QUADRATIC RELATIONS BETWEEN PERIODS OF CONNECTIONS

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Abstract. We prove the existence of quadratic relations between periods of meromorphic flat bundles on complex manifolds with poles along a divisor with normal crossings under the assumption of "goodness". In dimension one, for which goodness is always satisfied, we provide methods to compute the various pairings involved. In an appendix, we give details on the classical results needed for the proofs.

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1. Introduction

In a series of papers, Matsumoto et al. [10, 41, 32, 33, 42, 23, 28, 27, 37, 21, 22] proved that there exist natural relations between periods of various kinds of complex differential equations on a Riemann surface (mainly, the Riemann sphere) that generalize various functional equations for classical special functions. These relations are quadratic in the entries of the period matrix and arise from the comparison of the intersection matrix of the twisted cycles entering the period integrals with the de Rham pairing of the twisted differential 1-forms that are integrated along them. The aforementioned articles show many interesting examples where these quadratic relations can be effectively computed. However, they are not proved in the full generality that one needs for analyzing, say, quadratic relations between Bessel moments, which are periods of symmetric powers of the Bessel differential equation [19]. The goal of this article is to establish the existence of such quadratic relations in a general setting and to give some methods to compute them for differential equations on a Riemann surface, without any assumption on the kind of singularities it may have. The papers cited above deal mainly (but not only) with the case of a meromorphic connection with regular singularities twisted by a meromorphic form with arbitrary poles.

Section 2 considers the case of a differentiable manifold with corners and a differentiable vector bundle on it endowed with an integrable connection having poles along the boundary. No analyticity property is needed, but an assumption on the behaviour of the de Rham complexes near the boundary looks essential (Assumptions 2.1). Quadratic relations are obtained as Corollaries 2.13 and 2.25. We revisit the period pairings introduced by Bloch and Esnault [3] in the form considered by Hien [24, 25, 26]. Applications to period pairings similar to those of loc. cit. are given in Section 2.e. The main idea taken from [24, 25, 26] is to define local period pairings in the sheaf-theoretical sense, prove that they are perfect, and get for free the perfectness of the global period pairings by applying the Poincaré-Verdier duality theorem. In that case, the manifold with corners is nothing but the real oriented blow-up of a complex manifold along a divisor with normal crossings. Provided Assumptions 2.1 are satisfied, these ideas apply to the general framework for proving quadratic relations between periods. We emphasize *middle quadratic relations* since they tend to be the non-trivial part of quadratic relations.

The tools for proving the quadratic relations between periods are taken from classics in mathematics and could have been put to work more than fifty years ago. Nevertheless, the appendix recalls with details these classical results, which are taken from the Séminaire Cartan [5, 6] (we reinterpret the notion of "carapace" in terms of presheaves), de Rham's book [11, 12], Malgrange's book [38], and the book of Kashiwara and Schapira [29]. In particular, in Appendix C we make precise an "obvious result" (Corollary C.6) which we could not find in the literature. Let us emphasize that a similar approach has already been used by Kita and Yoshida [34] in order to define and compute the Betti intersection pairing for ordinary twisted homology. We improve their results by extending them to connections of any rank and possibly with irregular singularities.

Section 3 focuses on the case of meromorphic connections on Riemann surfaces. A formula "à la Čech" for computing the de Rham pairing is provided by Theorem 3.12. In the case of rank one bundles with meromorphic connection, this result already appears in Deligne's notes [13]. We also give a formula for computing period matrices (Proposition 3.43) that goes back, for connections with regular singularities, at least to [16, Rem. 2.16]. Quadratic relations in the sense of Matsumoto et al. are obtained in (3.49).

Needless to say, the results of this article are not essentially new, but we have tried to give them with enough generality, rigor, and details so that they can be used in various situations without reproving the intermediate steps.

2. Pairings for bundles with flat connection and quadratic relations

2.a. **Setting and notation.** We consider a C^∞ manifold M of real dimension m with corners, that we assume to be connected and orientable. The boundary of M is denoted by ∂M, and the inclusion of the interior $M^\circ = M \smallsetminus \partial M$ by $j\colon M^\circ \hookrightarrow M$. In the neighbourhood of each point of ∂M, the pair $(M, \partial M)$ is diffeomorphic to $(\mathbf{R}_+^p, \partial\mathbf{R}_+^p) \times \mathbf{R}^{m-p}$. The sheaf $\mathscr{C}_M^\infty$ of C^∞ functions (resp. $\mathfrak{Db}_M$ of distributions) on M is locally the sheaf-theoretic restriction to $(\mathbf{R}_+^p, \partial\mathbf{R}_+^p) \times \mathbf{R}^{m-p}$ of the sheaf of C^∞ functions (resp. distributions) on $\mathbf{R}^m$. We will also consider the sheaf $\mathscr{C}_M^\infty(*)$ of C^∞ functions on M° having at most poles along ∂M. In particular, the de Rham complex $(\mathscr{E}_M^\bullet, \mathrm{d})$ is a resolution of $\mathbf{C}_M$. The complex of currents $(\mathfrak{C}_{M,\bullet}, \partial)$, denoted homologically, can be regarded cohomologically as $(\mathfrak{Db}_M^{m-\bullet}, \mathrm{d})$. Recall (see Appendix B) that a current of homological index q is a linear form on test differential forms of degree q. Then the inclusion of complexes $(\mathscr{E}_M^{m-\bullet}, \mathrm{d}) \hookrightarrow (\mathfrak{Db}_M^{m-\bullet}, \mathrm{d})$ is a quasi-isomorphism. In other words, $(\mathfrak{Db}_M^{m-\bullet}, \mathrm{d})$ is a resolution of $\mathbf{C}_M[m]$. We refer the reader to Appendix B for the notation and results concerning currents with moderate growth. The results in this section are a mere adaptation of those contained in de Rham's book [11, 12] (see Chapter IV in [12]). In order to simplify the discussion, and since we are only interested in this case, we assume all along this section that M is moreover *compact*.

2.b. **De Rham formalism for quadratic relations.**

Setting and the main assumption. We keep the assumptions and notation of the previous section. Let $(\mathscr{V}, \nabla)$ be a locally free $\mathscr{C}_M^\infty(*)$-module on M endowed with a flat connection ∇ and a *flat non-degenerate* pairing $\langle\bullet,\bullet\rangle : \mathscr{V} \otimes \mathscr{V} \to \mathscr{C}_M^\infty(*)$, i.e., compatible with the connections. The pairing induces a self-duality $\lambda : \mathscr{V} \xrightarrow{\sim} \mathscr{V}^\vee$ which factorizes $\langle\bullet,\bullet\rangle$ as the composition

$$\mathscr{V} \otimes \mathscr{V} \xrightarrow{1 \otimes \lambda} \mathscr{V} \otimes \mathscr{V}^\vee \xrightarrow{(\bullet|\bullet)} \mathscr{C}_M^\infty(*),$$

where $(\bullet|\bullet)$ denotes the natural duality pairing. We can define the de Rham complexes of $(\mathscr{V}, \nabla)$ with respectively $\mathscr{C}_M^{\infty,\mathrm{rd}}$ and $\mathscr{C}_M^{\infty,\mathrm{mod}}$ coefficients. Since ∇ has only poles as possible singularities along ∂M, the complexes

$$\mathrm{DR}^{\mathrm{rd}}(\mathscr{V}, \nabla) = (\mathscr{E}_M^{\mathrm{rd},\bullet} \otimes \mathscr{V}, \nabla) \quad \text{and} \quad \mathrm{DR}^{\mathrm{mod}}(\mathscr{V}, \nabla) = (\mathscr{E}_M^{\mathrm{mod},\bullet} \otimes \mathscr{V}, \nabla)$$

are well-defined. The hypercohomologies of these complexes are the de Rham cohomologies with rapid decay $\mathrm{H}^r_{\mathrm{dR,rd}}(M, \mathscr{V})$ and the de Rham cohomologies with moderate growth $\mathrm{H}^r_{\mathrm{dR,mod}}(M, \mathscr{V})$. We make the following assumptions.

Assumption 2.1.

(1) The complexes $\mathrm{DR}^{\mathrm{rd}}(\mathscr{V}, \nabla)$ and $\mathrm{DR}^{\mathrm{mod}}(\mathscr{V}, \nabla)$ have cohomology concentrated in degree 0, denoted respectively by $\mathscr{V}^{\mathrm{rd}}$ and $\mathscr{V}^{\mathrm{mod}}$.

(2) The natural pairings defined by means of $\langle\bullet,\bullet\rangle : \mathscr{V} \otimes \mathscr{V} \to \mathscr{C}_M^\infty(*)$ and of the wedge product:

$$\mathsf{Q}_{\mathrm{rd,mod}} : \mathrm{DR}^{\mathrm{rd}}(\mathscr{V}, \nabla) \otimes \mathrm{DR}^{\mathrm{mod}}(\mathscr{V}, \nabla) \longrightarrow (\mathscr{E}_M^{\mathrm{rd},\bullet}, \mathrm{d}),$$
$$\mathsf{Q}_{\mathrm{mod,rd}} : \mathrm{DR}^{\mathrm{mod}}(\mathscr{V}, \nabla) \otimes \mathrm{DR}^{\mathrm{rd}}(\mathscr{V}, \nabla) \longrightarrow (\mathscr{E}_M^{\mathrm{rd},\bullet}, \mathrm{d})$$

are perfect, i.e., the pairings induced on the $\mathscr{H}^0$ are perfect:

$$\mathscr{V}^{\mathrm{rd}} \otimes \mathscr{V}^{\mathrm{mod}} \longrightarrow j_!\mathbf{C}_{M^\circ}[m],$$
$$\mathscr{V}^{\mathrm{mod}} \otimes \mathscr{V}^{\mathrm{rd}} \longrightarrow j_!\mathbf{C}_{M^\circ}[m].$$

Decoupling. One can drop the assumption on the existence of a non-degenerate pairing $\langle\bullet,\bullet\rangle$. One uses $(\bullet|\bullet)$ instead. One modifies Assumptions 2.1 as follows.

- In Assumption 2.1(1), one adds a similar property for $\mathrm{DR}^{\mathrm{rd}}(\mathscr{V}^\vee, \nabla)$ and $\mathrm{DR}^{\mathrm{mod}}(\mathscr{V}^\vee, \nabla)$.
- In Assumption 2.1(2), one uses the natural pairings induced by $(\bullet|\bullet)$.

The de Rham pairings. Verdier duality (Corollary C.6) implies that the natural pairings

$$\begin{aligned}\mathrm{H}^{m-r}(M,\mathscr{V}^{\mathrm{rd}})\otimes\mathrm{H}^{r}(M,\mathscr{V}^{\mathrm{mod}})&\longrightarrow\mathrm{H}^{m}_{\mathrm{c}}(M^{\circ},\mathbf{C})\simeq\mathbf{C},\\ \mathrm{H}^{m-r}(M,\mathscr{V}^{\mathrm{mod}})\otimes\mathrm{H}^{r}(M,\mathscr{V}^{\mathrm{rd}})&\longrightarrow\mathrm{H}^{m}_{\mathrm{c}}(M^{\circ},\mathbf{C})\simeq\mathbf{C}\end{aligned}$$

are non-degenerate. We will interpret these pairings in terms of the de Rham cohomologies. We note that the following diagram commutes in an obvious way:

$$\begin{array}{ccc}\mathrm{DR}^{\mathrm{rd}}(\mathscr{V},\nabla)\otimes\mathrm{DR}^{\mathrm{mod}}(\mathscr{V},\nabla) & \xrightarrow{\mathsf{Q}_{\mathrm{rd,mod}}} & (\mathscr{E}_M^{\mathrm{rd},\bullet},\mathrm{d})\\ \downarrow\qquad\qquad\uparrow & & \|\\ \mathrm{DR}^{\mathrm{mod}}(\mathscr{V},\nabla)\otimes\mathrm{DR}^{\mathrm{rd}}(\mathscr{V},\nabla) & \xrightarrow{\mathsf{Q}_{\mathrm{mod,rd}}} & (\mathscr{E}_M^{\mathrm{rd},\bullet},\mathrm{d})\end{array}\tag{2.2}$$

For any $r\geqslant 0$, let $\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}$ be the global de Rham pairing

$$\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}:\Gamma(M,\mathscr{E}_M^{\mathrm{rd},m-r}\otimes\mathscr{V})^{\nabla}\otimes\Gamma(M,\mathscr{E}_M^{\mathrm{mod},r}\otimes\mathscr{V})^{\nabla}\longrightarrow\mathbf{C}$$

obtained as the restriction to horizontal sections of the composition of $\Gamma(M,\mathsf{Q})$ with integration $\int_M:\mathscr{E}_M^{\mathrm{rd},m}\to\mathbf{C}$. Define $\mathsf{Q}^{m-r}_{\mathrm{mod,rd}}$ similarly. Since rapid decay forms vanish on ∂M, it follows from the Stokes formula that $\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}$ (resp. $\mathsf{Q}^{m-r}_{\mathrm{mod,rd}}$) vanishes when one of the terms is a coboundary. We thus obtain:

Corollary 2.3. *Under Assumptions* 2.1*, the induced pairing*

$$\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}:\mathrm{H}^{m-r}_{\mathrm{dR,rd}}(M,\mathscr{V})\otimes\mathrm{H}^{r}_{\mathrm{dR,mod}}(M,\mathscr{V})\longrightarrow\mathbf{C}$$

is perfect, and so is $\mathsf{Q}^{m-r}_{\mathrm{mod,rd}}$. □

Twisted currents with rapid decay and moderate growth. We consider the following complexes of currents with coefficients in $\mathscr{V}$ (see (B.3) for the boundary operator):

- twisted currents with rapid decay
$$(\mathfrak{C}^{\mathrm{rd}}_{M,\bullet}(\mathscr{V}),\partial)=(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet},\partial)\otimes\mathrm{DR}^{\mathrm{rd}}(\mathscr{V},\nabla)\simeq(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet},\partial)\otimes\mathscr{V}^{\mathrm{rd}},$$
- twisted currents with moderate growth
$$(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet}(\mathscr{V}),\partial)=(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet},\partial)\otimes\mathrm{DR}^{\mathrm{mod}}(\mathscr{V},\nabla)\simeq(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet},\partial)\otimes\mathscr{V}^{\mathrm{mod}}.$$

We define de Rham homology with rapid decay (resp. moderate growth) with coefficients in $\mathscr{V}$ as

$$\mathrm{H}^{\mathrm{rd}}_{\mathrm{dR},r}(M,\mathscr{V})=\mathrm{H}_r\big(\Gamma(M,\mathfrak{C}^{\mathrm{rd}}_{M,\bullet}(\mathscr{V}))\big),\quad\text{resp.}\quad\mathrm{H}^{\mathrm{mod}}_{\mathrm{dR},r}(M,\mathscr{V})=\mathrm{H}_r\big(\Gamma(M,\mathfrak{C}^{\mathrm{mod}}_{M,\bullet}(\mathscr{V}))\big),$$

(recall that M is assumed to be compact).[1]

The Poincaré isomorphisms. The quasi-isomorphism $\mathbf{C}_M[m]\xrightarrow{\sim}(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet},\partial)$ leads to Poincaré isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbf{C}_M)$, making the following diagram commutative

$$\begin{array}{ccc}\mathrm{DR}^{\mathrm{rd}}(\mathscr{V},\nabla)[m] & \xrightarrow[\sim]{\mathcal{P}^{\mathrm{rd}}(\mathscr{V})} & \mathfrak{C}^{\mathrm{rd}}_{M,\bullet}(\mathscr{V})=\mathfrak{C}^{\mathrm{mod}}_{M,\bullet}\otimes\mathrm{DR}^{\mathrm{rd}}(\mathscr{V},\nabla)\\ \downarrow & & \downarrow\\ \mathrm{DR}^{\mathrm{mod}}(\mathscr{V},\nabla)[m] & \xrightarrow[\sim]{\mathcal{P}^{\mathrm{mod}}(\mathscr{V})} & \mathfrak{C}^{\mathrm{mod}}_{M,\bullet}(\mathscr{V})=\mathfrak{C}^{\mathrm{mod}}_{M,\bullet}\otimes\mathrm{DR}^{\mathrm{mod}}(\mathscr{V},\nabla).\end{array}\tag{2.4}$$

By taking hypercohomologies (with compact support), we thus obtain global Poincaré isomorphisms

$$\begin{aligned}\mathcal{P}^{\mathrm{rd}}_r(\mathscr{V}):\mathrm{H}^{m-r}_{\mathrm{dR,rd}}(M,\mathscr{V})&\xrightarrow{\sim}\mathrm{H}^{\mathrm{rd}}_{\mathrm{dR},r}(M,\mathscr{V}),\\ \mathcal{P}^{\mathrm{mod}}_r(\mathscr{V}):\mathrm{H}^{m-r}_{\mathrm{dR,mod}}(M,\mathscr{V})&\xrightarrow{\sim}\mathrm{H}^{\mathrm{mod}}_{\mathrm{dR},r}(M,\mathscr{V}).\end{aligned}\tag{2.5}$$

[1] When $(\mathscr{V},\nabla)=(\mathscr{C}^{\infty}_M(*\partial M),\mathrm{d})$, rapid decay and moderate de Rham homologies are the homology and the Borel-Moore homology of M°. In general, both rapid decay and moderate de Rham cycles can have closed support.

De Rham period pairings. We define the de Rham period pairing

$$\mathsf{P}_{\mathrm{dR}}^{\mathrm{rd,mod}} : (\mathfrak{C}_{M,\bullet}^{\mathrm{rd}}(\mathscr{V}), \partial) \otimes \mathrm{DR}^{\mathrm{mod}}(\mathscr{V}, \nabla) \longrightarrow (\mathfrak{C}_{M,\bullet}^{\mathrm{rd}}, \partial) \tag{2.6}$$

so that, for local sections $T_q \otimes \eta^r \otimes v$ and $\omega^s \otimes w$ (where η^r has rapid decay and ω^s has moderate growth), $\mathsf{P}_{\mathrm{dR}}^{\mathrm{rd,mod}}(T_q \otimes \eta^r \otimes v, \omega^s \otimes w)$ is the current $T_q \otimes (\langle v, w\rangle \cdot \eta^r \wedge \omega^s)$ of dimension $q-r-s$. It has rapid decay since $\langle v, w\rangle$ has at most a pole (and hence moderate growth), and thus $\langle v, w\rangle \cdot \eta^r \wedge \omega^s$ has rapid decay. Note that, if T_q is C^∞, then we recover $\mathsf{Q}_{\mathrm{rd,mod}}$.

Similarly, we define the period pairing

$$\mathsf{P}_{\mathrm{dR}}^{\mathrm{mod,rd}} : (\mathfrak{C}_{M,\bullet}^{\mathrm{mod}}(\mathscr{V}), \partial) \otimes \mathrm{DR}^{\mathrm{rd}}(\mathscr{V}, \nabla) \longrightarrow (\mathfrak{C}_{M,\bullet}^{\mathrm{rd}}, \partial). \tag{2.7}$$

Then the following diagram commutes:

$$\begin{array}{ccc} (\mathfrak{C}_{M,\bullet}^{\mathrm{rd}}(\mathscr{V}), \partial) \otimes \mathrm{DR}^{\mathrm{mod}}(\mathscr{V}, \nabla) & \xrightarrow{\mathsf{P}_{\mathrm{dR}}^{\mathrm{rd,mod}}} & (\mathfrak{C}_{M,\bullet}^{\mathrm{rd}}, \partial) \\ {\scriptstyle \mathcal{P}^{\mathrm{rd}}(\mathscr{V})}\uparrow\wr \qquad \Vert & & \wr\uparrow{\scriptstyle \mathcal{P}^{\mathrm{rd}}} \\ [\mathrm{DR}^{\mathrm{rd}}(\mathscr{V}, \nabla) \otimes \mathrm{DR}^{\mathrm{mod}}(\mathscr{V}, \nabla)][m] & \xrightarrow{\mathsf{Q}_{\mathrm{rd,mod}}} & (\mathscr{E}_M^{\mathrm{rd},\bullet}, \partial) \end{array} \tag{2.8}$$

and a similar diagram for $\mathsf{Q}_{\mathrm{mod,rd}}$. It follows that the period pairings are also non-degenerate (here, we really work with Verdier duality in the derived category, since we have to invert the Poincaré quasi-isomorphisms).

For each $r \geqslant 0$, let $\mathsf{P}_{\mathrm{dR},r}^{\mathrm{rd,mod}}$ and $\mathsf{P}_{\mathrm{dR},r}^{\mathrm{mod,rd}}$ be the global period pairings

$$\begin{aligned} \mathsf{P}_{\mathrm{dR},r}^{\mathrm{rd,mod}} &: \Gamma(M, \mathfrak{C}_{M,r}^{\mathrm{rd}}(\mathscr{V}))^{\partial} \otimes \Gamma(M, \mathscr{E}_M^{\mathrm{mod},r} \otimes \mathscr{V})^{\nabla} \longrightarrow \mathbf{C} \\ \mathsf{P}_{\mathrm{dR},r}^{\mathrm{mod,rd}} &: \Gamma(M, \mathfrak{C}_{M,r}^{\mathrm{mod}}(\mathscr{V}))^{\partial} \otimes \Gamma(M, \mathscr{E}_M^{\mathrm{rd},r} \otimes \mathscr{V})^{\nabla} \longrightarrow \mathbf{C} \end{aligned}$$

obtained by evaluating on the constant function 1 the zero-dimensional rapid-decay current provided by $\mathsf{P}_{\mathrm{dR}}^{\mathrm{rd,mod}}$ and $\mathsf{P}_{\mathrm{dR}}^{\mathrm{mod,rd}}$ respectively. Then, according to Stokes formula, $\mathsf{P}_{\mathrm{dR},r}^{\mathrm{rd,mod}}$ and $\mathsf{P}_{\mathrm{dR},r}^{\mathrm{mod,rd}}$ vanish if one of the terms is a boundary, and hence induce pairings

$$\begin{aligned} \mathsf{P}_{\mathrm{dR},r}^{\mathrm{rd,mod}} &: \mathrm{H}_{\mathrm{dR},r}^{\mathrm{rd}}(M, \mathscr{V}) \otimes \mathrm{H}_{\mathrm{dR,mod}}^{r}(M, \mathscr{V}) \longrightarrow \mathbf{C} \\ \mathsf{P}_{\mathrm{dR},r}^{\mathrm{mod,rd}} &: \mathrm{H}_{\mathrm{dR},r}^{\mathrm{mod}}(M, \mathscr{V}) \otimes \mathrm{H}_{\mathrm{dR,rd}}^{r}(M, \mathscr{V}) \longrightarrow \mathbf{C}. \end{aligned}$$

It is then clear that the following diagram commutes (and similarly for $\mathsf{Q}_{\mathrm{mod,rd}}^{r}$):

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{dR},r}^{\mathrm{rd}}(M, \mathscr{V}) \otimes \mathrm{H}_{\mathrm{dR,mod}}^{r}(M, \mathscr{V}) & \xrightarrow{\mathsf{P}_{\mathrm{dR},r}^{\mathrm{rd,mod}}} & \mathbf{C} \\ {\scriptstyle \mathcal{P}_r^{\mathrm{rd}}(\mathscr{V})}\uparrow\wr \qquad \Vert & & \Vert \\ \mathrm{H}_{\mathrm{dR,rd}}^{m-r}(M, \mathscr{V}) \otimes \mathrm{H}_{\mathrm{dR,mod}}^{r}(M, \mathscr{V}) & \xrightarrow{\mathsf{Q}_{\mathrm{rd,mod}}^{m-r}} & \mathbf{C} \end{array} \tag{2.9}$$

We then deduce from Corollary 2.3:

Corollary 2.10. *The period pairings* $\mathsf{P}_{\mathrm{dR},r}^{\mathrm{rd,mod}}$ *and* $\mathsf{P}_{\mathrm{dR},m-r}^{\mathrm{mod,rd}}$ *are perfect.*

De Rham intersection pairing. We can apply once more the Poincaré isomorphisms to (2.8), and get the morphism $\mathsf{B}_{\mathrm{dR}}^{\mathrm{mod,rd}}$ (resp. $\mathsf{B}_{\mathrm{dR}}^{\mathrm{rd,mod}}$) making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{C}_{M,\bullet}^{\mathrm{mod}}(\mathscr{V}) \otimes \mathrm{DR}^{\mathrm{rd}}(\mathscr{V}, \nabla)[m] & \xrightarrow{\mathsf{P}_{\mathrm{dR}}^{\mathrm{mod,rd}}} & (\mathfrak{C}_{M,\bullet}^{\mathrm{rd}}, \partial) \\ \Vert \qquad \wr\downarrow{\scriptstyle \mathcal{P}^{\mathrm{rd}}(\mathscr{V})} & & \Vert \\ \mathfrak{C}_{M,\bullet}^{\mathrm{mod}}(\mathscr{V}) \otimes \mathfrak{C}_{M,\bullet}^{\mathrm{rd}}(\mathscr{V}) & \xrightarrow{\mathsf{B}_{\mathrm{dR}}^{\mathrm{mod,rd}}} & (\mathfrak{C}_{M,m-\bullet}^{\mathrm{rd}}, \partial) \end{array}$$

and therefore also the following:

$$
\begin{array}{ccc}
\mathrm{H}^{\mathrm{mod}}_{\mathrm{dR},m-r}(M,\mathscr{V})\otimes\mathrm{H}^{m-r}_{\mathrm{dR},\mathrm{rd}}(M,\mathscr{V}) & \xrightarrow{\ \mathsf{P}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}\ } & \mathbf{C}\\
\Big\| \qquad\qquad \wr\Big\downarrow \mathcal{P}^{\mathrm{rd}}_r(\mathscr{V}) & & \Big\|\\
\mathrm{H}^{\mathrm{mod}}_{\mathrm{dR},m-r}(M,\mathscr{V})\otimes\mathrm{H}^{\mathrm{rd}}_{\mathrm{dR},r}(\mathscr{V}) & \xrightarrow{\ \mathsf{B}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}\ } & \mathbf{C}
\end{array}
\tag{2.11}
$$

By definition, for cohomology classes $[\eta^r]\in\mathrm{H}^{m-r}_{\mathrm{dR,rd}}(M,\mathscr{V})$ and $[\omega^{m-r}]\in\mathrm{H}^r_{\mathrm{dR,mod}}(M,\mathscr{V})$, we have

$$\mathsf{Q}^r_{\mathrm{rd,mod}}([\eta^r],[\omega^{m-r}])=\mathsf{B}^{\mathrm{rd,mod}}_{\mathrm{dR},m-r}(\mathcal{P}^{\mathrm{rd}}_{m-r}([\eta^r]),\mathcal{P}^{\mathrm{mod}}_r([\omega^{m-r}])).$$

Corollary 2.12. *The de Rham intersection pairings $\mathsf{B}^{\mathrm{rd,mod}}_{\mathrm{dR},r},\mathsf{B}^{\mathrm{mod,rd}}_{\mathrm{dR},r}$ are perfect for each $r\geqslant 0$.*

Proof. This follows from the perfectness of the period pairings (see Corollary 2.10). □

Quadratic relations. We assume that the cohomology and homology vector spaces we consider are finite-dimensional and we fix bases of these spaces. We denote by the same letter a pairing and its matrix in the corresponding bases, and we use the matrix notation in which, for a pairing A, we also denote $\mathsf{A}(x\otimes y)$ by ${}^tx\cdot\mathsf{A}\cdot y$, and ${}^t\mathsf{A}$ denotes the transposed matrix.

Corollary 2.13 (Quadratic relations)**.** *The matrices of the pairings satisfy the following relations:*

$${}^t\mathsf{B}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}=\mathsf{P}^{\mathrm{rd,mod}}_{\mathrm{dR},r}\circ(\mathsf{Q}^{m-r}_{\mathrm{rd,mod}})^{-1}\circ{}^t\mathsf{P}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}.$$

Proof. In terms of matrices, the commutations (2.9) and (2.11) read

$${}^t\mathcal{P}^{\mathrm{rd}}_r(\mathscr{V})\cdot\mathsf{P}^{\mathrm{rd,mod}}_{\mathrm{dR},r}=\mathsf{Q}^{m-r}_{\mathrm{rd,mod}},\quad \mathsf{B}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}\cdot\mathcal{P}^{\mathrm{rd}}_r(\mathscr{V})=\mathsf{P}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r},$$

and eliminating $\mathcal{P}^{\mathrm{rd}}_r(\mathscr{V})$ we obtain the sought for relation. □

Remark 2.14 (Quadratic relations in presence of ±-symmetry)**.** Let us assume that $\langle\bullet,\bullet\rangle$ is ±-symmetric. Let us denote by

$${}^t\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}:\mathrm{H}^r_{\mathrm{dR,mod}}(M,\mathscr{V})\otimes\mathrm{H}^{m-r}_{\mathrm{dR,rd}}(M,\mathscr{V})\longrightarrow\mathbf{C}$$

the pairing defined (with obvious notation) by ${}^t\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}(h^r,h^{m-r})=\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}(h^{m-r},h^r)$, and let ${}^t\mathsf{Q}^{m-r}_{\mathrm{mod,rd}}$ be defined similarly. If η^{m-r} is a rapid decay $(m-r)$-form and ω^r an r-form with moderate growth, and if v,w are local sections of $\mathscr{V}$, we have

$$\begin{aligned}\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}(\eta^{m-r}\otimes v,\omega^r\otimes w)&=\langle v,w\rangle\cdot\eta^{m-r}\wedge\omega^r=\pm(-1)^{r(m-r)}\langle w,v\rangle\omega^r\wedge\eta^{m-r}\\&=\pm(-1)^{r(m-r)}\mathsf{Q}^r_{\mathrm{mod,rd}}(\omega^r\otimes w,\eta^{m-r}\otimes v).\end{aligned}$$

Passing to cohomology, we find

$${}^t\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}=\pm(-1)^{r(m-r)}\mathsf{Q}^r_{\mathrm{mod,rd}},\qquad {}^t\mathsf{Q}^{m-r}_{\mathrm{mod,rd}}=\pm(-1)^{r(m-r)}\mathsf{Q}^r_{\mathrm{rd,mod}}.$$

For the de Rham intersection pairing, we similarly find

$${}^t\mathsf{B}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}=\pm(-1)^{r(m-r)}\mathsf{B}^{\mathrm{rd,mod}}_{\mathrm{dR},r}.$$

As a consequence, the quadratic relations read

$$\pm(-1)^{r(m-r)}\mathsf{B}^{\mathrm{rd,mod}}_{\mathrm{dR},r}=\mathsf{P}^{\mathrm{rd,mod}}_{\mathrm{dR},r}\circ(\mathsf{Q}^{m-r}_{\mathrm{rd,mod}})^{-1}\circ{}^t\mathsf{P}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}.\tag{2.15}$$

2.c. **Middle quadratic relations.** For each degree r, we set

$$\begin{aligned}\mathrm{H}^{\mathrm{mid}}_{\mathrm{dR},r}(M,\mathscr{V})&=\mathrm{im}\big[\mathrm{H}^{\mathrm{rd}}_{\mathrm{dR},r}(M,\mathscr{V})\longrightarrow\mathrm{H}^{\mathrm{mod}}_{\mathrm{dR},r}(M,\mathscr{V})\big],\\ \mathrm{H}^{m-r}_{\mathrm{dR},\mathrm{mid}}(M,\mathscr{V})&=\mathrm{im}\big[\mathrm{H}^{m-r}_{\mathrm{dR},\mathrm{rd}}(M,\mathscr{V})\longrightarrow\mathrm{H}^{m-r}_{\mathrm{dR},\mathrm{mod}}(M,\mathscr{V})\big],\end{aligned}$$

where the maps are induced by the natural inclusion $\mathscr{C}^{\infty,\mathrm{rd}}_M\to\mathscr{C}^{\infty,\mathrm{mod}}_M$. From (2.2) one deduces that $\mathsf{Q}^{m-r}_{\mathrm{rd},\mathrm{mod}}$ and $\mathsf{Q}^{m-r}_{\mathrm{mod},\mathrm{rd}}$ induce the same *non-degenerate* pairing

$$\mathsf{Q}^{m-r}_{\mathrm{mid}}:\mathrm{H}^{m-r}_{\mathrm{dR},\mathrm{mid}}(M,\mathscr{V})\otimes\mathrm{H}^{r}_{\mathrm{dR},\mathrm{mid}}(M,\mathscr{V})\longrightarrow\mathbf{C}.$$

Similarly, according to (2.4), $\mathcal{P}^{\mathrm{rd}}_r(\mathscr{V})$ and $\mathcal{P}^{\mathrm{mod}}_r(\mathscr{V})$ induce the same isomorphism

$$\mathcal{P}^{\mathrm{mid}}_r(\mathscr{V}):\mathrm{H}^{m-r}_{\mathrm{dR},\mathrm{mid}}(M,\mathscr{V})\longrightarrow\mathrm{H}^{\mathrm{mid}}_{\mathrm{dR},r}(M,\mathscr{V}).$$

It follows that $\mathsf{P}^{\mathrm{rd},\mathrm{mod}}_{\mathrm{dR},r}$ and $\mathsf{P}^{\mathrm{mod},\mathrm{rd}}_{\mathrm{dR},r}$ induce the same *non-degenerate* period pairing

$$\mathsf{P}^{\mathrm{mid}}_{\mathrm{dR},r}:\mathrm{H}^{\mathrm{mid}}_{\mathrm{dR},r}(M,\mathscr{V})\otimes\mathrm{H}^{m-r}_{\mathrm{dR},\mathrm{mid}}(M,\mathscr{V})\longrightarrow\mathbf{C}.$$

In the same way, $\mathsf{B}^{\mathrm{rd},\mathrm{mod}}_{\mathrm{dR},r}$ and $\mathsf{B}^{\mathrm{mod},\mathrm{rd}}_{\mathrm{dR},r}$ induce the same *non-degenerate* intersection pairing

$$\mathsf{B}^{\mathrm{mid}}_{\mathrm{dR},r}:\mathrm{H}^{\mathrm{mid}}_{\mathrm{dR},r}(M,\mathscr{V})\otimes\mathrm{H}^{\mathrm{mid}}_{\mathrm{dR},m-r}(M,\mathscr{V})\longrightarrow\mathbf{C}.$$

We conclude:

Corollary 2.16 (Middle quadratic relations)**.** *These pairings are non-degenerate and their matrices satisfy the following relations:*

$${}^t\mathsf{B}^{\mathrm{mid}}_{\mathrm{dR},m-r}=\mathsf{P}^{\mathrm{mid}}_{\mathrm{dR},r}\circ(\mathsf{Q}^{m-r}_{\mathrm{mid}})^{-1}\circ{}^t\mathsf{P}^{\mathrm{mid}}_{\mathrm{dR},m-r}.$$

Remark 2.17 ($\pm$-Symmetry and quadratic relations)**.** Under the symmetry assumption of Remark 2.14, the following relations hold:

$${}^t\mathsf{B}^{\mathrm{mid}}_{\mathrm{dR},m-r}=\pm(-1)^{r(m-r)}\mathsf{B}^{\mathrm{mid}}_{\mathrm{dR},r},\quad {}^t\mathsf{Q}^{r}_{\mathrm{mid}}=\pm(-1)^{r(m-r)}\mathsf{Q}^{m-r}_{\mathrm{mid}},$$

and the relations of Corollary 2.16 read

$$\pm(-1)^{r(m-r)}\mathsf{B}^{\mathrm{mid}}_{\mathrm{dR},r}=\mathsf{P}^{\mathrm{mid}}_{\mathrm{dR},r}\circ(\mathsf{Q}^{m-r}_{\mathrm{mid}})^{-1}\circ{}^t\mathsf{P}^{\mathrm{mid}}_{\mathrm{dR},m-r}.$$

If $r=m-r$, we can regard these formulas as *algebraic relations of degree two* on the entries of the matrix $\mathsf{P}^{\mathrm{mid}}_{\mathrm{dR},r}$, with coefficients in the entries of $\mathsf{B}^{\mathrm{mid}}_{\mathrm{dR},r}$ and $\mathsf{Q}^{m-r}_{\mathrm{mid}}$.

Caveat 2.18 (On the notation mid)**.** The terminology "middle quadratic relation" and the associated notation mid could be confusing, as it is usually associated to the notion of intermediate (or middle) extension of a sheaf across a divisor. Here, we use it in the naive sense above, and we do not claim in general (that is, in the setting of Section 2.e below) any precise relation with the usual notion of intermediate extension. However, we will check that both notions coincide in complex dimension one (see (3.1)).

Decoupling. If we do not assume the existence of $\langle\bullet,\bullet\rangle$ but use $(\bullet|\bullet)$ instead, under the corresponding variant of Assumptions 2.1, we have the induced global de Rham, period and intersection pairings

$$\begin{aligned}\mathsf{Q}^{r}_{\mathrm{rd},\mathrm{mod}}(\mathscr{V})&:\mathrm{H}^{r}_{\mathrm{dR},\mathrm{rd}}(M,\mathscr{V})\otimes\mathrm{H}^{m-r}_{\mathrm{dR},\mathrm{mod}}(M,\mathscr{V}^\vee)\longrightarrow\mathbf{C},\\ \mathsf{P}^{\mathrm{rd},\mathrm{mod}}_{\mathrm{dR},r}(\mathscr{V})&:\mathrm{H}^{\mathrm{rd}}_{\mathrm{dR},r}(M,\mathscr{V})\otimes\mathrm{H}^{r}_{\mathrm{dR},\mathrm{mod}}(M,\mathscr{V}^\vee)\longrightarrow\mathbf{C},\\ \mathsf{B}^{\mathrm{rd},\mathrm{mod}}_{\mathrm{dR},r}(\mathscr{V})&:\mathrm{H}^{\mathrm{rd}}_{\mathrm{dR},r}(M,\mathscr{V})\otimes\mathrm{H}^{\mathrm{mod}}_{\mathrm{dR},m-r}(M,\mathscr{V}^\vee)\longrightarrow\mathbf{C},\end{aligned}$$

and $\mathsf{Q}^{r}_{\mathrm{mod},\mathrm{rd}}(\mathscr{V})$, etc. Reversing the roles of $\mathscr{V}$ and $\mathscr{V}^\vee$, we obtain pairings $\mathsf{Q}^{r}_{\mathrm{rd},\mathrm{mod}}(\mathscr{V}^\vee)$, etc.

Let us set $\mathscr{V}'=\mathscr{V}\oplus\mathscr{V}^\vee$ with the non-degenerate pairing $\langle\bullet,\bullet\rangle$ given by

$$\langle(v,\xi),(w,\eta)\rangle=(v|\eta)\pm(w|\xi),$$

which is $\pm$-symmetric and satisfies Assumptions 2.1. We further suppose that the cohomology $\mathrm{H}^r_{\mathrm{dR,rd}}(M,\mathscr{V})$ is finite-dimensional. Then in terms of matrices, we have

$$\mathsf{Q}^r_{\mathrm{rd,mod}} = \begin{pmatrix} 0 & \mathsf{Q}^r_{\mathrm{rd,mod}}(\mathscr{V}) \\ \pm\mathsf{Q}^r_{\mathrm{rd,mod}}(\mathscr{V}^\vee) & 0 \end{pmatrix}$$

and similarly for others. Thus the quadratic relations (2.15) yield

$$(-1)^{r(m-r)}\mathsf{B}^{\mathrm{rd,mod}}_{\mathrm{dR},r}(\mathscr{V}) = \mathsf{P}^{\mathrm{rd,mod}}_{\mathrm{dR},r}(\mathscr{V}) \circ \mathsf{Q}^{m-r}_{\mathrm{rd,mod}}(\mathscr{V})^{-1} \circ {}^t\mathsf{P}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}(\mathscr{V}^\vee). \tag{2.19}$$

Conversely, suppose the quadratic relations (2.19) hold in the decoupled setting under the variant of Assumptions 2.1 with $\mathrm{H}^r_{\mathrm{dR,rd}}(M,\mathscr{V})$ finite-dimensional. Let $(\mathscr{V}',\nabla,\langle\bullet,\bullet\rangle')$ be a connection satisfying Assumptions 2.1 with $\mathrm{H}^r_{\mathrm{dR,rd}}(M,\mathscr{V}')$ finite-dimensional. Let $\mathscr{V}''$ denote the same connection $\mathscr{V}'$ with the transposed self-pairing $\langle v,w\rangle'' = \langle w,v\rangle$. Let $\mathsf{Q}^r_{\mathrm{rd,mod}}(\mathscr{V}')$ be the global de Rham pairing considered before, and similarly for others. Then the relation (2.19) transforms via the isomorphism $\lambda : \mathscr{V}' \xrightarrow{\sim} \mathscr{V}'^\vee$ into

$$(-1)^{r(m-r)}\mathsf{B}^{\mathrm{rd,mod}}_{\mathrm{dR},r}(\mathscr{V}') = \mathsf{P}^{\mathrm{rd,mod}}_{\mathrm{dR},r}(\mathscr{V}') \circ \mathsf{Q}^{m-r}_{\mathrm{rd,mod}}(\mathscr{V}')^{-1} \circ {}^t\mathsf{P}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}(\mathscr{V}'').$$

If $\langle\bullet,\bullet\rangle$ is $\pm$-symmetric, then $\mathsf{P}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}(\mathscr{V}'') = \pm\mathsf{P}^{\mathrm{mod,rd}}_{\mathrm{dR},m-r}(\mathscr{V}')$, so we get back the relation (2.15).

2.d. **Betti formalism for quadratic relations.** In the topological setting we replace the complex of currents $(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet},\partial)$ with the complex of sheaves of relative singular chains $(\mathcal{C}_{M,\partial M,\bullet},\partial)$. We will consider coefficients in $\mathbf{C}$ since we will compare it with currents. We refer to Appendix A for basic properties of this complex, and we recall (see e.g. [25, p. 14]) that $(\mathcal{C}_{M,\partial M,\bullet},\partial)$, when regarded cohomologically with non-positive indices, is a homotopically fine resolution of $\mathbf{C}_M[m]$. Arguments similar to those of Proposition A.16 enable us to replace the complex of sheaves of singular chains with that of piecewise smooth singular chains, that we will denote in the same way.

Betti period pairings. Integration along a piecewise smooth singular chain in M of a test form with rapid decay along ∂M defines a morphism of chain complexes $(\mathcal{C}_{M,\partial M,\bullet},\partial) \to (\mathfrak{C}^{\mathrm{mod}}_{M,\bullet},\partial)$: indeed, if c is a q-dimensional chain and ω a form of degree q on M° with rapid decay, $\int_c\omega = 0$ if c is supported in ∂M; compatibility with ∂ follows from the Stokes formula. This morphism is a quasi-isomorphism since it induces an isomorphism on the unique non-zero homology sheaf of both complexes; it yields a morphism between two resolutions of $\mathbf{C}_M[m]$.

We introduce (see Section A.d) the following complexes of singular chains:

- with coefficients in $\mathscr{V}$ and rapid decay $(\mathcal{C}_{M,\partial M,\bullet}(\mathscr{V}^{\mathrm{rd}}),\partial) = (\mathcal{C}_{M,\partial M,\bullet},\partial)\otimes\mathscr{V}^{\mathrm{rd}}$,
- with coefficients in $\mathscr{V}$ and moderate growth $(\mathcal{C}_{M,\partial M,\bullet}(\mathscr{V}^{\mathrm{mod}}),\partial) = (\mathcal{C}_{M,\partial M,\bullet},\partial)\otimes\mathscr{V}^{\mathrm{mod}}$.

We also use the notation $\mathrm{H}^{\mathrm{rd}}_r(M,\mathscr{V})$ (resp. $\mathrm{H}^{\mathrm{mod}}_r(M,\mathscr{V})$) for the homology spaces $\mathrm{H}_r(M,\mathscr{V}^{\mathrm{rd}})$ (resp. $\mathrm{H}_r(M,\mathscr{V}^{\mathrm{mod}})$) (see Section A.d). We thus have quasi-isomorphisms of chain complexes

$$(\mathcal{C}_{M,\partial M,\bullet}(\mathscr{V}^{\mathrm{rd}}),\partial) \xrightarrow{\sim} (\mathfrak{C}^{\mathrm{rd}}_{M,\bullet}(\mathscr{V}),\partial), \quad (\mathcal{C}_{M,\partial M,\bullet}(\mathscr{V}^{\mathrm{mod}}),\partial) \xrightarrow{\sim} (\mathfrak{C}^{\mathrm{mod}}_{M,\bullet}(\mathscr{V}),\partial), \tag{2.20}$$

which induce isomorphisms

$$\mathrm{H}^{\mathrm{rd}}_r(M,\mathscr{V}) \xrightarrow{\sim} \mathrm{H}^{\mathrm{rd}}_{\mathrm{dR},r}(M,\mathscr{V}) \quad\text{and}\quad \mathrm{H}^{\mathrm{mod}}_r(M,\mathscr{V}) \xrightarrow{\sim} \mathrm{H}^{\mathrm{mod}}_{\mathrm{dR},r}(M,\mathscr{V}). \tag{2.21}$$

Composing (2.6) and (2.7) with these morphisms gives back the period pairings as those defined in [25], that we denote by $\mathsf{P}^{\mathrm{rd,mod}}_r$ and $\mathsf{P}^{\mathrm{mod,rd}}_r$. In particular, these Betti period pairings are perfect. Notice that, by means of these identifications, the Poincaré-de Rham isomorphisms (2.5) correspond to the Poincaré isomorphisms, denoted similary:

$$\begin{aligned} \mathcal{P}^{\mathrm{rd}}_r(\mathscr{V}) &: \mathrm{H}^{m-r}(M,\mathscr{V}^{\mathrm{rd}}) \xrightarrow{\sim} \mathrm{H}_r(M,\mathscr{V}^{\mathrm{rd}}), \\ \mathcal{P}^{\mathrm{mod}}_r(\mathscr{V}) &: \mathrm{H}^{m-r}(M,\mathscr{V}^{\mathrm{mod}}) \xrightarrow{\sim} \mathrm{H}_r(M,\mathscr{V}^{\mathrm{mod}}). \end{aligned} \tag{2.22}$$

According to Propositions A.13 and A.16, the homology spaces $\mathrm{H}_r^{\mathrm{rd}}(M,\mathscr{V})$ (resp. $\mathrm{H}_r^{\mathrm{mod}}(M,\mathscr{V})$) can be computed as the homology of the chain complexes

$$(\widetilde{\mathcal{C}}^{\mathrm{sm}}_{M,\partial M,\bullet}(M,\mathscr{V}^{\mathrm{rd}})),\partial)\quad(\text{resp. }(\widetilde{\mathcal{C}}^{\mathrm{sm}}_{M,\partial M,\bullet}(M,\mathscr{V}^{\mathrm{mod}})),\partial))$$

of piecewise smooth singular chains with coefficients in $\mathscr{V}^{\mathrm{rd}}$ (resp. $\mathscr{V}^{\mathrm{mod}}$).

On the one hand, let $\sigma:\Delta_r\to M$ be a piecewise smooth singular simplex (that we can assume not contained in ∂M since we only consider relative simplices), and let v be a section of $\mathscr{V}^{\mathrm{rd}}$ in the neighbourhood of $|\sigma|$, so that $\sigma\otimes v$ belongs to $\widetilde{\mathcal{C}}^{\mathrm{sm}}_{M,\partial M,\bullet}(M,\mathscr{V}^{\mathrm{rd}})$. On the other hand, note that, by using a partition of unity, any section in $\Gamma(M,\mathscr{E}_M^{\mathrm{mod},r}\otimes\mathscr{V})$ is a sum of terms $\omega^r\otimes w$, where w is a section of $\mathscr{V}$ in the neighbourhood of the support of ω^r. Then $\langle v,w\rangle$ is a C^∞ function on the open set where both sections are defined and has rapid decay along ∂M, so that $\langle v,w\rangle\cdot\omega^r$ is a C^∞-form with rapid decay, and hence integrable along σ. We have

$$\mathsf{P}_r^{\mathrm{rd,mod}}(\sigma\otimes v,\omega^r\otimes w)=\int_\sigma\langle v,w\rangle\cdot\omega^r=\int_{\Delta_r}\sigma^*\langle v,w\rangle\cdot\sigma^*\omega^r.$$

This interpretation makes the link between the approaches in [3] and [25]. A similar interpretation holds for $\mathsf{P}_r^{\mathrm{mod,rd}}$: now $\langle v,w\rangle$ has only moderate growth, but ω^r has rapid decay, so that $\langle v,w\rangle\cdot\omega^r$ remains a C^∞-form with rapid decay.

Existence of a $\boldsymbol{k}$-structure. Let us fix a subfield $\boldsymbol{k}$ of $\mathbf{C}$. We will make explicit the $\boldsymbol{k}$-structure on the various cohomology groups occurring in the de Rham model, provided that a $\boldsymbol{k}$-structure exists on the underlying sheaves. We enrich Assumptions 2.1 as follows.

Assumption 2.23.

(1) The sheaves $\mathscr{V}^{\mathrm{rd}}$ and $\mathscr{V}^{\mathrm{mod}}$ are endowed with a $\boldsymbol{k}$-structure and the natural inclusions $\mathscr{V}^{\mathrm{rd}}\hookrightarrow\mathscr{V}^{\mathrm{mod}}$ and $\mathscr{V}^{\mathrm{rd}}\hookrightarrow\mathscr{V}^{\mathrm{mod}}$ are defined over $\boldsymbol{k}$, as well as the flat pairing induced by $\langle\bullet,\bullet\rangle$ on $\mathscr{V}^\nabla\otimes\mathscr{V}^\nabla$.

(2) The perfect pairings $\mathscr{H}^0\mathsf{Q}_{\mathrm{rd,mod}}$ and $\mathscr{H}^0\mathsf{Q}_{\mathrm{mod,rd}}$ are defined over $\boldsymbol{k}$:

$$\mathscr{V}^{\mathrm{rd}}_{\boldsymbol{k}}\otimes\mathscr{V}^{\mathrm{mod}}_{\boldsymbol{k}}\longrightarrow j_!\boldsymbol{k}_{M^\circ}[m],$$
$$\mathscr{V}^{\mathrm{mod}}_{\boldsymbol{k}}\otimes\mathscr{V}^{\mathrm{rd}}_{\boldsymbol{k}}\longrightarrow j_!\boldsymbol{k}_{M^\circ}[m].$$

In particular, $\mathscr{H}^0$ of the commutative diagram (2.2) is defined over $\boldsymbol{k}$. As a consequence, the de Rham isomorphism induces a $\boldsymbol{k}$-structure on the de Rham cohomology groups, e.g. $\mathrm{H}^r_{\mathrm{dR,rd}}(M,\mathscr{V})_{\boldsymbol{k}}=\mathrm{H}^r(M,\mathscr{V}^{\mathrm{rd}}_{\boldsymbol{k}})$ via the de Rham isomorphism $\mathrm{H}^r_{\mathrm{dR,rd}}(M,\mathscr{V})\xrightarrow{\sim}\mathrm{H}^r(M,\mathscr{V}^{\mathrm{rd}})$. The pairings of Corollary 2.3 are defined over $\boldsymbol{k}$ and correspond to Poincaré-Verdier duality over $\boldsymbol{k}$.

The $\boldsymbol{k}$-structure on rapid decay and moderate growth cycles is obtained by means of the complexes $(\mathcal{C}_{M,\partial M,\bullet},\partial)\otimes\mathscr{V}^{\mathrm{rd}}_{\boldsymbol{k}}$ and $(\mathcal{C}_{M,\partial M,\bullet},\partial)\otimes\mathscr{V}^{\mathrm{mod}}_{\boldsymbol{k}}$. In other words, we set

$$\mathrm{H}_r^{\mathrm{rd}}(M,\mathscr{V})_{\boldsymbol{k}}=\mathrm{H}_r(M,\mathscr{V}^{\mathrm{rd}}_{\boldsymbol{k}}),\ \text{etc.}$$

The Poincaré isomorphisms (2.22) are then defined over $\boldsymbol{k}$.

Proposition 2.24. *Under Assumptions* 2.1 *and* 2.23*, let $(v_i)_i$ be a basis of $\mathrm{H}^{m-r}_{\mathrm{dR,rd}}(M,\mathscr{V})$ and let $(v_i^\vee)_i$ be the $\mathsf{Q}^{m-r}_{\mathrm{rd,mod}}$-dual basis of $\mathrm{H}^r_{\mathrm{dR,mod}}(M,\mathscr{V})$. The $\boldsymbol{k}$-vector space $\mathrm{H}^{m-r}_{\mathrm{dR,rd}}(M,\mathscr{V})_{\boldsymbol{k}}$ is the $\boldsymbol{k}$-subspace of $\mathrm{H}^{m-r}_{\mathrm{dR,rd}}(M,\mathscr{V})$ consisting of the elements*

$$\sum_i\mathsf{P}_r^{\mathrm{rd,mod}}(\alpha,v_i^\vee)\cdot v_i,$$

where α runs in $\mathrm{H}_r(M,\mathscr{V}^{\mathrm{rd}}_{\boldsymbol{k}})$. A similar assertion holds for $\mathrm{H}^{m-r}_{\mathrm{dR,mod}}(M,\mathscr{V})_{\boldsymbol{k}}$.

Proof. Let $e \in \mathrm{H}^{m-r}_{\mathrm{dR,rd}}(M,\mathscr{V})$ be an element written as

$$e = \sum_i \mathsf{Q}^{m-r}_{\mathrm{rd,mod}}(e, v_i^\vee)\cdot v_i.$$

Let us set $\alpha = \mathcal{P}^{\mathrm{rd}}_r(e) \in \mathrm{H}^{\mathrm{rd}}_{\mathrm{dR},r}(M,\mathscr{V})$. According to (2.9), we can rewrite e as

$$e = \sum_i \mathsf{P}^{\mathrm{rd,mod}}_{\mathrm{dR},r}(\alpha, v_i^\vee)\cdot v_i.$$

The class e belongs to $\mathrm{H}^{m-r}_{\mathrm{dR,rd}}(M,\mathscr{V})_{\boldsymbol{k}}$ if and only if, when regarded in $\mathrm{H}^{m-r}(M,\mathscr{V}^{\mathrm{rd}})$, it belongs to $\mathrm{H}^{m-r}(M,\mathscr{V}^{\mathrm{rd}}_{\boldsymbol{k}})$, equivalently, its image α by the Poincaré isomorphism $\mathcal{P}^{\mathrm{rd}}_r(\mathscr{V})$ belongs to $\mathrm{H}_r(M,\mathscr{V}^{\mathrm{rd}}_{\boldsymbol{k}})$. The assertion of the proposition follows from the above identifications. □

Middle quadratic relations. The Betti intersection pairing

$$\mathsf{B}^{\mathrm{rd,mod}}_r : \mathrm{H}^{\mathrm{rd}}_r(M,\mathscr{V}) \otimes \mathrm{H}^{\mathrm{mod}}_{m-r}(M,\mathscr{V}) \longrightarrow \mathbf{C}$$

is defined from the de Rham intersection pairing $\mathsf{B}^{\mathrm{rd,mod}}_{\mathrm{dR},r}$ (see Corollary 2.12) by composing it with the isomorphisms (2.21). It is thus a *perfect pairing*. A similar result holds for $\mathsf{B}^{\mathrm{mod,rd}}_r$.

From Corollary 2.16 we obtain immediately (and of course similarly for the rapid decay and moderate growth analogues):

Corollary 2.25. *The middle Betti period pairings satisfy the quadratic relations of Corollary* 2.16*, where the de Rham intersection pairing* $\mathsf{B}^{\mathrm{mid}}_{\mathrm{dR},r}$ *is replaced with the Betti intersection pairing* $\mathsf{B}^{\mathrm{mid}}_r$*. That is,*

$${}^t\mathsf{B}^{\mathrm{mid}}_{m-r} = \mathsf{P}^{\mathrm{mid}}_r \circ (\mathsf{Q}^{m-r}_{\mathrm{mid}})^{-1} \circ {}^t\mathsf{P}^{\mathrm{mid}}_{m-r}.$$

Under the symmetry assumption of Remark 2.17*, it reads*

$$\pm(-1)^{r(m-r)}\mathsf{B}^{\mathrm{mid}}_r = \mathsf{P}^{\mathrm{mid}}_r \circ (\mathsf{Q}^{m-r}_{\mathrm{mid}})^{-1} \circ {}^t\mathsf{P}^{\mathrm{mid}}_{m-r}.$$

Computation of the Betti intersection pairing. If $(M,\partial M)$ is endowed with a simplicial decomposition $\mathscr{T}$ such that the sheaves $\mathscr{V}^{\mathrm{rd}},\mathscr{V}^{\mathrm{mod}}$ satisfy Assumption A.17 and are locally constant on M°, we can replace the complex $(\widetilde{\mathcal{C}}^{\mathrm{sm}}_{M,\partial M,\bullet}(M,\mathscr{V}^{\mathrm{rd}}),\partial)$ (resp. the complex $(\widetilde{\mathcal{C}}^{\mathrm{sm}}_{M,\partial M,\bullet}(M,\mathscr{V}^{\mathrm{mod}}),\partial)$) with the corresponding simplicial complex $(\widetilde{\mathcal{C}}^{\triangle}_{M,\partial M,\bullet}(M,\mathscr{V}^{\mathrm{rd}}),\partial)$ resp. $(\widetilde{\mathcal{C}}^{\square}_{M,\partial M,\bullet}(M,\mathscr{V}^{\mathrm{mod}}),\partial)$ (see Proposition A.18 and Section A.f).

The Betti intersection pairing can easily be computed in the framework of simplicial chain complexes under this assumption. Indeed, choose an orientation for each simplex (it is natural to assume that the maximal-dimensional simplices have the orientation induced by that of M) and an orientation for each dual cell $\overline{D}(\sigma)$. Let $\sigma_r\otimes v$ be a simplex with coefficient v in $\mathscr{V}^{\mathrm{rd}}$ ($\sigma_r \not\subset \partial M$) and let $\overline{D}(\sigma'_r)\otimes w$ be a cell of codimension r with coefficient w in $\mathscr{V}^{\mathrm{mod}}$ for some simplex σ'_r of $\mathscr{T}$ (possibly $\sigma'_r\subset\partial M$). We regard them as currents with rapid decay and moderate growth respectively, according to (2.20).

Proposition 2.26 (Computation of the Betti intersection pairing)**.** *With these assumptions,*

$$\mathsf{B}^{\mathrm{rd,mod}}_r(\sigma_r\otimes v, \overline{D}(\sigma'_r)\otimes w) = \begin{cases} 0 & \text{if } \sigma'_r\neq\sigma_r,\\ \pm\langle v,w\rangle(\widehat{\sigma}_r) & \text{if } \sigma'_r=\sigma_r,\end{cases}$$

where $\widehat{\sigma}_r$ *is the barycenter of* σ_r *and* $\pm$ *is the orientation change between* $\sigma_r\times\overline{D}(\sigma_r)$ *and* M.

Proof. By definition, we have

$$\mathsf{B}^{\mathrm{rd,mod}}_r(\sigma_r\otimes v, \overline{D}(\sigma'_r)\otimes w) = \langle v,w\rangle\cdot\mathsf{B}_{\mathrm{dR},r}(\sigma_r,\overline{D}(\sigma'_r)),$$

where $\mathsf{B}_{\mathrm{dR},r}$ corresponds to de Rham's definition of the Kronecker index.

If $\sigma'_r\subset\partial M$, then $\overline{D}(\sigma'_r)$ does not physically intersect any simplex of $\mathscr{T}$ of dimension r, and if $\sigma'_r\not\subset\partial M$, then $\overline{D}(\sigma'_r)$ only intersects σ'_r among simplices of $\mathscr{T}$ dimension r. We are reduced to computing the value when $\sigma'_r = \sigma_r$. We are now in the framework of the computation of de Rham (see [12, p. 85–86]) which is recalled in the appendix (Proposition B.7). The current $\mathsf{B}_{\mathrm{dR},r}(\sigma_r, \overline{D}(\sigma_r))$ is supported on $\widehat{\sigma}_r = \sigma_r \cap \overline{D}(\sigma'_r)$ and has coefficient ± 1, so that

$$\langle v, w\rangle \cdot \mathsf{B}_{\mathrm{dR},r}(\sigma_r, \overline{D}(\sigma_r)) = \langle v, w\rangle(\widehat{\sigma}_r) \cdot \mathsf{B}_{\mathrm{dR},r}(\sigma_r, \overline{D}(\sigma'_r)) = \pm\langle v, w\rangle(\widehat{\sigma}_r). \qquad \square$$

2.e. **Quadratic relations for good meromorphic flat bundles.** We now consider the complex setup. Let X be a complex manifold of complex dimension n, let D be a divisor with normal crossings, and let $(\mathscr{V}, \nabla)$ be a coherent $\mathscr{O}_X(*D)$-module with integrable connection, endowed with a non-degenerate flat pairing $\langle\bullet,\bullet\rangle : \mathscr{V} \otimes \mathscr{V} \to \mathscr{O}_X(*D)$. Let $\varpi : \widetilde{X} \to X$ denote the oriented real blow-up of X along the irreducible components of D (that we assume to be smooth for the sake of simplicity). We set $\widetilde{D} = \varpi^{-1}(D) = \partial\widetilde{X}$, and $(\widetilde{X}, \widetilde{D})$ will play the role of $(M, \partial M)$ in the settings 2.a (see e.g. [51, §8.2] for the construction of $(\widetilde{X}, \varpi)$).

There is a natural Cauchy-Riemann operator on $\widetilde{X}$ and one defines the sheaves $\mathscr{A}_{\widetilde{X}}, \mathscr{A}_{\widetilde{X}}^{\mathrm{rd}}, \mathscr{A}_{\widetilde{X}}^{\mathrm{mod}}$ respectively as the subsheaves of $\mathscr{C}_{\widetilde{X}}^{\infty}, \mathscr{C}_{\widetilde{X}}^{\infty,\mathrm{rd}}, \mathscr{C}_{\widetilde{X}}^{\infty,\mathrm{mod}}$ annihilated by the Cauchy-Riemann operator (see e.g. [50]). Setting $\widetilde{\mathscr{V}} = \mathscr{A}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathscr{O}_X} \varpi^{-1}\mathscr{V}$, the connection ∇ lifts to a connection $\widetilde{\mathscr{V}} \to \varpi^{-1}\Omega^1_X \otimes_{\varpi^{-1}\mathscr{O}_X} \widetilde{\mathscr{V}}$. There is thus the corresponding notion of holomorphic de Rham complexes with growth conditions, that we denote by $\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}}, \nabla)$ and $\mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}}, \nabla)$ and which are quasi-isomorphic to the C^∞ ones, so there is no risk of confusion.

The goodness property. Let x be a point in D, and let $(x_1, \dots, x_n)$ be local complex coordinates of X centered at x such that the germ D_x is given by $x_1 \cdots x_\ell = 0$ ($\ell \leqslant n$). Let $\mathscr{V}_{\widehat{x}}$ be the formal germ of $\mathscr{V}$ at x, which is an $\mathscr{O}_{\widehat{x}}(*D)$-module of finite type with the integrable connection induced by ∇. We say that $(\mathscr{V}, \nabla)$ is *good* at x if

(a) there exists a finite ramification

$$\rho_x : (x'_1, \dots, x'_\ell, x_{\ell+1}, \dots, x_n) \longmapsto (x_1 = x_1'^p, \dots, x_\ell = x_\ell'^p, x_{\ell+1}, \dots, x_n)$$

such that the pullback $\rho_x^*(\mathscr{V}_{\widehat{x}}, \nabla)$ decomposes as the direct sum of free $\mathscr{O}_{\widehat{x}'}(*D')$-modules of finite rank with an integrable connection of the form $\nabla = \mathrm{d}\varphi + \nabla^{\mathrm{reg}}$, where ∇^{reg} has a regular singularities along $D' = \rho_x^{-1}D$ and φ belongs to a finite family $\Phi_x \subset \mathscr{O}_{X',x}(*D')/\mathscr{O}_{X',x}$,

(b) the finite family Φ_x is *good*, that is, for any $\varphi \neq \psi$ in $\Phi_x \cup \{0\}$, the divisor of $\varphi - \psi$ is effective.[2]

We say that $(\mathscr{V}, \nabla)$ is *good* if it is good at any $x \in X$. In particular, $\mathscr{V}$ is locally $\mathscr{O}_X(*D)$-free. Two important results should be emphasized.

(1) If $n = \dim X = 1$, then any $(\mathscr{V}, \nabla)$ is good. This is known as the Levelt-Turrittin theorem, which has various different proofs (see e.g. [53, 35, 48, 2, 15, 39, 52, 1]).
(2) If $n \geqslant 2$, goodness may fail, but there exists a finite sequence of complex blow-ups over a sufficiently small analytic neighbourhood of x, or over X if X is quasi-projective, that are isomorphisms away from D and such that the pullback of $(\mathscr{V}, \nabla)$ is good along the pullback of D, which can be made with normal crossings. This is the Kedlaya-Mochizuki theorem (see [50] for $n = 2$ and rank $\leqslant 5$, [44] for $n = 2$ in the projective case, [30] for $n = 2$ in the local formal and analytic case, [45] for $n \geqslant 2$ in the projective case, [31] for $n \geqslant 2$ in the local formal and analytic case, and also in the projective case).

Theorem 2.27. *If $(\mathscr{V}, \nabla)$ is good, then Assumptions* 2.1 *and* A.17 *hold for $(\widetilde{\mathscr{V}}, \nabla)$.*

Convention 2.28. In the case of complex manifolds, in order to make formulas independent of the choice of a square root of -1, we replace integration $\int_X$ of a form of maximal degree with the

[2] Goodness usually involves Φ_x rather than $\Phi_x \cup \{0\}$. This stronger condition is needed for Assumptions 2.1 to be satisfied.

trace $\operatorname{tr}_X = (1/2\pi\mathsf{i})^n \cdot \int_X$. The de Rham intersection pairing Q is replaced with the normalized de Rham intersection pairing:

$$\mathsf{S} = (1/2\pi\mathsf{i})^n\mathsf{Q}.$$

Let us for example consider the quadratic relations in middle dimension n if $\langle\bullet,\bullet\rangle$ is $\pm$-symmetric.

Corollary 2.29. *Assume that X is compact. The quadratic relations for the Betti period pairings*

$$\pm(-2\pi\mathsf{i})^n\,\mathsf{B}_n^{\mathrm{mid}} = \mathsf{P}_n^{\mathrm{mid}} \circ (\mathsf{S}^n_{\mathrm{mid}})^{-1} \circ {}^t\mathsf{P}_n^{\mathrm{mid}}$$

hold for a good meromorphic flat bundle on (X,D), and the Betti intersection pairing can be computed with a suitable simplicial decomposition of $(\widetilde{X},\widetilde{D})$ as in Proposition 2.26.

Proof of Theorem 2.27. Let us start with Assumption 2.1(1). In dimension one, the rapid decay case is the Hukuhara-Turrittin asymptotic expansion theorem (see [53] and [39, App. 1]). A variant of the moderate case (restricting moderate growth to Nilsson class functions) is also proved in [39, App. 1]. In dimension $\geqslant 2$, the proof is essentially a consequence of the work of Majima [36] on asymptotic analysis (see also [49, App.], [45, Chap. 20]). The rapid decay case is proved in [49] and the moderate case in [24, App.] with the assumption $n \leqslant 2$, but the argument extends to any $n \geqslant 2$ (see [25, Prop. 1]); the rapid decay case and the Nilsson-moderate case is also proved in [46, Prop. 5.1.3]. The proof of 2.1(2) is a simple consequence of these results (see e.g. the proof in dimension one of Proposition 3.41).

Let us now consider Assumption A.17. Due to the aforementioned theorems, the local structure of $\mathscr{H}^0\,\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}}) = \mathscr{V}^{\mathrm{rd}}$ and $\mathscr{H}^0\,\mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}}) = \mathscr{V}^{\mathrm{mod}}$ is easy to understand. Near $x \in D$ as above, the real blow-up $\widetilde{X}$ is diffeomorphic to $(S^1)^\ell \times [0,\varepsilon)^\ell \times \Delta(\varepsilon)^{n-\ell}$ for some $\varepsilon > 0$ and a disc $\Delta(\varepsilon)$ of radius ε. In this model, denoting by $(r_1,\dots,r_\ell)$ the coordinates on $[0,\varepsilon)^\ell$, $\widetilde{D}$ is described as $r_1\cdots r_\ell = 0$, giving it the structure of a real analytic set. The structure of manifold with corners is then clear.

From the set Φ_x of exponential factors defined after some ramification ρ_x, one constructs a local real semi-analytic stratification of $\widetilde{D}$: for any $\varphi \neq 0$ in Φ_x written as $\varphi = u_\varphi \prod_{i=1}^\ell x_i'^{-q_i(\varphi)} + \text{h.o.t.}$ with $u_\varphi \in \mathscr{O}(\Delta(\varepsilon)^{n-\ell})^*$ and, in the ramified coordinates of $\widetilde{X}'$, written as

$$\varphi = r_\varphi \prod_{i=1}^{\ell} r_i'^{-q_i(\varphi)} \cdot \exp\big[\mathsf{i}(a_\varphi - \textstyle\sum_i q_i(\varphi)\theta_i')\big] + \text{h.o.t.},$$

we consider the sets defined by

$$\cos\big(a_\varphi - \textstyle\sum_i q_i(\varphi)\theta_i'\big) = 0,$$

and their projections S_φ to $\widetilde{D}$, which are real semi-analytic subsets of $\widetilde{D}$ (the multi-dimensional Stokes directions); the desired stratification is any real semi-analytic stratification compatible with all these subsets. We also denote by $S_\varphi^{\leqslant 0}$ (resp. $S_\varphi^{>0}$) the corresponding subsets defined with $\leqslant 0$ (resp. > 0).

In the neighbourhood of any $\widetilde{x} \in \varpi^{-1}(x)$, $\mathscr{V}^{\mathrm{rd}}|_{\widetilde{D}}$ decomposes as $\bigoplus_\varphi \mathscr{V}_\varphi^{\mathrm{rd}}$, where $\mathscr{V}_\varphi^{\mathrm{rd}}$ is zero if $\varphi = 0$ and otherwise constant on $S_\varphi^{>0}$ and zero on $S_\varphi^{\leqslant 0}$. A similar property holds for $\mathscr{V}^{\mathrm{mod}}|_{\widetilde{D}}$, with the only change that $\mathscr{V}_\varphi^{\mathrm{mod}}$ is constant if $\varphi = 0$. In particular, $\mathscr{V}^{\mathrm{rd}}$ and $\mathscr{V}^{\mathrm{mod}}$ are locally $\mathbf{R}$-constructible in the sense of [29, Def. 8.4.3] (we can use a local stratification as above to check $\mathbf{R}$-constructibility).

In this local setting, $\mathscr{V}^{\mathrm{rd}}$ and $\mathscr{V}^{\mathrm{mod}}$, together with any simplicial structure compatible with any real semi-analytic stratification with respect to which they are $\mathbf{R}$-constructible,[3] satisfy thus Assumption A.17.

In order to conclude in the global setting, we note that the pair $(\widetilde{X},\widetilde{D})$ comes equipped with a real semi-analytic structure which induces the previous one in each local chart adapted to D,

[3]It is standard that such a simplicial structure exists, see e.g. [29, Prop. 8.2.5].

and that $\mathscr{V}^{\mathrm{rd}}$ and $\mathscr{V}^{\mathrm{mod}}$ are **R**-constructible, since **R**-constructibility is a local property on $\widetilde{D}$, according to [29, Th. 8.4.2]. Then the same conclusion as in the local setting holds. □

3. Algebraic computation of de Rham duality in dimension one

In this section, we restrict to the case of complex dimension one and we make explicit in algebraic terms the results mentioned in Section 2.e.

3.a. **Setting, notation, and objectives.** Let X be a connected smooth complex projective curve, let D be a *non-empty* finite set of points in X, and define $j : U = X \smallsetminus D \hookrightarrow X$ as the complementary inclusion, so that U is affine. Depending on the context, we work in the Zariski topology or the analytic topology on U and X, and hence $\mathscr{O}_U$ will have the corresponding meaning. We denote by $\mathscr{O}_{\widehat{D}}$ the structure sheaf of the formal neighbourhood of D in X, so that $\mathscr{O}_{\widehat{D}} = \bigoplus_{x\in D} \mathscr{O}_{\widehat{x}}$ and $\mathscr{O}_{\widehat{x}} \simeq \mathbf{C}[\![z]\!]$ for some local coordinate z centered at x. We fix an affine neighbourhood U_D of D in X. We set $U_D^\circ = U_D \smallsetminus D = U \cap U_D$ and, for a sheaf of $\mathscr{O}_X$-modules F, we denote by ι_D the pullback functor $F(U) \to F(U_D^\circ)$ attached to the inclusion $U_D^\circ \hookrightarrow U$. Similarly, we denote by $\iota_{\widehat{D}}$ the restriction functor $F(U) \to F(U_{\widehat{D}}^\circ)$.

We consider a locally free sheaf V on U endowed with a connection ∇ together with the associated $\mathscr{O}_X(*D)$-module with connection j_*V (one also finds the notation j_+V in the literature) that we denote by $\mathscr{V}$. It is thus a left $\mathscr{D}_X$-module, and thereby endowed with an action of meromorphic vector fields $\Theta_X(*D)$ with poles along D. We will mainly consider the action of the subsheaf $\Theta_X(-D)$ of logarithmic vector fields, i.e., vector fields vanishing on D.

On the other hand, let $(V^\vee, \nabla)$ be the dual bundle with connection and set $\mathscr{V}^\vee = j_*V^\vee$. The dual $\mathscr{D}_X$-module $\boldsymbol{D}(\mathscr{V}^\vee)$ is a holonomic $\mathscr{D}_X$-module that we denote by $\mathscr{V}(!D)$ (one also finds the notation $j_!V$ or $j_\dagger V$ in the literature). There exists a natural morphism $\mathscr{V}(!D) \to \mathscr{V}$ whose kernel K and cokernel C are supported on D and whose image is denoted by $\mathscr{V}(!{*}D)$.

We deduce natural morphisms of complexes $\mathrm{DR}\,\mathscr{V}(!D) \to \mathrm{DR}\,\mathscr{V}(!{*}D)$ and $\mathrm{DR}\,\mathscr{V}(!{*}D) \to \mathrm{DR}\,\mathscr{V}$. We denote the hypercohomologies on X of $\mathrm{DR}\,\mathscr{V}(!D)$, $\mathrm{DR}\,\mathscr{V}(!{*}D)$, and $\mathrm{DR}\,\mathscr{V}$ respectively by $\mathrm{H}^*_{\mathrm{dR,c}}(U, V)$, $\mathrm{H}^*_{\mathrm{dR,mid}}(U, V)$, and $\mathrm{H}^*_{\mathrm{dR}}(U, V)$.

We note that $\mathrm{DR}\,K$ and $\mathrm{DR}\,C$ have non-zero cohomology in degree one at most and are supported on D, whence exact sequences

$$\boldsymbol{H}^1(D, \mathrm{DR}\,K) \longrightarrow \mathrm{H}^1_{\mathrm{dR,c}}(U, V) \longrightarrow \mathrm{H}^1_{\mathrm{dR,mid}}(U, V) \longrightarrow 0$$
$$0 \longrightarrow \mathrm{H}^1_{\mathrm{dR,mid}}(U, V) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(U, V) \longrightarrow \boldsymbol{H}^1(D, \mathrm{DR}\,C).$$

In particular, we obtain an identification

$$\mathrm{H}^1_{\mathrm{dR,mid}}(U, V) = \mathrm{im}[\mathrm{H}^1_{\mathrm{dR,c}}(U, V) \to \mathrm{H}^1_{\mathrm{dR}}(U, V)]. \tag{3.1}$$

Moreover, when working in the analytic topology, one can define the analytic de Rham complex $\mathrm{DR}^{\mathrm{an}}\,\mathscr{V}$, which has constructible cohomology. Since X is compact, it is standard that the natural morphism $\mathrm{DR}\,\mathscr{V} \to \mathrm{DR}^{\mathrm{an}}\,\mathscr{V}$ is a quasi-isomorphism (and similarly for $\mathscr{V}(!D)$). This enables us to identify the algebraic de Rham cohomology with the analytic one. We will not use the exponent 'an' when the context is clear and we will use this GAGA theorem without mentioning it explicitly.

If $(V, \nabla) = (\mathscr{O}_U, \mathrm{d})$, one knows various forms of $\mathrm{DR}(\mathscr{O}_X(!D))$. Namely, $\mathrm{DR}(\mathscr{O}_X(!D))$ is quasi-isomorphic to one of the following complexes:

- $\mathrm{Cone}\big(\mathrm{DR}\,\mathscr{O}_X(*D) \to \iota_{D*}\,\mathrm{DR}\,\mathscr{O}_{\widehat{D}}\big)[-1]$,
- $\mathscr{O}_X(-D) \to \Omega^1_X$,

and, for the sake of computing cohomology, one can replace these complexes with their analytic counterparts. Let $\varpi : \widetilde{X} \to X$ be the real oriented blow-up of X along D, and let $\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}}$ be the sheaf on $\widetilde{X}$ of holomorphic functions on U having rapid decay along $\widetilde{D} = \varpi^{-1}(D)$ (see Section 3.g for details). The latter complex is also quasi-isomorphic to the direct image $\boldsymbol{R}\varpi_*$ of the de Rham complex with coefficients in $\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}}$.

The first goal of this section is to explain similar presentations for $\mathrm{DR}\,\mathscr{V}(!D)$ and, correspondingly, the presentation of $\mathrm{DR}\,\mathscr{V}$ in terms of the moderate de Rham complex on $\widetilde{X}$. Moreover, we make the de Rham duality pairing explicit in two ways:

- Theorem 3.12 gives an algebraic formula for the de Rham pairing S. It is a generalization to higher rank of a formula already obtained by Deligne [13] (see also [14]) in rank one.
- We compare this algebraic pairing with the de Rham pairing Q of Section 2.b in this context. In fact, perfectness of the algebraic duality is proved through this comparison result, which also needs the introduction of an $\mathscr{O}$-coherent version by means of weak Deligne lattices. The approach considered here is reminiscent of that of [46] in any dimension (see Cor. 5.2.7 from loc. cit.).

Once this material is settled, we translate into this setting the general results on quadratic relations obtained in Section 2.e, with the de Rham cohomology (resp. with compact support) instead of moderate (resp. rapid decay) cohomology. Also, we focus here our attention on the middle extension (co)homology.

3.b. **Čech computation of de Rham cohomologies.** Recall that, since U is affine, the de Rham cohomology of (V,∇) is computed as the cohomology of the complex

$$\Gamma(U,V)\xrightarrow{\ \nabla\ }\Gamma(U,\Omega^1_U\otimes V).$$

Therefore, any element of $\mathrm{H}^1_{\mathrm{dR}}(U,V)$ is represented by an element of $\Gamma(U,\Omega^1_U\otimes V)$ modulo the image of ∇.

For a sheaf F of $\mathscr{O}_X$-modules, we will compute the de Rham cohomology in terms of a Čech complex relative to the covering (U_D,U), whose differential is denoted by δ, so that for $(f,\varphi)\in F(U_D)\oplus F(U)$, we have $\delta(f,\varphi)=\iota_D\varphi-f\in F(U_D^\circ)$. We will implicitly identify $\Gamma(U_D^\circ,\iota_DV)$ with $\Gamma(U_D,\mathscr{V})=\mathscr{V}_D$.

Replacing X with the formal neighbourhood $\widehat{D}$ of D in X gives rise to $\mathscr{V}_{\widehat{D}}=\mathscr{O}_{\widehat{D}}\otimes_{\mathscr{O}_X}\mathscr{V}$, which is an $\mathscr{O}_{\widehat{D}}(*D)$-module with connection, and hence endowed with an action of $\Theta_{\widehat{D}}(-D)$.

Proposition 3.2. *The complex* $\mathrm{DR}\,\mathscr{V}(!D)$ *is quasi-isomorphic to the cone of the natural morphism* $\mathrm{DR}\,\mathscr{V}\to\mathrm{DR}\,\mathscr{V}_{\widehat{D}}$*, that is, to the simple complex associated with the double complex*

$$\begin{array}{ccc}j_*V & \xrightarrow{\ \iota_{\widehat{D}}\ } & \mathscr{V}_{\widehat{D}}\\ \nabla\big\downarrow & & \big\downarrow\widehat{\nabla}\\ j_*(\Omega^1_U\otimes V) & \xrightarrow{\ \iota_{\widehat{D}}\ } & \Omega^1_{\widehat{D}}\otimes\mathscr{V}_D.\end{array}$$

The complex $\boldsymbol{R}\Gamma(X,\mathrm{DR}\,\mathscr{V}(!D))$ *is quasi-isomorphic to the simple complex associated with the double complex*

$$\begin{array}{ccc}\Gamma(U,V) & \xrightarrow{\ \iota_{\widehat{D}}\ } & \mathscr{V}_{\widehat{D}}\\ \nabla\big\downarrow & & \big\downarrow\widehat{\nabla}\\ \Gamma(U,\Omega^1_U\otimes V) & \xrightarrow{\ \iota_{\widehat{D}}\ } & \Omega^1_{\widehat{D}}\otimes\mathscr{V}_{\widehat{D}}.\end{array}$$

Proof. Let us fix a section σ of the projection $D\times\mathbf{C}\to D\times(\mathbf{C}/\mathbf{Z})$ such that $\sigma(x,0)=(x,0)$ for each $x\in D$. Recall that any holonomic $\mathscr{D}_X$-module N is endowed with a canonical exhaustive filtration by coherent $\mathscr{D}_X(\log D)$-submodules $N_\bullet$ indexed by $\mathbf{Z}$, called the *Kashiwara-Malgrange filtration*, such that

- $N_{-k}=\mathscr{O}(-kD)N_0$ for any $k\geqslant 0$, and $N_{k+1}=N_k+\Theta_XN_k$ for any $k\geqslant 1$,

- for each $k \in \mathbf{Z}$, the eigenvalues of $\operatorname{Res}\nabla$ on N_k/N_{k-1}, as a function on D, belong to $k+\operatorname{im}(\sigma)$.

For each k, the connection ∇ defines a connection $\nabla : N_k \to \Omega^1_X \otimes N_{k+1}$. The above properties imply that, for $k \neq 0$, the induced morphism on graded pieces $\nabla : \operatorname{gr}_k N \to \Omega^1_X \otimes \operatorname{gr}_{k+1} N$ is bijective, so in particular the natural inclusion morphism is a quasi-isomorphism:

$$\{N_0 \xrightarrow{\nabla} \Omega^1_X \otimes N_1\} \xhookrightarrow{\sim} \operatorname{DR} N. \tag{3.3}$$

For example, the following also holds:

- If $N_{|U} = V$, then the natural morphism $N_k \to \mathscr{V}_k$ is an isomorphism for any $k \leqslant 0$.
- The formation of the Kashiwara-Malgrange filtration is compatible with tensoring with $\mathscr{O}_D$ and $\mathscr{O}_{\widehat{D}}$ and, for any k, $N/N_k \simeq N_D/N_{D,k} \simeq N_{\widehat{D}}/N_{\widehat{D},k}$.
- For $\mathscr{V}$ as above, we have $\mathscr{V}_k = \mathscr{O}(kD)\mathscr{V}_0$ for any $k \in \mathbf{Z}$.
- Among all N's such that $N_{|U} = V$, the $\mathscr{D}_X$-module $\mathscr{V}(!D)$ is characterized by the property that $\nabla : \operatorname{gr}_0 N \to \Omega^1_X \otimes \operatorname{gr}_1 N$ is bijective.

By the third point, the left-hand complex in (3.3) for $\mathscr{V}$ reads $\{\mathscr{V}_0 \xrightarrow{\nabla} \Omega^1_X(D) \otimes \mathscr{V}_0\}$, so (3.3) amounts to the quasi-isomorphism

$$\{\mathscr{V}_0 \xrightarrow{\nabla} \Omega^1_X(D) \otimes \mathscr{V}_0\} \xhookrightarrow{\sim} \operatorname{DR}\mathscr{V}. \tag{3.4}$$

On the other hand, the last point implies that the inclusion of complexes

$$\{\mathscr{V}(!D)_{-1} \xrightarrow{\nabla} \Omega^1_X \otimes \mathscr{V}(!D)_0\} \hookrightarrow \operatorname{DR}\mathscr{V}(!D)$$

is a quasi-isomorphism. The left-hand complex reads $\{\mathscr{V}_{-1} \xrightarrow{\nabla} \Omega^1_X \otimes \mathscr{V}_0\}$ according to the second point, so we obtain a quasi-isomorphism

$$\{\mathscr{V}_{-1} \xrightarrow{\nabla} \Omega^1_X(D) \otimes \mathscr{V}_{-1}\} \xrightarrow{\sim} \operatorname{DR}\mathscr{V}(!D). \tag{3.5}$$

The Čech resolution of this complex is the simple complex associated with the double complex

$$\begin{array}{ccc}
\mathscr{V}_{D,-1} \oplus j_*V & \xrightarrow{\quad\delta\quad} & \mathscr{V}_D \\
\nabla\big\downarrow & & \big\downarrow\nabla \\
\big(\Omega^1_{U_D}(D) \otimes \mathscr{V}_{D,-1}\big) \oplus j_*\big(\Omega^1_U \otimes V\big) & \xrightarrow{\ \delta\ } & \Omega^1_{U_D} \otimes \mathscr{V}_D.
\end{array}$$

Since $\mathscr{V}_{D,-1} \subset \mathscr{V}_D$, this complex is quasi-isomorphic to

$$\begin{array}{ccc}
j_*V & \xrightarrow{\quad\iota_D\quad} & \mathscr{V}_D/\mathscr{V}_{D,-1} \\
\nabla\big\downarrow & & \big\downarrow\nabla \\
j_*(\Omega^1_U \otimes V) & \xrightarrow{\ \iota_D\ } & \Omega^1_{U_D}(D) \otimes (\mathscr{V}_D/\mathscr{V}_{D,-1})
\end{array}
=
\begin{array}{ccc}
j_*V & \xrightarrow{\quad\iota_{\widehat{D}}\quad} & \mathscr{V}_{\widehat{D}}/\mathscr{V}_{\widehat{D},-1} \\
\nabla\big\downarrow & & \big\downarrow\nabla \\
j_*(\Omega^1_U \otimes V) & \xrightarrow{\ \iota_{\widehat{D}}\ } & \Omega^1_{\widehat{D}}(D) \otimes (\mathscr{V}_{\widehat{D}}/\mathscr{V}_{\widehat{D},-1}).
\end{array} \tag{3.6}$$

Reversing the reasoning, it is also isomorphic to the Čech complex

$$\begin{array}{ccc}
\mathscr{V}_{\widehat{D},-1} \oplus j_*V & \xrightarrow{\quad\delta\quad} & \mathscr{V}_{\widehat{D}} \\
\nabla\big\downarrow & & \big\downarrow\widehat{\nabla} \\
\big(\Omega^1_{\widehat{D}}(D) \otimes \mathscr{V}_{\widehat{D},-1}\big) \oplus j_*\big(\Omega^1_U \otimes V\big) & \xrightarrow{\ \delta\ } & \Omega^1_{\widehat{D}} \otimes \mathscr{V}_{\widehat{D}}.
\end{array} \tag{3.7}$$

We claim that the morphism $\nabla : \mathscr{V}_{\widehat{D},-1} \to \Omega^1_{\widehat{D}}(D) \otimes \mathscr{V}_{\widehat{D},-1}$ is bijective. Indeed, recall that $\mathscr{V}_{\widehat{D}}$ has a canonical decomposition

$$\mathscr{V}_{\widehat{D}} = \mathscr{V}^{\mathrm{reg}}_{\widehat{D}} \oplus \mathscr{V}^{\mathrm{irr}}_{\widehat{D}} \tag{3.8}$$

into its regular and irregular components and that $\mathscr{V}_{\widehat{D},\bullet}$ decomposes accordingly. Moreover, the filtration $\mathscr{V}^{\mathrm{irr}}_{\widehat{D},\bullet}$ is constant and equal to $\mathscr{V}^{\mathrm{irr}}_{\widehat{D}}$, and we know that

$$\widehat{\nabla}:\mathscr{V}^{\mathrm{irr}}_{\widehat{D}}\longrightarrow\Omega^1_{\widehat{D}}\otimes\mathscr{V}^{\mathrm{irr}}_{\widehat{D}}$$

is bijective (see for example [39, Th. (2.3) ii), p. 51] with $\mathscr{V}'=\mathscr{O}_{\widehat{D}}$ and $\mathscr{V}''=\mathscr{V}_{\widehat{D}}$). On the other hand, it is standard to check that, for the regular holonomic part,

$$\widehat{\nabla}:\mathscr{V}^{\mathrm{reg}}_{\widehat{D},-1}\longrightarrow\Omega^1_{\widehat{D}}(D)\otimes\mathscr{V}^{\mathrm{reg}}_{\widehat{D},-1}$$

is bijective. It follows from this claim that $\mathrm{DR}\,\mathscr{V}(!D)$ is quasi-isomorphic to the complex given in the proposition, and the statement for $\boldsymbol{R}\Gamma(X,\mathrm{DR}\,\mathscr{V}(!D))$ results immediately. □

Corollary 3.9. *An element of* $\mathrm{H}^1_{\mathrm{dR,c}}(U,V)$ *can be represented by a pair*

$$(\widehat{m},\omega)\in\mathscr{V}_{\widehat{D}}\oplus\Gamma(U,\Omega^1_U\otimes V)$$

satisfying $\widehat{\nabla}\widehat{m}=\iota_{\widehat{D}}\omega$*, modulo pairs of the form* $(\iota_{\widehat{D}}(v),\nabla v)$ *for* $v\in\Gamma(U,V)$*.*

Remark 3.10. The natural morphism $\mathrm{H}^1_{\mathrm{dR,c}}(U,V)\to\mathrm{H}^1_{\mathrm{dR}}(U,V)$ is described in terms of the previous representatives as $(\widehat{m},\omega)\mapsto\omega$.

(1) The image $\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)$ of this morphism (see (3.1)) consists of classes of sections $\omega\in\Gamma(U,V)$ such that $\iota_{\widehat{D}}\omega\in\mathrm{im}(\widehat{\nabla}:\mathscr{V}_{\widehat{D}}\to\mathscr{V}_{\widehat{D}})$. Therefore, any family consisting of $\dim\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)$ linearly independent classes $[\omega]$ in $\mathrm{H}^1_{\mathrm{dR}}(U,V)$ for which there exists $\widehat{m}\in\mathscr{V}_{\widehat{D}}$ satisfying $\widehat{\nabla}\widehat{m}=\iota_{\widehat{D}}\omega$ is a basis of $\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)$.

(2) The kernel of this morphism consists of pairs of the form $(\widehat{m},\nabla w)$ for some $w\in\Gamma(U,V)$, modulo pairs $(\iota_{\widehat{D}}(v),\nabla v)$. Each class has thus a representative of the form $(\widehat{m},0)$ with $\widehat{\nabla}\widehat{m}=0$, and there is a surjective morphism

$$\ker\bigl[\widehat{\nabla}:\mathscr{V}_{\widehat{D}}\to\mathscr{V}_{\widehat{D}}\bigr]\longrightarrow\!\!\!\!\!\rightarrow\ker\Bigl[\mathrm{H}^1_{\mathrm{dR,c}}(U,V)\to\mathrm{H}^1_{\mathrm{dR}}(U,V)\Bigr].$$

(3) If V has no constant subbundle with connection (e.g. if V is irreducible and non-constant), then a representative $(\widehat{m},0)$ with $\widehat{\nabla}\widehat{m}=0$ is unique. We thus conclude in such a case that

$$\ker\bigl[\widehat{\nabla}:\mathscr{V}_{\widehat{D}}\to\mathscr{V}_{\widehat{D}}\bigr]\simeq\ker\Bigl[\mathrm{H}^1_{\mathrm{dR,c}}(U,V)\to\mathrm{H}^1_{\mathrm{dR}}(U,V)\Bigr].$$

(4) With the assumption in (3), a basis of $\mathrm{H}^1_{\mathrm{dR,c}}(U,V)$ can thus be obtained as the union of the following sets:
- a basis of $\ker\widehat{\nabla}$ in $\mathscr{V}_{\widehat{D}}$,
- a family of $\dim\mathrm{H}^1_{\mathrm{dR}}(U,V)-\dim\ker\widehat{\nabla}$ representatives $(\widehat{m},\omega)$ for which the classes of ω in $\mathrm{H}^1_{\mathrm{dR}}(U,V)$ are linearly independent (and thus form a basis of $\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)$).

Remark 3.11 (Computation in the analytic topology)**.** By the same reasoning, we find that any element of $\mathrm{H}^1_{\mathrm{dR,c}}(U,V)\simeq\mathrm{H}^1_{\mathrm{dR}}(X,\mathrm{DR}^{\mathrm{an}}\,\mathscr{V}(!D))$ can be represented by a pair $(m,\omega^{\mathrm{an}})\in\Gamma(U^{\mathrm{an}}_D,\mathscr{V}^{\mathrm{an}})\otimes\Gamma(U^{\mathrm{an}},\Omega^1_{U^{\mathrm{an}}}\otimes V)$ such that $\nabla m=\omega^{\mathrm{an}}_{|U^\circ_D}$. Considering the analytic analogue of the Čech complex (3.7), we can also represent it as a triple $(m,(\omega^{\mathrm{an}}_{|U^\circ_D}-\nabla m_{|U^\circ_D}),\omega^{\mathrm{an}})$.

3.c. **Pairings.** For $\mathscr{V}=j_*V$ and $\mathscr{V}^\vee=j_*V^\vee$, the compatibility between duality of holonomic $\mathscr{D}_X$-modules and Poincaré-Verdier duality of their analytic de Rham complexes gives an isomorphism

$$\mathrm{DR}^{\mathrm{an}}\,\mathscr{V}^\vee\xrightarrow{\ \sim\ }\boldsymbol{D}'\,\mathrm{DR}^{\mathrm{an}}\,\mathscr{V}(!D),$$

where $\boldsymbol{D}'$ denotes the Verdier duality functor shifted by $-2\dim X$ ($=-2$ here). Passing to hypercohomology on X we obtain, through the GAGA isomorphism, an isomorphism

$$\mathrm{H}^1_{\mathrm{dR}}(U,V^\vee)\xrightarrow{\ \sim\ }\mathrm{H}^1_{\mathrm{dR,c}}(U,V)^\vee,$$

that we regard as a non-degenerate pairing

$$\mathrm{H}^1_{\mathrm{dR,c}}(U,V)\otimes\mathrm{H}^1_{\mathrm{dR}}(U,V^\vee)\longrightarrow\mathbf{C}.$$

We aim at giving two constructions of such a pairing, that we will not however compare with the one, given as a motivation, described above. On the one hand, one can define a pairing

$$\mathsf{S}^1:\mathrm{H}^1_{\mathrm{dR,c}}(U,V)\otimes\mathrm{H}^1_{\mathrm{dR}}(U,V^\vee)\longrightarrow\mathbf{C}$$

expressed on representatives $(\widehat{m},\omega)$ and $\omega^\vee$ by summing up the residues

$$(\widehat{m},\omega)\otimes\omega^\vee\longmapsto\operatorname{res}_D\langle\widehat{m},\iota_{\widehat{D}}\omega^\vee\rangle=\sum_{x\in D}\operatorname{res}_x\langle\widehat{m},\iota_{\widehat{x}}\omega^\vee\rangle.$$

Indeed, let us check that the formula does not depend on the choice of representatives of the de Rham cohomology classes.

- If $\omega^\vee=\nabla v^\vee$, we have

$$\operatorname{res}_D\langle\widehat{m},\iota_{\widehat{D}}\nabla v^\vee\rangle=\operatorname{res}_D\mathrm{d}\langle\widehat{m},\iota_{\widehat{D}}v^\vee\rangle-\operatorname{res}_D\iota_{\widehat{D}}\langle\omega,v^\vee\rangle=0$$

  since, on the one hand, $\operatorname{res}_x\mathrm{d}\widehat{\eta}=0$ for each x and any formal Laurent series $\widehat{\eta}$, and on the other hand, the residue theorem implies $\operatorname{res}_D\iota_{\widehat{D}}\langle\omega,v^\vee\rangle=0$.
- Similarly, if $(\widehat{m},\omega)=(\iota_{\widehat{D}}(v),\nabla v)$, we have $\operatorname{res}_D\iota_{\widehat{D}}\langle v,\omega^\vee\rangle=0$ by the residue theorem.

The pairing S^1 is decomposed as (see Appendix D for the isomorphism res_D)

$$\mathrm{H}^1_{\mathrm{dR,c}}(U,V)\otimes\mathrm{H}^1_{\mathrm{dR}}(U,V^\vee)\longrightarrow\mathrm{H}^2_{\mathrm{dR,c}}(U,\mathbf{C})\xrightarrow[\sim]{\operatorname{res}_D}\mathbf{C}.$$

On the other hand, by working on the real oriented blow-up of X along D, we define a Poincaré-Verdier pairing (see Appendix D for the notation $\operatorname{tr}_{U^{\mathrm{an}}}$)

$$\mathrm{PV}^1:\mathrm{H}^1_{\mathrm{dR,c}}(U,V)\otimes\mathrm{H}^1_{\mathrm{dR}}(U,V^\vee)\longrightarrow\mathrm{H}^2_{\mathrm{dR,c}}(U^{\mathrm{an}})\xrightarrow[\sim]{\operatorname{tr}_{U^{\mathrm{an}}}}\mathbf{C},$$

which is non-degenerate by the Verdier duality theorem (see Corollary 3.42).

Theorem 3.12. *The Poincaré-Verdier pairing* PV^1 *and the residue pairing* S^1 *are equal.*

In particular, the residue pairing S^1 is non-degenerate.

Corollary 3.13. *The pairing* S^1 *vanishes on* $\ker\widehat{\nabla}\otimes\mathrm{H}^1_{\mathrm{dR,mid}}(U,V^\vee)$ *and induces a non-degenerate pairing*

$$\mathsf{S}^1_{\mathrm{mid}}:\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)\otimes\mathrm{H}^1_{\mathrm{dR,mid}}(U,V^\vee)\longrightarrow\mathbf{C}.$$

Moreover, if $V\xrightarrow{\sim}V^\vee$ *is a* $\pm$*-symmetric isomorphism, it induces a* $\mp$*-symmetric non-degenerate pairing*

$$\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)\otimes\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)\longrightarrow\mathbf{C}.$$

Proof. If $\widehat{\nabla}\widehat{m}=0$ and $\omega^\vee$ satisfies $\iota_{\widehat{D}}\omega^\vee=\widehat{\nabla}\widehat{m}^\vee$, we have

$$\operatorname{res}_D\langle\widehat{m},\iota_{\widehat{D}}\omega^\vee\rangle=\operatorname{res}_D\mathrm{d}\langle\widehat{m},\widehat{m}^\vee\rangle=0,$$

hence the first assertion. That $\mathsf{S}^1_{\mathrm{mid}}$ is non-degenerate follows from the same property for S^1. Assume now that $\langle\bullet,\bullet\rangle$ is a $\pm$-symmetric pairing on V and let $\omega,\omega^\vee$ represent classes in $\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)$, so that there exist $\widehat{m},\widehat{m}'$ such that $\nabla\widehat{m}=\iota_{\widehat{D}}\omega$ and $\nabla\widehat{m}'=\iota_{\widehat{D}}\omega'$. Then

$$\operatorname{res}_D\langle\widehat{m},\iota_{\widehat{D}}\omega'\rangle=\operatorname{res}_D\langle\widehat{m},\nabla\widehat{m}'\rangle=-\operatorname{res}_D\langle\nabla\widehat{m},\widehat{m}'\rangle=\mp\operatorname{res}_D\langle\widehat{m}',\nabla\widehat{m}\rangle=\mp\operatorname{res}_D\langle\widehat{m}',\iota_{\widehat{D}}\omega\rangle.\qquad\square$$

The theorem can also be obtained for PV^1 by applying Remark 2.17 when working on the real blow-up space.

Remark 3.14. This theorem is of course a variant of the theorem asserting compatibility between duality of $\mathscr{D}$-modules and proper push-forward (see [43]), but more in the spirit of [46, Cor. 5.2.7]. The presentation given here owes much to [39, App. 2]. It can also be regarded as an extension of the duality theorem for logarithmic $\mathscr{D}$-modules (see [18, App. A]). Nevertheless, the present formulation is more precise and can lead to explicit computations.

In the present form, this theorem has a long history, starting (as far as we know) with [13] (see also [14]) in rank one. The computation that we perform with Čech complexes by using a formal neighbourhood of D is inspired from loc. cit. The result was used by Deligne for showing compatibility between duality and the irregular Hodge filtration introduced in loc. cit.

In a series of papers mentioned in the introduction, Matsumoto et al. also gave duality results of this kind, either in the case of regular singularities or in rank one cases with irregular singularities (possibly in higher dimensions).

Summary of the proof of Theorem 3.12.

(1) We choose a weak Deligne-Malgrange lattice $\mathscr{L}_0$ of $\mathscr{V}$ (see Section 3.d), whose dual gives rise to the logarithmic de Rham complex $\mathrm{DR}^{\log}(\mathscr{L}_1^\vee)$, which is a sub-complex of $\mathrm{DR}\,\mathscr{V}^\vee$ quasi-isomorphic to it. We also realize $\mathrm{DR}\,\mathscr{V}(!D)$ by the complex $\mathrm{DR}^{\log}(\mathscr{L}_0(-D))$.

(2) We consider the logarithmic de Rham complex (see Section 3.e)

$$\mathrm{DR}^{\log}(\mathscr{O}_X(-D)) = \{\mathscr{O}_X(-D) \xrightarrow{\mathrm{d}} \Omega^1_X\}$$

(we have identified Ω^1_X with $\Omega^1_X(D) \otimes \mathscr{O}_X(-D)$) and the natural pairing of complexes of sheaves

$$\mathrm{DR}^{\log}(\mathscr{L}_0(-D)) \otimes \mathrm{DR}^{\log}(\mathscr{L}_1^\vee) \longrightarrow \mathrm{DR}^{\log}(\mathscr{O}_X(-D)). \tag{3.15}$$

We show that the induced pairing on Čech cohomology

$$\mathrm{H}^1_{\mathrm{dR,c}}(U, V) \otimes \mathrm{H}^1_{\mathrm{dR}}(U, V^\vee) \longrightarrow \mathrm{H}^2_{\mathrm{dR,c}}(U) \tag{3.16}$$

can be expressed on representatives $(\widehat{m}, \omega)$ and $\omega^\vee$ as $\langle \widehat{m}, \iota_{\widehat{D}}\omega^\vee\rangle$.

(3) We express the analytification of (3.15) as coming by pushforward from a pairing on the real oriented blow-up $\widetilde{X}$ of X at D, where the logarithmic de Rham complexes have cohomology concentrated in degree zero (see (3.38)). We show that it is also identified with the de Rham pairing between rapid decay and moderate de Rham complexes of the pullback $\widetilde{\mathscr{V}}$ of $\mathscr{V}$ to $\widetilde{X}$ (see Section 3.h). Arguing in a way similar to [25, Th. 3] (done in the higher-dimensional case), we show that there exists a unique such pairing extending the natural one on U^{an}, and hence the de Rham pairing is identified with the Poincaré-Verdier duality pairing.

(4) Taking cohomology gives an identification between the analytification of (3.16) and the Poincaré-Verdier duality pairing with values in $\mathrm{H}^2_{\mathrm{c}}(U^{\mathrm{an}})$.

(5) The theorem now follows from the canonical identification between $\mathrm{H}^2_{\mathrm{dR,c}}(U)$ and $\mathrm{H}^2_{\mathrm{c}}(U^{\mathrm{an}})$ together with Proposition D.1. □

3.d. **Weak Deligne-Malgrange lattices.** By a *lattice* of $\mathscr{V}$ we mean an $\mathscr{O}_X$-locally free sheaf $\mathscr{L} \subset \mathscr{V}$ such that $j^*\mathscr{L} = V$. In order to construct a lattice $\mathscr{L}$ of $\mathscr{V}$, it is enough to construct it in the neighbourhood of each point of D. The global lattice $\mathscr{L}$ is then obtained by gluing the local ones with V. Moreover, since lattices of $\mathscr{V}$ on $\mathrm{nb}(D)$ are in one-to-one correspondence with lattices of $\mathscr{V}_{\widehat{D}}$, it is enough to construct such a lattice for $\mathscr{V}_{\widehat{D}}$. We conclude:

Lemma 3.17 ([40, Prop. 1.2])**.** *The map $\mathscr{L} \mapsto \mathscr{L}_{\widehat{D}}$ is a bijection from the set of lattices of $\mathscr{V}$ to the set of lattices of $\mathscr{V}_{\widehat{D}}$, whose inverse associates to $\mathscr{L}_{\widehat{D}}$ the unique lattice of $\mathscr{V}$ whose germ at D is $\mathscr{L}_{\widehat{D}} \cap \mathscr{V}_D$.*

If $\mathscr{L}$ is a lattice of $\mathscr{V}$, then $\mathscr{L}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$ is a lattice of $\mathscr{V}^\vee = j_*(V^\vee)$ through the identification

$$\mathscr{L}^\vee = \{\lambda : \mathscr{V} \to \mathscr{O}_X(*D) \mid \lambda(\mathscr{L}) \subset \mathscr{O}_X\}. \tag{3.18}$$

Definition 3.19 (of a weak Deligne-Malgrange lattice in the unramified case)**.** Let us fix a point $x \in D$ and choose a local coordinate z centered at this point, so that the germ $\mathscr{O}_{X,x}$ is identified with $\mathbf{C}\{z\}$. Let us assume that $\mathscr{V}_{\widehat{x}} = \mathscr{O}_{\widehat{x}} \otimes \mathscr{V}$ admits an unramified Levelt-Turrittin decomposition

$$(\mathscr{V}_{\widehat{x}}, \widehat{\nabla}) \simeq \bigoplus_{\varphi \in \Phi} (\widehat{R}_\varphi, \mathrm{d}\varphi + \widehat{\nabla}^{\mathrm{reg}})$$

for some subset $\Phi \subset \mathbf{C}(\{z\})/\mathbf{C}\{z\} = z^{-1}\mathbf{C}[z^{-1}]$ and some free $\mathbf{C}(\{z\})$-vector spaces with regular connection $(R_\varphi, \nabla^{\mathrm{reg}})$ of rank d_φ, having formalization $(\widehat{R}_\varphi, \nabla^{\mathrm{reg}})$. Let us set $E^\varphi = (\mathbf{C}[\![z]\!], \mathrm{d}{+}\mathrm{d}\varphi)$, where we regard $\mathrm{d} + \mathrm{d}\varphi : E^\varphi \to \Omega^1_{\widehat{0}} \otimes E^\varphi[1/z]$ as a formal meromorphic connection on E^φ. We can thus write the summands as $E^\varphi \otimes \widehat{R}_\varphi$.

We say that a lattice $\mathscr{L}$ of $\mathscr{V}$ is a *weak Deligne-Malgrange* lattice at $x \in D$ if

- $\mathscr{L}_{\widehat{x}} = \mathbf{C}[\![z]\!] \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$ decomposes according to the Levelt-Turrittin decomposition:
$$\mathscr{L}_{\widehat{x}} \simeq \bigoplus_{\varphi \in \Phi} (E^\varphi \otimes \widehat{R}^0_\varphi) \tag{3.20}$$
for some *logarithmic* lattice $\widehat{R}^0_\varphi$ of $\widehat{R}_\varphi$, and
- the eigenvalues of the residue of $\widehat{\nabla}^{\mathrm{reg}}$ on $\widehat{R}^0_0$ do not differ by a non-zero integer, and the only integral eigenvalue is 0: this is the 0*non-resonance condition*.

(The adjective "weak" is justified by the fact that we do not impose any condition on the residue of $\widehat{\nabla}^{\mathrm{reg}}$ on $\widehat{R}^0_\varphi$ for $\varphi \neq 0$.)[4]

The following lemmas may give a better idea of what a weak Deligne-Malgrange lattice is, when $\mathscr{V}$ is unramified.

Lemma 3.21 (Characterization of unramified weak Deligne-Malgrange lattices)**.** *Assume that $\mathscr{V}$ is unramified at $x \in D$. A lattice $\mathscr{L}$ of $\mathscr{V}$ is a weak Deligne-Malgrange lattice at x if and only if there exists a $\mathbf{C}\{z\}$-basis $\boldsymbol{v}$ of $\mathscr{L}_x$ such that the matrix of ∇ takes the form $\nabla\boldsymbol{v} = \boldsymbol{v}A(z)\mathrm{d}z/z$ with $A(z) = A_{<0}(z) + A_0 + A_{>0}(z)$, where*

1. *$A_{<0}(z) = \bigoplus_{\varphi\in\Phi} \varphi\,\mathrm{Id}_{d_\varphi} = z^{-q}D_q + z^{q-1}D_{q-1} + \cdots + z^{-1}D_1$, with D_i diagonal ($i \geqslant 1$),*
2. *A_0 is block-upper triangular, the blocks being indexed by the eigenvalues φ of $A_{<0}(z)$, and the eigenvalues of the block indexed by $0 \in \Phi$ are 0non-resonant,*
3. *$A_{>0}(z)$ has holomorphic entries vanishing at 0.*

Lemma 3.22 (A criterion for unramified weak Deligne-Malgrange lattices)**.** *Assume that there exists a $\mathbf{C}\{z\}$-basis $\boldsymbol{v}$ of $\mathscr{L}_x$ such that the matrix of ∇ takes the form $\nabla\boldsymbol{v} = \boldsymbol{v}A(z)\mathrm{d}z/z$ with $A(z) = A_{\leqslant 0}(z) + A_{>0}(z)$, where $A_{>0}(z)$ has holomorphic entries vanishing at 0 and $A_{\leqslant 0}(z)$ is decomposed into two blocks $A_{<0}(z) \oplus C_0$, with*

1. *$A_{<0}(z) = z^{-q}A_q + z^{q-1}A_{q-1} + \cdots + z^{-1}A_1$, with A_q invertible,*
2. *the eigenvalues of C_0 are 0non-resonant.*

Then $\mathscr{L}_x$ a weak Deligne-Malgrange of $\mathscr{V}$ at x.

Definition 3.23 (of a weak Deligne-Malgrange lattice in the ramified case)**.** We now relax the non-ramification condition at $x \in D$. There exists a ramification $\rho^* : \mathbf{C}\{z\} \to \mathbf{C}\{t\}$ with Galois group G such that $\mathscr{V}^\rho_x = \rho^*\mathscr{V}_x$ is unramified. We say that a lattice $\mathscr{L}$ of $\mathscr{V}$ is a *weak Deligne-Malgrange* lattice at $x \in D$ if there exists a weak Deligne-Malgrange lattice $\mathscr{L}^\rho_x$ of $\mathscr{V}^\rho_x$ such that $\mathscr{L}_x = (\mathscr{L}^\rho_x)^G$.

[4]In higher dimension, the proof of the global existence of a lattice is not as simple as in dimension one (see [40]). Fixing the eigenvalues of the residue also on each R^0_φ is necessary in order to glue along the divisor D the locally defined lattices to a global lattice; in dimension one, this is unnecessary since the singular set D is discrete.

Remark 3.24. If $\mathscr{L}$ is a weak Deligne-Malgrange lattice, $\mathscr{L}_{\widehat{x}}$ decomposes as $\mathscr{L}_{\widehat{x}} = \mathscr{L}_{\widehat{x}}^{\mathrm{reg}} \oplus \mathscr{L}_{\widehat{x}}^{\mathrm{irr}}$ according to (3.8) and the 0non-resonance condition only affects the logarithmic lattice $\mathscr{L}_{\widehat{x}}^{\mathrm{reg}}$ of $\mathscr{V}_{\widehat{x}}^{\mathrm{reg}}$. If the 0non-resonance holds for $\mathscr{L}_{\widehat{x}}^{\rho,\mathrm{reg}}$, then it also holds for $\mathscr{L}_{\widehat{x}}^{\mathrm{reg}}$, since the eigenvalues of the residue of $\widehat{\nabla}$ on $\mathscr{L}_{\widehat{x}}^{\mathrm{reg}}$ are of the form γ/r, where r is the order of ρ and γ is an eigenvalue of the residue of $\widehat{\nabla}$ on $\mathscr{L}_{\widehat{x}}^{\rho,\mathrm{reg}}$. Conversely, given a lattice $\mathscr{L}_{\widehat{x}}^{\mathrm{reg}}$, the 0non-resonance condition may hold for $\mathscr{L}_{\widehat{x}}^{\mathrm{reg}}$ without holding for $\mathscr{L}_{\widehat{x}}^{\rho,\mathrm{reg}}$.

Lemma 3.25 (Behaviour by duality)**.** *Suppose $\mathscr{L}_0$ is a weak Deligne-Malgrange lattice of $\mathscr{V}$ and set $\mathscr{L}_1 = \mathscr{L}_0 + \Theta_X(-D)\mathscr{L}_0$. Then the following hold.*

1. *$\mathscr{L}_1$ is also a weak Deligne-Malgrange lattice of $\mathscr{V}$.*
2. *$\mathscr{L}_0^\vee$ and $\mathscr{L}_1^\vee$ are weak Deligne-Malgrange lattices of $\mathscr{V}^\vee$.*
3. *We have $\mathscr{L}_0^\vee = \mathscr{L}_1^\vee + \Theta_X(-D)\mathscr{L}_1^\vee$.*

Proof. By Lemma 3.17, it is enough to prove the statements for $\mathscr{L}_{0,\widehat{D}}$.

(1) We have $\mathscr{L}_{1,\widehat{D}} = \mathscr{L}_{0,\widehat{D}} + \Theta_{\widehat{D}}(-D)\mathscr{L}_{0,\widehat{D}}$. It follows that $\mathscr{L}_{1,\widehat{D}}^{\mathrm{reg}} = \mathscr{L}_{0,\widehat{D}}^{\mathrm{reg}} + \Theta_{\widehat{D}}(-D)\mathscr{L}_{0,\widehat{D}}^{\mathrm{reg}}$ and similarly for the irregular part. Since $\mathscr{L}_{0,\widehat{D}}^{\mathrm{reg}}$ is logarithmic, the right-hand side above is nothing but $\mathscr{L}_{0,\widehat{D}}^{\mathrm{reg}}$. In the unramified irregular case, one checks that the action of $\Theta_{\widehat{D}}(-D)$ preserves the decomposition (3.20) and that

$$(E^\varphi \otimes \widehat{R}_\varphi^0) + \Theta_{\widehat{D}}(-D)(E^\varphi \otimes \widehat{R}_\varphi^0) = E^\varphi \otimes \widehat{R}_\varphi^0(qD),$$

where q is the order of the pole of φ, so the desired properties hold in the unramified case. The ramified case follows easily. The same argument shows that $z\partial_z : \mathscr{L}_{0,\widehat{x}}^{\mathrm{irr}} \to \mathscr{L}_{1,\widehat{x}}^{\mathrm{irr}}$ is bijective.

(2) We argue with $L = \mathscr{L}_0$ or $\mathscr{L}_1$. Duality is compatible with restriction to $\widehat{D}$ and transforms the decomposition (3.20) into the dual one (φ changed to $-\varphi$ and $\widehat{R}_\varphi^0$ to $(\widehat{R}_\varphi^0)^\vee$). We are thus left with showing that 0non-resonance of $\widehat{\nabla}$ on $(\mathscr{L}_{\widehat{x}}^{\mathrm{reg}})^\vee$ is fulfilled. If $A(z)\mathrm{d}z/z$ is the matrix of the connection on $\mathscr{L}_{\widehat{x}}^{\mathrm{reg}}$, with $A(z)$ having entries in $\mathbf{C}[\![z]\!]$, then $-{}^tA(z)\mathrm{d}z/z$ is that of $\nabla^\vee$ on $(\mathscr{L}_{\widehat{x}}^{\mathrm{reg}})^\vee$, so the eigenvalues of $\operatorname{Res}\nabla^\vee$ are opposite to those of $\operatorname{Res}\nabla$.

(3) The assertion is clear for the regular part, since $\mathscr{L}_{1,\widehat{x}}^{\mathrm{reg}} = \mathscr{L}_{0,\widehat{x}}^{\mathrm{reg}}$. For the irregular part, it is enough to prove the stronger property:

- $z\partial_z : \mathscr{L}_{1,\widehat{x}}^{\vee,\mathrm{irr}} \to \mathscr{L}_{0,\widehat{x}}^{\vee,\mathrm{irr}}$ is bijective.

Since $\mathscr{V}_{\widehat{x}}^{\mathrm{irr}}$ and $\mathscr{V}_{\widehat{x}}^{\mathrm{irr},\vee}$ are purely irregular, the action of $z\partial_z$ is bijective on each of them (in other words, $\mathrm{DR}\,\mathscr{V}_{\widehat{D}} \equiv 0$), and hence it is injective on $\mathscr{L}_{1,\widehat{x}}^{\vee,\mathrm{irr}}$. For the surjectivity we note that, by (3.18), for $\lambda \in \mathscr{V}_{\widehat{x}}^{\vee,\mathrm{irr}}$ we have $\lambda \in \mathscr{L}_{1,\widehat{x}}^{\vee,\mathrm{irr}}$ if and only if $\lambda(\mathscr{L}_{1,\widehat{x}}^{\mathrm{irr}}) \subset \mathscr{O}_{\widehat{x}}$, that is, $\lambda(z\partial_z\mathscr{L}_{0,\widehat{x}}^{\mathrm{irr}}) \subset \mathscr{O}_{\widehat{x}}$ (as seen above), and hence equivalently, $z\partial_z\lambda(\mathscr{L}_{0,\widehat{x}}^{\mathrm{irr}}) \subset \mathscr{O}_{\widehat{x}}$, which is also equivalent to $z\partial_z\lambda \in \mathscr{L}_{0,\widehat{x}}^{\vee,\mathrm{irr}}$. □

3.e. **Computation of $\mathrm{DR}\,\mathscr{V}(!D)$ with weak Deligne-Malgrange lattices.** Let $\mathscr{L}_0$ be a weak Deligne-Malgrange lattice of $\mathscr{V}$ and set $\mathscr{L}_1 = \mathscr{L}_0 + \Theta_X(-D)\mathscr{L}_0$. The logarithmic de Rham complex attached to $\mathscr{L}_0$ is defined as

$$\mathrm{DR}^{\log}(\mathscr{L}_0, \nabla) = \{\mathscr{L}_0 \xrightarrow{\nabla} \Omega_X^1(D) \otimes \mathscr{L}_1\}.$$

We have natural inclusions of complexes

$$\mathrm{DR}^{\log}(\mathscr{L}_0, \nabla) \hookrightarrow \mathrm{DR}^{\log}(\mathscr{V}) = \mathrm{DR}(\mathscr{V}), \tag{3.26}$$

which factorizes through

$$\{\mathscr{L}_0 \longrightarrow \Omega_X^1(D) \otimes \mathscr{L}_1\} \hookrightarrow \{\mathscr{V}_0 \xrightarrow{\nabla} \Omega_X^1(D) \otimes \mathscr{V}_0\} \underset{(3.4)}{\xrightarrow{\sim}} \mathrm{DR}\,\mathscr{V}.$$

We also have $\mathscr{L}_1(-D)=\mathscr{L}_0(-D)+\Theta_X(-D)(\mathscr{L}_0(-D))$ and, for the logarithmic de Rham complex $\mathrm{DR}^{\log}(\mathscr{L}_0(-D),\nabla)$, there is similarly a natural inclusion

$$\{\mathscr{L}_0(-D)\longrightarrow\Omega^1_X(D)\otimes\mathscr{L}_1(-D)\}\hookrightarrow\{\mathscr{V}_{-1}\xrightarrow{\nabla}\Omega^1_X(D)\otimes\mathscr{V}_{-1}\}\underset{(3.5)}{\xrightarrow{\sim}}\mathrm{DR}\,\mathscr{V}(!D).\tag{3.27}$$

Proposition 3.28. *For a weak Deligne-Malgrange lattice $\mathscr{L}_0$, the inclusions* (3.26) *and* (3.27) *are quasi-isomorphisms.*

Proof. We consider the Čech resolution of $\mathrm{DR}^{\log}(\mathscr{L}_0,\nabla)$ for the covering $(U_{\widehat{D}},U)$:

$$\begin{array}{ccc}\mathscr{L}_{0,\widehat{D}}\oplus j_*V & \xrightarrow{\quad\delta\quad} & \mathscr{V}_{\widehat{D}}\\ \nabla\big\downarrow & & \big\downarrow\nabla\\ (\Omega^1_{U_{\widehat{D}}}(D)\otimes\mathscr{L}_{1,\widehat{D}})\oplus(\Omega^1_U\otimes V) & \xrightarrow{\ \delta\ } & \Omega^1_{U_{\widehat{D}}}(*D)\otimes\mathscr{V}_{\widehat{D}}.\end{array}$$

It maps to the corresponding resolution of $\mathrm{DR}^{\log}(\mathscr{V}_0)$ and this morphism is a quasi-isomorphism if and only if the $\mathbf{C}$-linear morphism between finite-dimensional vector spaces $\widehat{\nabla}:\mathscr{V}_{\widehat{D},0}/\mathscr{L}_{0,\widehat{D}}\to\Omega^1_{U_{\widehat{D}}}(D)\otimes\mathscr{V}_{\widehat{D},0}/\mathscr{L}_{1,\widehat{D}}$ is an isomorphism. We can use the decomposition (3.8). For the regular part, we have $\mathscr{L}^{\mathrm{reg}}_{1,\widehat{D}}=\mathscr{L}^{\mathrm{reg}}_{0,\widehat{D}}$ and by definition 0 is not an eigenvalue of the endomorphism $z\partial_z:\mathscr{V}_{\widehat{x},0}/\mathscr{L}^{\mathrm{reg}}_{0,\widehat{x}}\to\mathscr{V}_{\widehat{x},0}/\mathscr{L}^{\mathrm{reg}}_{0,\widehat{x}}$ for any $x\in D$ and any local coordinate z at x. For the irregular part, we use that $z\partial_z$ is an isomorphism from $\mathscr{V}^{\mathrm{irr}}_{\widehat{x}}=\mathscr{V}^{\mathrm{irr}}_{\widehat{x},0}$ to itself and from $\mathscr{L}^{\mathrm{irr}}_{0,\widehat{x}}$ to $\mathscr{L}^{\mathrm{irr}}_{1,\widehat{x}}$. This gives the assertion for (3.26), and the proof for (3.27) is similar. ☐

Remark 3.29. In the Čech resolution of $\mathrm{DR}^{\log}(\mathscr{L}_0(-D),\nabla)$, the vertical arrow corresponding to $\mathrm{DR}^{\log}(\mathscr{L}_{0,\widehat{D}}(-D),\widehat{\nabla})$ is an isomorphism, so we can obviously identify it with the resolution of Proposition 3.2.

3.f. **Duality pairing for weak Deligne-Malgrange lattices.** Let $\mathscr{L}_0$ be a weak Deligne-Malgrange lattice in $\mathscr{V}$ and $\mathscr{L}_0^\vee$ the dual weak Deligne-Malgrange lattice in $\mathscr{V}^\vee$. The natural pairing (3.15) sends

- $\mathscr{L}_0(-D)\otimes\mathscr{L}_1^\vee$ to $\mathscr{O}_X(-D)$ (since $\mathscr{L}_0\subset\mathscr{L}_1$),
- $(\Omega^1_X(D)\otimes\mathscr{L}_1(-D))\otimes\mathscr{L}_1^\vee$ and $\mathscr{L}_0(-D)\otimes(\Omega^1_X(D)\otimes\mathscr{L}_0^\vee)$ to Ω^1_X,
- $(\Omega^1_X(D)\otimes\mathscr{L}_1(-D))\otimes(\Omega^1_X(D)\otimes\mathscr{L}_0^\vee)$ to 0.

By Proposition 3.28, the pairing (3.15) induces a cohomology pairing (3.16) that we make explicit in terms of Čech representatives:

- A class in $\mathrm{H}^1_{\mathrm{dR}}(X,\mathrm{DR}^{\log}(\mathscr{L}_0(-D)))$ is represented by a pair $(\widehat{m},\omega)$, with $\widehat{m}\in\mathscr{V}_{\widehat{D}}$ and $\omega\in\Gamma(U,\Omega^1_U\otimes V)$ such that $\widehat{\nabla}\widehat{m}=\iota_{\widehat{D}}\omega$.
- A class in $\mathrm{H}^1_{\mathrm{dR}}(X,\mathrm{DR}^{\log}(\mathscr{L}_1^\vee))$ is represented by a pair $(\widehat{m}^\vee,\omega^\vee+\widehat{\eta}_1^\vee)$, with $\widehat{m}^\vee\in\mathscr{V}^\vee_{\widehat{D}}$, $\widehat{\eta}_1^\vee\in\Omega^1_{\widehat{D}}(D)\otimes\mathscr{L}^\vee_{0,\widehat{D}}$, and $\omega^\vee\in\Gamma(U,\Omega^1_U\otimes V^\vee)$ such that $\widehat{\nabla}\widehat{m}^\vee=\iota_{\widehat{D}}\omega^\vee-\widehat{\eta}_1^\vee$.

The formula for the product $(\widehat{m},\omega)\cdot(\widehat{m}^\vee,\omega^\vee+\widehat{\eta}_1^\vee)\in\Omega^1_{\widehat{D}}(*D)$ in the Čech complexes is

$$(\widehat{m},\omega)\cdot(\widehat{m}^\vee,\omega^\vee+\widehat{\eta}_1^\vee)=\langle\widehat{m},\iota_{\widehat{D}}\omega^\vee\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing $\mathscr{V}_{\widehat{D}}\otimes\mathscr{V}^\vee_{\widehat{D}}\to\mathscr{O}_{\widehat{D}}(*D)$. The above formula makes it clear that (3.16) *is independent of the choice of the weak Deligne-Malgrange lattice.* This gives assertion (2) of the summary.

3.g. **Sheaves of holomorphic functions on the real blow-up.** Let $\varpi:\widetilde{X}\to X$ denote the real oriented blow-up of X at each point of D. Working on $\widetilde{X}$ makes proofs of local duality easier since all involved de Rham complexes are sheaves (up to a shift), i.e., have cohomology in only one degree. We denote by $\widetilde{\jmath}: U \hookrightarrow \widetilde{X}$ the open inclusion and by $\widetilde{\imath}: \varpi^{-1}(D) \hookrightarrow \widetilde{X}$ the complementary inclusion. *In this section, we use the analytic topology on U and $\widetilde{X}$.*

We start with a local study where X is the open unit disc Δ with coordinate z centered at the origin, D is equal to 0, and $\varpi : \widetilde{\Delta} \simeq S^1 \times [0,1) \to \Delta$ is the real oriented blow-up at the origin in Δ (in other words, we use polar coordinates). It identifies $\widetilde{\Delta} \smallsetminus (S^1 \times \{0\})$ with $\Delta^* = \Delta \smallsetminus \{0\}$. We will simply set $S^1 = \varpi^{-1}(0)$, so that $\widetilde{\Delta} = \Delta^* \sqcup S^1$. We consider the following sheaves on $\widetilde{X}$:

- $\mathscr{A}_{\widetilde{\Delta}}$ is the sheaf of holomorphic functions on $\widetilde{\Delta}$, that is, C^∞ functions on $\widetilde{\Delta}$ satisfying the Cauchy-Riemann equations on $\widetilde{\Delta}$. We thus have $\mathscr{A}_{\Delta^*} = \mathscr{O}_{\Delta^*}$. This is a $\varpi^{-1}\mathscr{O}_\Delta$-module and there is a Taylor map $\mathscr{A}_{\widetilde{\Delta}} \to \varpi^{-1}\mathscr{O}_{\widehat{0}}$.
- $\mathscr{A}^{\mathrm{rd}}_{\widetilde{\Delta}}$ is the kernel of $\mathscr{A}_{\widetilde{\Delta}} \to \varpi^{-1}\mathscr{O}_{\widehat{0}}$ and there is an exact sequence (see [39])

$$0 \longrightarrow \mathscr{A}^{\mathrm{rd}}_{\widetilde{\Delta}} \longrightarrow \mathscr{A}_{\widetilde{\Delta}} \longrightarrow \varpi^{-1}\mathscr{O}_{\widehat{0}} \longrightarrow 0. \tag{3.30}$$

It is easily seen that $\varpi_*\mathscr{A}_{\widetilde{\Delta}} = \mathscr{O}_\Delta$ and $\varpi_*\varpi^{-1}\mathscr{O}_{\widehat{0}} = \mathscr{O}_{\widehat{0}}$. However, $R^1\varpi_*\mathscr{A}_{\widetilde{\Delta}} \neq 0$. Indeed, from the exact sequence (3.30) and the Malgrange-Sibuya theorem (see e.g. [39, Prop. (1.1) p. 54]) one obtains an isomorphism

$$R^1\varpi_*\mathscr{A}_{\widetilde{\Delta}} \xrightarrow{\sim} R^1\varpi_*\varpi^{-1}\mathscr{O}_{\widehat{0}} \simeq \mathscr{O}_{\widehat{0}}.$$

We remedy this non-vanishing by considering instead the sheaf $\mathscr{A}^{\log}_{\widetilde{\Delta}}$, which is by definition the subsheaf of $\widetilde{\jmath}_*\mathscr{O}_{\Delta^*}$ generated, as a $\mathbf{C}$-algebra, by $\mathscr{A}_{\widetilde{\Delta}}$ and any local determination of $\log z$. Clearly, $\mathscr{A}^{\log}_{\widetilde{\Delta}}{}_{|\Delta^*} = \mathscr{O}_{\Delta^*}$. We will thus analyze the sheaf when restricted sheaf-theoretically to S^1.

Let $\mathbf{C}_{S^1}[\log z]$ be the subsheaf of $\mathbf{C}$-algebras of $\widetilde{\imath}^{-1}\widetilde{\jmath}_*\mathscr{O}_{\Delta^*}$ generated by any determination of $\log z$. Clearly, $\mathrm{H}^0(S^1, \mathbf{C}_{S^1}[\log z]) = \mathbf{C}$. On the other hand, if T denotes the monodromy operator, the formula $T((\log z)^k) = (\log z + 2\pi \mathrm{i})^k$ implies that any polynomial in $\log z$ of degree $\leqslant k$ is the image by $T - \mathrm{Id}$ of a polynomial in $\log z$ of degree $\leqslant k+1$. As a consequence, we have $\mathrm{H}^1(S^1, \mathbf{C}_{S^1}[\log z]) = 0$.

The natural morphism $\widetilde{\imath}^{-1}\mathscr{A}_{\widetilde{\Delta}} \otimes_{\mathbf{C}} \mathbf{C}_{S^1}[\log z] \to \widetilde{\imath}^{-1}\mathscr{A}^{\log}_{\widetilde{\Delta}}$ is an isomorphism: it is onto by definition, and one checks that if a polynomial in $\log z$ with coefficients in $\widetilde{\imath}^{-1}\mathscr{A}_{\widetilde{\Delta}}$ is zero as a function on some open set $(\theta - \delta, \theta + \delta) \times [0, \varepsilon)$ (for $\theta \in S^1$ and $0 < \varepsilon, \delta \ll 1$), then each coefficient is zero, by considering the order of growth along S^1.

Lemma 3.31. *The sheaf $\mathscr{A}^{\log}_{\widetilde{\Delta}}$ is $\varpi^{-1}\mathscr{O}_\Delta$-flat and*

$$\varpi_*\mathscr{A}^{\log}_{\widetilde{\Delta}} = \mathscr{O}_\Delta, \qquad R^k\varpi_*\mathscr{A}^{\log}_{\widetilde{\Delta}} = 0 \quad \textit{for } k \geqslant 1.$$

Proof. It is clear that $\mathscr{A}_{\widetilde{\Delta}}$ is $\varpi^{-1}\mathscr{O}_\Delta$-flat, i.e., torsion-free, and flatness of $\mathscr{A}^{\log}_{\widetilde{\Delta}}$ follows, since it is $\mathscr{A}_{\widetilde{\Delta}}$-locally free. The exact sequence (3.30) gives rise to the exact sequence

$$0 \longrightarrow \mathscr{A}^{\mathrm{rd}}_{\widetilde{\Delta}} \longrightarrow \mathscr{A}^{\log}_{\widetilde{\Delta}} \longrightarrow \varpi^{-1}\mathscr{O}_{\widehat{0}} \otimes_{\mathbf{C}} \mathbf{C}_{S^1}[\log z] \longrightarrow 0.$$

By the projection formula, $R\varpi_*(\varpi^{-1}\mathscr{O}_{\widehat{0}} \otimes_{\mathbf{C}} \mathbf{C}_{S^1}[\log z]) = \mathscr{O}_{\widehat{0}}$. The Malgrange-Sibuya theorem implies similarly the isomorphism

$$R^1\varpi_*\mathscr{A}^{\log}_{\widetilde{\Delta}} \simeq R^1\varpi_*(\varpi^{-1}\mathscr{O}_{\widehat{0}} \otimes_{\mathbf{C}} \mathbf{C}_{S^1}[\log z]) \simeq \mathscr{O}_{\widehat{0}} \otimes_{\mathbf{C}} R^1\varpi_*\mathbf{C}_{S^1}[\log z] = 0. \qquad \square$$

It is also useful to introduce the sheaf $\mathscr{A}^{\mathrm{mod}}_{\widetilde{\Delta}}$ of functions having moderate growth along $\varpi^{-1}(0)$. We have the inclusion

$$\mathscr{A}^{\log}_{\widetilde{\Delta}} \subset \mathscr{A}^{\mathrm{mod}}_{\widetilde{\Delta}},$$

and $\mathscr{A}^{\mathrm{mod}}_{\widetilde{\Delta}}$ also satisfies the properties in Lemma 3.31.

On the surface $\widetilde{X}$, the locally defined sheaves $\mathscr{A}^{\log}_{\widetilde{\Delta}_x}$ ($x \in D$) glue with $\mathscr{O}_U$ to give rise to the sheaf $\mathscr{A}^{\log}_{\widetilde{X}}$ with similar properties. The same remark applies to the moderate growth case.

Lemma 3.32. *The covering $(\widetilde{U}_D, U)$ of $\widetilde{X}$ is a Leray covering for $\mathscr{A}^{\log}_{\widetilde{X}}$ and $\mathscr{A}^{\log}_{\widetilde{X}}(-D)$.*

Proof. We have $\mathscr{A}^{\log}_{\widetilde{X}|U}=\mathscr{O}_U$, and similarly with U_D°, and hence both sheaves have no non-zero cohomology on these open sets, and Lemma 3.31 gives the remaining property on $\widetilde{U}_D$. □

The sheaf $\mathscr{A}^{\log}_{\widetilde{X}}$ is endowed with a logarithmic connection $\mathrm{d}:\mathscr{A}^{\log}_{\widetilde{X}}\to\mathscr{A}^{\log}_{\widetilde{X}}\otimes\varpi^{-1}\Omega^1_X(D)$. We denote this complex by $\mathrm{DR}^{\log}(\mathscr{A}^{\log}_{\widetilde{X}})$, and we will be mostly interested by the subcomplex $\mathrm{DR}^{\log}(\mathscr{A}^{\log}_{\widetilde{X}}(-D))$.

Lemma 3.33. *We have*

$$\mathrm{DR}^{\log}(\mathscr{A}^{\log}_{\widetilde{X}}(-D))\simeq\widetilde{\jmath}_!\mathbf{C}_U.$$

Proof. The question is local on $\widetilde{D}$, so we work in the local setting $0\in\Delta$ and we can fix a determination of $\log z$. We are reduced to proving that

$$z\partial_z+1:\mathscr{A}^{\log}_{\widetilde{\Delta}}\longrightarrow\mathscr{A}^{\log}_{\widetilde{\Delta}}$$

is an isomorphism. Since $z\partial_z$ strictly decreases the degree in $\log z$, the assertion follows from Lemma 3.34 below. □

Lemma 3.34. *The morphism $z\partial_z+1:\mathscr{A}_{\widetilde{\Delta}}\to\mathscr{A}_{\widetilde{\Delta}}$ is bijective.*

Proof. We write $z\partial_z+1=\partial_z z$ and it is enough to check the assertion on the extreme terms of (3.30). For the right term, it is enough to check the assertion on $\mathbf{C}[\![z]\!]$, for which it is obvious. For the left term, since z acts bijectively on $\mathscr{A}^{\mathrm{rd}}_{\widetilde{\Delta}}$, it is enough to prove that $\partial_z:\mathscr{A}^{\mathrm{rd}}_{\widetilde{\Delta}}\to\mathscr{A}^{\mathrm{rd}}_{\widetilde{\Delta}}$ is bijective. Injectivity is obvious, since non-zero constant functions do not have rapid decay. Surjectivity is proved e.g. in [39, Th. (1.3) p. 55]. □

3.h. **De Rham complexes on the real blow-up.** For $\mathscr{V}$ as in Section 3.a, we set

$$\widetilde{\mathscr{V}}=\mathscr{A}_{\widetilde{X}}\otimes_{\varpi^{-1}\mathscr{O}_X}\mathscr{V},$$

which is a locally free $\mathscr{A}_{\widetilde{X}}(*D)$-module of finite rank. The sheaf $\mathscr{A}_{\widetilde{X}}$ is acted on by $\varpi^{-1}\Theta_X$, so one can define a de Rham complex $\mathrm{DR}(\widetilde{\mathscr{V}})$ on $\widetilde{X}$ as

$$\widetilde{\mathscr{V}}\xrightarrow{\ \nabla\ }\varpi^{-1}\Omega^1_X\otimes_{\varpi^{-1}\mathscr{O}_X}\widetilde{\mathscr{V}}.$$

One defines similarly the de Rham complexes with growth conditions $\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}})$ and $\mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}})$. Recall (see e.g. [50, Cor. I.1.1.8]) that $\boldsymbol{R}\varpi_*\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}}=\mathscr{O}_X(*D)$. The following proposition gathers the main known properties of the de Rham complexes (see [39]).

Proposition 3.35 (De Rham complexes on $\widetilde{X}$ and X)**.**

(1) *The complexes* $\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}})$ *and* $\mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}})$ *have cohomology in degree zero at most.*
(2) *We have*

$$\boldsymbol{R}\varpi_*\,\mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}})\simeq\mathrm{DR}\,\mathscr{V}.$$

(3) $\boldsymbol{R}\varpi_*\,\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}})$ *is quasi-isomorphic to the cone of* $\mathrm{DR}\,\mathscr{V}\to\iota_{D*}\,\mathrm{DR}\,\mathscr{V}_{\widehat{D}}$ *(in particular,* $\boldsymbol{R}\varpi_*\,\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}})$ *is another realization of* $\mathrm{DR}^{\mathrm{an}}(\mathscr{V}(!D))$*, see Proposition* 3.2*).*

This proposition allows for a computation on $\widetilde{X}$ of the various algebraic de Rham cohomologies.

Corollary 3.36. *We have natural isomorphisms*

$$\mathrm{H}^k_{\mathrm{dR}}(U,V)\simeq\mathrm{H}^k_{\mathrm{dR,mod}}(\widetilde{X},\widetilde{\mathscr{V}}),\quad\mathrm{H}^k_{\mathrm{dR,c}}(U,V)\simeq\mathrm{H}^k_{\mathrm{dR,rd}}(\widetilde{X},\widetilde{\mathscr{V}}).$$

Computation with weak Deligne-Malgrange latices. In order to obtain similar properties for lattices, we consider de Rham complexes with coefficients in the sheaf $\mathscr{A}^{\log}_{\widetilde{X}}$.

Let $\mathscr{L}_0$ be a weak Deligne-Malgrange lattice of $\mathscr{V}$ and set

$$\widetilde{\mathscr{L}_0} = \mathscr{A}^{\log}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathscr{O}_X} \mathscr{L}_0,$$

which is a locally free $\mathscr{A}^{\log}_{\widetilde{X}}$-module of finite rank. From the projection formula and Lemma 3.31, we obtain

$$\varpi_*\widetilde{\mathscr{L}_0} = \mathscr{L}_0, \quad R^k\varpi_*\widetilde{\mathscr{L}_0} = 0 \quad \text{for } k \geqslant 1. \tag{3.37}$$

Since $\mathscr{A}^{\log}_{\widetilde{X}}$ is endowed with the logarithmic connection lifting d, one can naturally define a logarithmic de Rham complex $\mathrm{DR}^{\log}(\widetilde{\mathscr{L}_0}) = \{\widetilde{\mathscr{L}_0} \xrightarrow{\nabla} \varpi^{-1}\Omega^1_{U_D}(D) \otimes \widetilde{\mathscr{L}_1}\}$ on $\widetilde{X}$ which satisfies, according to (3.37) and Proposition 3.28,

$$\begin{aligned}\varpi_*\,\mathrm{DR}^{\log}(\widetilde{\mathscr{L}_0}) &\xrightarrow{\sim} \boldsymbol{R}\varpi_*\,\mathrm{DR}^{\log}(\widetilde{\mathscr{L}_0}) \simeq \mathrm{DR}^{\log}(\mathscr{L}_0) \simeq \mathrm{DR}^{\mathrm{an}}\,\mathscr{V}\\ \varpi_*\,\mathrm{DR}^{\log}(\widetilde{\mathscr{L}_0}(-D)) &\xrightarrow{\sim} \boldsymbol{R}\varpi_*\,\mathrm{DR}^{\log}(\widetilde{\mathscr{L}_0}(-D)) \simeq \mathrm{DR}^{\log}(\mathscr{L}_0(-D)) \simeq \mathrm{DR}^{\mathrm{an}}\,\mathscr{V}(!D).\end{aligned} \tag{3.38}$$

3.i. **Poincaré-Verdier duality on the real blow-up.** Let $\boldsymbol{k}$ be a field. We will mainly consider subfields of $\mathbf{C}$ such as $\boldsymbol{k} = \mathbf{Q}$. Recall that Verdier's dualizing complex on $\widetilde{X}$ in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{\widetilde{X}})$ is $\mathbf{D}_{\widetilde{X}} = \widetilde{\jmath}_!\boldsymbol{k}_{U^{\mathrm{an}}}[2]$ (see Appendix C for a reminder on Verdier duality). Given a bounded complex of sheaves $\mathscr{F}^\bullet$ of $\boldsymbol{k}$-vector spaces on $\widetilde{X}$, we set

$$\boldsymbol{D}\mathscr{F}^\bullet = \boldsymbol{R}\mathscr{H}om_{\boldsymbol{k}}(\mathscr{F}^\bullet, \widetilde{\jmath}_!\boldsymbol{k}_{U^{\mathrm{an}}}[2]), \quad \boldsymbol{D}'\mathscr{F}^\bullet = \boldsymbol{R}\mathscr{H}om_{\boldsymbol{k}}(\mathscr{F}^\bullet, \widetilde{\jmath}_!\boldsymbol{k}_{U^{\mathrm{an}}}).$$

There is a natural morphism

$$\mathscr{F}^\bullet \otimes \boldsymbol{D}'\mathscr{F}^\bullet \longrightarrow \widetilde{\jmath}_!\boldsymbol{k}_{U^{\mathrm{an}}} \tag{3.39}$$

in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ and Poincaré-Verdier duality theorem implies that the induced pairing

$$\boldsymbol{H}^i(\widetilde{X}, \mathscr{F}^\bullet) \otimes \boldsymbol{H}^{2-i}(\widetilde{X}, \boldsymbol{D}'\mathscr{F}^\bullet) \longrightarrow \mathrm{H}^2_{\mathrm{c}}(U^{\mathrm{an}}, \boldsymbol{k})$$

is non-degenerate for any i (see Corollary C.6).

Example 3.40 (of local computation of the Verdier dual)**.** Let $\theta_o \in S^1$. We consider an open neighbourhood $\mathrm{nb}(\theta_o) = (\theta_o - \delta, \theta_o + \delta) \times [0, \varepsilon)$ of θ_o in $\widetilde{X}$ and we set $\mathrm{nb}(\theta_o)^\circ = \mathrm{nb}(\theta_o) \cap U = (\theta_o - \delta, \theta_o + \delta) \times (0, \varepsilon)$. We consider the open inclusions

$$\mathrm{nb}(\theta_o)^\circ \xhookrightarrow{\alpha^+} \mathrm{nb}(\theta_o)^\circ \cup \big((\theta_o, \theta_o + \delta) \times \{0\}\big) \xhookrightarrow{\beta^+} \mathrm{nb}(\theta_o) = \mathrm{nb}(\theta_o)^\circ \cup \big((\theta_o - \delta, \theta_o + \delta) \times \{0\}\big)$$

and

$$\mathrm{nb}(\theta_o)^\circ \xhookrightarrow{\alpha^-} \mathrm{nb}(\theta_o)^\circ \cup \big((\theta_o - \delta, \theta_o) \times \{0\}\big) \xhookrightarrow{\beta^-} \mathrm{nb}(\theta_o).$$

Let $\mathcal{V}$ be a locally constant sheaf of $\boldsymbol{k}$-vector spaces on $\mathrm{nb}(\theta_o)^\circ$ and let $\mathcal{V}^\vee$ be the dual local system. The natural non-degenerate pairing

$$\mathcal{V} \otimes_{\boldsymbol{k}} \mathcal{V}^\vee \longrightarrow \boldsymbol{k}_{\mathrm{nb}(\theta_o)^\circ}$$

extends in a unique way to a pairing

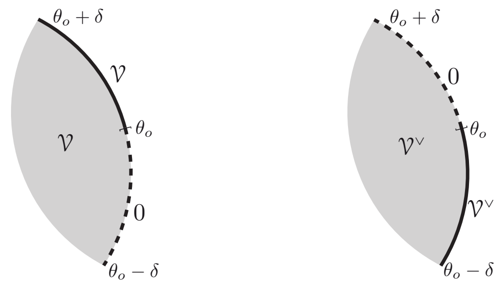
$$\beta^+_!\alpha^+_*\mathcal{V} \otimes_{\boldsymbol{k}} \beta^-_!\alpha^-_*\mathcal{V}^\vee \longrightarrow \widetilde{\jmath}_!\boldsymbol{k}_{\mathrm{nb}(\theta_o)^\circ}.$$

Indeed, at each point $(\theta, 0)$, the germ of one of the sheaves on the left-hand side is zero.

We claim that *this pairing is isomorphic to the Poincaré-Verdier pairing* (3.39), in the sense that it induces an isomorphism

$$\beta^-_!\alpha^-_*\mathcal{V}^\vee \simeq \boldsymbol{D}'\beta^+_!\alpha^+_*\mathcal{V}.$$

Indeed, it is standard to see that $\boldsymbol{D}'\beta^+_!\alpha^+_*\mathcal{V} \simeq \beta^+_*\alpha^+_!\mathcal{V}^\vee$. Both sheaves $\beta^-_!\alpha^-_*\mathcal{V}^\vee$ and $\beta^+_*\alpha^+_!\mathcal{V}^\vee$ are subsheaves of $\widetilde{\jmath}_*\mathcal{V}^\vee$ that clearly coincide away from $\theta_o \times \{0\}$, and one checks that the germ of both sheaves at $\theta_o \times \{0\}$ are zero.



$\beta_!^+\alpha_*^+\mathcal{V}$ $\qquad$ $\beta_!^-\alpha_*^-\mathcal{V}^\vee$

3.j. **De Rham realization of the duality on the real blow-up.** It follows from Proposition 3.35(1) that

$$\mathrm{DR}^{\mathrm{rd}}(\mathscr{A}_{\widetilde{X}}(*0)) \simeq \widetilde{\jmath}_!\mathbf{C}_{U^{\mathrm{an}}}.$$

Proposition 3.41. *There exists a unique isomorphism*

$$\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}}) \simeq \boldsymbol{D}'\,\mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}}^\vee)$$

extending the natural one on U^{an}. It is induced by the natural pairing

$$\widetilde{\mathrm{PV}} : \mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}}) \otimes \mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}}^\vee) \longrightarrow \mathrm{DR}^{\mathrm{rd}}(\mathscr{A}_{\widetilde{X}}(*D)),$$

which thus coincides with the Poincaré-Verdier pairing (3.39).

Proof. As the statement is local on $\widetilde{X}$, we work on an analytic neighbourhood Δ of $x \in D$ with coordinate z. The decomposition (3.8) and its refinement known as Levelt-Turrittin decomposition can be lifted locally on $S^1 \times \{0\}$ with coefficients in $\mathscr{A}_{\widetilde{\Delta}}$, so that the ramification ρ can be neglected by considering a local determination of $z^{1/r}$ and $\log z$, where r is the order of ramification. We are in this way reduced to proving the assertions in the case of an elementary model entering the Levelt-Turrittin decomposition. Locally on $S^1 \times \{0\}$ the situation is similar to that of Example 3.40, so the result follows from the identification with Poincaré-Verdier duality proved there. ☐

Corollary 3.42. *The natural pairing*

$$\widetilde{\mathrm{PV}} : \mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}}) \otimes \mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}}^\vee) \longrightarrow \mathrm{DR}^{\mathrm{rd}}(\mathscr{A}_{\widetilde{X}}(*D))$$

induces an isomorphism

$$\mathrm{PV} : \mathrm{DR}^{\mathrm{an}}(\mathscr{V}(!D)) \simeq \boldsymbol{R}\varpi_*\,\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}}) \xrightarrow{\;\sim\;} \boldsymbol{D}'\boldsymbol{R}\varpi_*\,\mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}}^\vee) \simeq \mathrm{DR}^{\mathrm{an}}\,\mathscr{V}^\vee.$$

In particular, it induces a non-degenerate pairing

$$\mathrm{PV}^1 : \mathrm{H}^1_{\mathrm{dR,c}}(U,V) \otimes \mathrm{H}^1_{\mathrm{dR}}(U,V^\vee) \longrightarrow \mathrm{H}^2_{\mathrm{dR,c}}(U^{\mathrm{an}}) \xrightarrow{\;\mathrm{tr}_{U^{\mathrm{an}}}\;} \mathbf{C}.$$

We have used the isomorphisms of Proposition 3.35(2) and (3).

Proof. This follows from compatibility between Verdier duality and proper pushforward by ϖ. ☐

End of the proof of Theorem 3.12. Let $\mathscr{L}_0$ be a weak Deligne-Malgrange lattice of $\mathscr{V}$. The natural pairing

$$\widetilde{\mathrm{PV}}^{\log} : \mathrm{DR}^{\log}(\widetilde{\mathscr{L}_0}(-D)) \otimes \mathrm{DR}^{\log}(\widetilde{\mathscr{L}_1^\vee}) \longrightarrow \mathrm{DR}^{\log}(\mathscr{A}^{\log}_{\widetilde{X}}(-D))$$

fits into a commutative diagram

$$\begin{array}{ccccc}
\mathrm{DR}^{\log}(\widetilde{\mathscr{L}_0}(-D)) & \otimes & \mathrm{DR}^{\log}(\widetilde{\mathscr{L}_1^\vee}) & \xrightarrow{\;\widetilde{\mathrm{PV}}^{\log}\;} & \mathrm{DR}^{\log}(\mathscr{A}^{\log}_{\widetilde{X}}(-D)) \\
\wr\big\uparrow & & \big\downarrow\wr & & \big\uparrow\wr \\
\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}}) & \otimes & \mathrm{DR}^{\mathrm{mod}}(\widetilde{\mathscr{V}}^\vee) & \xrightarrow{\quad\widetilde{\mathrm{PV}}\quad} & \mathrm{DR}^{\mathrm{rd}}(\mathscr{A}_{\widetilde{X}}(*D)).
\end{array}$$

According to (3.38) and Corollary 3.42, $\widetilde{\mathrm{PV}}^{\log}$ induces PV after applying ϖ_* to each term. On the other hand, also according to (3.38), it induces the analytic version of the pairing (3.15). This concludes assertion (4) of the summary, and hence the proof of the theorem. ☐

3.k. **Computation of Betti period pairings.** The formula for computing the period pairing

$$\mathsf{P}_1^{\mathrm{rd,mod}} : \mathrm{H}_1^{\mathrm{rd}}(\widetilde{X}, \widetilde{\mathscr{V}}) \otimes \mathrm{H}^1_{\mathrm{dR,mod}}(\widetilde{X}, \widetilde{\mathscr{V}}^\vee) \longrightarrow \mathbf{C}$$

(see Section 2.d) is the natural one: let $\sigma \in \mathrm{H}_1^{\mathrm{rd}}(\widetilde{X}, \widetilde{\mathscr{V}})$ be represented by a twisted cycle of the form $c\otimes v$, where $c : [0,1] \to \widetilde{X}$ is a piecewise smooth simplex satisfying $c((0,1)) \subset U$ and $c(o) \in \widetilde{D}$ ($o = 0,1$), and v is a horizontal section of $\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}} \otimes \widetilde{\mathscr{V}}$ on a neighbourhood of the support of c, and let $\omega^\vee \in \Gamma(U, \Omega^1_U \otimes V^\vee)$ be a representative of a de Rham class in $\mathrm{H}^1_{\mathrm{dR,mod}}(\widetilde{X}, \widetilde{\mathscr{V}}^\vee) = \mathrm{H}^1_{\mathrm{dR}}(U, V^\vee)$; then

$$\mathsf{P}_1^{\mathrm{rd,mod}}(\sigma, [(\widehat{m}, \omega)]) = \int_0^1 c^*(\langle v, \omega\rangle).$$

We will make explicit the formula for computing the period pairing

$$\mathsf{P}_1^{\mathrm{mod,rd}} : \mathrm{H}_1^{\mathrm{mod}}(\widetilde{X}, \widetilde{\mathscr{V}}) \otimes \mathrm{H}^1_{\mathrm{dR,rd}}(\widetilde{X}, \widetilde{\mathscr{V}}^\vee) \longrightarrow \mathbf{C}$$

starting from a representative $(\widehat{m}, \omega)$ of a de Rham class in $\mathrm{H}^1_{\mathrm{dR,rd}}(\widetilde{X}, \widetilde{\mathscr{V}}^\vee) = \mathrm{H}^1_{\mathrm{dR,c}}(U, V^\vee)$. With respect to the above formula, there is a supplementary regularization procedure to be performed.

Proposition 3.43.

- *Let $\sigma \in \mathrm{H}_1^{\mathrm{mod}}(\widetilde{X}, \widetilde{\mathscr{V}})$ be represented by a twisted cycle of the form $c\otimes v$, where $c : [0,1] \to \widetilde{X}$ is a piecewise smooth simplex satisfying $c((0,1)) \subset U$ and $c(o) \in \widetilde{D}$ ($o = 0,1$), and v is a horizontal section of $\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}} \otimes \widetilde{\mathscr{V}}$ on a neighbourhood of the support of c.*
- *Let $(\widehat{m}, \omega) \in \mathscr{V}_{\widehat{D}} \oplus \Gamma(U, \Omega^1_U \otimes V^\vee)$ be a representative of $[(\widehat{m}, \omega)] \in \mathrm{H}^1_{\mathrm{dR,c}}(U, V^\vee)$.*
- *Let $\widetilde{m}_o$ ($o = 0,1$) be a germ in $\widetilde{\mathscr{V}}^\vee_{c(o)}$ having $\widehat{m}_o$ as asymptotic expansion along $\widetilde{D}$ in the neighbourhood of $c(o)$.*

Then

$$\mathsf{P}_1^{\mathrm{mod,rd}}(\sigma, [(\widehat{m}, \omega)]) = \lim_{\varepsilon\to 0}\biggl[\biggl(\int_\varepsilon^{1-\varepsilon} c^*(\langle v, \omega\rangle)\biggr) + \langle v, \widetilde{m}_0\rangle(c(\varepsilon)) - \langle v, \widetilde{m}_1\rangle(c(1-\varepsilon))\biggr].$$

Remark 3.44. In practice, one only needs to know the first terms of the asymptotic expansion of $\widetilde{m}_o$ in order to compute the limit. The knowledge of $\widehat{m}_o$ up to a finite order is therefore enough.

Proof. We start with the C^∞ setting. We consider the sheaf

$$\widetilde{\mathscr{V}}^\vee_\infty = \mathscr{C}^\infty_{\widetilde{X}} \otimes_{\mathscr{A}_{\widetilde{X}}} \widetilde{\mathscr{V}}^\vee$$

and the corresponding rapid decay complex

$$\mathrm{DR}^{\mathrm{rd}}_\infty(\widetilde{\mathscr{V}}^\vee) = \{\mathscr{E}^{\mathrm{rd},0}_{\widetilde{X}} \otimes \widetilde{\mathscr{V}}^\vee \xrightarrow{\nabla + \overline{\partial}} \mathscr{E}^{\mathrm{rd},1}_{\widetilde{X}} \otimes \widetilde{\mathscr{V}}^\vee \xrightarrow{\nabla + \overline{\partial}} \mathscr{E}^{\mathrm{rd},2}_{\widetilde{X}} \otimes \widetilde{\mathscr{V}}^\vee\}.$$

This complex is a fine resolution of $\mathrm{DR}^{\mathrm{rd}}(\widetilde{\mathscr{V}}^\vee)$, so

$$\mathrm{H}^1_{\mathrm{dR,c}}(U, V^\vee) \simeq \mathrm{H}^1(\Gamma(\widetilde{X}, \mathrm{DR}^{\mathrm{rd}}_\infty(\widetilde{\mathscr{V}}^\vee))). \tag{3.45}$$

A class in the latter space can also be represented by a class in the Čech complex relative to the covering $(\widetilde{U}_D, U)$. It is then represented as the set of data $(m_\infty, \eta_\infty, \omega_\infty)$, where

- $m_\infty \in \Gamma(U^\circ_D, \widetilde{\mathscr{V}}^\vee_{\infty|U^\circ_D})$,
- $\eta_\infty \in \Gamma(\widetilde{U}_D, \mathscr{E}^{\mathrm{rd},1}_{\widetilde{U}_D} \otimes \widetilde{\mathscr{V}}^\vee)$ with $(\nabla + \overline{\partial})\eta_\infty = 0$,
- $\omega_\infty \in \Gamma(U, \mathscr{E}^1_U \otimes \widetilde{\mathscr{V}}^\vee_{|U})$ with $(\nabla + \overline{\partial})\omega_\infty = 0$,

related by

$$(\nabla+\overline{\partial})m_\infty=\omega_{\infty|U_D^\circ}-\eta_{\infty|U_D^\circ},$$

modulo data of the form $(\psi_{|U_D^\circ}-\varphi_{|U_D^\circ},(\nabla+\overline{\partial})\varphi,(\nabla+\overline{\partial})\psi)$, with $\varphi\in\Gamma(\widetilde{U}_D,\mathscr{E}_{\widetilde{X}}^{\mathrm{rd},0}\otimes\widetilde{\mathscr{V}}^\vee)$ and $\psi\in\Gamma(U,\widetilde{\mathscr{V}}_\infty^\vee)$. Equivalently, it consists of the data of a pair (m_∞,ω_∞) as above such that $\omega_{\infty|U_D^\circ}-(\nabla+\overline{\partial})m_\infty$ has rapid decay along $\widetilde{D}$, modulo data of the form $(\psi_{|U_D^\circ}-\varphi_{|U_D^\circ},(\nabla+\overline{\partial})\psi)$.

Lemma 3.46. *Let σ be as in the proposition and let (m_∞,ω_∞) be a representative of a class in $\mathrm{H}^1_{\mathrm{dR,c}}(U,V^\vee)$. Then*

$$\mathsf{P}_1^{\mathrm{mod,rd}}(\sigma,[(m_\infty,\omega_\infty)])=\lim_{\varepsilon\to0}\biggl[\biggl(\int_\varepsilon^{1-\varepsilon}c^*(\langle v,\omega_\infty\rangle)\biggr)+\langle v,m_\infty\rangle(c(\varepsilon))-\langle v,m_\infty\rangle(c(1-\varepsilon))\biggr].$$

Proof. By (3.45), for any class, there is a representative set of data defined from a closed global section in $\Gamma(\widetilde{X},\mathscr{E}_{\widetilde{X}}^{\mathrm{rd},1}\otimes\widetilde{\mathscr{V}}^\vee)$, that is, of the form $(0,\eta'_\infty,\omega'_\infty)$ with $(\nabla+\overline{\partial})\eta'_\infty=0$, $(\nabla+\overline{\partial})\omega'_\infty=0$, and $\eta_{\infty|U_D^\circ}=\omega_{\infty|U_D^\circ}$. Any other representative writes

$$(m_\infty,\eta_\infty,\omega_\infty)=(\psi_{|U_D^\circ}-\varphi_{|U_D^\circ},\eta'_\infty+(\nabla+\overline{\partial})\varphi,\omega'_\infty+(\nabla+\overline{\partial})\psi).\tag{3.47}$$

Let σ be as in the proposition. For the representative $(0,\eta'_\infty,\omega'_\infty)$ as above, the integral $\int_c\langle v,\omega'_\infty\rangle$ exists, since $\langle v,\omega'_\infty\rangle$ has rapid decay along the support of c in the neighbourhood of $\widetilde{D}$.

Since the complex of sheaves $\mathrm{DR}_\infty^{\mathrm{rd}}(\widetilde{\mathscr{V}}^\vee)$ has no $\mathscr{H}^1$ (Proposition 3.35), there exists a section $h'_{\infty,o}$ $(o=0,1)$ of $\mathscr{E}_{\widetilde{X}}^{\mathrm{rd},0}\otimes\widetilde{\mathscr{V}}^\vee$ in the neighbourhood of $c(o)$ such that $\eta'_\infty=(\nabla+\overline{\partial})h'_{\infty,o}$ on this neighbourhood. Let $\varepsilon>0$ be small enough so that $c([0,\varepsilon))$ resp. $c([1-\varepsilon,1])$ is contained in this neighbourhood of $c(0)$ resp. $c(1)$. Then, since $\langle v,h'_{\infty,o}\rangle$ vanishes on $\widetilde{D}$ (because it has rapid decay there), we have

$$\int_c\langle v,\omega'_\infty\rangle=\biggl(\int_\varepsilon^{1-\varepsilon}c^*\langle v,\omega'_\infty\rangle\biggr)+\langle v,h'_{\infty,0}\rangle(c(\varepsilon))-\langle v,h'_{\infty,1}\rangle(c(1-\varepsilon)).$$

Notice that there exists a unique C^∞ function $g_{\infty,o}$ with rapid decay on the neighbourhood of $c(o)$ such that $\mathrm{d}g_{\infty,o}=\langle v,\eta'_\infty\rangle$ on this neighbourhood (because the complex of sheaves $\mathrm{DR}_\infty^{\mathrm{rd}}(\mathscr{A}_{\widetilde{X}})\simeq\widetilde{\jmath}_!\mathbf{C}_{\widetilde{X}}$ is quasi-isomorphic to zero when restricted to $\widetilde{D}$). Hence, while fixing $h'_{\infty,o}$ can involve a choice, the expression above is independent of any such choice. It is also independent of the choice of ε, as long as it is sufficiently small.

In terms of the representative (3.47), and setting $h_{\infty,o}=h'_{\infty,o}+\varphi$, this integral reads

$$\begin{aligned}\int_c\langle v,\omega'_\infty\rangle&=\biggl(\int_\varepsilon^{1-\varepsilon}c^*\langle v,\omega_\infty\rangle\biggr)-\biggl(\int_\varepsilon^{1-\varepsilon}c^*\mathrm{d}\langle v,\psi\rangle\biggr)+\langle v,h_{\infty,0}\rangle(c(\varepsilon))-\langle v,h_{\infty,1}\rangle(c(1-\varepsilon))\\&\qquad-\langle v,\varphi\rangle(c(\varepsilon))+\langle v,\varphi\rangle(c(1-\varepsilon))\\&=\biggl(\int_\varepsilon^{1-\varepsilon}c^*\langle v,\omega_\infty\rangle\biggr)+\langle v,m_\infty+h_{\infty,0}\rangle(c(\varepsilon))-\langle v,m_\infty+h_{\infty,1}\rangle(c(1-\varepsilon)).\end{aligned}$$

Since $\langle v,h_{\infty,o}\rangle$ has rapid decay, we obtain

$$\int_c\langle v,\omega'_\infty\rangle=\lim_{\varepsilon\to0}\biggl[\biggl(\int_\varepsilon^{1-\varepsilon}c^*\langle v,\omega_\infty\rangle\biggr)+\langle v,m_\infty\rangle(c(\varepsilon))-\langle v,m_\infty\rangle(c(1-\varepsilon))\biggr].\qquad\square$$

Let us come back to the proposition. By Borel-Ritt's lemma, one can find $\mu_\infty\in\Gamma(U_D,\mathscr{V}_\infty^\vee)$ with Taylor series at D equal to $\widehat{m}$. Let $[\mu_\infty]$ denote the class of μ_∞ in $\Gamma(U_D,\mathscr{V}_\infty^\vee/\mathscr{V}_{-1,\infty}^\vee)$. If we regard ω as in $\Gamma(X,\Omega_X^1\otimes\mathscr{V}^\vee)\subset\Gamma(X,\mathscr{E}_X^1\otimes\mathscr{V}_\infty^\vee)$, the pair $([\mu_\infty],\omega)$ defines an element of $\mathrm{H}^1_{\mathrm{dR,c}}(X,V)$ through the C^∞ analogue of (3.6).

Lemma 3.48. *The class of $([\mu_\infty],\omega)$ is equal to the class of $([\widehat{m}],\omega)$*

Proof. Since $\mathscr{V}^\vee/\mathscr{V}_{-1}^\vee$ is supported on D, we have $\mathscr{V}_\infty^\vee/\mathscr{V}_{-1,\infty}^\vee=\mathscr{V}^\vee/\mathscr{V}_{-1}^\vee$, and thus $[\mu_\infty]=[\widehat{m}]$. $\square$

End of the proof of Proposition 3.43. Let m_∞ denote the lift of μ_∞ in $\widetilde{X}$. By the previous lemma, the formula of Lemma 3.46 computes $\mathsf{P}_1^{\mathrm{mod,rd}}(\sigma,[(\widehat{m},\omega)])$, with $\omega_\infty=\omega$. Let $\widetilde{m}_o$ be as in the proposition. Then $\widetilde{m}_o-m_\infty$ has rapid decay along $\widetilde{D}$ in the neighbourhood of $c(o)$, so we can replace m_∞ with $\widetilde{m}_o$ in the formula of Lemma 3.46, giving the desired formula. □

3.l. **Quadratic relations in dimension one.** We summarize the consequences of the identifications previously obtained in this section to the form of quadratic relations in dimension one. The setting is as in Section 3.a. We assume that (V,∇) is endowed with a non-degenerate pairing $\langle\bullet,\bullet\rangle:(V,\nabla)\otimes(V,\nabla)\to(\mathscr{O}_U,\mathrm{d})$. We assume that it is symmetric or skew-symmetric, that we denote by $\pm$-symmetric. We will make explicit the way to express middle quadratic relations (Corollaries 2.29, 2.16, and Remark 2.17) in the present setting.

Middle de Rham pairing. Let d be the dimension of $\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)=\mathrm{im}\big[\mathrm{H}^1_{\mathrm{dR,c}}(U,V)\to\mathrm{H}^1_{\mathrm{dR}}(U,V)\big]$.

- We choose d elements $\omega_1,\dots,\omega_d$ in $\mathrm{H}^1_{\mathrm{dR}}(U,V)$ so that one can solve $\nabla\widehat{m}_i=\iota_{\widehat{D}}\omega_i$ for each i and at each point of D, with $\widehat{m}_i\in V_{\widehat{D}}$. We choose such a solution $\widehat{m}_i$.
- The matrix $\mathsf{S}^1_{\mathrm{mid}}$ of the de Rham pairing with respect to these families has entries $\sum_{x\in D}\mathrm{res}_x\langle\widehat{m}_i,\iota_{\widehat{D}}\omega_j\rangle$. It is $\mp$-symmetric.
- If $\det\mathsf{S}^1_{\mathrm{mid}}\neq0$, then the family (ω_i) is a basis of $\mathrm{H}^1_{\mathrm{dR,mid}}(U,V)$.

Middle Betti pairing. $\mathrm{H}_1^{\mathrm{mid}}(U,V)=\mathrm{im}\big[\mathrm{H}_1^{\mathrm{rd}}(U,V)\to\mathrm{H}_1^{\mathrm{mod}}(U,V)\big]$ has also dimension d.

- We choose d elements $\beta_1,\dots,\beta_d$ in $\mathrm{H}_1^{\mathrm{mod}}(U,V)$ which are the images of elements $\alpha_1,\dots,\alpha_d$ in $\mathrm{H}_1^{\mathrm{rd}}(U,V)$.
- The matrix $\mathsf{B}_1^{\mathrm{mid}}$ of the Betti pairing with respect to these families is computed for example by means of Proposition 2.26. It is $\mp$-symmetric.
- If $\det\mathsf{B}_1^{\mathrm{mid}}\neq0$, then the family (β_i) is a basis of $\mathrm{H}_1^{\mathrm{mid}}(U,V)$.

Middle period pairing. Let us fix a triangulation of (X,D) that is induced by a triangulation of $\widetilde{X}$ such that $\mathrm{DR}^{\mathrm{rd}}\mathscr{V}$ and $\mathrm{DR}^{\mathrm{mod}}\mathscr{V}$ are constant on each open 1-simplex. Then all Stokes directions for $\mathscr{V}$ at a point of D are realized as the direction of a 1-simplex abutting to this point. We assume the simplices are given an orientation. Let us write each β_i as $\sum_\ell c_\ell\otimes v_{i,\ell}$, where $c_\ell:[0,1]\to X$ runs among the oriented 1-simplices of the chosen triangulation, and $v_{i,\ell}$ is a (possibly zero) section of $\mathscr{V}^{\mathrm{an}}$ with moderate growth in the neighbourhood of c_ℓ. The cycle condition reads:

$$\text{for each vertex } x\in U^{\mathrm{an}},\qquad \sum_{\ell\mid c_\ell(1)=x}v_{i,\ell}(x)-\sum_{\ell\mid c_\ell(0)=x}v_{i,\ell}(x)=0.$$

The middle period matrix $\mathsf{P}^1_{\mathrm{mid}}$ is the $d\times d$ matrix with entries

$$\mathrm{Pf}\sum_\ell\int_{c_\ell}\langle v_{i,\ell},\omega_j\rangle,$$

where the “finite part” Pf means that, equivalently,

- either we replace β_i with α_i that we realize as $\sum_\ell c_\ell\otimes v'_{i,\ell}$, where $v'_{i,\ell}$ has rapid decay near the boundary points of c_ℓ that are contained in D and satisfy the cycle condition, and
$$\mathrm{Pf}\int_{c_\ell}\langle v_{i,\ell},\omega_j\rangle=\int_{c_\ell}\langle v'_{i,\ell},\omega_j\rangle,$$
- or we replace ω_j with $(\widehat{m}_j,\omega_j)$, for each ℓ such that $c_\ell(o)\in D$ $(o=0,1)$, we choose a germ $\widetilde{m}^o_{j,\ell}$ as in Proposition 3.43, and we set
$$\mathrm{Pf}\int_{c_\ell}\langle v_{i,\ell},\omega_j\rangle=\lim_{\varepsilon\to0}\Big[\int_\varepsilon^{1-\varepsilon}c_\ell^*(\langle v_{i,\ell},\omega_j\rangle)+\langle v_{i,\ell},\widetilde{m}^0_{j,\ell}\rangle(c_\ell(0))-\langle v_{i,\ell},\widetilde{m}^1_{j,\ell}\rangle(c_\ell(1))\Big].$$

Middle quadratic relations. They now read (see Corollary 2.29)

$$\mp(2\pi\mathsf{i})\,\mathsf{B}_1^{\mathrm{mid}}=\mathsf{P}_1^{\mathrm{mid}}\cdot(\mathsf{S}^1_{\mathrm{mid}})^{-1}\cdot{}^t\mathsf{P}_1^{\mathrm{mid}}.\tag{3.49}$$

Appendix A. Twisted singular chains

In this section, we recall classical results from the Cartan Seminars [5, 6]. However, we do not use homology of cosheaves as in [4]. We assume that X is a compact topological space, so that all locally finite open coverings are finite. All sheaves are sheaves of vector spaces over some field $\boldsymbol{k}$, in order to avoid any problem with torsion.

A.a. **Presheaves and sheaves.** Let $\widetilde{\mathscr{F}}$ be a presheaf on X and let $\mathscr{F}$ be the associated sheaf. We have $\mathscr{F}_x = \varinjlim_{U\ni x}\widetilde{\mathscr{F}}(U)$. For every open set $U\subset X$, there is a natural morphism $\widetilde{\mathscr{F}}(U)\to\coprod_{x\in U}\mathscr{F}_x$. We denote by $\widetilde{s}_x\in\mathscr{F}_x$ the germ of a section $\widetilde{s}\in\widetilde{\mathscr{F}}(X)$. If $\widetilde{s}_x=0$, there exists U_x such that the image of $\widetilde{s}$ in $\widetilde{\mathscr{F}}(U_x)$ vanishes. It follows that the support of $\widetilde{s}$, that is, the subset $\{x\in X\mid\widetilde{s}_x=0\}$ is closed.

- We say that $\widetilde{\mathscr{F}}$ has the *surjectivity property* if

$$\text{for all } x\in X,\quad \widetilde{\mathscr{F}}(X)\longrightarrow\mathscr{F}_x \text{ is surjective.} \tag{A.1}$$

  This property is also called Φ-softness, for Φ the family of closed subsets of X consisting of points only.
- We say that $\widetilde{\mathscr{F}}$ has the *injectivity property* if

$$\widetilde{\mathscr{F}}(X)\longrightarrow\coprod_{x\in X}\mathscr{F}_x \text{ is injective.} \tag{A.2}$$

  The latter condition is equivalent to asking that the natural morphism $\widetilde{\mathscr{F}}(X)\to\Gamma(X,\mathscr{F})$ is injective, since (A.2) factorizes through the latter.
- We say that $\widetilde{\mathscr{F}}$ is a *fine presheaf* if, for any (locally) finite open covering $\mathscr{U}=(U_i)$ of X, there exist closed subsets $F_i\subset U_i$ and endomorphisms $\widetilde{\ell}_i$ of $\widetilde{\mathscr{F}}$ (i.e., each $\widetilde{\ell}_i$ is a family, indexed by open sets U, of endomorphisms $\widetilde{\ell}_i:\widetilde{\mathscr{F}}(U)\to\widetilde{\mathscr{F}}(U)$ compatible with restrictions $U'\subset U$ in an obvious way) such that
  (a) $\widetilde{\ell}_i=0$ on $\widetilde{\mathscr{F}}(U)$ if $U\cap F_i=\varnothing$,
  (b) for any open set U and each $\widetilde{s}\in\widetilde{\mathscr{F}}(U)$, the sum $\sum_i\widetilde{\ell}_i(\widetilde{s})$ exists in $\widetilde{\mathscr{F}}(U)$ and is equal to $\widetilde{s}$. Let $\ell_i:\mathscr{F}\to\mathscr{F}$ be the endomorphism associated with $\widetilde{\ell}_i$. If $x\notin F_i$, there exist $U\ni x$ such that $U\cap F_i=\varnothing$, and hence $\ell_{i,x}=0:\mathscr{F}_x\to\mathscr{F}_x$. Moreover, for each U and $s\in\Gamma(U,\mathscr{F})$, we have, for any $x\in U$, the relation $\sum_i\ell_{i,x}(s_x)=s_x$ (the sum is finite since the covering is finite).

Proposition A.3. *Assume that $\widetilde{\mathscr{F}}$ is a fine presheaf that satisfies the surjectivity property* (A.1)*. Then the natural morphism*

$$\widetilde{\mathscr{F}}(X)\longrightarrow\Gamma(X,\mathscr{F})$$

is onto. If it moreover satisfies the injectivity property (A.2)*, then it is an isomorphism.*

Proof. Once the first statement is proved, the second statement is obvious. Let $s\in\Gamma(X,\mathscr{F})$. By (A.1), for any $y\in X$, there exists a section $\widetilde{s}^y\in\widetilde{\mathscr{F}}(X)$ whose germ at y is s_y. Then there exists a neighbourhood U_y of y such that $\widetilde{s}^y_x=s_x$ for any $x\in U_y$. Let $\mathscr{U}$ be a finite open covering of X such that each open set U_i is of the form U_y as above. Let $\widetilde{s}^i\in\widetilde{\mathscr{F}}(X)$ be such that $\widetilde{s}^i_x=s_x$ for any $x\in U_i$. On the one hand, we have $\ell_{i,x}(s_x)=(\widetilde{\ell}_i(\widetilde{s}^i))_x$ for any $x\in U_i$, and on the other hand $s_x=\sum_{i\mid U_i\ni x}\ell_{i,x}(s_x)$, whence

$$s_x=\sum_{i\mid U_i\ni x}(\widetilde{\ell}_i(\widetilde{s}^i))_x=\sum_i(\widetilde{\ell}_i(\widetilde{s}^i))_x,$$

since $(\widetilde{\ell}_i(\widetilde{s}^i))_x=0$ for $x\notin U_i$ according to (a) above. We conclude, since the covering is finite,

$$s=\sum_i\operatorname{im}(\widetilde{\ell}_i(\widetilde{s}^i))=\operatorname{im}\Bigl(\sum_i\widetilde{\ell}_i(\widetilde{s}^i)\Bigr),$$

where we consider the image of the morphism $\widetilde{\mathscr{F}}(X) \to \Gamma(X, \mathscr{F})$. It follows that the natural morphism $\widetilde{\mathscr{F}}(X) \to \Gamma(X, \mathscr{F})$ is surjective. □

A.b. **Homotopically fine sheaves.** See [8, §8].

Definition A.4. Let $(\widetilde{\mathscr{F}}, \mathrm{d})$ be a **Z**-graded differential presheaf on X. We say that $\widetilde{\mathscr{F}}$ is homotopically fine if, for any (locally) finite open covering $\mathscr{U} = (U_i)$ of X, there exist closed subsets F_i contained in U_i and endomorphisms $\widetilde{\ell}_i$ and $\widetilde{k}$ of $\widetilde{\mathscr{F}}$ such that

(1) $\widetilde{\ell}_i = 0 : \widetilde{\mathscr{F}}(U) \to \widetilde{\mathscr{F}}(U)$ if $U \cap F_i = \varnothing$,
(2) $\sum_i \widetilde{\ell}_i = \mathrm{Id} + \widetilde{k}\mathrm{d} + \mathrm{d}\widetilde{k}$.

We have a similar definition for sheaves. We note the following properties:

(a) If a graded differential presheaf is homotopically fine, the associated graded differential sheaf is also homotopically fine.
(b) If $(\widetilde{\mathscr{F}}, \mathrm{d})$ is homotopically fine and if $\widetilde{\mathscr{G}}$ is any presheaf, then the graded differential presheaf $(\widetilde{\mathscr{F}} \otimes \widetilde{\mathscr{G}}, \mathrm{d} \otimes \mathrm{Id})$ is also homotopically fine. Recall that $\widetilde{\mathscr{F}} \otimes \widetilde{\mathscr{G}}$ is the presheaf $U \mapsto \widetilde{\mathscr{F}}(U) \otimes \widetilde{\mathscr{G}}(U)$. The same property holds for graded differential sheaves.

Theorem A.5 ([9, §3, Corollaire])**.** *Let $(\mathscr{F}, \mathrm{d})$ be a homotopically fine* **Z***-graded differential sheaf. For each p, the natural morphism*

$$\mathrm{H}^p(\Gamma(X, \mathscr{F}), \mathrm{d}) \longrightarrow \boldsymbol{H}^p(X, (\mathscr{F}, \mathrm{d}))$$

is an isomorphism.

The proof is obtained by choosing a resolution $C^\bullet$ of $\boldsymbol{k}_X$ by fine sheaves and realizing $\boldsymbol{H}^p(X, (\mathscr{F}, \mathrm{d}))$ as $\mathrm{H}^p(\Gamma(X, (C \otimes \mathscr{F})^\bullet)$, where $(C \otimes \mathscr{F})^\bullet$ stands for the simple complex associated with the double complex $C^p \otimes \mathscr{F}^q$. The assertion follows from the degeneration at E_1 of the second spectral sequence, since the second filtration is regular. We can complete this theorem in terms of presheaves as follows.

Proposition A.6. *Let $(\widetilde{\mathscr{F}}, \mathrm{d})$ be a homotopically fine* **Z***-graded differential presheaf satisfying* (A.1) *and* (A.2)*. Then the natural morphism*

$$\mathrm{H}^p(\widetilde{\mathscr{F}}(X), \mathrm{d}) \longrightarrow \boldsymbol{H}^p(X, (\mathscr{F}, \mathrm{d}))$$

is an isomorphism.

Proof. According to Theorem A.5, it is enough to prove that $(\widetilde{\mathscr{F}}(X), \mathrm{d}) \to (\Gamma(X, \mathscr{F}), \mathrm{d})$ induces an isomorphism in cohomology.

- Let $s \in \Gamma(X, \mathscr{F})$ be such that $\mathrm{d}s = 0$, and let $\widetilde{s}^i$ be as in the proof of Proposition A.3. Recall that $\ell_{i,x}(s_x) = (\widetilde{\ell}_i(\widetilde{s}^i))_x$ for any $x \in U_i$. Besides, $s_x = \sum_{i \mid U_i \ni x} \ell_{i,x}(s_x) - \mathrm{d}ks_x$ since s is d-closed. Hence

$$s_x + \mathrm{d}ks_x = \sum_{i \mid U_i \ni x} (\widetilde{\ell}_i(\widetilde{s}^i))_x = \sum_i (\widetilde{\ell}_i(\widetilde{s}^i))_x,$$

since $(\widetilde{\ell}_i(\widetilde{s}^i))_x = 0$ for $x \notin U_i$ according to (a) above. Since the covering is finite, we get

$$s + \mathrm{d}ks = \sum_i \mathrm{im}(\widetilde{\ell}_i(\widetilde{s}^i)) = \mathrm{im}\Big(\sum_i \widetilde{\ell}_i(\widetilde{s}^i)\Big).$$

Since $\mathrm{d}(s - \mathrm{d}ks) = 0$, (A.2) implies that $\sum_i \widetilde{\ell}_i(\widetilde{s}^i)$ is closed, which gives the surjectivity of the cohomological map.

- Let $\widetilde{s}$ be a d-closed section of $\widetilde{\mathscr{F}}(X)$ whose image $s \in \Gamma(X, \mathscr{F})$ satisfies $s = \mathrm{d}\sigma$ for some $\sigma \in \Gamma(X, \mathscr{F})$. Up to replacing σ with $\sigma + (\mathrm{d}k + k\mathrm{d})\sigma$ (and $\widetilde{s}, s$ with $\widetilde{s} + \mathrm{d}\widetilde{k}\widetilde{s}$, $s + \mathrm{d}ks$) we can assume that σ is the image of some $\widetilde{\sigma} \in \widetilde{\mathscr{F}}(X)$ as argued above. The image of $\widetilde{s} - \mathrm{d}\widetilde{\sigma}$ in $\Gamma(X, \mathscr{F})$ is zero, and hence $\widetilde{s} = \mathrm{d}\widetilde{\sigma}$, according to (A.2). This proves the injectivity of the cohomological map. □

A.c. **Homotopy operator for singular chains.** See [7, §3].

Let $(S_\bullet(X), \partial)$ be the complex of singular chains. Recall that $S_\bullet(X)$ is the infinite-dimensional vector space having as a basis the set of singular simplices (continuous maps from a simplex to X). The support of a singular simplex is its image, which is a compact subset of X. For a closed subset $Z \subset X$, we regard $S_\bullet(Z)$ as the subspace of $S_\bullet(X)$ generated by simplices with support in Z and $S_\bullet(X)/S_\bullet(Z)$ as the subspace generated by simplices whose support is not contained in Z. For any (locally) finite open covering $\mathscr{U} = (U_i)$, there exist (see loc. cit.) endomorphisms ℓ and k of $S_\bullet(X)$ (where ℓ preserves the grading and k has degree one) such that

(A.7)
- for any simplex $\sigma \in S_\bullet(X)$, $\ell(\sigma)$ is a sum of simplices, each one being contained in some $U_i \cap |\sigma|$,
- if $|\sigma|$ is contained in some U_i, then $\ell(\sigma) = \sigma$,
- $\ell = \mathrm{Id} + k\partial + \partial k$ (in particular, $\ell\partial = \partial\ell$).

For Z closed in X, let $(\widetilde{\mathcal{C}}_{X,Z,\bullet}, \partial)$ be the chain presheaf $U \mapsto S_\bullet(X)/S_\bullet((X \smallsetminus U) \cup Z)$ and let $(\mathcal{C}_{X,Z,\bullet}, \partial)$ be the associated differential sheaf (with the usual convention $(\mathcal{C}^{-\bullet}_{X,Z}, \mathrm{d}) = (\mathcal{C}_{X,Z,\bullet}, \partial)$, we obtain a $(-\mathbf{N})$-graded differential sheaf). For any $x \in X$, we have

$$\mathcal{C}_{X,Z,\bullet,x} = S_\bullet(X)/S_\bullet((X \smallsetminus \{x\}) \cup Z).$$

More precisely, $\mathcal{C}_{X,Z,\bullet,x}$ is identified with the subspace of $S_\bullet(X)$ with basis consisting of those singular simplices whose support contains x and is not contained in Z. Since $\widetilde{\mathcal{C}}_{X,Z,\bullet}(X) = S_\bullet(X)/S_\bullet(Z)$, properties (A.1) and (A.2) obviously hold.

Proposition A.8 ([8, p. 8])**.** *Given a finite open covering $\mathscr{U}$ of X, there exist a closed covering (F_i) with $F_i \subset U_i$ for all i and endomorphisms $\widetilde{\ell}_i$ and $\widetilde{k}$ of the presheaf $\widetilde{\mathcal{C}}_{X,Z,\bullet}$ such that*

1. *$\widetilde{\ell}_i = 0 : \widetilde{\mathscr{F}}(U) \to \widetilde{\mathscr{F}}(U)$ if $U \cap F_i = \varnothing$, and in particular $\widetilde{\ell}_i$ induces the zero map on each germ $\mathscr{F}_x$ with $x \notin F_i$,*
2. *for any singular simplex σ, each simplex component τ of $\widetilde{\ell}_i(\sigma)$ satisfies $|\tau| \subset F_i \cap |\sigma|$,*
3. *$\sum_i \widetilde{\ell}_i = \mathrm{Id} + \widetilde{k}\mathrm{d} + \mathrm{d}\widetilde{k}$.*

Corollary A.9. *The presheaf $(\widetilde{\mathcal{C}}_{X,Z,\bullet}, \partial)$ is homotopically fine.* □

It follows that the chain complex $(\mathcal{C}_{X,Z,\bullet}, \partial)$ is homotopically fine.

Proof of Proposition A.8. Let $(U_i)_{i \in I}$ be a finite open covering of X. One can find closed subsets $F_i \subset U_i$ such that $(F_i)_{i \in I}$ is a closed covering of X. We fix a total order on I and we write $I = \{1, \dots, n\}$. We also fix ℓ and k as above.

Let σ be a simplex in $S_\bullet(X)$. Then we define $\widetilde{\ell}_i(\sigma)$ inductively on $i = 1, \dots, n$ as the sum of components of the chain $\ell(\sigma) - \sum_{j<i} \widetilde{\ell}_j(\sigma)$ whose underlying simplices have support in F_i. The support is also contained in $|\sigma|$. We can then extend the definition to any finite chain $\sigma \in S_\bullet(X)$. Then $\ell(\sigma) = \sum_i \widetilde{\ell}_i(\sigma)$ in $S_\bullet(X)$ and $\sum_i \widetilde{\ell}_i(\sigma) = \sigma + k\partial\sigma + \partial k\sigma$.

Let U be an open subset of X. Since ℓ, k, and ∂ preserve the support, the above construction also preserves $S((X \smallsetminus U) \cup Z)$, and thus defines endomorphisms $\widetilde{\ell}_i$ and $\widetilde{k}$ of $\widetilde{\mathcal{C}}_{X,Z,\bullet}(U)$ satisfying the desired properties. □

A.d. **Singular chains with coefficients in a sheaf.** We make a statement of [8, p. 8] precise. Let $\mathscr{F}$ be any sheaf. We denote by $\widetilde{\mathcal{C}}_{X,Z,\bullet}(\mathscr{F})$ the presheaf $U \mapsto \widetilde{\mathcal{C}}_{X,Z,\bullet}(U) \otimes \Gamma(U,\mathscr{F})$ and by $\mathcal{C}_{X,Z,\bullet}(\mathscr{F})$ the associated sheaf $\mathcal{C}_{X,Z,\bullet} \otimes \mathscr{F}$. By Corollary A.9 and (a) and (b) after Definition A.4, both $\widetilde{\mathcal{C}}_{X,Z,\bullet}(\mathscr{F})$ and $\mathcal{C}_{X,Z,\bullet}(\mathscr{F})$ are homotopically fine. These are chain complexes indexed by $\mathbf{N}$, when equipped with the boundary $\partial \otimes \mathrm{Id}$, that we simply denote by ∂. As usual, we regard them as differential sheaves with grading indexed by $-\mathbf{N}$ by setting e.g. $\widetilde{\mathcal{C}}^{-p}_{X,Z}(\mathscr{F}) = \widetilde{\mathcal{C}}_{X,Z,p}(\mathscr{F})$ and with degree-one differential d identified with ∂. Let us set, by definition,

$$\mathrm{H}_p(X,Z;\mathscr{F}) = \boldsymbol{H}^{-p}(X, (\mathcal{C}^{\bullet}_{X,Z}(\mathscr{F}), \mathrm{d})).$$

Then, since each term of the complex $\mathcal{C}^{\bullet}_{X,Z}(\mathscr{F})$ is homotopically fine, Theorem A.5 implies

$$\mathrm{H}_p(X,Z;\mathscr{F}) \simeq \mathrm{H}_p(\Gamma(X, (\mathcal{C}_{X,Z,\bullet}(\mathscr{F}), \partial)).$$

In particular, given an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of sheaves, there is a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_p(X,Z;\mathscr{F}') \longrightarrow \mathrm{H}_p(X,Z;\mathscr{F}) \longrightarrow \mathrm{H}_p(X,Z;\mathscr{F}'') \longrightarrow \mathrm{H}_{p-1}(X,Z;\mathscr{F}') \longrightarrow \cdots \tag{A.10}$$

Besides, let $\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F})$ (not to be confused with $\widetilde{\mathcal{C}}_{X,Z,\bullet}(\mathscr{F})(X) = \widetilde{\mathcal{C}}_{X,Z,\bullet}(X) \otimes \mathscr{F}(X)$) denote the vector space which is the direct sum, indexed by the singular simplices σ whose support is not contained in Z, of the vector spaces $\boldsymbol{k}\sigma \otimes \Gamma(|\sigma|,\mathscr{F})$, with boundary induced by $\partial \otimes \mathrm{Id}$. More precisely, if $\partial\sigma = \sum_k \tau_k$, then $\partial(\sigma \otimes f) = \sum_k \tau_k \otimes f|_{|\tau_k|}$. A p-cycle is a sum $\sum_a (\sigma_a \otimes f_a)$ such that, for each $\tau \in S_{p-1}(X)$,

$$\sum_{\substack{a,k\\ \tau_{a,k}=\tau}} f_a|_{|\tau|} = 0 \in \Gamma(|\tau|,\mathscr{F}). \tag{A.11}$$

Such an element $\sigma \otimes f$, with $f \in \Gamma(|\sigma|,\mathscr{F})$, defines a global section in $\Gamma(X, \mathcal{C}_{X,Z,\bullet}(\mathscr{F}))$. Indeed, let U be an open neighbourhood of $|\sigma|$ on which f is defined. Then $\sigma \otimes f$ defines an element of $\widetilde{\mathcal{C}}_{X,Z,\bullet}(U) \otimes \Gamma(U,\mathscr{F})$, and hence of $\Gamma(U, \mathcal{C}_{X,Z,\bullet}(\mathscr{F}))$. Moreover, since the image of σ is zero in $\Gamma(U, \mathcal{C}_{X,Z,\bullet})$ if $U \cap |\sigma| = \varnothing$, this element is supported on $|\sigma|$, and hence extends in a unique way to a global section of $\mathcal{C}_{X,Z,\bullet}(\mathscr{F})$ on X.

The natural morphism

$$\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F}) \longrightarrow \Gamma(X, \mathcal{C}_{X,Z,\bullet}(\mathscr{F})) \tag{A.12}$$

is clearly compatible with ∂. Moreover, it is injective. Indeed, let $(\sigma_k)_k$ be distinct singular simplices and, for each k, let $f_k \in \Gamma(|\sigma_k|,\mathscr{F})$ be such that $\sum_k \sigma_{k,x} \otimes f_{k,x} = 0$ for all x, where $\sigma_{k,x}$ denotes the image of σ_k in $\mathcal{C}_{X,Z,\bullet,x}$. Let K_x be the set of indices k such that $x \in |\sigma_k|$. Then $(\sigma_{k,x})_{k\in K_x}$ is part of a basis of $\mathcal{C}_{X,Z,\bullet,x}$, and $\sum_{k\in K_x} \sigma_{k,x} \otimes f_{k,x} = 0$ implies that $f_{k,x} = 0$ for any $k \in K_x$. Since $\mathscr{F}$ is a sheaf, this implies the vanishing $f_k = 0$ in $\Gamma(|\sigma_k|,\mathscr{F})$ for all k.

Proposition A.13. *The chain map $(\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F}), \partial) \to (\Gamma(X, \mathcal{C}_{X,Z,\bullet}(\mathscr{F})), \partial)$ induces an isomorphism*

$$\mathrm{H}_p(\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F}), \partial) \simeq \mathrm{H}_p(X,Z;\mathscr{F}).$$

Proof. We argue as for Proposition A.6.

- Let $s \in \Gamma(X, \mathcal{C}_{X,Z,\bullet} \otimes \mathscr{F})$. There exists an open covering $\mathscr{U}$ and $\widetilde{s}^i \in \widetilde{\mathcal{C}}_{X,Z,\bullet}(U_i) \otimes \mathscr{F}(U_i)$ such that $s_x = \widetilde{s}^i_x$ for any $x \in U_i$. We decompose $\widetilde{s}^i = \sum_j \sigma_{ij} \otimes f_{ij}$, where σ_{ij} are singular simplices with support not contained in $(X \smallsetminus U_i) \cup Z$, and $f_{ij} \in \mathscr{F}(U_i)$. Let (F_i) be a closed covering such that $F_i \subset U_i$ for each i and let $(\widetilde{\ell}_i, \widetilde{k})$ be as in Proposition A.8, tensored with Id so that they act on the presheaf $\widetilde{\mathcal{C}}_{X,Z,\bullet} \otimes \mathscr{F}$. We denote by ℓ_i and k the induced sheaf morphisms. We have $s=\sum_i \ell_i(s)-(\partial k+k\partial)s$. For any x, $\ell_i(s)_x=0$ if $x\notin F_i$, according to Proposition A.8(1), and $\ell_i(s)_x=\widetilde{\ell}_i(\widetilde{s}^i)_x$ otherwise, and therefore

$$\begin{aligned}s_x=\sum_{i\mid U_i\ni x}\widetilde{\ell}_i(\widetilde{s}^i)_x-((k\partial+\partial k)s)_x&=\sum_{i\mid U_i\ni x}\sum_j\widetilde{\ell}_i(\sigma_{ij})_x\otimes f_{ij,x}-((k\partial+\partial k)s)_x\\&=\sum_i\sum_j\widetilde{\ell}_i(\sigma_{ij})_x\otimes f_{ij,x}-((k\partial+\partial k)s)_x,\end{aligned}$$

since $\widetilde{\ell}_i(\sigma_{ij})_x=0$ if $x\notin U_i$. Moreover, if we write $\widetilde{\ell}_i(\sigma_{ij})\otimes f_{ij}=\sum_a\sigma_a\otimes f_a$, where σ_a are singular simplices satisfying $|\sigma_a|\subset|\sigma_{ij}|\subset F_i$, and $f_a\in\mathscr{F}(U_i)$, we have $(\sigma_a\otimes f_a)_x=0$ if $x\notin|\sigma_a|$, and hence we can replace f_a with its restriction in $\Gamma(|\sigma_a|,\mathscr{F})$. Therefore,

$$s=\operatorname{im}\Big[\sum_i\sum_j\widetilde{\ell}_i(\sigma_{ij})\otimes f_{ij}\Big]-((k\partial+\partial k)s),$$

where the term between brackets belongs to $\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F})$. If s is closed, then $s+\partial ks$ is the image of $\sum_i\sum_j\widetilde{\ell}_i(\sigma_{ij})\otimes f_{ij}$ and the latter is closed since $\partial(s+\partial ks)=0$ and (A.12) is injective. This implies surjectivity of the homology map.

- Let $\widetilde{s}=\sum_a\sigma_a\otimes f_a$, where σ_a are singular simplices and $f_a\in\Gamma(|\sigma_a|,\mathscr{F})$, be closed and such that its image s is equal to $\partial s'$. The above formula shows that, up to replacing s' with $s'+(k\partial+\partial k)s'$ (and s with $s+\partial ks$), we can assume that s' lies in the image of $\widetilde{s}'\in\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F})$. Then $\widetilde{s}-\partial\widetilde{s}'$ has image zero, and hence is zero according to the injectivity of (A.12), so the homology map is injective. □

Proposition A.14 (Excision)**.** *Let Y be a closed subset contained in the interior Z° of Z. Then the natural morphism*

$$\mathrm{H}_p(X\smallsetminus Y,Z\smallsetminus Y;\mathscr{F})\longrightarrow\mathrm{H}_p(X,Z;\mathscr{F})$$

is an isomorphism for all p.

Proof. We consider the open covering $\mathscr{U}=(X\smallsetminus Y,Z^\circ)$. Let $(\widetilde{\mathcal{C}}^{\mathscr{U}}_{X,Z,\bullet}(X,\mathscr{F}),\partial)$ be the subcomplex of $(\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F}),\partial)$ for which the spaces $\boldsymbol{k}\sigma\otimes\Gamma(|\sigma|,\mathscr{F})$ occurring as components are those for which $|\sigma|$ is contained in one of the open sets of $\mathscr{U}$. By using (ℓ,k) adapted to this covering (see (A.7)), one obtains that any closed $s\in\widetilde{\mathcal{C}}_{X,Z,p}(X,\mathscr{F})$ can be written as $\ell(s)+\partial ks$, thus showing the surjectivity of the homology map. On the other hand, if $s'\in\widetilde{\mathcal{C}}^{\mathscr{U}}_{X,Z,p}(X,\mathscr{F})$ satisfies $s'=\partial s$ with $s\in\widetilde{\mathcal{C}}_{X,Z,p+1}(X,\mathscr{F})$, we have $s'=\ell(s')=\ell(\partial s)=\partial\ell(s)$, hence the injectivity. □

Remark A.15. The long exact sequence (A.10) is not easily seen in the model $\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F})$, while the excision property is better understood in that model, as well as the long exact sequence, for the closed inclusions $Z\subset Y\subset X$:

$$\cdots\longrightarrow\mathrm{H}_p(Y,Z;\mathscr{F})\longrightarrow\mathrm{H}_p(X,Z;\mathscr{F})\longrightarrow\mathrm{H}_p(X,Y;\mathscr{F})\longrightarrow\mathrm{H}_{p-1}(Y,Z;\mathscr{F})\longrightarrow\cdots$$

On the other hand, if $\mathscr{F}$ is the constant sheaf with fiber F (assume that X is connected), then for each singular simplex σ, the support $|\sigma|$ is connected, so $\Gamma(|\sigma|,\mathscr{F})$ is canonically identified with F by the restriction morphism $F=\Gamma(X,\mathscr{F})\to\Gamma(|\sigma|,\mathscr{F})$, so $(\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F}),\partial)\simeq(\widetilde{\mathcal{C}}_{X,Z,\bullet}(X)\otimes F,\partial)$ and $\mathrm{H}_p(X,Z;\mathscr{F})$ is the usual singular homology with coefficients in F.

A.e. **Piecewise smooth and simplicial chains.** Assume that X is a manifold (possibly with boundary). We denote by $(\widetilde{\mathcal{C}}^{\mathrm{sm}}_{X,Z,\bullet},\partial)$ the subcomplex of $(\widetilde{\mathcal{C}}_{X,Z,\bullet},\partial)$ consisting of piecewise smooth singular chains (i.e., having a basis formed of piecewise smooth singular simplices).

Proposition A.16. *The inclusion of chain complexes*

$$(\widetilde{\mathcal{C}}^{\mathrm{sm}}_{X,Z,\bullet}(X,\mathscr{F}),\partial)\hookrightarrow(\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F}),\partial)$$

induces an isomorphism in homology.

Proof. Let $s=\sum_a \sigma_a\otimes f_a$, with $f_a\in\Gamma(|\sigma_a|,\mathscr{F})$, be a p-cycle in $\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F})$. We can assume that f_a is the germ along $|\sigma_a|$ of $f_a\in\Gamma(U_a,\mathscr{F})$ for some open neighbourhood U_a of $|\sigma_a|$. The cycle condition is as in (A.11) and for each τ there exists U_τ such that the corresponding sum $\sum f_a$ is zero on U_τ. We can approximate each σ_a by piecewise smooth simplices with image contained in U_a and more precisely construct a family of approximations $\Sigma_a(t)=\sigma_a^t$ ($t\in[0,1]$) with $\sigma_a^0=\sigma_a$ and σ_a^t piecewise smooth for each $t\in(0,1]$ such that $\tau\times[0,1]$ has image in U_τ for each face τ of σ_a. We can then subdivide $\Delta_p\times[0,1]$ to regard each Σ_a as a $(p+1)$-chain and the total family $S=\sum_a\Sigma_a\otimes f_a$ as an element of $\widetilde{\mathcal{C}}_{X,Z,p+1}(X,\mathscr{F})$ (we restrict f_a to $|\Sigma_a|\subset U_a$). The boundary chain ∂S reads $\sum_{a,k}T_{a,k}\otimes f_a|_{|T_{a,k}|}+\sigma_a^1\otimes f_a-\sigma_a^0\otimes f_a$, where $T_{a,k}$ is a map $\Delta_{p-1}\times[0,1]\to U_a$ satisfying $T_{a,k}^0=\tau_{a,k}$ and $|T_{a,k}|\subset U_{\tau_{a,k}}$. The cycle condition on s implies that, for each $\tau:\Delta_{p-1}\to X$, we have $\sum_{a,k|\tau_{a,k}=\tau}f_a|_{U_\tau}=0$, so that $\partial S=\sigma_a^1\otimes f_a-\sigma_a^0\otimes f_a$. This shows the surjectivity of the homology map, and its injectivity follows. □

On the other hand, assume that X is endowed with a simplicial structure $\mathscr{T}$ compatible with Z. By this, we mean that X is the support of a simplicial complex $\mathscr{T}$ and Z is the support of a subcomplex $\mathscr{T}_Z$ of $\mathscr{T}$. We denote by $(\widetilde{\mathcal{C}}^{\triangle}_{\mathscr{T},\mathscr{T}_Z,\bullet},\partial)$ the subcomplex of $(\widetilde{\mathcal{C}}_{X,Z,\bullet},\partial)$ consisting of simplicial chains of the simplicial structure. We define correspondingly the simplicial chain complex $(\widetilde{\mathcal{C}}^{\triangle}_{\mathscr{T},\mathscr{T}_Z,\bullet}(X,\mathscr{F}),\partial)$. We make the following assumption on $\mathscr{F}$ and the simplicial structure.

Assumption A.17. For any simplex σ of the simplicial complex $\mathscr{T}$, there exists an open neighbourhood U_σ of σ that retracts onto σ and such that the sheaf $\mathscr{F}_{|U_\sigma}$ can be decomposed as a direct sum of sheaves $\mathscr{F}_{\sigma,i}$, each of which is constant of finite rank on $U_\sigma\smallsetminus T_{\sigma,i}$ and zero on $T_{\sigma,i}$, for some closed subset $T_{\sigma,i}$ of U_σ that intersects σ along a (possibly empty or full) closed face τ_i of σ.

We note that, if $\mathscr{T}'$ is a subdivision of $\mathscr{T}$ and $\mathscr{T}'_Z$ is the corresponding subdivision of $\mathscr{T}'_Z$, then $(\mathscr{F},\mathscr{T}',\mathscr{T}'_Z)$ satisfies Assumption A.17 if $(\mathscr{F},\mathscr{T},\mathscr{T}_Z)$ does so.

Proposition A.18. *Under Assumption* A.17*, the inclusion of chain complexes*

$$(\widetilde{\mathcal{C}}^{\triangle}_{\mathscr{T},\mathscr{T}_Z,\bullet}(X,\mathscr{F}),\partial)\hookrightarrow(\widetilde{\mathcal{C}}_{X,Z,\bullet}(X,\mathscr{F}),\partial)$$

induces an isomorphism in homology.

We immediately note the following consequence, which justifies that we now denote by $(\widetilde{\mathcal{C}}^{\triangle}_{X,Z,\bullet}(X,\mathscr{F}),\partial)$ the complex $(\widetilde{\mathcal{C}}^{\triangle}_{\mathscr{T},\mathscr{T}_Z,\bullet}(X,\mathscr{F}),\partial)$ when Assumption A.17 holds.

Corollary A.19. *The homology of $(\widetilde{\mathcal{C}}^{\triangle}_{\mathscr{T},\mathscr{T}_Z,\bullet}(X,\mathscr{F}),\partial)$ is independent of $\mathscr{T}$ provided Assumption* A.17 *is satisfied. In particular, it is equal to the homology of $(\widetilde{\mathcal{C}}^{\triangle}_{\mathscr{T}',\mathscr{T}'_Z,\bullet}(X,\mathscr{F}),\partial)$ for any subdivision $\mathscr{T}'$ of $\mathscr{T}$.* □

Proof of Proposition A.18. We can argue separately with X and Z and obtain the result for the pair (X,Z) due to compatibility between long exact sequences of pairs. So we argue with X, as in [47, §34], by induction on the (finite) number of simplices in the simplicial decomposition X. Let σ_o be a simplex of maximal dimension (that we can assume $\geqslant 1$) and let X' be the simplicial set X with σ_o deleted (but the boundary simplices of σ_o kept in X'). The underlying topological space X' is X with the interior of σ_o deleted. Then $(X',\mathscr{F})$ also satisfies Assumption A.17. We consider, in both simplicial and singular homology, the long exact sequences of pairs

$$\cdots\longrightarrow \mathrm{H}_p(\sigma_o;\mathscr{F})\longrightarrow \mathrm{H}_p(X;\mathscr{F})\longrightarrow \mathrm{H}_p(X,\sigma_o;\mathscr{F})\longrightarrow \mathrm{H}_{p-1}(\sigma_o;\mathscr{F})\longrightarrow\cdots$$

and the natural morphism between them. The assertion follows from Lemma A.20 below. □

Lemma A.20. *With the notation and assumption from Proposition* A.18*,*

(1) *the natural morphism*

$$(\widetilde{\mathcal{C}}^{\triangle}_{\sigma_o,\bullet}(\sigma_o,\mathscr{F}),\partial) \hookrightarrow (\widetilde{\mathcal{C}}_{\sigma_o,\bullet}(\sigma_o,\mathscr{F}),\partial)$$

induces an isomorphism in homology;

(2) *both natural inclusions*

$$\begin{aligned}(\widetilde{\mathcal{C}}^{\triangle}_{X',\partial\sigma_o,\bullet}(X',\mathscr{F}),\partial) &\hookrightarrow (\widetilde{\mathcal{C}}^{\triangle}_{X,\sigma_o,\bullet}(X,\mathscr{F}),\partial),\\ (\widetilde{\mathcal{C}}_{X',\partial\sigma_o,\bullet}(X',\mathscr{F}),\partial) &\hookrightarrow (\widetilde{\mathcal{C}}_{X,\sigma_o,\bullet}(X,\mathscr{F}),\partial),\end{aligned}$$

induce an isomorphism in homology.

Proof.

(1) Let us decompose $\mathscr{F}_{|\sigma_o}$ as in Assumption A.17, let us fix a corresponding component $\mathscr{F}_{\sigma_o,i}$ of $\mathscr{F}$ that we still call $\mathscr{F}$, and let τ_o be the face of σ_o on which it is zero. The assertion is trivial if $\tau_o=\sigma_o$. If $\tau_o=\varnothing$, then $\mathscr{F}$ is constant and the proof is done in [47, §34].

Assume now that τ_o is non-empty and different from σ_o. For any face τ of σ_o intersecting τ_o, $\mathscr{F}_{|\tau}$ is constant on $\tau\smallsetminus(\tau\cap\tau_o)$ and zero on $\tau\cap\tau_o$, so that $\Gamma(\tau,\mathscr{F})=0$. Let τ'_o be the union of faces of σ_o not intersecting τ_o. It is a single face of σ_o, and hence a simplex. Now, $\mathscr{F}$ is constant on τ'_o and, obviously, $\mathrm{H}^{\triangle}_p(\sigma_o;\mathscr{F})=\mathrm{H}^{\triangle}_p(\tau'_o;\mathscr{F})$.

Let us now compute $\mathrm{H}_p(\sigma_o;\mathscr{F})$. Let $\sigma:\Delta_p\to\sigma_o$ be a singular p-simplex of σ_o, with image $|\sigma|$. Note that the natural morphism $\Gamma(|\sigma|,\mathscr{F})\to\Gamma(\Delta_p,\sigma^{-1}\mathscr{F})$ is injective (since it preserves germs). If $|\sigma|\cap\tau_o\neq\varnothing$, the argument above shows that $\Gamma(\Delta_p,\sigma^{-1}\mathscr{F})=0$, and hence so is $\Gamma(|\sigma|,\mathscr{F})$. Thus, $\widetilde{\mathcal{C}}_{\sigma_o,p}(\sigma_o,\mathscr{F})$ only involves singular simplices σ not intersecting τ_o. Let F be the constant value of $\mathscr{F}$ on $\sigma_o\smallsetminus\tau_o$. For such a simplex σ, we thus have $\Gamma(|\sigma|,\mathscr{F})\simeq\Gamma(\Delta_p,\sigma^{-1}\mathscr{F})=F$, and then $\widetilde{\mathcal{C}}_{\sigma_o,p}(\sigma_o,\mathscr{F})=\widetilde{\mathcal{C}}_{\sigma_o\smallsetminus\tau_o,p}(\sigma_o\smallsetminus\tau_o)\otimes F$. In other words, $\mathrm{H}_p(\sigma_o;\mathscr{F})=\mathrm{H}_p(\sigma_o\smallsetminus\tau_o)\otimes F$. Since $(\sigma_o\smallsetminus\tau_o)$ retracts to τ'_o, this group is also equal to $\mathrm{H}_p(\tau'_o;\mathscr{F})$, and we conclude with [47, §34].

(2) The assertion for the simplicial complexes is obvious since it is already an equality. We thus consider the singular chain complexes.

Let us prove surjectivity of the homology map. Let s be a closed singular chain in $\widetilde{\mathcal{C}}_{X,\sigma_o,p}(X,\mathscr{F})$. We can cover X by U_o and U', where $U_o=U_{\sigma_o}$ is given by Assumption A.17 and $U'\subset X'$. Choosing (ℓ,k) as in (A.7), we write $s+\partial ks=s_o+s'$, with s_o,s' closed in $\widetilde{\mathcal{C}}_{U_o,\sigma_o,p}(X,\mathscr{F})$ and $\widetilde{\mathcal{C}}_{U',p}(X,\mathscr{F})\subset\widetilde{\mathcal{C}}_{X',\sigma_o,p}(X',\mathscr{F})$ respectively. We decompose s_o according to Assumption A.17. For the component corresponding to "$\mathscr{F}$ constant on σ_o", it is proved in [47, §34] that it is homologous to a chain in $\widetilde{\mathcal{C}}_{U_o\cap X',\partial\sigma_o,p}(X,\mathscr{F})$. Assume thus that $\mathscr{F}$ is zero on T_o with $\tau_o=T_o\cap\sigma_o\neq\varnothing$, and constant on $U_o\smallsetminus T_o\neq\varnothing$. Arguing as in Case (1), we show that singular simplices occurring with a non-zero coefficient in s_o do not intersect T_o. There is a projection $\sigma_o\smallsetminus\tau_o\to\tau'_o$ (with τ'_o as above), so that if we compose s_o with the corresponding retraction, we obtain a homologous closed p-chain in $\widetilde{\mathcal{C}}_{U_o\cap X',\partial\sigma_o,p}(X,\mathscr{F})$.

Assume now that $\widetilde{s}\in\widetilde{\mathcal{C}}_{X',\partial\sigma_o,p}(X,\mathscr{F})$ is equal to ∂s with s in $\widetilde{\mathcal{C}}_{X,\sigma_o,p+1}(X,\mathscr{F})$. It follows that $\partial\widetilde{s}=0$ in $\widetilde{\mathcal{C}}_{X,\sigma_o,p-1}(X,\mathscr{F})$, and hence in $\widetilde{\mathcal{C}}_{X',\partial\sigma_o,p-1}(X,\mathscr{F})$. With U_o,U', and (ℓ,k) as above, we have $\widetilde{s}+\partial k\widetilde{s}=\ell(\widetilde{s})=\partial\ell(s)=\partial s_o+\partial s'$. The above argument shows that s_o is homologous to a $(p+1)$-chain in $\widetilde{\mathcal{C}}_{U_o\cap X',\partial\sigma_o,p+1}(X,\mathscr{F})$, and hence $\widetilde{s}$ is the boundary of a chain in $\widetilde{\mathcal{C}}_{X',\partial\sigma_o,p+1}(X,\mathscr{F})$, which shows injectivity. ☐

A.f. **The dual chain complex with coefficient in a sheaf.** In this section, we set $(X,Z)=(M,\partial M)$, where M is a manifold with corners, and we consider a simplicial decomposition $\mathscr{T}$ of $(M,\partial M)$. Let $\mathscr{T}'$ denote the first barycentric subdivision of $\mathscr{T}$. For any simplex σ in $\mathscr{T}$, let $\widehat{\sigma}$ denote its barycenter and let $D(\sigma)$ be the open cell dual to σ, which is the sum of open simplices of $\mathscr{T}'$ having $\widehat{\sigma}$ as their final vertex (see [47, §64] for definitions and details). If σ is not contained in ∂M, then the closure $\overline{D}(\sigma)$ does not intersect ∂M. If σ is contained in ∂M, then $D(\sigma)$ is contained in M°. In any case, if we regard $\overline{D}(\sigma)$ as a chain of $\mathscr{T}'$ relative to ∂M, then $\partial\overline{D}(\sigma)$ is a sum of terms $\partial\overline{D}(\sigma_i)$ for σ_i in $\mathscr{T}$. The following is clear.

Lemma A.21. *Let σ be a chain of $\mathscr{T}$ of dimension p. Then σ is the only chain of dimension p of $\mathscr{T}$ intersected by $D(\sigma)$, and the intersection is transversal.*

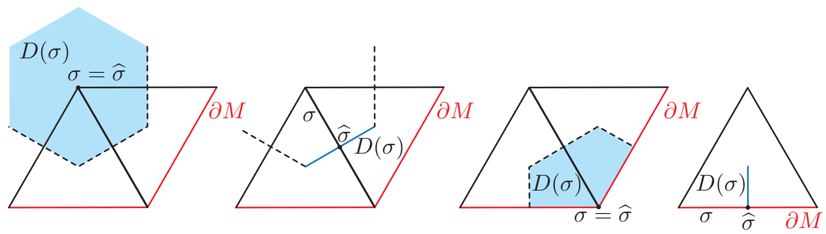


Figure 1. Various examples of dual cells

We consider a sheaf $\mathscr{F}$ on M satisfying Assumption A.17 with respect to $\mathscr{T}$, with the supplementary condition that $\mathscr{F}$ is locally constant on $M^\circ = M \smallsetminus \partial M$, that is, the closed subsets $T_{\sigma,i}$ of Assumption A.17 are contained in ∂M. We note that $\mathscr{F}$ also satisfies the previous properties with respect to $\mathscr{T}'$. The chain complex $(\widetilde{\mathcal{C}}^{\square}_{\mathscr{T}^\vee,\partial M}(M,\mathscr{F}),\partial)$ whose term of index p consists of the direct sum of the terms $\Gamma(\overline{D}(\sigma),\mathscr{F})$ with σ of codimension p, is thus a subcomplex of $(\widetilde{\mathcal{C}}^{\triangle}_{\mathscr{T}',\partial M}(M,\mathscr{F}),\partial)$.

Lemma A.22. *Under the previous assumptions, the inclusion of chain complexes*

$$(\widetilde{\mathcal{C}}^{\square}_{\mathscr{T}^\vee,\partial M}(M,\mathscr{F}),\partial) \hookrightarrow (\widetilde{\mathcal{C}}^{\triangle}_{\mathscr{T}',\partial M}(M,\mathscr{F}),\partial)$$

induces an isomorphism in homology.

Proof. Similar to that of [47, Th. 64.2]. □

It follows from Proposition A.18 that the chain complex $(\widetilde{\mathcal{C}}^{\square}_{\mathscr{T}^\vee,\partial M}(M,\mathscr{F}),\partial)$ also computes the relative homology $\mathrm{H}_\bullet(M,\partial M;\mathscr{F})$.

Appendix B. Poincaré lemma for currents

In this section, we work in a local setting. We let U be a convex open subset of $\mathbf{R}^m$ (with coordinates $x_1,\dots,x_m$) containing the origin, we fix r such that $1 \leqslant r \leqslant m$, we set $g(x)=\prod_{i=1}^r x_i$ and

$$M=\{g(x)\geqslant 0\}\cap U,\quad \partial M=\{g(x)=0\}\cap U,\quad M^\circ=\{g(x)>0\}\cap U \quad\text{and}\quad U^\circ=\{g(x)\neq 0\}.$$

The sheaves of functions or distributions on M that we consider below are by definition the sheaf-theoretic restriction to M of the corresponding sheaves on U.

B.a. Functions with moderate growth and rapid decay along ∂M

Definition B.1.

(1) A C^∞ function f on M° has moderate growth along ∂M if, for any relatively compact open subset $W \Subset M$ and any multi-index α, there exists $N \geqslant 0$ and $C>0$ such that $\partial^\alpha f_{|W^\circ}$ is bounded by $C|g|^{-N}$.

(2) A C^∞ function f on M has rapid decay along ∂M if all its derivatives vanish at any point of ∂M. Equivalently, the function f/g^N is C^∞ for every N.

There are similar definitions on U° and a C^∞ function with moderate growth on M° (resp. with rapid decay on M) can be extended by 0 to a similar function on U° resp. U.

The corresponding sheaves are denoted respectively by $\mathscr{C}_M^{\infty,\mathrm{mod}}$ and $\mathscr{C}_M^{\infty,\mathrm{rd}}$, and the corresponding de Rham complexes by $(\mathscr{E}_M^{\mathrm{mod},\bullet},\mathrm{d})$ and $(\mathscr{E}_M^{\mathrm{rd},\bullet},\mathrm{d})$ (we omit the boundary ∂M in the notation). On the other hand, we will also consider the sheaf of C^∞ functions on M with poles along ∂M, that we simply denote by $\mathscr{C}_M^\infty(*)$. It is a subsheaf of $\mathscr{C}_M^{\infty,\mathrm{mod}}$.

Proposition B.2 (Poincaré lemma). *The complexes $(\mathscr{E}_M^{\mathrm{mod},\bullet},\mathrm{d})$ and $(\mathscr{E}_M^{\mathrm{rd},\bullet},\mathrm{d})$ have cohomology in degree zero only, and are a resolution of $j_*\mathbf{C}_{M^\circ}=\mathbf{C}_M$ and $j_!\mathbf{C}_{M^\circ}$ respectively.*

Proof. It is enough to prove the assumptions on $(U,\partial M)$ and then restrict them sheaf-theoretically to $(M,\partial M)$. We prove the corresponding statements on global sections.

Let $\eta\in\mathscr{E}^{\mathrm{rd},p}(U)$ (resp. $\eta\in\mathscr{E}^{\mathrm{mod},p}(U^\circ)$) be such that $\mathrm{d}\eta=0$ and $p\geqslant 1$. There is an explicit formula (see for example [17, (1.23)]) computing $\psi\in\mathscr{E}^{p-1}(U)$ such that $\mathrm{d}\psi=\eta$:

$$\psi(x)=\sum_{\substack{|I|=p\\ k\in[1,p]}}\Big(\int_0^1 t^{p-1}\eta_I(tx)\mathrm{d}t\Big)(-1)^{k-1}x_{i_k}\mathrm{d}x_{i_1}\wedge\cdots\wedge\widehat{\mathrm{d}x_{i_k}}\wedge\cdots\wedge\mathrm{d}x_{i_p}.$$

This formula shows that ψ belongs to $\mathscr{E}^{\mathrm{rd},p-1}(U)$ (resp. to $\mathscr{E}^{\mathrm{mod},p-1}(U^\circ)$), hence the first assertion. The second assertion in both cases is clear. □

B.b. **Poincaré lemma for currents.** As in [11, 12], we define the chain complex of currents $(\mathfrak{C}_{M,\bullet},\partial)$. By definition, it is the sheaf-theoretic restriction to M of the complex $\mathfrak{C}_{U,\bullet}$ that we consider now. A current T of dimension q on U can be paired with a test q-form φ on U (C^∞ with compact support) and the Stokes formula holds: $\langle\partial T,\varphi\rangle=\langle T,\mathrm{d}\varphi\rangle$. If ω is a C^∞ $(m-q)$-form, it defines a q-dimensional current by the formula $\langle T_\omega,\varphi\rangle=\int_U\omega\wedge\varphi$. We can consider a current as a differential form with distributional coefficients, i.e., we can write $T=\sum_{\#J=m-q}\theta_J\mathrm{d}x_J$ with $\theta_J\in\mathfrak{Db}(U)$. When considered as a form, the differential d on currents extends that on C^∞ forms. The relation between d and ∂ is given, for a current of dimension q (i.e., degree $m-q$) by

$$\partial T_q=(-1)^{m-q+1}\mathrm{d}T_q.$$

For a tensor product taken over $\mathbf{C}$, like $(\mathfrak{C}_{M,\bullet},\partial)\otimes(\mathscr{E}_M^{\mathrm{rd},\bullet},\mathrm{d})$, the boundary is given by the formula, for a current $T_q\otimes\omega^p$ (of dimension $q-p$):

$$\partial(T_q\otimes\omega^p)=(-1)^p(\partial T_q\otimes\omega^p-T_q\otimes\mathrm{d}\omega^p)=(-1)^p\partial T_q\otimes\omega^p+T_q\otimes\partial\omega^p. \tag{B.3}$$

Proposition B.4 (Poincaré lemma for $(\mathfrak{C}_{M,\bullet},\partial)$). *The complex of currents on M satisfies*

$$\mathscr{H}_q(\mathfrak{C}_{M,\bullet},\partial)=\begin{cases}0 & \text{if } q\neq m,\\ \mathbf{C}_M & \text{if } q=m.\end{cases}$$

More precisely, the inclusion $(\mathscr{E}_M^{m-\bullet}\mathrm{d})\hookrightarrow(\mathfrak{C}_{M,\bullet},\partial)$ is a quasi-isomorphism.

Proof. We prove the result on an open subset U of $\mathbf{R}^m$, and we obtain the proposition by sheaf-theoretic restriction to $M\cap U$. We refer to [17, §2.D.4] for the proof of the lemma below, originally in [11, 12]. Let $T\in\mathfrak{C}_q(U)$.

Lemma B.5. *For any $\varepsilon\in(0,1)$, there exist $\mathbf{C}$-linear morphisms $R_\varepsilon:\mathfrak{C}_q(U)\to\mathfrak{C}_q(U)$ and $S_\varepsilon:\mathfrak{C}_{q+1}(U)\to\mathfrak{C}^{m-q-1}(U)$ such that*

1. *R_ε takes values in $\mathscr{E}^{m-q}(U)$,*
2. *$R_\varepsilon(T)-T=\partial S_\varepsilon(T)+S_\varepsilon(\partial T)$,*
3. *$R_\varepsilon(\partial T)=\partial R_\varepsilon(T)$ and $\lim_{\varepsilon\to0}R_\varepsilon(T)=T$ weakly.*

In particular, R_ε is a morphism of complexes $(\mathfrak{C}_\bullet(U),\partial)\to(\mathscr{E}^{m-\bullet}(U),\mathrm{d})$, and (2) implies that it is a quasi-isomorphism. Since $(\mathscr{E}^{m-\bullet}(U),\mathrm{d})$ has homology concentrated in degree zero, it follows that the same holds for $(\mathfrak{C}_\bullet(U),\partial)$. Then we can sheafify the construction. □

Remark B.6. Arguing as in [12, §15, Th. 12], we can globalize the construction on a manifold with corners. More precisely, given a covering $(U_i)_{i\in I}$ of M by charts, and $\varepsilon_i>0$ small enough $(i\in I)$, one can construct morphisms $R_{\boldsymbol\varepsilon}:\mathfrak{C}_{M,\bullet}\to\mathscr{E}_M^{m-\bullet}$ and $S_{\boldsymbol\varepsilon}:\mathfrak{C}_{M,\bullet}\to\mathfrak{C}_{M,\bullet+1}$ such that the conclusion of Lemma B.5 holds for any $T\in\Gamma(M,\mathfrak{C}_{M,\bullet})$, up to replacing ε with $\boldsymbol\varepsilon=(\varepsilon_i)_{i\in I}$. In particular, $R_{\boldsymbol\varepsilon}:(\mathfrak{C}_{M,\bullet},\partial)\to(\mathscr{E}_M^{m-\bullet},\mathrm{d})$ is a quasi-isomorphism.

Let us end by recalling the computation of the Kronecker index made by de Rham. Let T_p and $T^\vee_{m-p}$ be two currents of complementary dimension on U. If both currents are closed, let us choose a decomposition $T_p=\Theta^{m-p}+\partial S_{p+1}$ and $T^\vee_{m-p}=\Theta^{\vee,p}+\partial S^\vee_{m-p+1}$, where Θ^{m-p} and $\Theta^{\vee,p}$ are C^∞ closed forms. Then $\mathsf{B}_{\mathrm{dR}}(T_p,T^\vee_{m-p})$ is by definition $\mathsf{Q}(\Theta^{m-p},\Theta^{\vee,p})$ and is independent of the choices. Without the closedness assumption, but if one of the supports is compact, the intersection $\mathsf{B}_{\mathrm{dR}}(T_p,T^\vee_{m-p})$ is defined by the limit, when it exists

$$\lim_{\varepsilon\to0}\langle T_p,R_\varepsilon(T^\vee_{m-p})\rangle.$$

We now consider the setting of Section A.f. Let us choose a simplex σ_p in M not contained in ∂M and let $\overline{D}(\sigma_p)$ denote the dual cell in M°, which intersects σ_p at its barycenter $\widehat{\sigma}_p$. Let us set $T_p=\int_{\sigma_p}$, $T^\vee_{m-p}=\int_{\overline{D}(\sigma_p)}$ and $\mathsf{B}_{\mathrm{dR}}(\sigma_p,\overline{D}(\sigma_p))=\mathsf{B}_{\mathrm{dR}}(T_p,T^\vee_{m-p})$.

Proposition B.7 ([12, p. 85–86])**.** *With these assumptions, the intersection* $\mathsf{B}_{\mathrm{dR}}(\sigma_p,\overline{D}(\sigma_p))$ *exists and is given by the formula*

$$\mathsf{B}_{\mathrm{dR}}(\sigma_p,\overline{D}(\sigma_p))=\pm1,$$

where $\pm$ *is the orientation change between* $\sigma_p\times\overline{D}(\sigma_p)$ *and* M°*.*

Proof. Let us choose coordinates $(x',x'')=(x_1,\dots,x_p,x_{p+1},\dots,x_m)$ so that σ_p (with orientation) (resp. $\overline{D}(\sigma_p)$) is contained in the coordinate plane $(x_1,\dots,x_p)$ (resp. $(x_{p+1},\dots,x_m)$). The formula for R_ε in Lemma B.5 is:

$$R_\varepsilon(T^\vee_{m-p})=\theta_{m-p,\varepsilon}(x)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_p,\quad\text{with }\theta_{m-p,\varepsilon}(x)=\int_{\overline{D}(\sigma_p)}\frac{\chi_\varepsilon((x-y)/\psi(y))}{\psi(y)^m}\,\mathrm{d}y_{p+1}\wedge\cdots\wedge\mathrm{d}y_m,$$

where $\chi:B(0,1)\to\mathbf{R}_+$ is a C^∞-function with compact support and integral equal to 1 and, for $\varepsilon>0$, let $\chi_\varepsilon(v)=\varepsilon^{-m}\chi(v/\varepsilon)$ with support in $B(0,\varepsilon)$. Hence,

$$\begin{aligned}\mathsf{B}_{\mathrm{dR}}(\sigma_p,\overline{D}(\sigma_p))&=\lim_{\varepsilon\to0}\int_{\sigma_p}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_p\cdot\theta_{m-p,\varepsilon}(x')\\&=\lim_{\varepsilon\to0}\int_{\sigma_p\times\overline{D}(\sigma_p)}\frac{\chi_\varepsilon((x',-x'')/\psi(x''))}{\psi(x'')^m}\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_m=\pm1\end{aligned}$$

by the properties of χ, if we assume that χ is even with respect to the variables x''. □

B.c. **Poincaré lemma for current with moderate growth.** We consider the sheaf $\mathfrak{Db}_M^{\mathrm{mod}}$ defined as the subsheaf of $j_*\mathfrak{Db}_{M^\circ}$ consisting of distributions extendable to M. Its space of sections on an open set is the topological dual of the space of C^∞ functions with rapid decay along ∂M, endowed with its usual family of seminorms. Let us first consider the similar objects on U. It follows from [38, Chap. VII] that $\mathfrak{Db}_U^{\mathrm{mod}}=\mathscr{C}_U^\infty(*)\otimes_{\mathscr{C}_U^\infty}\mathfrak{Db}_U$, where we have set $\mathscr{C}_U^\infty(*)=\mathscr{C}_U^\infty[1/g]$. We also have an exact sequence

$$0\longrightarrow\mathfrak{Db}_{[\partial M]}\longrightarrow\mathfrak{Db}_U\longrightarrow\mathfrak{Db}_U^{\mathrm{mod}}\longrightarrow0,\tag{B.8}$$

where $\mathfrak{Db}_{[\partial M]}$ denotes the subsheaf of distributions supported on ∂M.

Definition B.9 (Currents with moderate growth)**.** We set $\mathfrak{C}^{\mathrm{mod}}_{U,p} = \mathscr{E}^{m-p}_U \otimes_{\mathscr{C}^\infty_U} \mathfrak{Db}^{\mathrm{mod}}_U$ and $\mathfrak{C}^{\mathrm{mod}}_{M,p} = \mathscr{E}^{m-p}_M \otimes_{\mathscr{C}^\infty_M} \mathfrak{Db}^{\mathrm{mod}}_M$.

In a way similar to currents, a current T of dimension q with moderate growth on M can be paired with a test q-form φ on M with rapid decay and the Stokes formula holds.

Proposition B.10 (Poincaré lemma for $(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet}, \partial)$)**.** *The chain complex $(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet}, \partial)$ satisfies Poincaré lemma as in Proposition* B.4*.*

Proof. We will work with the cohomological version $(\mathfrak{C}^{\mathrm{mod},\bullet}_M, \mathrm{d})$, that we wish to prove to be a resolution of its $\mathscr{H}^0$, that is, $\mathbf{C}_M$. In other words, we wish to prove that the natural morphism $(\mathfrak{C}^\bullet_M, \mathrm{d}) \to (\mathfrak{C}^{\mathrm{mod},\bullet}_M, \mathrm{d})$ is a quasi-isomorphism.

Lemma B.11. *The complex $(\mathfrak{C}^\bullet_{[\partial M]}, \mathrm{d})$ has cohomology in degree one only and, for x_o in the intersection of p smooth components of ∂M, we have* $\dim \mathscr{H}^1(\mathfrak{C}^\bullet_{[\partial M]}, \mathrm{d})_{x_o} = 2^p - 1$.

Proof. Let $Z \subset U$ denote the p-th intersection defined by $x_1 = \cdots = x_p = 0$, so that $\mathfrak{Db}_{[Z]} \simeq \mathfrak{Db}_Z[\partial_{x_1}, \dots, \partial_{x_p}]$. We write $(\mathfrak{C}^\bullet_{[Z]}, \mathrm{d})$ as the simple complex associated to the p-cube with i-th edges all equal to

$$\mathfrak{C}^\bullet_Z[\partial_{x_1}, \dots, \partial_{x_p}] \xrightarrow{\partial_{x_i}} \mathfrak{C}^\bullet_Z[\partial_{x_1}, \dots, \partial_{x_p}],$$

which is quasi-isomorphic to $\mathfrak{C}^{\bullet-p}_Z$, so $\mathfrak{C}^\bullet_{[Z]}$ has cohomology in degree p only, equal to the constant sheaf $\mathbf{C}_Z$ (by the Poincaré lemma for Z).

Let us set $Z_i = \{x_i = 0\}$ and $Z^{(p)} = \bigcup_{i=1}^p Z_i$. We prove by induction on p that the lemma hods for $Z^{(p)}$, the case $p = 1$ begin proved in the first point. According to [38, Prop. VII.1.4], the natural sequence of complexes

$$0 \longrightarrow (\mathfrak{C}^\bullet_{[Z_p^{(p-1)}]}, \mathrm{d}) \longrightarrow (\mathfrak{C}^\bullet_{[Z_p]}, \mathrm{d}) \oplus (\mathfrak{C}^\bullet_{[Z^{(p-1)}]}, \mathrm{d}) \longrightarrow (\mathfrak{C}^\bullet_{[Z^{(p)}]}, \mathrm{d}) \longrightarrow 0$$

is exact. Let us restrict the complexes on $Z = \bigcap_{i=1}^p Z_i$. By induction, the middle complex has cohomology in degree one only, of dimension $1 + (2^{p-1} - 1) = 2^{p-1}$. Also by induction, the left complex has cohomology in degree 2 only, of dimension $2^{p-1} - 1$. This completes the proof. ☐

The exact sequence (B.8) gives rise to an exact sequence of complexes

$$0 \longrightarrow \mathfrak{C}^\bullet_{[\partial M]} \longrightarrow \mathfrak{C}^\bullet_U \longrightarrow \mathfrak{C}^{\mathrm{mod},\bullet}_U \longrightarrow 0$$

and the lemma above reduces it to an exact sequence $0 \to \mathbf{C}_U \to \mathscr{H}^0(\mathfrak{C}^{\mathrm{mod},\bullet}_U) \to \mathscr{H}^1(\mathfrak{C}^\bullet_{[\partial M]}) \to 0$. At a point $x_o \in \partial M$ at the intersection of exactly p smooth components, $\mathscr{H}^0(\mathfrak{C}^{\mathrm{mod},\bullet}_U)_{x_o}$ has thus dimension 2^p. If $U^\circ_{x_o,a}$ are the 2^p local connected components of $U^\circ_{x_o}$, then

$$\mathscr{H}^0(\mathfrak{C}^{\mathrm{mod},\bullet}_U)_{x_o} = \mathrm{H}^0(\mathfrak{C}^{\mathrm{mod},\bullet}(U^\circ_{x_o})) = \bigoplus_a \mathrm{H}^0(\mathfrak{C}^{\mathrm{mod},\bullet}(U^\circ_{x_o,a})),$$

so that each term in the sum is isomorphic to $\mathbf{C}$. In particular, considering the component $M^\circ_{x_o}$, the natural composed morphism (after sheaf-theoretically restricting to M) $\mathfrak{C}^\bullet_U \to \mathfrak{C}^{\mathrm{mod},\bullet}_U \to \mathfrak{C}^{\mathrm{mod},\bullet}_M$ induces an isomorphism on the $\mathscr{H}^0$ at each x_o, and hence is an isomorphism. ☐

We define the complex of currents with rapid decay as $(\mathfrak{C}^{\mathrm{mod}}_{M,\bullet}, \partial) \otimes (\mathscr{E}^{\mathrm{rd},\bullet}_M, \mathrm{d})$, with boundary operator given by (B.4). From Propositions B.10 and B.2, we thus obtain:

Proposition B.12 (Poincaré lemma for $(\mathfrak{C}^{\mathrm{rd}}_{M,\bullet}, \partial)$)**.** *The chain complex $(\mathfrak{C}^{\mathrm{rd}}_{M,\bullet}, \partial)$ has homology in degree m only, which is equal to $j_!\mathbf{C}_{M^\circ}$.*

Remark B.13. If M is compact, the homology of the complex $(\Gamma(M, \mathfrak{C}^{\mathrm{rd}}_{M,\bullet}), \partial)$ is isomorphic to $\mathrm{H}_\bullet(M^\circ, \mathbf{C})$, while that of $(\Gamma(M, \mathfrak{C}^{\mathrm{mod}}_{M,\bullet}), \partial)$ is isomorphic to the Borel-Moore homology $\mathrm{H}^{\mathrm{BM}}_\bullet(M^\circ, \mathbf{C})$.

Appendix C. Remarks on Verdier duality

To pass from local results on pairings to global ones, we use compatibility of Verdier duality with proper pushforward. In this section is made precise an "obvious result" (Corollary C.6) which we could not find in the literature. We refer to [29, Chap. 2 & 3] for standard results of sheaf theory.

We fix a field $\boldsymbol{k}$ and we work in the category of sheaves of $\boldsymbol{k}$-vector spaces. All topological spaces we consider are assumed to be locally compact, and all maps are assumed to have finite local cohomological dimension. Let $\mathbf{D}_X$ be the dualizing complex. If $a_X : X \to \mathrm{pt}$ denotes the constant map, one has $\mathbf{D}_X = a_X^! \boldsymbol{k}$.

C.a. **The duality isomorphism.** Let $f : X \to Y$ be a continuous map, let $\mathscr{F}$ be an object of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ and $\mathscr{G}$ an object of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$. There is a bifunctorial isomorphism

$$\boldsymbol{R}f_*\boldsymbol{R}\mathscr{H}om(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \boldsymbol{R}\mathscr{H}om(\boldsymbol{R}f_!\mathscr{F}, \mathscr{G}) \tag{C.1}$$

in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$ (see [29, Prop. 3.1.10]). By applying $\boldsymbol{R}\Gamma(Y, \bullet)$, one obtains

$$\boldsymbol{R}\operatorname{Hom}(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}(\boldsymbol{R}f_!\mathscr{F}, \mathscr{G}),$$

and taking cohomology in degree 0,

$$\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(X)}(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(Y)}(\boldsymbol{R}f_!\mathscr{F}, \mathscr{G}).$$

There exists a natural morphism of functors $\boldsymbol{R}f_!f^! \mapsto \mathrm{Id}$. Indeed, taking $\mathscr{F} = f^!\mathscr{G}$ above, one finds

$$\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(X)}(f^!\mathscr{G}, f^!\mathscr{G}) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(Y)}(\boldsymbol{R}f_!f^!\mathscr{G}, \mathscr{G})$$

and the desired morphism is the image of the identity.

One can thus write the duality isomorphism (C.1) as the composition of two bifunctorial morphisms (see [29, (2.6.25)] for the first one)

$$\boldsymbol{R}f_*\boldsymbol{R}\mathscr{H}om(\mathscr{F}, f^!\mathscr{G}) \longrightarrow \boldsymbol{R}\mathscr{H}om(\boldsymbol{R}f_!\mathscr{F}, \boldsymbol{R}f_!f^!\mathscr{G}) \longrightarrow \boldsymbol{R}\mathscr{H}om(\boldsymbol{R}f_!\mathscr{F}, \mathscr{G}).$$

C.b. **Poincaré-Verdier duality.** We set $\boldsymbol{D}\mathscr{F} = \boldsymbol{R}\mathscr{H}om(\mathscr{F}, \mathbf{D}_X)$. One then has an isomorphism

$$\boldsymbol{R}\Gamma(X, \boldsymbol{D}\mathscr{F}) \xrightarrow{\sim} \operatorname{Hom}(\boldsymbol{R}\Gamma_{\mathrm{c}}(X, \mathscr{F}), \boldsymbol{k}),$$

and taking hypercohomology,

$$\boldsymbol{H}^\ell(X, \boldsymbol{D}\mathscr{F}) \xrightarrow{\sim} \boldsymbol{H}^{-\ell}_{\mathrm{c}}(X, \mathscr{F})^\vee.$$

Via this isomorphism and by using the natural duality pairing, we deduce a perfect pairing

$$\boldsymbol{H}^{-\ell}_{\mathrm{c}}(X, \mathscr{F}) \otimes \boldsymbol{H}^\ell(X, \boldsymbol{D}\mathscr{F}) \longrightarrow \boldsymbol{k}. \tag{C.2}$$

On the other hand, one has a natural morphism $\mathscr{F} \otimes \boldsymbol{D}\mathscr{F} \to \mathbf{D}_X$ and, in addition, according to [29, (2.6.23)] for $f = a_X$, a morphism

$$\boldsymbol{R}\Gamma_{\mathrm{c}}(X, \mathscr{F}) \otimes \boldsymbol{R}\Gamma(X, \boldsymbol{D}\mathscr{F}) \longrightarrow \boldsymbol{R}\Gamma_{\mathrm{c}}(X, \mathscr{F} \otimes \boldsymbol{D}\mathscr{F}).$$

By composing, one obtains a morphism

$$\boldsymbol{R}\Gamma_{\mathrm{c}}(X, \mathscr{F}) \otimes \boldsymbol{R}\Gamma(X, \boldsymbol{D}\mathscr{F}) \to \boldsymbol{R}\Gamma_{\mathrm{c}}(X, \mathscr{F} \otimes \boldsymbol{D}\mathscr{F}) \to \boldsymbol{R}\Gamma_{\mathrm{c}}(X, \mathbf{D}_X) = \boldsymbol{R}a_{X!}a_X^!\boldsymbol{k} \to \boldsymbol{k}.$$

By taking hypercohomology, one obtains for each $\ell \in \mathbf{Z}$ a pairing

$$\boldsymbol{H}^{-\ell}_{\mathrm{c}}(X, \mathscr{F}) \otimes \boldsymbol{H}^\ell(X, \boldsymbol{D}\mathscr{F}) \longrightarrow \boldsymbol{k}. \tag{C.3}$$

Proposition C.4. *The pairings* (C.2) *and* (C.3) *coincide. In particular,* (C.3) *is a perfect pairing.*

For the sake of completeness, we will prove the following lemma at the end of this section.

Lemma C.5. *Let $\mathscr{F}$ and $\mathscr{G}$ be objects of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ and let $f : X \to Y$ be a morphism. The two following composed natural morphisms coincide:*

$$\boldsymbol{R}f_!\mathscr{F} \otimes \boldsymbol{R}f_*\boldsymbol{R}\mathscr{H}om(\mathscr{F},\mathscr{G}) \longrightarrow \boldsymbol{R}f_!\big(\mathscr{F} \otimes \boldsymbol{R}\mathscr{H}om(\mathscr{F},\mathscr{G})\big) \longrightarrow \boldsymbol{R}f_!\mathscr{G},$$
$$\boldsymbol{R}f_!\mathscr{F} \otimes \boldsymbol{R}f_*\boldsymbol{R}\mathscr{H}om(\mathscr{F},\mathscr{G}) \longrightarrow \boldsymbol{R}f_!\mathscr{F} \otimes \boldsymbol{R}\mathscr{H}om(\boldsymbol{R}f_!\mathscr{F},\boldsymbol{R}f_!\mathscr{G}) \longrightarrow \boldsymbol{R}f_!\mathscr{G}.$$

Proof of Proposition C.4. Taking $f = a_X$ and $\mathscr{G} = \mathbf{D}_X$, we have equality of the morphisms

$$\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}) \otimes \boldsymbol{R}\Gamma(X,\boldsymbol{D}\mathscr{F}) \longrightarrow \boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F} \otimes \boldsymbol{D}\mathscr{F}) \longrightarrow \boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathbf{D}_X)$$

and

$$\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}) \otimes \boldsymbol{R}\Gamma(X,\boldsymbol{D}\mathscr{F}) \longrightarrow \boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}) \otimes \mathrm{Hom}(\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}),\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathbf{D}_X)) \longrightarrow \boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathbf{D}_X).$$

Besides, we have a commutative diagram, by considering the natural morphism $\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathbf{D}_X) \to \boldsymbol{k}$,

$$\begin{array}{ccc}
\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}) \otimes \mathrm{Hom}(\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}),\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathbf{D}_X)) & \longrightarrow & \boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathbf{D}_X). \\
\downarrow & & \downarrow \\
\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}) \otimes \mathrm{Hom}(\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}),\boldsymbol{k}) & \longrightarrow & \boldsymbol{k}
\end{array}$$

Hence, both composed natural morphisms

$$\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}) \otimes \boldsymbol{R}\Gamma(X,\boldsymbol{D}\mathscr{F}) \longrightarrow \boldsymbol{R}\Gamma_{\mathrm{c}}(X,\boldsymbol{D}\mathscr{F} \otimes \mathscr{F}) \longrightarrow \boldsymbol{k}$$

and

$$\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}) \otimes \boldsymbol{R}\Gamma(X,\boldsymbol{D}\mathscr{F}) \longrightarrow \boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}) \otimes \mathrm{Hom}(\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F}),\boldsymbol{k}) \longrightarrow \boldsymbol{k}$$

coincide, and we deduce the proposition for every ℓ. ☐

Let now $\mathscr{F},\mathscr{G}$ in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_X)$. We have a natural isomorphism (see [29, (2.6.8)])

$$\mathrm{Hom}(\mathscr{F} \otimes \mathscr{G},\mathbf{D}_X) \simeq \mathrm{Hom}(\mathscr{F},\boldsymbol{D}\mathscr{G}) \simeq \mathrm{Hom}(\mathscr{G},\boldsymbol{D}\mathscr{F}).$$

Giving a pairing $\varphi : \mathscr{F} \otimes \mathscr{G} \to \mathbf{D}_X$ amounts thus to giving a morphism $\mathscr{F} \to \boldsymbol{D}\mathscr{G}$, or as well a morphism $\mathscr{G} \to \boldsymbol{D}\mathscr{F}$. We say that the pairing φ is *perfect* if the corresponding morphism $\mathscr{F} \to \boldsymbol{D}\mathscr{G}$ (or as well $\mathscr{G} \to \boldsymbol{D}\mathscr{F}$) is an isomorphism.

A pairing φ induces in a natural way a pairing

$$\boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{G}) \otimes \boldsymbol{R}\Gamma(X,\mathscr{F}) \longrightarrow \boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathscr{F} \otimes \mathscr{G}) \longrightarrow \boldsymbol{R}\Gamma_{\mathrm{c}}(X,\mathbf{D}_X) \longrightarrow \boldsymbol{k}$$

(and a similar pairing by permuting the roles of $\mathscr{F}$ and $\mathscr{G}$), and thus, for every $\ell \in \mathbf{Z}$, pairings

$$\varphi^\ell : \boldsymbol{H}^{-\ell}_{\mathrm{c}}(X,\mathscr{G}) \otimes \boldsymbol{H}^\ell(X,\mathscr{F}) \longrightarrow \boldsymbol{H}^0_{\mathrm{c}}(X,\mathscr{F} \otimes \mathscr{G}) \longrightarrow \boldsymbol{H}^0_{\mathrm{c}}(X,\mathbf{D}_X) = \boldsymbol{k}.$$

Corollary C.6. *If φ is a perfect pairing, the pairings φ^ℓ are also perfect.*

Proof. By construction, the composition of φ^ℓ and the isomorphism

$$\boldsymbol{H}^\ell(X,\mathscr{F}) \xrightarrow{\sim} \boldsymbol{H}^\ell(X,\boldsymbol{D}\mathscr{G})$$

induced by φ is the pairing (C.3) for $\mathscr{G}$. ☐

Proof of Lemma C.5. We first show that, for sheaves $\mathscr{F}$ and $\mathscr{G}$ of $\boldsymbol{k}$-vector spaces, the natural morphisms

$$f_*\mathscr{F} \otimes f_*\mathscr{H}om(\mathscr{F},\mathscr{G}) \longrightarrow f_*\big(\mathscr{F} \otimes \mathscr{H}om(\mathscr{F},\mathscr{G})\big) \longrightarrow f_*\mathscr{G}, \tag{C.7}$$
$$f_*\mathscr{F} \otimes f_*\mathscr{H}om(\mathscr{F},\mathscr{G}) \longrightarrow f_*\mathscr{F} \otimes \mathscr{H}om(f_*\mathscr{F},f_*\mathscr{G}) \longrightarrow f_*\mathscr{G} \tag{C.8}$$

coincide. Let us first recall the construction of the natural morphism $\mathscr{F} \otimes \mathscr{H}om(\mathscr{F},\mathscr{G}) \to \mathscr{G}$. We denote by $[\mathscr{F} \otimes \mathscr{H}om(\mathscr{F},\mathscr{G})]^\sim$ the presheaf $U \mapsto \Gamma(U,\mathscr{F}) \otimes \Gamma(U,\mathscr{H}om(\mathscr{F},\mathscr{G}))$ whose associated sheaf is $\mathscr{F} \otimes \mathscr{H}om(\mathscr{F},\mathscr{G})$ by definition. Recall that $\Gamma(U,\mathscr{H}om(\mathscr{F},\mathscr{G})) = \mathrm{Hom}(\mathscr{F}_{|U},\mathscr{G}_{|U})$ (compatible families of morphisms $\mathrm{Hom}(\Gamma(U',\mathscr{F}),\Gamma(U',\mathscr{G}))$ for U' open in U), so that there is a forgetful morphism $\Gamma(U,\mathscr{H}\!om(\mathscr{F},\mathscr{G}))\to\mathrm{Hom}(\Gamma(U,\mathscr{F}),\Gamma(U,\mathscr{G}))$. By composition, we obtain a morphism

$$[\mathscr{F}\otimes\mathscr{H}\!om(\mathscr{F},\mathscr{G})]^{\sim}(U)\longrightarrow\Gamma(U,\mathscr{F})\otimes\mathrm{Hom}(\Gamma(U,\mathscr{F}),\Gamma(U,\mathscr{G}))\longrightarrow\Gamma(U,\mathscr{G}).$$

Since $\mathscr{G}$ is a sheaf, it induces a sheaf morphism $\mathscr{F}\otimes\mathscr{H}\!om(\mathscr{F},\mathscr{G})\to\mathscr{G}$. Considering only open sets of the form $f^{-1}(V)$ with V open in Y gives rise to the natural morphism $f_*(\mathscr{F}\otimes\mathscr{H}\!om(\mathscr{F},\mathscr{G}))\to f_*\mathscr{G}$. But the presheaf $V\mapsto[\mathscr{F}\otimes\mathscr{H}\!om(\mathscr{F},\mathscr{G})]^{\sim}(f^{-1}(V))$ is identified with the presheaf giving rise to $f_*\mathscr{F}\otimes f_*\mathscr{H}\!om(\mathscr{F},\mathscr{G})$, so $f_*(\mathscr{F}\otimes\mathscr{H}\!om(\mathscr{F},\mathscr{G}))=f_*\mathscr{F}\otimes f_*\mathscr{H}\!om(\mathscr{F},\mathscr{G})$. We thus have obtained (C.7). Finally, the morphism above reads $[f_*\mathscr{F}\otimes f_*\mathscr{H}\!om(\mathscr{F},\mathscr{G})]^{\sim}(V)\to\Gamma(V,f_*\mathscr{G})$, and its sheafification is nothing but (C.8).

By suitably restricting to f-proper supports, we obtain the non-derived version of the identification of morphisms in the lemma. In order to get the general case, one replaces $\mathscr{G}$ with an injective resolution $\mathscr{G}^\bullet\in\mathsf{D}^+(\boldsymbol{k}_X)$. It follows that, for $\mathscr{F},\mathscr{G}\in\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_X)$, $\boldsymbol{R}\mathscr{H}\!om(\mathscr{F},\mathscr{G})=\mathscr{H}\!om(\mathscr{F},\mathscr{G}^\bullet)$ is flabby (see [29, Prop. 2.4.6(vii)]) and, by considering a flabby resolution $\mathscr{F}^\bullet\in\mathsf{D}^+(\boldsymbol{k}_X)$ of $\mathscr{F}$, we replace e.g. $\mathscr{F}\otimes\boldsymbol{R}\mathscr{H}\!om(\mathscr{F},\mathscr{G})$ with $\mathscr{F}^\bullet\otimes\mathscr{H}\!om(\mathscr{F},\mathscr{G}^\bullet)$, so that we can apply the result for sheaves (we use flabbiness of $\mathscr{H}\!om(\mathscr{F},\mathscr{G}^\bullet)$ since we cannot consider $\mathscr{H}\!om(\mathscr{F}^\bullet,\mathscr{G}^\bullet)$, as $\boldsymbol{R}\mathscr{H}\!om$ is not defined on $\mathsf{D}^+(\boldsymbol{k}_X)\times\mathsf{D}^+(\boldsymbol{k}_X)$). □

Appendix D. Cohomology with compact support

In this section, we apply the setting of Section 3.a. We consider two de Rham realizations of the cohomology with compact support $\mathrm{H}^*_{\mathrm{c}}(U^{\mathrm{an}},\mathbf{C})$, namely the algebraic one $\mathrm{H}^*_{\mathrm{dR,c}}(U)$ and the C^∞ one $\mathrm{H}^*(\Gamma(X,\mathscr{E}_X^{\mathrm{rd},\bullet}))$. More precisely, we will recall that there exists a *canonical* isomorphism

$$\mathrm{can}:\mathrm{H}^*_{\mathrm{dR,c}}(U)\xrightarrow{\ \sim\ }\mathrm{H}^*(\Gamma(X,\mathscr{E}_X^{\mathrm{rd},\bullet})).$$

Let U_D be an affine neighbourhood of D in X. We will see that, by computing with a Čech complex relative to the covering (U_D,U), the cohomology space $\mathrm{H}^2_{\mathrm{dR,c}}(U)$ is a quotient of $\Omega^1(U_D^\circ)$, with $U_D^\circ=U_D\smallsetminus D$. Let $\mathrm{res}_D:\mathrm{H}^2_{\mathrm{dR,c}}(U)\to\mathbf{C}$ be induced by $\sum_{x\in D}\mathrm{res}_x:\Omega^1(U_D^\circ)\to\mathbf{C}$.

On the other hand, we consider the trace morphism

$$\mathrm{tr}_X:\frac{1}{2\pi\mathsf{i}}\int_X:\mathrm{H}^2(\Gamma(X,\mathscr{E}_X^{\mathrm{rd},\bullet}))\longrightarrow\mathbf{C}.$$

Proposition D.1. *The morphisms* tr_X *and* res_D *are isomorphims and the following diagram commutes:*

$$\begin{array}{ccc}\mathrm{H}^2_{\mathrm{dR,c}}(U) & \xrightarrow{\ \mathrm{res}_D\ } & \\ {\scriptstyle\mathrm{can}}\downarrow\wr & & \mathbf{C} \\ \mathrm{H}^2(\Gamma(X,\mathscr{E}_X^{\mathrm{rd},\bullet})) & \xrightarrow{\ \mathrm{tr}_X\ } & \end{array}$$

D.a. **Analytic comparison.** In this section, we endow X with its analytic topology, and we omit the exponent 'an'. We consider the (analytic) logarithmic de Rham complex

$$\mathrm{DR}^{\log}(\mathscr{O}_X(-D))=\Big\{\mathscr{O}_X(-D)\xrightarrow{\ \mathrm{d}\ }\Omega^1_X(D)\otimes\mathscr{O}_X(-D)\Big\}$$

attached to the (analytic) logarithmic lattice $\mathscr{O}_X(-D)$ of $\mathscr{O}_X$. This complex is quasi-isomorphic to $j_!\mathbf{C}_U$, as can be seen in the local setting, where the assertion is reduced to checking that $z\partial_z+1:\mathbf{C}\{z\}\to\mathbf{C}\{z\}$ is an isomorphism. Let Δ_D be the disjoint union of complex analytic discs Δ_x $(x\in D)$. Then the covering (Δ_D,U) is acyclic with respect to each term of the logarithmic complex. The corresponding Čech complex will simply be denoted by $\check{\mathrm{C}}^\bullet(X,\mathrm{DR}^{\log}(\mathscr{O}_X(-D)))$.

This is the simple complex attached to the double complex[5]

$$\begin{array}{ccc}\Gamma(\Delta_D,\mathscr{O}_{\Delta_D}(-D))\oplus\Gamma(U,\mathscr{O}_U) & \xrightarrow{\ \delta\ } & \Gamma(\Delta^*_D,\mathscr{O}_{\Delta^*_D})\\ \Big\downarrow{\scriptstyle\mathrm{d}} & & \Big\downarrow{\scriptstyle\mathrm{d}}\\ \Gamma(\Delta_D,\Omega^1_{\Delta_D})\oplus\Gamma(U,\Omega^1_U) & \xrightarrow{\ \delta\ } & \Gamma(\Delta^*_D,\Omega^1_{\Delta^*_D})\end{array}$$

that we regard as defined from the morphism between the first horizontal line to the second one. There exists thus a canonical isomorphism (see e.g. [20, p. 208], see also [4, §II.2])

$$\mathrm{H}^*\big(\check{\mathrm{C}}^\bullet(X,\mathrm{DR}^{\log}(\mathscr{O}_X(-D)))\big)\xrightarrow{\ \sim\ }\mathrm{H}^*_{\mathrm{c}}(U,\mathbf{C}). \tag{D.2}$$

Since $\mathrm{d}:\Gamma(\Delta_D,\mathscr{O}_{\Delta_D}(-D))\to\Gamma(\Delta_D,\Omega^1_{\Delta_D})$ is bijective, the above double complex is isomorphic to

$$\begin{array}{ccc}\Gamma(U,\mathscr{O}_U) & \xrightarrow{\ \rho\ } & \Gamma(\Delta^*_D,\mathscr{O}_{\Delta^*_D})\\ \Big\downarrow{\scriptstyle\mathrm{d}} & & \Big\downarrow{\scriptstyle\mathrm{d}}\\ \Gamma(U,\Omega^1_U) & \xrightarrow{\ \rho\ } & \Gamma(\Delta^*_D,\Omega^1_{\Delta^*_D})\end{array}$$

where ρ is the restriction from U to Δ^*_D, and also to

$$\begin{array}{ccc}\Gamma(U,\mathscr{O}_U) & \xrightarrow{\ \rho\ } & \Gamma(\Delta_D,j_*\mathscr{O}_{\Delta^*_D}/\mathscr{O}_{\Delta_D}(-D)).\\ \Big\downarrow{\scriptstyle\mathrm{d}} & & \Big\downarrow{\scriptstyle\mathrm{d}}\\ \Gamma(U,\Omega^1_U) & \xrightarrow{\ \rho\ } & \Gamma(\Delta_D,j_*\Omega^1_{\Delta^*_D}/\Omega^1_{\Delta})\end{array} \tag{D.3}$$

On the other hand, the C^∞ de Rham complex $(\mathscr{E}^{\mathrm{rd},\bullet}_X,\mathrm{d})$ with rapid decay at D is also a resolution of $j_!\mathbf{C}_U$: this follows for example from the similar result on the real bow-up space $\widetilde{X}$ (Proposition B.4) after taking the direct image by $\widetilde{X}\to X$. Since each $\mathscr{E}^{\mathrm{rd},\bullet}_X$ is fine, we obtain canonical isomorphisms (see [20, Th. 4.7.1, p. 181])

$$\mathrm{H}^*\big(\Gamma(X,\mathscr{E}^{\mathrm{rd},\bullet}_X),\mathrm{d}\big)\xrightarrow{\ \sim\ }\mathrm{H}^*_{\mathrm{c}}(U,\mathbf{C}). \tag{D.4}$$

In particular, by composing (D.2) and (D.4) we get a canonical isomorphism

$$\mathrm{can}:\mathrm{H}^2\big(\check{\mathrm{C}}^\bullet(X,\mathrm{DR}^{\log}(\mathscr{O}_X(-D)))\big)\xrightarrow{\ \sim\ }\mathrm{H}^2\big(\Gamma(X,\mathscr{E}^{\mathrm{rd},\bullet}_X),\mathrm{d}\big). \tag{D.5}$$

The group on the left-hand side is a quotient of $\Gamma(\Delta^*_D,\Omega^1_{\Delta^*_D})$ while that on the right is a quotient of $\Gamma(X,\mathscr{E}^{\mathrm{rd},2}_X)$. Let $\mu\in\Gamma(\Delta^*_D,\Omega^1_{\Delta^*_D})$. Its image in $\Gamma(\Delta^*_D,\mathscr{E}^{1,0}_{\Delta^*_D})$ writes $\psi_{|\Delta^*_D}-\varphi_{|\Delta^*_D}$, with $\varphi\in\Gamma(U,\mathscr{E}^{1,0}_U)$ and $\psi\in\Gamma(\Delta_D,\mathscr{E}^{\mathrm{rd},1,0}_{\Delta_D})$: indeed, the Čech cohomology $\mathrm{H}^1((\Delta_D,U),\mathscr{E}^{\mathrm{rd},1,0}_X)$ is zero since $\mathscr{E}^{\mathrm{rd},1,0}_X$ is fine.

Lemma D.6. *For μ as above, we have* $\mathrm{can}([\mu])=[\mathrm{d}\varphi]$.

The proof will be given below.

Proof of Proposition D.1 *in the analytic setting.* With the above notation, recall that $\mathrm{d}\varphi$ has rapid decay, so $\int_X\mathrm{d}\varphi$ is well-defined. We have

$$\frac{1}{2\pi\mathsf{i}}\int_X\mathrm{d}\varphi=\frac{1}{2\pi\mathsf{i}}\int_{X\smallsetminus\Delta_D}\mathrm{d}\varphi+\frac{1}{2\pi\mathsf{i}}\int_{\Delta_D}\mathrm{d}\psi=\frac{1}{2\pi\mathsf{i}}\int_{\partial\Delta_D}(\psi-\varphi)=\frac{1}{2\pi\mathsf{i}}\int_{\partial\Delta_D}\mu=\mathrm{res}_D\,\mu. \qquad\square$$

[5]Given a morphism $\varphi:(K^\bullet_0,d_0)\to(K^\bullet_1,d_1)$ of complexes, we associate with it the double complex $(K^{\bullet,\bullet},d',d'')$ with $K^{0,j}=K^j_0$, $K^{1,j}=K^j_1$, $K^{i,j}=0$ for $i\geqslant 2$, $d'=\varphi$, $d''=(-1)^jd_j$, so that $d'd'=d''d''=d'd''+d''d'=0$.

Proof of Lemma D.6. We denote by $(\mathscr{E}_X^{\log,\bullet}(-D),\mathrm{d})$ the de Rham complex with terms

$$\mathscr{E}_X^{\log,0}(-D)=\mathscr{E}_X^0(-D),\quad \mathscr{E}_X^{\log,1}(-D)=\mathscr{E}_X^{1,0}\oplus\mathscr{E}_X^{0,1}(-D),\quad \mathscr{E}_X^{\log,2}(-D)=\mathscr{E}_X^2.$$

The quasi-isomorphisms of complexes

$$\mathrm{DR}^{\log}\mathscr{O}_X(-D)\xrightarrow{\sim}(\mathscr{E}_X^{\log,\bullet}(-D),\mathrm{d})\xleftarrow{\sim}(\mathscr{E}_X^{\mathrm{rd},\bullet},\mathrm{d})$$

give rise to quasi-isomorphisms of Čech complexes (with respect to (Δ_D,U))

$$\begin{array}{ccccc}\check{\mathcal{C}}^\bullet(X,\mathrm{DR}^{\log}(\mathscr{O}_X(-D))) & \xrightarrow{\sim} & \check{\mathcal{C}}^\bullet(X,\mathscr{E}_X^{\log,\bullet}(-D)) & \xleftarrow{\sim} & \check{\mathcal{C}}^\bullet(X,\mathscr{E}_X^{\mathrm{rd},\bullet})\\ & & \uparrow\wr & & \uparrow\wr\\ & & \Gamma(X,\mathscr{E}_X^{\log,\bullet}(-D)) & \xleftarrow{\sim} & \Gamma(X,\mathscr{E}_X^{\mathrm{rd},\bullet})\end{array}$$

and the diagram commutes. The isomorphisms induced on cohomology are the canonical ones, defined similarly to (D.5).

Let μ be as in the lemma. In particular, $\mathrm{d}\mu=0$. We regard μ as defining a class in $\mathrm{H}^2_{\mathrm{c}}(U)$. In the realization by $\check{\mathcal{C}}^\bullet(X,\mathscr{E}_X^{\log,\bullet}(-D))$, it is written

$$(0,0,\mu)\in\Gamma(\Delta_D,\mathscr{E}_X^{\log,2}(-D))\oplus\Gamma(U,\mathscr{E}_U^2)\oplus\Gamma(\Delta_D^*,\mathscr{E}_U^1).$$

Any choice of $\varphi\in\Gamma(U,\mathscr{E}_U^1)$ and $\psi\in\Gamma(\Delta_D,\mathscr{E}_{\Delta_D}^{\log,1}(-D))$ such that $\psi-\varphi=\mu$ on Δ_D^* gives rise to another representative, namely $(\mathrm{d}\psi,\mathrm{d}\varphi,0)$, since the total differential of (ψ,φ) is equal to $(\mathrm{d}\psi,\mathrm{d}\varphi,\varphi-\psi)$. Such φ and ψ exist since $\mathrm{H}^1(X,\mathscr{E}_{\Delta_D}^{\log,1}(-D))=0$. Moreover, $\mathrm{d}\psi-\mathrm{d}\varphi=\mathrm{d}\mu=0$ on Δ_D^*, so that $(\mathrm{d}\psi,\mathrm{d}\varphi)$ glue as $\omega\in\Gamma(X,\mathscr{E}_X^2)$. The image of $[\mu]$ by the canonical morphism $\mathrm{H}^2_{\mathrm{dR,c}}(U)\to\mathrm{H}^2(\Gamma(X,\mathscr{E}_X^{\log,\bullet}(-D))=\mathrm{H}^2(X)$ is equal to $[\omega]$.

Since $\mathrm{H}^1(X,\mathscr{E}_X^{\mathrm{rd},1})=0$, we can also choose $\psi\in\Gamma(\Delta_D,\mathscr{E}_X^{\mathrm{rd},1})$ (up to changing φ). Then $[\mathrm{d}\varphi]\in\mathrm{H}^2(\Gamma(X,\mathscr{E}_X^{\mathrm{rd},\bullet})$ maps to (the corresponding) $[\omega]$ in $\mathrm{H}^2(\Gamma(X,\mathscr{E}_X^{\log,\bullet}(-D))$, which concludes the proof. □

D.b. **Algebraic comparison.** We now use the Zariski topology on X and indicate with an exponent 'an' the analytic topology. We choose a covering (U_D,U) of X by affine open subsets. Its Čech resolution relative to (U_D,U) is

$$\begin{array}{ccc}\mathscr{O}_{U_D}(-D)\oplus\mathscr{O}_U & \xrightarrow{\delta} & \mathscr{O}_{U_D^\circ}.\\ \mathrm{d}\downarrow & & \downarrow\mathrm{d}\\ \Omega^1_{U_D}\oplus\Omega^1_U & \xrightarrow{\delta} & \Omega^1_{U_D^\circ}\end{array}$$

The de Rham cohomology $\mathrm{H}^2_{\mathrm{dR,c}}(U)$ is computed as the cohomology of the Čech complex

$$\begin{array}{ccc}\Gamma(U_D,\mathscr{O}_{U_D}(-D))\oplus\Gamma(U,\mathscr{O}_U) & \xrightarrow{\delta} & \Gamma(U_D^\circ,\mathscr{O}_{U_D^\circ})\\ \mathrm{d}\downarrow & & \downarrow\mathrm{d}\\ \Gamma(U_D,\Omega^1_{U_D})\oplus\Gamma(U,\Omega^1_U) & \xrightarrow{\delta} & \Gamma(U_D^\circ,\Omega^1_{U_D^\circ})\end{array}$$

or equivalently

$$\begin{array}{ccc}\Gamma(U,\mathscr{O}_U) & \xrightarrow{\delta} & \Gamma(U_D,\mathscr{O}_{U_D}(*D)/\mathscr{O}_{U_D}(-D))\\ \mathrm{d}\downarrow & & \downarrow\mathrm{d}\\ \Gamma(U,\Omega^1_U) & \xrightarrow{\delta} & \Gamma(U_D,\Omega^1_{U_D}(*D)/\Omega^1_{U_D}(-D)).\end{array}$$

This expression shows the independence with respect to the choice of the affine neighbourhood U_D, since the sheaves on the right-hand side are supported on D. The above complex maps canonically to the corresponding analytic complex (D.3), defining the canonical map $\mathrm{H}^*_{\mathrm{dR,c}}(U) \to \mathrm{H}^*_{\mathrm{dR,c}}(U^{\mathrm{an}})$. The canonical morphism of Proposition D.1 is composed from the latter and (D.5). Proposition D.1 then follows from its analytic version. □

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