CM values of higher Green's functions

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Abstract. In this paper we study the values of higher Green's functions at points of complex multiplication (CM points). Higher Green's functions are real-valued functions of two variables on the upper half-plane which are bi-invariant under the action of a congruence subgroup, have a logarithmic singularity along the diagonal and satisfy $\Delta f = k(1-k)f$, where Δ is the hyperbolic Laplace operator and k is a positive integer. B. Gross and D. Zagier conjectured in "Heegner points and derivatives of L-series" (1986) that certain explicit linear combinations of CM values of a Green's function are equal to the logarithm of the absolute value of an algebraic number. In this paper we prove the conjecture for any pair of CM points lying in the same imaginary quadratic field.

Contents

This paper is based on the author's PhD thesis [13] written at the Max Planck Institute for Mathematics and the University of Bonn.

1. Introduction

For any integer $k > 1$ there is a unique function G_k on the product of two upper half-planes $\mathfrak{H} \times \mathfrak{H}$ which satisfies the following conditions:

(i) G_k is a smooth function on $\mathfrak{H} \times \mathfrak{H} \setminus \{(\tau, \gamma\tau), \tau \in \mathfrak{H}, \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$ with values in $\mathbb{R}$.

(ii) $G_k(\tau_1, \tau_2) = G_k(\gamma_1\tau_1, \gamma_2\tau_2)$ for all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$.

(iii) $\Delta_i G_k = k(1-k)G_k$, where $\Delta_i = -4\Im(\tau_i)^2 \frac{\partial^2}{\partial\tau_i \partial\bar{\tau}_i}$ is the hyperbolic Laplacian with respect to the i-th variable, $i = 1, 2$.

(iv) $G_k(\tau_1, \tau_2) = s \log|\tau_1 - \tau_2| + O(1)$ when τ_1 tends to τ_2 (s is the order of the stabilizer of τ_2, which is almost always 1).

(v) $G_k(\tau_1, \tau_2)$ tends to 0 when τ_1 tends to a cusp.

This function is called the *k-th Green's function.*

The significant arithmetic properties of these functions were discovered by B. Gross and D. Zagier [10]. In particular, it was conjectured that certain explicit linear combinations of CM values of a Green's function are equal to the logarithm of the absolute value of an algebraic number. To state the conjecture, we need the following definitions.

Let f be a modular function. Then the action of the *Hecke operator* T_m on f is given by

$$\big(f \mid T_m\big)(\tau) = m^{-1} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{M}_m} f\left(\frac{a\tau + b}{c\tau + d}\right),$$

where $\mathcal{M}_m$ denotes the set of 2×2 integral matrices of determinant m.

The Green's functions G_k have the property

$$G_k(\tau_1, \tau_2) \mid T_m^{\tau_1} = G_k(\tau_1, \tau_2) \mid T_m^{\tau_2},$$

where $T_m^{\tau_i}$ denotes the Hecke operator with respect to the variable τ_i, $i = 1, 2$. Therefore, we will simply write $G_k(\tau_1, \tau_2) | T_m$.

Denote by $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ the space of cusp forms of weight $2k$ on the full modular group.

Proposition 1.1. *Let $k > 1$ and $\boldsymbol{\lambda} = \{\lambda_m\}_{m=1}^{\infty} \in \bigoplus_{m=1}^{\infty} \mathbb{Z}$. Then the following are equivalent:*

(i) $\sum_{m=1}^{\infty} \lambda_m a_m = 0$ *for any cusp form*

$$f = \sum_{m=1}^{\infty} a_m q^m \in S_{2k}(\mathrm{SL}_2(\mathbb{Z})).$$

(ii) There exists a weakly holomorphic modular form

$$g_{\boldsymbol{\lambda}}(\tau) = \sum_{m=1}^{\infty} \lambda_m q^{-m} + O(1) \in M^!_{2-2k}(\mathrm{SL}_2(\mathbb{Z})).$$

The proof of this proposition can be found, for example, in [3, Section 3]. Let us outline the proof. The space of obstructions to finding modular forms of weight $2-2k$ with given singularity at the cusp and the space of holomorphic modular forms of weight $2k$ can be both identified with cohomology groups of line bundles over a modular curve. The statement follows from Serre duality between these spaces.

We call a $\boldsymbol{\lambda}$ with the properties given in the above proposition a *relation* for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. Note that the function $g_{\boldsymbol{\lambda}}$ in (ii) is unique and has integral Fourier coefficients.

For a relation $\boldsymbol{\lambda}$ denote

$$G_{k,\boldsymbol{\lambda}} := \sum_{m=1}^{\infty} \lambda_m \, m^{k-1} \, G_k(\tau_1, \tau_2) \mid T_m.$$

The following conjecture was formulated in [10] and [9].

Conjecture 1.2. *Suppose $\boldsymbol{\lambda}$ is a relation for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. Then for any two CM points $\mathfrak{z}_1$, $\mathfrak{z}_2$ of discriminants D_1, D_2 there is an algebraic number α such that*

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1, \mathfrak{z}_2) = (D_1 D_2)^{\frac{1-k}{2}} \log|\alpha|.$$

This conjecture was verified numerically by D. Zagier [9], [15]. In the case $k = 2$, $D_1 = -4$ and D_2 arbitrary the conjecture was proven by A. Mellit in his doctoral dissertaion [11]. In this paper we prove the conjecture for any pair of points $\mathfrak{z}_1, \mathfrak{z}_2$ lying in the same imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$.

Theorem 1.3. *Let $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{H}$ be two CM points in the same quadratic imaginary field $\mathbb{Q}(\sqrt{-D})$ and let $\boldsymbol{\lambda}$ be a relation for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ for some integer $k > 1$. Then there is an algebraic number α such that*

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1, \mathfrak{z}_2) = \log|\alpha|.$$

Let us briefly explain the idea of the proof. The key technique used in our paper is the regularized theta lift introduced in [2]. Following ideas given in [4] we prove in Theorem 8.2 that the Green's function is equal to the regularized theta lift of an eigenfunction of the hyperbolic Laplace operator. This allows us to extend a method given in [12], that is to analyze CM values of Green's functions using rich functoriality properties of theta lifts. In Theorem 9.1 we prove that a CM-value of a higher Green's function is equal to the regularized Petersson product of a weakly holomorphic modular form of weight 1 and a binary theta series. In Theorem 10.1 we show that the regularized Petersson product of any weakly holomorphic modular form of weight 1 and a binary theta series is equal to the logarithm of a CM-value of a certain meromorphic modular function. Thus, from Theorems 9.1 and 10.1 we see that a CM-value of a higher Green's function is equal to the logarithm of a CM-value of a meromorphic modular function with algebraic Fourier coefficients. This proves Theorem 1.3.

2. Vector-valued modular forms

Recall that the group $\mathrm{SL}_2(\mathbb{Z})$ has a double cover $\mathrm{Mp}_2(\mathbb{Z})$ called the *metaplectic group* whose elements can be written in the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau+d}\right)$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\sqrt{c\tau+d}$ is considered as a holomorphic function of τ in the upper half plane whose square is $c\tau+d$. The multiplication is defined by

$$(A, f(\tau))(B, g(\tau)) = (AB, f(B(\tau))g(\tau))$$

for $A, B \in \mathrm{SL}_2(\mathbb{Z})$ and f, g suitable functions on $\mathfrak{H}$.

Suppose that V is a vector space over $\mathbb{Q}$ and that $(\ ,\ )$ is a bilinear form on $V \times V$ with signature (b^+, b^-). For an element $x \in V$ we will write $q(x) = \frac{1}{2}(x,x)$. Let $L \subset V$ be a lattice. The dual lattice of L is defined as $L' = \{x \in V | (x, L) \subseteq \mathbb{Z}\}$. We say that L is even if $q(\ell) \in \mathbb{Z}$ for all $\ell \in L$. In this case L is contained in L' and L'/L is a finite abelian group.

We let the elements e_ν for $\nu \in L'/L$ be the standard basis of the group ring $\mathbb{C}[L'/L]$, so that $e_\mu e_\nu = e_{\mu+\nu}$. Complex conjugation acts on $\mathbb{C}[L'/L]$ by $\overline{e_\mu} = e_\mu$. Consider the hermitian scalar product on $\mathbb{C}[L'/L]$ given by

$$\langle e_\mu, e_\nu \rangle = \delta_{\mu,\nu} \tag{2.1}$$

and extended to $\mathbb{C}[L'/L]$ by sesquilinearity. Recall that there is a unitary representation ρ_L of the double cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(\widetilde{T})(e_\nu) = \mathrm{e}\big(q(\nu)\big)\, e_\nu \tag{2.2}$$

$$\rho_L(\widetilde{S})(e_\nu) = i^{(b^-/2-b^+/2)}\, |L'/L|^{-1/2} \sum_{\mu \in L'/L} \mathrm{e}\big(-(\mu,\nu)\big) e_\mu, \tag{2.3}$$

where

$$\widetilde{T} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right), \text{and } \widetilde{S} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right) \tag{2.4}$$

are the standard generators of $\mathrm{Mp}_2(\mathbb{Z})$ and we use the notation $\mathrm{e}(a) := e^{2\pi i a}$.

For an integer $n \in \mathbb{Z}$ we denote by $L(n)$ the lattice L equipped with a quadratic form $q^{(n)}(\ell) := nq(\ell)$. In the case $n = -1$ the lattices $L'(-1)$ and $(L(-1))'$ coincide and hence the groups L'/L and $L(-1)'/L(-1)$ are equal. Both representations ρ_L and $\rho_{L(-1)}$ act on $\mathbb{C}[L'/L]$ and for $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$ we have $\rho_{L(-1)}(\gamma) = \overline{\rho_L(\gamma)}$.

A vector valued modular form of half-integral weight k and representation ρ_L is a function $f\colon \mathfrak{H} \to \mathbb{C}[L'/L]$ that satisfies the following transformation law:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \sqrt{c\tau+d}^{2k} \rho_L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau+d}\right) f(\tau).$$

We will use the notation $\mathfrak{M}_k(\rho_L)$ for the space of real analytic, $M_k(\rho_L)$ for the space of holomorphic, $\widehat{M}_k(\rho_L)$ for the space of almost holomorphic, and $M^!_k(\rho_L)$ for the space of weakly holomorphic modular forms of weight k and representation ρ_L.

Now we recall some standard maps between the spaces of vector valued modular forms associated to different lattices [6].

If $M \subset L$ is a sublattice of finite index then a vector valued modular form $f \in \mathfrak{M}_k(\rho_L)$ can be naturally viewed as a vector valued modular form $f \in \mathfrak{M}_k(\rho_M)$. Indeed, we have the inclusions

$$M \subset L \subset L' \subset M'$$

and therefore

$$L/M \subset L'/M \subset M'/M.$$

We have the natural map $L'/M \to L'/L$, $\mu \to \bar{\mu}$.

Lemma 2.1. *For $\mathcal{M} = \mathfrak{M}, M, \widehat{M}$ or $M^!$ there are two natural maps*

$$\mathrm{res}_{L/M} \colon \mathcal{M}_k(\rho_L) \to \mathcal{M}_k(\rho_M)$$
$$\mathrm{tr}_{L/M} \colon \mathcal{M}_k(\rho_M) \to \mathcal{M}_k(\rho_L)$$

given by

$$\big(\mathrm{res}_{L/M}(f)\big)_\mu = \begin{cases} f_{\bar{\mu}} & \text{if } \mu \in L'/M \\ 0 & \text{if } \mu \notin L'/M \end{cases} \qquad \big(f \in \mathcal{M}_k(\rho_L),\ \mu \in M'/M\big) \tag{2.5}$$

$$\big(\mathrm{tr}_{L/M}(g)\big)_\lambda = \sum_{\substack{\mu \in L'/M \\ \bar{\mu} = \lambda}} g_\mu \qquad \big(g \in \mathcal{M}_k(\rho_M),\ \ \lambda \in L'/L\big). \tag{2.6}$$

Now suppose that M and N are two even lattices and $L = M \oplus N$. Then we have

$$L'/L \cong (M'/M) \oplus (N'/N).$$

Moreover

$$\mathbb{C}[L'/L] \cong \mathbb{C}[M'/M] \otimes \mathbb{C}[N'/N]$$

as unitary vector spaces and naturally

$$\rho_L = \rho_M \otimes \rho_N.$$

Lemma 2.2. *For two modular forms $f \in \mathcal{M}_k(\rho_L)$ and $g \in \mathcal{M}_l(\rho_{M(-1)})$ the function*

$$h := \langle f, g \rangle_{\mathbb{C}[M'/M]} = \sum_{\nu \in N'/N} e_\nu \sum_{\mu \in M'/M} f_{\mu \oplus \nu}\, g_\mu$$

belongs to $\mathcal{M}_{k+l}(\rho_N)$.

3. Regularized theta lift

In this section we recall the definition of the regularized theta lift given by R. Borcherds in the paper [2].

We let (L, q) be an even lattice of signature $(2, b)$ with dual L'. The (positive) Grassmannian $\mathrm{Gr}^+(L)$ is the set of positive definite two-dimensional subspaces v^+ of $L\otimes\mathbb{R}$. We write v^- for the orthogonal complement of v^+, so that $L\otimes\mathbb{R}$ is the orthogonal direct sum of the positive definite subspace v^+ and the negative definite subspace v^-. The projection of a vector $\ell \in L\otimes\mathbb{R}$ into the subspaces v^+ and v^- is denoted by ℓ_{v^+} and ℓ_{v^-} respectively, so that $\ell = \ell_{v^+} + \ell_{v^-}$.

The vector valued Siegel theta function $\Theta_L\colon \mathfrak{H}\times\mathrm{Gr}^+(L) \to \mathbb{C}[L'/L]$ of L is defined by

$$\Theta_L(\tau, v^+) = \Im(\tau)^{b/2} \sum_{\ell\in L'} \mathrm{e}\big(q(\ell_{v^+})\tau + q(\ell_{v^-})\bar{\tau}\big)\, e_{\ell+L}. \tag{3.1}$$

Remark 3.1. Our definition of Θ_L differs from the one given in [2] by the multiple $\Im(\tau)^{b/2}$.

Theorem 4.1 in [2] says that $\Theta_L(\tau, v^+)$ is a real analytic vector valued modular form of weight $1 - b/2$ and representation ρ_L with respect to the variable τ.

We suppose that f is some $\mathbb{C}[L'/L]$-valued function on the upper half plane $\mathfrak{H}$ transforming under $\mathrm{SL}_2(\mathbb{Z})$ with weight $1-b/2$ and representation ρ_L. Define a *regularized theta integral* as

$$\Phi_L(v^+, f) := \int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} \langle f(\tau), \Theta_L(\tau, v^+)\rangle\, y^{-1-b/2}\, dx\, dy, \quad \tau = x + iy. \tag{3.2}$$

The integral is often divergent and has to be regularized. In [2] Borcherds suggested the following method. We integrate over the region

$$\mathcal{F}_t = \{\tau\in\mathfrak{H} \mid -1/2 < \Re(\tau) < 1/2, \text{ and } |\tau| > 1, \text{ and } \Im\tau < t\}.$$

Note that $\mathcal{F}_\infty := \bigcup_{t>0}\mathcal{F}_t$ is a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$. Suppose that for $\Re(s) \gg 0$ the limit

$$\lim_{t\to\infty}\int_{\mathcal{F}_t} \langle f(\tau), \Theta_L(\tau, v^+)\rangle\, y^{-1-b/2-s}\, dx\, dy$$

exists and can be continued to a meromorphic function defined for all complex s. Then we define

$$\int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} \langle f(\tau), \Theta_L(\tau, v^+)\rangle\, y^{-1-b/2}\, dx\, dy$$

to be the constant term of the Laurent expansion of this function at $s = 0$. In all the cases considered in this paper the limit as t goes to infinity already exists at $s = 0$.

Denote by $\mathrm{Aut}(L)$ the group of those isometries of $(L \otimes \mathbb{Q}, q)$ which map L to itself. The action of $\mathrm{Aut}(L)$ on f is given by the action on L'/L. We define $\mathrm{Aut}(L, f)$ to be the subgroup of $\mathrm{Aut}(L)$ fixing f. The regularized integral $\Phi_L(v^+, f)$ is a function on the Grassmannian $\mathrm{Gr}^+(L)$ that is invariant under $\mathrm{Aut}(L, f)$. Let $\mathrm{Aut}(L, L')$ be the group of those elements in $\mathrm{Aut}(L)$ which act trivially on L'/L. Clearly, $\mathrm{Aut}(L, L')$ is a subgroup of $\mathrm{Aut}(L, f)$.

Suppose that $f \in \widehat{M}^!(\mathrm{Mp}_2(\mathbb{Z}), \rho_L)$ has a Fourier expansion

$$f_\mu(\tau) = \sum_{n\in\mathbb{Q}} \sum_{t\in\mathbb{Z}} c_\mu(n,t)\, \mathrm{e}(n,\tau)\, y^{-t}$$

and the coefficients $c_\mu(n, t)$ vanish whenever $n \ll 0$ or $t < 0$ or $t \gg 0$.

We will say that a function f has singularities of type g at a point if $f - g$ can be redefined on a set of codimension at least 1 so that it becomes real analytic near the point.

Then the following theorem, which is proved in [2], describes the singularities of regularized theta lift $\Phi_L(v^+, f)$.

Theorem B1 ([2, Theorem 6.2])**.** *Near the point $v_0^+ \in \mathrm{Gr}^+(L)$, the function $\Phi_L(v^+, f)$ has a singularity of type*

$$\sum_{t\geq 0} \sum_{\substack{\ell\in L'\cap v_0^- \\ \ell\neq 0}} -c_{\ell+L}\big(q(\ell), t\big) \left(-4\pi\, q(\ell_{v^+})\right)^t \log\big(q(\ell_{v^+})\big)/t!\ .$$

In particular Φ_L is nonsingular (real analytic) except along a locally finite set of codimension 2 sub-Grassmannians (isomorphic to $\mathrm{Gr}^+(2, b-1)$) of $\mathrm{Gr}^+(L)$ of the form $\ell^\perp$ for some negative norm vectors $\ell \in L$.

The Grassmannians of signature $(2, b)$ are of particular interest for us since they can be equipped with a complex structure. A complex structure can be introduced as follows. The open subset

$$\mathcal{P} = \{[Z] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (Z, Z) = 0 \text{ and } (Z, \overline{Z}) > 0\}$$

is isomorphic to two copies of $\mathrm{Gr}^+(L)$ by mapping $[Z]$ to the oriented subspace spanned by $\Re(Z)$, $\Im(Z)$.

Next, we recall a convenient coordinate system on $\mathrm{Gr}^+(L)$ introduced in [2]. To this end we need the additional assumption that L contains a primitive vector of length 0. We choose $m \in L$, $m' \in L'$ such that $m^2 = 0$ and $(m, m') = 1$. Denote $V_0 := L \otimes \mathbb{Q} \cap m^\perp \cap m'^\perp$. The tube domain

$$\mathcal{H} = \{z \in V_0 \otimes_{\mathbb{R}} \mathbb{C} \mid (\Im(z), \Im(z)) > 0\} \tag{3.3}$$

is isomorphic to $\mathcal{P}$ by mapping $z \in \mathcal{H}$ to the class in $\mathbb{P}(L \otimes \mathbb{C})$ of

$$Z(z) = z + m' - \frac{1}{2}((z, z) + (m', m'))m.$$

We consider the lattices $M = L \cap m^\perp$ and $K = (L \cap m^\perp)/\mathbb{Z}m$, and we identify $K \otimes \mathbb{R}$ with the subspace $L \otimes \mathbb{R} \cap m^\perp \cap m'^\perp$.

We write N for the smallest positive value of the inner product (m, ℓ) with $\ell \in L$, so that $|L'/L| = N^2|K'/K|$.

Suppose that $f = \sum_\mu e_\mu f_{L+\mu}$ is a modular form of type ρ_L and half integral weight k. Define a $\mathbb{C}[K'/K]$-valued function

$$f_K(\tau) = \sum_{\kappa \in K'/K} f_{K+\kappa}(\tau)\, e_\kappa$$

by putting

$$f_{K+\kappa}(\tau) = \sum_{\substack{\mu \in L'/L \\ \mu|M = \kappa}} f_{L+\mu}(\tau)$$

for $\kappa \in K$. The notation $\lambda|M$ means the restriction of $\lambda \in \mathrm{Hom}(L, \mathbb{Z})$ to M, and $\gamma \in \mathrm{Hom}(K, \mathbb{Z})$ is considered as an element of $\mathrm{Hom}(M, \mathbb{Z})$ using the quotient map from M to K. The elements of L' whose restriction to M is 0 are exactly the integer multiples of m/N.

For $z \in \mathcal{H}$ denote by w^+ the following positive-definite subspace of V_0

$$w^+(z) = \mathbb{R}\Im(z) \in \mathrm{Gr}^+(K).$$

Theorem 7.1 in [2] gives the Fourier expansion of the regularized theta lift. In the case where the lattice L has signature $(2, b)$ this theorem can be reformulated in the following form:

Theorem B2 **(**[2, Theorem 7.1]**).** *Let L, K, m, m' be defined as above. Suppose*

$$f = \sum_{\mu \in L'/L} e_\mu \sum_{m \in \mathbb{Q}} c_\mu(m, y)\, \mathrm{e}(mx)$$

is a modular form of weight $1 - b$ and type ρ_L with at most exponential growth as $y \to \infty$. Assume that each function $c_\mu(m, y)\exp(-2\pi|m|y)$ has an asymptotic expansion as $y \to \infty$ whose terms are constants times products of complex powers of y and nonnegative integral powers of $\log(y)$. Let $z = u + iv$ be an element of a tube domain $\mathcal{H}$. If (v, v) is sufficiently large then the Fourier expansion of $\Phi_L(v^+(z), f)$ is given by the constant term of the Laurent expansion at $s = 0$ of the analytic continuation of

$$\sqrt{q(v)}\Phi_K(w^+(z), f_K) + \frac{1}{\sqrt{q(v)}} \sum_{\ell \in K'} \sum_{\substack{\mu \in L'/L \\ \mu|M = \ell}} \sum_{n > 0} \mathrm{e}\big((n\ell, u - m') + (n\mu, m')\big) \times$$
$$\times \int\limits_{y>0} c_\mu\left(q(\ell), y\right) \exp\Big(-\frac{\pi n^2 q(v)}{y} - \pi y\Big(\frac{(\ell, v)^2}{q(v)} - 2q(\ell)\Big)\Big)\, y^{-s-3/2}\, dy \tag{3.4}$$

(which converges for $\Re(s) \gg 0$ to a holomorphic function of s which can be analytically continued to a meromorphic function of all complex s).

The lattice K has signature $(1, b-1)$, so $\mathrm{Gr}^+(K)$ is a real hyperbolic space of dimension $b-1$ and the singularities of Φ_K lie on hyperplanes of codimension 1. Then the set of points where Φ_K is real analytic is not connected. The components of the points where Φ_K is real analytic are called the Weyl chambers of Φ_K. If W is a Weyl chamber and $\ell \in K$ then $(\ell, W) > 0$ means that ℓ has positive inner product with all elements in the interior of W.

4. Infinite products

We see from Theorem B1 that the theta lift of a weakly holomorphic modular form has logarithmic singularities along special divisors. The following theorem relates regularized theta lifts with infinite products introduced in Borcherds's earlier paper [1].

Theorem B3 ([2, Theorem 13.3]). *Suppose that $f \in M^!_{1-b/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_L)$ has a Fourier expansion*

$$f(\tau) = \sum_{\lambda \in L'/L} \sum_{n \gg -\infty} c_\lambda(n)\, \mathrm{e}(n\tau)\, e_\lambda$$

and the Fourier coefficients $c_\lambda(n)$ are integers for $n \leq 0$. Then there is a meromorphic function $\Psi_L(Z, f)$ on $\mathcal{L}$ with the following properties:

1. *Ψ is an automorphic form of weight $c_0(0)/2$ for the group* $\mathrm{Aut}(L, f)$ *with respect to some unitary character of* $\mathrm{Aut}(L, f)$.
2. *The only zeros and poles of Ψ_L lie on the rational quadratic divisors $\ell^\perp$ for $\ell \in L$, $q(\ell) < 0$ and are zeros of order*
$$\sum_{\substack{x \in \mathbb{R}^+ \\ x\ell \in L'}} c_{x\ell}\big(q(x\ell)\big).$$
3. *The following equality holds:*
$$\Phi_L(Z, f) = -4 \log |\Psi_L(Z, f)| - 2c_0(0)(\log |Y| + \Gamma'(1)/2 + \log \sqrt{2\pi}).$$
4. *For each primitive norm 0 vector m of L and for each Weyl chamber W of K the restriction $\Psi_m(Z(z), f)$ has an infinite product expansion converging when z is in a neighborhood of the cusp of m and $\Im(z) \in W$ which is some constant of absolute value*
$$\prod_{\substack{\delta \in \mathbb{Z}/N\mathbb{Z} \\ \delta \neq 0}} (1 - \mathrm{e}(\delta/N))^{c_{\delta m/N}(0)/2}$$
times
$$\mathrm{e}((Z, \rho(K, W, f_K))) \prod_{\substack{k \in K' \\ (k, W) > 0}} \prod_{\substack{\mu \in L'/L \\ \mu | M = k}} (1 - \mathrm{e}((k, Z) + (\mu, m')))^{c_\mu(k^2/2)}.$$

The vector $\rho(K, W, f_K)$ is the Weyl vector, which can be evaluated explicitly using the theorems in [2, Section 10].

Remark 4.1. In the case where L has no primitive norm 0 vectors Fourier expansions of Ψ do not exist.

Remark 4.2. We say that $c_0(0)$ is the constant term of f.

5. Differential operators

For $k \in \mathbb{Z}$ denote by R_k and L_k the Maass rasing and lowering differential operators

$$R_k = \frac{1}{2\pi i}\left(\frac{\partial}{\partial\tau} + \frac{k}{\tau - \bar\tau}\right), \;\; L_k = \frac{1}{2\pi i}\,(\tau - \bar\tau)^2\,\frac{\partial}{\partial\bar\tau}.$$

The the weight k Laplace operator is given by

$$\Delta_k = -4\pi^2\,R_{k-2}\,L_k = -4\pi^2\,(L_{k+2}\,R_k - k) = (\tau - \bar\tau)^2\frac{\partial^2}{\partial\tau\partial\bar\tau} + k(\tau - \bar\tau)\frac{\partial}{\partial\bar\tau}.$$

For integers l, k we denote by $F_{l,k}$ the space of functions of weight k satisfying

$$\Delta_k f = \left(l(1-l) + \frac{k(k-2)}{4}\right) f.$$

We will use the following well-known properties of the spaces $F_{l,k}$ and differential operators L_k, R_k.

Proposition 5.1. *The spaces $F_{l,k}$ satisfy the following properties:*

(i) The space $F_{l,k}$ is invariant under the action of the group $\mathrm{SL}_2(\mathbb{R})$*,*

(ii) The operator R_k maps $F_{l,k}$ to $F_{l,k+2}$,

(iii) The operator L_k maps $F_{l,k}$ to $F_{l,k-2}$.

For a modular form f of weight k we will use the notation

$$R^r f = R^r_k(f) = R_{k+2r-2}\circ\cdots\circ R_k f.$$

For simplicity we drop the weight subscript in R^r_k.

Denote $f^{(s)} := \frac{1}{(2\pi i)^s}\frac{\partial^s}{\partial\tau^s} f$. We have [5, equation (56)]

$$R^r(f) = \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}\frac{(k+s)_{r-s}}{(4\pi y)^{r-s}}\,f^{(s)}, \tag{5.1}$$

where $(a)_m = a(a+1)\cdots(a+m-1)$ is the Pochhammer symbol. For modular forms f and g of weight k and l the Rankin–Cohen bracket is defined by

$$[f, g] = l f' g - k f g',$$

and more generally

$$[f, g]_r = \sum_{s=0}^{r} (-1)^s \binom{k+r-1}{s}\binom{l+r-1}{r-s} f^{(r-s)}\,g^{(s)}.$$

The function $[f,g]_r$ is a modular form of weight $k+l+2r$. Note that

$$\binom{k}{s} = \frac{(k-s+1)_s}{s!}$$

is defined for $s \in \mathbb{N}$ and arbitrary k.

We will need the following proposition.

Proposition 5.2. *Suppose that f and g are modular forms of weight k and l respectively. Then, for integer $r \geq 0$ we have*

$$R^r(f)\,g = a[f,g]_r + R\biggl(\sum_{s=0}^{r-1} b_s\, R^s(f)\, R^{r-s-1}(g)\biggr)$$

where

$$a = \binom{k+l+2r-2}{r}^{-1}$$

and b_s are some rational numbers.

Proof. The operator R satisfies the following property:

$$R(fg) = R(f)g + fR(g).$$

Thus, the sum

$$\sum_{i+j=r} a_i R^i(f) R^j(g)$$

can be written as

$$R\biggl(\sum_{i+j=r-1} b_i\, R^i(f)\, R^j(g)\biggr)$$

for some numbers b_i if and only if $\sum_{i=0}^{r}(-1)^i a_i = 0$. For the Rankin–Cohen brackets the following identity holds:

$$[f,g]_r = \sum_{s=0}^{r} (-1)^s \binom{k+r-1}{s}\binom{l+r-1}{r-s} R^{(r-s)}(f) R^s(g). \tag{5.2}$$

We will use the following standard identity:

$$\sum_{s=0}^{r} \binom{k+r-1}{s}\binom{l+r-1}{r-s} = \binom{k+l+2r-2}{r}.$$

It follows from the above formula and (5.2) that the sum

$$\binom{k+l+2r-2}{r} R^r(f)g - [f,g]_r$$

can be written in the form

$$R\biggl(\sum_{i+j=r-1} b_i\, R^i(f)\, R^j(g)\biggr).$$

This finishes the proof. ∎

Proposition 5.3. *Suppose that f is a real analytic modular form of weight $k-2$ and g is a holomorphic modular form of weight k. Then, for a compact region $F \subset \mathfrak{H}$ we have*

$$\int_F R_{k-2}(f)\, \bar{g}\, y^{k-2}\, dx\, dy = \int_{\partial F} f\, \bar{g}\, y^{k-2}\, (dx - idy).$$

Proof. The proposition follows immediately from Stokes's theorem. □

Denote by K_ν the Bessel K-function

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \quad K_\nu(x) = \frac{\pi}{2}\, \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}.$$

The function K_ν becomes elementary for $\nu \in \mathbb{Z} + 1/2$. We have

$$K_{k+\frac{1}{2}}(x) = \frac{(\pi/2)^{\frac{1}{2}}}{x^{k+\frac{1}{2}}}\, e^{-x}\, h_k(x),$$

for $k \in \mathbb{Z}_{\geq 0}$, where h_k is the polynomial

$$h_k(x) = \sum_{r=0}^{k} \frac{(k+r)!}{2^r r!(k-r)!} x^{k-r}.$$

The following statement follows immediately from equation (5.1):

Proposition 5.4. *For $k \in \mathbb{Z}_{>0}$ the following identity holds:*

$$R_{-2k}^{k}(\mathrm{e}(n\tau)) = 2\, y^{\frac{1}{2}}\, n^{k+\frac{1}{2}}\, K_{k+1/2}(2\pi n y)\, \mathrm{e}(nx).$$

6. A see-saw identity

Suppose that (V, q) is a rational quadratic space of signature $(2, b)$ and $L \subset V$ is an even lattice. Let $V = V_1 \oplus V_2$ be the rational orthogonal splitting of (V, q) such that the space V_1 has signature $(2, b-d)$ and the space V_2 has signature $(0, d)$. Consider two lattices $N := L \cap V_1$ and $M := L \cap V_2$. We have two orthogonal projections

$$\mathrm{pr}_M \colon L \otimes \mathbb{R} \to M \otimes \mathbb{R} \quad \text{and} \quad \mathrm{pr}_N \colon L \otimes \mathbb{R} \to N \otimes \mathbb{R}.$$

Let M' and N' be the dual lattices of M and N. We have the following inclusions

$$M \subset L,\ N \subset L,\ M \oplus N \subseteq L \subseteq L' \subseteq M' \oplus N',$$

and equalities of the sets

$$\mathrm{pr}_M(L') = M',\ \mathrm{pr}_N(L') = N'.$$

Consider a rectangular $|L'/L| \times |N'/N|$-dimensional matrix $T_{L,N}$ with entries

$$\vartheta_{\lambda,\nu}(\tau) = \sum_{\substack{m \in M' \\ m+\nu \in \lambda + L}} \mathrm{e}\big(-q(m)\tau\big)$$

where $\lambda \in L'/L, \nu \in N'/N, \tau \in \mathfrak{H}$. This sum is well defined since $N \subset L$. Note that the lattice M is negative definite and hence the series converge.

Theorem 6.1. *Suppose that the lattices L, M and N are as above. Then there is a map $T_{L,N}\colon M_k(\rho_L) \to M_{k+d/2}(\rho_N)$ sending a function $f = (f_\lambda)_{\lambda \in L'/L}$ to the function $g = (g_\nu)_{\nu \in N'/N}$ defined as*

$$g_\nu(\tau) = \sum_{\lambda \in L'/L} \vartheta_{\lambda,\nu}(\tau)\, f_\lambda(\tau). \tag{6.1}$$

In other words,

$$g = T_{L,N} f$$

where f and g are considered as column vectors.

Proof. Consider the function

$$\Theta_{M(-1)}(\tau) = \Im(\tau)^{-d/2}\overline{\Theta_M(\tau)} = \sum_{\mu \in M'/M} e_\mu \sum_{m \in M+\mu} \mathrm{e}(-q(m)\tau)$$

that belongs to $M_{d/2}(\rho_{M(-1)})$. It follows from (6.1) that

$$T_{L,N}(f) = \big\langle \mathrm{res}_{L/M\oplus N}(f), \Theta_{M(-1)} \big\rangle_{\mathbb{C}[M'/M]}.$$

Thus, from Lemma 2.2 we deduce that $T_{L,N}(f)$ is in $M_{k+d/2}(\rho_N)$. □

Theorem 6.2. *Let L, M, N be as above. Denote by $i\colon \mathrm{Gr}^+(N) \to \mathrm{Gr}^+(L)$ the natural embedding induced by the inclusion $N \subset L$. Then, for $v^+ \in \mathrm{Gr}^+(N)$ the theta lift of a function $f \in \widehat{M}^!_{1-b/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_L)$ the following holds:*

$$\Phi_L(i(v^+), f) = \Phi_N(v^+, T_{L,N}(f)). \tag{6.2}$$

Proof. For a vector $\ell \in L'$ denote $m = \mathrm{pr}_M(\ell)$ and $n = \mathrm{pr}_N(\ell)$. Recall that $m \in M'$ and $n \in N'$. Since v^+ is an element of $\mathrm{Gr}^+(N)$ it is orthogonal to M. We have

$$q(\ell_{v^+}) = q(n_{v^+}), \quad q(\ell_{v^-}) = q(m) + q(n_{v^-}).$$

Thus for $\lambda \in L'/L$ we obtain

$$\begin{aligned}\Theta_{\lambda+L}(\tau, v^+) &= \sum_{\ell \in \lambda+L} \mathrm{e}\big(q(\ell_{v^+})\tau + q(\ell_{v^-})\bar{\tau}\big) \\ &= \sum_{\substack{m \in M',\, n \in N' \\ m+n \in \lambda+L}} \mathrm{e}\big(q(n_{v^+})\tau + q(n_{v^-})\bar{\tau} + q(m)\bar{\tau}\big).\end{aligned}$$

Since $N \subset L$ we can rewrite this sum as

$$\Theta_{\lambda+L}(\tau, v^+) = \sum_{\nu \in N'/N} \Theta_{\nu+N}(\tau, v^+)\, \overline{\vartheta_{\nu,\lambda}(\tau)}.$$

Thus, we see that for $f = (f_\lambda)_{\lambda \in L'/L}$ the following scalar products are equal:

$$\langle f, \Theta_L(\tau, v^+) \rangle = \langle T_{L,N}(f), \Theta_N(\tau, v^+) \rangle.$$

So, the regularized integrals (3.2) of both sides of the equality are also equal. □

Remark 6.3. Theorem 6.2 works even in the case where v^+ is a singular point of $\Phi_L(v^+, f)$. If the constant terms of f and $T_{L,N}(f)$ are different, then the subvariety $\mathrm{Gr}^+(N)$ lies in singular locus of $\Phi_L(v^+, f)$. On the other hand, if the constant terms of f and $T_{L,N}(f)$ are equal, then the singularities cancel at the points of $\mathrm{Gr}^+(N)$.

Remark 6.4. The map $T_{M,N}$ is essentially the *contraction map* defined in [12, Paragraph 3.2].

7. The lattice $M_2(\mathbb{Z})$

Consider the lattice of integral 2×2 matrices denoted by $M_2(\mathbb{Z})$. Equipped with the quadratic form $q(x) := -\det x$ it becomes an even unimodular lattice.

The Grassmannian $\mathrm{Gr}^+(M_2(\mathbb{Z}))$ is isomorphic to $\mathfrak{H} \times \mathfrak{H}$ as a complex manifold. This isomorphism can be constructed in the following way. For a pair of points $(\tau_1, \tau_2) \in \mathfrak{H} \times \mathfrak{H}$ consider the element of norm zero

$$Z = \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \in M_2(\mathbb{Z}) \otimes \mathbb{C}.$$

Define $v^+(\tau_1, \tau_2)$ to be the vector subspace of $M_2(\mathbb{Z}) \otimes \mathbb{R}$ spanned by two vectors $X = \Re(Z)$ and $Y = \Im(Z)$. The group $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ acts on $M_2(\mathbb{Z})$ by $(\gamma_1, \gamma_2)(x) = \gamma_1 x \gamma_2^t$ and preserves the norm. The action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ on the Grassmannian agrees with the action on $\mathfrak{H} \times \mathfrak{H}$ by fractional linear transformations

$$(\gamma_1, \gamma_2)(v^+(\tau_1, \tau_2)) = v^+(\gamma_1(\tau_1), \gamma_2(\tau_2)).$$

We have

$$X^2 = Y^2 = \frac{1}{2}(Z, \overline{Z}) = -\frac{1}{2}(\tau_1 - \bar{\tau}_1)(\tau_2 - \bar{\tau}_2),$$

$$(X, Y) = Z^2 = 0.$$

For $\ell = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ and $v^+ = v^+(\tau_1, \tau_2)$ we have

$$q(\ell_{v^+}) = \frac{(\ell, Z)(\ell, \overline{Z})}{(Z, \overline{Z})} = \frac{|\, d\tau_1\tau_2 - c\tau_1 - b\tau_2 + a\,|^2}{-(\tau_1 - \bar{\tau}_1)(\tau_2 - \bar{\tau}_2)}.$$

Denote

$$\Theta(\tau; \tau_1, \tau_2) := \Theta_{M_2(\mathbb{Z})}\big(\tau, v^+(\tau_1, \tau_2)\big)$$

where $\tau = x + iy$. Considered as a function of τ, the kernel Θ belongs to $\mathfrak{M}_0(\mathrm{SL}_2(\mathbb{Z}))$ and we can explicitly write this function as

$$\Theta(\tau;\tau_1,\tau_2)=y\sum_{a,b,c,d\in\mathbb{Z}} \mathrm{e}\left(\frac{|a\tau_1\tau_2+b\tau_1+c\tau_2+d|^2}{-(\tau_1-\bar{\tau}_1)(\tau_2-\bar{\tau}_2)}(\tau-\bar{\tau})-(ad-bc)\bar{\tau}\right)$$
$$=y\sum_{a,b,c,d\in\mathbb{Z}} \mathrm{e}\left(\frac{|a\tau_1\tau_2+b\tau_1+c\tau_2+d|^2}{-(\tau_1-\bar{\tau}_1)(\tau_2-\bar{\tau}_2)}\tau-\frac{|a\tau_1\bar{\tau}_2+b\tau_1+c\bar{\tau}_2+d|^2}{-(\tau_1-\bar{\tau}_1)(\tau_2-\bar{\tau}_2)}\bar{\tau}\right).$$

8. Higher Green's functions as theta lifts

The key point of our proof is the following observation:

Proposition 8.1. *Denote by Δ^z the hyperbolic Laplacian with respect to the variable z. For the function Θ defined in the previous section the following identities hold:*

$$\Delta^{\tau}\Theta(\tau;\tau_1,\tau_2)=\Delta^{\tau_1}\Theta(\tau;\tau_1,\tau_2)=\Delta^{\tau_2}\Theta(\tau;\tau_1,\tau_2).$$

A similar identity can be found in [4].

Suppose that $\boldsymbol{\lambda}=\{\lambda_m\}_{m=1}^{\infty}$ is a relation for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ (the definition is given in the introduction). Then there exists a unique weakly holomorphic modular form $g_{\boldsymbol{\lambda}}$ of weight $2-2k$ with Fourier expansion of the form

$$\sum_m \lambda_m\, q^{-m}+O(1).$$

Consider the function $h_{\boldsymbol{\lambda}}:=R^{k-1}(g_{\boldsymbol{\lambda}})$ which belongs to $\widehat{M}_0^!(\mathrm{SL}_2(\mathbb{Z}))$.

Theorem 8.2. *The following identity holds:*

$$G_{k,\boldsymbol{\lambda}}(\tau_1,\tau_2)=\Phi_{M_2(\mathbb{Z})}(v^+(\tau_1,\tau_2),h_{\boldsymbol{\lambda}}).$$

Here

$$\Phi_{M_2(\mathbb{Z})}(v^+(\tau_1,\tau_2),h_{\boldsymbol{\lambda}})=\lim_{t\to\infty}\int_{F_t} h_{\boldsymbol{\lambda}}(\tau)\,\overline{\Theta(\tau;\tau_1,\tau_2)}\,y^{-2}dx\,dy. \tag{8.1}$$

Proof. We verify that the function $\Phi_{M_2(\mathbb{Z})}(v^+(\tau_1,\tau_2),h_{\boldsymbol{\lambda}})$ satisfies conditions (i) to (v) listed at the introduction.

Firstly, we verify property (i). Let $T_m\subset \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}\times\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ be the Hecke correspondence. We recall that T_m is defined as

$$T_m=\{(\tau_1,\tau_2)\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}\times\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}\mid \tau_2=\frac{a\tau_1+b}{c\tau_1+d}\text{ for some }\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathcal{M}_m\}.$$

For a relation $\boldsymbol{\lambda}$ consider a divisor

$$D_{\boldsymbol{\lambda}}:=\sum_m \lambda_m T_m.$$

Denote by $S_{\boldsymbol{\lambda}}$ the support of $D_{\boldsymbol{\lambda}}$. It follows from properties (i) and (iv) of Green's function given at the introduction that the singular locus of $G_{k,\boldsymbol{\lambda}}$ is equal to $S_{\boldsymbol{\lambda}}$. It follows from Theorem B1 ([2, Theorem 6.2, p. 24]) that the limit (8.1) exists for all $\tau_1, \tau_2 \in \mathfrak{H} \times \mathfrak{H} \setminus S_{\boldsymbol{\lambda}}$; moreover, it defines a real analytic function on this set. Next, we apply the argument given in [2] to our setting. The function $h_{\boldsymbol{\lambda}}$ has Fourier expansion

$$h_{\boldsymbol{\lambda}}(\tau) = \sum_{\substack{n\in\mathbb{Z} \\ n\gg-\infty}} c(n,y)\, \mathrm{e}(n\tau).$$

Fix $v^+ = v^+(\tau_1, \tau_2)$ for some $\tau_1, \tau_2 \in \mathfrak{H} \times \mathfrak{H}$. For each $t > 1$ the set F_t can be decomposed into two parts $F_t = F_1 \cup \Pi_t$ where Π_t is a rectangle $\Pi_t = [-1/2, 1/2] \times [1, t]$. It suffices to show that the limit

$$\lim_{t\to\infty} \int_{\Pi_t} h_{\boldsymbol{\lambda}}(\tau)\, \overline{\Theta_{M_2(\mathbb{Z})}(\tau; v^+)}\, y^{-2} dx\, dy$$

exists for all $(\tau_1, \tau_2) \notin S_{\boldsymbol{\lambda}}$. This can be seen from the following computation:

$$\begin{aligned}
&\int\limits_{\Pi_t} h_{\boldsymbol{\lambda}}(\tau)\, \overline{\Theta_{M_2(\mathbb{Z})}(\tau; v^+)}\, y^{-2}\, dx\, dy \\
&\quad = \int\limits_{\Pi_t} \sum_{n\in\mathbb{Z}} \sum_{\ell\in M_2(\mathbb{Z})} c(n,y)\, \mathrm{e}(n\tau)\, \overline{\mathrm{e}\big(q(\ell_{v^+})\tau + q(\ell_{v^-})\big)}\, y^{-1}\, dx\, dy \\
&\quad = \int\limits_{-1/2}^{1/2} \int\limits_{1}^{t} \sum_{n\in\mathbb{Z}} \sum_{\ell\in M_2(\mathbb{Z})} c(n,y)\, \mathrm{e}\big(nx - q(\ell)x\big) \exp\big(-4\pi\, q(\ell_{v^+})\, y\big)\, y^{-1}\, dx\, dy \\
&\quad = \int\limits_{1}^{t} \sum_{\ell\in M_2(\mathbb{Z})} c\big(q(\ell), y\big) \exp\big(-4\pi\, q(\ell_{v^+})\, y\big)\, y^{-1}\, dy.
\end{aligned}$$

Properties (i) and (iv) follow from Theorem B1 ([2, Theorem 6.2, p. 24]).

Property (ii) is obvious since the function $\Theta(\tau; \tau_1, \tau_2)$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant in the variables τ_1 and τ_2.

Property (iii) formally follows from the property of theta kernel given in Proposition 8.1 and the fact that the Laplace operator is self adjoint with respect to the Petersson scalar product. More precisely, we have

$$\Delta^{\tau_1} \Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol{\lambda}}, v^+(\tau_1, \tau_2)) = \lim_{t\to\infty} \int\limits_{F_t} h_{\boldsymbol{\lambda}}(\tau) \overline{\Delta^{\tau_1} \Theta(\tau; \tau_1, \tau_2)} y^{-2} dx\, dy.$$

Using Proposition 8.1 we arrive at

$$\Delta^{\tau_1} \Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol{\lambda}}, v^+(\tau_1, \tau_2)) = \lim_{t\to\infty} \int\limits_{F_t} h_{\boldsymbol{\lambda}}(\tau) \overline{\Delta^{\tau} \Theta(\tau; \tau_1, \tau_2)} y^{-2} dx dy.$$

It follows from Stokes's theorem that

$$\int_{F_t} h_{\boldsymbol\lambda}(\tau)\overline{\Delta^\tau\Theta(\tau;\tau_1,\tau_2)}y^{-2}dxdy - \int_{F_t} \Delta h_{\boldsymbol\lambda}(\tau)\overline{\Theta(\tau;\tau_1,\tau_2)}y^{-2}dxdy$$

$$= \int_{-1/2}^{1/2} (h_{\boldsymbol\lambda}\,\overline{L_0(\Theta)} - L_0(h_{\boldsymbol\lambda})\,\overline{\Theta})y^{-2}dx\Bigg|_{y=t}.$$

This expression tends to zero as t tends to infinity. Since $g_{\boldsymbol\lambda} \in F_{k,2-2k}$ it follows from Proposition 5.1 that $\Delta h_{\boldsymbol\lambda} = k(1-k)h_{\boldsymbol\lambda}$. Thus, we see that the theta lift $\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol\lambda}, v^+)$ satisfies the desired differential equation

$$\Delta^{\tau_i}\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol\lambda}, v^+(\tau_1,\tau_2)) = k(1-k)\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol\lambda}, v^+(\tau_1,\tau_2)) \quad (i=1,2).$$

It remains to prove (v). To this end we compute the Fourier expansion of $\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol\lambda}, v^+(\tau_1,\tau_2))$. This can be done using Theorem B2. We select a primitive norm zero vector $m := \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and choose $m' := \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ so that $(m,m')=1$. For this choice of vectors m, m' the tube domain $\mathcal{H}$ defined by equation (3.3) is isomorphic to $\mathfrak{H}\times\mathfrak{H}$ and the map between $\mathfrak{H}\times\mathfrak{H}$ and the Grassmannian $\mathrm{Gr}^+(M_2(\mathbb{Z})$ is given by

$$(\tau_1,\tau_2)\to v^+(\tau_1,\tau_2).$$

The lattice $K=(M\cap m^\perp)/m$ can be identified with

$$M\cap m^\perp\cap m'^\perp = \left\{ \begin{pmatrix}0&b\\c&0\end{pmatrix} \middle| b,c\in\mathbb{Z}\right\}.$$

Set $x_i=\Re(\tau_i)$ and $y_i=\Im(\tau_i)$ for $i=1,2$. The subspace $w^+(\tau_1,\tau_2)\in\mathrm{Gr}^+(K)$ is equal to

$$\mathbb{R}\begin{pmatrix}0&y_1\\y_2&0\end{pmatrix}.$$

Suppose that the function $g_{\boldsymbol\lambda}$ has Fourier expansion

$$g_{\boldsymbol\lambda} = \sum_{n\in\mathbb{Z}} a(n)\mathrm{e}(n\tau).$$

It follows from Proposition 5.4 that

$$h_{\boldsymbol\lambda}(\tau) = \sum_{n\in\mathbb{Z}} c(n,y)\,\mathrm{e}(n\tau)$$

where

$$c(n,y) = a(n)\,y^{1/2}\,n^{k-1/2}\,K_{k-1/2}(2\pi ny)\exp(2\pi ny).$$

We write

$$c(n,y) = \sum_{t\geq 0} b(n,t)y^{-t}$$

for some complex numbers $b(n,t)$. We can rewrite equality (3.4) as

$$\Phi_M(v^+,h_{\boldsymbol\lambda}) = \frac{1}{\sqrt{2}|m_{v^+}|}\Phi_K(w^+,h) + \frac{\sqrt{2}}{|m_{v^+}|}\sum_{\ell\in K}\sum_{n>0} \mathrm{e}((nl,m''_v))\times$$
$$\times\int_0^\infty c\big(q(\ell),y\big)\exp\big(-\pi n^2/4q(m_{v^+})y - 4\pi q(\ell_{w^+})y\big)y^{-3/2}dy$$
$$= \sqrt{y_1y_2}\Phi_K(w^+(\tau_1,\tau_2),f_K) + \frac{1}{\sqrt{y_1y_2}}\sum_{\ell\in K}\sum_{n>0}\mathrm{e}((nl,u))\times$$
$$\times\int_0^\infty c\big(q(\ell),y\big)\exp\Big(-\frac{\pi n^2y_1y_2}{y} - \pi y\frac{(\ell,v)^2}{y_1y_2}\Big)\,y^{-3/2}dy, \qquad (8.2)$$

where $u=\Re\left(\begin{smallmatrix}0 & \tau_1\\ \tau_2 & 0\end{smallmatrix}\right)$ and $v=\Im\left(\begin{smallmatrix}0 & \tau_1\\ \tau_2 & 0\end{smallmatrix}\right)$. We choose a primitive norm zero vector $r=\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)\in K$. It follows from [2, Theorem 10.2] that

$$\Phi_K(w^+,h_{\boldsymbol\lambda}) = \sum_t b(0,t)(2r_{w^+}^2)^{t+1/2}\pi^{-t-1}\Gamma(t+1)(-2\pi i)^{2t+2}B_{2t+2}/(2t+2)!$$
$$= \sum_t b(0,t)(y_2/y_1)^{t+1/2}\pi^{-t-1}\Gamma(t+1)(-2\pi i)^{2t+2}B_{2t+2}/(2t+2)!. \qquad (8.3)$$

In the case where $\ell_{w^+}\neq 0$ it follows from [2, Lemma 7.2] that

$$\int_{y>0} c(\ell^2/2,y)\exp(-\pi n^2/2ym_{v^+}^2 - \pi y\ell_{w^+}^2)y^{-3/2}dy$$
$$= \sum_t 2b(\ell^2/2,t)(2|m_{v^+}|\,|\ell_{w^+}|/n)^{t+1/2}K_{-t-1/2}(2\pi n|\ell_{w^+}|/|m_{v^+}|). \qquad (8.4)$$

In case $\ell_{w^+}=0$ it follows from [2, Lemma 7.3] that

$$\int_{y>0} c(\ell^2/2,y)\exp(-\pi n^2/2ym_{v^+}^2 - 2\pi y\ell_{w^+}^2)y^{-3/2}dy$$
$$= \sum_t b(\ell^2/2,t)(2m_{v^+}^2/\pi n^2)^{t+1/2}\Gamma(t+1/2). \qquad (8.5)$$

Substituting formulas (8.3) to (8.5) into (8.2) we obtain

$$\Phi_M(v^+(\tau_1,\tau_2),h_{\boldsymbol\lambda}) = -\sum_t \frac{y_2^{t+1}}{y_1^t}b(0,t)(-4\pi)^{t+1}\zeta(-2t-1)\frac{t!}{(2t+1)!}$$
$$+4\sum_t (y_1y_2)^{-t}b(0,t)(4\pi)^{-t}\zeta(2t+1)\frac{2t!}{t!}$$
$$+4\sum_t\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}}\sum_{n>0}(y_1y_2)^{-t}b(cd,t)n^{-2t-1}\times$$
$$\times\,\mathrm{e}(ncx_1+ndx_2)|ncy_1+ndy_2|^{t+1/2}K_{-t-1/2}(2\pi|ncy_1+ndy_2|). \qquad (8.6)$$

We see from (8.6) that $\Phi_M(v^+(\tau_1,\tau_2),h_{\boldsymbol\lambda})\to 0$ as $y_1\to\infty$. This finishes the proof. □

Remark 8.3. The Fourier expansion of higher Green's functions is computed using a different method by Zagier in an unpublished note [15]. Also, we should say that Theorem 8.2 could be deduced from the results of Bruinier on theta lifts of real analytic Poincaré series [4].

9. CM values as regularized Petersson products

Now we can analyze the CM values of $G_{k,\boldsymbol\lambda}$ using the see-saw identity (6.2).

Let $\mathfrak{z}_1,\mathfrak{z}_2\in\mathfrak{H}$ be two CM points lying in the same quadratic imaginary field $\mathbb{Q}(\sqrt{-D})$. Let $v^+(\mathfrak{z}_1,\mathfrak{z}_2)$ be a two-dimensional positive definite subspace of $M_2(\mathbb{R})$ defined as

$$v^+(\mathfrak{z}_1,\mathfrak{z}_2)=\mathbb{R}\Re\begin{pmatrix}\mathfrak{z}_1\mathfrak{z}_2 & \mathfrak{z}_1\\ \mathfrak{z}_2 & 1\end{pmatrix}+\mathbb{R}\Im\begin{pmatrix}\mathfrak{z}_1\mathfrak{z}_2 & \mathfrak{z}_1\\ \mathfrak{z}_2 & 1\end{pmatrix}. \tag{9.1}$$

In the case where $\mathfrak{z}_1$ and $\mathfrak{z}_2$ lie in the same quadratic imaginary field the subspace $v^+(\mathfrak{z}_1,\mathfrak{z}_2)$ defines a rational splitting of $M_2(\mathbb{Z})\otimes\mathbb{Q}$. So, we can consider two lattices

$$N:=v^+(\mathfrak{z}_1,\mathfrak{z}_2)\cap M_2(\mathbb{Z})\quad\text{and}\quad M:=v^-(\mathfrak{z}_1,\mathfrak{z}_2)\cap M_2(\mathbb{Z}).$$

The Grassmannian $\mathrm{Gr}^+(N)$ consists of a single point $N\otimes\mathbb{R}$ and its image in $\mathrm{Gr}^+(M_2(\mathbb{Z}))$ is $v^+(\mathfrak{z}_1,\mathfrak{z}_2)$.

Since the lattice N has signature $(2,0)$ the theta lift of a function $f\in\widehat{M}^!_1(\mathrm{SL}_2(\mathbb{Z}),\rho_N)$ is just a number, which is equal to the regularized integral

$$\Phi_N(f)=\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}}\langle f(\tau),\Theta_N(\tau)\rangle\, y^{-1}\,dx\,dy. \tag{9.2}$$

Here Θ_N is the usual (vector valued) theta function of the lattice N. The matrix $T_{M_2(\mathbb{Z}),N}=(\vartheta_{0,\nu})_{\nu\in N'/N}$ becomes a vector in this case, given by

$$\vartheta_{0,\nu}(\tau)=\sum_{m\in M'\cap(-\nu+M_2(\mathbb{Z}))}\mathrm{e}(-\tau m^2/2).$$

Till the end of this section we will simply write $\vartheta_\nu(\tau)$ for $\vartheta_{0,\nu}(\tau)$.

Theorem 9.1. *Suppose given two CM-points $\mathfrak{z}_1,\mathfrak{z}_2$ and a lattice $N\subset M_2(\mathbb{Z})$ as above. Let $\boldsymbol\lambda$ be a relation for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ and let $g_{\boldsymbol\lambda}\in M^!_{2-2k}(\mathrm{SL}_2(\mathbb{Z}))$ be the corresponding weakly holomorphic modular form defined in Proposition 1.1. Then, if $(\mathfrak{z}_1,\mathfrak{z}_2)\notin S_{\boldsymbol\lambda}$ we have*

$$\Phi_{M_2(\mathbb{Z})}(v^+(\mathfrak{z}_1,\mathfrak{z}_2),R^{k-1}(g_{\boldsymbol\lambda}))=\Phi_N(f),$$

where $f=(f_\nu)_{\nu\in N'/N}\in M^!_1(\mathrm{SL}_2(\mathbb{Z}),\rho_N)$ is given by

$$f_\nu=[g_{\boldsymbol\lambda},\vartheta_\nu]_{k-1}.$$

Proof. For $(\mathfrak{z}_1, \mathfrak{z}_2) \notin S_{\boldsymbol{\lambda}}$ the constant term (with respect to $\mathrm{e}(x)$) of the product

$$\langle R^{k-1}(g_{\boldsymbol{\lambda}})(\tau), \Theta(\tau; \mathfrak{z}_1, \mathfrak{z}_2)\rangle$$

is equal to

$$\sum_{\ell \in M_2(\mathbb{Z})} y\, c\big(q(\ell), y\big) \exp(-2\pi y \ell_{v+}^2)$$

where $c\big(q(\ell), y\big)$ are the Fourier coefficients of $R^{k-1}(g_{\boldsymbol{\lambda}})$. This constant term decays as $O(y^{2-k})$ as $y \to \infty$. Thus,

$$\Phi_{M_2(\mathbb{Z})}(v^+(\mathfrak{z}_1, \mathfrak{z}_2), R^{k-1}(g_{\boldsymbol{\lambda}})) = \lim_{t\to\infty} \int_{F_t} R^{k-1}(g_{\boldsymbol{\lambda}})(\tau)\, \overline{\Theta(\tau; \mathfrak{z}_1, \mathfrak{z}_2)}\, y^{-2} dx\, dy.$$

It follows from the see-saw identity (6.2) that

$$\Phi_{M_2(\mathbb{Z})}(v^+(\mathfrak{z}_1, \mathfrak{z}_2), R^{k-1}(g_{\boldsymbol{\lambda}})) = \lim_{t\to\infty} \int_{F_t} \langle R^{k-1}(g_{\boldsymbol{\lambda}})\, \vartheta, \Theta_N\rangle y^{-1}\, dx\, dy.$$

By Proposition 5.2,

$$R^{k-1}(g_{\boldsymbol{\lambda}})\, \vartheta_\nu = (-1)^{k-1}[g_{\boldsymbol{\lambda}}, \vartheta_\nu]_{k-1} + R(\sum_{s=0}^{k-2} b_s R^s(g_{\boldsymbol{\lambda}})\, R^{k-2-s}(\vartheta_\nu)) \qquad (9.3)$$

where b_s are some rational numbers. For $\nu \in N'/N$ denote

$$\psi_\nu(\tau) := \sum_{s=0}^{k-2} b_s R^s(g_{\boldsymbol{\lambda}})\, R^{k-2-s}(\vartheta_\nu).$$

Using identity (9.3) we write

$$\begin{aligned}\lim_{t\to\infty} \int_{F_t} &\langle R^{k-1}(g_{\boldsymbol{\lambda}})\, \vartheta, \Theta_N\rangle y^{-1}\, dx\, dy \\ &= (-1)^{k-1} \lim_{t\to\infty} \int_{F_t} \langle [g_{\boldsymbol{\lambda}}, \vartheta]_{k-1}, \Theta_N\rangle y^{-1}\, dx\, dy \\ &\quad + \lim_{t\to\infty} \int_{F_t} \langle R(\psi), \Theta_N\rangle y^{-1}\, dx\, dy.\end{aligned}$$

It follows from Proposition 5.3 that

$$\lim_{t\to\infty} \int_{F_t} \langle R(\psi), \Theta_N\rangle y^{-1}\, dx\, dy = \lim_{t\to\infty} \int_{-1/2}^{1/2} \langle \psi(x+it), \Theta_N(x+it)\rangle t^{-1}\, dx = 0,$$

which finishes the proof □

10. Algebraicity of weight one Petersson products

Theorem 10.1. *Let N be an even lattice of signature $(2,0)$ and let $f \in M_1^!(\rho_N)$ be a modular form with zero constant term and integer Fourier coefficients. There exists an even lattice P of signature $(2,1)$ and a function $h \in M_{1/2}^!(\rho_P)$ such that*

1. *There is an inclusion $N \subset P$.*

2. *The lattice P contains a primitive norm zero vector.*
3. *The function h has rational Fourier coefficients. Moreover, $720h$ has integer Fourier coefficients.*
4. *The constant term of h is zero.*
5. *We have $T_{P,N}(h) = f$ for the map $T_{P,N}$ defined in Theorem 6.2.*

Proof. We adopt the method explained in [2, Lemma 8.1].

Consider two even unimodular definite lattices of dimension 24, say three copies $E_8 \oplus E_8 \oplus E_8$ of the E_8 root lattice and the Leech lattice Λ_{24}. We can embed both lattices into $\frac{1}{16}\mathbb{Z}^{24}$ (equipped with the standard Euclidean norm). To this end we use the standard representation of E_8 in which all vectors have half integral coordinates, the standard representation of the Leech lattice and the norm doubling map defined in [7, Chapter 8, p. 242]

Denote by M_1 and M_2 the negative definite lattices obtained from $E_8 \oplus E_8 \oplus E_8$ and Λ_{24} by multiplying the norm by -1 and assume that they are embedded into $\frac{1}{16}\mathbb{Z}^{24}$. Denote by M the negative definite lattice $16\mathbb{Z}^{24}$. The theta functions of the lattices M_1 and M_2 are modular forms of level 1 and weight 12 and their difference is 720Δ, where

$$\Delta(\tau) = q - 24q^2 + 252q^3 + O(q^4)$$

is the unique cusp form of level 1 and weight 12.

Consider the function g in $M^!_{-11}(\mathrm{SL}_2(\mathbb{Z}), \rho_{N\oplus M})$ defined as

$$g := \mathrm{res}_{(N\oplus M_1)/N\oplus M}(f/\Delta) - \mathrm{res}_{(N\oplus M_2)/N\oplus M}(f/\Delta).$$

The maps

$$\mathrm{res}_{(N\oplus M_i)/N\oplus M} \colon M^!_{-11}(\mathrm{SL}_2(\mathbb{Z}), \rho_{N\oplus M_i}) \to M^!_{-11}(\mathrm{SL}_2(\mathbb{Z}), \rho_{N\oplus M})$$

for $i = 1, 2$ are defined as in Lemma 2.1. It is easy to see from the definitions (2.5) and (6.1) that

$$\begin{aligned} T_{N\oplus M,N}(g) &= T_{N\oplus M,N}\big(\mathrm{res}_{(N\oplus M_1)/N\oplus M}(f/\Delta) - \mathrm{res}_{(N\oplus M_2)/N\oplus M}(f/\Delta)\big) \\ &= T_{N\oplus M_1,N}(f/\Delta) - T_{N\oplus M_2,N}(f/\Delta) \\ &= \frac{f}{\Delta}(\bar{\Theta}_{M_1} - \bar{\Theta}_{M_2}) \\ &= 720f. \end{aligned}$$

Suppose that g has Fourier expansion

$$g_\mu(\tau) = \sum_{m\in\mathbb{Q}} c_\mu(m)\, \mathrm{e}(m\tau),\ \mu \in (N' \oplus M')/(N \oplus M).$$

By construction, the constant term of g is zero. Consider the following finite set of vectors in M'

$$S := \{\ell \in M' | c_{(0,\ell+M)}(q(\ell)) \neq 0\},$$

where $(0, \ell + M)$ denotes an element in $(N' \oplus M')/(N \oplus M)$. Note that this set is finite and does not contain the zero vector. We claim that there exists a vector $p \in M$ such that

1) the lattice $N \oplus \mathbb{Z}p$ contains a primitive norm 0 vector;
2) $(p, \ell) \neq 0$ for all $\ell \in S$.

Indeed, condition 1) is equivalent to the property that there exists a vector $n \in N \otimes \mathbb{Q}$ such that $q(p) = -q(n)$. Condition 2) is equivalent to the requirement that p avoids finitely many hyperplanes $H_\ell := \{w \in M \otimes \mathbb{Q} \mid (w, \ell) = 0\}$. Since the dimension of M is 24 we know that all numbers in $-16^2\mathbb{Z}_{\geq 0}$ are represented by (M, q). Moreover, for $k \in 16^2\mathbb{Z}_{\geq 0}$ the points

$$X_k := \left\{ \frac{s}{-q(s)} \mid s \in M, q(s) = -k \right\}$$

are uniformly distributed on the unit sphere S^{23} as k tends to infinity. Namely, for a continuous function $f\colon S^{23} \to \mathbb{R}$ we have

$$\lim_{k \to \infty} \frac{1}{|X_k|} \sum_{x \in X_k} f(x) = \int_{S^{23}} f(y)\, d\mu(y),$$

where μ is the normalized Lebesque measure. Therefore, for a sufficiently large $k \in 16^2\mathbb{Z}_{\geq 0}$ the set $X_k \setminus (\cup_{\ell \in S} H_l)$ is non-empty. In particular, this condition holds for some k represented by $(N \otimes \mathbb{Q}, q)$. Then each point in $X_k \setminus (\cup_{\ell \in S} H_l)$ corresponds to a vector p satisfying conditions 1) and 2).

Consider the lattice $P := N \oplus \mathbb{Z}p$. It follows from Theorem B1 that the subvariety $\mathrm{Gr}^+(P)$ of $\mathrm{Gr}^+(N \oplus M)$ is not contained in the singular locus of $\Phi_{N \oplus M}(v^+, g)$. Moreover, the restriction of $\Phi_{N \oplus M}(v^+, g)$ to $\mathrm{Gr}^+(P)$ is nonsingular at the point $\mathrm{Gr}^+(N)$.

Define $h := \frac{1}{720} T_{N \oplus M, P}(g)$. The constant term of h is zero and h has rational (with denominator bounded by 720) Fourier coefficients. We have

$$T_{P,N}(h) = \frac{1}{720} T_{P,N}\big(T_{N \oplus M, P}(g)\big) = \frac{1}{720} T_{N \oplus M, N}(g) = f.$$

This finishes the proof. □

Corollary 10.2. *Let f, N, h, P be defined as in Theorem 10.1. Then*

$$\Phi_N(f) = \Phi_P(\mathrm{Gr}^+(N), h) = -4 \log |\Psi_P(\mathrm{Gr}^+(N), h)|,$$

where $\Psi_P(\,\cdot\,, h)$ is the meromorphic infinite product defined in Theorem B3.

Remark 10.3. By abuse of notation we write

$$\log |\Psi_P(\,\cdot\,, h)| = \frac{1}{n} \log |\Psi_P(\,\cdot\,, nh)|$$

where n is an integer big enought such that nh has integer Fourier coefficients and therefore the meromorphic infinite product $\Psi_P(\,\cdot\,, nh)$ is well defined.

Proof. The corollary follows immediately from the Theorem 10.1 and the see-saw identity of Theorem 6.2. □

Theorem 10.4. *Let N be an even lattice of signature $(2,0)$ and let $f \in M^!(N)$ be a modular form with zero constant term and rational Fourier coefficients. Then*

$$\Phi_N(f) = \log \alpha$$

for some $\alpha \in \overline{\mathbb{Q}}$.

Proof. We recall that if a weakly holomorphic modular form has rational coefficients then the denominators of the coefficients are bounded. Therefore we may assume that the Fourier coefficients of f are integers. Let P and h be as in Theorem 10.1. Since the constant term of h is zero, we know from Theorem B3 that there exists an integer $m \in \mathbb{Z}\backslash\{0\}$ such that $\Psi_P^m(v,h)$ is a meromorphic function on the complex manifold $\mathrm{Gr}^+(P)/\mathrm{Aut}(P,P')$. The theory of Shimura varieties of orthogonal type implies that the complex variety $\mathrm{Gr}^+(P)/\mathrm{Aut}(P,P')$, the point in this variety defined by $\mathrm{Gr}^+(N)$, and the meromorphic function $\Psi_P^m(v,h)$ can be defined over a certain number field. Therefore, the value $\Psi_P(\mathrm{Gr}^+(N),h)$ is algebraic. Below we give a more detailed proof of this statement, which uses only the theory of modular curves.

Recall that by Theorem 10.1 the lattice P has signature $(2,1)$ and contains a primitive vector of norm 0. It is easy to see that the rational quadratic space $(P\otimes\mathbb{Q}, q)$ is isomorphic to the space

$$V = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{Q} \right\}$$

of symmetric 2×2 matrices with rational coefficients equipped with the quadratic form $q(v) = -a\det(v)$ for some $a \in \mathbb{Q}_{>0}$. Indeed, let p be a nonzero element of P such that $q(p)=0$. The dimension of the orthogonal subspace $p^\perp$ is 2 and a simple computation shows that its signature is $(1,0,1)$ (i.e every Sylvester basis of this quadratic space over $\mathbb{R}$ contains 1 vector of positive norm, no vectors of negative norm, and 1 vector of zero norm). To see this we consider a basis (p,p',p'') of $P\otimes\mathbb{R}$ such that p' is in $p^\perp$. In this basis the Gram matrix of the symmetric bilinear form β associated to q takes the form

$$\begin{pmatrix} 0 & 0 & \beta(p,p'') \\ 0 & \beta(p',p') & \beta(p',p'') \\ \beta(p'',p) & \beta(p'',p') & \beta(p'',p'') \end{pmatrix}.$$

The determinant of this matrix is negative since the signature of $P\otimes\mathbb{R}$ is $(2,1)$. Therefore we see that $\beta(p',p')>0$. Let r be a vector of $P\otimes\mathbb{Q}$ in $p^\perp$ such that $q(r)>0$. Then $r^\perp$ is an isotropic quadratic space of signature (1,1). Therefore, $r^\perp$ has a basis (p,s) such that $q(p)=0$, $q(s)=0$, and $\beta(p,s)=-a$ for some $a\in\mathbb{Q}_{>0}$. Note that (p,r,s) is a basis of $P\otimes\mathbb{Q}$ and

$$q(x_1\,p + x_2\,r + x_3\,s) = -a\det\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, \qquad x_1,x_2,x_3\in\mathbb{Q}.$$

Therefore, the quadratic space $(P\otimes\mathbb{Q},q)$ is isomorphic to V.

Next, we observe that $\mathrm{SL}_2(\mathbb{Q})$ acts on V by

$$v \mapsto gvg^t, \qquad v\in V, g\in\mathrm{SL}_2(\mathbb{Q}) \tag{10.1}$$

and preserves the quadratic form q. Moreover, there exists a natural number n such that the lattice

$$U = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \mid x_1, x_2, x_3 \in n\mathbb{Z} \right\}$$

is contained in P and the dual of P, the lattice P', is contained in the dual of U

$$U' = \left\{ \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \mid y_1, y_2, y_3 \in \frac{1}{2an}\mathbb{Z} \right\}.$$

Therefore, the group $\mathrm{Aut}(P, P')$ contains the principal congruence subgroup of level $2an^2$

$$\Gamma_{2an^2} = \left\{ g \in \mathrm{SL}_2(\mathbb{Z}) | g \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod 2an^2 \right\}.$$

Now we consider the Grassmannian $\mathrm{Gr}^+(P)$ in more detail. We have explained in Section 8 that $\mathrm{Gr}^+(P)$ is isomorphic to a tube domain. Since the signature of P is $(2, 1)$, this tube domain is isomorphic to the upper half-plane $\mathfrak{H}$. The isomorphism is given by mapping $\tau \in \mathfrak{H}$ to the two-dimensional vector subspace of $V \otimes \mathbb{R}$ spanned by the vectors $\Re \left(\begin{smallmatrix} \tau^2 & \tau \\ \tau & 1 \end{smallmatrix}\right)$ and $\Im \left(\begin{smallmatrix} \tau^2 & \tau \\ \tau & 1 \end{smallmatrix}\right)$. The corresponding action of $\mathrm{SL}_2(\mathbb{Q})$ on $\mathfrak{H}$ induced by (10.1) is given by linear-fractional transformations. Moreover, the point $\mathrm{Gr}^+(N)$ in $\mathrm{Gr}^+(P)$ corresponds to a CM point $\tau_N \in \mathfrak{H}$. By Corollary 10.2 we have

$$\Phi_N(f) = -4 \log |\Psi_P(\tau_N, h)|,$$

where $\Psi_P(\tau, h)$ is a meromorphic modular function on $\mathfrak{H}$ for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and some unitary character. Theorem 4.1 of [3] says that this character has finite order. Part 4 of Theorem B3 implies that $\Psi_P(\tau, h)$ has rational Fourier coefficients with bounded denominators. Thus, it follows from the q-expansion principle and the theory of complex multiplication that $\alpha := \Psi_P(\tau_N, h)$ is an algebraic number. □

Remark 10.5. Regularized Petersson products of weight one modular forms are studied in detail in [8] and [14]. In particular, the field of definition of α and its prime factorization are discussed in these papers.

11. Proof of Theorem 1.3

Proof. Let $g_{\boldsymbol{\lambda}}$ be the weakly holomorphic modular form of weight $2 - 2k$ defined by Proposition 1.1. Consider a function $h_{\boldsymbol{\lambda}} = R^{k-1}(g_{\boldsymbol{\lambda}})$. In Theorem 8.2 we show that

$$G_{k,\boldsymbol{\lambda}}(\tau_1, \tau_2) = \Phi_{M_2(\mathbb{Z})}(v^+(\tau_1, \tau_2), h_{\boldsymbol{\lambda}}) \tag{11.1}$$

for $(\tau_1, \tau_2) \in \mathfrak{H} \times \mathfrak{H} \setminus S_{\boldsymbol{\lambda}}$.

Let $v^+(\mathfrak{z}_1, \mathfrak{z}_2)$ be the two-dimensional positive definite subspace of $M_2(\mathbb{R})$ defined in (9.1). In the case where $\mathfrak{z}_1$ and $\mathfrak{z}_2$ lie in the same quadratic imaginary field the subspace $v^+(\mathfrak{z}_1, \mathfrak{z}_2)$ defines a rational splitting of $M_2(\mathbb{Z}) \otimes \mathbb{R}$. So, the lattice $N := v^+(\tau_1, \tau_2) \cap M_2(\mathbb{Z})$ has signature $(2, 0)$.

It follows from Theorem 9.1 that

$$\Phi_{M_2(\mathbb{Z})}(v^+(\mathfrak{z}_1,\mathfrak{z}_2), R^{k-1}(g_{\boldsymbol{\lambda}})) = \Phi_N(f), \tag{11.2}$$

where $f = (f_\nu)_{\nu \in N'/N} \in M_1^!(\mathrm{SL}_2(\mathbb{Z}), \rho_N)$ is given by

$$f_\nu = [\, g_{\boldsymbol{\lambda}}, \vartheta_\nu]_{k-1}.$$

Let P and h be as in Theorem 10.1. Corollary 10.2 implies that

$$\Phi_N(f) = \Phi_P(\mathrm{Gr}^+(N), h) = -4\log|\Psi_P(\mathrm{Gr}^+(N), h)|.$$

Thus, from the theory of complex multiplication we know that

$$\Phi_N(f) = \log|\alpha| \tag{11.3}$$

for some $\alpha \in \overline{\mathbb{Q}}$. The statement of the theorem follows from equations (11.1) to (11.3). □

Acknowledgments

I thank D. Zagier for introducing me to the subject and for his valuable comments on the manuscript. Also I would like to thank Max Planck Institute for Mathematics for hospitality and for excellent working conditions. Finally, I thank anonymous referees for their comments on the manuscript.

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