}$:

$$\delta \in \mathrm{rad}_{\mathcal{C}}(M_{\geqslant n}, M_{\leqslant n-1}[1]).$$

Recall [18] that the radical of an F-linear category $\mathfrak{A}$ is the ideal $\mathrm{rad}_{\mathfrak{A}}$ which associates to each pair of objects A, B of $\mathfrak{A}$ the subset

$$\mathrm{rad}_{\mathfrak{A}}(A,B) := \{f \in \mathrm{Hom}_{\mathfrak{A}}(A,B) \mid \forall\, g \in \mathrm{Hom}_{\mathfrak{A}}(B,A),\ \mathrm{id}_A - gf \text{ invertible}\}$$

of $\mathrm{Hom}_{\mathfrak{A}}(A,B)$. It can be checked that $\mathrm{rad}_{\mathfrak{A}}$ is indeed a two-sided ideal of $\mathfrak{A}$ in the sense of [2, Sect. 1.3], *i.e.*, for each pair of objects A, B, $\mathrm{rad}_{\mathfrak{A}}(A,B)$ is an F-submodule of $\mathrm{Hom}_{\mathfrak{A}}(A, B)$, and for each pair of morphisms $h\colon A' \to A$ and $k\colon B \to B'$ in $\mathfrak{A}$,

$$k \operatorname{rad}_{\mathfrak{A}}(A, B) h \subset \operatorname{rad}_{\mathfrak{A}}(A', B').$$

Any two minimal weight filtrations of the same object $M \in \mathcal{C}$ are related by an isomorphism (which in general is *not* unique) [37, proof of Thm. 2.2 (b)]. Minimal weight filtrations do not necessarily exist [37, Ex. 2.3 (c)].

Remark 2.2. (a) Minimal weight filtrations *do* exist if the *heart*

$$\mathcal{C}_{w=0} := \mathcal{C}_{w\leqslant 0} \cap \mathcal{C}_{w\geqslant 0}$$

of w is pseudo-Abelian and *semi-primary* [37, Thm. 2.2 (a)], *i.e.* [2, Déf. 2.3.1] if

(1) for all objects M of $\mathcal{C}_{w=0}$, the radical $\operatorname{rad}_{\mathcal{C}_{w=0}}(M, M)$ is nilpotent,

(2) the F-linear quotient category $\mathcal{C}_{w=0}/\operatorname{rad}_{\mathcal{C}_{w=0}}$ is semi-simple.

(b) In practice, given $M \in \mathcal{C}$, the existence of minimal weight filtrations of M is assured once M can be shown to belong to a full, triangulated sub-category $\mathcal{C}'$ of $\mathcal{C}$, such that w induces a weight structure on $\mathcal{C}'$, and such that $\mathcal{C}'_{w=0} = \mathcal{C}' \cap \mathcal{C}_{w=0}$ is pseudo-Abelian and semi-primary.

Example 2.3. According to [5, Sect. 6], the category $\mathcal{C} := DM_{gm}(k)_F$ of geometrical motives over k carries a bounded weight structure w, if k is of characteristic zero. This claim still holds for arbitrary perfect fields (remember that F is supposed to be a $\mathbb{Q}$-algebra), as can be seen from the proof of [31, Thm. 1.13], using the main results from [19, Sect. 5.5]. We refer to this weight structure as *motivic*. It identifies $CHM(k)_F$ with the heart $DM_{gm}(k)_{F,w=0}$. The category $CHM(k)_F$ is pseudo-Abelian. It is expected, but not known to be semi-primary. Consider the full sub-category $CHM^{Ab}(k)_F$ of $CHM(k)_F$ of *Chow motives of Abelian type over* k, *i.e.* [34, Def. 1.1 (b)], the sub-category of Chow motives whose base change to an algebraic closure $\bar{k}$ of k can be constructed out of shifts of Tate motives $\mathbb{1}(m)[2m]$, for $m \in \mathbb{Z}$, and of Chow motives of Abelian varieties over $\bar{k}$. Let $\mathcal{C}'$ denote the full triangulated sub-category $DM^{Ab}_{gm}(k)_F$ of $DM_{gm}(k)_F$ generated by $CHM^{Ab}(k)_F$. The motivic weight structure induces a weight structure, still denoted w, on $DM^{Ab}_{gm}(k)_F$ [34, Prop. 1.2], and $DM^{Ab}_{gm}(k)_{F,w=0} = CHM^{Ab}(k)_F$. According to [34, Prop. 1.8], the category $CHM^{Ab}(k)_F$ is (pseudo-Abelian and) semi-primary. We may thus apply Remark 2.2 (b): any geometrical motive belonging to $DM^{Ab}_{gm}(k)_F$ admits minimal weight filtrations.

In the sequel of this section, we shall assume in addition that $\mathcal{C}$ carries a bounded t-structure $t = (\mathcal{C}^{t\leqslant 0}, \mathcal{C}^{t\geqslant 0})$, which is *transversal* to w [7, Def. 1.2.2]. (Note that in the motivic context discussed in Example 2.3, such a t-structure is expected, but not known to exist.) Denote by $\mathcal{C}^{t=m}_{w=n}$, $m, n \in \mathbb{Z}$, the full sub-categories given by the intersections

$$\mathcal{C}^{t=m}_{w=n} := \mathcal{C}_{w=n} \cap \mathcal{C}^{t=m}$$

(where $\mathcal{C}_{w=n} := \mathcal{C}_{w\leqslant n} \cap \mathcal{C}_{w\geqslant n} = \mathcal{C}_{w=0}[n]$ and $\mathcal{C}^{t=m} := \mathcal{C}^{t\leqslant m} \cap \mathcal{C}^{t\geqslant m} = \mathcal{C}^{t=0}[-m]$).

Proposition 2.4 ([7, Thm. 1.2.1, Rem. 1.2.3, Prop. 1.2.4]). *Assume $\mathcal{C}$ to be equally equipped with a bounded t-structure, which is transversal to w.*

(a) The categories $\mathcal{C}_{w=n}^{t=m}$, $m, n \in \mathbb{Z}$, are Abelian semi-simple, and any object of $\mathcal{C}_{w=0}$ is isomorphic to a finite direct sum of objects in $\mathcal{C}_{w=0}^{t=m}$, $m \in \mathbb{Z}$.

(b) For fixed $m \in \mathbb{Z}$, and $n_1 \neq n_2$, there are no non-zero morphisms between objects of $\mathcal{C}_{w=n_1}^{t=m}$ and of $\mathcal{C}_{w=n_2}^{t=m}$.

(c) For fixed $m \in \mathbb{Z}$, any object of $\mathcal{C}^{t=m}$ admits weight filtrations by objects of $\mathcal{C}^{t=m}$, which are exact sequences in $\mathcal{C}^{t=m}$.

(d) The t-truncation functors $\tau^{t\leqslant m}$, $\tau^{t\geqslant m}$, $m \in \mathbb{Z}$, respect the sub-categories $\mathcal{C}_{w\geqslant n}$ and $\mathcal{C}_{w\leqslant n}$, $n \in \mathbb{Z}$.

Corollary 2.5. *Assume $\mathcal{C}$ to be equally equipped with a bounded t-structure, which is transversal to w.*

(a) The heart $\mathcal{C}_{w=0}$ is pseudo-Abelian and semi-primary.

(b) Any object of $\mathcal{C}$ admits minimal weight filtrations.

Proof. As recalled in Remark 2.2 (a), part (b) is implied by (a).

The second claim of part (a) follows from Proposition 2.4 (a), and from [2, proof of Prop. 2.3.4 c)]. It remains to show that $\mathcal{C}_{w=0}$ is pseudo-Abelian. Let $M = \oplus_{m\in\mathbb{Z}} M^m$ be an object of $\mathcal{C}_{w=0}$, with $M^m \in \mathcal{C}_{w=0}^{t=m}$, almost all M^m being zero (Proposition 2.4 (a)). Let e be an idempotent endomorphism of M. In order to show that e admits a kernel, we apply induction on the number of non-zero components M^m, the initial case $M = M^m$ resulting from Proposition 2.4 (a). For the induction step, take m to be minimal such that $M^m \neq 0$, and write

$$M = M^m \oplus N.$$

Orthogonality for the t-structure tells us that with respect to this direct sum,

$$e = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

with $A \in \mathrm{End}_{\mathcal{C}_{w=0}}(M^m)$, $B \in \mathrm{Hom}_{\mathcal{C}_{w=0}}(N, M^m)$, and $D \in \mathrm{End}_{\mathcal{C}_{w=0}}(N)$. The relation $e^2 = e$ is equivalent to the system of relations

$$A^2 = A, \quad D^2 = D, \quad AB + BD = B. \qquad (*)$$

By our induction hypothesis, $\ker(A) \subset M^m$ and $\ker(D) \subset N$ exist (and so do $\ker(\mathrm{id}_{M^m} - A)$ and $\ker(\mathrm{id}_N - D)$). We leave it as an exercice to the reader to prove, using $(*)$, that the morphism

$$\begin{pmatrix} \mathrm{id}_{\ker(A)} & -B \\ 0 & \mathrm{id}_{\ker(D)} \end{pmatrix}$$

from $\ker(A) \oplus \ker(D)$ to M is a kernel of e. □

Although it will not be needed in the sequel of the present article, let us mention the following.

Corollary 2.6. *Assume $\mathcal{C}$ to be equally equipped with a bounded t-structure, which is transversal to w. Then $\mathcal{C}$ is pseudo-Abelian.*

Proof. This follows from Corollary 2.5 (a), and from [5, Lemma 5.2.1]. □

Write

$$H^m : \mathcal{C} \longrightarrow \mathcal{C}^{t=0}, \quad m \in \mathbb{Z},$$

for the cohomology functors associated to t. The following holds even in the absence of a weight structure.

Proposition 2.7. *Let M and N be objects of $\mathcal{C}$, and $\alpha\colon M \to N$ a morphism. If $H^m(\alpha) = 0$ for all $m \in \mathbb{Z}$, then $\alpha \in \mathrm{rad}_{\mathcal{C}}(M,N)$.*

Proof. For any morphism $\beta\colon N \to M$, and any $m \in \mathbb{Z}$, we have

$$H^m\big(\mathrm{id}_M - \beta\alpha\big) = \mathrm{id}_{H^m(M)},$$

meaning that $H^*(\mathrm{id}_M - \beta\alpha)$ is an automorphism. Hence so is $\mathrm{id}_M - \beta\alpha$. □

Example 2.8. The converse of Proposition 2.7 is not true in general. Let N be a simple object of $\mathcal{C}^{t=0}_{w=0}$ and M a non-trivial extension in $\mathcal{C}^{t=0}$

$$0 \longrightarrow N_- \longrightarrow M \stackrel{\alpha}{\longrightarrow} N \longrightarrow 0$$

of N by some object N_- of $\mathcal{C}^{t=0}$ of strictly negative weights. Schur's Lemma and Proposition 2.4 (c), (b) imply that

$$\mathrm{Hom}_{\mathcal{C}}(N,M) = 0.$$

Therefore, $\alpha \in \mathrm{rad}_{\mathcal{C}}(M,N)$, but $0 \neq \alpha = H^0(\alpha)$.

It turns out that the converse of Proposition 2.7 *is* true, once we restrict the weights of M and N. Here is the main result of this section.

Theorem 2.9. *Assume $\mathcal{C}$ to be equally equipped with a bounded t-structure, which is transversal to w. Let $n \in \mathbb{Z}$, $M \in \mathcal{C}_{w \geqslant n}$, and $N \in \mathcal{C}_{w \leqslant n}$. Let $\alpha\colon M \to N$ be a morphism. Then the following are equivalent.*

(1) $H^m(\alpha) = 0$ *for all* $m \in \mathbb{Z}$.

(2) $\alpha \in \mathrm{rad}_{\mathcal{C}}(M,N)$.

Proof. Given Proposition 2.7, we may assume that α belongs to the radical, and need to establish that it is zero on cohomology.

Let us first treat the case where both M and N are objects of some $\mathcal{C}^{t=r}$, and show the following claim (∗): if a morphism $\gamma\colon M \to N$ is non-zero, then there is $\beta\colon N \to M$ such that $\beta\gamma\colon M \to M$ is a projector onto a non-zero direct factor of M.

Indeed, let

$$0 \longrightarrow W_n M \longrightarrow M \longrightarrow W^{n+1} M \longrightarrow 0$$

and

$$0 \longrightarrow W_{n-1} N \longrightarrow N \longrightarrow W^n N \longrightarrow 0$$

be weight filtrations of M and N within $\mathcal{C}^{t=r}$ (Proposition 2.4 (c)), with $W_n M$ and $W^n N$ in $\mathcal{C}^{t=r}_{w=n}$, $W^{n+1}M \in \mathcal{C}_{w \geqslant n+1}$, and $W_{n-1}N \in \mathcal{C}_{w \leqslant n-1}$. Write γ_n for the composition

$$W_n M \hookrightarrow M \xrightarrow{\gamma} N \twoheadrightarrow W^n N;$$

Proposition 2.4 (c), (b) imply that $\gamma_n \neq 0$. According to Proposition 2.4 (a), there is $\beta_n : W^n N \to W_n M$ such that $\beta_n \gamma_n$ is a projector onto a complement of $\ker(\gamma_n) \subset W_n M$. Define β as the composition

$$N \twoheadrightarrow W^n N \xrightarrow{\beta_n} W_n M \hookrightarrow M.$$

Now let us come back to $\alpha \in \mathrm{rad}_{\mathcal{C}}(M, N)$. In order to show that $H^m(\alpha) = 0$ for all $m \in \mathbb{Z}$, let us apply induction on the sum s of the number of non-zero cohomology objects of M and of N. The claim is trivial if $s \leqslant 1$, or if there is no degree r for which both $H^r(M)$ and $H^r(N)$ are non-zero. In order to treat the case $s = 2$, we may thus assume that $M, N \in \mathcal{C}^{t=r}$, for some $r \in \mathbb{Z}$. Claim $(*)$ then implies that indeed

$$0 = \alpha = H^r(\alpha).$$

For the induction step, let $r \in \mathbb{Z}$ be maximal such that $H^r(M)$ and $H^r(N)$ are both non-zero. This takes trivially care of $H^m(\alpha)$, for all $m \geqslant r + 1$.

Using t-truncations $\tau^{t \leqslant \bullet}$, $\tau^{t \geqslant \bullet}$, we see that the induction hypothesis can be applied unless $M \in \mathcal{C}^{t \leqslant r}$ and $N \in \mathcal{C}^{t \geqslant r}$. In that case, $H^m(\alpha) = 0$ for all $m \neq r$, and α equals the composition

$$M \longrightarrow H^r(M)[-r] \xrightarrow{\gamma} H^r(N)[-r] \longrightarrow N,$$

with $\gamma := H^r(\alpha)[-r]$. The objects $H^r(M)[-r]$ and $H^r(N)[-r]$ belong to $\mathcal{C}_{w \geqslant n}$ and $\mathcal{C}_{w \leqslant n}$, respectively (Proposition 2.4 (d)). If γ were non-zero, then claim $(*)$ would show that there is $\beta' : H^r(N)[-r] \to H^r(M)[-r]$ such that $\beta'\gamma$ is a projector onto a non-trivial direct factor of $H^r(M)[-r]$. The maps

$$\mathrm{Hom}_{\mathcal{C}}(N, M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(H^r(N)[-r], M)$$

and

$$\mathrm{Hom}_{\mathcal{C}}(H^r(N)[-r], M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(H^r(N)[-r], H^r(M)[-r])$$

are both surjective, as $\tau^{t \geqslant r+1}(N)[-1] \in \mathcal{C}_{w \leqslant n-1}$ and $\tau^{t \leqslant r-1}(M)[1] \in \mathcal{C}_{w \geqslant n+1}$ by Proposition 2.4 (d). Thus, the morphism β' can be extended to yield $\beta : N \to M$. By construction, the composition $\beta\alpha : M \to M$ is a projector onto a non-trivial direct factor of M, implying in particular that $\mathrm{id}_M - \beta\alpha$ is not an automorphism of M. But this contradicts the assumption on α. □

Theorem 2.9 has the following important consequences.

Corollary 2.10. *Assume $\mathcal{C}$ to be equally equipped with a bounded t-structure, which is transversal to w. Let*

$$M_{\leqslant n-1} \longrightarrow M \longrightarrow M_{\geqslant n} \xrightarrow{\delta} M_{\leqslant n-1}[1]$$

be a minimal weight filtration concentrated at some $n \in \mathbb{Z}$ of an object M of $\mathcal{C}$ (according to Corollary 2.5(b), such minimal weight filtrations exist). Then for all $m \in \mathbb{Z}$, the sequence

$$0 \longrightarrow H^m\big(M_{\leqslant n-1}\big) \longrightarrow H^m(M) \longrightarrow H^m\big(M_{\geqslant n}\big) \longrightarrow 0$$

is exact, and yields the weight filtration of $H^m(M)$ concentrated at $m+n$. In particular, the sequence does not depend on the choice of minimal weight filtration.

Proof. Apply Theorem 2.9 to $\delta\colon M_{\geqslant n} \to M_{\leqslant n-1}[1]$. Thus, the long exact cohomology sequence yields short exact sequences

$$0 \longrightarrow H^m\big(M_{\leqslant n-1}\big) \longrightarrow H^m(M) \longrightarrow H^m\big(M_{\geqslant n}\big) \longrightarrow 0,$$

for $m \in \mathbb{Z}$. According to Proposition 2.4 (d), $H^m(M_{\leqslant n-1})[-m]$ is indeed of weights $\leqslant n-1$, and $H^m(M_{\geqslant n})[-m]$, of weights $\geqslant n$. □

Corollary 2.11. *Assume $\mathcal{C}$ to be equally equipped with a bounded t-structure, which is transversal to w. Let $M, N \in \mathcal{C}_{w=0}$. Then*

$$\mathrm{rad}_{\mathcal{C}_{w=0}}(M,N) = \{\alpha \in \mathrm{Hom}_{\mathcal{C}_{w=0}}(M,N) \mid H^m(\alpha) = 0 \ \textit{for all} \ m \in \mathbb{Z}\}.$$

Example 2.12. Assume F to be a field contained in $\mathbb{R}$, and $\mathcal{C} := D^b(\mathbf{MHS}_F)$, the bounded derived category of *mixed graded-polarizable F-Hodge structures* [3, Def. 3.9, Lemma 3.11]. According to [7, Prop. 2.3.1 I] (with $X = \mathbf{Spec}\,\mathbb{C}$), $\mathcal{C}$ carries a canonical weight structure w, which is bounded: indeed, the class of a bounded complex K of mixed graded-polarizable F-Hodge structures lies in $D^b(\mathbf{MHS}_F)_{w\leqslant 0}$ (resp., in $D^b(\mathbf{MHS}_F)_{w\geqslant 0}$) if and only if the m-th cohomology object $H^m(K)$ is a Hodge structure of weights $\leqslant m$ (resp., $\geqslant m$), for all $m \in \mathbb{Z}$. Furthermore, the canonical t-structure on $D^b(\mathbf{MHS}_F)$ is transversal to w. Therefore, the theory developed in the present section applies. In particular (Corollary 2.5 (a)), $D^b(\mathbf{MHM}_F\, X)_{w=0}$ is pseudo-Abelian and semi-primary.

Example 2.13. Examples 2.3 and 2.12 are related by the *Hodge theoretic realization* ([17, Sect. 2.3 and Corrigendum]; see [14, Sect. 1.5] for a simplification of this approach). The field k is assumed to be embedded into $\mathbb{C}$ *via* $\sigma\colon k \hookrightarrow \mathbb{C}$, and F is assumed to be a sub-field of $\mathbb{R}$. Denote by

$$R_\sigma\colon\ DM_{gm}(k)_F \longrightarrow D^b(\mathbf{MHS}_F)$$

the Hodge theoretic realization of [loc. cit.], and recall that it is a *contravariant* tensor functor mapping the pure Tate motive $\mathbb{1}(m)$ to the pure Hodge structure $\mathbb{Q}(-m)$ [17, Thm. 2.3.3]. Chow motives are mapped to objects of $D^b(\mathbf{MHS}_F)$, which are pure of weight zero. Since the motivic weight structure is bounded, it follows formally that

$$R_\sigma\big(DM_{gm}(k)_{F,w\leqslant 0}\big) \subset D^b(\mathbf{MHS}_F)_{w\geqslant 0},$$

and that

$$R_\sigma\big(DM_{gm}(k)_{F,w\geqslant 0}\big) \subset D^b(\mathbf{MHS}_F)_{w\leqslant 0}.$$

In the setting of Example 2.13, we have the following result.

Theorem 2.14. *Consider the restriction*

$$R_\sigma^{Ab}\colon\ DM_{gm}^{Ab}(k)_F \longrightarrow D^b(\mathbf{MHS}_F)$$

of R_σ *to* $DM_{gm}^{Ab}(k)_F \subset DM_{gm}(k)_F$.

(a) Let $n \in \mathbb{Z}$, $M \in DM_{gm}^{Ab}(k)_{F,w \geqslant n}$, *and* $N \in DM_{gm}^{Ab}(k)_{F,w \leqslant n}$. *Then*

$$R_\sigma^{Ab}\big(\mathrm{rad}_{DM_{gm}^{Ab}(k)_F}(M,N)\big) \subset \mathrm{rad}_{D^b(\mathbf{MHS}_F)}\big((R_\sigma(N), R_\sigma(M)\big).$$

(b) The functor R_σ^{Ab} *maps minimal weight filtrations (concentrated at* n*) to minimal weight filtrations (concentrated at* $-n+1$*).*

Proof. Part (a) follows from [34, Lemma 1.11] and Proposition 2.7.

Given (a), part (b) follows from the definition (and the contravariance of R_σ^{Ab}). □

Remark 2.15. The importance of the hypothesis "of Abelian type" is twofold. First, it guarantees the existence of minimal weight filtrations of objects of $DM_{gm}^{Ab}(k)_F$. Note that this would still be true if we replaced $DM_{gm}^{Ab}(k)_F$ by the full triangulated sub-category of $DM_{gm}(k)_F$ generated by Chow motives, which are *finite-dimensional* [20, Def. 3.7]. Second, and more seriously, Theorem 2.14 (a) implies that the restriction of R_σ to $CHM^{Ab}(k)_F$ is *radicial* [2, Déf. 1.4.6]. Indeed, this latter statement is the vital ingredient of the proof of Theorem 2.14 (a); see [34, Thm. 1.9]. But as shown by the proof of [loc. cit.], that statement is intimately related to the identification of homological and numerical equivalence, which thanks to [22] is known for motives associated to Abelian varieties.

3. Definitions

Let us come back to the situation from the beginning of Section 2, *i.e.*, an F-linear triangulated category $\mathcal{C}$, equipped with a bounded weight structure $w = (\mathcal{C}_{w\leqslant 0}, \mathcal{C}_{w\geqslant 0})$ (but not necessarily with a t-structure). Let us fix a morphism $u\colon M_- \to M_+$ in $\mathcal{C}$ between $M_- \in \mathcal{C}_{w\leqslant 0}$ and $M_+ \in \mathcal{C}_{w\geqslant 0}$.

Construction 3.1. First choose a cone C of u, *i.e.*, an exact triangle

$$C[-1] \longrightarrow M_- \stackrel{u}{\longrightarrow} M_+ \longrightarrow C$$

in $\mathcal{C}$. Then choose a weight filtration of C

$$C_{\leqslant 0} \longrightarrow C \longrightarrow C_{\geqslant 1} \stackrel{\delta}{\longrightarrow} C_{\leqslant 0}[1]$$

(with $C_{\leqslant 0} \in \mathcal{C}_{w \leqslant 0}$ and $C_{\geqslant 1} \in \mathcal{C}_{w \geqslant 1}$). Consider the diagram, which we shall denote by the symbol (1), of exact triangles

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_{\geqslant 1}[-1] & = & C_{\geqslant 1}[-1] & \longrightarrow & 0 \\
\downarrow & & & & \downarrow{\scriptstyle \delta[-1]} & & \downarrow \\
M_- & & & & C_{\leqslant 0} & \longrightarrow & M_-[1] \\
\| & & & & \downarrow & & \| \\
M_- & \xrightarrow{u} & M_+ & \longrightarrow & C & \longrightarrow & M_-[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_{\geqslant 1} & = & C_{\geqslant 1} & \longrightarrow & 0
\end{array}$$

which according to axiom TR4' of triangulated categories [4, Sect. 1.1.6] can be completed to give a diagram, denoted by (2)

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_{\geqslant 1}[-1] & = & C_{\geqslant 1}[-1] & \longrightarrow & 0 \\
\downarrow & & {\scriptstyle \delta_+[-1]}\downarrow & & \downarrow{\scriptstyle \delta[-1]} & & \downarrow \\
M_- & \xrightarrow{i} & M & \xrightarrow{\delta_-[-1]} & C_{\leqslant 0} & \longrightarrow & M_-[1] \\
\| & & {\scriptstyle \pi}\downarrow & & \downarrow & & \| \\
M_- & \xrightarrow{u} & M_+ & \longrightarrow & C & \longrightarrow & M_-[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_{\geqslant 1} & = & C_{\geqslant 1} & \longrightarrow & 0
\end{array}$$

with $M \in \mathcal{C}$. By the second row, and the second column of diagram (2), the object M is simultaneously an extension of objects of weights $\leqslant 0$, and an extension of objects of weights $\geqslant 0$. It follows easily (*c.f.* [5, Prop. 1.3.3 3]) that M belongs to both $\mathcal{C}_{w \leqslant 0}$ and $\mathcal{C}_{w \geqslant 0}$, and hence to $\mathcal{C}_{w=0}$.

Note that even for a fixed choice of C, diagram (2) necessitates further choices of M and of factorizations $u = \pi i$ and $\delta = \delta_- \delta_+$. In general, the object M is unique up to possibly non-unique isomorphism.

Very precisely, we have the following.

Proposition 3.2. *The map*

$$\{(C_{\leqslant 0}, C_{\geqslant 1})\}/\cong \;\longrightarrow\; \{(M, i, \pi)\}/\cong$$

from Construction 3.1 is a bijection between

(1) the isomorphism classes of cones C of u, together with weight filtrations of C,

(2) the isomorphism classes of objects M of $\mathcal{C}_{w=0}$, together with a factorization

$$M_- \stackrel{i}{\longrightarrow} M \stackrel{\pi}{\longrightarrow} M_+$$

of the morphism $u \colon M_- \to M_+$, such that both i and π can be completed to give weight filtrations of M_- and of M_+, respectively.

Definition 3.3. Assume that a cone C of $u\colon M_- \to M_+$ (and hence *any* cone of u) admits a minimal weight filtration

$$C_{\leqslant 0} \longrightarrow C \longrightarrow C_{\geqslant 1} \xrightarrow{\delta} C_{\leqslant 0}[1]$$

concentrated at $n = 1$. A *minimal factorization of* $u\colon M_- \to M_+$ is a triple (M, i, π) consisting of an object M of $\mathcal{C}_{w=0}$, together with a factorization

$$M_- \xrightarrow{i} M \xrightarrow{\pi} M_+$$

of u, whose isomorphism class corresponds to $(C_{\leqslant 0}, C_{\geqslant 1})$ under the bijection of Proposition 3.2.

Thus, given two minimal factorizations

$$M_- \xrightarrow{i_1} M_1 \xrightarrow{\pi_1} M_+ \quad \text{and} \quad M_- \xrightarrow{i_2} M_2 \xrightarrow{\pi_2} M_+$$

of u, there is an isomorphism (which in general is not unique) $\alpha\colon M_1 \xrightarrow{\sim} M_2$ such that the diagram

$$\begin{array}{ccccc} M_- & \xrightarrow{i_1} & M_1 & \xrightarrow{\pi_1} & M_+ \\ \| & & \alpha\downarrow & & \| \\ M_- & \xrightarrow{i_2} & M_2 & \xrightarrow{\pi_2} & M_+ \end{array}$$

commutes.

Remark 3.4. (a) The correspondence from Proposition 3.2 is not *a priori* functorial, meaning that a morphism between weight filtrations of C does not yield a morphism of the corresponding objects M, not even up to automorphism.

(b) Minimality of the weight filtration

$$C_{\leqslant 0} \longrightarrow C \longrightarrow C_{\geqslant 1} \xrightarrow{\delta} C_{\leqslant 0}[1]$$

of C does in general not imply minimality of the weight filtrations

$$C_{\leqslant 0}[-1] \longrightarrow M_- \xrightarrow{i} M \xrightarrow{\delta_-[-1]} C_{\leqslant 0}$$

of M_- and

$$M \xrightarrow{\pi} M_+ \longrightarrow C_{\geqslant 1} \xrightarrow{\delta_+} M[1]$$

of M_+.

Applying the theory to the examples of Section 2, we get the following.

Definition 3.5. Assume that the field k admits resolution of singularities. Let $X \in Sm/k$. Assume that the *boundary motive* $\partial M_{gm}(X)$ of X [30, Def. 2.1] admits a minimal weight filtration concentrated at $n = 0$. A Chow motive $M^{!*}(X)$, together with a factorization

$$M_{gm}(X) \xrightarrow{i} M^{!*}(X) \xrightarrow{\pi} M_{gm}^c(X)$$

of the canonical morphism $u\colon M_{gm}(X) \to M_{gm}^c(X)$ [28, pp. 223–224], is called an *absolute intersection motive of* X if it is a minimal factorization of u.

Recall [30, Prop. 2.2, Cor. 2.3] that $\partial M_{gm}(X)$ fits into a canonical exact triangle

$$\partial M_{gm}(X) \longrightarrow M_{gm}(X) \xrightarrow{u} M^c_{gm}(X) \longrightarrow \partial M_{gm}(X)[1]$$

in $DM_{gm}(k)_F$. Therefore, $C := \partial M_{gm}(X)[1]$ is a cone of u. According to [31, Cor. 1.14], the motive $M_{gm}(X)$ belongs to $DM_{gm}(k)_{F,w \leqslant 0}$, and the motive with compact support $M^c_{gm}(X)$ to $DM_{gm}(k)_{F,w \geqslant 0}$. Given Example 2.3, the assumptions of Definition 3.3 are satisfied if the boundary motive $\partial M_{gm}(X)$ belongs to the category $DM^{Ab}_{gm}(k)_F$.

Definition 3.6. Assume F to be a field contained in $\mathbb{R}$, and let $X \in Sm/\mathbb{C}$. An object $R\Gamma_{!*}(X, F(0))$ of $D^b(\mathbf{MHS}_F)_{w=0}$, together with a factorization

$$R\Gamma_c(X, F(0)) \xrightarrow{i} R\Gamma_{!*}(X, F(0)) \xrightarrow{\pi} R\Gamma(X, F(0))$$

of the canonical morphism $u\colon R\Gamma_c(X, F(0)) \to R\Gamma(X, F(0))$, is called a *Hodge theoretic absolute intersection complex of* X if it is a minimal factorization of u.

Here, the objects $R\Gamma_c(X, F(0))$ and $R\Gamma(X, F(0))$ of $D^b(\mathbf{MHS}_F)$ are the canonical quasi-isomorphism classes computing singular cohomology of $X(\mathbb{C})$ with and without compact support, respectively, and together with their Hodge structures [13, Sect. 8.1], [3, Sect. 4]. They satisfy the assumption on weights thanks to [12, Cor. (3.2.15) (ii)], [3, Sect. 4.2]. According to Example 2.12 and Corollary 2.5, any object of $D^b(\mathbf{MHS}_F)$ admits minimal weight filtrations.

Variant 3.7. *(a) Assume that the field k admits resolution of singularities. Let $X \in Sm/k$, and fix an idempotent endomorphism e of the triangle*

$$\partial M_{gm}(X) \longrightarrow M(X) \longrightarrow M^c_{gm}(X) \longrightarrow \partial M_{gm}(X)[1],$$

that is, fix idempotent endomorphisms of each of the motives $M(X)$, $M^c_{gm}(X)$ and $\partial M_{gm}(X)$, which yield an endomorphism of the triangle. Denote by $M(X)^e$, $M^c_{gm}(X)^e$ and $\partial M_{gm}(X)^e$ the images of e on $M(X)$, $M^c_{gm}(X)$ and $\partial M_{gm}(X)$, respectively, and consider the canonical morphism $u\colon M(X)^e \to M^c_{gm}(X)^e$. The object $M(X)^e$ belongs to $DM_{gm}(k)_{F,w \leqslant 0}$, and $M^c_{gm}(X)^e$ to $DM_{gm}(k)_{F,w \geqslant 0}$. Assuming that $\partial M_{gm}(X)^e$ admits a minimal weight filtration concentrated at $n = 0$, Definition 3.3 allows for the definition of the notion of e-part of the absolute intersection motive of X, *which is a triple $(M^{!*}(X)^e, i, \pi)$, with $M^{!*}(X)^e \in CHM(k)_F$, and a minimal factorizaion*

$$M_{gm}(X)^e \xrightarrow{i} M^{!*}(X)^e \xrightarrow{\pi} M^c_{gm}(X)^e$$

of u. The hypothesis on $\partial M_{gm}(X)^e$ is satisfied as soon as $\partial M_{gm}(X)^e$ belongs to $DM^{Ab}_{gm}(k)_F$ (which is the case in particular if the whole of $\partial M_{gm}(X)$ belongs to $DM^{Ab}_{gm}(k)_F$).

(b) Similarly, for F a field contained in $\mathbb{R}$, and $X \in Sm/\mathbb{C}$, any pair e of idempotent endomorphisms of $R\Gamma_c(X, F(0))$ and of $R\Gamma(X, F(0))$ commuting with the canonical morphism $R\Gamma_c(X, F(0)) \to R\Gamma(X, F(0))$ allows one to

define the notion of e-part of the absolute intersection complex of X*, which is a triple* $(R\Gamma_{!*}(X,F(0))^e,i,\pi)$*, with* $R\Gamma_{!*}(X,F(0))^e \in D^b(\mathbf{MHS}_F)_{w=0}$*, and a minimal factorizaion*

$$R\Gamma_c(X,F(0))^e \xrightarrow{i} R\Gamma_{!*}(X,F(0))^e \xrightarrow{\pi} R\Gamma(X,F(0))^e$$

of $u\colon R\Gamma_c(X,F(0))^e \to R\Gamma(X,F(0))^e$*.*

(c) In a similar vein, there are variants "with coefficients" of the absolute intersection motive and the Hodge theoretic absolute intersection complex. The scheme X *need no longer be smooth, and the constant coefficients* $F(0)$ *are replaced by a* Chow motive $N \in CHM(X)_F$ [32, Def. 1.5] *in the motivic context, and by a* complex of Hodge modules $N \in D^b(\mathbf{MHM}_F\,X)$ [25, Sect. 4]*, which is pure of weight zero* [25, Sect. 4.5] *in the Hodge theoretic context. Write* a *for the structural morphism of* X*, and consider the canonical morphism* $u\colon a_!(N) \to a_*(N)$ *(*[8, Thm. 2.4.50 (2)], [25, proof of Thm. 4.3]*). The hypotheses on weights are satisfied (*[16, Thm. 3.8 (i_c), (i'_c)], [25, (4.5.2)]*). In the motivic context, one needs to assume in addition that a cone of* u *admits a minimal weight filtration concentrated at* $n=1$*, which as before is the case if it belongs to* $DM^{Ab}_{gm}(k)_F$*. The result is an isomorphism class of a Chow motive* $M^{!*}(X,N)$ *in the motivic context, and of an object* $R\Gamma_{!*}(X,N)$ *of* $D^b(\mathbf{MHS}_F)_{w=0}$ *in the Hodge theoretic context, together with a minimal factorization of* u*.*

Note that in the motivic context, following [8, Cor. 16.1.6], we have identified $DM_{gm}(k)_F$ with the triangulated category $DM_{\mathrm{B},c}(\mathbf{Spec}\,k)_F$ of *constructible Beilinson motives* over $\mathbf{Spec}\,k$ [8, Def. 15.1.1]. Under that identification, the canonical morphism $M(X) \to M^c_{gm}(X)$ equals the dual of the canonical morphism $a_!(\mathbb{1}_X) \to a_*(\mathbb{1}_X)$ (see *e.g.* [36, Prop. 2.8]). This point of view also allows one to drop the hypothesis on resolution of singularities in Definition 3.5 and Variant 3.7 (a).

4. Absolute intersection cohomology

The purpose of this section is to identify, up to isomorphism, the Hodge structure on the cohomology objects of a Hodge theoretic absolute intersection complex $R\Gamma_{!*}(X,F(0))$ of a smooth, separated scheme X over $\mathbb{C}$. For $n \in \mathbb{Z}$, denote by $H^n(X,F(0))$ the n-th singular cohomology group, and by $H^n_c(X,F(0))$ the n-th cohomology group with compact support of $X(\mathbb{C})$.

Definition 4.1. Assume F to be a field contained in $\mathbb{R}$, and let $X \in Sm/\mathbb{C}$. *Absolute intersection cohomology of* X is defined as the collection

$$\big(H^n_{!*}(X,F(0)),i,\pi\big)_{n\in\mathbb{Z}},$$

where for $n \in \mathbb{Z}$, we denote by $H^n_{!*}(X, F(0))$ the n-th cohomology object of a Hodge theoretic absolute intersection complex $R\Gamma_{!*}(X, F(0))$ of X (Definition 3.6), and by

$$H^n_c(X, F(0)) \xrightarrow{i} H^n_{!*}(X, F(0)) \xrightarrow{\pi} H^n(X, F(0))$$

the factorization of the canonical morphism $H^n_c(X, F(0)) \to H^n(X, F(0))$ induced by the factorization

$$R\Gamma_c(X, F(0)) \xrightarrow{i} R\Gamma_{!*}(X, F(0)) \xrightarrow{\pi} R\Gamma(X, F(0)).$$

The relation of absolute intersection cohomology of X to intersection cohomology of X with respect to a compactification will be spelled out in Corollary 4.11.

From Definition 4.1, we deduce the following.

Proposition 4.2. *Assume F to be a field contained in $\mathbb{R}$, and let $X \in Sm/\mathbb{C}$.*

(a) Absolute intersection cohomology of X is equipped with a pure polarizable F-Hodge structure. More precisely, $H^n_{!*}(X, F(0))$ is pure (and polarizable) of weight n, for all $n \in \mathbb{Z}$.

(b) Absolute intersection cohomology of X is well-defined up to isomorphism of F-Hodge structures.

Remember that for $n \in \mathbb{Z}$, the mixed graded-polarizable F-Hodge structures on $H^n_c(X, F(0))$ and on $H^n(X, F(0))$ are of weights $\leqslant n$ and $\geqslant n$, respectively. Denote by W_r the r-th filtration step of the weight filtration of a mixed Hodge structure, and by Gr^W_r the quotient of W_r by W_{r-1}, $r \in \mathbb{Z}$.

Corollary 4.3. *Assume F to be a field contained in $\mathbb{R}$, let $X \in Sm/\mathbb{C}$ and $n \in \mathbb{Z}$. Then $i\colon H^n_c(X, F(0)) \to H^n_{!*}(X, F(0))$ factors uniquely over*

$$\mathrm{Gr}^W_n H^n_c(X, F(0)) = \frac{H^n_c(X, F(0))}{W_{n-1} H^n_c(X, F(0))} :$$

$$i\colon H^n_c(X, F(0)) \twoheadrightarrow \mathrm{Gr}^W_n H^n_c(X, F(0)) \xrightarrow{i_n} H^n_{!*}(X, F(0)),$$

and $\pi\colon H^n_{!*}(X, F(0)) \to H^n(X, F(0))$ factors uniquely over

$$\mathrm{Gr}^W_n H^n(X, F(0)) = W_n H^n(X, F(0)) :$$

$$\pi\colon H^n_{!*}(X, F(0)) \xrightarrow{\pi_n} \mathrm{Gr}^W_n H^n(X, F(0)) \hookrightarrow H^n(X, F(0)).$$

Definition 4.4. Assume F to be a field contained in $\mathbb{R}$, let $X \in Sm/\mathbb{C}$ and $n \in \mathbb{Z}$. Denote by u_n the canonical morphism

$$\mathrm{Gr}^W_n H^n_c(X, F(0)) \longrightarrow \mathrm{Gr}^W_n H^n(X, F(0))$$

induced by $H^n_c(X, F(0)) \to H^n(X, F(0))$.

Thus, any choice of absolute intersection cohomology of X yields a factorization

$$\mathrm{Gr}_n^W H_c^n(X, F(0)) \xrightarrow{i_n} H_{!*}^n(X, F(0)) \xrightarrow{\pi_n} \mathrm{Gr}_n^W H^n(X, F(0))$$

of u_n, for all $n \in \mathbb{Z}$. As we shall see, this factorization can be used to characterize absolute intersection cohomology up to isomorphism.

It turns out that the most appropriate context to formulate the result is purely abstract. Fix a semi-simple Abelian category $\mathfrak{A}$.

Definition 4.5. Let $v\colon S \to T$ be a morphism in $\mathfrak{A}$. Define

$$\mathcal{H}(v) := \ker(v) \oplus \mathrm{im}(v) \oplus \mathrm{coker}(v).$$

Monomorphisms and epimorphisms being split in $\mathfrak{A}$,

$$\mathcal{H}(v) \cong S \oplus \mathrm{coker}(v) \quad \text{and} \quad \mathcal{H}(v) \cong \ker(v) \oplus T,$$

the isomorphisms being in general non-canonical. In particular, $\mathcal{H}(v) \cong S$ if v is an epimorphism, and $\mathcal{H}(v) \cong T$ if it is a monomorphism. The inclusion of $\mathrm{im}(v)$ into $\mathcal{H}(v)$, or equivalently, the projection from $\mathcal{H}(v)$ to $\mathrm{im}(v)$ is an isomorphism if and only if v is an isomorphism.

Proposition 4.6. *Let $v\colon S \to T$ be a morphism in $\mathfrak{A}$. There exists a factorization*

$$S \xrightarrow{i^{\mathcal{H}}} \mathcal{H}(v) \xrightarrow{\pi^{\mathcal{H}}} T$$

of v with the following properties:

(1) $i^{\mathcal{H}}$ is a monomorphism, and $\pi^{\mathcal{H}}$ is an epimorphism.

(2) The restriction of $i^{\mathcal{H}}$ to $\ker(v)$ equals the inclusion i_1 of the first component into $\mathcal{H}(v) = \ker(v) \oplus \mathrm{im}(v) \oplus \mathrm{coker}(v)$, and the composition of $\pi^{\mathcal{H}}$ with the quotient map to $\mathrm{coker}(v)$ equals the projection π_3 from $\mathcal{H}(v) = \ker(v) \oplus \mathrm{im}(v) \oplus \mathrm{coker}(v)$ to the last component. In particular,

$$\pi_3 \circ i^{\mathcal{H}} = 0\colon S \longrightarrow \mathrm{coker}(v) \quad \textit{and} \quad \pi^{\mathcal{H}} \circ i_1 = 0\colon \ker(v) \longrightarrow T.$$

Proof. Choose a left inverse s of the inclusion of $\ker(v)$ into S, and a right inverse t of the quotient map from T to $\mathrm{coker}(v)$. Define

$$i^{\mathcal{H}} := (s, q, 0)\colon S \longrightarrow \mathcal{H}(v) = \ker(v) \oplus \mathrm{im}(v) \oplus \mathrm{coker}(v),$$

where q is the quotient map from S to $\mathrm{im}(v)$, and

$$\pi^{\mathcal{H}} := 0 + \iota + t\colon \mathcal{H}(v) = \ker(v) \oplus \mathrm{im}(v) \oplus \mathrm{coker}(v) \longrightarrow T,$$

where ι is the inclusion of $\mathrm{im}(v)$ into T. □

Again, factorizations as in Proposition 4.6 are in general not unique (they *are* unique for trivial reasons, if v is an isomorphism). Still, any given choice satisfies a “versal” property, as we are about to see.

Theorem 4.7. *Let $v\colon S \to T$ be a morphism in $\mathfrak{A}$. Fix a factorization*

$$S \xrightarrow{i^{\mathcal{H}}} \mathcal{H}(v) \xrightarrow{\pi^{\mathcal{H}}} T$$

of v as in Proposition 4.6. Let

$$S \xrightarrow{k} H \xrightarrow{p} T$$

be any factorization of v, with a monomorphism k and an epimorphism p. Then there is an object H' of $\mathfrak{A}$ and an isomorphism

$$H \cong \mathcal{H}(v) \oplus H'$$

compatible with the factorizations. In other words, denoting by ι the inclusion of, and by q the quotient map to $\mathcal{H}(v)$, the diagrams

$$\begin{array}{ccccc} S & \xrightarrow{i^{\mathcal{H}}} & \mathcal{H}(v) & \xrightarrow{\pi^{\mathcal{H}}} & T \\ \| & & \downarrow{\scriptstyle\iota} & & \| \\ S & \xrightarrow{k} & H & \xrightarrow{p} & T \end{array}$$

and

$$\begin{array}{ccccc} S & \xrightarrow{k} & H & \xrightarrow{p} & T \\ \| & & \downarrow{\scriptstyle q} & & \| \\ S & \xrightarrow{i^{\mathcal{H}}} & \mathcal{H}(v) & \xrightarrow{\pi^{\mathcal{H}}} & T \end{array}$$

commute.

Proof. Denote by s the composition

$$S \xrightarrow{i^{\mathcal{H}}} \mathcal{H}(v) \xrightarrow{\pi_1} \ker(v),$$

where π_1 denotes the projection from $\mathcal{H}(v)$ to the first component $\ker(v)$. According to property 4.6 (2), the morphism s is a left inverse of the inclusion of $\ker(v)$ into S. Let us show first that there is a left inverse of the inclusion of $\ker(p)$ into H, which is compatible with s in the sense that the diagram

$$\begin{array}{ccc} S & \xrightarrow{s} & \ker(v) \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k} \\ H & \longrightarrow & \ker(p) \end{array}$$

commutes; we shall denote such a left inverse by the same letter s.

Choose a complement H' of $k(\ker(v))$ in $\ker(p)$. As $k^{-1}(\ker(p)) = \ker(v)$, the intersection of $\mathrm{im}(k)$ and H' is trivial. Therefore,

$$\ker(p) = k(\ker(v)) \oplus H' \hookrightarrow \mathrm{im}(k) \oplus H' \subset H.$$

Choose a complement H'' of $\mathrm{im}(k) \oplus H'$ in H, and define

$$s := (k(s), \mathrm{id}_{H'}, 0)\colon H = \mathrm{im}(k) \oplus H' \oplus H'' \longrightarrow \ker(p) = k(\ker(v)) \oplus H' \oplus 0.$$

Then, using s, we have

$$\mathcal{H}(v) = \ker(v) \oplus T \quad \text{and} \quad H = \ker(p) \oplus T,$$

T being in both cases identified with $\ker(s)$. We define

$$\iota := (k, \mathrm{id}_T) \colon \mathcal{H}(v) \longrightarrow H$$

and

$$q := (k^{-1} \circ \pi_1', \mathrm{id}_T) \colon H \longrightarrow \mathcal{H}(v),$$

where π_1' is the left inverse of $k \colon \ker(v) \hookrightarrow \ker(p)$ with kernel equal to H'. $\square$

Let us get back to the geometric situation considered in the beginning of this section.

Theorem 4.8. *Assume F to be a field contained in $\mathbb{R}$, let $X \in Sm/\mathbb{C}$ and $n \in \mathbb{Z}$. Recall the morphism $u_n \colon \mathrm{Gr}_n^W H_c^n(X, F(0)) \to \mathrm{Gr}_n^W H^n(X, F(0))$ from Definition 4.4. Fix a choice of $H_{!*}^n(X, F(0))$, together with the associated factorization*

$$\mathrm{Gr}_n^W H_c^n(X, F(0)) \xrightarrow{i_n} H_{!*}^n(X, F(0)) \xrightarrow{\pi_n} \mathrm{Gr}_n^W H^n(X, F(0))$$

of u_n.

(a) The morphism i_n is injective, and π_n is surjective.

(b) In the semi-simple Abelian category of polarizable F-Hodge structures, which are pure of weight n,

$$H_{!*}^n(X, F(0)) \cong \mathcal{H}(u_n),$$

and the isomorphism can be chosen such that $i_n = i^{\mathcal{H}}$ and $\pi_n = \pi^{\mathcal{H}}$.

Proof. According to Definition 4.1, absolute intersection cohomology is the collection of cohomology objects of a choice of Hodge theoretic absolute intersection complex $R\Gamma_{!*}(X, F(0))$. This choice induces a factorization

$$R\Gamma_c(X, F(0)) \xrightarrow{i} R\Gamma_{!*}(X, F(0)) \xrightarrow{\pi} R\Gamma(X, F(0)),$$

which on the level of cohomology yields the factorization

$$H_c^n(X, F(0)) \longrightarrow H_{!*}^n(X, F(0)) \longrightarrow H^n(X, F(0)).$$

According to Definition 3.6, the above objects and morphisms fit into a diagram

$$\begin{CD}
0 @>>> C_{\geqslant 1}[-1] @= C_{\geqslant 1}[-1] @>>> 0 \\
@VVV @VVV @VV{\delta[-1]}V @VVV \\
R\Gamma_c(X, F(0)) @>{i}>> R\Gamma_{!*}(X, F(0)) @>>> C_{\leqslant 0} @>>> R\Gamma_c(X, F(0))[1] \\
@| @VV{\pi}V @VVV @| \\
R\Gamma_c(X, F(0)) @>{u}>> R\Gamma(X, F(0)) @>>> C @>>> R\Gamma_c(X, F(0))[1] \\
@VVV @VVV @VVV @VVV \\
0 @>>> C_{\geqslant 1} @= C_{\geqslant 1} @>>> 0
\end{CD}$$

of exact triangles in $D^b(\mathbf{MHS}_F)$, where $\delta \colon C_{\geqslant 1} \to C_{\leqslant 0}[1]$ belongs to the radical.

On the level of cohomology, the above induces a diagram of exact sequences, the part of interest of which looks as follows:

$$\begin{array}{ccccccc}
H^{n-2}(C_{\geqslant 1}) & \longrightarrow & 0 & \longrightarrow & H^{n-1}(C_{\geqslant 1}) & = & H^{n-1}(C_{\geqslant 1}) \\
\downarrow{\scriptstyle H^{n-2}(\delta)} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle H^{n-1}(\delta)} \\
H^{n-1}(C_{\leqslant 0}) & \longrightarrow & H^n_c(X,F(0)) & \xrightarrow{i} & H^n_{!*}(X,F(0)) & \longrightarrow & H^n(C_{\leqslant 0}) \\
\downarrow & & \| & & \downarrow{\scriptstyle \pi} & & \downarrow \\
H^{n-1}(C) & \longrightarrow & H^n_c(X,F(0)) & \longrightarrow & H^n(X,F(0)) & \longrightarrow & H^n(C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{n-1}(C_{\geqslant 1}) & \longrightarrow & 0 & \longrightarrow & H^n(C_{\geqslant 1}) & = & H^n(C_{\geqslant 1})
\end{array}$$

According to Corollary 2.10, the first and last columns equal the weight filtrations of $H^{n-1}(C)$ and $H^n(C)$, concentrated at n and $n+1$, respectively:

$$\begin{array}{ccccccc}
 & & & & \frac{H^{n-1}(C)}{W_{n-1}H^{n-1}(C)} & & \\
 & & & & \downarrow & & \\
W_{n-1}H^{n-1}(C) & \longrightarrow & H^n_c(X,F(0)) & \xrightarrow{i} & H^n_{!*}(X,F(0)) & \longrightarrow & W_nH^n(C) \\
\downarrow & & \| & & \downarrow{\scriptstyle \pi} & & \downarrow \\
H^{n-1}(C) & \longrightarrow & H^n_c(X,F(0)) & \longrightarrow & H^n(X,F(0)) & \longrightarrow & H^n(C) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \frac{H^n(C)}{W_nH^n(C)} & = & \frac{H^n(C)}{W_nH^n(C)}
\end{array}$$

Applying Gr_n^W, the sequences remain exact, yielding

$$\begin{array}{ccccc}
 & & & & \mathrm{Gr}_n^W H^{n-1}(C) \\
 & & & & \downarrow \\
0 & \longrightarrow & \mathrm{Gr}_n^W H^n_c(X,F(0)) & \xrightarrow{i_n} & H^n_{!*}(X,F(0)) \\
\downarrow & & \| & & \downarrow{\scriptstyle \pi_n} \\
\mathrm{Gr}_n^W H^{n-1}(C) & \longrightarrow & \mathrm{Gr}_n^W H^n_c(X,F(0)) & \xrightarrow{u_n} & \mathrm{Gr}_n^W H^n(X,F(0)) \\
 & & & & \downarrow \\
 & & & & 0
\end{array}$$

This diagram shows that i_n is injective, and that π_n is surjective, proving part (a) of our claim.

The last column also shows that $\ker(\pi_n)$ is the image of $\mathrm{Gr}_n^W H^{n-1}(C)$ in $H^n_{!*}(X,F(0))$. The morphism $\mathrm{Gr}_n^W H^{n-1}(C) \to H^n_{!*}(X,F(0))$ factors over $\mathrm{Gr}_n^W H^n_c(X,F(0))$. The image of $\mathrm{Gr}_n^W H^{n-1}(C) \to \mathrm{Gr}_n^W H^n_c(X,F(0))$ equals $\ker(u_n)$, according to the second line. The morphism i_n being injective, we get a canonical short exact sequence

$$0 \longrightarrow \ker(u_n) \longrightarrow H^n_{!*}(X,F(0)) \xrightarrow{\pi_n} \mathrm{Gr}_n^W H^n(X,F(0)) \longrightarrow 0,$$

which is (in general non-canonically) split. This shows that

$$H^n_{!*}(X, F(0)) \cong \ker(u_n) \oplus \mathrm{Gr}^W_n H^n(X, F(0)),$$

which in turn is isomorphic to $\mathcal{H}(u_n)$. Applying Theorem 4.7 to

$$\mathrm{Gr}^W_n H^n_c(X, F(0)) \xrightarrow{i_n} H^n_{!*}(X, F(0)) \xrightarrow{\pi_n} \mathrm{Gr}^W_n H^n(X, F(0))$$

(using part (a)), we see that the isomorphism $H^n_{!*}(X, F(0)) \xrightarrow{\sim} \mathcal{H}(u_n)$ can be chosen in a way compatible with the factorizations. □

Recall that for $X \in Sm/\mathbb{C}$ and $n \in \mathbb{Z}$, *interior cohomology* $H^n_!(X, F(0))$ is defined as the image of the canonical morphism $H^n_c(X,F(0)) \to H^n(X,F(0))$. Given that $H^n_c(X, F(0))$ and $H^n(X, F(0))$ are of weights $\leqslant n$ and $\geqslant n$, respectively,

$$H^n_!(X, F(0)) = \mathrm{im}(u_n),$$

where $u_n\colon \mathrm{Gr}^W_n H^n_c(X, F(0)) \to \mathrm{Gr}^W_n H^n(X, F(0))$ is as before.

Corollary 4.9. *Assume F to be a field contained in $\mathbb{R}$, let $X \in Sm/\mathbb{C}$ and $n \in \mathbb{Z}$. Then interior cohomology $H^n_!(X, F(0))$ is a direct factor of any choice of absolute intersection cohomology $H^n_{!*}(X, F(0))$. The two are canonically isomorphic if and only if $u_n\colon \mathrm{Gr}^W_n H^n_c(X, F(0)) \to \mathrm{Gr}^W_n H^n(X, F(0))$ is an isomorphism.*

Proof. On the one hand, $H^n_!(X, F(0)) = \mathrm{im}(u_n)$. On the other hand, $\mathcal{H}(u_n)$ contains $\mathrm{im}(u_n)$ as a direct factor, with complement $\ker(u_n) \oplus \mathrm{coker}(u_n)$. Now apply Theorem 4.8 (b). □

Note that a sufficient condition for u_n to be an isomorphism is that *boundary cohomology* $\partial H^r(X, F(0))$ avoids weight n in degrees $r = n - 1$ and $r = n$.

Remark 4.10. It happens rarely that $\partial H^r(X, F(0))$ avoids weights r and $r+1$ in all degrees r. Note however that all constructions and results concerning Hodge structures on cohomology established in this section admit obvious analogues in the context considered in Variant 3.7 (b), *i.e.*, in the presence of an idempotent endomorphism e, and that there exist non-trivial examples [33, 34, 9, 29] where $\partial H^r(X, F(0))^e$ *does* avoid weights r and $r + 1$ in all degrees.

Corollary 4.11. *Assume F to be a field contained in $\mathbb{R}$, let $X \in Sm/\mathbb{C}$ and $j\colon X \hookrightarrow \widetilde{X}$ any compactification of X. Then absolute intersection cohomology $H^n_{!*}(X, F(0))$ is a direct factor of intersection cohomology of X with respect to j,* i.e., *of $H^n(\widetilde{X}, j_{!*} F(0))$, for all $n \in \mathbb{Z}$.*

A word of explanation is in order. Since X is smooth, the variation of Hodge structure $F(0)$ on X can be seen as a *Hodge module*, up to a shift by $-\dim_X$:

$$F(0) = (F(0)[\dim_X])[-\dim_X],$$

see [25, Thm. 3.27]. By slight abuse of notation, we define

$$j_{!*}\, F(0) := (j_{!*}\, F(0)[\dim_X])[-\dim_X]$$

[25, Sect. 4.5]; this definition, together with the formalism of six operations on the bounded derived category of Hodge modules, in particular [25, Thm. 4.3], yields the Hodge structure on $H^n(\widetilde{X}, j_{!*}\, F(0))$.

Proof of Corollary 4.11. The morphism u_n factors canonically through the pure polarizable Hodge structure on $H^n(\widetilde{X}, j_{!*}\, F(0))$:

$$\mathrm{Gr}_n^W\, H_c^n(X, F(0)) \stackrel{k}{\longrightarrow} H^n(\widetilde{X}, j_{!*}\, F(0)) \stackrel{p}{\longrightarrow} \mathrm{Gr}_n^W\, H^n(X, F(0)) \,.$$

Denote by i the closed immersion of the complement of X into $\widetilde{X}$, and by $\widetilde{a}$ the structural morphism of $\widetilde{X}$. The exact localization triangle [25, (4.4.1)]

$$j_{!*}\, F(0) \longrightarrow j_*F(0) \longrightarrow i_*i^!\, j_{!*}\, F(0)[1] \longrightarrow j_{!*}\, F(0)[1]$$

is a weight filtration concentrated at 1 of $j_*F(0)$ [25, (4.5.9), (4.5.2)]. The morphism $\widetilde{a}$ being proper, the direct image $\widetilde{a}_*$ preserves weights [25, (4.5.2)]. Therefore,

$$\widetilde{a}_*\, j_{!*}\, F(0) \longrightarrow \widetilde{a}_*j_*F(0) \longrightarrow \widetilde{a}_*i_*i^!\, j_{!*}\, F(0)[1] \longrightarrow \widetilde{a}_*\, j_{!*}\, F(0)[1]$$

is a weight filtration concentrated at 1 of $\widetilde{a}_*j_*F(0)$.

It follows that the morphism $p\colon H^n(\widetilde{X}, j_{!*}\, F(0)) \to \mathrm{Gr}_n^W\, H^n(X, F(0))$ is surjective for all integers n. Dually, the morphism k is injective. Now apply Theorems 4.8 (b) and 4.7. □

Remark 4.12. Both complexes $R\Gamma_{!*}(X, F(0))$ and $R\Gamma(\widetilde{X}, j_{!*}\, F(0))$ being pure of weight zero, they are (in general, non-canonically) isomorphic to the direct sum of their n-th cohomology objects, shifted by n. According to Corollary 4.11, the complex $R\Gamma_{!*}(X, F(0))$ is therefore isomorphic to a direct factor of $R\Gamma(\widetilde{X}, j_{!*}\, F(0))$. We do not know whether the isomorphism can be chosen in a way compatible with the factorizations of

$$R\Gamma_c(X, F(0)) \longrightarrow R\Gamma(X, F(0))$$

through $R\Gamma_{!*}(X, F(0))$ and through $R\Gamma(\widetilde{X}, j_{!*}\, F(0))$ (see Remark 3.4 (a)).

Corollary 4.13. *Assume F to be a field contained in $\mathbb{R}$, let $X \in Sm/\mathbb{C}$ and $\widetilde{X}$ any smooth compactification of X. Then absolute intersection cohomology $H^n_{!*}(X, F(0))$ is a direct factor of $H^n(\widetilde{X}, F(0))$, for all $n \in \mathbb{Z}$.*

5. Examples

Let us begin by recalling a special case of a result of Voevodsky.

Theorem 5.1. *Let r and s be two integers. Then*

$$\mathrm{Hom}_{DM_{gm}(k)_F}\big(\mathbb{1}(r)[2r], \mathbb{1}(s)[2s]\big) = 0 \quad \textit{if} \quad r \neq s.$$

Proof. This is a special case of [27, Cor. 2] ([28, Prop. 4.2.9] if k admits resolution of singularities). $\square$

In the sequel, we assume that the base field k admits resolution of singularities. For $X \in Sm/k$, repeat Construction 3.1 for the canonical morphism $u\colon M_{gm}(X) \to M^c_{gm}(X)$ and (the shift by $+1$ of) a weight filtration

$$\partial M_{\leqslant -1} \longrightarrow \partial M_{gm}(X) \longrightarrow \partial M_{\geqslant 0} \stackrel{d}{\longrightarrow} \partial M_{\leqslant -1}[1]$$

of $\partial M_{gm}(X)$ (with $\partial M_{\leqslant -1} \in \mathcal{C}_{w \leqslant -1}$ and $\partial M_{\geqslant 0} \in \mathcal{C}_{w \geqslant 0}$):

$$\begin{array}{ccccccc}
0 & \longrightarrow & \partial M_{\geqslant 0} & = & \partial M_{\geqslant 0} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle d_+} & & \downarrow{\scriptstyle d} & & \downarrow \\
M_{gm}(X) & \stackrel{i}{\longrightarrow} & M & \stackrel{d_-}{\longrightarrow} & \partial M_{\leqslant -1}[1] & \longrightarrow & M_{gm}(X)[1] \\
\| & & \downarrow{\scriptstyle \pi} & & \downarrow & & \| \\
M_{gm}(X) & \stackrel{u}{\longrightarrow} & M^c_{gm}(X) & \longrightarrow & \partial M_{gm}(X)[1] & \longrightarrow & M_{gm}(X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \partial M_{\geqslant 0}[1] & = & \partial M_{\geqslant 0}[1] & \longrightarrow & 0
\end{array}$$

The motive M is pure of weight zero, *i.e.*, it is a Chow motive,

$$M_{gm}(X) \stackrel{i}{\longrightarrow} M \stackrel{\pi}{\longrightarrow} M^c_{gm}(X)$$

is a factorization of u, and

$$\partial M_{\leqslant -1} \longrightarrow M_{gm}(X) \stackrel{i}{\longrightarrow} M \stackrel{d_-}{\longrightarrow} \partial M_{\leqslant -1}[1]$$

and

$$M \stackrel{\pi}{\longrightarrow} M^c_{gm}(X) \longrightarrow \partial M_{\geqslant 0}[1] \stackrel{d_+[1]}{\longrightarrow} M[1]$$

are weight filtrations of $M_{gm}(X)$ and of $M^c_{gm}(X)$, respectively. Proposition 3.2 tells us that, up to isomorphism, the process is reversible: a triple (M, i, π) leads to a weight filtration $(\partial M_{\leqslant -1}, \partial M_{\geqslant 0})$ of $\partial M_{gm}(X)$.

According to Definition 3.5, the triple (M, i, π) is an absolute intersection motive of X (in which case we write $M = M^{!*}(X)$) if and only if it is a minimal factorization of u, *i.e.*, if and only if the morphism

$$\partial M_{\geqslant 0} \stackrel{d}{\longrightarrow} \partial M_{\leqslant -1}[1]$$

belongs to the radical $\mathrm{rad}_{DM_{gm}(k)_F}(\partial M_{\geqslant 0}, \partial M_{\leqslant -1}[1])$.

Example 5.2. Put $X = \mathbb{A}^1_k$. The morphism $u\colon M_{gm}(\mathbb{A}^1_k) \to M^c_{gm}(\mathbb{A}^1_k)$ factors canonically through the Chow motive $M_{gm}(\mathbb{P}^1_k)$:

$$u\colon M_{gm}(\mathbb{A}^1_k) \stackrel{j_*}{\longrightarrow} M_{gm}(\mathbb{P}^1_k) = M^c_{gm}(\mathbb{P}^1_k) \stackrel{j^*}{\longrightarrow} M^c_{gm}(\mathbb{A}^1_k),$$

where j^* and j_* denote the morphisms induced by the open immersion j of $\mathbb{A}^1_k$ into $\mathbb{P}^1_k$. We claim that $(M_{gm}(\mathbb{P}^1_k), j_*, j^*)$ is a minimal factorization of u.

Indeed, both j^* and j_* can be completed to give weight filtrations of $M_{gm}(\mathbb{A}^1_k)$ and of $M^c_{gm}(\mathbb{A}^1_k)$, respectively: the inclusion of the point at infinity of $M_{gm}(\mathbb{P}^1_k)$ yields exact (*purity* and *localization*) triangles

$$\mathbb{1}(1)[1] \longrightarrow M_{gm}(\mathbb{A}^1_k) \xrightarrow{j_*} M_{gm}(\mathbb{P}^1_k) \xrightarrow{d_-} \mathbb{1}(1)[2]$$

[28, Prop. 3.5.4] and

$$M_{gm}(\mathbb{P}^1_k) \xrightarrow{j^*} M^c_{gm}(\mathbb{A}^1_k) \longrightarrow \mathbb{1}(0)[1] \xrightarrow{d_+[1]} M_{gm}(\mathbb{P}^1_k)[1]$$

[28, Prop. 4.1.5]. We thus get a diagram of exact triangles

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{1}(0) & = & \mathbb{1}(0) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle d_+} & & \downarrow{\scriptstyle d} & & \downarrow \\
M_{gm}(\mathbb{A}^1_k) & \xrightarrow{j_*} & M_{gm}(\mathbb{P}^1_k) & \xrightarrow{d_-} & \mathbb{1}(1)[2] & \longrightarrow & M_{gm}(\mathbb{A}^1_k)[1] \\
\| & & \downarrow{\scriptstyle j^*} & & \downarrow & & \| \\
M_{gm}(\mathbb{A}^1_k) & \xrightarrow{u} & M^c_{gm}(\mathbb{A}^1_k) & \longrightarrow & \partial M_{gm}(\mathbb{A}^1_k)[1] & \longrightarrow & M_{gm}(\mathbb{A}^1_k)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{1}(0)[1] & = & \mathbb{1}(0)[1] & \longrightarrow & 0
\end{array}$$

The morphism $d\colon \mathbb{1}(0) \to \mathbb{1}(1)[2]$ is zero according to Theorem 5.1. *A fortiori*, it belongs to $\mathrm{rad}_{DM_{gm}(k)_F}(\mathbb{1}(0), \mathbb{1}(1)[2])$. Therefore, $(M_{gm}(\mathbb{P}^1_k), j_*, j^*)$ is minimal; in particular,

$$M^{!*}(\mathbb{A}^1_k) = M_{gm}(\mathbb{P}^1_k).$$

The third column of the above diagram also shows that

$$\partial M_{gm}(\mathbb{A}^1_k) \cong \mathbb{1}(1)[1] \oplus \mathbb{1}(0).$$

Exercice 5.3. More generally, if X is the complement of a zero-dimensional sub-scheme Z in a smooth, proper k-scheme Y, of pure dimension $n \geqslant 1$, then

$$M^{!*}(X) = M_{gm}(Y)$$

and

$$\partial M_{gm}(X) \cong M_{gm}(Z)(n)[2n-1] \oplus M_{gm}(Z).$$

Example 5.4. To generalize further, assume that X is the complement of a smooth closed sub-scheme Z in a smooth, proper k-scheme Y, and that the immersion of Z into Y is of pure codimension c. Again, we have weight filtrations of $M_{gm}(X)$ and of $M^c_{gm}(X)$, respectively, coming from purity and localization:

$$M_{gm}(Z)(c)[2c-1] \longrightarrow M_{gm}(X) \xrightarrow{j_*} M_{gm}(Y) \xrightarrow{d_-} M_{gm}(Z)(c)[2c]$$

[28, Prop. 3.5.4] and

$$M_{gm}(Y) \xrightarrow{j^*} M^c_{gm}(X) \longrightarrow M_{gm}(Z)[1] \xrightarrow{d_+[1]} M_{gm}(Y)[1]$$

[28, Prop. 4.1.5]. We get

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_{gm}(Z) & = & M_{gm}(Z) & \longrightarrow & 0 \\
\downarrow & & {\scriptstyle d_+}\downarrow & & \downarrow{\scriptstyle d} & & \downarrow \\
M_{gm}(X) & \xrightarrow{j_*} & M_{gm}(Y) & \xrightarrow{d_-} & M_{gm}(Z)(c)[2c] & \longrightarrow & M_{gm}(X)[1] \\
\| & & {\scriptstyle j^*}\downarrow & & \downarrow & & \| \\
M_{gm}(X) & \xrightarrow{u} & M_{gm}^c(X) & \longrightarrow & \partial M_{gm}(X)[1] & \longrightarrow & M_{gm}(X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_{gm}(Z)[1] & = & M_{gm}(Z)[1] & \longrightarrow & 0
\end{array}$$

The (shift by -1 of the) third column is clearly a weight filtration of the boundary motive $\partial M_{gm}(X)$; the question is to determine whether it is minimal, *i.e.*, whether the morphism

$$M_{gm}(Z) \xrightarrow{d} M_{gm}(Z)(c)[2c]$$

belongs to the radical $\mathrm{rad}_{DM_{gm}(k)_F}(M_{gm}(Z), M_{gm}(Z)(c)[2c])$. Observe that according to [28, Thm. 4.3.7 3., Prop. 4.2.9],

$$\mathrm{Hom}_{DM_{gm}(k)_F}\big(M_{gm}(Z), M_{gm}(Z)(c)[2c]\big) = CH^{\dim_X}(Z \times_k Z)_F .$$

(a) Assume that the dimension of Z is strictly smaller than c. Then

$$CH^{\dim_X}(Z \times_k Z)_F = 0.$$

Therefore, the morphism d is zero, $(M_{gm}(Y), j_*, j^*)$ is minimal,

$$M^{!*}(X) = M_{gm}(Y),$$

and

$$\partial M_{gm}(X) \cong M_{gm}(Z)(c)[2c-1] \oplus M_{gm}(Z).$$

These four statements remain true, up to replacing $M_{gm}(Z)(c)[2c-1]$ by $M_{gm}(Z)^*(\dim_X)[2\dim_X - 1]$, if the smoothness assumption on Z is dropped. (Hint: use induction on the dimension of $T \in Sch/k$ to show that, for all non-negative integers j,

$$\mathrm{Hom}_{DM_{gm}(k)_F}(M_{gm}(T), \mathbb{1}(r)[2r-j]) = 0 \quad \text{if} \quad r > \dim_T + j.$$

For the induction step, use [28, Prop. 4.1.3]. Apply this to $T = Z \times_k Z$, $j = 0$ and $r = \dim_X$.)

(b) By contrast, without the condition on the codimension from (a), the morphism

$$M_{gm}(Z) \xrightarrow{d} M_{gm}(Z)(c)[2c],$$

which corresponds to a class in the Chow group

$$CH^{\dim_X}(Z \times_k Z)_F,$$

can *a priori* be non-zero. Actually, it does happen that d does not belong to the radical, meaning that $(M_{gm}(Y), j_*, j^*)$ is not minimal. This is the case for $X = \mathbb{A}^n_k$ and $Y = \mathbb{P}^n_k$, $n \geqslant 2$; we refer to Example 5.10 below.

(c) Consider the identification

$$\mathrm{Hom}_{DM_{gm}(k)_F}\big(M_{gm}(Z), M_{gm}(Z)(c)[2c]\big) = CH^{\dim X}(Z \times_k Z)_F.$$

According to [11, Cor. 2.33], the morphism $d\colon M_{gm}(Z) \to M_{gm}(Z)(c)[2c]$ corresponds to the *Lefschetz operator* $L_{i^*[Z]}$ associated to the "self intersection" $i^*[Z]$, defined as the image under

$$\Delta_{Z,*}\colon CH^c(Z)_F \longrightarrow CH^{\dim X}(Z \times_k Z)_F$$

of $i^*[Z]$, where $[Z]$ denotes the class of the cycle Z in $CH^c(Y)_F$, and i^* the pull-back from $CH^c(Y)_F$ to $CH^c(Z)_F$.

Alternatively, given the definition of composition of cycles, we see that d corresponds to the image of the class

$$[\Delta_Z] \in CH^{\dim X}(Z \times_k Y)_F$$

of the diagonal $\Delta_Z \subset Z \times_k Z \subset Z \times_k Y$ under the pull-back

$$CH^{\dim X}(Z \times_k Y)_F \longrightarrow CH^{\dim X}(Z \times_k Z)_F,$$

which by [21, Lemma 1.1 (i)]) equals $L_{i^*[Z]}$.

(d) Assume that $c = 1$, *i.e.*, that Z is a smooth divisor. The morphism

$$M_{gm}(Z) \xrightarrow{\ d\ } M_{gm}(Z)(1)[2]$$

corresponds to

$$L_{i^*[Z]} \in CH^{\dim X}(Z \times_k Z)_F.$$

Assume that $i^*[Z]$ or its opposite is ample. Assume also that $M_{gm}(Z)$ satisfies the following *weak form of the Lefschetz decomposition* for $L_{i^*[Z]}$: there exist two decompositions

$$M_{gm}(Z) = P_1 \oplus M_{gm}(Z)_1^s \quad \text{and} \quad M_{gm}(Z) = P_2 \oplus M_{gm}(Z)_2^s,$$

such that $L_{i^*[Z]}$ equals zero on P_1, maps the sub-motive $M_{gm}(Z)_1^s$ of $M_{gm}(Z)$ to the sub-motive $M_{gm}(Z)_2^s(1)[2]$ of $M_{gm}(Z)(1)[2]$, and induces an isomorphism $M_{gm}(Z)_1^s \xrightarrow{\ \sim\ } M_{gm}(Z)_2^s(1)[2]$.

This means that the zero morphism $P_1 \to P_2(1)[2]$ and d have isomorphic cones. Thus,

$$P_2(1)[1] \longrightarrow \partial M_{gm}(X) \longrightarrow P_1 \xrightarrow{\ 0\ } P_2(1)[2]$$

is a minimal weight filtration of $\partial M_{gm}(X)$. The absolute intersection motive $M^{!*}(X)$ can be constructed out of $M_{gm}(Y)$ as follows. Start with the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_{gm}(Z) & = & M_{gm}(Z) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle d_+} & & \downarrow{\scriptstyle d} & & \downarrow \\
M_{gm}(X) & \xrightarrow{j_*} & M_{gm}(Y) & \xrightarrow{d_-} & M_{gm}(Z)(1)[2] & \longrightarrow & M_{gm}(X)[1] \\
\| & & \downarrow{\scriptstyle j^*} & & \downarrow & & \| \\
M_{gm}(X) & \xrightarrow{u} & M^c_{gm}(X) & \longrightarrow & \partial M_{gm}(X)[1] & \longrightarrow & M_{gm}(X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_{gm}(Z)[1] & = & M_{gm}(Z)[1] & \longrightarrow & 0
\end{array}$$

The motives $M_{gm}(Z)_1^s$ and $M_{gm}(Z)_2^s(1)[2]$ are direct factors of $M_{gm}(Z)$ and $M_{gm}(Z)(1)[2]$, respectively, and the morphism $d = d_- \circ d_+$ induces an isomorphism between them. Therefore, the kernel $M'_{gm}(Y)$ of the composition

$$M_{gm}(Y) \xrightarrow{d_-} M_{gm}(Z)(1)[2] \twoheadrightarrow M_{gm}(Z)_2^s(1)[2]$$

(exists and) is a direct factor of $M_{gm}(Y)$, with complement equal to the image of the composition

$$M_{gm}(Z)_1^s \hookrightarrow M_{gm}(Z) \xrightarrow{d_+} M_{gm}(Y).$$

By definition, the morphisms j_* and j^* factor through $M'_{gm}(Y)$. We thus get the following direct factor of the above diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & P_1 & = & P_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow \\
M_{gm}(X) & \xrightarrow{j_*} & M'_{gm}(Y) & \longrightarrow & P_2(1)[2] & \longrightarrow & M_{gm}(X)[1] \\
\| & & {\scriptstyle j^*}\downarrow & & \downarrow & & \| \\
M_{gm}(X) & \xrightarrow{u} & M^c_{gm}(X) & \longrightarrow & \partial M_{gm}(X)[1] & \longrightarrow & M_{gm}(X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_1[1] & = & P_1[1] & \longrightarrow & 0
\end{array}$$

The zero morphism belonging to the radical, we see that $(M'_{gm}(Y), j_*, j^*)$ is an absolute intersection motive of X. In particular,

$$M^{!*}(X) = M'_{gm}(Y).$$

Remark 5.5. (a) Recall the classical notion of *Lefschetz decomposition for* L_D associated to an ample divisor D on a smooth, proper k-scheme Z of pure dimension r: there is a decomposition

$$M_{gm}(Z) = \bigoplus_{i=0}^{2r} \bigoplus_{m=max(0,i-r)}^{[\frac{i}{2}]} L^m P^{i-2m}$$

into Chow motives $L^m P^{i-2m}$, such that

(1) for all $0 \leqslant m \leqslant r - i - 1$, the morphism L_D induces an isomorphism between the sub-motive $L^m P^i$ of $M_{gm}(Z)$ and the sub-motive $L^{m+1} P^i(1)[2]$ of $M_{gm}(Z)(1)[2]$,

(2) for all $i \leqslant r$, the morphism L_D is zero on $L^{r-i}P^i$.

Such decompositions exist if Z is an Abelian variety [21, Thm. 5.1].

Putting

$$P_1 := \bigoplus_{i=0}^{r} L^{r-i} P^i \quad \text{and} \quad P_2 := \bigoplus_{i=0}^{r} P^i$$

(and $M_{gm}(Z)_1^s$, $M_{gm}(Z)_2^s$ equal to the sums of the respective remaining direct factors in the Lefschetz decomposition), we get

$$M_{gm}(Z) = P_1 \oplus M_{gm}(Z)_1^s \quad \text{and} \quad M_{gm}(Z) = P_2 \oplus M_{gm}(Z)_2^s,$$

such that L_D equals zero on P_1, and induces an isomorphism

$$M_{gm}(Z)_1^s \xrightarrow{\sim} M_{gm}(Z)_2^s(1)[2].$$

(b) We leave it to the reader to formulate a generalization of Example 5.4 (d) for smooth sub-schemes $i\colon Z \hookrightarrow Y$ of pure codimension c, such that $i^*[Z]$ is the c-fold self intersection of the class of an ample divisor on Z.

Let us come back to the situation considered in Example 5.4.

Theorem 5.6. *Let Z be a smooth, geometrically connected closed sub-scheme of pure dimension $c \geqslant 1$ of a smooth, proper k-scheme Y of pure dimension $2c$. Denote by i the closed immersion of Z, and by j the open immersion of its complement X into Y. Consider the morphism*

$$M_{gm}(Z) \xrightarrow{d} M_{gm}(Z)(c)[2c]$$

corresponding to

$$L_{i^*[Z]} \in CH^{2c}(Z \times_k Z)_F = CH_0(Z \times_k Z)_F.$$

(a) The morphism d belongs to $\mathrm{rad}_{DM_{gm}(k)_F}(M_{gm}(Z), M_{gm}(Z)(c)[2c])$ if and only if the self intersection number $Z \cdot Z$ of Z in Y equals zero.

(b) If $Z \cdot Z \neq 0$, then there exist unique decompositions

$$M_{gm}(Z) = M_1^r \oplus M_1^s, \quad M_{gm}(Z)(c)[2c] = M_2^r \oplus M_2^s,$$

such that

(1) the decompositions are respected by d:

$$d = d^r \oplus d^s \in \mathrm{Hom}_{DM_{gm}(k)_F}\big(M_{gm}(Z), M_{gm}(Z)(c)[2c]\big),$$

(2) the morphism d^r belongs to $\mathrm{rad}_{DM_{gm}(k)_F}(M_1^r, M_2^r)$,

(3) and the morphism d^s is an isomorphism $M_1^s \xrightarrow{\sim} M_2^s$.

Furthermore, the motives M_1^s and M_2^s are canonically isomorphic to $\mathbb{1}(c)[2c]$, and under these isomorphisms,

(4) the composition $\mathbb{1}(c)[2c] \xrightarrow{\sim} M_1^s \hookrightarrow M_{gm}(Z)$ corresponds to the inclusion into $M_{gm}(Z)$ of the canonical sub-motive $M_{gm}^{2c}(Z)$ [26, Sect. 1.13]*,*

(5) the composition $M_{gm}(Z)(c)[2c] \twoheadrightarrow M_2^s \xrightarrow{\sim} \mathbb{1}(c)[2c]$ corresponds to the twist by c of the shift by $2c$ of the projection from $M_{gm}(Z)$ onto the canonical quotient $M_{gm}^0(Z)$ [26, Sect. 1.11]*,*

(6) and the morphism d^s corresponds to multiplication by $Z \cdot Z$.

Proof. We have

$$\mathrm{Hom}_{DM_{gm}(k)_F}\big(M_{gm}(Z)(c)[2c], M_{gm}(Z)\big) = CH^0(Z \times_k Z)_F$$

[28, Thm. 4.3.7 3., Prop. 4.2.9]. The group $CH^0(Z \times_k Z)$ is generated by the class of $Z \times_k Z$. According to [21, Lemma 1.1 (i)], the composition

$$e' := L_{i^*[Z]} \circ [Z \times_k Z]$$

equals $Z \cdot Z$ times an idempotent non-zero endomorphism of $M_{gm}(Z)$. As cycles classes on $Z \times_k Z$, both $L_{i^*[Z]}$ and $[Z \times_k Z]$ are symmetric; therefore, the composition $[Z \times_k Z] \circ L_{i^*[Z]}$ equals ${}^t e'$, which is again $Z \cdot Z$ times an idempotent non-zero endomorphism. Thus, if $Z \cdot Z \neq 0$, then

$$\mathrm{id}_{M_{gm}(Z)} - \frac{1}{Z \cdot Z}[Z \times_k Z] \circ L_{i^*[Z]}$$

is not an automorphism, meaning that $L_{i^*[Z]}$ is not in the radical. If $Z \cdot Z = 0$, then $[Z \times_k Z] \circ L_{i^*[Z]} = 0$, *i.e.*,

$$\mathrm{id}_{M_{gm}(Z)} - g \circ L_{i^*[Z]} = \mathrm{id}_{M_{gm}(Z)}$$

for any morphism $g \in \mathrm{Hom}_{DM_{gm}(k)_F}(M_{gm}(Z)(c)[2c], M_{gm}(Z))$.

This shows part (a) of our claim. As for part (b), assume that $Z \cdot Z \neq 0$. Put $f := \frac{1}{Z \cdot Z}[Z \times_k Z] \in CH^0(Z \times_k Z)_F$. As the reader will verify, both

$$e := f \circ d \in \mathrm{End}_{DM_{gm}(k)_F}\big(M_{gm}(Z)\big)$$

and

$$g := d \circ f \in \mathrm{End}_{DM_{gm}(k)_F}\big(M_{gm}(Z)(c)[2c]\big)$$

are idempotent; more precisely, the image M_1^s of e equals $M_{gm}^{2c}(Z)$, and the image M_2^s of f projects isomorphically onto $M_{gm}^0(Z)(c)[2c]$. Put

$$M_1^r := \ker(e) \quad \text{and} \quad M_2^r := \ker(f).$$

Properties (1) to (6) then follow from our construction, and from the fact that any morphism in $\mathrm{Hom}_{DM_{gm}(k)_F}(M_{gm}(Z)(c)[2c], M_{gm}(Z))$ is a scalar multiple of f; hence

$$\mathrm{Hom}_{DM_{gm}(k)_F}\big(M_2^r, M_1^r\big) = 0.$$

The same argument shows the unicity of the decompositions. □

Corollary 5.7. *In the situation of Theorem 5.6, assume that $Z \cdot Z = 0$. Then $(M_{gm}(Y), j_*, j^*)$ is an absolute intersection motive of X. In particular,*

$$M^{!*}(X) = M_{gm}(Y).$$

Remark 5.8. The above setting provides examples for non-zero morphisms in the radical. Namely, if $i^*[Z]$ is of degree zero, without being rationally equivalent to zero, then $0 \neq d\colon M_{gm}(Z) \to M_{gm}(Z)(c)[2c]$ belongs to the radical. In this case,

$$\partial M_{gm}(X) \not\cong M_{gm}(Z)(c)[2c-1] \oplus M_{gm}(Z).$$

Corollary 5.9. *In the situation of Theorem 5.6, assume that $Z \cdot Z \neq 0$.*

(a) There is a canonical direct factor $M'_{gm}(Y)$ of $M_{gm}(Y)$, admitting a canonical complement, which is isomorphic to $\mathbb{1}(c)[2c]$.

(b) The morphism $j_*\colon M_{gm}(X) \to M_{gm}(Y)$ factors through the sub-motive $M'_{gm}(Y)$, and the morphism $j^*\colon M_{gm}(Y) \to M_{gm}^c(X)$ factors through the quotient $M'_{gm}(Y)$.

(c) *The triplet* $(M'_{gm}(Y), j_*, j^*)$ *is an absolute intersection motive of* X. *In particular,*

$$M_{gm}(Y) \cong \mathbb{1}(c)[2c] \oplus M^{!*}(X).$$

Proof. Recall the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_{gm}(Z) & = & M_{gm}(Z) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle d_+} & & \downarrow{\scriptstyle d} & & \downarrow \\
M_{gm}(X) & \xrightarrow{j_*} & M_{gm}(Y) & \xrightarrow{d_-} & M_{gm}(Z)(c)[2c] & \longrightarrow & M_{gm}(X)[1] \\
\| & & \downarrow{\scriptstyle j^*} & & \downarrow & & \| \\
M_{gm}(X) & \xrightarrow{u} & M^c_{gm}(X) & \longrightarrow & \partial M_{gm}(X)[1] & \longrightarrow & M_{gm}(X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_{gm}(Z)[1] & = & M_{gm}(Z)[1] & \longrightarrow & 0
\end{array}$$

According to Theorem 5.6 (b), the motive $\mathbb{1}(c)[2c]$ can be identified with direct factors of both $M_{gm}(Z)$ and $M_{gm}(Z)(c)[2c]$, and the morphism $d = d_- \circ d_+$ induces an isomorphism between them. Therefore, the kernel $M'_{gm}(Y)$ of the composition

$$M_{gm}(Y) \xrightarrow{d_-} M_{gm}(Z)(c)[2c] \twoheadrightarrow \mathbb{1}(c)[2c]$$

(exists and) is a direct factor of $M_{gm}(Y)$, with complement equal to the image of the composition

$$\mathbb{1}(c)[2c] \hookrightarrow M_{gm}(Z) \xrightarrow{d_+} M_{gm}(Y).$$

By definition, the morphisms j_* and j^* factor through $M'_{gm}(Y)$. With the notation of Theorem 5.6 (b), we thus get the following direct factor of the above diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_1^r & = & M_1^r & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle d^r} & & \downarrow \\
M_{gm}(X) & \xrightarrow{j_*} & M'_{gm}(Y) & \longrightarrow & M_2^r & \longrightarrow & M_{gm}(X)[1] \\
\| & & \downarrow{\scriptstyle j^*} & & \downarrow & & \| \\
M_{gm}(X) & \xrightarrow{u} & M^c_{gm}(X) & \longrightarrow & \partial M_{gm}(X)[1] & \longrightarrow & M_{gm}(X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_1^r[1] & = & M_1^r[1] & \longrightarrow & 0
\end{array}$$

But according to Theorem 5.6 (b) (2),

$$d^r \in \mathrm{rad}_{DM_{gm}(k)_F}(M_1^r, M_2^r).$$

□

Example 5.10. Let $n \geqslant 1$, and put $X = \mathbb{A}^n_k$. We have $M_{gm}(\mathbb{A}^n_k) = \mathbb{1}(0)$ and $M^c_{gm}(\mathbb{A}^n_k) = \mathbb{1}(n)[2n]$ [28, Cor. 4.1.8]. In particular, the exact triangle

$$M^c_{gm}(\mathbb{A}^n_k)[-1] \longrightarrow \partial M_{gm}(\mathbb{A}^n_k) \longrightarrow M_{gm}(\mathbb{A}^n_k) \xrightarrow{u} M^c_{gm}(\mathbb{A}^n_k)$$

is a weight filtration of $\partial M_{gm}(\mathbb{A}^n_k)$. By Theorem 5.1, the morphism u is zero; therefore, the above weight filtration is minimal. Construction 3.1 ensures the existence of a Chow motive M, and of morphisms i, π, d_+ and d_- such that

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_{gm}(\mathbb{A}^n_k) & = & M_{gm}(\mathbb{A}^n_k) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle d_+} & & \downarrow{\scriptstyle 0} & & \downarrow \\
M_{gm}(\mathbb{A}^n_k) & \xrightarrow{i} & M & \xrightarrow{d_-} & M^c_{gm}(\mathbb{A}^n_k) & \longrightarrow & M_{gm}(\mathbb{A}^n_k)[1] \\
\| & & \downarrow{\scriptstyle \pi} & & \downarrow & & \| \\
M_{gm}(\mathbb{A}^n_k) & \xrightarrow{0} & M^c_{gm}(\mathbb{A}^n_k) & \longrightarrow & \partial M_{gm}(\mathbb{A}^n_k)[1] & \longrightarrow & M_{gm}(\mathbb{A}^n_k)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_{gm}(\mathbb{A}^n_k)[1] & = & M_{gm}(\mathbb{A}^n_k)[1] & \longrightarrow & 0
\end{array}$$

is a diagram of exact sequences. Observe that both morphisms $M^c_{gm}(\mathbb{A}^n_k) \to M_{gm}(\mathbb{A}^n_k)[1]$ in the second row and in the second column are zero since their source is of weight zero, and their target, of weight one. Thus, we see that $M := M_{gm}(\mathbb{A}^n_k) \oplus M^c_{gm}(\mathbb{A}^n_k)$, $i := d_+ :=$ the inclusion of the first direct factor, and $\pi := d_- :=$ the projection to the second direct factor, is a solution. Therefore, $(M_{gm}(\mathbb{A}^n_k) \oplus M^c_{gm}(\mathbb{A}^n_k), i, \pi)$ is minimal. In particular,

$$M^{!*}(\mathbb{A}^n_k) = M_{gm}(\mathbb{A}^n_k) \oplus M^c_{gm}(\mathbb{A}^n_k) = \mathbb{1}(0) \oplus \mathbb{1}(n)[2n];$$

we may think of this as the "motive of the $2n$-sphere" $M_{gm}(\mathbb{S}^{2n})$. The minimal weight filtration of $\partial M_{gm}(\mathbb{A}^n_k)$ shows that

$$\partial M_{gm}(\mathbb{A}^n_k) \cong M^c_{gm}(\mathbb{A}^n_k)[-1] \oplus M_{gm}(\mathbb{A}^n_k) = \mathbb{1}(n)[2n-1] \oplus \mathbb{1}(0).$$

Remark 5.11. Example 5.10 shows in particular that, in general, the absolute intersection motive of a product is unequal to the tensor product of the absolute intersection motives of the factors.

Example 5.12. Let X^* be a proper surface over k, which we assume to be normal. The singular locus X^*_{sing} of X^* is of dimension zero; denote by X its complement. We claim that the absolute intersection motive of X (exists and) is isomorphic to the *intersection motive of* X^* [10, 35].

Choose a cartesian diagram

$$\begin{array}{ccccc}
X & \overset{j}{\hookrightarrow} & Y & \overset{i}{\hookleftarrow} & Z \\
\| & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
X & \hookrightarrow & X^* & \hookleftarrow & X^*_{sing}
\end{array}$$

where π is proper and birational, Y is smooth (and proper), and Z is a divisor with normal crossings, whose irreducible components Z_m are smooth (this is possible according to [24, Theorem] and the discussion in [23, pp. 191–194]).

As before, localization gives a weight filtration of $M^c_{gm}(X)$:

$$M_{gm}(Y) \xrightarrow{j^*} M^c_{gm}(X) \longrightarrow M_{gm}(Z)[1] \xrightarrow{d_+[1]} M_{gm}(Y)[1].$$

Dualizing it, twisting by 2 and shifting by 4, we get a weight filtration of $M_{gm}(X)$, according to [28, Thm. 4.3.7 3.]:

$$M_{gm}(Z)^*(2)[3] \longrightarrow M_{gm}(X) \xrightarrow{j_*} M_{gm}(Y) \xrightarrow{d_-} M_{gm}(Z)^*(2)[4].$$

We get

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_{gm}(Z) & = & M_{gm}(Z) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle d_+} & & \downarrow{\scriptstyle d} & & \downarrow \\
M_{gm}(X) & \xrightarrow{j_*} & M_{gm}(Y) & \xrightarrow{d_-} & M_{gm}(Z)^*(2)[4] & \longrightarrow & M_{gm}(X)[1] \\
\| & & \downarrow{\scriptstyle j^*} & & \downarrow & & \| \\
M_{gm}(X) & \xrightarrow{u} & M^c_{gm}(X) & \longrightarrow & \partial M_{gm}(X)[1] & \longrightarrow & M_{gm}(X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_{gm}(Z)[1] & = & M_{gm}(Z)[1] & \longrightarrow & 0
\end{array}$$

There is a canonical morphism

$$\iota_* \colon \bigoplus_m M^2_{gm}(Z_m) \hookrightarrow \bigoplus_m M_{gm}(Z_m) \longrightarrow M_{gm}(Z)$$

[26, Sect. 1.13]. According to [35, Thm. 2.2 (i)] (see also [10, Sect. 2.5]), the composition

$$\bigoplus_m M^2_{gm}(Z_m) \xrightarrow{\iota_*} M_{gm}(Z) \xrightarrow{d} M_{gm}(Z)^*(2)[4] \xrightarrow{(\iota_*)^*(2)[4]} \bigoplus_m M^2_{gm}(Z_m)^*(2)[4]$$

is an isomorphism. Therefore, the kernel of the composition

$$M_{gm}(Y) \xrightarrow{d_-} M_{gm}(Z)^*(2)[4] \xrightarrow{(\iota_*)^*(2)[4]} \bigoplus_m M^2_{gm}(Z_m)^*(2)[4]$$

(exists and) is a direct factor of $M_{gm}(Y)$, with complement equal to the image of the composition

$$\bigoplus_m M^2_{gm}(Z_m) \xrightarrow{\iota_*} M_{gm}(Z) \xrightarrow{d_+} M_{gm}(Y).$$

But by definition [35, Def. 2.3, Ex. 5.2], that kernel equals $M^{!*}(X^*)$, the intersection motive of M^*. The morphisms j_* and j^* factor through $M^{!*}(X^*)$. Denoting by $M^{\leqslant 1}_{gm}(Z)$ the complement of $\bigoplus_m M^2_{gm}(Z_m)$ in $M_{gm}(Z)$ (see [35, Lemma 5.4]), we thus get the following direct factor of the above diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M^{\leqslant 1}_{gm}(Z) & = & M^{\leqslant 1}_{gm}(Z) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle d^{\leqslant 1}} & & \downarrow \\
M_{gm}(X) & \xrightarrow{j_*} & M^{!*}(X^*) & \longrightarrow & M^{\leqslant 1}_{gm}(Z)^*(2)[4] & \longrightarrow & M_{gm}(X)[1] \\
\| & & \downarrow{\scriptstyle j^*} & & \downarrow & & \| \\
M_{gm}(X) & \xrightarrow{u} & M^c_{gm}(X) & \longrightarrow & \partial M_{gm}(X)[1] & \longrightarrow & M_{gm}(X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M^{\leqslant 1}_{gm}(Z)[1] & = & M^{\leqslant 1}_{gm}(Z)[1] & \longrightarrow & 0
\end{array}$$

Here, we set $d^{\leqslant 1}$:= the restriction of d to $M_{gm}^{\leqslant 1}(Z)$. We claim that

$$d^{\leqslant 1} \in \mathrm{rad}_{DM_{gm}(k)_F}\left(M_{gm}^{\leqslant 1}(Z), M_{gm}^{\leqslant 1}(Z)^*(2)[4]\right).$$

In fact, *all* morphisms $M_{gm}^{\leqslant 1}(Z) \to M_{gm}^{\leqslant 1}(Z)^*(2)[4]$ belong to the radical, since

$$\mathrm{Hom}_{DM_{gm}(k)_F}\left(M_{gm}^{\leqslant 1}(Z)^*(2)[4], M_{gm}^{\leqslant 1}(Z)\right) = 0.$$

To prove this latter claim, observe that the closed covering by the Z_m induces a weight filtration of $M_{gm}^{\leqslant 1}(Z)$ (see the first part of the proof of [35, Prop. 6.5 (i)]):

$$\bigoplus_{n<m} M_{gm}(Z_{n,m}) \longrightarrow \bigoplus_m M_{gm}^{\leqslant 1}(Z_m) \longrightarrow M_{gm}^{\leqslant 1}(Z) \longrightarrow \bigoplus_{n<m} M_{gm}(Z_{n,m})[1],$$

where we denote by $Z_{n,m}$ the intersection $Z_n \cap Z_m$. Whence the dual weight filtration

$$\bigoplus_{n<m} M_{gm}(Z_{n,m})^*[-1] \longrightarrow M_{gm}^{\leqslant 1}(Z)^* \longrightarrow \bigoplus_m M_{gm}^{\leqslant 1}(Z_m)^* \longrightarrow \bigoplus_{n<m} M_{gm}(Z_{n,m})^*$$

of $M_{gm}^{\leqslant 1}(Z)^*$. Orthogonality for weight structures shows that any morphism from $M_{gm}^{\leqslant 1}(Z)^*(2)[4]$ to a motive of non-negative weights factors through $\bigoplus_m M_{gm}^{\leqslant 1}(Z_m)^*(2)[4]$, and that any morphism from a motive of weight zero to $M_{gm}^{\leqslant 1}(Z)$ factors through $\bigoplus_m M_{gm}^{\leqslant 1}(Z_m)$. Therefore,

$$\mathrm{Hom}_{DM_{gm}(k)_F}\left(M_{gm}^{\leqslant 1}(Z)^*(2)[4], M_{gm}^{\leqslant 1}(Z)\right)$$

is a direct factor of

$$\bigoplus_{n,m} \mathrm{Hom}_{DM_{gm}(k)_F}\left(M_{gm}^{\leqslant 1}(Z_n)^*(2)[4], M_{gm}^{\leqslant 1}(Z_m)\right).$$

We claim that

$$\mathrm{Hom}_{DM_{gm}(k)_F}\left(M_{gm}^{\leqslant 1}(Z_n)^*(2)[4], M_{gm}^{\leqslant 1}(Z_m)\right) = 0$$

for all n and m. Indeed, the split monomorphisms $M_{gm}^2(Z_m) \hookrightarrow M_{gm}(Z_m)$ induce an isomorphism between

$$\mathrm{Hom}_{DM_{gm}(k)_F}\left(M_{gm}^2(Z_n)^*(2)[4], M_{gm}^2(Z_m)\right)$$

and

$$\mathrm{Hom}_{DM_{gm}(k)_F}\left(M_{gm}(Z_n)^*(2)[4], M_{gm}(Z_m)\right),$$

as can be seen from the comparison wih Chow theory: both groups are identified with

$$CH^0(Z_n \times_k Z_m)_F$$

[28, Thm. 4.3.7 3., Prop. 4.2.9], and the morphism between them corresponds to the identity.

Altogether, we proved that $(M^{!*}(X^*), j_*, j^*)$ is minimal. In particular,

$$M^{!*}(X) \cong M^{!*}(X^*).$$

Remark 5.13. For $k = \mathbb{C}$, and F a field contained in $\mathbb{R}$, the reader may choose to compute absolute intersection cohomology in the geometric situations treated in this section, taking into account the results from Section 4, in particular, Theorem 4.8. The computations are *a priori* compatible with the above under the Hodge theoretic realization.

6. A question

In this section, the base k equals the field $\mathbb{C}$ of complex numbers. The coefficients F are assumed to be contained in $\mathbb{R}$. Let $X \in Sm/\mathbb{C}$. The examples treated in Section 5 seem to suggest the following.

Question 6.1. Does there exist an open, dense immersion j of $X(\mathbb{C})$ into a compact topological space $X^{!*}$ satisfying the hypothesis of [4, 2.1.16], and equipped with a stratification into topological manifolds of even (real) dimension, among which $X(\mathbb{C})$, such that "intersection cohomology of $X^{!*}$ equals absolute intersection cohomology of X" in the following sense: for all $n \in \mathbb{Z}$, there is an isomorphism of F-vector spaces

$$H^n(X^{!*}, j_{!*}F) \xrightarrow{\sim} H^n_{!*}(X, F)$$

from cohomology of $X^{!*}$ with coefficients in the intermediate extension $j_{!*}F$ to the F-vector space $H^n_{!*}(X, F)$ underlying absolute intersection cohomology $H^n_{!*}(X, F(0))$, respecting the factorizations

$$H^n_c(X, F) \xrightarrow{H^n j_*} H^n(X^{!*}, j_{!*}F) \xrightarrow{H^n j^*} H^n(X, F)$$

and

$$H^n_c(X, F(0)) \xrightarrow{i} H^n_{!*}(X, F(0)) \xrightarrow{\pi} H^n(X, F(0)) \ ?$$

Note that since X is smooth, the constant sheaf F on X can be seen as a perverse sheaf up to a shift by $-\dim_X$:

$$F = (F[\dim_X])[-\dim_X],$$

where $F[\dim_X]$ belongs to the heart of the perverse t-structure. By slight abuse of notation, we define

$$j_{!*}F := (j_{!*}F[\dim_X])[-\dim_X].$$

Note that $j_{!*}F[\dim_X]$ is defined, and that if $X^{!*}$ is a topological manifold, then $H^n(X^{!*}, j_{!*}F)$ equals $H^n(X^{!*}, F)$ for all $n \in \mathbb{Z}$ [4, Prop. 2.1.17]. Note also that

$$H^n j_* \colon H^n_c(X, F) \longrightarrow H^n(X^{!*}, j_{!*}F)$$

factors through the F-vector space underlying $\mathrm{Gr}^W_n H^n_c(X, F(0))$, and

$$H^n j^* \colon H^n(X^{!*}, j_{!*}F) \longrightarrow H^n(X, F)$$

factors through the F-vector space underlying $\mathrm{Gr}^W_n H^n(X, F(0))$ (Corollary 4.3).

Examples 6.2. The answer to Question 6.1 is affirmative in the following cases:

(a) For $X = \mathbb{A}^1_{\mathbb{C}}$, put $X^{!*} = \mathbb{P}^1(\mathbb{C})$ (Example 5.2).

(b) More generally, for X equal to the complement of a finite number of closed points in a smooth, proper $\mathbb{C}$-scheme Y, which is of pure dimension $n \geqslant 1$, put $X^{!*} = Y(\mathbb{C})$ (Example 5.3).

(c) Even more generally, let X be equal to the complement of a closed sub-scheme Z in a smooth, proper $\mathbb{C}$-scheme Y. Assuming that the dimension of Z is strictly smaller than its codimension in Y, we may put $X^{!*} = Y(\mathbb{C})$ (Example 5.4 (a)).

(d) Let X be equal to the complement of a smooth closed sub-scheme Z of pure dimension $c \geqslant 1$ in a smooth, proper $\mathbb{C}$-scheme Y of pure dimension $2c$. Assume that Z is connected, and that the self intersection number $Z \cdot Z$ of Z in Y is non-zero. We know (Corollary 5.9) that $X^{!*}$ cannot be chosen to be equal to $Y(\mathbb{C})$.

(e) Let X be equal to the complement of a smooth divisor Z in a smooth, proper $\mathbb{C}$-scheme Y. Assume that the pull-back $i^*[Z]$ of the divisor to the sub-scheme Z is ample, and that $M_{gm}(Z)$ satisfies the weak form of the Lefschetz decomposition for $L_{i^*[Z]}$. We know (Example 5.4 (d)) that $X^{!*}$ cannot be chosen to be equal to $Y(\mathbb{C})$.

(f) For $X = \mathbb{A}^n_{\mathbb{C}}$, $n \geqslant 1$, put $X^{!*} = \mathbb{S}^{2n}$ (Example 5.10).

(g) For X equal to the complement of the singular locus in a proper normal surface X^* over $\mathbb{C}$, we may put $X^{!*} = X^*(\mathbb{C})$ (Example 5.12).

We omitted to specify $X^{!*}$ in Examples 6.2 (d) and (e). Before doing so, let us treat the remaining example from Section 5; as we shall see (Proposition 6.4), it is fundamentally different from those treated in Example 6.2.

Example 6.3. Let X be equal to the complement of a smooth closed sub-scheme Z of pure dimension $c \geqslant 1$ in a smooth, proper $\mathbb{C}$-scheme Y of pure dimension $2c$. Assume that Z is connected, and that $Z \cdot Z = 0$. We may then put $X^{!*} = Y(\mathbb{C})$ (Corollary 5.7).

Denote by X^+ the Alexandrov one-point compactification of $X(\mathbb{C})$.

Proposition 6.4. *(a) In all of Examples 6.2, the compactification $X^{!*}$ of $X(\mathbb{C})$ may be chosen to be equal to X^+.*

(b) In Example 6.3, exactly one of the following cases occurs: either

$$\dim_F H^n(X^+, j_{!*} F) > \dim_F H^n_{!*}(X, F)$$

for $n = 2c - 1$ and $n = 2c + 1$ or there is no isomorphism

$$H^{2c}(X^+, j_{!*} F) \longrightarrow H^{2c}_{!*}(X, F)$$

respecting the factorizations of $H^{2c}_c(X, F) \to H^{2c}(X, F)$. In particular, the compactification $X^{!*}$ of $X(\mathbb{C})$ cannot be chosen to be equal to X^+.

Proposition 6.4 is obvious in certain cases. For the treatment of the others, the following observation turns out to be useful.

Proposition 6.5. *Let $X \in Sm/\mathbb{C}$ of pure dimension. Denote by j the immersion of $X(\mathbb{C})$ into X^+. Then*

$$\begin{aligned} H^n(X^+, j_{!*}\, F) &= H^n(X,F) \quad \textit{for} \quad n \leqslant \dim_X -1 \\ H^{\dim_X}(X^+, j_{!*}\, F) &= H_!^{\dim_X}(X,F), \\ H^n(X^+, j_{!*}\, F) &= H_c^n(X,F) \quad \textit{for} \quad n \geqslant \dim_X +1. \end{aligned}$$

Each of the identifications is understood with the canonical factorization of the morphism $H_c^n(X,F) \to H^n(X,F)$.

Remark 6.6. (a) In particular, intersection cohomology of X^+ carries a canonical Hodge structure (which in general is mixed).

(b) Given the formulae from Proposition 6.5, we see that in those cases where intersection cohomology of X^+ and absolute intersection cohomology of X are isomorphic, the isomorphisms $H^n(X^+, j_{!*}\, F) \xrightarrow{\sim} H_{!*}^n(X,F)$, $n \in \mathbb{Z}$, are unique.

Indeed, for $n \leqslant \dim_X -1$, to say that $H^n(X,F)$ is (abstractly) isomorphic to $H_{!*}^n(X,F)$ implies that $H^n(X,F)$ is pure of weight n. The only morphism α making the diagram

$$\begin{array}{ccccc} \operatorname{Gr}_n^W H_c^n(X,F) & \xrightarrow{i_n} & H_{!*}^n(X,F) & \xrightarrow{\pi_n} & H^n(X,F) = \operatorname{Gr}_n^W H^n(X,F) \\ \| & & \downarrow{\scriptstyle\alpha} & & \| \\ \operatorname{Gr}_n^W H_c^n(X,F) & \xrightarrow{u_n} & H^n(X,F) & = & H^n(X,F) = \operatorname{Gr}_n^W H^n(X,F) \end{array}$$

commute is $\alpha = \pi_n$. Since we suppose the existence of *some* isomorphism making the diagram commute, this isomorphism equals α. The case where $n \geqslant \dim_X -1$ is dual. For $n = \dim_X$, consider the diagram

$$\begin{array}{ccccc} \operatorname{Gr}_{\dim_X}^W H_c^{\dim_X}(X,F) & \xhookrightarrow{i_{\dim_X}} & H_{!*}^{\dim_X}(X,F) & \xrightarrow{\pi_{\dim_X}} & \operatorname{Gr}_{\dim_X}^W H^{\dim_X}(X,F) \\ \| & & & & \| \\ \operatorname{Gr}_{\dim_X}^W H_c^{\dim_X}(X,F) & \twoheadrightarrow & H_!^{\dim_X}(X,F) & \hookrightarrow & \operatorname{Gr}_{\dim_X}^W H^{\dim_X}(X,F). \end{array}$$

If there is an isomorphism α making the diagram commute, then all horizontal morphisms are necessarily isomorphisms (cmp. Corollary 4.9). Therefore, the isomorphism α is unique.

Furthermore, the isomorphisms $H^n(X^+, j_{!*}\, F) \xrightarrow{\sim} H_{!*}^n(X,F)$, $n \in \mathbb{Z}$, respect the Hodge structures (which are pure).

Proof of Proposition 6.5. Denote by $\tau^{t\leqslant \bullet}$ and $\tau^{t\geqslant \bullet}$ the truncation functors with respect to the t-structure on the derived category of sheaves on the complement $\{*\}$ of $X(\mathbb{C})$ in X^+, in other words, on the derived category of F-vector spaces. Denote by i the inclusion of $\{*\}$ into X^+. According to [4, Prop. 2.1.11], there is an exact triangle

$$i_*\tau^{t\geqslant \dim_X} i^* Rj_* F[-1] \longrightarrow j_{!*}\, F \longrightarrow Rj_*F \longrightarrow i_*\tau^{t\geqslant \dim_X} i^* Rj_* F.$$

Applying the direct image a_* under the structure map a of X^+, we get isomorphisms

$$H^n(X^+, j_{!*}\, F) \xrightarrow{\sim} H^n(X, F) \quad \text{for} \quad n < \dim_X,$$

and a monomorphism

$$H^{\dim_X}(X^+, j_{!*}\, F) \hookrightarrow H^{\dim_X}(X, F).$$

Dually, we obtain an epimorphism

$$H_c^{\dim_X}(X, F) \twoheadrightarrow H^{\dim_X}(X^+, j_{!*}\, F),$$

and isomorphisms

$$H_c^n(X, F) \xrightarrow{\sim} H^n(X^+, j_{!*}\, F) \quad \text{for} \quad n > \dim_X. \qquad \square$$

Using Proposition 6.5, the reader may choose to provide a direct proof of Proposition 6.4. The one we shall give is a consequence of the next result.

Theorem 6.7. *Let $X \in Sm/\mathbb{C}$ of pure dimension. Then the following are equivalent.*

(1) The compactification $X^{!*}$ of $X(\mathbb{C})$ (exists and) may be chosen to be equal to X^+.

(2) For all integers $n \leqslant \dim_X - 1$, the Hodge structure on $\partial H^n(X, F(0))$ is of weights $\leqslant n$.

(3) For all integers $n \geqslant \dim_X$, the Hodge structure on $\partial H^n(X, F(0))$ is of weights $\geqslant n+1$.

(4) There exists an open immersion of X into a smooth, proper $\mathbb{C}$-scheme Y, with complement $i\colon Z \hookrightarrow Y$, such that the maps

$$H^n(Z, i^! F) \longrightarrow H^n(Z, F)$$

($i^!$ = the exceptional inverse image under the immersion i) are injective in the range $2 \cdot \mathrm{codim}_Y Z \leqslant n \leqslant \dim_X$.

(5) Whenever X is represented as the complement of a closed sub-scheme $i\colon Z \hookrightarrow Y$ in a smooth, proper $\mathbb{C}$-scheme Y, the maps

$$H^n(Z, i^! F) \longrightarrow H^n(Z, F)$$

are injective in the range $2 \cdot \mathrm{codim}_Y Z \leqslant n \leqslant \dim_X$.

If the equivalent conditions (1)–(5) are satisfied, then the following also hold.

(6) For all integers $n \leqslant \dim_X - 1$, the Hodge structure on $H^n(X, F(0))$ is pure of weight n.

(7) For all integers $n \geqslant \dim_X + 1$, the Hodge structure on $H_c^n(X, F(0))$ is pure of weight n.

Proof. Proposition 6.5 shows that claim (1) implies claims (6) and (7).

The Hodge structures $H_c^n(X, F(0))$ and $H^{2\dim_X - n}(X, F(0))(\dim_X)$ are dual to each other, and under this duality, the collection of morphisms

$$\big(u\colon H_c^n(X, F(0)) \longrightarrow H^n(X, F(0))\big)_{n\in\mathbb{Z}}$$

is auto-dual up to a twist by $\dim_X$. Given the long exact sequence

$$\cdots \longrightarrow \partial H^{n-1}(X, F(0)) \longrightarrow H_c^n(X, F(0)) \xrightarrow{u} H^n(X, F(0)) \longrightarrow \cdots,$$

the Hodge structures $\partial H^n(X, F(0))$ and $\partial H^{2\dim_X - n-1}(X, F(0))(\dim_X)$ are therefore dual to each other. Thus, claims (2) and (3) are equivalent.

We leave it as an exercice to the reader to show, using Theorem 4.8 (b) and Proposition 6.5, that (2) and (3) together (and hence individually) are equivalent to (1).

To finish the proof, note that we have nothing to prove if X is proper. Else, let $i\colon Z \hookrightarrow Y$ be a closed immersion, with Y smooth and proper over $\mathbb{C}$, and such that $X = Y - Z$. Write j for the open immersion of X into Y, and $a\colon Y \to \mathbf{Spec}\,\mathbb{C}$ for the structure morphism of Y. The morphism $u\colon R\Gamma_c(X, F(0)) \to R\Gamma(X, F(0))$ in $D^b(\mathbf{MHS}_F)$ is the result of applying the direct image a_* to the morphism

$$v\colon j_!F(0) \longrightarrow j_*F(0)$$

in $D^b(\mathbf{MHM}_F\, Y)$ [25, proof of Thm. 4.3]. A canonical choice of cone of v is given by $i_*i^*j_*F(0)$ (apply [25, (4.4.1)] to $j_*F(0)$). Thus, $a_*i_*i^*j_*F(0)$ is a cone of u; its cohomology objects are thus equal to boundary cohomology.

Consider the localization exact triangle

$$i_*i^!F(0) \longrightarrow F(0) \longrightarrow j_*F(0) \longrightarrow i_*i^!F(0)[1]$$

in $D^b(\mathbf{MHM}_F\, Y)$ [25, (4.4.1)]. Applying the functor $a_*i_*i^*$ yields

$$a_*i_*i^!F(0) \longrightarrow a_*i_*i^*F(0) \longrightarrow a_*i_*i^*j_*F(0) \longrightarrow a_*i_*i^!F(0)[1].$$

We thus see that boundary cohomology is part of a long exact sequence

$$\cdots \longrightarrow \partial H^n(X, F(0)) \longrightarrow H^{n+1}(Z, i^!F(0)) \longrightarrow H^{n+1}(Z, F(0)) \longrightarrow \cdots.$$

The Hodge structures on $H^{n+1}(Z, i^!F(0))$ and on $H^{n+1}(Z, F(0))$ are of weights $\geqslant n+1$ and $\leqslant n+1$, respectively [25, (4.5.2)]. Therefore, for a fixed integer n, the Hodge structure on $\partial H^n(X, F(0))$ is of weights $\leqslant n$ if and only if $H^{n+1}(Z, i^!F) \to H^{n+1}(Z, F)$ is injective. We leave it to the reader to show, for example by using duality, that

$$H^n(Z, i^!F) = 0 \quad \text{for} \quad n < 2\cdot \operatorname{codim}_Y Z. \qquad \square$$

Example 6.8. For the surface $X = \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$, the Hodge structure $H^n(X, F(0))$ is pure of weight n for $n = 0, 1$, meaning that property 6.7 (6) is satisfied. But according to Proposition 6.4, the compactification $X^{!*}$ of $X(\mathbb{C})$ may not be chosen to be equal to X^+.

Proof of Proposition 6.4. The claim is obvious in cases 6.2 (a) and 6.2 (f). Cases 6.2 (b) and 6.2 (g) follow from invariance of intersection cohomology under direct images of finite morphisms.

To treat cases 6.2 (c), 6.2 (d) and 6.3, we apply criteria 6.7 (4) (for 6.2 (c) and 6.2 (d)) and 6.7 (5) (for 6.3) to the respective choice of $i\colon Z \hookrightarrow Y$. In case 6.2 (c), it is trivially satisfied since $2 \cdot \operatorname{codim}_Y Z > \dim_X$. In cases 6.2 (d) and 6.3, the only degree that needs to be verified is $n = 2c$. By purity, the source $H^{2c}(Z, i^! F_Y)$ of the map in question is identified with $H^0(Z, F(-c)) = F(-c)$. The trace map gives the same type of identification for the target $H^{2c}(Z, F)$. Under these identifications,

$$H^{2c}(Z, i^! F_Y) \longrightarrow H^{2c}(Z, F)$$

corresponds to multiplication by $Z \cdot Z$. According to Theorem 6.7, we may put $X^{!*} = X^+$ in case 6.2 (d), but not in case 6.3. In the latter case, note that by duality,

$$\dim_F H^{2c-1}(X^+, j_{!*} F) > \dim_F H^{2c-1}_{!*}(X, F)$$

if and only if

$$\dim_F H^{2c+1}(X^+, j_{!*} F) > \dim_F H^{2c+1}_{!*}(X, F),$$

and recall that for all integers n,

$$H^n_{!*}(X, F) = H^n(Y, F).$$

The sequence

$$H^{2c}(Z, F) \xrightarrow{\partial} H^{2c+1}_c(X, F) = H^{2c+1}(X^+, j_{!*} F) \longrightarrow H^{2c+1}(Y, F) \longrightarrow 0$$

is exact. Thus, if $\partial \neq 0$, then $\dim_F H^{2c+1}(X^+, j_{!*} F) > \dim_F H^{2c+1}_{!*}(X, F)$. If $\partial = 0$, then

$$H^{2c}_c(X, F) \longrightarrow H^{2c}(Y, F)$$

is not surjective. This means that interior cohomology

$$H^{2c}_!(X, F) = H^{2c}(X^+, j_{!*} F)$$

and $H^{2c}(Y, F)$ cannot be identified in a way compatible with the factorizations of $H^{2c}_c(X, F) \to H^{2c}(X, F)$.

To treat case 6.2 (e), note that the maps

$$H^n(Z, i^! F(0)) \longrightarrow H^n(Z, F(0))$$

are identified with the cohomological Lefschetz operators

$$L_{i^*[Z]}\colon H^{n-2}(Z, F(-1)) \longrightarrow H^n(Z, F(0)).$$

According to the weak Lefschetz theorem, these operators are injective for all $n \leqslant \dim_Z + 1 = \dim_X$. In other words, criterion (4) from Theorem 6.7 is satisfied. □

Exercice 6.9. For $X = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$, compute $M^{!*}(X)$ and identify a choice of $X^{!*}$. Prove that $X^{!*}$ can neither be chosen to be equal to X^+, nor to the space of $\mathbb{C}$-valued points of a smooth algebraic compactification of X. Relate this observation to the rank of the intersection matrix of the irreducible components Z_i of the complement Z of X in a smooth compactification.

Acknowledgments

Part of this work was done while I was enjoying a *délégation auprès du CNRS*, and during a visit to Institut Mittag-Leffler (Djursholm). I am grateful to both institutions. I also wish to thank the referees for a thorough reading of, and thoughtful comments on a first version of this article.

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