



Fixed-point statistics from spectral measures on tensor envelope categories

Arthur Forey¹ · Javier Fresán² · Emmanuel Kowalski³

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Abstract

We prove old and new convergence statements for fixed-points statistics and characters of symmetric groups using tensor envelope categories, such as the Deligne–Knop category of representations of the “symmetric group” S_t for an indeterminate t . We also speculate on a generalization of Chebotarev’s density theorem to pseudopolynomials.

Keywords Spectral measure · Moment problem · Tensor category · Tensor envelope · Fixed-point statistics · Random permutations · Problème des rencontres · Chebotarev’s density theorem

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1 Introduction

Spectral measures associated to operators on Hilbert spaces are key tools in functional analysis and its applications, for instance to quantum mechanics and ergodic theory. Recall that a continuous normal linear operator $u: E \rightarrow E$ on a Hilbert space E has a compact spectrum and that, for each vector $x \in E$, there exists a unique bounded positive Radon measure μ_x on the set of complex numbers \mathbb{C} such that the equality

$$\int_{\mathbb{C}} f d\mu_x = \langle x | f(u)x \rangle$$

✉ Emmanuel Kowalski
kowalski@math.ethz.ch

Arthur Forey
arthur.forey@univ-lille.fr

Javier Fresán
javier.fresan@imj-prg.fr

¹ Univ. Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, Lille 59000, France

² Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, Paris 75005, France

³ D-MATH, ETH Zürich, Rämistrasse 101, Zürich 8092, Switzerland

holds for all continuous functions $f: \mathbb{C} \rightarrow \mathbb{C}$; see, for instance, [4, IV, p. 190, déf. 2]. This measure is supported on the spectrum of u and is called the *spectral measure of u relative to x* . In particular, we have

$$\int_{\mathbb{C}} z^a \bar{z}^b d\mu_x(z) = \langle x | u^a (u^*)^b(x) \rangle$$

for all non-negative integers a and b . In this paper, we consider an analogue of this last relation for objects in symmetric monoidal categories.

Definition 1.1 (*Spectral measure*) Let \mathcal{C} be a symmetric monoidal category with an endofunctor D , and let i be a complex-valued invariant of \mathcal{C} , by which we mean a map from the set of isomorphism classes of objects of \mathcal{C} to \mathbb{C} . Let M be an object of \mathcal{C} .

A positive measure $\mu = \mu(i, M)$ on \mathbb{C} is called a *spectral measure of M relative to i* if the equality

$$\int_{\mathbb{C}} z^a \bar{z}^b d\mu(z) = i(M^{\otimes a} \otimes D(M)^{\otimes b})$$

holds for all non-negative integers a and b .

We will think of D as a duality functor on \mathcal{C} , although no extra condition is required for this definition. In general, the measure μ depends on i and M and might not be unique (see Remark 3.8). The basic motivation is provided by the following example.

Example 1.2 Let $r \geq 1$ be an integer, and let $G \subset \mathbf{GL}_r(\mathbb{C})$ be a compact group with probability Haar measure ν . Let \mathcal{C} be the category of finite-dimensional continuous complex representations of G , and D the contragredient endofunctor of \mathcal{C} . By representation theory of compact groups, the direct image of ν by the trace $\text{Tr}: G \rightarrow \mathbb{C}$ is a spectral measure $\mu = \text{Tr}_*(\nu)$ of the “tautological” object of \mathcal{C} corresponding to the inclusion of G in $\mathbf{GL}_r(\mathbb{C})$, relative to the invariant given on a representation ϱ by

$$i(\varrho) = \dim_{\mathbb{C}}(\varrho^G) = \dim_{\mathbb{C}} \text{Hom}(1, \varrho),$$

where 1 denotes the trivial one-dimensional representation of G . In number theory, measures of this kind are often called *Sato–Tate measures*, the original example being the one associated to $\text{SU}_2 \subset \mathbf{GL}_2(\mathbb{C})$.

By a somewhat ad hoc construction, one can also see the spectral measures from functional analysis as instances of spectral measures in the sense of Definition 1.1.

Example 1.3 Let E be a Hilbert space, and let A be a commutative unitary C^* -subalgebra of $\text{End}(E)$, e.g. the closure of the span of all u^a and $(u^*)^b$ for some normal endomorphism u . We consider the category \mathcal{C} with objects the elements of A and morphisms given by

$$\text{Hom}_{\mathcal{C}}(u, v) = \begin{cases} \mathbb{C} \cdot 1_u & \text{if } u = v, \\ \{0\} & \text{else.} \end{cases}$$

We endow \mathcal{C} with the tensor product given on objects by $u \otimes v = u \circ v$, and by the obvious rule on morphisms, i.e., the tensor product of non-zero morphisms $f = s1_{u_1}$ and $g = t1_{u_2}$ is equal to $f \otimes g = st1_{u_1 \otimes u_2}$. The unit object is the identity map $1_E \in A$, and the category is symmetric monoidal because A is commutative. We consider the “duality” functor $D(u) = u^*$.

Let us now fix a vector $x \in E$ and consider the invariant i on \mathcal{C} defined by the assignment $i_x(u) = \langle x | u(x) \rangle$ for all $u \in A$. Then the equality

$$i_x(u^{\otimes a} \otimes D(u)^{\otimes b}) = \langle x | u^a (u^*)^b(x) \rangle$$

holds for all non-negative integers a and b , and hence the functional-analytic spectral measure μ_x is a spectral measure of u relative to the invariant i_x .

Our first main result is a new proof of a statement that goes back to the very early studies of probability theory through the analysis of card games and the like (see the historical paper of Takács [27] for references). Interestingly, the tensor categories that will arise in the proof are the categories of representations of the “symmetric group” S_t for an indeterminate t of Deligne [10] and Knop [18]. A rather different construction of this category has recently been given by Harman and Snowden [15, § 15].

Theorem 1.4 (“Problème des rencontres”; Montmort [8]; N. Bernoulli I; de Moivre [7]) *Let $(X_n)_{n \geq 1}$ be a sequence of random variables with X_n a uniformly chosen random permutation in the symmetric group S_n . The sequence $(|\text{Fix}(X_n)|)_{n \geq 1}$, where $\text{Fix}(\sigma)$ denotes the set of fixed points of $\sigma \in S_n$, converges in law to a Poisson distribution with parameter 1.*

Recall that the Poisson distribution with parameter a positive real number λ is the measure P_λ supported on non-negative integers given by

$$P_\lambda(\{r\}) = e^{-\lambda} \frac{\lambda^r}{r!}$$

for all integers $r \geq 0$. The meaning of the statement (and how it was originally proved) is therefore that the formula

$$\lim_{n \rightarrow +\infty} \frac{1}{n!} |\{\sigma \in S_n \text{ with } |\text{Fix}(\sigma)| = r\}| = \frac{1}{e} \frac{1}{r!} \quad (1.1)$$

holds for all integers $r \geq 0$. Neither categories nor spectral measures appear in the statement; the link comes from the fact that the limit Poisson distribution arises as the spectral measure of a suitable object in the Deligne–Knop category $\mathcal{C}_t = \underline{\text{Rep}}(S_t)$. This is a $\mathbb{C}(t)$ -linear semisimple tensor category that “interpolates” the categories of representations of the symmetric groups S_n . From that point of view, an interesting feature of our proof is that it shows how the Poisson distribution (maybe the most natural measure on non-negative integers) is some kind of analogue of the Sato–Tate measures from Example 1.2. In Section 3.3, we will see a similar statement for the complex gaussian distribution. By Chebotarev’s density theorem, the probability that

a random permutation in S_n has r fixed points governs the asymptotic density of the set of primes p such that a generic polynomial of degree n with integer coefficients has r roots modulo p . In Section 5, we will speculate on a generalization of Chebotarev's density theorem that could explain the occurrence of the limit (1.1) in numerical experiments involving certain pseudopolynomials.

Our second main result is the following new theorem.

Theorem 1.5 *Let $m \geq 1$ be an integer and let λ be a partition of m with parts*

$$\lambda_1 \geq \lambda_2 \geq \dots.$$

For $n \geq m + \lambda_1$, let $\pi_{\lambda,n}$ be the representation of S_n corresponding to the partition

$$\lambda^{(n)} = (n - m, \lambda_1, \lambda_2, \dots),$$

and let $\chi_{\lambda,n}: S_n \rightarrow \mathbb{C}$ be its character. Then the measures $(\chi_{\lambda,n}(X_n))_{n \geq m + \lambda_1}$, where as before X_n is a uniformly distributed random permutation on S_n , converges in law as $n \rightarrow +\infty$ to a spectral measure of the simple object $x_{\lambda,t}$ of the Deligne–Knop category \mathcal{C}_t associated to the partition λ , relative to the invariant

$$i(M) = \dim_{\mathbb{C}(t)} \operatorname{Hom}(1_t, M),$$

where 1_t is the unit object of \mathcal{C}_t .

This second result has a corollary which relates it to the theory of FI-modules of Church, Ellenberg and Farb [6]. An FI-module over a field k is a functor V from the category with objects finite sets and morphisms injective maps to the category of k -vector spaces. For each $n \geq 0$, we write V_n for the image of the set $\{1, \dots, n\}$; note that V_n has a natural structure of representation of S_n . An FI-module V is called *finitely generated* if there exists a finite set of elements $x_i \in V_{n_i}$ that do not “lie” on a proper subfunctor of V . We refer the reader to the introduction of [6] for a list of examples of finitely-generated FI-modules arising in algebra, geometry and topology.

Corollary 1.6 *Let $V = (V_n)_{n \geq 0}$ be a finitely-generated FI-module over \mathbb{C} . For $n \geq 0$, let ξ_n be the character of V_n as an S_n -representation. The sequence of measures $(\xi_n(X_n))_{n \geq 0}$ converges in law as $n \rightarrow +\infty$ to a combination of spectral measures of objects of \mathcal{C}_t relative to the invariant i from Theorem 1.5.*

Proof A fundamental result of the theory of FI-modules [6, Prop. 3.3.3] implies the existence of a polynomial $Q \in \mathbb{C}[(T_\lambda)_\lambda]$ (in indeterminates parameterized by all partitions of all integers $m \geq 1$) satisfying $\xi_n = Q((\chi_{\lambda,n})_\lambda)$ for all large enough n , and hence the corollary follows immediately from Theorem 1.5. \square

This paper is intended as a first glimpse on a subject that deserves further exploration. We hope that the simple example of an application of spectral measures to classical problems will motivate the reader's interest in this notion.

Conventions

Let X be a set. A *partition* of X is a set of non-empty subsets of X that are pairwise disjoint and whose union is equal to X . Note that this definition contrasts with that of Bourbaki (E, II, p. 29, déf. 7), where a partition is a *family* of subsets of X , allowing the empty set.

2 Existence and uniqueness of spectral measures in tensor categories

From now on, we only consider k -linear tensor categories, for some field k , in the sense of Deligne [9, 1.2]. We always use the duality functor of such a category as the endofunctor D in Definition 1.1. An object M is called *self-dual* if there exists an isomorphism $M \simeq D(M)$.

Proposition 2.1 (Spectral measures for self-dual objects) *Let \mathcal{C} be a tensor category in which every object is self-dual, and let i be an \mathbb{R} -valued additive invariant of \mathcal{C} . If*

$$2i(M \otimes N) \leq i(M \otimes M) + i(N \otimes N) \quad (2.1)$$

holds for all objects M and N of \mathcal{C} , then every object of \mathcal{C} admits a spectral measure relative to i which is supported on \mathbb{R} .

Proof By the solution of the Hamburger moment problem (see, e.g., [26, Th.3.8]), a sequence $(\mu_a)_{a \geq 0}$ of real numbers is the sequence of moments of a positive Borel measure μ on \mathbb{R} if and only if the inequality

$$\sum_{1 \leq a, b \leq A} \alpha_a \alpha_b \mu_{a+b} \geq 0 \quad (2.2)$$

holds for all integers $A \geq 1$ and all real numbers α_a . Therefore, M admits a real spectral measure relative to i if and only if the values $\mu_a = i(M^{\otimes a})$ satisfy this condition.

We first consider the case where α_a are integers. Setting

$$P = \bigoplus_{\alpha_a \geq 0} \alpha_a M^{\otimes a} \quad \text{and} \quad N = \bigoplus_{\alpha_a \leq -1} (-\alpha_a) M^{\otimes a},$$

we then get

$$\sum_{1 \leq a, b \leq A} \alpha_a \alpha_b \mu_{a+b} = i(P \otimes P) + i(N \otimes N) - 2i(P \otimes N),$$

and hence the assumption (2.1) implies the inequality

$$\sum_{1 \leq a, b \leq A} \alpha_a \alpha_b \mu_{a+b} \geq 0.$$

This extends to \mathbb{Q} by homogeneity, and to \mathbb{R} by continuity. \square

If not all objects are self-dual (as it often happens), then the situation is more subtle, because the moment problem on \mathbb{C} is more challenging than that on \mathbb{R} : the analogue of the positivity condition above is not sufficient to ensure the existence of a positive measure on \mathbb{C} with given moments. However, under an extra growth condition, one obtains both existence and uniqueness of the spectral measure.

Proposition 2.2 (Spectral measures for general objects) *Let \mathcal{C} be a tensor category. Let i be a \mathbb{C} -valued additive invariant of \mathcal{C} . Suppose that the inequality*

$$i(M \otimes D(N)) + i(D(M) \otimes N) \leq i(M \otimes D(M)) + i(N \otimes D(N)) \quad (2.3)$$

holds for all objects M and N of \mathcal{C} . Let M be an object of \mathcal{C} satisfying the Carleman condition

$$\sum_{a \geq 1} i((M \otimes D(M))^{\otimes a})^{-1/(2a)} = +\infty. \quad (2.4)$$

Then there exists a unique spectral measure for M relative to i .

Proof This follows as above (*mutatis mutandis* using complexification) from the fact (due to Nussbaum) that the Carleman condition (2.4) combined with the analogue of (2.2) is a sufficient condition for the existence and uniqueness of a measure on \mathbb{C} with given moments; see for instance [26, Th. 15.11]. \square

Remark 2.3 The Carleman condition holds in particular if there exist $c \geq 0$ and $r \geq 0$ such that the inequality $i((M \otimes D(M))^{\otimes n}) \leq cr^n$ holds for all non-negative integers n . This is a frequent occurrence, but it corresponds to measures with compact support (compare with Deligne’s “subexponential growth theorem”; see [12, Th. 9.11.4]).

Definition 2.4 (Positive invariants) An additive invariant i on \mathcal{C} is called a *positive invariant* if it satisfies (2.3) for all objects M and N of \mathcal{C} .

The following result gives a usable criterion to check that certain invariants are positive.

Proposition 2.5 *Let \mathcal{C} be an essentially small tensor category. Let $\widehat{\mathcal{C}}$ be a set of objects of \mathcal{C} such that every object of \mathcal{C} is isomorphic to a finite direct sum of objects from $\widehat{\mathcal{C}}$. We write $\mathbb{Z}^{(\widehat{\mathcal{C}})}$ for the set of functions $n: \widehat{\mathcal{C}} \rightarrow \mathbb{Z}$ with finite support and $n_V = n(V)$ for all objects V of $\widehat{\mathcal{C}}$. Let i be an additive invariant of \mathcal{C} .*

Then i is positive if the bilinear form

$$b(n, m) = \sum_{V, W \in \widehat{\mathcal{C}}} n_V m_W i(V \otimes D(W))$$

on $\mathbb{Z}^{(\widehat{\mathcal{C}})}$ is positive, i.e., $b(n, n) \geq 0$ for all functions $n: \widehat{\mathcal{C}} \rightarrow \mathbb{Z}$ with finite support.

Proof Let M and N be objects of \mathcal{C} , and represent them as direct sums

$$M = \bigoplus_{V \in \widehat{\mathcal{C}}} m_V V, \quad N = \bigoplus_{W \in \widehat{\mathcal{C}}} n_W W$$

with only finitely many non-zero integers m_V, n_W . By additivity, we obtain the formulas

$$\begin{aligned} i(M \otimes D(N)) + i(D(M) \otimes N) &= \sum_{V, W} m_V n_W i(V \otimes D(W)) + \sum_{V, W} m_V n_W i(D(V) \otimes W), \\ i(M \otimes D(M)) + i(N \otimes D(N)) &= \sum_{V, W} m_V m_W i(V \otimes D(W)) + \sum_{V, W} n_V n_W i(V \otimes D(W)), \end{aligned}$$

so that we get

$$\left(i(M \otimes D(M)) + i(N \otimes D(N)) \right) - \left(i(M \otimes D(N)) + i(D(M) \otimes N) \right) = b(m - n, m - n),$$

and the result then follows from Proposition 2.2. \square

As a special case, we deduce:

Corollary 2.6 *Let k be a field. Let \mathcal{C} be any essentially small k -linear semisimple tensor category with unit object $1_{\mathcal{C}}$ in which the Hom spaces are finite-dimensional. The formula*

$$i(M) = \dim_k \operatorname{Hom}(1_{\mathcal{C}}, M)$$

defines a positive invariant on \mathcal{C} .

Proof We apply Proposition 2.5 to the set $\widehat{\mathcal{C}}$ of isomorphism classes of simple objects of \mathcal{C} . Then $i(V \otimes D(W)) = 0$ for V and W in $\widehat{\mathcal{C}}$, unless V is equal to W , so that the bilinear form b in the statement is diagonal in the canonical basis of $\mathbb{Z}^{(\widehat{\mathcal{C}})}$, with diagonal coefficients equal to $i(V \otimes D(V)) = \dim_k \operatorname{Hom}(1_{\mathcal{C}}, V \otimes D(V)) \geq 0$. \square

Remark 2.7 (1) The remainder of this paper will concentrate on the invariant of Corollary 2.6. However, there are other natural potential invariants that may be considered. One which seems quite interesting is the *length* of an object in a semisimple category (where all objects have finite length). In the simplest case of the category of finite-dimensional complex representations of a finite group G , it is an elementary exercise that the length is a positive invariant (and every object has a unique spectral measure relative to the length) if and only if the sum (without multiplicity) of the irreducible characters of G is non-negative. This holds for instance for all symmetric groups, but not all alternating groups (the latter experimentally). We hope to come back to this example in a later paper.

(2) We emphasize that it is essential to impose the positivity of the spectral measure in Definition 1.1: it is known by independent work of Boas and Pólya (see, e.g., [3])

that any sequence of complex numbers is the sequence of moments of infinitely many complex measures on \mathbb{R} .

(3) In general, spectral measures are not uniquely determined given the object of interest, and only their moments are unambiguously known (see the conclusion of Remark 3.8 for a simple example where the spectral measure is not unique).

We conclude this section with a simple observation.

Proposition 2.8 *Let k be a field. Let \mathcal{C} be a k -linear tensor category with unit object $1_{\mathcal{C}}$ and let i be a positive invariant on \mathcal{C} . Let M be an object of \mathcal{C} . Let μ be a spectral measure for M relative to i .*

- (1) *For any non-negative integers a and b , the image measure $(z \mapsto z^a \bar{z}^b)_* \mu$ is a spectral measure for $M^{\otimes a} \otimes D(M)^{\otimes b}$ relative to i .*
- (2) *The image measure $(z \mapsto 2 \operatorname{Re}(z))_* \mu$ is a spectral measure for $M \oplus D(M)$ relative to i .*

Proof We prove the second statement, the first being similar. Since the object $N = M \oplus D(M)$ is self-dual, it suffices to consider $i(N^{\otimes a})$ for all integers $a \geq 0$. From the isomorphism

$$N^{\otimes a} \simeq \bigoplus_{0 \leq b \leq a} \binom{a}{b} M^{\otimes b} \otimes D(M)^{\otimes (a-b)},$$

and the definition of spectral measures, we get the equality

$$i(N^{\otimes a}) = \sum_{0 \leq b \leq a} \binom{a}{b} \int_{\mathbb{C}} z^b \bar{z}^{a-b} d\mu(z) = \int_{\mathbb{C}} (z + \bar{z})^a d\mu(z),$$

which means that $(z \mapsto 2 \operatorname{Re}(z))_* \mu$ is a spectral measure for N . \square

Remark 2.9 With obvious conventions, this proposition can be phrased and generalized as follows: for any polynomial $Q \in \mathbb{N}[z, \bar{z}]$, the measure $Q_* \mu$ is a spectral measure for the object $Q(M, D(M))$.

More generally, one can raise the following natural question, for which we do not have good answers at the moment: given an object M and some spectral measure $\mu(M)$ for M relative to some invariant i , is there a “natural” definition of spectral measures $\mu(N)$, for all objects N of the tensor category generated by M , such that $\mu(N)$ coincides with the measure $Q_* \mu_M$ when $N = Q(M, D(M))$ as above? Already when considering simple examples of Schur functors, such as symmetric powers (when they are defined), the answer is not clear.

3 Tensor envelopes and fixed-point statistics

Let k be a field of characteristic zero and $t \in k$ an element. Deligne [10, Th. 2.18] defined a rigid k -linear pseudo-abelian symmetric monoidal category $\underline{\operatorname{Rep}}(S_t, k)$ by generators and relations, relying on some stability properties of representations of

symmetric groups. If t is not a non-negative integer $n \geq 0$, then $\text{Rep}(S_t, k)$ is abelian and semisimple. If $t = n$, then the semisimplification of $\text{Rep}(S_t, k)$ is equivalent to the category $\text{Rep}(S_n)$ of k -linear representations of the symmetric group S_n . We will mainly deal with the case where $k = \mathbb{C}(t)$ and t is the indeterminate of k , which we simply denote by $\text{Rep}(S_t)$.

Knop [18] discovered an alternative approach to constructing new rigid symmetric monoidal categories which is *a priori* independent of ideas of interpolating other categories; this leads to many more examples, and happens to recover in a special case the categories of Deligne. The input data in Knop's construction is a base category \mathcal{A} satisfying some regularity conditions, a field k and a degree function δ which associates to every surjective morphism e in \mathcal{A} an element $\delta(e)$ of k , again subject to some conditions. The resulting category is denoted $\mathcal{T}(\mathcal{A}, \delta)$ by Knop, and is called the *tensor envelope* of \mathcal{A} with respect to δ . For the moment, it is sufficient for us to recall that every object x of \mathcal{A} defines an object $[x]$ of $\mathcal{T}(\mathcal{A}, \delta)$, which is always self-dual, and that the k -linear space of morphisms from $[x]$ to $[y]$ admits as a basis the set of all *relations* from x to y , i.e., the set of all subobjects of the product $x \times y$. To give some context, we spell out in Appendix A the construction of $\mathcal{T}(\mathcal{A}, \delta)$ in the special case relevant to Theorem 1.4, namely when \mathcal{A} is the *opposite* of the category of finite sets.

3.1 Proof of Theorem 1.4

Let P_1 denote the Poisson distribution with parameter 1. By the so-called *Dobiński's formula* (see, e.g., [25]), for each integer $k \geq 0$, the k -th moment

$$\mathbb{E}(P_1^k) = \frac{1}{e} \sum_{r=0}^{\infty} \frac{r^k}{r!}$$

agrees with the k -th Bell number, i.e., the number of partitions of a set with k elements (indeed, both sequences satisfy $a_0 = 0$ and the recurrence relation $a_{k+1} = \sum_{r=0}^k \binom{k}{r} a_r$). In particular, $\mathbb{E}(P_1^k) \leq k^k$, so that the Carleman condition holds and P_1 is determined by its moments (as is well-known). Thanks to the method of moments (see, e.g., [2, Th. 30.2]), to prove the convergence in law $|\text{Fix}(X_n)| \rightarrow P_1$ as $n \rightarrow +\infty$, it suffices to prove that, for each integer $k \geq 0$, the sequence of moments $(\mathbb{E}(|\text{Fix}(X_n)|^k))_{n \geq 1}$ converges to $\mathbb{E}(P_1^k)$.

We first observe the equality $|\text{Fix}(X_n)| = \chi_n(X_n)$, where χ_n is the character of the “standard” permutation representation Std_n of S_n acting on \mathbb{C}^n . By basic representation theory of finite groups, we then get the expression

$$\mathbb{E}(|\text{Fix}(X_n)|^k) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_n(\sigma)^k = \dim_{\mathbb{C}} \text{Hom}_{\text{Rep}(S_n)}(1_n, \text{Std}_n^{\otimes k}), \quad (3.1)$$

where 1_n is the trivial one-dimensional representation of S_n .

We now appeal to the Deligne–Knop category $\mathcal{C}_t = \underline{\text{Rep}}(S_t)$, first in the situation where t is the indeterminate in the field $\mathbb{C}(t)$. Before pursuing the proof, we summarize the properties of \mathcal{C}_t that will be useful for us:

- (a) Each finite set X defines a self-dual object $[X]$ of \mathcal{C}_t , and these objects satisfy

$$\text{Hom}_{\mathcal{C}_t}([X], [Y]) = \mathbb{C}(t)\langle \text{partitions of } X \sqcup Y \rangle.$$

- (b) The tensor product of $[X]$ and $[Y]$ is the object $[X] \otimes [Y] = [X \sqcup Y]$.

- (c) The category \mathcal{C}_t is a semisimple $\mathbb{C}(t)$ -linear tensor category.

In particular, \mathcal{C}_t contains objects $1_t = [\emptyset]$ and $\text{Std}_t = [\{1\}]$, the first being the unit object for the tensor product. By Corollary 2.6, the assignment

$$i(M) = \dim_{\mathbb{C}(t)} \text{Hom}_{\mathcal{C}_t}(1_t, M)$$

defines a positive invariant on \mathcal{C}_t .

Lemma 3.1 *The object Std_t admits a unique spectral measure with respect to i , and this measure is equal to the Poisson distribution P_1 . In particular,*

$$\mathbb{E}(P_1^k) = \dim_{\mathbb{C}(t)} \text{Hom}_{\mathcal{C}_t}(1_t, \text{Std}_t^{\otimes k}). \quad (3.2)$$

Proof Recall that the object Std_t is self-dual. For each integer $k \geq 0$, the k -th tensor product $\text{Std}_t^{\otimes k}$ is the object $[\{1, \dots, k\}]$ of \mathcal{C}_t . Hence,

$$i(\text{Std}_t^{\otimes k}) = \dim_{\mathbb{C}(t)} \text{Hom}_{\mathcal{C}_t}(1_t, \text{Std}_t^{\otimes k})$$

is the number of partitions of the set $\{1, \dots, k\}$, which is also the k -th moment of P_1 . \square

Combining (3.1) and (3.2), the proof of Theorem 1.4 then reduces to showing the equality

$$\lim_{n \rightarrow +\infty} \dim_{\mathbb{C}} \text{Hom}_{\text{Rep}(S_n)}(1_n, \text{Std}_n^{\otimes k}) = \dim_{\mathbb{C}(t)} \text{Hom}_{\underline{\text{Rep}}(S_t)}(1_t, \text{Std}_t^{\otimes k}).$$

For this, we use the variant $\mathcal{C}_z = \underline{\text{Rep}}(S_z, \mathbb{C})$ of the Deligne–Knop category obtained by “specializing” the indeterminate t to some fixed complex number z . Properties (a) and (b) from above still hold, and in particular

$$\dim_{\mathbb{C}(t)} \text{Hom}_{\underline{\text{Rep}}(S_t)}(1_t, \text{Std}_t^{\otimes k}) = \dim_{\mathbb{C}} \text{Hom}_{\mathcal{C}_z}(1_z, \text{Std}_z^{\otimes k})$$

since a basis of both vector spaces is given by the partitions of $\{1, \dots, k\}$. Unless z is an integer $n \geq 0$, the category \mathcal{C}_z is still semisimple.

For integer values $z = n$, the semisimplification of \mathcal{C}_n is equivalent, as a tensor category, to the category of finite-dimensional complex representations of S_n , an equivalence being given by the functor that maps an object of the form

$[X]$ to the permutation representation on the space V_X of functions $X \rightarrow \mathbb{C}^n$ (see [18, Th. 9.8, Example 1, p. 606]). In particular, such a functor sends the object $[\emptyset]$ to the trivial one-dimensional representation 1_n , and the object $[\{1\}]$ to the standard permutation representation Std_n on \mathbb{C}^n . The semisimplification of \mathcal{C}_n is the quotient category $\overline{\mathcal{C}}_n = \mathcal{C}_n / \mathcal{N}_n$, where \mathcal{N}_n denotes the tensor radical of \mathcal{C}_n (see [18, § 4.1]). It is a semisimple abelian tensor category by [18, Th. 6.1].

Lemma 3.2 *Let $n \geq 1$ be an integer. For each integer $k \geq 0$, the inequality*

$$\dim_{\mathbb{C}} \text{Hom}_{\text{Rep}(S_n)}(1_n, \text{Std}_n^{\otimes k}) \leq \dim_{\mathbb{C}} \text{Hom}_{\mathcal{C}_n}(1_t, \text{Std}_n^{\otimes k})$$

holds, with equality if and only if $n \geq k$.

Proof Let X and Y be finite sets. By definition of the quotient category, the objects of $\overline{\mathcal{C}}_n$ are the same as those of \mathcal{C}_n , and the morphisms between the representations V_X and V_Y of S_n corresponding to $[X]$ and $[Y]$ via the equivalence of categories are given by

$$\text{Hom}_{\text{Rep}(S_n)}(V_X, V_Y) = \text{Hom}_{\overline{\mathcal{C}}_n}([X], [Y]) = \text{Hom}_{\mathcal{C}_n}([X], [Y]) / \mathcal{N}_n([X], [Y]).$$

Therefore, we obtain an inequality

$$\dim_{\mathbb{C}} \text{Hom}_{\text{Rep}(S_n)}(V_X, V_Y) \leq \dim_{\mathbb{C}} \text{Hom}_{\mathcal{C}_n}([X], [Y]),$$

with equality if and only if $\mathcal{N}_n([X], [Y])$ is reduced to the zero morphism. Taking $X = \emptyset$ and $Y = \{1, \dots, k\}$, so that $V_X = 1_n$ and $V_Y = \text{Std}_n^{\otimes k}$, proves the first part of the statement.

It remains to see when $\mathcal{N}_n(1_n, \text{Std}_n^{\otimes k})$ is zero. By a result of Knop [18, Cor. 8.5], this holds if and only if certain invariants ω_e in \mathbb{C} are non-zero for all indecomposable surjective morphisms $e: u \rightarrow v$ in the category Set^{opp} such that u is a subquotient of $1_t \otimes \text{Std}_t^{\otimes k} = \text{Std}_t^{\otimes k}$. By [18, Ex. 1, p. 596], this invariant is equal to $\omega_e = n - |v|$ for such morphisms; since indecomposable surjective morphisms $u \rightarrow v$ in Set^{opp} are injective maps of sets $v \hookrightarrow u$ satisfying $|v| = |u| - 1$, and u is a subquotient of $\text{Std}_n^{\otimes k} = [\{1, \dots, k\}]$, we have $|v| \leq k - 1$, and hence $\omega_e = n - |v| \geq 1$ is non-zero for all $n \geq k$. \square

This concludes the proof of Theorem 1.4.

Remark 3.3 (1) In comparison with other proofs, this abstract argument has the advantage of explaining, to some extent, where the Poisson distribution comes from.

(2) It is natural to ask if similar ideas can be used to reprove other statements in the theory of random permutations, such as the fact that the sequence $(\ell_i(X_n))_{n \geq 1}$, where $\ell_i(\sigma)$ denotes the number of i -cycles in the decomposition of a permutation σ , converges in law to the Poisson distribution $P_{1/i}$. More ambitiously, one can try to count the number of cycles in a random permutation.

(3) To the best of our knowledge, the fact that the first moments of $|\text{Fix}(X_n)|$ coincide with those of the Poisson distribution first appears in the work of Diaconis–Shahshahani [11, Th. 7].

3.2 Fixed-point statistics for vector and affine spaces over finite fields

Knop's approach yields many more instances of tensor categories, and the principles above are applicable. As an example, we recover a result of Fulman (proved in his 1997 unpublished thesis) which appears in a paper of Fulman and Stanton [13, Th. 4.1].

Proposition 3.4 (Fulman) *Let E be a finite field and let $(X_n)_{n \geq 1}$ be a sequence of random variables with X_n uniformly distributed in $\mathbf{GL}_n(E)$. The sequence $(|\text{Fix}(X_n)|)_{n \geq 1}$, where $\text{Fix}(g)$ is the 1-eigenspace of $g \in \mathbf{GL}_n(E)$, converges in law as $n \rightarrow +\infty$. For $k \geq 0$, the k -th moment of the limiting distribution is equal to the number of vector subspaces of E^k . Moreover, the k -th moment of $|\text{Fix}(X_n)|$ is equal to the limiting moment for $n \geq k$.*

Proof We argue as in the proof of Theorem 1.4, using instead the base category $\text{Vec}(E)$ of finite-dimensional E -vector spaces and the degree function

$$\delta(e: U \rightarrow V) = t^{\dim_E(\ker(e))}$$

for a surjective E -linear map to construct Knop's category \mathcal{C}_t . We use as before the unit object $1_t = [\{0\}]$ and the standard object $\text{Std}_t = [E]$, which is self-dual.

Specializing to $t = |E|^n$ for some integer $n \geq 1$, the quotient $\overline{\mathcal{C}}_{|E|^n}$ is naturally equivalent to the category of finite-dimensional complex representations of $\mathbf{GL}_n(E)$ (see [18, Example 5, p. 606]). We obtain

$$\dim_{\mathbb{C}} \text{Hom}_{\mathbf{GL}_n(E)}(1_n, \text{Std}_n^{\otimes k}) \leq \dim_{\mathbb{C}(t)} \text{Hom}_{\mathcal{C}_t}(1_t, \text{Std}_t^{\otimes k}),$$

where Std_n is the $|E|^n$ -dimensional permutation representation of $\mathbf{GL}_n(E)$ associated to its natural action on E^n . As before, there is equality if the numerical invariants ω_e are non-zero for indecomposable surjective E -linear maps $e: U \rightarrow V$, where U is a subquotient of $\text{Std}_n^{\otimes k}$ in $\mathcal{C}_{|E|^n}$. We have $\omega(e) = |E|^n - |V|$, and hence there is equality if $n \geq k$ (note that in $\mathcal{C}_{|E|^n}$, the tensor product is defined using the *direct sum* of finite-dimensional E -vector spaces).

On the one hand, for all $n \geq 1$, the function $g \mapsto |\text{Fix}(g)|$ is the character of the standard representation, and on the other hand, by Knop's construction, the dimension

$$\dim_{\mathbb{C}(t)} \text{Hom}_{\mathcal{C}_t}(1_t, \text{Std}_t^{\otimes k})$$

is the number of subspaces of E^k . Thus, $\mathbb{E}(|\text{Fix}(X_n)|^k)$ converges to this number. To conclude, we need however to apply Lemma 3.5 below, since in this case the size of the moments does not satisfy the Carleman condition, but it is known that

$$|\{\text{subspaces of } E^k\}| \ll |E|^{k(k+1)/4}. \quad \square$$

Lemma 3.5 (Heath–Brown) *Let $q \geq 1$ be an integer, and let $(m_k)_{k \geq 0}$ be a sequence of real numbers such that $m_k \ll q^{k(k+1)/4}$ for $k \geq 0$. Let $(Z_n)_{n \geq 1}$ be a sequence of random variables such that*

- (1) For all n , the support of Z_n is contained in the set of powers q^r for $r \geq 0$.
 (2) For all $k \geq 0$, we have $\mathbb{E}(Z_n^k) \rightarrow m_k$.

Then (Z_n) converges in law to a random variable Z supported on powers of q with moments m_k for all $k \geq 0$.

Proof This is implicit in [16, Lemmas 17 and 18]. More precisely, it follows from standard results in the method of moments that the second assumption implies that any subsequence of $(Z_n)_{n \geq 0}$ which converges in law has a limit with moments m_k , and it is elementary from the first assumption that all such limits are supported on powers of q . Heath–Brown’s result (proved in [16] in the case $q = 4$, but with immediate generalization) is that there is a unique probability measure on \mathbb{R} with these two properties. Since the convergence of moments implies uniform integrability (or tightness), this means that the sequence $(Z_n)_{n \geq 0}$ is relatively compact and has a unique limit point, and hence converges. The stated properties of the limit are then clear. \square

Remark 3.6 A result of Christiansen [5] (also cited by Fulman and Stanton) shows that the limiting measure of Proposition 3.4, as a measure on \mathbb{R} , is not characterized by its moments. Thus, some extra condition is necessary to ensure uniqueness, and this is provided by the assumption that the support is restricted to powers of q .

Considering another example of Knop leads by the same method to a similar result which is new, to the best of our knowledge.

Proposition 3.7 Let E be a finite field and let $(X_n)_{n \geq 1}$ be a sequence of random variables with X_n uniformly distributed in the affine-linear group $\mathbf{Aff}_n(E)$ of E^n .

The sequence $(|\mathrm{Fix}(X_n)|)_{n \geq 1}$, where $\mathrm{Fix}(g)$ is the set of fixed points of $g \in \mathbf{Aff}_n(E)$, converges in law as $n \rightarrow +\infty$. For $k \geq 0$, the k -th moment of the limiting distribution is equal to the number of affine subspaces of E^{k-1} .

Moreover, the k -th moment of $|\mathrm{Fix}(X_n)|$ is equal to the limiting moment for $n \geq k$.

Proof We argue as above with the base category \mathcal{A} of (non-empty) affine spaces over E (see [18, p. 597, Ex. 6; p. 607, Ex. 7]). \square

3.3 Fixed-point statistics for complex vector spaces

It is also natural to consider the category $\mathrm{Rep}(\mathrm{GL}_t)$ of Deligne and Milne (discussed in [10, § 10, Déf. 10.2]), interpolating the categories of representations of $\mathrm{GL}_n(\mathbb{C})$. Indeed, the argument applies rather similarly, and leads to the analogue of Theorem 1.4 in this context: the direct image under the trace $\mathrm{Tr}: U_n \rightarrow \mathbb{C}$ of the probability Haar measure on the unitary group U_n converges as $n \rightarrow +\infty$ to a standard complex gaussian. This was first proved by Diaconis and Shahshahani [11]; see also Larsen’s paper [23] for the case of the symplectic or orthogonal groups and real gaussians.

First, by Corollary 2.6, the assignment

$$i(M) = \dim_{\mathbb{C}(t)} \mathrm{Hom}_{\mathrm{Rep}(\mathrm{GL}_t)}(1_t, M)$$

defines a positive invariant on $\text{Rep}(\text{GL}_t)$. One can then show that there exists an object Std_t , which for $t = n$ corresponds to the standard representation of $\text{GL}_n(\mathbb{C})$ through the equivalence from the semisimplification of $\text{Rep}(\text{GL}_t)$ to the category of representations of $\text{GL}_n(\mathbb{C})$, satisfying

$$i(\text{Std}_t^{\otimes a} \otimes \text{D}(\text{Std}_t)^{\otimes b}) = \dim_{\mathbb{C}(t)} \text{Hom}(1_t, \text{Std}_t^{\otimes a} \otimes \text{D}(\text{Std}_t)^{\otimes b}) = \begin{cases} 0 & \text{if } a \neq b, \\ a! & \text{if } a = b. \end{cases}$$

More precisely, with the notation of [10, Déf. 10.2], the object Std_t corresponds to the pair of finite sets $(\{1\}, \emptyset)$ and is denoted by $X_0^{\otimes \{1\}}$. Thus, $\text{Std}_t^{\otimes a} \otimes \text{D}(\text{Std}_t)^{\otimes b}$ corresponds to the pair $(\{1, \dots, a\}, \{1, \dots, b\})$ and the value of $i(\text{Std}_t^{\otimes a} \otimes \text{D}(\text{Std}_t)^{\otimes b})$ is the dimension of the space

$$\text{Hom}((\emptyset, \emptyset), (\{1, \dots, a\}, \{1, \dots, b\})),$$

which is by definition the number of bijections $\{1, \dots, b\} \rightarrow \{1, \dots, a\}$. These values are known to be equal to the moments

$$\frac{1}{\pi} \int_{\mathbb{C}} z^a \bar{z}^b e^{-|z|^2} dz$$

of a standard complex gaussian random variable, which is therefore the spectral measure associated to Std_t . Using a stabilization property of the corresponding invariants for $\text{GL}_n(\mathbb{C})$ when $n > a + b$, one gets convergence as before (see [10, Prop. 10.6]).

This proof is not as satisfactory as that of Theorem 1.4, because Deligne and Milne's definition of $\text{Rep}(\text{GL}_t)$ involves some *a priori* knowledge of stability properties of representations and linear invariants of $\text{GL}_n(\mathbb{C})$. The argument does show, however, that the convergence to the gaussian can be interpreted in terms of spectral measures, and that the standard gaussian can also be interpreted as a “generalized” Sato–Tate measure. Moreover, it suggests the question: what are the spectral measures for other objects of $\text{Rep}(\text{GL}_t)$?

Remark 3.8 Since Berg [1] proved that the third power of a real gaussian random variable is not determined by its moments, the third tensor power of $\text{Std}_t \oplus \text{D}(\text{Std}_t)$ (which, thanks to Proposition 2.8, has spectral measure the cube of a real gaussian), gives an example of an object of $\text{Rep}(\text{GL}_t)$ whose spectral measure is not unique. Once a spectral measure is not unique, it is a classical fact from the solution of the Hamburger moment problem that the set of all possible $\mu(i, M)$ has a rather complicated structure, as explained in [26, Ch. 7].

One can argue similarly with the category $\text{Rep}(\text{O}(t))$ of Deligne [10, § 9, Déf. 9.2], which interpolates representations of orthogonal groups: the standard object Std_t in this category is self-dual and satisfies

$$\dim_{\mathbb{C}(t)} \text{Hom}_{\text{Rep}(\text{O}(t))}(1_t, \text{Std}_t^{\otimes a}) = |\{\text{partitions of } \{1, \dots, a\} \text{ with all parts of size } 2\}|$$

by definition, which coincides with the a -th moment of the standard real gaussian (namely, it is 0 when a is odd, and equal to $a!/(2^{a/2}(a/2)!)$ when a is even). Thus, the standard real gaussian is the (unique) spectral measure of Std_t in $\text{Rep}(\text{O}(t))$ relative to the invariant $i(M) = \dim_{\mathbb{C}(t)} \text{Hom}(1_t, M)$. The corresponding convergence theorem is that of the direct image under the trace of the probability Haar measure of $\text{O}(n)$ to the standard real gaussian.

4 Proof of Theorem 1.5

Let $m \geq 1$ be an integer and let λ be a partition of m with parts $\lambda_1 \geq \lambda_2 \geq \dots$. Recall that the statement concerns the limiting behavior of the measures $(\chi_{\lambda,n}(X_n))_{n \geq m+\lambda_1}$, where X_n is a uniformly distributed random permutation in S_n and $\chi_{\lambda,n} : S_n \rightarrow \mathbb{C}$ is the character of the representation of S_n corresponding to the partition $(n-m, \lambda_1, \lambda_2, \dots)$.

The argument will consist of two stages:

- We prove *a priori* that the sequence of measures $(\chi_{\lambda,n}(X_n))_{n \geq m+\lambda_1}$ converges in law as $n \rightarrow +\infty$ to some measure μ_λ .
- We compute the moments of the limiting measure and show that they coincide with those of a spectral measure of the object $x_{\lambda,t}$ of the Deligne–Knop category.

Lemma 4.1 *The sequence $(\chi_{\lambda,n}(X_n))_{n \geq m+\lambda_1}$ converges in law as $n \rightarrow +\infty$ to a measure μ_λ .*

Proof For each $i \geq 1$, let $\ell_i(\sigma)$ denote the number of i -cycles (fixed points if $i = 1$) in the representation of σ as a product of cycles with disjoint support. It is known from the theory of symmetric functions that there exists a so-called character polynomial $q_\lambda \in \mathbb{Q}[(L_i)_{i \geq 1}]$ such that, for all large enough n , the equality

$$\chi_{\lambda,n}(\sigma) = q_\lambda(\ell_1(\sigma), \dots, \ell_i(\sigma), \dots)$$

holds for all $\sigma \in S_n$ (see, for instance, [24, Ex.I.7.14]). Since the sequences $(\ell_i(X_n))_{i \geq 1}$ are also known to converge in law as $n \rightarrow +\infty$ to a sequence $(P_{1/i})_{i \geq 1}$ of independent Poisson random variables with parameters $1/i$ (see, e.g., [11, Th. 7]), the sequence $(\chi_{\lambda,n}(X_n))_{n \geq m+\lambda_1}$ converges in law to $\mu_\lambda = q_\lambda(P_1, \dots, P_{1/i}, \dots)$. \square

Remark 4.2 In the spirit of Remark 3.3 (2), it would be interesting to find a proof of Lemma 4.1 without using character polynomials.

The second step will rely on Deligne’s construction of the category of representations of S_t , which enjoys some functoriality properties that have not been explicitly established by Knop. Since Theorem 1.5 is new, the fact that Deligne’s definition involves some *a priori* knowledge of representations of the symmetric groups is not an instance of circular reasoning.

Let A be a commutative ring and $t \in A$. We use the A -linear category $\text{Rep}(S_t, A)$ of Deligne [10, Déf. 2.17], keeping the notation $\text{Rep}(S_t)$ for $A = \mathbb{C}(t)$ and the indeterminate t . The basic objects of this category are associated to finite sets U and denoted

by¹ [U]; their Hom spaces are introduced in [10, Déf. 2.12]. For each integer $N \geq 0$, we consider the full subcategory $\underline{\text{Rep}}(S_t, A)^{(N)}$ whose objects are the direct factors of sums of [U] for U of cardinality $\leq N$. Deligne [10, Prop. 5.1] proved that $\underline{\text{Rep}}(S_t)^{(N)}$ is a semisimple abelian category if t is not an integer between 0 and $2N - 2$. Moreover, under the assumptions

$$t - k \in A^\times \text{ for } 0 \leq k \leq 2N - 2 \text{ and } N! \in A^\times, \quad (4.1)$$

he associated to any pair (y, ϱ) consisting of a finite set y with $|y| \leq 2N$ and an irreducible representation ϱ of the symmetric group S_y , an object $x_{y, \varrho, A}$ of $\underline{\text{Rep}}(S_t, A)^{(N)}$; see [10, Prop. 5.1 and Rem. 5.6]. (This object is independent, up to isomorphism, of the choice of N , provided (4.1) holds, and hence the value of N is omitted from the notation.)

The objects $x_{y, \varrho, A}$ are functorial with respect to A under the natural base-change functor

$$T_{A, B}: \underline{\text{Rep}}(S_t, A) \longrightarrow \underline{\text{Rep}}(S_t, B)$$

when B is an A -algebra (see [10, Déf. 2.17]), i.e., there are isomorphisms

$$x_{y, \varrho, B} \simeq T_{A, B}(x_{y, \varrho, A}).$$

If B is a field of characteristic zero and the image of t is not a non-negative integer, then the full category $\underline{\text{Rep}}(S_t, B)$ is a semisimple abelian category, and its simple objects are precisely those of the form $x_{y, \varrho, B}$, for a unique pair (y, ϱ) , up to isomorphism.

From now on, we fix an integer $N \geq 1$ and consider the ring

$$A = \mathbb{C}[t] \left[\left(\frac{1}{t - k} \right)_{0 \leq k \leq 2N-2} \right],$$

which is a principal ideal domain (being a localization of the principal ideal domain $\mathbb{C}[t]$) and satisfies the assumption (4.1).

Let $m \geq 1$ be an integer and λ a partition of m . We then set $x_{\lambda, A} = x_{y, \varrho, A}$, where $y = \{1, \dots, m\}$ and ϱ is the irreducible representation of S_m associated to the partition λ . We denote by $x_{\lambda, t}$ the base change of $x_{\lambda, A}$ to $\underline{\text{Rep}}(S_t)$ under the natural inclusion $A \hookrightarrow \mathbb{C}(t)$. Furthermore, if $n > 2N - 2$, then we denote by $x_{\lambda, n}$ the base change of $x_{\lambda, A}$ to $\underline{\text{Rep}}(S_n)$ under the morphism $A \rightarrow \mathbb{C}(t)$ that maps t to n .

We begin with a lemma generalizing the first step of the proof of Lemma 3.2 (in the sense that it shows that certain Hom spaces have the same dimension in all the categories $\underline{\text{Rep}}(S_t)$, even when t is a non-negative integer, provided it is “large enough”).

Lemma 4.3 *Let λ be a partition of an integer $m \geq 1$ and let $a \geq 0$ be an integer. For any integer $n \geq 4am - 1$, the following equality holds:*

$$\dim_{\mathbb{C}(t)} \text{Hom}_{\underline{\text{Rep}}(S_t)}(1_t, x_{\lambda, t}^{\otimes a}) = \dim_{\mathbb{C}} \text{Hom}_{\underline{\text{Rep}}(S_n)}(1_n, x_{\lambda, n}^{\otimes a}).$$

¹ Deligne’s basic generators [U] are not the same as the basic objects in Knop’s definition, but the precise relation between them is explained by Knop in [19, Rem. 1.2].

Proof Let $N \geq 1$ be an integer such that $N \geq 2am$. Then both $x_{\lambda,A}$ and $x_{\lambda,A}^{\otimes a}$ are objects of $\underline{\text{Rep}}(S_t, A)^{(N)}$ (this follows from the fact that the tensor product of two basic objects $[U]$ and $[V]$ is a direct sum of objects $[W]$ with $|W| \leq |U| + |V|$, as explained in [10, § 5.10]). Consequently, by [10, Rem. 5.6] and the fact that A is principal, there is a direct sum decomposition

$$x_{\lambda,A}^{\otimes a} \simeq \bigoplus_{|\mu| \leq N} v(\mu) x_{\mu,A} \quad (4.2)$$

for some non-negative integers $v(\mu)$, where the sum is over partitions of integers $\leq N$. Assume $n > 2N - 2$. Applying base-change to $\mathbb{C}(t)$ and to \mathbb{C} as above ($t \mapsto n$), we derive from (4.2) direct sum decompositions

$$x_{\lambda,t}^{\otimes a} \simeq \bigoplus_{|\mu| \leq N} v(\mu) x_{\mu,t} \quad x_{\lambda,n}^{\otimes a} \simeq \bigoplus_{|\mu| \leq N} v(\mu) x_{\mu,n}.$$

By [10, Rem. 5.6], the objects $x_{\mu,A}$ have the property that

$$\text{Hom}(x_{\mu,A}, x_{\nu,A}) = \begin{cases} 0 & \text{if } \mu \neq \nu, \\ A & \text{if } \mu = \nu. \end{cases}$$

Since the unit objects of $\underline{\text{Rep}}(S_t)$ and $\underline{\text{Rep}}(S_n)$ are $x_{\mu,t}$ and $x_{\mu,n}$, respectively, for the partition $\mu = (m)$ corresponding to the trivial representation of S_m , we therefore deduce from these decompositions that the equalities

$$\dim_{\mathbb{C}(t)} \text{Hom}_{\underline{\text{Rep}}(S_t)}(1_t, x_{\lambda,t}^{\otimes a}) = v((m)) = \dim_{\mathbb{C}} \text{Hom}_{\underline{\text{Rep}}(S_n)}(1_n, x_{\lambda,n}^{\otimes a})$$

hold for all $n \geq 4am - 1$, which concludes the proof. \square

Remark 4.4 A combinatorial formula for

$$\dim_{\mathbb{C}(t)} \text{Hom}_{\underline{\text{Rep}}(S_t)}(1_t, x_{\lambda,t}^{\otimes a})$$

has been obtained (in the generality of tensor envelopes) by Knop [20, Cor. 5.4, Ex. 5.6].

End of the proof of Theorem 1.5 Let $a \geq 0$ be an integer. By Lemma 4.3, the equalities

$$i(x_{\lambda,t}^{\otimes a}) = \dim_{\mathbb{C}(t)} \text{Hom}_{\underline{\text{Rep}}(S_t)}(1_t, x_{\lambda,t}^{\otimes a}) = \dim_{\mathbb{C}} \text{Hom}_{\underline{\text{Rep}}(S_n)}(1_n, x_{\lambda,n}^{\otimes a})$$

hold for all large enough integers n . Besides, Deligne [10, Prop. 6.4] has shown that, provided $n > 2m$, the semisimplification functor

$$\underline{\text{Rep}}(S_n) \rightarrow \underline{\text{Rep}}(S_n)/\mathcal{N}_n = \text{Rep}(S_n)$$

maps the object $x_{\lambda,n}$ to the representation $\pi_{\lambda,n}$ of S_n associated to the partition $\lambda^{(n)}$. Thus, we obtain the lower bound

$$i(x_{\lambda,t}^{\otimes a}) = \dim_{\mathbb{C}} \operatorname{Hom}_{\operatorname{Rep}(S_n)}(1_n, x_{\lambda,n}^{\otimes a}) \geq \dim_{\mathbb{C}} \operatorname{Hom}_{\operatorname{Rep}(S_n)}(1_n, \pi_{\lambda,n}^{\otimes a}),$$

with equality if and only if $\mathcal{N}(1_n, x_{n,\lambda}^{\otimes a}) = 0$. For all large enough (depending on a and λ) integers n , we have $\mathcal{N}(1_n, x_{\lambda,n}^{\otimes a}) = 0$ (e.g., by Knop's criterion), and hence for such n , we get

$$\int_{\mathbb{R}} x^a \mu_{\lambda}(x) = i(x_{\lambda,t}^{\otimes a}) = \dim_{\mathbb{C}} \operatorname{Hom}_{\operatorname{Rep}(S_n)}(1_n, \pi_{\lambda,n}^{\otimes a}) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{\lambda,n}(\sigma)^a,$$

as we wanted to show. This concludes the proof of Theorem 1.5. \square

5 Arithmetic speculations

The distribution of the number of fixed points of random permutations in S_n for a given integer $n \geq 1$ occurs naturally in number theory as a limiting distribution for the number of zeros modulo a prime number p of a fixed polynomial with integer coefficients $f \in \mathbb{Z}[T]$ of degree n and Galois group S_n . Indeed, let $\varrho_f(p)$ be this number. A special case of Chebotarev's density theorem states in that case² that the limit formula

$$\lim_{x \rightarrow +\infty} \frac{1}{\pi(x)} |\{p \leq x \mid \varrho_f(p) = r\}| = \frac{1}{n!} |\{\sigma \in S_n \text{ with } |\operatorname{Fix}(\sigma)| = r\}|$$

holds for all integers $r \geq 0$, where $\pi(x)$ denotes the number of primes $p \leq x$ (this was already observed by Kronecker [22] in 1880, who also pointed out the limiting behavior as $n \rightarrow +\infty$).

One may ask if a similar framework can give rise to the Poisson distribution, viewed as the number of fixed points of a “random element” of S_t for an indeterminate t . Some work of Kowalski and Soundararajan [21, § 2.4] involving *pseudopolynomials* might be related. Indeed, they have formulated the following conjecture:

Conjecture 5.1 (Kowalski–Soundararajan) *Let $F(n) = \sum_{k=0}^n n!/k!$ for integers $n \geq 0$. For any prime number p , let $\varrho_F(p)$ be the number of integers x satisfying $0 \leq x \leq p-1$ and $F(x) \equiv 0 \pmod{p}$. Then, for each integer $r \geq 0$, the following limit formula holds:*

$$\lim_{x \rightarrow +\infty} \frac{1}{\pi(x)} |\{p \leq x \mid \varrho_F(p) = r\}| = \frac{1}{e} \frac{1}{r!}.$$

² For an arbitrary irreducible polynomial $f \in \mathbb{Z}[T]$, the corresponding limit would be the probability that a uniformly distributed random element of the Galois group of the splitting field of f , viewed as a permutation of the complex roots of f , has r fixed points.

A *pseudopolynomial* in the sense of Hall [14] is a sequence $(a_n)_{n \geq 0}$ of integers such that $m - n$ divides $a_m - a_n$ for all $m > n$. Setting $G(n) = a_n$, this condition guarantees that the value $G(x) \pmod{p}$ is well-defined for $x \in \mathbb{Z}/p\mathbb{Z}$, independently of the choice of a representative to compute it. Besides the sequences $(f(n))_n$ of values of a polynomial with integer coefficients $f \in \mathbb{Z}[X]$, a standard example is $F(n)$ as in Conjecture 5.1. This function can also be written as $e \int_1^\infty x^n e^{-x} dx$ for all $n \geq 0$ (an incomplete gamma function), or $\lfloor en! \rfloor$ for $n \geq 1$.

Numerical evidence in favour of Conjecture 5.1 is quite convincing [21, § 2.4]. We speculate that, if true, this limiting behaviour might be explained by appealing to the properties of S_t and some avatar of Chebotarev's density theorem.

Another tantalizing experimental parallel observation is the following. It results from Deligne's equidistribution theorem and the work of Katz (see [17, Th. 7.10.6]) that, given a polynomial $f \in \mathbb{Z}[X]$ of degree $n \geq 6$ whose derivative f' has Galois group S_{n-1} , the exponential sums

$$W_f(a; p) = \frac{1}{\sqrt{p}} \sum_{x \pmod{p}} \exp\left(2\pi i \frac{af(x)}{p}\right)$$

for $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ become equidistributed as $p \rightarrow +\infty$ like the traces of random matrices in a compact group $K \subset U_n$ which contains SU_n .

By analogy and comparison with the results of Diaconis–Shahshahani and Larsen, we are then led to expect the following:

Conjecture 5.2 *Let $F(n) = \sum_{k=0}^n n!/k!$ for integers $n \geq 0$. For a prime number p and $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, set*

$$W_F(a; p) = \frac{1}{\sqrt{p}} \sum_{x \pmod{p}} \exp\left(2\pi i \frac{aF(x)}{p}\right).$$

Then the values $(W_F(a; p))_{a \in (\mathbb{Z}/p\mathbb{Z})^\times}$ become equidistributed as $p \rightarrow +\infty$ like a standard complex gaussian, i.e., for any continuous bounded function $\varphi: \mathbb{C} \rightarrow \mathbb{C}$, the following holds:

$$\lim_{p \rightarrow +\infty} \frac{1}{p-1} \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} \varphi(W_F(a; p)) = \frac{1}{\pi} \int_{\mathbb{C}} \varphi(z) e^{-|z|^2} dz.$$

Numerical evidence is again very convincing here. A potential link suggests itself with the category $\underline{\text{Rep}}(\text{GL}_t)$, and even more tantalizing is the suggestion of a form of Schur–Weyl duality relating the categories $\underline{\text{Rep}}(S_t)$ and $\underline{\text{Rep}}(\text{GL}_t)$.

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Appendix A Knop's construction of the category $\text{Rep}(S_t)$

In this section, we recall the steps of Knop's construction of tensor envelopes, specialized to the case of the opposite of the category of finite sets which leads to Deligne's category of "representations" of S_t .

Given sets X , Y and Z with maps $f: Y \rightarrow X$ and $g: Y \rightarrow Z$, we define the gluing $X \sqcup_Y Z$ as the quotient of the disjoint union $X \sqcup Z$ by the smallest equivalence relation that identifies $f(y) \in X$ with $g(y) \in Z$ for all $y \in Y$.

Recall that a partition of a set X is a set of non-empty subsets of X , pairwise disjoint and with union equal to X ; we will identify partitions with equivalence relations on X .

Given sets X , Y and Z , and partitions α of $X \sqcup Y$ and β of $Y \sqcup Z$, one defines a partition $\beta \odot \alpha$ of $X \sqcup Z$ as follows:

- the equivalence class of an element $x \in X$ is the union of the α -equivalence class of x and of the set of $z \in Z$ such that there exists $y \in Y$ which is α -equivalent to x and β -equivalent to z ;
- the equivalence class of an element $z \in Z$ is the union of the β -equivalence class of z and of the set of $x \in X$ such that there exists $y \in Y$ which is α -equivalent to x and β -equivalent to z .

Using the quotient maps

$$Y \rightarrow (X \sqcup Y)/\alpha, \quad Y \rightarrow (Y \sqcup Z)/\beta,$$

we define the gluing $(X \sqcup Y)/\alpha \sqcup_Y (Y \sqcup Z)/\beta$ as above. There is an injective map

$$j: (X \sqcup Z)/\beta \odot \alpha \rightarrow (X \sqcup Y)/\alpha \sqcup_Y (Y \sqcup Z)/\beta,$$

and we define $\gamma(\alpha, \beta)$ as the cardinality of the complement of the image of j . Concretely, this is the number of equivalence classes of elements of Y which are neither α -equivalent to an element of X nor β -equivalent to an element of Z .

We fix a ring k and an element t of k . The category

$$\mathcal{C}_t = \underline{\text{Rep}}(S_t)$$

is constructed in three steps. One first defines a k -linear category \mathcal{C}_t^0 : its objects are finite sets, and the morphism space $\text{Hom}_{\mathcal{C}_t^0}(X, Y)$ is the free k -module generated by

partitions of the finite set $X \sqcup Y$. The composition maps are the k -bilinear maps

$$\begin{aligned} \text{Hom}_{\mathcal{C}_t^0}(Y, Z) \times \text{Hom}_{\mathcal{C}_t^0}(X, Y) &\longrightarrow \text{Hom}_{\mathcal{C}_t^0}(X, Z) \\ (\beta, \alpha) &\longmapsto \beta \circ \alpha = t^{\gamma(\alpha, \beta)} \beta \odot \alpha. \end{aligned}$$

Associativity is not obvious, and relates to basic properties of the function γ .

If $f: X \rightarrow Y$ is a map of finite sets, then there is an associated morphism $Y \rightarrow X$ in \mathcal{C}_t^0 given by the smallest equivalence relation α_f on $Y \sqcup X$ that identifies $x \in X$ with $f(x) \in Y$ for all $x \in X$. This construction gives rise to a contravariant functor from the category of finite sets to the category \mathcal{C}_t^0 (because it is elementary that $\gamma(\beta, \alpha) = 0$ whenever α and β are equivalence relations associated to maps, and hence $\alpha_{g \circ f} = \alpha_g \circ \alpha_f$ holds for composable maps f and g); this functor is faithful.

From \mathcal{C}_t^0 , a category \mathcal{C}_t' is constructed as the category of formal finite direct sums of objects of \mathcal{C}_t^0 , with morphisms given by matrices in the obvious way. Finally, Knop's tensor envelope category \mathcal{C}_t is defined by “adding images of projectors”: an object is a pair (X, p) of an object X of \mathcal{C}_t' and an endomorphism p of X such that $p \circ p = p$, and morphisms between two such pairs are given by

$$\text{Hom}_{\mathcal{C}_t}((X, p), (Y, q)) = q \circ \text{Hom}_{\mathcal{C}_t'}(X, Y) \circ p \subset \text{Hom}_{\mathcal{C}_t'}(X, Y).$$

The category \mathcal{C}_t^0 admits a monoidal structure in the sense of [12, Def. 2.1.1]. The tensor product bifunctor is defined on objects as $X \otimes Y = X \sqcup Y$ for finite sets X and Y . As for morphisms, the tensor product $\alpha \otimes \beta \in \text{Hom}_{\mathcal{C}_t^0}(X \otimes Y, X' \otimes Y')$ of $\alpha \in \text{Hom}_{\mathcal{C}_t^0}(X, X')$ and $\beta \in \text{Hom}_{\mathcal{C}_t^0}(Y, Y')$ is the equivalence relation on

$$(X \otimes Y) \sqcup (X' \otimes Y') = (X \sqcup Y) \sqcup (X' \sqcup Y')$$

which “coincides” with α on $X \sqcup X'$ and with β on $Y \sqcup Y'$. The commutativity constraint $X \otimes Y \xrightarrow{\sim} Y \otimes X$ and the associativity constraint $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ are given, respectively, by the morphisms associated to the obvious identifications $X \sqcup Y = Y \sqcup X$ and $(X \sqcup Y) \sqcup Z = X \sqcup (Y \sqcup Z)$. The unit object $\mathbf{1}$ is the empty set, along with the unique morphism $\mathbf{1} \otimes \mathbf{1} \rightarrow \mathbf{1}$.

It is elementary that if p and q are projectors, then $p \otimes q$ is also one, so the rules

$$(X, p) \otimes (Y, q) = (X \otimes Y, p \otimes q)$$

and bilinearity define a symmetric monoidal structure on \mathcal{C}_t .

The monoidal category \mathcal{C}_t^0 is rigid ([12, Def. 2.10.1]). Indeed, the dual of a finite set X is defined to be $D(X) = X$ itself, and the evaluation and coevaluation morphisms

$$\text{ev}_X: D(X) \otimes X \rightarrow \mathbf{1}, \quad \text{coev}_X: \mathbf{1} \rightarrow X \otimes D(X)$$

are both identified with the equivalence relation on $X \sqcup X$ associated to the identity map on X . For a morphism $\alpha \in \text{Hom}_{\mathcal{C}_t^0}(X, Y)$, the transpose ${}^t\alpha \in \text{Hom}_{\mathcal{C}_t^0}(Y, X)$ is

defined as the composition

$$\begin{aligned} D(Y) = Y = Y \otimes \mathbf{1} &\xrightarrow{\text{id} \otimes \text{coev}_X} Y \otimes (X \otimes X) \simeq (Y \otimes X) \otimes X \\ &\xrightarrow{(\text{id} \otimes \alpha) \otimes \text{id}} (D(Y) \otimes Y) \otimes X \xrightarrow{\text{ev}_Y \otimes \text{id}} \mathbf{1} \otimes X = X = D(X), \end{aligned}$$

and corresponds to the obvious equivalence relation on $X \sqcup Y$ which is “the same” as α on $Y \sqcup X$.

This duality functor extends by linearity to \mathcal{C}'_t , and finally to \mathcal{C}_t : we have

$$D(X, p) = (D(X), \text{Id}_{D(X)} - {}^t p)$$

for an object (X, p) of \mathcal{C}_t , and ${}^t(q \circ \alpha \circ p) = {}^t p \circ {}^t \alpha \circ {}^t q$ for a morphism $q \circ \alpha \circ p \in \text{Hom}_{\mathcal{C}_t}((X, p), (Y, q))$.

Thus, \mathcal{C}_t has the structure of a rigid symmetric monoidal k -linear category.

Suppose that the ring k is a field of characteristic 0 and t is not a non-negative integer. Then Knop proved that the category \mathcal{C}_t is semisimple [18, Th. 6.1 along with Ex. 1, p. 596].

Let A be a fixed finite set and $G = \text{Aut}(A)$ the corresponding symmetric group. An element $g \in G$ acts by precomposition with g^{-1} on the sets of maps from A to any finite set X . The contravariant functor

$$h_A(X) = \text{Hom}_{\text{Set}}(A, X)$$

from the category of finite sets to the category Set_G of finite sets with a G -action can be extended to a tensor functor $T_A: \mathcal{C}_t \rightarrow \text{Rep}_k(G)$, where $\text{Rep}_k(G)$ is the category of finite-dimensional k -linear representations of G . So that the diagram

$$\begin{array}{ccc} \text{Set}^{opp} & \xrightarrow{h_A} & \text{Set}_G \\ \downarrow & & \downarrow \\ \mathcal{C}_t & \xrightarrow{T_A} & \text{Rep}_k(G) \end{array}$$

commutes [18, proof of Th. 9.4, (9.23)], where the functor $\text{Set}_G \rightarrow \text{Rep}_k(G)$ associates to a finite set Y with a G -action the permutation representation of G on the free k -module with basis Y .

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