EULER'S CONSTANT AND EXPONENTIAL MOTIVES

(Pisa, Regulators V, 3/6/24)

1 Exponential motives

$k \subset \mathbb{C}$ (e.g. $k = \mathbb{Q}$)

X smooth algebraic variety over k (e.g. X affine)

$f : X \longrightarrow \mathbb{A}^1$ regular function

- Twisted de Rham cohomology

$$H^n_{dR}(X, f) = \mathbb{H}^n_{Zar}(X, \mathcal{O}_X \to \Omega^1_X \to \cdots)$$

$$\omega \mapsto d\omega - df \wedge \omega$$

- Rapid decay homology

$$H^{rd}_n(X, f) = \lim_{t \to \infty} H^{sing}_n(X(\mathbb{C}), \{\mathrm{Re} f \geq t\}; \mathbb{Q})$$

and an integration pairing

$$H^{Rd}_n(X, f) \otimes H^n_{dR}(X, f) \longrightarrow \mathbb{C}$$

$$[\omega] \otimes [\sigma] \longmapsto \int_\sigma e^{-f}\omega$$

which is perfect by Bloch-Esnault and Hien-Roucairol (Commelin-Habegger-Huber)

Variants: $H^n_{dR}(X, Y, f)$ for $Y \subset X$ a simple normal crossing divisor

$H^{Rd}_n(X, Y, f)$

When $k \subset \bar{\mathbb{Q}}$, the coefficients of this pairing are called $\boxed{\text{exponential periods}}$

(joint work with P. Jossen)

construction of a $\mathbb{Q}$-linear neutral Tannakian category $M^{exp}(k)$ of exponential motives over k with fiber functors

$$\begin{tikzcd} & \mathrm{Vect}_K & H^n_{dR}(X,Y,f) \\ M^{exp}(k) \arrow[ur, "w_{dR}"] \arrow[dr, "w_{Rd}"'] & & \\ H^n(X,Y,f) & \mathrm{Vect}_{\mathbb{Q}} & H^n_{Rd}(X,Y,f) \end{tikzcd}$$

$= $ linear dual of $H^{Rd}_n(X,Y,f)$

$$\boxed{\text{Conjecture: } \operatorname{tr\,deg} \overline{\mathbb{Q}}\begin{pmatrix}\text{exponential}\\ \text{periods of } M\end{pmatrix} = \dim G_M}$$

This category contains Nori's category of mixed motives over k as a full subcategory

$$M(k) \subset M^{exp}(k)$$

→ There is a rather simple criterion to decide whether an object of $M^{exp}(k)$ belongs to $M(k)$, i.e. is a classical motive

$\mathrm{Perv}_0 \subset \mathrm{Perv}(\mathbb{A}^1(\mathbb{C}), \mathbb{Q})$ defined by the condition $R\pi_* = 0$ $\quad \pi: \mathbb{A}^1 \to \operatorname{Spec} \mathbb{C}$

(Kontsevich - Soibelman)

e.g. $j: \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \quad E(0) = j_! \mathbb{Q}[1]$

There is a projection $\Pi: \mathrm{Perv} \to \mathrm{Perv}_0$

left adjoint to the inclusion

$$A \longmapsto A * E(0)$$

where $*$ is additive convolution

Perverse realisation functor

$$M^{exp}(k) \longrightarrow \mathrm{Perv}_0$$

$$H^n(X, f) \longmapsto \Pi({}^p\mathcal{H}^n(Rf_* \mathbb{Q}))$$

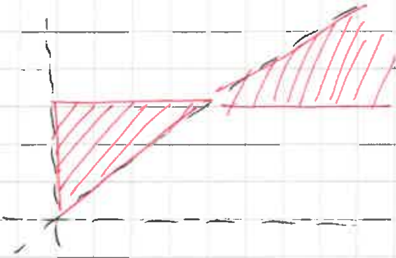
$$= R^{n+1} p_* \mathbb{Q}_{[X \times \mathbb{A}^1, \Gamma_f]}$$

where $p : X \times \mathbb{A}^1 \to \mathbb{A}^1$ is the projection and Γ_f is the graph of f.

Theorem: An object of $M^{exp}(k)$ lies in the essential image of $M(k)$ $\Longleftrightarrow$ its perverse realisation is trivial (e.g. isomorphic to $E(0)^{\oplus S}$).

② Euler's constant

The relation with the Γ-function gives the integral representation



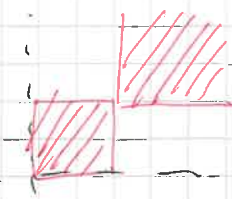
$$\gamma = -\int_0^\infty \log(x)\, e^{-x}\, dx$$

$$= -\int_0^\infty e^{-x} \int_1^x \frac{dy}{y}\, dx = \int_0^1 e^{-x} \int_x^1 \frac{dy}{y}\, dx$$

$$- \int_1^\infty e^{-x} \int_1^x \frac{dy}{y}\, dx$$

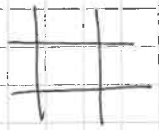
$x = uv$
$y = v$

$$= \int_0^1\!\!\int_0^1 e^{-xy}\, dx\, dy - \int_1^\infty\!\!\int_1^\infty e^{-xy}\, dx\, dy$$



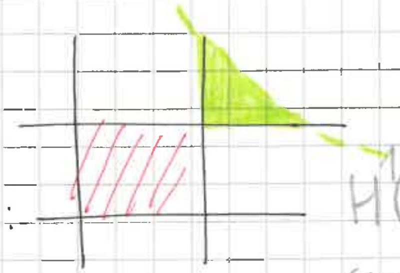
This suggests considering the exponential motive $H^2(X, Y, f)$ where

$X = \mathbb{A}^2_{\mathbb{Q}}$

$Y =$  $xy(x-1)(y-1) = 0$

$f = xy$

One can compute it and it turns out to have dimension 3 and period matrix with respect to suitable basis:



$H^1(xy = t)$ contributes with a class as well

$$\begin{pmatrix} \boxed{1} & 0 & \boxed{\gamma} \\ 0 & 1/e & E_1(1) \\ \boxed{0} & 0 & \boxed{2\pi i} \end{pmatrix}$$

exponential integral function $E_1(z) = \int_1^{\infty} e^{-tz} \frac{dt}{t}$

The shape of the matrix suggests the existence of a 2-dim quotient $M(\gamma)$ with period matrix $\begin{pmatrix} 1 & \gamma \\ 0 & 2\pi i \end{pmatrix}$, and indeed this can be constructed as the image by the morphism induced by the blow-up

$$\pi : \mathrm{Bl}_{(1,1)} \mathbb{A}^2 \longrightarrow \mathbb{A}^2$$

* $M(\gamma)$ is an extension

$$0 \to \boxed{\mathbb{Q}(0)} \to M(\gamma) \to \boxed{\mathbb{Q}(-1)} \to 0$$

which is not isomorphic to a classical motive

(i.e. $M(K)$ is not closed under extension!)

* Indeed, the perverse realisation of $M(\gamma)$ is a non-trivial extension of $E(0)$ by itself, namely $j_! L[1]$ where L is the rank 2 local system on $\mathbb{G}_m$ with unipotent monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

[Intuition: $(X, Y) = (\mathbb{A}^2, \#)$

$$\begin{array}{c} (X,Y) \\ \downarrow f \\ \mathbb{A}^1 \end{array} \qquad f^{-1}(z) \simeq (\mathbb{G}_m, \{1, z\}) \quad z \neq 0 \;]$$

Using this, we can show that the expected period conjecture implies:

Conjecture: γ is transcendental over the field generated by all classical periods.

(indeed, let M be a classical motive. Then

$$\begin{array}{ccc} \mathbb{G}_a = G_{R_{perv}(M(\gamma))} & \longrightarrow & 0 \\ \cap & & \cap \\ G_{M \oplus M(\gamma)} & \twoheadrightarrow & G_M \end{array}$$

so $\dim G_{M \oplus M(\gamma)} > \dim G_M$).

③ Monodromy features of E-functions

(Siegel, 1929) An E-function is a power series

$$E(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$$

which solves a non-zero differential equation with coefficients in $\overline{\mathbb{Q}}[z]$ and satisfies the growth condition

$$\max_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| < C^n$$

$$\mathrm{den}(a_0, \dots, a_n) < C^n$$

e.g. the modified Bessel function

$$I_0(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$$

$$= \frac{1}{2\pi i} \int_{|z|=1} e^{-\frac{z}{2}(x + \frac{1}{x})} \frac{dx}{x}$$

which is an exponential period function for the family $H^1(\mathbb{G}_m, \frac{z}{2}(x+\frac{1}{x}))(1)$. Integrating over the rapid decay cycle $[0, \infty]$ instead we get the function

$$K_0(z) = \frac{1}{2} \int_0^{\infty} e^{-\frac{z}{2}(x+\frac{1}{x})} \frac{dx}{x}$$

$$= -\left(\log\left(\frac{z}{2}\right) + \gamma\right) I_0(z) + \underbrace{\sum_{n=1}^{\infty} \frac{H_n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}}_{\text{another E-function!}}$$

Theorem (with P. Jossen) X smooth affine of dimension d, $f : X \to \mathbb{A}^1$, $\omega \in H^d_{dR}(X, f)$

$\sigma \in H^{Rd}_d(X, f)$

$\int_\sigma e^{-zf}\omega$ is a linear combination of

$z^{a} (\log z)^{b} E(z)$ with coefficients in the ring $\overline{\mathbb{Q}}\left[\text{classical periods}, \gamma, \Gamma(r)_{r \in \mathbb{Q}\setminus\mathbb{Z}_{\leq 0}}\right]$.

$\mathbb{Q}$ $\qquad \mathbb{Z}_{\geq 0}$ $\qquad$ E-function

A key point is that the differential equation of $z \mapsto \int_\sigma e^{-zf}\omega$ (which is the $\mathcal{D}_{\mathbb{G}_m}$-module $\pi_+ \mathcal{E}^{zf}$ for $\pi : X \times \mathbb{G}_m \to \mathbb{G}_m$ the projection) has quasi-unipotent monodromy and $z = 0$ (a regular singularity) and there is a tensor structure on $M(\gamma)$ and $H^1(\mathbb{A}^1, x^n)$ whose ~~perse~~ realisation has the same monodromy.

4 A question

The computation of extension groups is out of reach in the category $M^{exp}(k)$. However, we can compute

$$\mathrm{Hom}_{DExp(k)}(\mathbb{Q}(0), \mathbb{Q}(n)[j])$$

in the subcategory $DExp(k) \subset DM_{gm}(\mathbb{A}^1_k)$ of motives satisfying $\pi_* M = 0$. (Gallauer Pepin-Lehalleur)

For k a number field, we find:

$$\begin{cases} \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n)) & n \neq 1 \qquad M(\gamma) \\ \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \oplus \mathbb{Q} & n = 1 \\ \mathrm{Ext}^2(\mathbb{Q}(0), \mathbb{Q}(n)) = \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n-1)) \end{cases}$$

$\Longleftarrow$ cup product with $M(\gamma)(1)$

of $M(1)$

- What are these new mixed Tate motives over k? What are their periods which are not polynomials in γ, $J(2)$, ... ?

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Q}(n) \arrow[r] & M(1) \arrow[r] \arrow[dr] & M(\gamma)(1) \arrow[r] & \mathbb{Q}(0) \arrow[r] & 0 \\ & 0 \arrow[rr] & & \mathbb{Q}(1) \arrow[u] \arrow[dr] & & \\ & & & & 0 & \end{tikzcd}$$