Multiple zeta values: from numbers to motives

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Preface

Multiple zeta values (MZVs for short) are real numbers of the form

$$\zeta(s_1, s_2, \ldots, s_\ell) = \sum_{n_1 > n_2 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}},$$

(0.1)

where all the exponents $s_i$ are integers greater than or equal to 1 and $s_1 \geq 2$, to ensure that the sum converges. For $\ell = 1$, these are the special values at integers $s \geq 2$ of the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$ 

Euler proved that, when $s$ is even, $\zeta(s)$ is a rational multiple of $\pi^s$; for example, $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. The values at odd integers are much more mysterious. Indeed, a folklore conjecture asserts that they are all “new” transcendental numbers:

**Transcendence conjecture.** The numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

This conjecture seems completely out of reach: at the time of writing, we do not even know whether $\zeta(3)$ is transcendental, let alone the algebraic independence with $\pi$, or whether $\zeta(5)$ is irrational!

The case $\ell = 2$ was also considered by Euler, back in his 1776 paper *Meditationes circa singulare serierum genus* (“Meditations about a singular type of series”) [Eul76]. In an attempt to find a closed formula for $\zeta(3)$, he looked for linear relations with integer coefficients among the numbers $\pi^3$, $\pi^2 \log 2$, and $(\log 2)^3$. This led him to the discovery of remarkable identities involving double zeta values, the simplest being $\zeta(3) = \zeta(2, 1)$.

After more than two centuries of oblivion, multiple zeta values were independently rediscovered in the 1990s by Hoffman and Zagier. It was soon realized that these numbers appear in a wealth of different contexts, including Witten’s zeta functions, deformation quantization, Vassiliev knot invariants, and the theory of mixed Tate motives. Most of these topics share a physics flavour. In fact, roughly at the same time, the physicists Broadhurst and Kreimer found that a lot of Feynman amplitudes in quantum field theory are given by multiple zeta values. The next two decades saw extensive work by Brown, Cartier, Deligne, Drinfeld, Écalle, Goncharov, Hain, Hoffman, Kontsevich, Terasoma, Zagier, and many others. Major progress was made, but fundamental questions remain open and multiple zeta values are still nowadays an active, rapidly moving field of research.

The product of two multiple zeta values is a linear combination, with integral coefficients, of multiple zeta values. For instance,

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2),$$
an identity already known to Euler. The \( \mathbb{Q} \)-subvector space \( \mathcal{Z} \subseteq \mathbb{R} \) spanned by all multiple zeta values has thus an algebra structure. Contrary to the algebra generated by Riemann zeta values, which according to the transcendence conjecture should simply be a polynomial algebra in \( \zeta(2), \zeta(3), \zeta(5) \ldots \), multiple zeta values satisfy a plethora of relations that endow \( \mathcal{Z} \) with a rich combinatorial structure. One can argue that the main goal of the theory is to understand all linear relations among these numbers.

To make this more precise, we attach to each \( \zeta(s_1, \ldots, s_\ell) \) the integer \( s_1 + \ldots + s_\ell \), which is called the weight. Let \( \mathcal{Z}_k \subseteq \mathcal{Z} \) be the vector subspace generated by multiple zeta values of weight \( k \), with the convention that \( \mathcal{Z}_0 = \mathbb{Q} \) and \( \mathcal{Z}_1 = \{0\} \). Based on a mix of numerical evidence and pure thought, Zagier conjectured “after many discussions with Drinfeld, Kontsevich, and Goncharov” that there is a direct sum decomposition

\[ \mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k, \]

and that the dimension of each graded piece is given by a Fibonacci-like sequence of integers

\[ \dim_{\mathbb{Q}} \mathcal{Z}_k = d_k. \quad (0.2) \]

Precisely, \((d_k)_{k \geq 0}\) is defined recursively by setting \( d_0 = d_2 = 1, d_1 = 0 \), and \( d_k = d_{k-2} + d_{k-3} \) for all \( k \geq 3 \), so that the generating series is equal to

\[ \sum_{k \geq 0} d_k t^k = \frac{1}{1 - t^2 - t^3}. \quad (0.3) \]

This conjecture would imply that \( \dim_{\mathbb{Q}} \mathcal{Z}_k \) grows like a constant multiple of \( r^k \), where \( r = 1.3247 \ldots \) is the real root of \( x^3 - x - 1 \), which is much smaller than the number \( 2^{k-2} \) of multi-indices \((s_1, \ldots, s_\ell)\) of weight \( k \).

**Plan.** The goal of these notes is to give a reasonably self-contained proof of the following results towards Zagier’s conjecture:

**Theorem A** (Deligne–Goncharov [DG05], Terasoma [Ter02]). The integers \( d_k \) are upper bounds for the dimensions of \( \mathcal{Z}_k \):

\[ \dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k. \]

**Theorem B** (Brown, [Bro12]). Each multiple zeta value can be written as a \( \mathbb{Q} \)-linear combination of multiple zeta values with only 2s and 3s as exponents, i.e. the following family generates the \( \mathbb{Q} \)-vector space \( \mathcal{Z} \):

\[ \{ \zeta(s_1, \ldots, s_\ell) \mid s_i \in \{2, 3\} \}. \quad (0.4) \]

In fact, Hoffman conjectured that (0.4) forms a *basis* of \( \mathcal{Z} \). By a simple counting argument, this would imply equality (0.2). Theorem B addresses the “algebraic” part of this conjecture, which suffices to deduce Theorem A. It is also worth mentioning that, taking these results for granted, the algebraic independence of \( \pi, \zeta(3), \zeta(5), \ldots \) is a consequence of Zagier’s conjecture. In a sense, we have “linearized” the transcendence conjecture. On the
negative side, let us emphasize that, despite the progress made thus far, we still do not know a single $k$ for which $\dim_{\mathbb{Q}} \mathcal{Z}_k$ is bigger than one!

Surprisingly enough, the proofs of these easy-to-state theorems use the machinery of motives. Kontsevich noticed that multiple zeta values of weight $k$ admit a representation as iterated integrals

$$\zeta(s_1, \ldots, s_\ell) = \int_{\Delta^k} \omega_0(t_1) \cdots \omega_0(t_{s_1-1}) \omega_1(t_{s_1}) \omega_0(t_{s_1+1}) \cdots \omega_1(t_k), \quad (0.5)$$

where $\omega_0(t) = dt/t$ and $\omega_1(t) = dt/(1-t)$ are differential forms on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and the integration domain is the simplex

$$\Delta^k = \{(t_1, \ldots, t_k) \in [0, 1]^k \mid 1 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 0\}.$$

This integral representation exhibits multiple zeta values as periods of algebraic varieties. In the words of Deligne, “whereas the notion of infinite sum is unfamiliar (étrangère) to algebraic geometry, the study of integrals of algebraic quantities is one of its sources.” Thanks to the identity (0.5), “algebraic geometry, and more precisely the theory of mixed Tate motives, is useful for the study of multiple zeta values” [Del13, p. 3].

Usually, the philosophy of motives represents a powerful tool to predict all algebraic relations between periods. However, when it comes to proving them, one is confronted with the problem that even the first step in this program—getting a category of motives with all the desired properties—remains conjectural. In contrast, for mixed Tate motives over a number field, there is an unconditional theory that relies ultimately on Borel’s deep results about the $K$-theory of number fields. This gives good control over the group governing the symmetries of multiple zeta values. Using this group, one can construct a pro-algebraic variety, together with an action of $\mathbb{G}_m$, in such a way that the Hilbert–Poincaré series of its graded algebra of functions $\mathcal{H}$ coincides with (0.3). The raison d’être of this construction is the existence of a surjective map $\mathcal{H} \to \mathcal{Z}$ compatible with the weight; we shall refer to elements of $\mathcal{H}$ as “motivic multiple zeta values”. This immediately implies Theorem A. To prove Theorem B, one exploits the motivic coaction, a new structure of $\mathcal{H}$, invisible at the level of numbers, that allows one to get relations among motivic multiple zeta values in a systematic way. A variant of the Grothendieck period conjecture asserts that the algebras $\mathcal{H}$ and $\mathcal{Z}$ are isomorphic, from which Zagier’s conjecture would follow.

Outline. Let us now give a more detailed description of the contents of each chapter. The word cloud on the next page should also give a quick idea of the main concepts involved.

Chapter 1 lays out what could be called the “minimal theory” of multiple zeta values. We first define them as infinite series and prove that the product of two multiple zeta values is a linear combination of multiple zeta values by decomposing the indexation domain. This so-called stuffle product makes $\mathcal{Z}$ into a $\mathbb{Q}$-algebra, conjecturally graded by the weight. We discuss Zagier’s
conjecture for the dimension of the graded pieces, as well as refinements due to Hoffmann, and Broadhurst and Kreimer. That progress has been made towards these conjectures relies very much on the existence of the integral representation \((0.5)\). We prove that the decomposition of the product of two simplices yields a new algebra structure on \(Z\), the shuffle product. Comparing the stuffle and the shuffle product, one gets many relations among multiple zeta values but not all of them. As we explain in the last section of the chapter, to conjecturally describe the full algebraic structure, one needs to introduce a regularization process that assigns a finite value to the divergent series \(\zeta(1, s_2, \ldots, s_\ell)\).

The goal of Chapter 2 is to show that multiple zeta values are periods of algebraic varieties. To begin with, we briefly recall the definition of singular cohomology of a differential manifold and de Rham’s theorem, according to which it can be computed using analytic differential forms. Grothendieck’s breakthrough was to realize that, if we are dealing with algebraic varieties, then algebraic differential forms suffice; this gives rise to algebraic de Rham cohomology and the period isomorphism. After introducing these concepts, we give a first interpretation, due to Goncharov and Manin, of multiple zeta values as periods of the moduli spaces \(\mathcal{M}_{0,n}\) of stable genus zero curves. We then move to mixed Hodge structures (a first approximation to the notion of motive), discuss a number of examples, and compute the extension groups of \(\mathbb{Q}(0)\) by \(\mathbb{Q}(n)\). We end the chapter with a discussion of the problem of finding a geometric construction of these extensions, as well as a potential application to irrationality proofs following Brown.

Chapter 3 introduces iterated integrals, a second way to interpret multiple zeta values as periods. We first present the basic definitions and tackle
the question of which iterated integrals are homotopy invariant. We then recall the notions of affine group scheme and Hopf and Lie algebras, which will be extensively used in the sequel. We define the pro-unipotent completion of a group and we construct it, under some finiteness assumptions, following work of Quillen. One of the main results of the chapter is Chen’s $\pi_1$-de Rham theorem, which roughly says that functions on the pro-unipotent completion of the fundamental group of a differential manifold $M$ are given by homotopy invariant iterated integrals. A consequence, due to Hain, is that when $M$ underlies an algebraic variety, this pro-unipotent completion carries a mixed Hodge structure. The general formalism being settled, we specialize everything to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Multiple zeta values are iterated integrals along the straight path from 0 to 1. Since the endpoints do not belong to the space, this forces us to work with tangential base points. The last section examines in detail all the structures carried by the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, including Goncharov’s coproduct.

In Chapter 4, we study the category of mixed Tate motives over $\mathbb{Z}$. The first two sections contain reminders of the Tannakian formalism, triangulated categories, and $t$-structures. We then sketch a construction of Voevodsky’s triangulated category of mixed motives over a field $k$. It is unknown how to extract an abelian category with good properties from it. However, it was observed by Levine that, when $k$ is a number field, Borel’s results on $K$-theory enable one to extract an abelian category of mixed Tate motives over $k$, which is moreover Tannakian. Even for $k = \mathbb{Q}$, this category is too large for the purposes of studying multiple zeta values. To remedy this, one defines the subcategory of mixed Tate motives over $\mathbb{Z}$. We determine the structure of its Tannaka group and show, after Deligne and Goncharov, that it contains a pro-object whose Hodge realization is the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Finally, in Chapter 5 we pull everything together to prove the main results. In the first section, we construct the graded algebra $H$ of motivic multiple zeta values and a surjective map $H \rightarrow \mathbb{Z}$ compatible with the grading. Using the structure of the Tannaka group of the category of mixed Tate motives over $\mathbb{Z}$, we derive Theorem A. We then present the proof of Theorem B, following closely Brown’s original paper.

The notes are supplemented by an appendix where we discuss some results in homological algebra. It contains an introduction to abelian categories, triangulated categories and $t$-structures, derived functors, filtrations and spectral sequences and sheaf cohomology.

**Warning.** Before continuing, we should warn the reader that the literature contains two competing conventions for multiple zeta values, sometimes in the same paper! Other authors, including Brown, define $\zeta(s_1, \ldots, s_\ell)$, for
\[ s_i \geq 1 \text{ and } s_\ell \geq 2, \text{ as the sum } \]
\[ \sum_{1 \leq n_1 < n_2 < \cdots < n_\ell} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}. \]

In fact, one needs to fix conventions for the order of composition of paths, the definition of iterated integrals, and the expression of multiple zeta values as iterated integrals. Things get simpler if they are compatible. We have chosen those conventions for which the monodromy of a local system is a group morphism.

**Prerequisites.** The difficulty of the exposition increases as the notes progress. In Chapter 1, besides a couple of digressions, the emphasis is mainly on combinatorial aspects and very little background is required. From Chapter 2 on, we assume some familiarity with algebraic varieties, the language of schemes and cohomology of sheaves, at the level of any introductory book, for instance [Har77]. Chapter 3 contains a crash course on algebraic groups and Lie and Hopf algebras, which will play an important role in the sequel. However, we do not treat topics such as Lie algebra cohomology or Galois cohomology that will only appear in some proofs of the following chapter. Finally, in Chapter 4 we freely use basic notions from category theory and homological algebra, for example abelian categories. We have done our best to present all the materials in the most clear and accessible way, but occasionally we were unable to prevent the text from being sketchy. Unfortunately, Borel’s theorem about the \( K \)-theory of number fields is used as a black box. For the convenience of the reader, in the appendix we gather several results in homological algebra needed through the book.

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1. Classical theory of multiple zeta values
(by J. I. Burgos Gil, J. Fresán, and U. Kühn)

In this chapter, we introduce multiple zeta values and begin to study their basic properties. These are the real numbers
\[ \zeta(s_1, \ldots, s_\ell) = \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}} \]
associated with tuples of integers \( s = (s_1, \ldots, s_\ell) \) such that \( s_i \geq 1 \) and \( s_1 \geq 2 \). The quantity \( s_1 + \cdots + s_\ell \) is called the weight and \( \ell \) is referred to as the length. Of great importance is that multiple zeta values cannot only be written as infinite series, as above, but also as integrals. This gives two different ways of showing that the product of \( \zeta(s) \) and \( \zeta(s') \) is a linear combination, with integral coefficients, of multiple zeta values or, in more algebraic terms, that the \( \mathbb{Q} \)-vector space \( \mathbb{Z} \subseteq \mathbb{R} \) generated by multiple zeta values has an algebra structure. From the series representation one obtains the stuffle product, whereas the integral representation gives the shuffle product. Comparing both products yields many relations amongst multiple zeta values. However, Euler’s identity \( \zeta(3) = \zeta(2,1) \) cannot be obtained by this method since the product of multiple zeta values has always weight at least 4. A way to solve this problem is to introduce a regularization process which assigns a finite value to the divergent series corresponding to multi-indices with \( s_1 = 1 \). There will be, in fact, two kinds of regularizations, modelled on the stuffle and the shuffle product. Conjecturally, all relations amongst multiple zeta values come from comparing them.

Good references for this chapter are the survey articles [Car02], [Wal12] and [Zud03], as well as Chapter 3 of the book [Zha16].

1.1. Riemann zeta values. The Riemann zeta function is one of the most famous objects in mathematics. It is said that it encodes all arithmetic properties of prime numbers: our task is to extract them.

Definition 1.1. The Riemann zeta function is defined, on the half-plane of complex numbers \( s \) with \( \text{Re}(s) > 1 \), by the absolute convergent series
\[ \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \] (1.2)
and extended to a meromorphic function on the whole complex plane with a single pole at \( s = 1 \).

The Riemann zeta function still keeps many mysteries. The most impenetrable of them is undoubtedly the Riemann hypothesis (the conjecture that all the non-trivial zeros of \( \zeta(s) \) lie in the line \( \text{Re}(s) = 1/2 \)), which has many far-reaching consequences in number theory.

The aim of these notes is to glimpse at other aspects of this function, namely, what numbers do we get when evaluating \( \zeta \) at integers? In fact, the
story began 120 years before Riemann’s paper\(^1\), with Euler’s solution to the so-called *Basel problem*, that is, the computation of the value
\[
\zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

Indeed, Euler showed much more:

**Theorem 1.3 (Euler, 1735).** The values of the zeta function at even positive integers are given by
\[
\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}.
\]  
(1.4)

Here \(B_{2k}\) are rational numbers, called *Bernoulli numbers* and defined by the power series identity
\[
\frac{t}{e^t - 1} = 1 + \sum_{k \geq 1} B_k \frac{t^k}{k!}.
\]  
(1.5)

**Remark 1.6.** Note that the function
\[
f(t) = \frac{t}{e^t - 1} + \frac{1}{2} t = \frac{t(1 + e^t)}{2(e^t - 1)}
\]
is even, i.e. satisfies \(f(t) = f(-t)\). It follows that \(B_1 = -\frac{1}{2}\) and \(B_k = 0\) for all odd integers \(k \geq 3\). The first Bernoulli numbers are easily computed:

\[
\begin{array}{c|cccccc}
  k & 2 & 4 & 6 & 8 & 10 & 12 \\
  B_k & \frac{1}{6} & -\frac{1}{30} & \frac{1}{42} & -\frac{1}{30} & \frac{5}{66} & -\frac{691}{2730} \\
\end{array}
\]

**Proof of Theorem 1.3.** The key ingredient is an identity for the cotangent function, also due to Euler (see Exercise 1.19). For \(x \in \mathbb{C} \setminus \mathbb{Z}\),
\[
\pi \cot(\pi x) = \frac{1}{x} + \sum_{n \geq 1} \frac{2x}{\pi^2 - n^2}.
\]  
(1.7)

Expanding the quotient inside the summation sign as a geometric series and interchanging the order of summation, we obtain
\[
\pi \cot(\pi x) = \frac{1}{x} - 2 \sum_{k \geq 1} \zeta(2k)x^{2k-1}.
\]  
(1.8)

Besides, we have
\[
\frac{1}{e^t - 1} = \frac{e^{-\frac{1}{2}}}{e^t - e^{-\frac{1}{2}}} \quad \text{and} \quad \frac{1}{e^{-t} - 1} = \frac{e^{\frac{1}{2}}}{e^t - e^{-\frac{1}{2}}}.
\]

\(^1\)For the prehistory of the Riemann zeta function we refer the reader to Weil’s beautiful account [Wei89].
from which the identity
\[
\frac{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}{e^{\frac{1}{2}} - e^{-\frac{1}{2}}} = 2 + 2 \sum_{k \geq 1} B_{2k} t^{2k-1}/(2k)!
\]
follows, using (1.5) and the vanishing of \(B_k\) for odd \(k \geq 3\). Therefore,
\[
\pi \cot(\pi x) = \pi i \frac{e^{\frac{2\pi i x}{2}} + e^{-\frac{2\pi i x}{2}}}{e^{\frac{2\pi i x}{2}} - e^{-\frac{2\pi i x}{2}}} = \frac{1}{x} + \sum_{k \geq 1} \frac{(2\pi i)^{2k} B_{2k}}{(2k)!} x^{2k-1},
\]
and we conclude by identifying the coefficients in (1.8) and (1.9). □

Remarks 1.10.

(1) Euler’s formula (1.4) implies the equality
\[
Q[\zeta(2), \zeta(4), \ldots] = Q[\pi^2]
\]
of subrings of the real numbers.

(2) Thanks to the functional equation
\[
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),
\]
where \(\Gamma\) is the gamma function, we deduce the values of the Riemann zeta function at negative integers:
\[
\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}
\]
for all \(k \geq 1\). In particular, \(\zeta(-2k) = 0\) for all \(k \geq 1\); these are the “trivial zeros”. One can also compute \(\zeta(0) = -\frac{1}{2}\).

1.1.1. Odd zeta values. By contrast, despite the many efforts of the mathematical community, nobody has been able to give closed formulas for the values of the Riemann zeta function at odd integers \(s = 3, 5, \ldots\) in terms of previously known numbers like \(\pi\). This led to the following conjecture:

**Conjecture 1.12 (Transcendence conjecture).** The numbers
\[
\pi, \zeta(3), \zeta(5), \ldots
\]
are algebraically independent over \(\mathbb{Q}\), that is, for each integer \(k \geq 0\) and each non-zero polynomial \(P \in \mathbb{Z}[x_0, \ldots, x_k]\), one has \(P(\pi, \zeta(3), \ldots, \zeta(2k+1)) \neq 0\).

This conjecture seems completely out of reach of the current techniques in transcendence theory. The transcendence of \(\pi\) was proved by Lindemann in 1882 [Lin82]. Combined with Euler’s formula (1.4), it implies that all the numbers \(\zeta(2k)\) are transcendental. But we do not even know whether \(\zeta(3)\) is transcendental—not to speak of the algebraic independence with \(\pi\)—or whether \(\zeta(5)\) is irrational. The few known results, as the moment of writing, are summarized below. The Bourbaki seminar [Fis04] is an excellent survey of the developments prior to 2004.
• Apéry proved the irrationality of $\zeta(3)$ in 1978, see [Apé78] and [vdP79] for a more detailed account. Different proofs by Beukers [Beu79, Beu87], Nesterenko [Nes96], Sorokin [Sor98], and Prévost [Pré96] are now available, but none of them seems to generalize to other odd zeta values.

• Rivoal [Riv00] and Ball and Rivoal [BR01] proved that, if $n \geq 3$ is an odd integer, then
  \[ \dim_{\mathbb{Q}} \langle 1, \zeta(3), \zeta(5), \ldots, \zeta(n) \rangle \geq \frac{1}{3} \log(n). \]
In particular, infinitely many odd zeta values $\zeta(2k + 1)$ are irrational. A proof “by elementary means” of this corollary was recently given by Sprang [Spr18] building on ideas of Zudilin [Zud18].

• Zudilin [Zud01] proved that at least one out of the four numbers $\zeta(5), \zeta(7), \zeta(9),$ and $\zeta(11)$ is irrational.

**Remark 1.13.** Recently, Brown has suggested in [Bro16] a common geometric framework for these irrationality proofs. The approach is based on the study of periods of the moduli spaces $M_{0,n}$ of curves of genus zero with $n$ marked points (see paragraph 2.8.3).

**Digression 1.14.** Despite their “simplicity”, special values of the Riemann zeta function are linked to much interesting mathematics. For instance, $K$-groups and regulators provide an explanation of why the values at even integers are easier to understand.

Let $F$ be a number field and $\mathcal{O}_F$ its ring of integers. The *Dedekind zeta function* of $F$ is defined, for $\text{Re}(s) > 1$, by the convergent series
  \[ \zeta_F(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}, \]
where $\mathfrak{a}$ runs through all non-zero ideals of $\mathcal{O}_F$ and $N(\mathfrak{a})$ denotes the absolute norm. In particular, $\zeta_{\mathbb{Q}}$ agrees with the Riemann zeta function (1.2).

The Dedekind zeta function extends to a meromorphic function on the complex plane, with a simple pole at $s = 1$. Its residue is given by the celebrated *class number formula*
  \[ \lim_{s \to 1} (s - 1) \zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}h_FR_F}{w_F\sqrt{|d_F|}}, \]
where $r_1$ (resp. $2r_2$) denotes the number of real (resp. complex) embeddings of $F$, $h_F$ is the class number, $w_F$ is the number of roots of unity contained in $F$, and $d_F$ stands for the discriminant.

The remaining term $R_F$ is defined using the *Dirichlet regulator* map
  \[ \rho: \mathcal{O}_F^\times \longrightarrow \mathbb{R}^{r_1 + r_2} \]
  \[ u \mapsto (\log \|u\|_v)_v. \]
Here $v$ runs over all archimedean places of $F$ and we write

$$||u||_v = \begin{cases} |\sigma(u)| & \text{if } v = \sigma \text{ is a real place}, \\ |\sigma(u)|^2 & \text{if } v = \{\sigma, \sigma\} \text{ is a complex place}. \end{cases}$$

The product formula $\prod_v ||u||_v = 1$ implies that $\rho$ lands in the hyperplane of points whose coordinates sum to zero. In fact, Dirichlet showed that the image of $\rho$ is a lattice in $\mathbb{R}^{r_1+r_2-1}$, that is, a subgroup of the form $\mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_{r_1+r_2-1}$ for linearly independent vectors $v_1, \ldots, v_{r_1+r_2-1}$. By definition, its covolume is the Lebesgue measure of the set

$$\{x_1v_1 + \cdots + x_{r_1+r_2-1}v_{r_1+r_2-1} \mid x_i \in \mathbb{R}, \ 0 \leq x_i < 1\}.$$ 

The covolume of the lattice $\rho(\mathcal{O}_F^\times)$ is a real number $R_F$, abusively called Dirichlet regulator as well.

Borel generalized this picture to other values of the Dedekind zeta function. The role of the units $\mathcal{O}_F^\times$ is replaced by the higher $K$-groups $K_n(\mathcal{O}_F)$, certain finitely generated abelian groups which carry a lot of information about the “hidden” arithmetic of $F$. Borel computed the rank of these groups and defined, for each $n \geq 2$, a map from $K_{2n-1}(\mathcal{O}_F)$ to a suitable finite-dimensional real vector space, the Borel regulator map, whose image is again a lattice. Its covolume is a real number $R_n$, also called Borel regulator. Letting $\zeta^*_F(1-n)$ denote the first non-vanishing coefficient in the Taylor expansion of the Dedekind zeta function at $s = 1-n$, he proved that there exists a rational number $q_n$ such that

$$\zeta^*_F(1-n) = q_n R_n.$$ 

The Dedekind zeta function satisfies a functional equation similar to (1.11). Using it, it follows that $\zeta_F(n)$ is, up to some easy factor involving the square root of the discriminant and powers of $\pi$, a rational multiple of $R_n$.

When $F = \mathbb{Q}$, the $K$-group $K_{2n-1}(\mathbb{Z})$ has rank one if $n \geq 3$ is odd, and zero otherwise (see Section 4.3 below). Therefore, $R_n = 1$ for even $n$. Thus, $\zeta(n)$ is given by a rational number times a power of $\pi$ for even $n$, while it involves the “mysterious” Borel regulator for odd $n$. This result will play a pivotal role in the motivic approach to multiple zeta values. For more details, we refer the reader to the original papers [Bor74] and [Bor77], the monograph [BG02] or the short survey [Sou10].

1.1.2. Double zeta values. In order to investigate possible relations among zeta values, Euler looked at the algebraic structure of these numbers. If we
multiply two Riemann zeta values, we obtain a new kind of interesting sum:

\[
\zeta(s_1) \cdot \zeta(s_2) = \left( \sum_{n_1 \geq 1} \frac{1}{n_1^{s_1}} \right) \cdot \left( \sum_{n_2 \geq 1} \frac{1}{n_2^{s_2}} \right) \\
= \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} \\
= \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_2 > n_1 \geq 1} \frac{1}{n_2^{s_2} n_1^{s_1}} + \sum_{n=n_1=n_2 \geq 1} \frac{1}{n^{s_1+s_2}}. 
\]

(1.16)

The first two terms in the last line are called double zeta values and admit the various representations

\[
\zeta(s_1, s_2) = \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} = \sum_{n \geq 2} \frac{1}{n^{s_1}} \left( 1 + \frac{1}{2^{s_2}} + \cdots + \frac{1}{(n-1)^{s_2}} \right) = \sum_{m,n \geq 1} \frac{1}{(n+m)^{s_1} n^{s_2}}.
\]

With this notation, equation (1.16) can be rewritten as

\[
\zeta(s_1) \cdot \zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2). 
\]

This identity already appears in Euler’s work [Eul76, p.144] under the name of “prima methodus”.

**Example 1.18.** One has \( \zeta(2)^2 = 2 \zeta(2, 2) + \zeta(4) \), and hence the equality \( \zeta(2, 2) = \pi^4/120 \) holds by Euler’s formula (1.4). Similarly, \( \zeta(2k, 2k) \) is a rational multiple of \( \pi^{4k} \).

As we have seen, products of two Riemann zeta values are linear combinations of zeta and double zeta values. To handle products of more factors, multiple zeta values of higher length are needed. These new numbers satisfy many linear relations with rational coefficients, and one can argue that the main goal of the theory is to fully understand them.

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**Exercise 1.19.** Prove that the logarithmic derivative of Euler’s product expansion for the sine function

\[
\frac{\sin \pi z}{\pi z} = \prod_{n \geq 1} \left( 1 - \frac{x^2}{n^2} \right)
\]
yields the identity
\[ \pi \cot(\pi x) = \frac{1}{x} + \sum_{n \geq 1} \frac{2x}{x^2 - n^2} \quad (x \in \mathbb{C}\setminus\mathbb{Z}), \]
and deduce formula (1.8) in the proof of Theorem 1.3.

**Exercise 1.20.** Prove that the Taylor expansion of the logarithm of the gamma function at \( z = 0 \) is given by
\[ \log \Gamma(1 - z) = \gamma z + \sum_{n \geq 2} \zeta(n) \frac{z^n}{n}, \]
where \( \gamma \) is the Euler-Mascheroni constant
\[ \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n) \right). \]
[Hint: use Weierstrass's factorization formula for the gamma function].

**Exercise 1.21 (Tornheim sums).** Given three integers \( a, b, c \geq 0 \), consider the series
\[ S(a, b, c) = \sum_{m,n \geq 1} \frac{1}{m^a n^b (m + n)^c}, \]
which is sometimes called *Tornheim sum*, in reference to [Tor50].

(a) Prove that \( S(a, b, c) \) converges if and only if \( a + c > 1 \), \( b + c > 1 \), and \( a + b + c > 2 \).

(b) Show that the following Pascal triangle-like recurrence holds
\[ S(a, b, c) = S(a - 1, b, c + 1) + S(a, b - 1, c + 1). \]

(c) Deduce that \( S(a, b, c) \) is a linear combination, with integral coefficients, of double zeta values, e.g. \( S(1, 1, 1) = 2\zeta(2, 1) \).

(d) Prove by direct computation that
\[ S(1, 1, 1) = \zeta(2, 1) + \zeta(3) \]
and deduce Euler's identity \( \zeta(3) = \zeta(2, 1) \). [Hint: use the equality \( \frac{1}{mn(m+n)} = \frac{1}{m^2} \left( \frac{1}{n} - \frac{1}{m+n} \right) \) to transform the sum over \( n \) into a telescoping series].

**1.2. Definition of multiple zeta values.** We now introduce multiple zeta values, the main character of these notes. In doing so, it will be convenient to use the following terminology:

**Definition 1.22.** A multi-index
\[ s = (s_1, \ldots, s_\ell) \in \mathbb{Z}^\ell \]
is called *positive* if \( s_i \geq 1 \) for all \( i = 1, \ldots, \ell \) and *admissible* if it is positive and, in addition, satisfies \( s_1 \geq 2 \). By convention, the empty multi-index \((\ell = 0)\) will also be considered to be admissible.
Lemma 1.23. Let $s = (s_1, s_2, \ldots, s_\ell)$ be an admissible multi-index. Then the following series converges:

$$\zeta(s) = \zeta(s_1, s_2, \ldots, s_\ell) = \sum_{n_1 > n_2 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.$$  

Proof. Since $\zeta(\emptyset) = 1$, we may assume that the multi-index $s$ is non-empty. In view of the inequality

$$\zeta(s) \leq \zeta(2, 1, \ldots, 1),$$

it suffices to show that $\zeta(2, 1, \ldots, 1)$ converges. Using the estimate

$$\sum_{k=1}^{n} \frac{1}{k} \leq 1 + \log(n),$$

which is obtained by comparison with the integral $\int_{1}^{n} \frac{dx}{x}$, one gets:

$$\zeta(2, 1, \ldots, 1) = \sum_{n_1 > n_2 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}} \leq \sum_{n \geq 1} \frac{1}{n^2} \left( \sum_{k=1}^{n} \frac{1}{k} \right)^{\ell-1} \leq \sum_{n \geq 1} \frac{(1 + \log(n))^{\ell-1}}{n^2}.$$  \hspace{1cm} (1.24)

The last series converges, as can be seen as follows: since

$$\lim_{n \to +\infty} \frac{\log(1 + \log(n))}{\log(n)} = 0,$$

there exists an integer $n_0$ such that $(1 + \log(n))^{\ell-1} < \sqrt{n}$ for all $n \geq n_0$. The tail of the series (1.24) is thus bounded by the convergent series $\sum_{n \geq n_0} n^{-3/2}$.

Definition 1.25. The multiple zeta value associated with an admissible multi-index $s = (s_1, \ldots, s_\ell)$ is the real number

$$\zeta(s) = \sum_{n_1 > n_2 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.$$  

The weight of $\zeta(s)$ is the sum of the exponents $s_1 + \cdots + s_\ell$, and $\ell$ is called its length.\footnote{Also called depth in the literature.} We write:

$$\text{wt}(\zeta(s)) = \text{wt}(s) = s_1 + \cdots + s_\ell,$$  \hspace{1cm} (1.26)

$$\ell(\zeta(s)) = \ell(s) = \ell.$$  \hspace{1cm} (1.27)

We shall adopt the convention that $\zeta(\emptyset) = 1$, so $\text{wt}(1) = \ell(1) = 0.$
Remark 1.28. Strictly speaking, only the weight and the length of \( s \) are well defined, since we may have \( \zeta(s) = \zeta(s') \) for different multi-indices. Conjecturally, when such an equality holds \( s \) and \( s' \) have the same weight, and hence the notation (1.26) makes sense. By contrast, the length is only well defined at the level of multi-indices, as Euler’s relation \( \zeta(2,1) = \zeta(3) \) already shows that the same value can be represented by multi-indices of different lengths (see Exercise 1.21 or Corollary 1.57 below).

Example 1.29. Let \( 2^{(n)} \) be the admissible multi-index of length \( n \) whose entries are all equal to 2. We compute the value of \( \zeta(2^{(n)}) \) using the method of generating series and Euler’s product expansion

\[
\frac{\sin \pi x}{\pi x} = \prod_{n \geq 1} \left( 1 - \frac{x^2}{n^2} \right). \tag{1.30}
\]

Plugging the definition of \( \zeta(2^{(n)}) \) into the power series below we get:

\[
\sum_{n \geq 0} \zeta(2^{(n)})(-x^2)^n = \sum_{n \geq 0} \sum_{m_1 > \cdots > m_n \geq 1} \left( -\frac{x^2}{m_1} \right) \cdots \left( -\frac{x^2}{m_n} \right) = \prod_{m \geq 1} \left( 1 - \frac{x^2}{m^2} \right) = \sum_{n \geq 0} (-1)^n \frac{\pi^{2n}}{(2n+1)!} x^{2n}.
\]

The second equality above comes from the elementary observation that, in the development of the infinite product, the terms of degree \( 2n \) correspond bijectively to choices of \( n \) integers \( m_1 > m_2 > \cdots > m_n \geq 1 \). The third equality is the combination of (1.30) and the power series expansion of the sine function. Now, identification of the coefficients yields

\[
\zeta(2^{(n)}) = \frac{\pi^{2n}}{(2n+1)!}. \tag{1.31}
\]

Note that this agrees with the result \( \zeta(2,2) = \frac{\pi^4}{120} \) from Example 1.18.

1.2.1. The algebra of multiple zeta values.

Definition 1.32. We will write \( \mathcal{Z} \) for the \( \mathbb{Q} \)-subvector space of \( \mathbb{R} \) generated by all multiple zeta values

\[
\mathcal{Z} = \langle 1, \zeta(2), \zeta(3), \zeta(2,1), \zeta(4), \ldots \rangle_\mathbb{Q}.
\]

Given integers \( k, \ell \geq 0 \), we also consider the subvector spaces of \( \mathcal{Z} \):

\[
\mathcal{Z}_k = \langle \zeta(s) \mid \text{wt}(s) = k \rangle_\mathbb{Q},
\]

\[
F_\ell \mathcal{Z} = \langle \zeta(s) \mid l(s) \leq \ell \rangle_\mathbb{Q},
\]

\[
F_\ell \mathcal{Z}_k = \langle \zeta(s) \mid \text{wt}(s) = k, \ell(s) \leq \ell \rangle_\mathbb{Q}.
\]

In particular, \( \mathcal{Z}_0 = \mathbb{Q} \) and \( \mathcal{Z}_1 = \{0\} \).
Remark 1.33. The subspaces $F_\ell Z$ define an increasing filtration of $Z$:
\[ \mathbb{Q} = F_0 Z \subseteq F_1 Z \subseteq F_2 Z \subseteq \ldots. \]
There is an obvious inclusion $F_\ell Z_k \subseteq F_\ell Z \cap Z_k$. This is actually expected to be an equality, but not known so far.

Equation (1.17) is the first indication that the $\mathbb{Q}$-vector space $Z$ has the richer structure of an algebra. Recall that this simply means that $Z$ is equipped with a bilinear “multiplication” $Z \times Z \to Z$.

Theorem 1.34. The multiplication of real numbers induces an algebra structure on $Z$ which is compatible with the weight and the length filtration in that, for all non-negative integers $\ell_1, \ell_2, k_1$ and $k_2$, one has:
\[ F_{\ell_1} Z_{k_1} \cdot F_{\ell_2} Z_{k_2} \subseteq F_{\ell_1 + \ell_2} Z_{k_1 + k_2}. \]

The theorem affirms, in particular, that every product of multiple zeta values can be written as a linear combination of MZVs.

Corollary 1.35. Every polynomial relation between Riemann zeta values $\zeta(k)$ gives rise to a linear relation between multiple zeta values.

Thus, finding algebraic relations among zeta values amounts to finding linear relations among multiple zeta values; this is a first interpretation of what we meant by “linearizing Conjecture 1.12” in the preface.

1.2.2. Proof of Theorem 1.34. The result will directly follow from lemmas 1.40 and 1.41 below. Before stating them, we need to introduce the stuffle of two multi-indices.

Construction 1.36. Given positive multi-indices
\[ s = (s_1, s_2, \ldots, s_\ell), \quad s' = (s'_1, s'_2, \ldots, s'_{\ell'}), \]
consider the set of all $2 \times \ell''$-matrices, for integers $\ell'' = \max(\ell, \ell'), \ldots, \ell + \ell'$, satisfying the following properties:

1. the entries of the first row are the numbers $s_i$, $1 \leq i \leq \ell$, in this order, plus some interlaced zeros;
2. the entries of the second row are the numbers $s'_i$, $1 \leq i \leq \ell'$, in this order, plus some interlaced zeros;
3. no column has two zeros.

Each such matrix defines a new positive multi-index $s'' = (s''_1, \ldots, s''_{\ell''})$ by adding the two entries of each column.

An equivalent construction will be given in Exercise 1.47.

Example 1.37. For the multi-indices $s = (2, 1, 1)$ and $s' = (2, 3)$, two possible choices of such a matrix are
\[
\begin{pmatrix}
0 & 2 & 1 & 1 \\
2 & 0 & 3 & 0
\end{pmatrix},
\]
from which we get the multi-index \( s'' = (2, 2, 4, 1) \), and
\[
\begin{pmatrix}
2 & 1 & 1 \\
2 & 0 & 3
\end{pmatrix},
\]
which gives \( s'' = (4, 1, 4) \). Observe that the length of \( s'' \) varies.

**Definition 1.38.** Let \( s, s' \) and \( s'' \) be positive multi-indices. The *stuffle multiplicity* \( \text{st}(s, s'; s'') \) is the number of times that the multi-index \( s'' \) appears in the previous construction.

By definition, the stuffle multiplicity is a non-negative integer.

**Example 1.39.** In the easy case \( s = (2) \) and \( s' = (2) \), all possible matrices are
\[
\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},
\]
from which one gets multi-indices \((4)\), \((2, 2)\) and \((2, 2)\). Hence
\[
\text{st}(s, s'; s'') = \begin{cases} 1 & s'' = (4) \\ 2 & s'' = (2, 2) \\ 0 & \text{otherwise} \end{cases}.
\]

From conditions (1)–(3) above, we immediately deduce the following properties of the stuffle multiplicity:

**Lemma 1.40.** Let \( s, s' \) and \( s'' \) be three positive multi-indices such that \( \text{st}(s, s'; s'') > 0 \). Then the following holds:

1. \( \text{wt}(s'') = \text{wt}(s) + \text{wt}(s') \);
2. \( \ell(s'') \leq \ell(s) + \ell(s') \);
3. if \( s \) and \( s' \) are admissible, then so is \( s'' \).

The main reason to introduce the stuffle index is the following result which, together with the previous lemma, implies Theorem 1.34.

**Lemma 1.41.** Let \( s = (s_1, s_2, \ldots, s_\ell) \) and \( s' = (s_1', s_2', \ldots, s_{\ell'}') \) be admissible multi-indices. Then
\[
\zeta(s) \cdot \zeta(s') = \sum_{s''} \text{st}(s, s'; s'') \zeta(s'').
\]

**Proof.** Multiplying the series
\[
\zeta(s) = \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}} \text{ and } \zeta(s') = \sum_{m_1 > \cdots > m_{\ell'} \geq 1} \frac{1}{m_1^{s_1'} \cdots m_{\ell'}^{s_{\ell'}'}},
\]
on one gets
\[
\zeta(s) \zeta(s') = \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell} m_1^{s_1'} \cdots m_{\ell'}^{s_{\ell'}'}}. \tag{1.42}
\]
We now decompose the sum (1.42) according to the possible orderings of the terms of the sequence \(n_1, \ldots, n_\ell, m_1, \ldots, m_\ell\). For instance, if \(\ell = \ell' = 1\), we distinguish the three cases \(n_1 > m_1, n_1 = m_1\) and \(n_1 < m_1\), which results in the decomposition

\[
\sum_{n_1 \geq 1 \atop m_1 \geq 1} \frac{1}{n_1^{s_1} m_1^{s_1'}} = \sum_{n_1 > m_1 \geq 1} \frac{1}{n_1^{s_1} m_1^{s_1'}} + \sum_{n_1 \geq 1 \atop m_1 > m_1 \geq 1} \frac{1}{n_1^{s_1} m_1^{s_1'}} + \sum_{m_1 > m_1 \geq 1} \frac{1}{n_1^{s_1} m_1^{s_1'}}.
\]

By construction, the number of times a given sum \(\zeta(s'') = \sum_{k_1 > \cdots > k_{\ell''} \geq 1} k_1^{s_1} \cdots k_{\ell''}^{s_{\ell''}}\) appears in this process is precisely the stuffle multiplicity \(st(s, s'; s'')\).

\[\square\]

**Example 1.43.** Let \(a, b, c\) be integers such that \(a, c \geq 2\) and \(b \geq 1\). We decompose the product \(\zeta(a, b) \zeta(c)\):

\[
\zeta(a, b) \zeta(c) = \sum_{n_1 > n_2 \geq 1 \atop m_1 \geq 1} \frac{1}{n_1^a n_2^b m^c} = \sum_{m > n_1 > n_2 \geq 1} \frac{1}{m^c n_1^a n_2^b} + \sum_{m = n_1 > n_2 \geq 1} \frac{1}{n_1^{a+c} n_2^b} + \sum_{n_1 > m > n_2 \geq 1} \frac{1}{n_1^a m^c n_2^b} + \sum_{n_1 > n_2 > m \geq 1} \frac{1}{n_1^a n_2^{b+c} m^c}.
\]

\[
= \zeta(c, a, b) + \zeta(a + c, b) + \zeta(a, c, b) + \zeta(a, b + c) + \zeta(a, b, c).
\]

More examples will be presented in the next sections.


ggg

**Exercise 1.44.** It would have been possible, as Euler did in length two (see Figure 1 below), to define multiple zeta values as

\[
\zeta^*(s_1, s_2, \ldots, s_\ell) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.
\]

Find the relation between \(\zeta(s_1, s_2, \ldots, s_\ell)\) and \(\zeta^*(s_1, s_2, \ldots, s_\ell)\).

\[
\zeta^*(s_1, s_2, \ldots, s_\ell) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.
\]

**Figure 1.** Euler’s definition of double zeta values in [Eul76].

**Exercise 1.45.** Given an integer \(s \geq 2\), let \(s^{(n)}\) be the length \(n\) multi-index \((s, \ldots, s)\).
(a) Adapt the argument from Example 1.29 to prove that
\[ \sum_{n \geq 0} \zeta(s^{(n)}) x^n = \exp \left( \sum_{k \geq 1} (-1)^{k-1} \zeta(s^k) x^k \right). \]

(b) Deduce that \( \zeta(s^{(n)}) \) belongs to the ring \( \mathbb{Q}[\zeta(s), \zeta(2s), \zeta(3s), \ldots] \). More precisely, consider an infinite collection of weighted variables \((x_k)_{k \geq 1}\), where \( x_k \) is given weight \( sk \). Then, for each \( n \geq 1 \), there exists a polynomial with rational coefficients \( P_n(x_1, \ldots, x_n) \), homogeneous of weight \( sn \), such that
\[ \zeta(s^{(n)}) = P_n(\zeta(s), \zeta(2s), \ldots, \zeta(ns)). \]
Combined with this, Euler’s formula (1.4) implies that, for even \( s \), the multiple zeta value \( \zeta(s^{(n)}) \) is a rational multiple of \( \pi^{ns} \).

(c) Some explicit formulas:
\[ \zeta(4^{(n)}) = \frac{(2\pi)^{4n}}{2^{2n-1}(4n+2)!}, \quad \zeta(6^{(n)}) = \frac{6(2\pi)^{6n}}{(6n+3)!}, \]
\[ \zeta(8^{(n)}) = \frac{(2\pi)^{8n}}{2^{2n-2}(8n+4)!} \left( (\sqrt{2} + 1)^{4n+2} + (\sqrt{2} - 1)^{4n+2} \right). \]
Note that the last factor is rational despite its appearance.

Exercise 1.46. Use the stuffle product to prove that, for each pair of integers \( n, k \geq 1 \), the following holds:
\[ \zeta(2k+1) \zeta(2^{(n-k)}) = \sum_{i=0}^{n-k} \zeta(2^{(i)}, 2k+1, 2^{(n-k-i)}) + \sum_{i=0}^{n-k-1} \zeta(2^{(i)}, 2k+3, 2^{(n-k-1-i)}) . \]

Exercise 1.47. Let \( \text{st}(\ell, \ell'; r) \) denote the set of surjective maps
\[ \sigma : \{1, 2, \ldots, \ell + \ell'\} \rightarrow \{1, 2, \ldots, \ell + \ell' - r\} \]
satisfying \( \sigma(1) < \sigma(2) < \cdots < \sigma(\ell) \) and \( \sigma(\ell + 1) < \cdots < \sigma(\ell + \ell') \).

(a) Determine the cardinality of \( \text{st}(\ell, \ell'; r) \) and show how to get from \( \sigma \) a matrix like the ones in Construction 1.36.

(b) Prove the identity
\[ \sum_{s''} \sum_{r=0}^{\min (\ell(s), \ell(s'))} \zeta(s') = \sum_{\sigma \in \text{st}(\ell, \ell'; r)} \sum_{s''(\sigma)} \zeta(s''(\sigma)_1, \ldots, s''(\sigma)_{\ell+\ell'-r}) . \]
where $s''(\sigma)$ is the multi-index with

$$s''(\sigma)_k = \begin{cases} s_i, & \text{if } \sigma^{-1}(k) = \{i\}, \ i \leq \ell, \\ s'_j, & \text{if } \sigma^{-1}(k) = \{\ell + j\}, \\ s_i + s'_j, & \text{if } \sigma^{-1}(k) = \{i, \ell + j\}. \end{cases}$$

1.3. Relations among double zeta values. We now undertake the task of finding linear relations among multiple zeta values by elementary methods. Historically, one of the first techniques consisted of reordering multiple sums by means of a partial fraction decomposition. In what follows, we show how this yields linear relations among double zeta values.

1.3.1. Partial fraction expansions. For $a, b$ integers with $b \geq 0$, we shall use the standard convention for binomial numbers:

$$\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!}. \quad (1.48)$$

In particular, $\binom{a}{0} = 1$ for all $a$ and, if $b > a \geq 0$, then $\binom{a}{b} = 0$.

**Lemma 1.49.** Let $i, j \geq 1$ be integers. The following equality of rational functions holds:

$$\frac{1}{x^iy^j} = \sum_{r=1}^{i+j-1} \left[ \frac{(r-1)}{(x+y)^{r+1}y^{i+j-r}} + \frac{(r-1)}{(x+y)^{r+1}x^{i+j-r}} \right]. \quad (1.50)$$

**Proof.** We proceed by induction on $i$ and $j$. The proof in the case $i = j = 1$ is a simple check. Assume that (1.50) holds for a given pair $(i, j)$. Derivating with respect to $x$, we find that $\frac{1}{x^iy^j}$ is equal to

$$\frac{1}{i} \sum_{r=1}^{i+j-1} \left[ \frac{(r-1)}{(x+y)^{r+1}y^{i+j-r}} + \frac{(r-1)}{(x+y)^{r+1}x^{i+j-r}} \right] = \frac{1}{i} \sum_{r=2}^{i+j-1} \left[ \frac{(r-1)}{(x+y)^r y^{i+1+j-r}} + \frac{(r-1)}{(x+y)^r x^{i+1+j-r}} \right] + \frac{1}{i} \sum_{r=1}^{i+j-1} \frac{(i+j-r)}{(x+y)^r x^{i+j+1-r}}.$$

Thanks to the identities

$$(r-1)(r-2)_{i-1} = i(r-1), \quad (r-1)(r-2)_{j-1} = (r-j)(r-1)_{j-1}$$

and the convention (1.48), the previous expression becomes

$$\sum_{r=1}^{i+j} \left[ \frac{(r-1)}{(x+y)^r y^{i+1+j-r}} + \frac{(r-1)}{(x+y)^r x^{i+1+j-r}} \right].$$
which agrees with the right-hand side of (1.50) for \((i+1, j)\). The induction step from \((i, j)\) to \((i, j+1)\) is completely symmetric. \(\square\)

**Corollary 1.51.** Let \(p, q \geq 1\) be integers. For any non-zero complex number \(a\), the following equality of rational functions holds:

\[
\frac{1}{u^p(u-a)^q} = (-1)^q \sum_{k=0}^{p-1} \frac{q+k-1}{u^{p-k}a^{q+k}} + \sum_{k=0}^{q-1} (-1)^k \frac{(p+k-1)}{a^{p+k}(u-a)^{q-k}}.
\] (1.52)

**Proof.** Take \(y = u\) and \(x = a-u\) in (1.50). To transform the obtained expression into (1.52), one notes that the binomial number \((r-1)_{q-1}\) vanishes unless \(q \leq r \leq p+q-1\), and hence \(r\) can be written as \(r = q + k\) for \(k = 0, \ldots, p-1\). The same holds for \((r-1)_{p-1}\). \(\square\)

1.3.2. **Applications.** A straightforward consequence of the partial fraction decomposition of Lemma 1.49 is the **shuffle relation**

\[
\zeta(j)\zeta(k-j) = \sum_{r=2}^{k-1} \left[\binom{j-1}{j-1} + \binom{k-1}{j-1} \right] \zeta(r, k-r)
\] (1.53)

for any \(k \geq 4\) and \(2 \leq j \leq k-2\). Replacing the product in the left-hand side of (1.53) by the stuffle formula (1.17) we get the linear identity

\[
\zeta(j, k-j) + \zeta(k-j, j) + \zeta(k) = \sum_{r=2}^{k-1} \left[\binom{j-1}{j-1} + \binom{k-1}{j-1} \right] \zeta(r, k-r),
\] (1.54)

which is called a **double shuffle relation**. The reason for these names will be apparent in Section 1.5.

A more sophisticated application of partial fraction decompositions gives the following result, essentially what Euler calls “tertia methodus” in [Eul76]. We refer the reader to [Har18] for a nice exposition of his techniques.

**Theorem 1.55** (Euler, 1776). Given integers \(p \geq 2\) and \(q \geq 1\), the following equality holds:

\[
\zeta(p, q) = \sum_{k=0}^{q-2} (-1)^k \binom{p+k-1}{p-1} \zeta(q-k)\zeta(p+k)
\]

\[
+ (-1)^q \sum_{k=0}^{p-2} \binom{q+k-1}{q-1} \zeta(p-k, q+k)
\]

\[
+ (-1)^{q-1} \binom{p+q-2}{p-1} \left[\zeta(p+q) + \zeta(p+q-1, 1)\right].
\]

**Remark 1.56.** The assumptions \(p \geq 2\) and \(q \geq 1\) ensure that all the terms in the formula are convergent series. Euler also allowed the case \(p = 1\). Then the sum contains divergent terms such as \(\zeta(1)\) or \(\zeta(1, 1)\) that one needs to regularize, see [Har18] for a rigorous treatment of Euler’s method.
Making \( q = 1 \) we immediately get:

**Corollary 1.57** (Euler’s sum formula). If \( s \geq 3 \), then

\[
\zeta(s) = \sum_{j=1}^{s-2} \zeta(s-j,j).
\] (1.58)

In particular, \( \zeta(3) = \zeta(2,1) \).

**Proof of Theorem 1.55.** We follow [Nie65, III, §18, p. 48]. Let us first observe that

\[
\zeta(p,q) = \sum_{n>m} \frac{1}{n^p m^q} = \sum_{n\geq 2} \left( \sum_{a=1}^{n-1} \frac{1}{n^p(n-a)^q} \right).
\] (1.59)

Applying the partial fraction expansion of Corollary 1.51 to each summand in the right-hand side and separating the terms coming from \( k = p-1 \) and \( k = q-1 \), gives:

\[
\sum_{a=1}^{n-1} \frac{1}{n^p(n-a)^q} = (-1)^q \sum_{k=0}^{p-2} \sum_{a=1}^{n-1} \frac{(q+k-1)}{n^p-k a^q+k} \sum_{k=0}^{q-2} \sum_{a=1}^{n-1} (-1)^k \frac{(p+k-1)}{a^p+k(n-a)^q-k} \sum_{a=1}^{n-1} \left[ \frac{1}{n a^p+q-1} - \frac{1}{a^p+q-1(n-a)} \right].
\]

The sum over \( n \) of the first two terms in the above expression converges, whereas the sum of each individual summand of the third term diverges. We will show later that the sum over \( n \) of the third term is also convergent.

Applying equation (1.59) to the first term we obtain

\[
\sum_{n\geq 2} (-1)^q \sum_{k=0}^{p-2} \sum_{a=1}^{n-1} \frac{(q+k-1)}{n^p-k a^q+k} = (-1)^q \sum_{k=0}^{p-2} \frac{(q+k-1)}{q-1} \zeta(p-k,q+k).
\]

We next observe that

\[
\zeta(p)\zeta(q) = \sum_{n\geq 2} \sum_{a=1}^{n-1} \frac{1}{(n-a)^p a^q},
\]

which implies that the sum over \( n \) of the second term is equal to

\[
\sum_{n\geq 2} \sum_{k=0}^{q-2} \sum_{a=1}^{n-1} \frac{(-1)^k (p+k-1)}{a^{p+k}(n-a)^{q-k}} = \sum_{k=0}^{q-2} (-1)^k (p+k-1) \zeta(q-k) \zeta(p+k).
\]
For the last term we use the identity
\[
\sum_{a=1}^{n-1} \frac{1}{a^{p+q-1}(n-a)} =
\begin{cases}
\sum_{a=1}^{\lfloor n/2 \rfloor} \frac{1}{(n-a)a^{p+q-1}} + \sum_{a=\lceil n/2 \rceil}^{n-1} \frac{1}{a^{p+q-1}(n-a)} + \left( \frac{1}{ \zeta(p+q)}, \quad n \text{ even,} \\
0, \quad n \text{ odd.}
\end{cases}
\]
We note that
\[
\sum_{n\geq 2} \frac{1}{\frac{n}{2} q+p} = \zeta(p+q)
\]
and
\[
\sum_{n\geq 2} \sum_{a=\lceil n/2 \rceil}^{n-1} \frac{1}{a^{p+q-1}(n-a)} = \zeta(p+q-1, 1).
\]
We finally estimate the remaining term. For \(N > 2\), one has:
\[
\sum_{n=2}^{N} \left( \sum_{a=1}^{n-1} \frac{1}{n(n-a)^{p+q-1}} - \sum_{a=\lceil n/2 \rceil}^{n-1} \frac{1}{(n-a)a^{p+q-1}} \right) = \sum_{n=\lfloor N/2 \rfloor + 1}^{N} \sum_{a=N-n+1}^{n-1} \frac{1}{n a^{p+q-1}}.
\]
Using that \(p+q-1 \geq 2\), one sees that the last term converges to zero as \(N\) goes to \(\infty\). The theorem follows from summing up all the computations. \(\Box\)

**Corollary 1.60 (Nielsen).** If \(n \geq 2\), the following equalities hold:
\[
\sum_{r=1}^{n-1} \zeta(2r, 2n-2r) = \frac{3}{4} \zeta(2n),
\]
\[
\sum_{r=1}^{n-1} \zeta(2r+1, 2n-2r-1) = \frac{1}{4} \zeta(2n).
\]

**Proof.** We follow [Nie65, III, §19, p. 49]. We shall use the following identity, which follows from the decomposition (1.17) of the product of two zeta values and Euler’s sum formula (1.58):
\[
\sum_{r=2}^{p-1} \zeta(r)\zeta(p-r+1) = p\zeta(p+1) - 2\zeta(p, 1).
\]

Theorem 1.55 for \(p = 2\) and \(q = 2n-2\) yields the equality
\[
(2n-2) [\zeta(2n) + \zeta(2n-1, 1)] = \sum_{k=0}^{2n-4} (-1)^k (k+1) \zeta(k+2) \zeta(2n-k-2).
\]
Note that the term $\zeta(k + 2)\zeta(2n - k - 2)$ is invariant under the substitution $k \mapsto 2n - k - 4$ and that it appears with multiplicity $(-1)^k(2n - 2)$ in the sum in the right-hand side. Therefore,

$$2 \left[ \zeta(2n) + \zeta(2n - 1, 1) \right] = \sum_{k=0}^{2n-4} (-1)^k \zeta(k + 2)\zeta(2n - k - 2)$$

$$= \sum_{r=1}^{n-1} \zeta(2r)\zeta(2n - 2r) - \sum_{r=1}^{n-2} \zeta(2r + 1)\zeta(2n - 2r - 1). \quad (1.62)$$

Summing and subtracting equations (1.62) and (1.61) for $p = 2n - 1$ yields the recursion formulas

$$\sum_{r=1}^{n-1} \zeta(2r)\zeta(2n - 2r) = \frac{2n + 1}{2} \zeta(2n), \quad n \geq 2,$$

$$\sum_{r=1}^{n-2} \zeta(2r + 1)\zeta(2n - 2r - 1) = \frac{2n - 3}{2} \zeta(2n) - 2\zeta(2n - 1, 1), \quad n \geq 3.$$

The statement is proved by replacing the products of zeta values in the left hand sides by (1.17). \hfill \Box

**Remark 1.63.** The previous corollary was rediscovered by Gangl, Kaneko and Zagier, see [GKZ06, Thm. 1] and Exercise 1.67.

### 1.3.3. Relations in low weight.

We now show how to use the above results to get linear relations among multiple zeta values of low weight.

**Corollary 1.64.** The following relations hold in $\mathcal{Z}$:

1. **in weight 3:**
   $$\zeta(3) = \zeta(2, 1).$$

2. **in weight 4:**
   $$\zeta(4) = 4\zeta(3, 1),$$
   $$\zeta(2, 2) = 3\zeta(3, 1).$$

3. **in weight 5:**
   $$\zeta(5) = -4\zeta(4, 1) + 2\zeta(2, 3),$$
   $$\zeta(3, 2) = -5\zeta(4, 1) + \zeta(2, 3).$$

4. **in weight 6:**
   $$\zeta(6) = 4\zeta(5, 1) + 4\zeta(3, 3),$$
   $$\zeta(2, 4) = \frac{13}{3} \zeta(5, 1) + \frac{7}{3} \zeta(3, 3),$$
   $$\zeta(4, 2) = -\frac{4}{3} \zeta(5, 1) + \frac{2}{3} \zeta(3, 3).$$
Proof. All the relations follow from Theorem 1.55 together with the decomposition (1.17). We have already seen that the equality \( \zeta(3) = \zeta(2, 1) \) is the first instance of Euler’s sum formula.

Let us now derive the two relations in weight 4. On the one hand, Theorem 1.55 for \( p = q = 2 \) gives
\[
\zeta(2) = 2 \zeta(4) + 2 \zeta(3, 1).
\]
Combining this with the identity \( \zeta(2) = 2 \zeta(2, 2) + \zeta(4) \), we obtain
\[
\zeta(4) + 2 \zeta(3, 1) = 2 \zeta(2, 2).
\]
On the other hand, by Euler’s sum formula, \( \zeta(4) = \zeta(3, 1) + \zeta(2, 2) \), hence the equalities \( \zeta(4) = 4 \zeta(3, 1) \) and \( \zeta(2, 2) = 3 \zeta(3, 1) \).

The remaining identities are left as an exercise. \( \square \)

1.3.4. An upper bound for the dimension of \( F_2 \mathbb{Z}_k \). Putting together all the identities of this section, one gets upper bounds for the dimension of the \( \mathbb{Q} \)-vector space generated by zeta and double zeta values of a given weight. However, as we will see in the next section, these bounds are not expected to be optimal in general (see Remark 1.94).

Proposition 1.65. If \( k \geq 4 \), then the \( \mathbb{Q} \)-vector space of zeta and double zeta values of weight \( k \) satisfies
\[
\dim_\mathbb{Q} F_2 \mathbb{Z}_k \leq \left\lfloor \frac{k - 2}{2} \right\rfloor.
\]
Proof. The space \( F_2 \mathbb{Z}_k \) is generated by the \( k - 1 \) elements \( \zeta(k) \) and \( \zeta(j, k - j) \) for \( j = 2, \ldots, k - 1 \). Recall from Corollary 1.57 that they satisfy Euler’s sum formula
\[
\zeta(2, k - 2) + \cdots + \zeta(k - 1, 1) - \zeta(k) = 0,
\]
as well as the double shuffle relations (1.54)
\[
\zeta(j, k - j) + \zeta(k - j, j) + \zeta(k) = \sum_{r=2}^{k-1} \left( \binom{r-1}{j-1} + \binom{r-1}{k-j-1} \right) \zeta(r, k - r), \quad j = 2, \ldots, k - 2.
\]
Since the latter are symmetric with respect to \( j \mapsto k - j \), it suffices to consider the equations for \( j \leq k - j \), that is \( j \leq \left\lfloor \frac{k}{2} \right\rfloor \).

One gets one equation from Euler’s sum formula and \( \left\lfloor \frac{k}{2} \right\rfloor - 1 \) equations from the double shuffle relations. We claim that these \( \left\lfloor \frac{k}{2} \right\rfloor \) equations are linearly independent. As \( k - 1 - \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k-2}{2} \right\rfloor \), this implies the statement. Indeed, by the convention (1.48), the double shuffle relations take the form
\[
\sum_{r=j+1}^{k-1} a_r \zeta(r, k - r) - \zeta(k) = 0, \quad j = 2, \ldots, k - 2,
\]
with \( a_r \) positive integers. The matrix of relations is thus upper triangular with non-zero entries in the diagonal, and hence invertible. \( \square \)
Exercise 1.66. Derive the remaining relations of Corollary 1.64.

Exercise 1.67 (Gangl–Kaneko–Zagier). Define the generating function of double zeta values of weight $k$ as the formal power series

$$T_k(X,Y) = \sum_{r+s=k, r,s \geq 1} \zeta(r,s) X^{r-1} Y^{s-1}.$$

(a) Use the double shuffle relation (1.54) to show that the following functional equation holds for all integers $k \geq 3$:

$$T_k(X+Y,Y) + T_k(X+Y,X) = T_k(X,Y) + T_k(Y,X) + \zeta(k) \frac{X^{k-1} - Y^{k-1}}{X - Y}.$$

(b) Give an alternative proof of Corollary 1.60 using the above functional equation for $(X,Y) = (1, 0)$ and $(1, -1)$.

1.4. The Zagier and the Broadhurst–Kreimer conjectures. As we have seen in the previous section, there are many linear relations between multiple zeta values. In order to elucidate the structure of the algebra $\mathcal{Z}$, one can start by performing numerical experiments.

1.4.1. Numerical experiments. The first step is to use clever techniques to accelerate the convergence of the infinite series defining multiple zeta values. With these techniques, one can compute them with very high precision (for instance 800 significant digits) in reasonable time. Then we can apply lattice algorithms such as the LLL algorithm or, more efficiently, the PSLQ algorithm to find linear relations with integral coefficients among the computed multiple zeta values. At a given precision, we will find many spurious relations (as we are only working with rational approximations), but we can easily distinguish between true relations and spurious ones. The true relations should have small coefficients compared to the inverse of the used precision. Moreover, the true relations will survive after doubling the precision, say from 100 to 200 significant digits.

After extensive experimentation by many mathematicians, no non-trivial linear relations between multiple zeta values of different weight have been found: all known relations are homogeneous. Moreover, we can write a table with the “experimental” dimension of each vector space $\mathcal{Z}_k$. Below, $k$ is the weight, $d_k^{\text{exp}}$ is the apparent dimension of $\mathcal{Z}_k$ given by the experiments and $2^{k-2}$ is the number of admissible multi-indices of weight $k$, that is, the dimension $\mathcal{Z}_k$ would have had if there were no $\mathbb{Q}$-linear relations at all.

---

3See e.g. §4 of [Bro96] for a description of such techniques, as well as [BBV10] for the state of the art some years ago.
Of course, the experiments are not conclusive. There may exist linear relations with “big” coefficients that we have not yet found; then the dimension of $\mathbb{Z}_k$ would be smaller than $d_k^{\text{exp}}$. In fact, there is not even a single $k$ for which the inequality $\dim_{\mathbb{Q}} \mathbb{Z}_k > 1$ is known.

Many of the relations obtained experimentally can be proved theoretically. For instance, Euler’s sum formula (1.58) gives

$$\zeta(3) = \zeta(2, 1),$$

the expected relation in weight 3. In weight 4, there are four admissible multi-indices but $d_4^{\text{exp}} = 1$; we thus need to find three independent relations. Indeed, according to Corollary 1.64 and Example 1.132 below, we have

$$\zeta(3, 1) = \frac{1}{4}\zeta(4), \quad \zeta(2, 2) = \frac{3}{4}\zeta(4), \quad \zeta(2, 1, 1) = \zeta(4).$$

In weight 5, we expect six relations. In fact, by Corollary 1.64 and Exercise 1.138 below, we have the linear relations

$$\zeta(5) = \frac{4}{5}\zeta(3, 2) + \frac{6}{5}\zeta(2, 3), \quad \zeta(4, 1) = -\frac{1}{5}\zeta(3, 2) + \frac{1}{5}\zeta(2, 3),$$

$$\zeta(5) = \zeta(2, 1, 1), \quad \zeta(4, 1) = \zeta(3, 1, 1),$$

(1.68)

$$\zeta(2, 1, 2) = \zeta(2, 3), \quad \zeta(2, 2, 1) = \zeta(3, 2).$$

However, given the lack of a theoretical proof, it is conceivable that experimental relations survive up to the number of significant digits that we have used but fail with higher precision.

1.4.2. Does the weight define a grading? The fact that all known relations between multiple zeta values are homogeneous led to the following:

**Conjecture 1.69.** The subspaces $\mathbb{Z}_k \subseteq \mathbb{Z}$ are in direct sum:

$$\mathbb{Z} = \bigoplus_{k \geq 0} \mathbb{Z}_k.$$  

Together with the fact that $\mathbb{Z}_k_1 \cdot \mathbb{Z}_k_2 \subseteq \mathbb{Z}_{k_1+k_2}$ (Theorem 1.34), the conjecture would be reformulated below as the statement that the weight defines a grading on the $\mathbb{Q}$-algebra $\mathbb{Z}$.

**Remark 1.70.** Assuming Conjecture 1.69, we immediately deduce that all multiple zeta values of positive weight are transcendental numbers. Indeed, let $s$ be an admissible multi-index of weight $w > 0$. If $\zeta(s)$ were algebraic, it would satisfy a polynomial equation of the form $\sum_{k=0}^{d} a_k \zeta(s)^k = 0$, 

<table>
<thead>
<tr>
<th>$k$</th>
<th>2 3 4 5 6 7 8 9 10 11 12 13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{k-2}$</td>
<td>1 2 4 8 16 32 64 128 256 512 1024 2048</td>
</tr>
<tr>
<td>$d_k^{\text{exp}}$</td>
<td>1 1 1 2 2 3 4 5 7 9 12 16</td>
</tr>
</tbody>
</table>

**Table 1.1.** Experimental dimension
where the $a_k$ are rational numbers. But then one would have
\[ a_d \zeta(s)^d \in \mathcal{Z}_{wd} \cap \bigoplus_{d' < d} \mathcal{Z}_{wd'}, \]
and hence $a_d = 0$ since subspaces of different weights intersect only at 0.

1.4.3. Zagier’s conjecture. In order to give the conjectural dimension of the $\mathbb{Q}$-vector spaces $\mathcal{Z}_k$, we need to introduce a Fibonacci-like sequence of integers. Set $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and, for $k \geq 3$,$$
 d_k = d_{k-2} + d_{k-3}.
$$
These numbers fit together into the generating series
\[
\sum_{k \geq 0} d_k t^k = \frac{1}{1 - t^2 - t^3}.
\]

There is an overwhelming amount of numerical evidence for the following conjecture, stated by Zagier in [Zag94, p. 509] “after many discussions with Drinfel’d, Kontsevich and Goncharov”.

**CONJECTURE 1.71 (Zagier).** The equality $\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$ holds.

Hoffman proposed the following refinement of Zagier’s conjecture, in which not only the dimension of $\mathcal{Z}_k$ but also a particular $\mathbb{Q}$-basis is postulated [Hof97, Conj. C, p. 493]:

**CONJECTURE 1.72 (Hoffman).** For each weight $k$, multiple zeta values $\zeta(s_1, \ldots, s_\ell)$ with $s_i \in \{2, 3\}$ form a $\mathbb{Q}$-basis of $\mathcal{Z}_k$.

This would imply the following representations of the spaces $\mathcal{Z}_k$:
\[
\begin{align*}
\mathcal{Z}_2 &= \langle \zeta(2) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_3 &= \langle \zeta(3) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_4 &= \langle \zeta(2, 2) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_5 &= \langle \zeta(2, 3), \zeta(3, 2) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_6 &= \langle \zeta(2, 2, 2), \zeta(3, 3) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_7 &= \langle \zeta(2, 2, 3), \zeta(2, 3, 2), \zeta(3, 2, 2) \rangle_{\mathbb{Q}}
\end{align*}
\]

**REMARKS 1.73.**

1. The previous discussion shows that $\mathcal{Z}_5$ is generated by $\zeta(2, 3)$ and $\zeta(3, 2)$. Thus, the first step towards the conjecture would be to prove that these numbers are $\mathbb{Q}$-linearly independent.

2. Having the right number of elements does not mean finding a basis. For instance, one could have thought that the elements
\[ \zeta(2n_1 + 1, \ldots, 2n_r + 1)\zeta(2)^k, \]
for \( r \geq 0, k \geq 0, n_i \geq 1 \), form a basis of \( \mathcal{Z} \), since their number in a given weight agrees with the conjectural dimension (see Exercise 1.99). However, Gangl, Kaneko and Zagier [GKZ06, p. 74] discovered the relation

\[
28\zeta(9, 3) + 150\zeta(7, 5) + 168\zeta(5, 7) = \frac{5197}{691}\zeta(12),
\]
which disproves such an expectation.

### 1.4.4. Algebra generators of multiple zeta values.

In what follows, by a \( \mathbb{Q} \)-algebra (without any further qualifier) we will mean an associative commutative algebra with unit.

**Definition 1.74.** A **graded** \( \mathbb{Q} \)-algebra is a \( \mathbb{Q} \)-algebra \( A \), together with a direct sum decomposition (called grading)

\[
A = \bigoplus_{k \in \mathbb{Z}} A_k
\]

into \( \mathbb{Q} \)-vector spaces \( A_k \) such that \( A_k \cdot A_{k'} \subseteq A_{k+k'} \). Note that the unit of the algebra then belongs necessarily to \( A_0 \), hence a map \( \eta: \mathbb{Q} \to A_0 \). A graded \( \mathbb{Q} \)-algebra is said to be **connected** if \( A_k = 0 \) for all \( k < 0 \) and \( \eta \) is an isomorphism. Moreover, \( A \) is said to be **free** if it is isomorphic to a polynomial algebra \( \mathbb{Q}[X_1, \ldots, X_n, \ldots] \) with \( X_i \) homogenous of some degree.

**Definition 1.75.** Assume that all \( A_k \) are finite-dimensional. Then the **Hilbert-Poincaré series** of \( A \) is defined as

\[
H_A(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Q}} A_k t^k.
\]

If \( A \) is connected, then its Hilbert-Poincaré series has only positive degrees and the constant coefficient is equal to 1.

**Lemma 1.76.** Let \( A \) be a connected graded free \( \mathbb{Q} \)-algebra, and let \( D_k \) denote the number of generators in degree \( k \). Then

\[
H_A(t) = \prod_{k \geq 1} (1 - t^k)^{-D_k}.
\]  

(1.77)

**Proof.** Let \( X_{1,1}, \ldots, X_{1,D_1}, \ldots, X_{k,1}, \ldots, X_{k,D_k}, \ldots \) be a set of homogeneous generators of \( A \), with \( X_{i,j} \) of degree \( i \geq 1 \). It suffices to observe that the coefficient of \( t^k \) in the power series expansion of the product (1.77) agrees with the number of monomials of degree \( k \) in the variables \( X_{i,j} \), and hence with the dimension of \( A_k \) since we are dealing with a free algebra.

We now explain how to compute the number of algebra generators in terms of the logarithm of the Hilbert-Poincaré series. Let us keep the assumption that \( A \) is connected, and write

\[
\log H_A(t) = \sum_{n \geq 1} c_n t^n.
\]  

(1.78)
Recall that the M"obius function $\mu$ takes the value 1 (resp. $-1$) on square-free integers with an even (resp. odd) number of prime factors, and 0 on non-square-free integers. In particular, $\mu(1) = 1$. The M"obius inversion formula is the statement that, if two sequences of complex numbers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ are related by the equality $a_n = \sum_{d|n} b_d$ for all $n \geq 1$, then

$$b_n = \sum_{d|n} \mu(d)a_{n/d}.$$

**Lemma 1.79.** Let $A$ be a connected graded free $\mathbb{Q}$-algebra, let $D_k$ denote the number of generators in degree $k$ and let $c_k$ be the coefficients of $\log H_A(t)$ as in (1.78). Then the following equality holds:

$$D_k = \sum_{d|k} \frac{\mu(d)}{d} c_{k/d}. \quad (1.80)$$

**Proof.** Taking the logarithm of the identity (1.77) and using the formal power series expansion $-\log(1-x) = \sum_{n \geq 1} \frac{x^n}{n}$, one gets

$$\log H_A(t) = -\sum_{k \geq 1} D_k \log(t^k) = \sum_{k \geq 1} D_k \sum_{d \geq 1} \frac{t^{kd}}{d} = \sum_{n \geq 1} \left( \sum_{d|n} \frac{D_{n/d}}{d} \right) t^n.$$

Comparison of coefficients then yields

$$c_n = \sum_{d|n} D_{n/d} \frac{1}{d},$$

and the equality (1.80) follows from M"obius inversion. \qed

Let us specialize the above discussion to the algebra $\mathcal{Z}$ of multiple zeta values. According to Zagier’s conjecture, its Hilbert-Poincaré series is

$$H_{\mathcal{Z}}(t) = \frac{1}{1 - t^2 - t^3}.$$

**Conjecture 1.81.** $\mathcal{Z}$ is a graded free algebra.

Assuming this and Zagier’s conjecture, we would like to compute the number $D_k$ of algebra generators in weight $k$. For this, we define a sequence of integers $(P_d)_{d \geq 1}$ by the equality

$$\sum_{d \geq 1} P_d t^d = \sum_{d \geq 1} d c_d t^d = t \frac{d}{dt} \log H_{\mathcal{Z}}(t) = \frac{2t^2 + 3t^3}{1 - t^2 - t^3}.$$

Then $P_1 = 0$, $P_2 = 2$, $P_3 = 3$ and $P_d = P_{d-2} + P_{d-3}$ for all $d \geq 4$. Therefore, Lemma 1.79 gives

$$D_k = \frac{1}{k} \sum_{d|k} \mu(k/d)P_d.$$

The first values of $P_k$ and $D_k$ are given in Table 1.2.
Recall that Hoffman’s conjecture 1.72 predicts that multiple zeta values with exponents equal to 2 and 3 form a graded \( \mathbb{Q} \)-basis of \( \mathcal{Z} \). It is only natural to try to extract from these elements a set of algebra generators; this is done through the theory of Lyndon words.

**Definition 1.82.** Let \( X \) be the alphabet \( \{a, b\} \) and equip the set \( X^* \) of words in \( X \) with the lexicographic order for which \( a < b \). A **Lyndon word** is a non-empty word \( w \in X^* \) such that, for each non-trivial decomposition \( w = uv \), the inequality \( w < v \) holds.

For example, \( ab \) is a Lyndon word because \( ab < b \), but none of the words \( aa, ba, bb \) is Lyndon.

**Conjecture 1.83.** \( \mathcal{Z} \) is the free \( \mathbb{Q} \)-algebra generated by Lyndon words on the alphabet \( \{2, 3\} \) with the order \( 2 < 3 \).

Assuming that the conjecture holds, the algebra generators in weights up to 13 are listed in Table 1.3.

<table>
<thead>
<tr>
<th>weight</th>
<th>generators</th>
<th>weight</th>
<th>generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \zeta(2) )</td>
<td>8</td>
<td>( \zeta(2, 3) )</td>
</tr>
<tr>
<td>3</td>
<td>( \zeta(3) )</td>
<td>9</td>
<td>( \zeta(2, 2, 2, 3) )</td>
</tr>
<tr>
<td>4</td>
<td>( \emptyset )</td>
<td>10</td>
<td>( \zeta(2, 2, 3) )</td>
</tr>
<tr>
<td>5</td>
<td>( \zeta(2, 3) )</td>
<td>11</td>
<td>( \zeta(2, 2, 2, 3), \zeta(2, 3, 3) )</td>
</tr>
<tr>
<td>6</td>
<td>( \emptyset )</td>
<td>12</td>
<td>( \zeta(2, 2, 3, 3), \zeta(2, 2, 3, 2) )</td>
</tr>
<tr>
<td>7</td>
<td>( \zeta(2, 2, 3) )</td>
<td>13</td>
<td>( \zeta(2, 2, 2, 3), \zeta(2, 3, 3, 3), \zeta(2, 3, 2, 3) )</td>
</tr>
</tbody>
</table>

**Table 1.3.** First Lyndon words on the alphabet \( \{2, 3\} \)

1.4.5. **The Broadhurst–Kreimer conjecture.** So far we have only taken into account the weight of multiple zeta values. To add the length to the picture, the first difficulty one needs to face is that the length is only expected to induce a filtration and not a grading, as it is already evident from the existence of relations such as \( \zeta(3) = \zeta(2, 1) \).

**Definition 1.84.**

1. A **filtered** \( \mathbb{Q} \)-algebra is a \( \mathbb{Q} \)-algebra \( A \), together with an increasing collection of vector subspaces

\[
\ldots \subseteq F_{\ell-1} A \subseteq F_{\ell} A \subseteq F_{\ell+1} A \subseteq \ldots
\]
indexed by $\ell \in \mathbb{Z}$ and such that $F_\ell A \cdot F_\ell' A \subseteq F_{\ell + \ell'} A$. The filtration is called separated if $\bigcap \ell F_\ell A = 0$, and exhaustive if $\bigcup \ell F_\ell A = A$.

(2) Given a filtered algebra $(A, F_\bullet)$, the associated graded algebra is

$$\text{Gr}^F A = \bigoplus_{\ell \in \mathbb{Z}} F_\ell A / F_{\ell - 1} A.$$  

Note that the compatibility of the product and the filtration guarantees that $\text{Gr}^F A$ inherits an algebra structure.

(3) A filtered graded $\mathbb{Q}$-algebra is a $\mathbb{Q}$-algebra $A$ with a filtration $F_\bullet A$ and a grading $A = \bigoplus_{k \in \mathbb{Z}} A_k$ which are compatible in the sense that

$$F_\ell A = \bigoplus_{k \in \mathbb{Z}} F_\ell A_k.$$  

Given such an algebra, we set

$$A_{k, \ell} = \text{Gr}^F_{\ell} A_k = F_\ell A_k / F_{\ell - 1} A_k$$

and form the associated bigraded algebra $\bigoplus_{k, \ell \in \mathbb{Z}} A_{k, \ell}$.

Returning to the algebra of multiple zeta values, we see that the length defines a separated and exhaustive filtration

$$F_\ell \mathcal{Z} = \langle \zeta(s) \mid \ell(s) \leq \ell \rangle_{\mathbb{Q}}.$$  

Assuming Conjecture 1.69, $\mathcal{Z}$ is hence a filtered graded algebra.

The associated bigraded algebra is not free, since $\zeta(2) \cdot \zeta(2) = \frac{5}{2} \zeta(4)$ implies that $\zeta(2)^2$ vanishes in $\mathcal{Z}_{4,2}$. To remedy this, we consider the quotient by the ideal generated by $\zeta(2)$:

$$\mathcal{Z}^\circ = \mathcal{Z} / \langle \zeta(2) \rangle.$$  

It is a graded filtered algebra as well. Moreover, we equip $\mathbb{Q}[\zeta(2)]$ with the filtration $F_0 = \mathbb{Q} \subset F_1 = \mathbb{Q}[\zeta(2)]$, and the grading that gives $\zeta(2)$ weight 2.

The following is a refinement of Conjecture 1.81.

**Conjecture 1.85.**

(1) $\text{Gr}^F \mathcal{Z}^\circ$ is a free bigraded algebra.

(2) By the first part of the conjecture, there exists a morphism of filtered graded algebras $\mathcal{Z}^\circ \to \mathcal{Z}$ that is a section of the quotient $\mathcal{Z} \to \mathcal{Z}^\circ$. Then the induced map $\mathcal{Z}^\circ \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta(2)] \to \mathcal{Z}$ is an isomorphism of filtered graded algebras.

Definition 1.75 and lemmas 1.76 and 1.79 extend to bigraded algebras. In particular, if $A = \bigoplus_{k, \ell} A_{k, \ell}$ is a connected free bigraded algebra, then

$$H_A(x, y) = \sum_{k, \ell \geq 0} (\dim_{\mathbb{Q}} A_{k, \ell}) x^k y^\ell = \prod_{k, \ell \geq 1} (1 - x^k y^\ell)^{-D_{k, \ell}},$$  

where $D_{k, \ell}$ is the number of generators in bidegree $(k, \ell)$. 

Extensive numerical experiments support the following refinement of Zagier’s conjecture, due to Broadhurst and Kreimer [BK97, §2]:

**Conjecture 1.86 (Broadhurst–Kreimer).** Define integers $(D_{k,\ell})_{k \geq 3, \ell \geq 1}$ by the product expansion formula

$$
\prod_{k \geq 3} \prod_{\ell \geq 1} (1 - x^k y^\ell)^{-D_{k,\ell}} = \frac{1}{1 - O(x)y + S(x)y^2 - S(x)y^4},
$$

(1.87)

where $O(x)$ and $S(x)$ are the formal series

$$
O(x) = \frac{x^3}{1 - x^2} = x^3 + x^5 + x^7 + x^9 + \ldots,
$$

$$
S(x) = \frac{x^{12}}{(1 - x^4)(1 - x^6)} = x^{12} + x^{16} + x^{18} + x^{20} + x^{22} + 2x^{24} + \ldots.
$$

Then $D_{k,\ell}$ is the number of generators of $\mathbb{Z}^\circ$ of weight $k$ and length $\ell$.

For shorthand, write $BK^0(x, y)$ for the power series expansion of

$$
\frac{1}{1 - O(x)y + S(x)y^2 - S(x)y^4}.
$$

Arguing as in Lemma 1.79, the numbers $D_{k,\ell}$ are given by the formula

$$
D_{k,\ell} = \sum_{d \mid (k,\ell)} \frac{\mu(d)}{d} \cdot \text{coefficient of } x^{k/d} y^{\ell/d} \text{ in } \log BK^0(x, y),
$$

(1.88)

where $(k, \ell)$ denotes the greatest common divisor of $k$ and $\ell$.

Taking Conjecture 1.85 for granted, the multiplicative formula (1.87) becomes equivalent to the following additive version, which is the one usually found in the literature:

**Conjecture 1.89 (Broadhurst–Kreimer).** Define non-negative integers $(d_{k,\ell})_{k,\ell \geq 0}$ by the generating series

$$
\sum_{k,\ell \geq 0} d_{k,\ell} x^k y^\ell = \frac{1 + E(x)y}{1 - O(x)y + S(x)y^2 - S(x)y^4},
$$

where

$$
E(x) = \frac{x^2}{1 - x^2} = x^2 + x^4 + x^6 + x^8 + \ldots.
$$

Then $d_{k,\ell}$ coincides with the dimension of the space of multiple zeta values of (precisely) weight $k$ and length $\ell$, that is

$$
d_{k,\ell} = \dim \mathbb{Q} \mathcal{Z}_{k,\ell}.
$$

**Remark 1.90.** The series $E(x)$ counts even zeta values, while $O(x)$ counts the odd ones. More interestingly, Zagier realized that $S(x)$ agrees with the generating series

$$
S(x) = \sum_{k \geq 1} (\dim \mathbb{Q} S_k)x^k,
$$
where $S_k$ stands for the vector space of cuspidal modular forms of weight $k$ for the full modular group $SL_2(\mathbb{Z})$. It is a classical result that

$$\dim_{\mathbb{Q}} S_k = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & k \text{ even, } k \not\equiv 2 \mod 12, \\ \left\lfloor \frac{k}{12} \right\rfloor - 1 & k \equiv 2 \mod 12, \\ 0 & \text{otherwise} \end{cases}$$

(see e.g. §1.3 and §2.1 of [Zag08] for an elementary proof).

Let us denote by $BK(x, y)$ the power series expansion of

$$1 + E(x)y \over 1 - O(x)y + S(x)y^2 - S(x)y^4.$$  

Expanding the fraction as a geometric series and collecting the terms with lower powers of $y$, we obtain

$$BK(x, y) = 1 + [E(x) + O(x)]y + [(E(x) + O(x))O(x) - S(x)]y^2 + [(O(x)^2 - 2S(x))O(x) + (O(x)^2 - S(x))E(x)]y^3 + \ldots.$$  

**Remark 1.91.** Observe that $d_{k,1} = 1$ for all $k \geq 2$. Since $F_0 \mathbb{Z} = \mathbb{Q}$, the Broadhurst–Kreimer conjecture holds in this case if and only if $\zeta(k)$ is irrational, which is only known for even $k$ and $k = 3$.

The first values of $d_{k,2}$ and $d_{k,3}$ are given in Table 1.4.

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<th>12</th>
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<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{k,2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>5</td>
</tr>
<tr>
<td>$d_{k,3}$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>3</td>
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<td>6</td>
<td>9</td>
<td>8</td>
<td>14</td>
<td>13</td>
</tr>
</tbody>
</table>

**Table 1.4.** First values of $d_{k,2}$ and $d_{k,3}$

Similarly, we derive

$$\log BK^0(x, y) = -\log(1 - O(x)y + S(x)y^2 - S(x)y^4)$$

$$= O(x)y + \left(\frac{1}{2}O(x)^2 - S(x)\right)y^2 + \left(\frac{1}{3}O(x)^3 - O(x)S(x)\right)y^3 + \ldots.$$  

**Remark 1.92.** Note that $D_{k,\ell} = 0$ if $k$ and $\ell$ have different parity. Indeed, in this case the integers $d$ contributing to formula (1.88) are all odd, so $k/d$ and $\ell/d$ have again different parity. However, it is clear from the above expression for $\log BK^0(x, y)$ that only monomials in which the degree of $x$ and the degree of $y$ have the same parity appear.

**Lemma 1.93.**

(1) If $k$ is even, then $D_{k,2} = \left\lfloor \frac{k-2}{6} \right\rfloor$. 
(2) If \( k \) is odd, then \( D_{k,3} = \left\lfloor \frac{(k-3)^2 - 1}{48} \right\rfloor \).

**Proof.** Specializing (1.88) to the case \( \ell = 2 \), we get

\[
D_{k,2} = \text{coeff. of } x^k y^2 - \frac{1}{2} \text{coeff. of } x^k y^2 \text{ in } \log BK^0(x, y)
= \text{coeff. of } x^k \text{ in } \left( \frac{1}{2} O(x)^2 - S(x) \right) - \frac{1}{2} \text{coeff. of } x^k \text{ in } O(x).
\]

Since \( O(x)^2 = \sum_{k \geq 6, \text{even}} \frac{k-4}{2} x^k \), we find that

\[
D_{k,2} = \begin{cases} 
\frac{k-4}{4} - \left\lfloor \frac{k}{12} \right\rfloor & k \equiv 0 \mod 4 \\
\frac{k-6}{4} - \left\lfloor \frac{k}{12} \right\rfloor & k \equiv 2 \mod 4, k \not\equiv 2 \mod 12 \\
\frac{k-2}{4} - \left\lfloor \frac{k}{12} \right\rfloor & k \equiv 2 \mod 12,
\end{cases}
\]

and it is a simple matter to check that this quantity agrees with \( \left\lfloor \frac{k-2}{6} \right\rfloor \). The proof of the second assertion follows the same pattern (Exercise 1.102). \( \square \)

**Remarks 1.94.**

(1) The numbers \( D_{k,2} \) and \( D_{k,3} \) are known to be upper bounds for the number of generators of length 2 and 3, see [Zag93, §3] for \( \ell = 2 \) and [Gon98, Thm. 1.5] for \( \ell = 3 \). From this it follows that, in lengths \( \ell = 1, 2, 3 \), one has the inequality:

\[
\dim_{\mathbb{Q}} (F_\ell \mathbb{Z}_k/F_{\ell-1} \mathbb{Z}_k) \leq d_{k,\ell}.
\]

(2) In particular, for double zeta values we get

\[
\dim_{\mathbb{Q}} F_2 \mathbb{Z}_k - 1 \leq d_{k,2}.
\]

By contrast, Proposition 1.65 yields the upper bound

\[
\dim_{\mathbb{Q}} F_2 \mathbb{Z}_k - 1 \leq \left\lceil \frac{k-4}{2} \right\rceil.
\]

The right-hand side of this last inequality agrees with the coefficient of degree \( k \) of the power series \((E(x) + O(x))O(x)\), while \( d_{k,2} \) is, by definition, the coefficient of degree \( k \) in \((E(x) + O(x))O(x) - S(x)\). Therefore, the bound of Proposition 1.65 is not optimal for those weights \( k \) such that \( S_k \) is non-trivial.

(3) Brown reformulated the Broadhurst–Kreimer conjecture in terms of the homology of a certain Lie algebra [Bro13a].

1.4.6. Known results. Not much is known about these conjectures, especially the last one. The goal of these notes is to explain in detail the following two results towards Zagier’s and Hoffman’s conjectures. In spite of their elementary formulation, this will carry us far away since the only known proofs are based on the theory of motives.
Theorem 1.95 (Terasoma [Ter02], Deligne-Goncharov [DG05]). The number $d_k$ is an upper bound for the dimension of the $\mathbb{Q}$-vector space of multiple zeta values of weight $k$, that is 
$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k.$$  

Theorem 1.96 (Brown, [Bro12]). Every multiple zeta value can be written as a $\mathbb{Q}$-linear combination of $\zeta(s_1, \ldots, s_l)$ with $s_i \in \{2, 3\}$.

Remark 1.97. As we will see at the very end of the text, in paragraph 5.5.1, a corollary of these two theorems is that Zagier’s conjecture implies the algebraic independence of odd zeta values (Conjecture 1.12).

Exercise 1.98. Prove that the sequence $(d_k)_{k \geq 0}$ satisfies 
$$\lim_{k \to \infty} (d_k - \kappa r^k) = 0$$
where $\kappa = \frac{r+1}{2r+3}$ and $r$ is the real root of $x^3 - x - 1$.

Exercise 1.99. Let $\delta_k$ denote the number of ordered tuples of integers $(s, n_1, \ldots, n_r)$ such that $s \geq 0$, $r \geq 0$, $n_i \geq 1$, and 
$$k = 2s + 2n_1 + 1 + \cdots + 2n_r + 1.$$  
Show that $\delta_0 = 1$, $\delta_1 = 0$, $\delta_2 = 1$ and $\delta_k = \delta_{k-2} + \delta_{k-3}$ for all $k \geq 3$. Therefore, $\delta_k = d_k$.

Exercise 1.100. Assume that the numbers $\zeta(2), \zeta(3), \zeta(5), \ldots$ are algebraically independent, so that $\mathbb{Q}[\zeta(2), \zeta(3), \ldots]$ is a free graded algebra. Apply Lemma 1.79 to compute the dimensions of the graded pieces, and compare them to the conjectural dimensions of multiple zeta values. Then find an example of a multiple zeta value which is not expected to be in the algebra generated by Riemann zeta values.

Exercise 1.101. Show that either Hoffman’s or the Broadhurst–Kreimer conjecture implies Zagier’s conjecture.

Exercise 1.102. Prove the equality $D_{k,3} = \lfloor \frac{(k-3)^2 - 1}{48} \rfloor$.

1.5. Integral representation of multiple zeta values. We have defined multiple zeta values as sums of infinite series. Using this representation, we proved that the vector space generated by these numbers forms an algebra under the stuffle product. We also derived some linear relations among multiple zeta values by means of the partial fraction method. Kontsevich found a different representation in terms of integrals. This way of writing multiple zeta values is central to the theory. From a combinatorial point of view, it yields a new structure, the stuffle product, from which many other linear relations are obtained in a systematic way. More importantly,
from a conceptual point of view, the integral representation shows that multiple zeta values are \textit{periods} of algebraic varieties. This will allow us to use algebro-geometric tools to study them, and paves the road for applications to a wealth of different areas such as knot theory or quantum field theory.

1.5.1. \textit{Two examples.}

\textbf{Example 1.103.} The following identity holds:

\[
\zeta(2) = \int_{1\geq t_1\geq t_2\geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} = \int_0^1 \left( \frac{1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \right) dt_1.
\] (1.104)

Indeed, for \(0 \leq t_2 < 1\) we have the geometric series expansion

\[
\frac{1}{1-t_2} = \sum_{n\geq 1} t_2^{n-1},
\]

and thus

\[
\int_0^{t_1} \frac{dt_2}{1-t_2} = \sum_{n\geq 1} \int_0^{t_1} t_2^{n-1} dt_2 = \sum_{n\geq 1} \frac{t_1^n}{n}.
\]

Therefore we get

\[
\int_0^1 \frac{1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} dt_1 = \int_0^1 \sum_{n\geq 1} \frac{t_1^n}{n} \frac{dt_1}{t_1} = \sum_{n\geq 1} \int_0^1 t_1^{n-1} dt_1 = \sum_{n\geq 1} \frac{1}{n^2}.
\]

\textbf{Example 1.105.} The identity

\[
\zeta(2, 1) = \int_{1\geq t_1\geq t_2\geq t_3\geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3}
\]

holds. Indeed,

\[
\int_{1\geq t_1\geq t_2\geq t_3\geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3} = \int_{1\geq t_1\geq t_2\geq 0} \frac{1}{t_1} \sum_{n\geq 1} \frac{t_2^n}{n} \frac{dt_1}{1-t_2} dt_2
\]

\[
= \int_{1\geq t_1\geq t_2\geq 0} \frac{1}{t_1} \sum_{n,m\geq 1} \frac{t_2^{n+m-1}}{n} dt_1 dt_2
\]

\[
= \int_{1\geq t_1\geq 0} \sum_{n,m\geq 1} \frac{t_1^{n+m}}{(n+m)t_1} dt_1
\]

\[
= \sum_{n,m\geq 1} \frac{1}{(n+m)^2n}
\]

\[
= \zeta(2, 1).
\]

\textbf{Remark 1.106.} As we will see in Section 3.7, the above integrals are particular cases of \textit{iterated} integrals, but for the moment we will think of them just as ordinary integrals over a simplex.
1.5.2. **The integral representation.** A piece of notation is needed to describe the general integral representation of multiple zeta values.

**Notation 1.107.** Given a real number $0 \leq t \leq 1$, we define

$$\Delta^p(t) = \{(t_1, \ldots, t_p) \in \mathbb{R}^p \mid t \geq t_1 \geq t_2 \geq \cdots \geq t_p \geq 0\}.$$ 

When $t = 1$, we will simply write $\Delta^p = \Delta^p(1)$. Furthermore, consider the measures on the open interval $]0, 1[$

$$\omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}.$$ 

If $s = (s_1, \ldots, s_l) \in \mathbb{Z}^l$ is a positive multi-index (i.e. all $s_i \geq 1$), we write $r_i = s_1 + \cdots + s_i$ for each $i = 1, \ldots, l$. In particular, $r_1 = s_1$ and $r_l$ is the weight of $s$. For convenience, we also set $r_0 = 0$. Let $\omega_s$ be the measure on the interior of the simplex $\Delta^{wt(s)}$ given by

$$\omega_s = \prod_{i=1}^l \omega_0(t_{r_i-1}+1) \cdots \omega_0(t_{r_i}) \omega_1(t_i) \text{ s_i-1 times}.$$ 

For example, one has:

$$\omega(2) = \frac{dt_1 \ dt_2}{t_1 \ 1-t_2},$$

$$\omega(2,2) = \frac{dt_1 \ dt_2 \ dt_3 \ dt_4}{t_1 \ 1-t_2 \ t_3 \ 1-t_4},$$

$$\omega(2,1) = \frac{dt_1 \ dt_2 \ dt_3}{t_1 \ 1-t_2 \ 1-t_3},$$

$$\omega(1,3) = \frac{dt_1 \ dt_2 \ dt_3 \ dt_4}{1-t_1 \ t_2 \ t_3 \ 1-t_4}.$$ 

The following result is attributed to Kontsevich:

**Theorem 1.108.** Let $s = (s_1, \ldots, s_l)$ be an admissible multi-index. The multiple zeta value $\zeta(s)$ can be obtained by a convergent improper integral:

$$\zeta(s) = \zeta(s_1, \ldots, s_l) = \int_{\Delta^{wt(s)}} \omega_s. \quad (1.109)$$

In order to easily prove this theorem, we introduce the polylogarithm functions, which will also be of use later in Chapter 3.

**Definition 1.110.** Let $s = (s_1, \ldots, s_l)$ be a positive multi-index and $t$ a complex number with $|t| < 1$. We define

$$\text{Li}_s(t) = \sum_{n_1 > n_2 > \cdots > n_l \geq 1} \frac{t^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_l^{s_l}}.$$ 

We call $\text{Li}_s$ the (multiple) polylogarithm function (of one variable).
Remark 1.111. Similarly, one can define multiple polylogarithms of several variables by
\[ Li_s(t_1, \ldots, t_\ell) = \sum_{n_1 > n_2 > \cdots > n_\ell \geq 1} \frac{t_1^{n_1} \cdots t_\ell^{n_\ell}}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}} \]
whenever the complex numbers \( t_i \) satisfy \( |t_1| < 1 \) and \( |t_i| \leq 1 \) for \( i = 2, \ldots, \ell \).

The following proposition is a straightforward consequence of basic results in complex analysis:

Proposition 1.112. If \( s \) is a positive multi-index, then the function \( Li_s \) is holomorphic on the open unit disc \( |t| < 1 \). Moreover, if \( s \) is admissible, then \( Li_s \) can be extended continuously to the closed disc \( |t| \leq 1 \) and \( Li_s(1) = \zeta(s) \).

For instance,
\[ Li_1(t) = \sum_{n \geq 1} \frac{t^n}{n} = -\log(1 - t) = \int_0^t \frac{dt_1}{1 - t_1}. \] (1.113)

An important property of polylogarithms is that they satisfy many functional equations, the easiest being:

Proposition 1.114. The following identities hold for all \( |t| < 1 \):
\[ \int_0^t Li_{s_1, \ldots, s_\ell}(t_1) \frac{dt_1}{t_1} = Li_{s_1 + 1, \ldots, s_\ell}(t), \] (1.115)
\[ \int_0^t Li_{s_1, \ldots, s_\ell}(t_1) \frac{dt_1}{1 - t_1} = Li_{1, s_1, \ldots, s_\ell}(t). \] (1.116)

Proof. We first prove equation (1.115):
\[ \int_0^t Li_{s_1, \ldots, s_\ell}(t_1) \frac{dt_1}{t_1} = \int_0^t \sum_{n_1 > n_2 > \cdots > n_\ell \geq 1} \frac{t_1^{n_1} \cdots t_\ell^{n_\ell}}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}} \frac{dt_1}{t_1} = \sum_{n_1 > n_2 > \cdots > n_\ell \geq 1} \frac{t_1^{n_1} \cdots t_\ell^{n_\ell}}{n_1^{s_1 + 1} n_2^{s_2} \cdots n_\ell^{s_\ell}} = Li_{s_1 + 1, \ldots, s_\ell}(t). \]

Similarly, equation (1.116) follows from the manipulations
\[ \int_0^t Li_{s_1, \ldots, s_\ell}(t_1) \frac{dt_1}{1 - t_1} = \int_0^t \sum_{n_1 > n_2 > \cdots > n_\ell \geq 1} \frac{t_1^{n_1} \cdots t_\ell^{n_\ell}}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}} \sum_{m \geq 0} t_1^m dt_1 = \sum_{n_0 > n_1 > \cdots > n_\ell \geq 1} \frac{t_1^{n_0} \cdots t_\ell^{n_\ell}}{n_0 n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}} = Li_{1, s_1, \ldots, s_\ell}(t), \]
where we have written \( n_0 = n_1 + m + 1 > n_1 \). \( \square \)
Now Theorem 1.108 is a particular case of the next result.

**Theorem 1.117.** If \( s \) is a positive multi-index and \( 0 < t < 1 \) a real number, then the following identity holds:

\[
\text{Li}_s(t) = \int_{\Delta^{\text{wt}(s)(t)}} \omega_s.
\]

**Proof.** The proof is by induction on the weight of \( s \). If \( \text{wt}(s) = 1 \), then \( s = (1) \) and the statement is just formula (1.113). The inductive step follows from the functional equations in Proposition 1.114. Indeed, let \( s = (s_1, \ldots, s_\ell) \) be a positive multi-index and assume that the result is true for all multi-indices of lower weight. If \( s_1 > 1 \), we write \( s' = (s_1 - 1, \ldots, s_\ell) \).

Then, by the identity (1.115) and induction,

\[
\text{Li}_s(t) = \int_0^t \text{Li}_{s'}(t_1) \frac{dt_1}{t_1} = \int_0^t \int_{\Delta^{\text{wt}(s')(t_1)}} \frac{\omega_{s'}}{t_1} dt_1 = \int_{\Delta^{\text{wt}(s)(t)}} \omega_s.
\]

The case \( s_1 = 1 \) is similar, using equation (1.116) instead. \( \square \)

### 1.5.3. The shuffle product.

Since multiple zeta values are integrals along simplices, certain combinatorial properties of the latter translate into relations among the former. Let us first illustrate this with an example.

**Example 1.118.** We have the following equalities:

\[
\zeta(2)^2 = \int_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1 dt_2}{t_1(1-t_2)} \cdot \int_{1 \geq u_1 \geq u_2 \geq 0} \frac{du_1 du_2}{u_1(1-u_2)}
\]

\[
= \int_{1 \geq t_1 \geq t_2 \geq 1 \geq u_1 \geq u_2 \geq 0} \frac{dt_1 dt_2 du_1 du_2}{t_1(1-t_2)u_1(1-u_2)}
\]

\[
= \sum_{i=1}^6 \int_{U_i} \frac{dt_1 dt_2 du_1 du_2}{t_1(1-t_2)u_1(1-u_2)}
\]

\[
= \zeta(3,1) + \zeta(3,1) + \zeta(2,2) + \zeta(3,1) + \zeta(3,1) + \zeta(2,2)
\]

\[
= 4\zeta(3,1) + 2\zeta(2,2),
\]

where the sets \( U_i, i = 1, \ldots, 6 \), are defined by

\[
U_1 = \{1 \geq t_1 \geq u_1 \geq t_2 \geq u_2 \geq 0\},
\]

\[
U_2 = \{1 \geq t_1 \geq u_1 \geq u_2 \geq t_2 \geq 0\},
\]

\[
U_3 = \{1 \geq t_1 \geq t_2 \geq u_1 \geq u_2 \geq 0\},
\]

\[
U_4 = \{1 \geq u_1 \geq t_1 \geq u_2 \geq t_2 \geq 0\},
\]

\[
U_5 = \{1 \geq u_1 \geq t_1 \geq t_2 \geq u_2 \geq 0\},
\]

\[
U_6 = \{1 \geq u_1 \geq u_2 \geq t_1 \geq t_2 \geq 0\}.
\]
The third equality comes from the decomposition
\[
\{(t_1, t_2, u_1, u_2) \mid 1 \geq t_1 \geq t_2 \geq 0, \ 1 \geq u_1 \geq u_2 \geq 0\} = \bigcup_{i=1}^{6} U_i,
\]
and the fourth one from Theorem 1.108.

Remark 1.119. This expression of \(\zeta(2)^2\) as linear combination of double zeta values is different from the one obtained by means of the series representation in Example 1.18. Combining both, we recover one of the relations which was proved in Corollary 1.64 using the method of partial fraction expansions, namely:
\[
\zeta(4) = 4\zeta(3, 1)
\]

To generalize the previous example, we consider shuffles:

Definition 1.120. A permutation \(\sigma\) of the set \(\{1, 2, \ldots, r+s\}\) is called a shuffle of type \((r, s)\) if the following two conditions are satisfied:
\[
\sigma(1) < \sigma(2) < \cdots < \sigma(r),
\]
\[
\sigma(r+1) < \sigma(r+2) < \cdots < \sigma(r+s).
\]
We denote the set of all shuffles of type \((r, s)\) by \(\underline{w}(r, s)\).

Remark 1.121. By definition, a shuffle is a permutation that respects the ordering of two distinguished subsets. The name comes from the way gamblers shuffle a deck of cards in western saloons.

Example 1.122. The set of shuffles of type \((2, 2)\) consists of \(\mathrm{Id}, (23), (243), (123), (1243), (13)(24)\).

Shuffles allow us to decompose a product of two simplices into a union of simplices, and therefore to express a product of integrals over simplices as a linear combination of integrals.

Proposition 1.123. Let \(r, s \geq 0\) be integers, \(0 < t < 1\) a real number and \(\mu_i \in \{\omega_0, \omega_1\}\) for \(i = 1, \ldots, r+s\). Then
\[
\int_{\Delta^r(t)} \mu_1(t_1) \cdots \mu_r(t_r) \int_{\Delta^s(t)} \mu_{r+1}(t_{r+1}) \cdots \mu_{r+s}(t_{r+s})
\]
\[
= \sum_{\sigma \in \underline{w}(r, s)} \int_{\Delta^{r+s}(t)} \mu_{\sigma^{-1}(1)}(t_1) \cdots \mu_{\sigma^{-1}(r+s)}(t_{r+s}).
\]

Proof. Using the decomposition
\[
\Delta^r(t) \times \Delta^s(t)
\]
\[
= \bigcup_{\sigma \in \underline{w}(r, s)} \{(t_1, \ldots, t_{r+s}) \mid t \geq t_{\sigma^{-1}(1)} \geq \cdots \geq t_{\sigma^{-1}(r+s)} \geq 0\},
\]
together with the fact that the intersection of two simplices on the right-hand side is a set of measure zero we obtain
\[
\int_{\Delta^r(t)} \mu_1(t_1) \cdots \mu_r(t_r) \int_{\Delta^s(t)} \mu_{r+1}(t_{r+1}) \cdots \mu_{r+s}(t_{r+s})
\]
\[
= \int_{\Delta^r(t) \times \Delta^s(t)} \mu_1(t_1) \cdots \mu_r(t_r) \cdot \mu_{r+s}(t_{r+s})
\]
\[
= \sum_{\sigma \in \omega(r,s)} \int_{t_{\sigma-1(1)} \geq \cdots \geq t_{r+s} \geq 0} \mu_1(t_1) \cdots \mu_{r+s}(t_{r+s}),
\]
where, in the last equality we have made the change of variables \(t_i = t_{\sigma-1(i)}\) to put the set \(t_{\sigma-1(1)} \geq \cdots \geq t_{r+s} \geq 0\) as \(\Delta^{r+s}(t)\).

1.5.4. Multi-indices and binary sequences. To easily exploit the preceding proposition to derive relations among polylogarithms, and in particular among multiple zeta values, we need a new notation.

**Definition 1.124.** A binary sequence is an element \(\alpha \in \{0,1\}^k\). We call \(k\) the weight of \(\alpha\), while its length is defined as the number of ones in the sequence. A sequence is called positive if it ends in one and admissible if it ends in one and starts with zero.

We will use the following notation to go from multi-indices to binary sequences and the other way around.

**Notation 1.125.** To each positive multi-index \(s = (s_1, \ldots, s_\ell)\) we attach the positive binary sequence
\[
bs(s) = (0^{s_1-1}, 1, \ldots, 0^{s_\ell-1}, 1)
\]
where \(0^s\) means that the entry zero is repeated \(s\) times. By convention, the empty binary sequence is admissible of weight and length both equal to zero. Clearly, \(bs\) is a bijection between the set of positive multi-indices and the set of positive binary sequences which respects the weight and the length. Moreover, it restricts to a bijection between the subsets of admissible objects on both sides.

If \(\alpha = (\varepsilon_1, \ldots, \varepsilon_r)\) is a binary sequence we will set
\[
\omega_\alpha = \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}.
\]
In particular, if \(s\) is a positive multi-index then
\[
\omega_s = \omega_{bs(s)}.
\]
Moreover, if $\alpha$ is positive we will denote $\text{Li}_\alpha(t) = \text{Li}_{bs^{-1}(\alpha)}(t)$ and if it is also admissible, we write $\zeta(\alpha) = \zeta(bs^{-1}(\alpha))$.

1.5.5. The shuffle product.

**Definition 1.126.** Let $\alpha = (\varepsilon_1, \ldots, \varepsilon_r)$, $\alpha' = (\varepsilon_{r+1}, \ldots, \varepsilon_{r+s})$ and $\alpha''$ be three binary sequences of lengths $r$, $s$ and $t$ respectively. Then the shuffle index $\shuffle(\alpha, \alpha'; \alpha'')$ is the number of shuffles of type $(r, s)$ which send $\alpha \alpha'$ to $\alpha''$. That is,

$$\shuffle(\alpha, \alpha'; \alpha'') = \# \{ \sigma \in \shuffle(r, s) \mid \alpha'' = (\varepsilon_{\sigma^{-1}(1)}, \ldots, \varepsilon_{\sigma^{-1}(r+s)}) \}.$$ 

Clearly, $\shuffle(\alpha, \alpha'; \alpha'') = 0$ unless $t = r + s$.

The next result is the analogue of Lemma 1.40 for the shuffle index; it follows directly from the definition as well.

**Lemma 1.127.** Let $\alpha$, $\alpha'$ and $\alpha''$ be three binary sequences such that $\shuffle(\alpha, \alpha'; \alpha'') > 0$. Then

1. $\text{wt}(\alpha'') = \text{wt}(\alpha) + \text{wt}(\alpha')$;
2. $\ell(\alpha'') = \ell(\alpha) + \ell(\alpha')$;
3. if both $\alpha$ and $\alpha'$ are positive (resp. admissible), then so is $\alpha''$.

With this notation, Proposition 1.123 translates into the following result, which is the analogue of Lemma 1.41 for shuffles.

**Lemma 1.128.** Let $\alpha$ and $\alpha'$ be positive binary sequences. Then

$$\text{Li}_\alpha(t) \text{Li}_{\alpha'}(t) = \sum_{\alpha''} \shuffle(\alpha, \alpha'; \alpha'') \text{Li}_{\alpha''}(t).$$

Moreover, if $\alpha$ and $\alpha'$ are admissible, then

$$\zeta(\alpha) \cdot \zeta(\alpha') = \sum_{\alpha''} \shuffle(\alpha, \alpha'; \alpha'') \zeta(\alpha'').$$

1.5.6. An involution. Another useful identity comes from exploiting the symmetry $t \mapsto 1 - t$.

**Proposition 1.129.** Let $\alpha = (\varepsilon_1, \ldots, \varepsilon_r)$ be an admissible binary sequence. Then

$$\int_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} \omega_{\varepsilon_1}(t_1) \cdots \omega_{\varepsilon_r}(t_r) = \int_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} \tilde{\omega}_{\varepsilon_r}(t_1) \cdots \tilde{\omega}_{\varepsilon_1}(t_r),$$

where $\tilde{\omega}_0 = \omega_1$ and $\tilde{\omega}_1 = \omega_0$. 

Proof. The change of variables \( s_i = 1 - t_i \) transforms the measure \( \omega_0(t_i) \) into \( \omega_1(s_i) = \tilde{\omega}_0(s_i) \), and \( \omega_1(t_i) \) into \( \omega_0(s_i) = \tilde{\omega}_1(s_i) \). Hence

\[
\int_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} \omega_{\varepsilon_1}(t_1) \cdots \omega_{\varepsilon_r}(t_r) = \int_{0 \leq s_1 \leq \cdots \leq s_r \leq 1} \tilde{\omega}_{\varepsilon_1}(s_1) \cdots \tilde{\omega}_{\varepsilon_r}(s_r).
\]

The sought formula follows after renaming the variables in the right as \( s_i = t_{r-i} \).

Definition 1.130. For a binary sequence \( \alpha = (\varepsilon_1, \ldots, \varepsilon_r) \), we write

\[
\tau(\alpha) = (1 - \varepsilon_r, \ldots, 1 - \varepsilon_1).
\]

If \( \alpha \) is admissible, then so is \( \tau(\alpha) \).

As a consequence of Proposition 1.129 and Theorem 1.108, we deduce

Corollary 1.131. If \( \alpha \) is an admissible binary sequence, then

\[
\zeta(\alpha) = \zeta(\tau(\alpha)).
\]

Example 1.132. We have:

\[
\zeta(4) = \zeta((0, 0, 0, 1)) = \zeta((0, 1, 1, 1)) = \zeta(2, 1, 1).
\]

\[\star\star\star\]

Exercise 1.133. Justify the exchange of the integral and the summation sign in the computations of examples 1.103 and 1.105.

Exercise 1.134. Show that the number of shuffles of type \((r, s)\) is the binomial number \( \binom{r+s}{r} \).

Exercise 1.135. Manipulating series, show directly that

\[
\zeta(3) = \int_{1 \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1 dt_2 dt_3}{t_1 t_2 (1 - t_3)}
\]

and, more generally,

\[
\zeta(s) = \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_s \geq 0} \frac{dt_1 \cdots dt_{s-1} dt_s}{t_1 \cdots t_{s-1} (1 - t_s)}.
\]

Exercise 1.136. Use Lemma 1.128 to check the shuffle relation (1.53) for \( \zeta(2)\zeta(3) \). Same in the general case \( \zeta(i)\zeta(j) \).

Exercise 1.137. Find a formula for \( \zeta(s)\zeta(p, q) \) with shuffles.

Exercise 1.138. Check the identities

\[
\begin{align*}
\zeta(5) &= \zeta(2, 1, 1, 1), & \zeta(4, 1) &= \zeta(3, 1, 1), \\
\zeta(2, 1, 2) &= \zeta(2, 3), & \zeta(2, 2, 1) &= \zeta(3, 2)
\end{align*}
\]

with the help of Proposition 1.129.
1.6. Quasi-shuffle products and the Hoffman algebra. In the previous sections, we have seen two methods to express a product of multiple zeta values as a linear combination of MZVs. The first, using the series representation, is called the stuffle product, and the second, using the integral representation, is called the shuffle product. As seen in examples 1.18 and 1.118, both methods may give different linear combinations for the same product of MZVs leading to linear relations among MZVs. The stuffle product is easily expressed in terms of multi-indices as in Lemma 1.41, while the shuffle product is expressed more conveniently using binary sequences as in Lemma 1.128. We now want to put a little order to make clearer the combinatorial structure of MZVs.

1.6.1. Alphabets and the quasi-shuffle product.

Notation 1.139. Let $A = \{a_i\}_{i \in S}$ be a countable set. The elements of $A$ will be called *letters* and $A$ is called an alphabet. Let $\mathbb{Q}A$ be the $\mathbb{Q}$-vector space with $A$ as a basis. Let $\mathbb{Q}\langle A \rangle$ be the non-commutative polynomial algebra over $A$, i.e. $\mathbb{Q}\langle A \rangle = \langle a_{i_1}, a_{i_2}, \ldots, a_{i_n} \rangle$ is the vector space with the set of *words* in the letters of $A$ as a basis, which is equipped with the concatenation product

$$a_{i_1} \cdots a_{i_n} \cdot a_{j_1} \cdots a_{j_m} = a_{i_1} \cdots a_{i_n} a_{j_1} \cdots a_{j_m}.$$  

We say that a word $w = a_1 \cdots a_n$ has *length* $\ell(w) = n$ and set $\ell(1) = 0$, as we consider $1$ as the empty word.

Definition 1.140. Let $A$ be an alphabet and let $\diamond : \mathbb{Q}A \times \mathbb{Q}A \to \mathbb{Q}A$ be a commutative and associative product. We define a new product $\ast_\diamond$ on $\mathbb{Q}\langle A \rangle$ recursively by setting $1 \ast_\diamond w = w \ast_\diamond 1 = w$ and

$$aw \ast_\diamond bv = a(w \ast_\diamond bv) + b(aw \ast_\diamond v) + (a \diamond b)(w \ast_\diamond v),$$

for any pair $a, b \in A$ of letters, and $w, v \in \mathbb{Q}\langle A \rangle$ of words. This product is extended to $\mathbb{Q}\langle A \rangle$ by $\mathbb{Q}$-linearity and is called the quasi-shuffle product associated with $\diamond$.

Theorem 1.141 (Hoffman [Hof00]). The vector space $\mathbb{Q}\langle A \rangle$ equipped with the product $\ast_\diamond$ is a commutative $\mathbb{Q}$-algebra.

Proof. Let us check the commutativity

$$u_1 \ast_\diamond u_2 = u_2 \ast_\diamond u_1$$

(1.142)

by induction on $\ell(u_1) + \ell(u_2)$. If one of $u_1$ or $u_2$ is the empty word, then (1.142) holds trivially. Thus let $u_1 = aw$ and $u_2 = bv$ with letters $a, b \in A$ and words $w, v \in \mathbb{Q}\langle A \rangle$. Then, by definition of the product $\ast_\diamond$ and the induction hypothesis, we get

$$u_1 \ast_\diamond u_2 - u_2 \ast_\diamond u_1 = (a \diamond b)(w \ast_\diamond v) - (b \diamond a)(v \ast_\diamond w).$$

Since $\diamond$ is assumed to be commutative, (1.142) follows from induction.
The proof of the associativity is similar and is left as an exercise. □

1.6.2. **Stuffle product.** We first introduce the stuffle product. Let $Y$ be the alphabet with letters $y_1, y_2, y_3, \ldots$, together with the product

$$\hat{\odot}: \mathbb{Q}Y \times \mathbb{Q}Y \to \mathbb{Q}Y, \quad y_i \hat{\odot} y_j = y_{i+j}.$$  

The product $\hat{\odot}$ is commutative and associative. The product $\odot$ on $\mathbb{Q}\langle Y \rangle$ will be denoted by $*$ and called the **stuffle product**. By definition, it is given by

$$y_i w * y_j v = y_i(w * y_j v) + y_j(y_i w * v) + y_{i+j}(w * v). \quad (1.143)$$

**Example 1.144.** We have $y_i y_j = y_i y_j + y_j y_i + y_{i+j}$ and

$$y_2 * y_3 y_4 = y_2(y_3 y_4) + y_3(y_2 y_4) + y_5(y_4)$$

$$= y_2 y_3 y_4 + y_3(y_2 y_4) + y_4 y_2 + y_6 \quad (1.145)$$

$$= y_2 y_3 y_4 + y_3 y_2 y_4 + y_3 y_4 y_2 + y_3 y_6 + y_5 y_4.$$

**Notation 1.145.** A positive multi-index $s = (s_1, \ldots, s_\ell)$ defines a word

$$y_s = y_{s_1} \cdots y_{s_\ell}.$$  

In fact, the set of positive multi-indices and the set of words in the alphabet $Y$ are in bijection. We will use this bijection to identify both sets.

**Lemma 1.146.** The stuffle product is given by

$$y_s * y_{s'} = \sum_{s''} \text{st}(s, s'; s'')y_{s''}.$$

**Proof.** Let $s = (s_1, \ldots)$ and $s' = (s'_1, \ldots)$ be two positive multi-indices. Thus $y_s = y_{s_1} v$ and $y'_s = y_{s'_1} w$. The matrices used to define the stuffle indices $\text{st}(s, s'; s'')$ in Definition 1.38 fall into three types.

$$\begin{pmatrix} s_1 & \cdots \\ 0 & \cdots \end{pmatrix}, \quad \begin{pmatrix} 0 & \cdots \\ s'_1 & \cdots \end{pmatrix}, \quad \begin{pmatrix} s_1 & \cdots \\ s'_1 & \cdots \end{pmatrix}.$$  

The matrices of the first type give rise to the term $y_{s_1}(v * y_{s'})$, the matrices of the second type to the term $y_{s'_1}(y_s * w)$ and the matrices of the third type to the term $y_{s_1 + s'_1}(v * w)$. □

Since the words of the alphabet $Y$ are related to multi-indices and the product of $\mathbb{Q}\langle Y \rangle$ is the stuffle product, one may expect to have a morphism of $\mathbb{Q}$-algebras

$$(\mathbb{Q}\langle Y \rangle, *) \longrightarrow (\mathbb{Z}, \cdot)$$  

$$y_{s_1} \cdots y_{s_\ell} \mapsto \zeta(s_1, \ldots, s_\ell).$$

But since multiple zeta values are defined only when $s_1 > 1$ we have to restrict the source of this map. Later, in Section 1.7 we will see how to extend the evaluation map to the whole $(\mathbb{Q}\langle Y \rangle, *)$.  


DEFINITION 1.147. A word \( w = y_{s_1} \ldots y_{s_l} \) is called admissible if \( s_1 > 1 \), i.e. if it corresponds to an admissible multi-index. We will denote by \( \mathbb{Q}(Y)^0 \) the subspace of \( \mathbb{Q}(Y) \) generated by admissible words.

PROPOSITION 1.148.

(1) \((\mathbb{Q}(Y)^0, *)\) is a subalgebra of \((\mathbb{Q}(Y), *)\).

(2) We have a morphism of \( \mathbb{Q} \)-algebras

\[ \mathbb{Q}(Y)^0 \to \mathbb{Z} \]

determined by the assignment

\[ y_{s_1} \ldots y_{s_l} \mapsto \zeta(s_1, \ldots, s_l). \]

Proof. The first statement can be checked directly from the definition of the product \( * \). Alternatively, it follows from Lemma 1.146 and part (3) of Lemma 1.40. The second statement follows from lemmas 1.146 and 1.41. \( \square \)

Since we have identified positive multi-indices with words in the alphabet \( Y \), we often just write \( \zeta(w) \) instead of \( \zeta(s_1, \ldots, s_l) \) for \( w = y_{s_1} \ldots y_{s_l} \), thus

\[ \zeta(w * v) = \zeta(w) \zeta(v) \] \hspace{1cm} (1.149)

for all words \( w, v \in \mathbb{Q}(Y)^0 \).

1.6.3. Shuffle product. We now introduce the shuffle product. Let \( X \) be the alphabet in two letters \( X = \{x_0, x_1\} \), equipped with the trivial product

\[ a \triangleleft 2 b = 0. \]

We will denote by \( \triangleleft 2 \) the corresponding product \( * \triangleleft 2 \) and call it the shuffle product. \(^4\)

DEFINITION 1.150. We call \( S = (\mathbb{Q}(X), \triangleleft 2) \) the Hoffman algebra.

PROPOSITION 1.151. Given two words \( x_{\varepsilon_1} \ldots x_{\varepsilon_r} \) and \( x_{\varepsilon_{r+1}} \ldots x_{\varepsilon_{r+s}} \) on the alphabet \( X \), their shuffle product is given by

\[ x_{\varepsilon_1} \ldots x_{\varepsilon_r} \triangleleft 2 x_{\varepsilon_{r+1}} \ldots x_{\varepsilon_{r+s}} = \sum_{\sigma \in \Omega(r,s)} x_{\varepsilon_{\sigma^{-1}(1)}} \ldots x_{\varepsilon_{\sigma^{-1}(r+s)}}. \]

Proof. Exercise 1.163. \( \square \)

EXAMPLE 1.152. We have

\[ x_0 x_1 \triangleleft 2 x_0 x_1 = 2x_0 x_1 x_0 x_1 + 4x_0^2 x_1^2 \]
\[ x_0 x_1 \triangleleft 2 x_0^2 x_1 = x_0 x_1 x_0^2 x_1 + 3x_0^2 x_1 x_0 x_1 + 6x_0^3 x_1^2. \]

NOTATION 1.153. There is an obvious bijection between binary sequences and words in the alphabet \( X \): to a binary sequence \( \alpha = (\varepsilon_1, \ldots, \varepsilon_r) \) we associate the word \( x_\alpha = x_{\varepsilon_1} \ldots x_{\varepsilon_r} \). Using this bijection, we can transfer the shuffle index, as introduced in Definition 1.126, to words in the alphabet \( X \). The resulting index will be denoted by \( \triangledown(u, v; w) \).

\(^4\)This justifies the name quasi-shuffle.
With this notation, Proposition 1.151 can be rewritten as
\[ u \sqcup v = \sum_w \sqcup(u, v; w)w. \tag{1.154} \]

This equation hints at the existence of an algebra morphism from \( \mathcal{H} \) to multiple zeta values. As in the case of the alphabet \( Y \), one needs to restrict to the space where the series are convergent.

**Definition 1.155.** A word in the alphabet \( X \) is said to be positive if it ends in \( x_1 \) and is said to be admissible if it ends in \( x_1 \) and starts in \( x_0 \).

Let \( \mathcal{H}^1 \) (resp. \( \mathcal{H}^0 \)) be the subspace generated by positive (resp. admissible) words, so that
\[ \mathcal{H} \supset \mathcal{H}^1 \supset \mathcal{H}^0. \]

**Proposition 1.156.**

1. \( (\mathcal{H}^0, \sqcup) \) and \( (\mathcal{H}^1, \sqcup) \) are subalgebras of \( (\mathcal{H}, \sqcup) \).

2. There is a morphism of \( \mathbb{Q} \)-algebras
\[ \zeta : \mathcal{H}^0 \to \mathbb{Z} \]
given by the assignment
\[ x_\alpha \mapsto \zeta(\alpha). \]
(Recall that the multiple zeta value corresponding to an admissible binary sequence was defined as \( \zeta(\text{bs}^{-1}(\alpha)) \) in Notation 1.125).

**Proof.** Exercise 1.164. \( \square \)

Since we are identifying binary sequences and words in the alphabet \( X \), we will often write \( \zeta(x_\alpha) \) instead of \( \zeta(\alpha) \). With this notation, Proposition 1.156 says that the following identity holds for all \( w, v \in \mathcal{H}^0 \):
\[ \zeta(w \sqcup v) = \zeta(w)\zeta(v). \tag{1.157} \]

### 1.6.4. Double shuffle relations

In the same way that positive multi-indices can be translated into binary sequences, there is a map between \( \mathbb{Q}\langle Y \rangle \) and \( \mathcal{H} \). This map does not preserve the product structure, the stuffle product on one side, the shuffle product on the other. We can define a second product on \( \mathcal{H} \) that is compatible with the stuffle product in \( \mathbb{Q}\langle Y \rangle \).

**Definition 1.158.** The **stuffle product** in \( \mathcal{H} \), denoted \( \ast \), is defined inductively as follows:
\[ 1 \ast w = w \ast 1 = w \quad \forall w \in \mathcal{H} \]
\[ x_0^p \ast w = w \ast x_0^p = wx_0^p \quad \forall p > 0, \forall w \in \mathcal{H} \]
\[ z_p w \ast z_q v = z_p(w \ast z_q v) + z_q(z_p w \ast v) + z_{p+q}(w \ast v) \quad \forall w, v \in \mathcal{H}, \]
where \( z_p = x_0^{p-1} x_1 \).
Proposition 1.159.

a) \((\mathcal{H}, \ast)\) is a commutative and associative \(\mathbb{Q}\)-algebra.

b) The map
\[
(\mathbb{Q}\langle Y \rangle, \ast) \hookrightarrow (\mathcal{H}, \ast) \quad y_i \mapsto z_i = x_0^{i-1} x_1.
\]
is an algebra monomorphism.

Proof. Exercise 1.165.

Theorem 1.160. Let \(\zeta : \mathcal{H}^0 \to \mathbb{R}\) be as before. Then we have
\[
\zeta(w \shuffle v - w \ast v) = 0.
\]

Proof. This follows from equations (1.149) and (1.157).

This theorem is a source of relations among MZVs called double shuffle relations. Nevertheless, it is clear that they are not enough to describe all relations among MZVs. For instance, we do not obtain any relation in weight 3, while we know the Euler relation, and we can only produce one relation in weight 4, while there are at least 3 independent relations in weight 4. In order to obtain the needed relations we will need to consider products with non-admissible words. This will be done in the next section.

**

Exercise 1.161. Show that
\[
x^r \shuffle x^s = x_0 \ldots x_0 \shuffle x_0 \ldots x_0 = \frac{(r + s)!}{r!s!} x^{r+s}.
\]

Exercise 1.162. Prove that, for a letter \(a\) and words \(u\) and \(v\), the following identity holds:
\[
a \shuffle uv = (a \shuffle u)v + u(a \shuffle v) - uav.
\]

Exercise 1.163. Prove Proposition 1.151.

Exercise 1.164. Prove Proposition 1.156.

Exercise 1.165. Prove Proposition 1.159.

Exercise 1.166. Given a multi-index \(s\) and an integer \(M \geq 0\), we set
\[
\zeta_M(s) = \sum_{M > m_1 > m_2 > \cdots > m_i > 0} \frac{1}{m_1^{s_1} \cdots m_i^{s_i}}.
\]

(a) Show that, if \(s\) is admissible, then \(\lim_{M \to \infty} \zeta_M(s) = \zeta(s)\).
(b) Recall that we identified words and multi-indices. Prove that
\[ \zeta_M : (\mathcal{F}_1, *) \to \mathbb{Q} \]
is a group morphism, i.e. for all \( w, v \in \mathcal{F}_1 \)
\[ \zeta_M(w * v) = \zeta_M(w) \zeta_M(v). \]

**Exercise 1.167.** Using the identification between words in the alphabet \( X \) and binary sequences, we obtain a map
\[ L_i : \mathcal{F}_1 \to C^\infty(\mathbb{N}). \]
Prove that this map is a homomorphism, that is, for all \( w, v \in \mathcal{F}_1 \),
\[ L_i_{w \circ v}(t) = L_i_{w}(t) \cdot L_i_{v}(t). \]

1.7. Regularization and the Ihara–Kaneko–Zagier theorem. In this section, we discuss how to extend multiple zeta values to non-admissible words and use this extension to derive relations among them. Conjecturally, all relations can be obtained in this way. We follow the paper [IKZ06].

1.7.1. The stuffle algebra as a polynomial ring.

**Theorem 1.168.** The map of \((\mathbb{Q}\langle Y \rangle^0, \ast)\)-algebras
\[ \varphi : \mathbb{Q}\langle Y \rangle^0[T] \to \mathbb{Q}\langle Y \rangle \]
is an isomorphism.

**Proof.** We first show that \( \varphi \) is surjective, which amounts to saying that any \( w \in \mathbb{Q}\langle Y \rangle \) can be written as a polynomial in \( y_1 \) with coefficients in \( \mathbb{Q}\langle Y \rangle^0 \). The bijection between the sets of multi-indices and words in the alphabet \( Y \) induces a grading by the weight \( \text{wt} \) and a filtration by the length \( l \) on the space \( \mathbb{Q}\langle Y \rangle \) given by
\[ \text{wt}(y_{s_1} \cdots y_{s_\ell}) = s_1 + \cdots + s_\ell \]
\[ \ell(y_{s_1} \cdots y_{s_\ell}) = \ell. \]
If we show that, for a fixed length \( \ell \) and word \( w \in F_\ell \mathbb{Q}\langle Y \rangle \), there are elements \( v_1 \in F_\ell \mathbb{Q}\langle Y \rangle^0 \) and \( v_2, v_3 \in F_{\ell-1} \mathbb{Q}\langle Y \rangle \) such that
\[ w = v_1 + v_2 \ast y_1 + v_3, \tag{1.169} \]
then the claim follows by induction on \( \ell \).

Any word of length \( \ell \) can be written as
\[ w = y_1 \cdots y_1 y_{s_1} \cdots y_{s_{\ell-m}} = \{y_1\}^m y_{s_1} \cdots y_{s_{\ell-m}} \]
with \( s_1 \neq 1 \) and \( m \geq 0 \).
We next prove, by induction on $m$, that $w$ can be written as in (1.169). For $m = 0$, we have $w \in \mathbb{Q}\langle Y \rangle^0$. Thus we can choose $v_1 = w$, $v_2 = v_3 = 0$. For the induction step we compute

$$\{y_1\}^{m-1} y_{s_1} \cdots y_{s_{t-m}} * y_1 = m \cdot w + \sum_{i=1}^{\ell-m} \{y_1\}^{m-1} y_{s_1} \cdots y_{s_i} y_{s_{i+1}} \cdots y_{s_{\ell-m}} - v_3$$

with $v_3 \in F_{\ell-1}\mathbb{Q}(Y)$. Applying the induction hypothesis with respect to $m$ we deduce that $w$ can be written as in (1.169). It follows that $\varphi$ is surjective.

To prove the injectivity of $\varphi$, we write each non-zero $P \in \mathbb{Q}\langle Y \rangle^0[T]$ as

$$P = w_1 T^m + w_2$$

with $0 \neq w_1 \in \mathbb{Q}\langle Y \rangle^0$ and $w_2$ of degree less than $m$ in the variable $T$. Then

$$\varphi(P) = m! y_1^m w_1 + v_2$$

where all the words in $v_2$ have less than $m$ factors $y_1$ in the front. Thus $\varphi(P) \neq 0$ and $\varphi$ is injective.

1.7.2. The shuffle algebra as a polynomial ring. Mutatis mutandis, one can prove the analogous result for the shuffle product.

**Theorem 1.170.**

(1) The map of $(\mathfrak{S}^0,\shuffle)$-algebras

$$\psi_1: \mathfrak{S}^0[T] \longrightarrow \mathfrak{S}^1$$

is an isomorphism.

(2) The map of $(\mathfrak{S}^1,\shuffle)$-algebras

$$\psi_2: \mathfrak{S}^1[U] \longrightarrow \mathfrak{S}$$

is an isomorphism.

Therefore, the map of $(\mathfrak{S}^0,\shuffle)$-algebras

$$\psi: \mathfrak{S}^0[T,U] \longrightarrow \mathfrak{S}$$

is an isomorphism.

**Proof.** Exercise 1.203. □
1.7.3. Regularized zeta values. Using the previous theorems we define the stuffle and shuffle regularization maps.

**Definition 1.171.** The stuffle regularization map
\[ \text{reg}^T_s : \mathcal{S}^1 \rightarrow \mathcal{S}^0 \] is defined as \( \text{reg}^T_s = \varphi^{-1} \), while the shuffle regularization maps
\[ \text{reg}^T_{su} : \mathcal{S}^1 \rightarrow \mathcal{S}^0, \quad \text{and} \]
\[ \text{reg}^{T,U}_{su} : \mathcal{S} \rightarrow \mathcal{S}^0 \] as \( \text{reg}^T_{su} = \psi^{-1} \) and \( \text{reg}^{T,U}_{su} = \psi^{-1} \).

Theorems 1.168 and 1.170 allow us to extend the function \( \zeta \) in a formal way.

**Definition 1.172.** The stuffle regularized zeta map, denoted \( \zeta^T_s \), is the composition
\[ \mathbb{Q}\langle Y \rangle \xrightarrow{\text{reg}^T_s} \mathbb{Q}\langle Y \rangle^0[T] \xrightarrow{\zeta} \mathbb{Z}[T] \subset \mathbb{R}[T]. \]
We denote by \( \zeta_s \) the composition of \( \zeta^T_s \) with the evaluation at \( T = 0 \).

The shuffle regularized zeta map, denoted by \( \zeta^T_{su} \), is the composition
\[ \mathcal{S}^1 \xrightarrow{\text{reg}^T_{su}} \mathcal{S}^0[T] \xrightarrow{\zeta} \mathbb{Z}[T] \subset \mathbb{R}[T]. \]
Similarly, we write \( \zeta^{T,U}_{su} \) for the composition
\[ \mathcal{S} \xrightarrow{\text{reg}^{T,U}_{su}} \mathcal{S}^0[T,U] \xrightarrow{\zeta} \mathbb{Z}[T,U] \subset \mathbb{R}[T,U]. \]
We denote by \( \zeta_{su} \) the composition of \( \zeta^{T,U}_{su} \) with the evaluation at \( T = U = 0 \). We will also denote by \( \zeta_{su} \) its restriction to \( \mathcal{S}^1 \).

By identifying \( (\mathbb{Q}\langle Y \rangle, \ast) \) with \( (\mathcal{S}^1, \ast) \), we will also consider \( \zeta^T_s \) as a map from \( (\mathcal{S}^1, \ast) \) to \( \mathbb{R}[T] \). This map is characterized by the conditions
\[ \zeta^T_s(w) = \zeta(w) \in \mathbb{R}, \quad \text{if } w \in \mathcal{S}^0, \]
\[ \zeta^T_s(x_1) = T, \]
\[ \zeta^T_s(v \ast w) = \zeta^T_s(v)\zeta^T_s(w). \]
In the same way, the map \( \zeta^{T}_{su} \) is characterized by the identities
\[ \zeta^{T}_{su}(w) = \zeta(w) \in \mathbb{R}, \quad \text{if } w \in \mathcal{S}^0, \]
\[ \zeta^{T}_{su}(x_1) = T, \]
\[ \zeta^{T}_{su}(v \ast w) = \zeta^{T}_{su}(v)\zeta^{T}_{su}(w). \]
The maps \( \zeta_s, \zeta_{su} \) and \( \zeta^{T,U}_{su} \) are determined by similar conditions. For future reference we single out the properties characterizing \( \zeta_{su} \).
Proposition 1.173. The map $\zeta: \mathcal{H} \to \mathbb{R}$ is the only map satisfying

$$
\zeta(w) = \zeta(w) \in \mathbb{R}, \quad \text{if } w \in \mathcal{H}^0, \tag{1.174}
$$

$$
\zeta(x_0) = 0, \quad \zeta(x_1) = 0, \tag{1.175}
$$

$$
\zeta(v \shuffle w) = \zeta(v) \zeta(w). \tag{1.176}
$$

Corollary 1.177. The image of $\zeta$ agrees with $\mathcal{Z}$.

Proof. By Theorem 1.170, every element $w \in \mathcal{H}$ can be written as a polynomial in $x_0$ and $x_1$ with coefficients in $\mathcal{H}^0$ with respect to the shuffle product. By Proposition 1.173, we deduce $\zeta(w) \in \mathcal{Z}$. \hfill \square

Example 1.178. On the one hand, we have

$$
\zeta^T_*(1, 2) = \frac{1}{2} \zeta^T_*(y_1 y_2)
= \frac{1}{2} \zeta^T_*(y_2 * y_1 - y_2 y_1 - y_3)
= \zeta(2)T - \zeta(2, 1) - \zeta(3),
$$

which yields $\zeta_*(1, 2) = -\zeta(2, 1) - \zeta(3)$. On the other hand,

$$
\zeta^T_*(1, 2) = \frac{1}{2} \zeta^T_* (x_1 x_0 x_1)
= \frac{1}{2} \zeta^T_*(x_0 x_1 \shuffle x_1 - 2 x_0 x_1 x_1)
= \zeta(2)T - 2\zeta(2, 1).
$$

Therefore, $\zeta_*(1, 2) = -2\zeta(2, 1)$.

1.7.4. Comparing the shuffle and the stuffle regularizations. As we just saw in the previous example, the regularizations $\zeta^T_*(w)$ and $\zeta^T_*(w)$ are in general different from each other. In order to compare them, we introduce the formal power series

$$
A(u) = e^{u \Gamma(1 + u)} = \exp \left( \sum_{n \geq 2} \frac{(-1)^n}{n} \zeta(n) u^n \right),
$$

where $\gamma$ is the Euler-Mascheroni constant, and the second identity follows from Exercise 1.20. We write

$$
A(u) = \sum_{k \geq 0} \gamma_k u^k.
$$

Observe that $\gamma_k$ is a linear combination, with rational coefficients, of multiple zeta values of weight $k$. Here are the first values:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_k$</td>
<td>0</td>
<td>$\frac{\zeta(2)}{2}$</td>
<td>$-\frac{\zeta(3)}{3}$</td>
<td>$\frac{\zeta(2, 2)}{4}$</td>
<td>$\frac{3\zeta(4)}{8}$</td>
<td>$-\frac{\zeta(2, 3)}{6}$</td>
</tr>
</tbody>
</table>

We define an $\mathbb{R}$-linear map $\varrho : \mathbb{R}[T] \to \mathbb{R}[T]$ by

$$
\varrho(T^n) = \frac{d^n}{du^n} \left. (A(u)e^{Tu}) \right|_{u=0} = n! \sum_{k=0}^n \gamma_k \frac{T^{n-k}}{(n-k)!}, \tag{1.179}
$$
Theorem 1.180 (Ihara–Kaneko–Zagier, [IKZ06]). The following identity holds for all words \( w \in \mathcal{Y}^1 \):
\[
\zeta^T_m(w) = \varrho(\zeta^T_m(w)).
\]

Example 1.181. Since \( \gamma_0 = 1 \) and \( \gamma_1 = 0 \), we have \( \varrho(1) = 1 \) and \( \varrho(T) = T \). Combining this with Example 1.178 we find
\[
\varrho(\zeta^T_m(1, 2)) = \varrho(\zeta(2)T - \zeta(2, 1) - \zeta(3)) = \zeta(2)T - \zeta(2, 1) - \zeta(3).
\]

On the other hand,
\[
\zeta^T_m(1, 2) = \zeta(2)T - 2\zeta(2, 1),
\]
and hence we recover Euler’s relation \( \zeta(2, 1) = \zeta(3) \).

Proof of Theorem 1.180. The idea is to view \( \zeta^T_m(w) = \varrho(\zeta^T_m(w)) \) as an identity of functions in \( T \). Let \( M > 0 \) be an integer and \( w = (y_1 \cdots y_{s_1}) \) a word in the alphabet \( Y \). We write
\[
\zeta_M(w) = \sum_{M > m_1 > m_2 > \cdots > m_\ell > 0} \frac{1}{m_1^{s_1} \cdots m_\ell^{s_\ell}}.
\]
Note that, if \( w \) is admissible, then \( \lim_{M \to \infty} \zeta_M(w) = \zeta(w) \). We extend \( \zeta_M \) to a map \( \mathbb{Q}(Y) \to \mathbb{R} \) by linearity. Then \( \zeta_M \) satisfies the stuffle relation
\[
\zeta_M(w_1)\zeta_M(w_2) = \zeta_M(w_1 \ast w_2).
\]

From the approximation of the harmonic series
\[
\zeta_M(y_1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{M-1} = \log M + \gamma + O \left( \frac{1}{M} \right)
\]
and the representation of \( \zeta^T_m(w) \) as a polynomial on \( \zeta_m(1) \), it follows that there exists \( j \geq 0 \) such that, for \( M \) large enough, one has
\[
\zeta_M(w) = \zeta_m^{\log^M M + \gamma}(w) + O(M^{-1} \log^j M),
\]
where \( \zeta_m^{\log^M M + \gamma}(w) \) means the evaluation at \( T = \log M + \gamma \) of \( \zeta^T_m(w) \).

Recall that, with each positive multi-index \( s \) we associated a polylogarithm function. Using the identification of positive multi-indices with words in the alphabet \( Y \) and linearity, we attach, to each element \( w \in \mathcal{Y}^1 \) a function \( \text{Li}_w \) on the segment \((0, 1)\). If \( w \in \mathcal{Y}^0 \), then
\[
\lim_{t \to 1^-} \text{Li}_w(t) = \zeta(w).
\]

Moreover we have for all \( w, w' \) and \( t \in (0, 1) \)
\[
\text{Li}_w(t) \cdot \text{Li}_{w'}(t) = \text{Li}_{w \shuffle w'}(t).
\]

Since
\[
\text{Li}_{y_1}(t) = \log \left( \frac{1}{1-t} \right),
\]
when \( t \to 1^{-} \)
\[
\text{Li}_w(t) = \zeta^{\log\left(\frac{1}{1-t}\right)}(w) + O\left((1 - t) \log^j \left(\frac{1}{1-t}\right)\right) \tag{1.183}
\]
for some \( j \geq 0 \) depending on \( w \). Here \( \zeta^{\log\left(\frac{1}{1-t}\right)}(w) \) means the evaluation at \( T = \log\left(\frac{1}{1-t}\right) \) of \( \zeta^T(w) \).

By explicit calculations,
\[
\text{Li}_w(t) = \sum_{m \geq 2} \zeta_m(w) t^{m-1}.
\]

In the last equality we use that \( \zeta_1(w) = 0 \).

We now need Lemma 1.184 below. We apply it to the polynomials \( P(T) = \zeta^T_w(s) \) and \( Q(T) = \varrho(\zeta^T_w(s)) \). We derive
\[
\text{Li}_w(t) = (1 - t) \sum_{m \geq 2} \zeta_m(w) t^{m-1}
\]
\[
= (1 - t) \sum_{m \geq 2} \zeta_m(w) t^{m-1} + (1 - t) \sum_{m \geq 1} O\left(\frac{\log^j m}{m}\right) t^{m-1}
\]
\[
= Q\left(\log \frac{1}{1-t}\right) + O\left((1 - t) \log^j \left(\frac{1}{1-t}\right)\right).
\]

Comparing with the asymptotic expansion we get the claimed identity
\[
\zeta^T_w(s) = \varrho(\zeta^T(s)). \quad \Box
\]

The next lemma is used in the proof of Theorem 1.180.

**Lemma 1.184.**

1. Let \( P(T) \in \mathbb{R}[T] \) and \( Q(T) = \varrho(P(T)) \). Then
\[
\sum_{m \geq 2} P(\log(m) + \gamma) t^{m-1} = \frac{1}{1-t} Q\left(\log \frac{1}{1-t}\right) + O\left(\log^j \left(\frac{1}{1-t}\right)\right) \tag{1.185}
\]

for some \( j \in \mathbb{N} \), as \( t \to 1^{-} \).
(2) As \( t \to 1^- \), we have
\[
\sum_{m \geq 2} \frac{\log^j m}{m} t^{m-1} = O \left( \log^{j+1} \left( \frac{1}{1-t} \right) \right). \tag{1.186}
\]

**Proof.** Let us prove (1.186) first. Since
\[
\sum_{m \geq 2} \frac{1}{m} t^{m-1} = -1 - \frac{1}{t} \log(1-t),
\]
for \( j = 0 \) the left hand side of (1.186) is of type \( O \left( \log \left( \frac{1}{1-t} \right) \right) \) as \( t \to 1^- \), which proves the statement in this case. Now we proceed by induction on \( j \). We have
\[
\log^{j+1}(m) \leq c_j \sum_{n=1}^{m} \frac{\log^j n}{n}
\]
for \( m \geq 1, j \geq 0 \). This follows easily from the integral
\[
\int_1^m \frac{\log^j(x)}{x} \, dx = \frac{\log^{j+1}(m)}{j+1}.
\]
Hence, for \( t < 1 \), we obtain
\[
\sum_{m \geq 1} \frac{\log^{j+1}(m)}{m} t^{m-1} \leq c_j \sum_{m \geq 1} \frac{t^{m-1}}{m} \sum_{n=1}^{m} \frac{\log^j(n)}{n} \leq c_j \left( \sum_{n \geq 1} \frac{\log^j(n)}{n} t^{n-1} \right) \left( \frac{1}{t} \log \left( \frac{1}{1-t} \right) \right).
\]
Now (1.186) follows by induction on \( j \) for all \( j \geq 0 \).

We now establish the identity (1.185). By construction, \( \varrho \) is a linear map on \( \mathbb{R}[T] \) and it therefore suffices to prove (1.185) for \( P(T) = (T-\gamma)^n \).

Thus we put \( Q(T) = \varrho((T-\gamma)^n) \). Then, by equation (1.179),
\[
Q(T) = \frac{d^n}{du^n} \left( A(u)e^{(T-\gamma)u} \right) \bigg|_{u=0} = \frac{d^n}{du^n} \left( \Gamma(1+u)e^{Tu} \right) \bigg|_{u=0}.
\]
Hence,
\[
\frac{1}{1-t} Q \left( \log \left( \frac{1}{1-t} \right) \right) = \frac{d^n}{du^n} \left( \frac{\Gamma(1+u)}{(1-t)^{1+u}} \right) \bigg|_{u=0} = \frac{d^n}{du^n} \left( \sum_{m \geq 1} \frac{\Gamma(m+u)}{\Gamma(m)} t^{m-1} \right) \bigg|_{u=0} = \sum_{m \geq 1} \frac{\Gamma^{(n)}(m)}{\Gamma(m)} t^{m-1},
\]
where $\Gamma^{(n)}(m)$ is the $n$-th derivative of the $\Gamma$ function evaluated at $m$. Now we use that, for $m \to \infty$ and all $n$, we have the estimate

$$\frac{\Gamma^{(n)}(m)}{\Gamma(m)} = \log(m)^n + O\left(\frac{\log^{n-1}(m)}{m}\right).$$

(1.187)

Using this and (1.186) we obtain

$$\sum_{m \geq 1} \frac{\Gamma^{(n)}(m)}{\Gamma(m)} t^{m-1} = \sum_{m \geq 1} \log(m)^n t^{m-1} + O\left(\log^n\left(\frac{1}{1-t}\right)\right) = \sum_{m \geq 1} P(\log(m) + \gamma) t^{m-1} + O\left(\log^n\left(\frac{1}{1-t}\right)\right),$$

concluding the proof of the lemma. $\square$

1.7.5. The extended double shuffle relations. We now introduce the extended double shuffle relations. We first recall the two commutative diagrams

$$(\mathcal{S}^1, \sqcup) \xrightarrow{\text{reg}_\sqcup^T} (\mathcal{S}^0, \sqcup)[T]$$

$$(\mathcal{S}^1, \ast) \xrightarrow{\text{reg}_\ast^T} (\mathcal{S}^0, \ast)[T]$$

$$(\zeta_\ast \circ \zeta_\sqcup) \downarrow \downarrow$$

$$(\zeta_\ast \circ \zeta_\sqcup) \downarrow \downarrow$$

$$\mathbb{R}[T]$$

$$\mathbb{R}[T]$$

(1.188)

**Definition 1.189.** Let $(R, \cdot)$ be a $\mathbb{Q}$-algebra and $Z_R : \mathcal{S}^0 \to R$ a map. We say that $(R, Z_R)$ satisfies the finite double shuffle relations if $Z_R$ is an algebra homomorphism $Z_R : (\mathcal{S}^0, \sqcup) \to (R, \cdot)$, as well as an algebra homomorphism $Z_R : (\mathcal{S}^0, \ast) \to (R, \cdot)$, that is:

$$Z_R(w \sqcup v) = Z_R(w) \cdot Z_R(v) = Z_R(w \ast v).$$

(1.190)

Composing $Z_R$ with the regularization maps $\text{reg}_\sqcup^T$ and $\text{reg}_\ast^T$, we obtain extensions

$$Z_{\mathcal{S}_\sqcup}^T : (\mathcal{S}^1, \sqcup) \to R[T],$$

$$Z_{\mathcal{S}_\ast}^T : (\mathcal{S}^1, \ast) \to R[T].$$

Since $R$ is a $\mathbb{Q}$-algebra that receives a map from $\mathcal{S}^0$, we can define the formal power series

$$A_R(u) = \exp\left(\sum_{n \geq 2} \frac{(-1)^n}{n} Z_R(y_n) u^n\right).$$

By analogy with $\rho$, we can define a linear map $\varrho_R : R[T] \to R[T]$ by

$$\varrho_R(e^{Tu}) = A_R(u)e^{Tu}.$$

(1.191)
Definition 1.192. Assume that \((R, Z_R)\) satisfies the finite double shuffle relations. We say \((R, Z_R)\) satisfies the extended double shuffle relations, if in addition, for all \(w \in \hat{H}^1\), one has

\[ Z_{R, w}^T = \varrho_R(Z_{R, w}^T(w)). \quad (1.193) \]

Combining theorems 1.160 and 1.180 we obtain the main result of this section.

Theorem 1.194. The pair \((\mathbb{R}, \zeta)\) satisfies the extended double shuffle relations.

In particular, since for \(w_0 \in \hat{H}^0\) we have that \(x_1 \Join w_0 - x_1 * w_0 \in \hat{H}^0\) (Exercise 1.198), we deduce the Hoffman relation

\[ \zeta(x_1 \Join w_0 - x_1 * w_0) = 0. \quad (1.195) \]

Moreover, the following holds for all \(w_0 \in \hat{H}^0\)

\[ \zeta_{T=0}^T(x_1^m * w_0) = 0. \]

1.7.6. The universal algebra satisfying the extended double shuffle relations. Let \(\varphi: R \to R'\) be a morphism of \(Q\)-algebras. If \((R, Z_R)\) satisfies the extended double shuffle relations, then so does \((R', \varphi \circ Z_R)\). Let \(R_{EDS}\) be the universal algebra with this property. Thus \(R_{EDS}\) is a quotient of \(\hat{H}^0\) by certain relations and, for any \((R, Z_R)\) satisfying the extended double shuffle relations, there exists a map \(\varphi_R: R_{EDS} \to R\) such that the following diagram commutes

\[ \begin{array}{ccc}
\hat{H}^0 & \xrightarrow{Z_R} & R_{EDS} \\
\parallel & \downarrow \varphi_R & \\
\varphi & \downarrow & R.
\end{array} \]

The following conjecture describes the structure of the algebra of multiple zeta values.

Conjecture 1.196. The map \(\varphi_R\) is injective, that is the algebra \(\mathcal{Z}\) of MZVs is isomorphic to \(R_{EDS}\).

Remark 1.197. The finite double shuffle relations are linear and homogeneous with respect to the weight. Moreover the extended double shuffle relations are also homogeneous (Exercise 1.201). Since the coefficients of the power series \(A_R\) are polynomials in zeta values, the extended double shuffle relations relations are polynomial in the MZVs. Since products of MZVs can be reduced to linear combinations of MZVs using either the shuffle or the stuffle product, we can reduce the extended double shuffle relations relations to linear ones. Hence all possible relations among MZVs are conjectured to be generated by homogeneous linear relations.

\[ \star \star \star \]
Exercise 1.198. Show that, if \( w \in \mathcal{H}^0 \), then \( x_1 \sqcup w - x_1 \ast w \in \mathcal{H}^0 \).

Exercise 1.199. Deduce Euler’s sum formula (1.58) from the Hoffman relation. [Hint: take \( w = z_p \).]

Exercise 1.200. Show that \( \gamma_k \) is a polynomial in \( \zeta(2), \zeta(3), \ldots \), that is homogeneous of weight \( k \).

Exercise 1.201. Use Exercise 1.200 to prove that the EDS relations are homogeneous.

Exercise 1.202. What identities do we get from a comparison of \( \zeta_s(1, 1, 2) \) and \( \zeta_w(1, 1, 2) \)?

Exercise 1.203. Prove Theorem 1.170.

Exercise 1.204. Verify

\[
\zeta_M(s) = \sum_{k=0}^{n} a_k \left( \log M + \gamma + O \left( \frac{1}{M} \right) \right)^k
= \sum_{k=0}^{n} a_k (\log M + \gamma)^k + O \left( \frac{1}{M} \log^{n-1}(M) \right).
\]

Exercise 1.205. Prove (1.187).
2. Periods of mixed Hodge structures

In this chapter, we introduce the first tools from algebraic geometry that will be needed for the study of multiple zeta values. The main goal is to show that all these numbers can be obtained by integrating an algebraic differential form over a topological cycle on an algebraic variety defined over the field of rational numbers; the extra structures carried by cohomology will then give non-trivial information about multiple zeta values. With this in mind, we begin by recalling the definition of singular homology and cohomology of a topological space \(M\) in Section 2.1. It is a classical theorem of de Rham that, whenever \(M\) is a differentiable manifold, singular cohomology can be computed using differential forms. More precisely, the map that sends a differential form to the integration functional on singular homology induces an isomorphism between de Rham cohomology and singular cohomology of \(M\). If \(M\) underlies a complex algebraic variety \(X\), this cohomology can be computed using differential forms with polynomial coefficients: it is isomorphic to algebraic de Rham cohomology. A remarkable consequence is that, when \(X\) is defined over \(\mathbb{Q}\), we get two different rational structures on the same complex vector space which are not compatible. This is not bad news, quite the opposite: the comparison between them produces an interesting class of complex numbers called \textit{periods}. Another important consequence is that the cohomology of \(X\) is equipped with two filtrations whose interaction gives rise to a mixed Hodge structure. We explain the definition and give many examples, in particular of Hodge structures of mixed Tate type. Conjecturally, mixed Hodge structures of algebraic varieties over \(\mathbb{Q}\) capture all algebraic relations between periods. As an illustration, we explain in detail how to interpret \(\zeta(2)\) as a period of an algebraic variety and how this may be used to prove “by pure thought” that it is a rational multiple of \(\pi^2\).

2.1. Singular homology and cohomology. We begin by briefly recalling the definition of singular homology and cohomology of a topological space. For each integer \(n \geq 0\), let

\[
\Delta^n_{st} = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for } i = 0, \ldots, n \right\}
\]

be the standard \(n\)-dimensional simplex. For each integer \(i = 0, \ldots, n + 1\), there is a \textit{face} map \(\delta^i : \Delta^n_{st} \rightarrow \Delta^{n+1}_{st}\) given by

\[
\delta^i(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n).
\]

Let \(M\) be a topological space. A continuous map \(\sigma : \Delta^n_{st} \rightarrow M\) is called a \textit{singular \(n\)-simplex}. For each \(n \geq 0\), define

\[
C_n(M) = \bigoplus_{\sigma : \Delta^n_{st} \rightarrow M} \mathbb{Z} \sigma
\]
as the free abelian group generated by singular $n$-simplices on $M$. Elements of $C_n(M)$ are thus finite linear combinations with integral coefficients of continuous maps $\sigma: \Delta^n_{st} \to M$; they are called singular $n$-chains, or simply singular chains when $n$ is clear from the context. For example, a singular 0-chain is a linear combination of points of $M$ and a singular 1-chain is a linear combination of paths in $M$:

$$C_0(M) = \bigoplus_{p \in M} \mathbb{Z} p, \quad C_1(M) = \bigoplus_{\gamma: [0,1] \to M} \mathbb{Z} \gamma.$$ 

To have uniform notation in what follows, we also set $C_{-1}(M) = 0$.

For each $n \geq 1$, we define a boundary homomorphism

$$\partial_n: C_n(M) \longrightarrow C_{n-1}(M)$$

\[
\sigma \longmapsto \sum_{i=0}^{n} (-1)^i (\sigma \circ \delta^i) \tag{2.1}
\]

(Indeed, $\sigma \circ \delta^i: \Delta^n_{st} \to \Delta^n_{st} \to M$ is a singular $(n-1)$-simplex on $M$.) We also set $\partial_0 = 0$ on $C_0(M)$. Thanks to the alternating signs, these maps satisfy $\partial_{n-1} \circ \partial_n = 0$ for all $n \geq 1$ (Exercise 2.21), thus making

$$(C_\ast(M), \partial_\ast) = \left[ \cdots \partial_2 \longrightarrow C_1(M) \xrightarrow{\partial_1} C_0(M) \right]$$

into a chain complex of abelian groups in the sense of Definition A.6 from the appendix: the singular chain complex of $M$. Elements in the kernel of the boundary map $\partial_n$ are called closed singular chains, or cycles, and elements in the image of $\partial_{n+1}$ are called boundaries.

**Definition 2.2.** The singular homology in degree $n$ of $M$ is the abelian group $H_n(M, \mathbb{Z})$ defined as the $n$-th homology group of the singular chain complex of $M$, that is:

$$H_n(M, \mathbb{Z}) = \frac{\text{Ker}(\partial_n: C_n(M) \longrightarrow C_{n-1}(M))}{\text{Im}(\partial_{n+1}: C_{n+1}(M) \longrightarrow C_n(M))}.$$ 

In other words, $H_n(M, \mathbb{Z})$ is the group of cycles modulo boundaries.

For example, $H_0(M, \mathbb{Z}) = \mathbb{Z}^{\pi_0(M)}$, where $\pi_0(M)$ denotes the set of path-connected components of $M$. Indeed, $C_0(M)$ is the free abelian group generated by the points of $M$ and two points $p$ and $q$ define the same element in $H_0(M, \mathbb{Z})$ if and only if their difference $p - q$ is a boundary, that is, if there exists a path $\gamma: [0,1] \to M$ such that $\gamma(0) = q$ and $\gamma(1) = p$.

The construction of singular homology is functorial. If $f: M_1 \to M_2$ is a continuous map between two topological spaces, sending a singular $n$-simplex $\sigma: \Delta^n_{st} \to M_1$ on $M_1$ to the singular $n$-simplex $f \circ \sigma: \Delta^n_{st} \to M_2$ on $M_2$ obtained by composition with $f$ induces homomorphisms

$$f_*: C_n(M_1) \to C_n(M_2) \tag{2.3}$$
compatible with the boundary maps, hence a homomorphism
\[ f_\ast: H_n(M_1, \mathbb{Z}) \to H_n(M_2, \mathbb{Z}). \]

**Example 2.4.** Let \( M = \mathbb{C} \setminus \{0\} \) be the punctured complex plane. Consider the singular simplices
\[
\sigma_0: \Delta^0_0 \to M, \quad 1 \mapsto 1,
\sigma_1: \Delta^1_{st} \to M, \quad (t, 1-t) \mapsto \exp(2\pi it).
\]
Since \( M \) is connected, \( H_0(M, \mathbb{Z}) \) is the free abelian group generated by \( \sigma_0 \).
The group \( H_1(M, \mathbb{Z}) \) is also free of rank one, generated by \( \sigma_1 \) respectively. All other homology groups vanish.

Dualizing the definitions of singular chains and boundary maps, we find the free abelian group of singular \( n \)-cochains
\[ C^n(M) = \text{Hom}_\mathbb{Z}(C_n(M), \mathbb{Z}), \]
as well as coboundary maps \( d^n: C^n(M) \to C^{n+1}(M) \) for each \( n \geq -1 \). They form a cochain complex of abelian groups
\[ (C^n(M), d^*) = \left[ C^0(M) \xrightarrow{d^0} C^1(M) \xrightarrow{d^1} \cdots \right] \]
called the singular cochain complex of \( M \).

**Definition 2.5.** The singular cohomology in degree \( n \geq 0 \) of \( M \) is the abelian group \( H^n(M, \mathbb{Z}) \) defined as the \( n \)-th cohomology group of the singular cochain complex of \( M \), that is:
\[ H^n(M, \mathbb{Z}) = \frac{\text{Ker}(d^n: C^n(M) \to C^{n+1}(M))}{\text{Im}(d^{n-1}: C^{n-1}(M) \to C^n(M))}. \]

A continuous map of topological spaces \( f: M_1 \to M_2 \) defines a homomorphism \( f^*: C^n(M_2) \to C^n(M_1) \) by sending a cochain \( \varphi: C_n(M_2) \to \mathbb{Z} \) on \( M_2 \) to the cochain \( \varphi \circ f_\ast: C_n(M_1) \to \mathbb{Z} \) on \( M_1 \) obtained by composition with (2.3). The map \( f^* \) being compatible with the coboundaries \( d^n \), it induces a homomorphism
\[ f^*: H^*(M_2, \mathbb{Z}) \to H^*(M_1, \mathbb{Z}). \]

**Remarks 2.6.**

1. We have defined singular homology and cohomology with integral coefficients, but the same construction extends to other coefficient rings such as \( \mathbb{Q} \) or \( \mathbb{R} \). Most of the time it will be enough for our purposes to work with rational coefficients.

2. Singular homology and cohomology are invariants defined for any topological space. When \( M \) is the topological space underlying a differentiable manifold, instead of continuous maps \( \sigma: \Delta^n_{st} \to M \) we may use smooth maps, that is, maps admitting a \( C^\infty \) extension.
to an open neighborhood of $\Delta^n_k$ in $\mathbb{R}^{n+1}$. The resulting groups are the same, see e.g. [War83, 5.32] or [Lee13, Thm. 18.7].

(3) Working with rational coefficients, we can identify singular cohomology with the linear dual of singular homology

$$H^n(M, \mathbb{Q}) \simeq \text{Hom}_\mathbb{Q}(H_n(M, \mathbb{Q}), \mathbb{Q}),$$

(2.7)

and think of cohomology classes as linear functionals on homology. This point of view will be useful to discuss periods. However, the isomorphism (2.7) cannot hold with integral coefficients since the group $\text{Hom}(H_n(M, \mathbb{Z}), \mathbb{Z})$ is always torsion free, while $H^n(M, \mathbb{Z})$ may have torsion (see Exercise 2.23).

In the sequel, we will mainly consider singular cohomology with rational coefficients of topological spaces given by the complex points of algebraic varieties defined over subfields of $\mathbb{C}$. It deserves a special name:

**Definition 2.8.** Let $k$ be a subfield of $\mathbb{C}$ and $X$ an algebraic variety over $k$. The Betti cohomology $H^*_B(X)$ is the singular cohomology of the space of complex points $X(\mathbb{C})$ equipped with the analytic topology:

$$H^*_B(X) = H^*(X(\mathbb{C}), \mathbb{Q}).$$

As $X(\mathbb{C})$ has the homotopy type of a finite CW-complex, the Betti cohomology of $X$ is a finite-dimensional graded $\mathbb{Q}$-vector space by the first property below.

2.1.1. *Properties of singular homology and cohomology.* Singular homology and cohomology have many useful properties such as

(1) **Homotopy invariance:** if $M_1$ and $M_2$ are topological spaces with the same homotopy type, then

$$H_*(M_1, \mathbb{Z}) \simeq H_*(M_2, \mathbb{Z}), \quad H^*(M_1, \mathbb{Z}) \simeq H^*(M_2, \mathbb{Z}).$$

(2) **Mayer–Vietoris:** for any two open subspaces $U$ and $V$ such that $M = U \cup V$, there is a long exact sequence

$$\cdots \longrightarrow H_n(U \cap V, \mathbb{Z}) \xrightarrow{\alpha} H_n(U, \mathbb{Z}) \oplus H_n(V, \mathbb{Z}) \xrightarrow{\beta} H_n(M, \mathbb{Z}) \longrightarrow H_{n-1}(U \cap V, \mathbb{Z}) \longrightarrow \cdots$$

(2.9)

where, if $\iota_{U \cap V, U}: U \cap V \hookrightarrow U$ denotes the inclusion, and similarly for other pairs of a space and a subspace, the maps $\alpha$ and $\beta$ are given by

$$\alpha = (\iota_{U \cap V, U})_* \oplus (\iota_{U \cap V, V})_*, \quad \beta = ((\iota_{U, M})_*, -(\iota_{V, M})_*).$$
Dually, there is a long exact sequence in cohomology

\[ \cdots \rightarrow H^n(U \cap V, \mathbb{Z}) \rightarrow H^n(U, \mathbb{Z}) \oplus H^n(V, \mathbb{Z}) \rightarrow H^n(M, \mathbb{Z}) \rightarrow H^{n-1}(U \cap V, \mathbb{Z}) \rightarrow \cdots \] (2.10)

(3) Künneth formula: there are natural isomorphisms

\[ H_n(M_1 \times M_2, \mathbb{Q}) \cong \bigoplus_{i+j=n} H_i(M_1, \mathbb{Q}) \otimes_{\mathbb{Q}} H_j(M_2, \mathbb{Q}), \] (2.11)

\[ H^m(M_1 \times M_2, \mathbb{Q}) \cong \bigoplus_{i+j=n} H^i(M_1, \mathbb{Q}) \otimes_{\mathbb{Q}} H^j(M_2, \mathbb{Q}). \] (2.12)

Note that for the Künneth formula to be true as stated we need rational coefficients (Exercise 2.24). A more sophisticated formula involving Tor groups is true with integral coefficients, see for instance [Hat02, Thm 3B.6].

2.1.2. Relative homology and cohomology. There are also relative versions of homology and cohomology. If \( \iota : N \hookrightarrow M \) is a topological subspace, the morphism of complexes \( \iota_* : C_*(N) \rightarrow C_*(M) \) induced by functoriality is injective. Recall e.g. from [Wei94, 1.5] that the cone of \( \iota_* \) is the complex \( C_*(M,N) = \text{cone}(\iota_*) \) given by

\[ C_n(M,N) = C_{n-1}(N) \oplus C_n(M) \]

in degree \( n \), together with the differential

\[ \partial(a,b) = (-\partial a, -\iota_*(a) + \partial b). \]

**Definition 2.13.** The relative homology of a pair \((M, N)\) consisting of a topological space \( M \) and a subspace \( N \) is the abelian group defined as the homology of the complex \( C_*(M,N) = \text{cone}(\iota_*) \), that is:

\[ H_*(M,N; \mathbb{Z}) = H_*(C_*(M,N)). \]

We refer the reader to Exercise 2.26 for an alternative construction.

By design, \( C_*(M,N) \) fits into a short exact sequence of complexes

\[ 0 \rightarrow C_*(M) \rightarrow C_*(M,N) \rightarrow C_*(N)[-1] \rightarrow 0, \]

where the left map sends \( b \) to \( (0,b) \), and the right map sends \( (a,b) \) to \( -a \). Above, the shifted complex \( C_*(N)[-1] \) has \( C_{n-1}(N) \) as degree \( n \) term, with differential \( -\partial_{n-1} \), so that the relation \( H_n(C_*(N)[-1]) = H_{n-1}(N, \mathbb{Z}) \) holds. The associated long exact sequence then reads

\[ \cdots \rightarrow H_n(M, \mathbb{Z}) \rightarrow H_n(M,N; \mathbb{Z}) \rightarrow H_{n-1}(N, \mathbb{Z}) \rightarrow H_{n-1}(M, \mathbb{Z}) \rightarrow \cdots \] (2.14)
Figure 2. A basis of $H_1(\mathbb{C} \setminus \{0\}, \{p, q\}; \mathbb{Z})$

and the connecting morphisms are nothing other than the maps

$$\iota_*: H_*(N, \mathbb{Z}) \to H_*(M, \mathbb{Z})$$

induced by the inclusion $\iota: N \hookrightarrow M$.

**Remark 2.15.** An element of relative homology $H_n(M, N; \mathbb{Z})$ is represented by a pair $(\sigma_N, \sigma_M)$ consisting of singular chains $\sigma_N \in C_{n-1}(N)$ and $\sigma_M \in C_n(M)$ such that $\partial \sigma_N = 0$ and $\partial \sigma_M = -\iota_* \sigma_N$. Since $\iota_*$ is injective, the singular chain $\sigma_N$ is determined by the latter condition, which implies the former. In other words, relative homology classes are represented by singular chains in $M$ which are not necessarily closed but whose boundary is constrained to lie in $N$.

**Example 2.16.** Consider $M = \mathbb{C} \setminus \{0\}$ and let $N = \{p, q\} \subseteq M$ be a subspace consisting of two distinct points. Let $\sigma_2: \Delta^1_{st} \to M$ be any continuous map such that $\sigma_2((0, 1)) = p$ and $\sigma_2((1, 0)) = q$. Then

$$\partial \sigma_2 = p - q \in C_0(N),$$

so $\sigma_2$ defines a relative chain. It follows from the long exact sequence (2.14) that the only non-trivial relative homology group is $H_1(M, N; \mathbb{Z})$, which has a basis given by the chain $\sigma_1$ from Example 2.4 and $\sigma_2$ (see Figure 2).

In a similar way, one defines relative cohomology groups. Let $\iota: N \hookrightarrow M$ be the inclusion of a topological subspace and $\iota^*: C^*(M) \to C^*(N)$ the induced map on cochain complexes. We consider the complex

$$C^*(M, N) = \text{cone}(\iota^*)[-1].$$

Explicitly, this is the complex given in degree $n$ by

$$C^n(M, N) = C^n(M) \oplus C^{n-1}(N)$$

together with the differential

$$d(a, b) = (da, \iota^*(a) - db),$$
so that there is an exact sequence of complexes
\[ 0 \to C^*(M, N) \to C^*(M) \xrightarrow{\iota^*} C^*(N) \to 0. \]

**Definition 2.17.** The *relative cohomology* of the pair \((M, N)\) is the abelian group defined as the cohomology of the complex \(C^*(M, N)\):
\[ \text{rel} \quad H^*(M, N; \mathbb{Z}) = H^*(C^*(M, N)). \]

By construction, relative cohomology fits into a long exact sequence
\[ \cdots \xrightarrow{} H^n(M, N; \mathbb{Z}) \xrightarrow{} H^n(M, \mathbb{Z}) \xrightarrow{\iota^*} H^n(N, \mathbb{Z}) \xrightarrow{} H^{n+1}(M, N; \mathbb{Z}) \xrightarrow{} H^{n+1}(M, \mathbb{Z}) \xrightarrow{} \cdots \]
(2.18)

**2.1.3. Sheaf cohomology.** Under mild assumptions on the topological space \(M\), singular cohomology can be identified with sheaf cohomology of the constant sheaf. More precisely, let \(\mathbb{Z}\) be the sheaf that assigns to each open subset \(U \subseteq M\) the group \(\mathbb{Z}(U) = \mathbb{Z}^{\pi_0(U)}\), where \(\pi_0(U)\) stands for the number of connected components of \(U\). Note that \(\mathbb{Z}(U)\) can be identified with the group of locally constant functions \(U \to \mathbb{Z}\). If \(M\) is a locally contractible topological space, then
\[ H^*(M, \mathbb{Z}) = H^*(M, \mathbb{Z}), \]
where the left-hand group is singular cohomology and the right-hand group is sheaf cohomology. The same result is true for other rings of coefficients.

In the same spirit, relative cohomology as in 2.1.2 can be written as the hypercohomology of a complex of sheaves. Let \(X\) be a topological space and \(\iota : Y \to X\) a closed immersion. The sheaf \(\iota_* \mathbb{Z}_Y\) is defined as
\[ \iota_* \mathbb{Z}_Y(U) = \mathbb{Z}_Y(U \cap Y). \]
There is a morphism of sheaves \(\mathbb{Z}_X \to \iota_* \mathbb{Z}_Y\). If \(X\) and \(Y\) are both locally contractible, then
\[ H^*(Y, \mathbb{Z}) = H^*(X, \iota_* \mathbb{Z}_Y) \]
(2.19)
\[ H^*(X, Y; \mathbb{Z}) = \mathbb{H}^*(X, \mathbb{Z}_X \to \iota_* \mathbb{Z}_Y). \]
(2.20)

**Exercise 2.21.** Prove that the boundary map (2.1) in the definition of singular homology satisfies \(\partial_{n-1} \circ \partial_n = 0\) for all integers \(n \geq 1\).

**Exercise 2.22.** Combine the Mayer–Vietoris long exact sequences (2.9) and (2.10) with Example 2.4 to compute the homology and the cohomology of the Riemann sphere \(\mathbb{P}^1(\mathbb{C})\).

**Exercise 2.23.** In this exercise, we compute the homology and the cohomology of the real projective plane.
Let $M$ be the M"obius band defined as the quotient $M = [0, 1] \times (0, 1)/ \sim$ where $\sim$ is the equivalence relation generated by $(0, x) \sim (1, 1 - x)$. Prove that the first homology group of $M$ is spanned by the singular simplex $\sigma: \Delta^1_{st} \to M$, $\sigma(t, 1 - t) = (t, 1/2)$ and that the simplex

$$
\sigma_1(t, 1 - t) = \begin{cases} 
(2t, 3/4), & \text{if } t \leq 1/2 \\
(2t - 1, 1/4), & \text{if } t \geq 1/2,
\end{cases}
$$

is closed and represents the same class as $2\sigma$ in $H_1(M, \mathbb{Z})$.

Let $P^2(\mathbb{R})$ be the real projective plane viewed as the quotient $P^2(\mathbb{R}) = S^2/\sim$ of the two-dimensional sphere $S^2$ by the equivalence relation generated by $x \sim -x$. Show that $P^2(\mathbb{R})$ can be covered by an open subset homeomorphic to the unit disc in $\mathbb{R}^2$ and an open subset homeomorphic to the M"obius band.

Use the Mayer–Vietoris sequences in homology and cohomology to compute

$$
H_i(P^2(\mathbb{R}), \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & \text{if } i = 0, \\
\mathbb{Z}/2\mathbb{Z}, & \text{if } i = 1, \\
0, & \text{if } i = 2.
\end{cases}
$$

$$
H^i(P^2(\mathbb{R}), \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & \text{if } i = 0, \\
0, & \text{if } i = 1, \\
\mathbb{Z}/2\mathbb{Z}, & \text{if } i = 2.
\end{cases}
$$

Exercise 2.24. Compute the homology and the cohomology of the product $P^2(\mathbb{R}) \times P^2(\mathbb{R})$ and deduce that the K"unneth formulas (2.11) and (2.12) do not hold if the coefficients $\mathbb{Q}$ are replaced by $\mathbb{Z}$.

Exercise 2.25. Prove that there is a natural short exact sequence

$$
0 \to \text{Ext}(H_{n-1}(M, \mathbb{Z}), \mathbb{Z}) \to H^n(M, \mathbb{Z}) \to \text{Hom}(H_n(M, \mathbb{Z}), \mathbb{Z}) \to 0.
$$

Whenever $H_{n-1}(M, \mathbb{Z})$ is torsion-free, the Ext group vanishes and we get an isomorphism between $H^n(M, \mathbb{Z})$ and the linear dual of $H_n(M, \mathbb{Z})$.

Exercise 2.26 (An alternative definition of relative homology). We keep the notation from paragraph 2.1.2. Given a topological space $M$ and a subspace $N$, show that the boundary maps $\partial_n$ yield a complex

$$
\cdots \to C_n(M) \to C_{n-1}(M) \to \cdots
$$

which is quasi-isomorphic to $C_\bullet(M, N)$. Therefore, one can also define the relative homology of the pair $(M, N)$ as the homology of (2.27).
2.2. Algebraic de Rham cohomology. Inspired by ideas of Atiyah and Hodge, Grothendieck introduced de Rham cohomology of algebraic varieties over fields of characteristic zero in his paper [Gro66], written shortly after Hironaka’s proof of resolution of singularities. In this section, we explain the definition and give a few examples of how to compute it.

2.2.1. Motivation: de Rham’s theorem. Before going into Grothendieck’s construction, we shall give a quick review of the more familiar objects in differential geometry. The reader is encouraged to consult [BT82] for a very nice exposition of the subject.

Let $M$ be a differentiable manifold of dimension $d$. Recall that a differential $p$-form on $M$ can be written in local coordinates as

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq d} f_{i_1, \ldots, i_p}(x_1, \ldots, x_d) dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \quad (2.28)$$

where $f_{i_1, \ldots, i_p}(x_1, \ldots, x_d)$ are $C^\infty$-functions on $M$. Let $E^p(M)$ denote the real vector space of differential $p$-forms and set

$$E^*(M) = \bigoplus_{p=0}^d E^p(M).$$

The exterior derivative $d: E^*(M) \to E^*(M)$ is the unique $\mathbb{R}$-linear map which sends $p$-forms to $(p+1)$-forms and satisfies the following three axioms:

(a) If $f$ is a smooth function, then $df$ is given in local coordinates by

$$df = \sum_{i=1}^d \frac{\partial f}{\partial x_i} dx_i.$$

(b) The equality $d^2 = 0$ holds.

(c) If $\alpha$ is a $p$-form, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

for every differential form $\beta \in E^*(M)$.

We thus get a complex

$$0 \to E^0(M) \xrightarrow{d} E^1(M) \xrightarrow{d} \cdots \xrightarrow{d} E^d(M) \to 0,$$

whose cohomology $H^*_\text{dR}(M)$ is called the de Rham cohomology of $M$.

A classical theorem of de Rham asserts that singular cohomology with real coefficients $H^*(M, \mathbb{R})$ can be computed using differential forms. As was mentioned in part (2) of Remark 2.6, if one replaces singular chains with smooth singular chains in the definition of singular homology the resulting groups are the same. We denote by $C^*_{\text{sm}}(M, \mathbb{R})$ the complex of smooth singular chains with real coefficients and we set

$$C^*_\text{sm}(M, \mathbb{R}) = \text{Hom}_\mathbb{R}(C^*_\text{sm}(M, \mathbb{R}), \mathbb{R}).$$
The groups $C^n_{\text{sm}}(M, \mathbb{R})$ form a complex that computes the singular cohomology with real coefficients of $M$. Working with smooth chains allows one to integrate differential forms over them, thus getting a map

$$E^\ast(M) \to C^n_{\text{sm}}(M, \mathbb{R})$$

(2.29)

that associates with each differential form $\omega \in E^\ast(M)$ the integration functional $\int_\sigma \omega: C^\ast_{\text{sm}}(M, \mathbb{R}) \to \mathbb{R}$ given by

$$\sigma \mapsto \int_\sigma \omega.$$

In this setting, the classical Stokes theorem

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

becomes the statement that the map (2.29) is a morphism of complexes. Therefore, it descends to a map in cohomology, which we shall still call the integration functional. De Rham’s theorem is the statement that the resulting map is an isomorphism or, equivalently, that the original morphism of complexes is a quasi-isomorphism.

**Theorem 2.30 (de Rham).** Let $M$ be a differentiable manifold of dimension $d$. For each $0 \leq j \leq d$, the map

$$H^j_{\text{dR}}(M, \mathbb{R}) \to H^j(M, \mathbb{R})$$

that sends the class of a differential form $\omega$ to the integration functional $\int_\sigma \omega: H^j(M, \mathbb{R}) \to \mathbb{R}$ is an isomorphism.

Remarkably enough, when $M$ is the underlying topological space of a complex algebraic variety $X$, it suffices to consider differential forms with regular functions on $X$ as coefficients to capture all de Rham cohomology classes. In this way, one obtains a purely algebraic definition of cohomology.

### 2.2.2. Kähler differentials.

From now on, we will assume that the reader is familiar with the rudiments of the language of schemes, as can be found e.g. in the first sections of Chapter II of [Har77]. We first recall the notion of Kähler differentials, the substitute for the differential forms (2.28) that will allow for a purely algebraic definition of de Rham cohomology.

Let $k$ be a field of characteristic zero and let $A$ be a finitely generated reduced $k$-algebra. Recall that reduced means that there are no non-zero nilpotent elements in $A$, i.e. if $x^n = 0$ for some integer $n \geq 1$, then $x = 0$. The spectrum $X = \text{Spec}(A)$ is then an affine algebraic variety over $k$.

**Definition 2.31.** Let $M$ be an $A$-module. A $k$-linear derivation of $A$ into $M$ is a $k$-linear map $D: A \to M$ satisfying the Leibniz rule

$$D(ab) = aD(b) + bD(a)$$

(2.32)

for all elements $a, b \in A$. Note that (2.32) implies that $D(r) = 0$ for all $r \in k$, that is, elements of $k$ are “constants” for the derivation.
**Definition 2.33.** The $A$-module of *Kähler differentials* $\Omega^1_{A/k}$ is the quotient of the free $A$-module generated by symbols $da$, for all $a \in A$, by the submodule spanned by the elements
$$dr, \quad d(a+b) - da - db, \quad d(ab) - adb - bda,$$
for all $r \in k$ and all $a, b \in A$.

By construction, the map $d: A \to \Omega^1_{A/k}$ sending $a$ to $da$ is a $k$-linear derivation. It is actually the universal one, in the sense that, given any $k$-linear derivation $D: A \to M$, there exists a unique morphism of $A$-modules $\varphi: \Omega^1_{A/k} \to M$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{d} & \Omega^1_{A/k} \\
\downarrow{D} & & \downarrow{\varphi} \\
M & & 
\end{array}
$$

(2.34)

The following example is at the base of many computations in algebraic de Rham cohomology.

**Example 2.35.** Let $A = k[x_1, \ldots, x_n]$. Then $\Omega^1_{A/k}$ is the free $A$-module generated by $dx_1, \ldots, dx_n$. Indeed, let $D: A \to M$ be any $k$-linear derivation. It follows from the Leibniz rule (2.32) that the image of a polynomial $f \in A$ is equal to
$$D(f) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} D(x_i),$$
where $\partial f/\partial x_i \in A$ stands for the partial derivative in the usual sense. Thus, $D$ is completely determined by the images of the $x_i$. More generally, if
$$A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m),$$
then the $A$-module of Kähler differentials $\Omega^1_{A/k}$ has generators $dx_1, \ldots, dx_n$ and relations $df_j = \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i$ for $j = 1, \ldots, m$.

**2.2.3. Algebraic de Rham cohomology of smooth affine varieties.**

**Proposition 2.36.** If $X = \text{Spec}(A)$ is smooth of dimension $d$, then the module of Kähler differentials $\Omega^1_{A/k}$ is locally free of rank $d$.

Recall that an $O_X$-module $\mathcal{F}$ is locally free of some rank $d$ if $X$ can be covered by open subsets $U$ such that the restriction $\mathcal{F}|_U$ is isomorphic to $O_U^{\oplus d}$. We refer to Exercise 2.83 for an example illustrating why the smoothness condition is necessary for Proposition 2.36 to hold.

For each integer $p \geq 0$, let
$$\Omega^p_{A/k} = \Lambda^p \Omega^1_{A/k}$$
be the $p$-th exterior power of $\Omega^1_{A/k}$ over $A$. In particular, $\Omega^0_{A/k} = A$ and $\Omega^p_{A/k} = 0$ for $p > n$. For each $1 \leq p \leq n - 1$, the $A$-module $\Omega^p_{A/k}$ is the quotient of the free $A$-module generated by elements $\omega_1 \wedge \cdots \wedge \omega_p$, with $\omega_i \in \Omega^1_{A/k}$, by the submodule spanned by

$$(a\omega_1 + b\omega'_1) \wedge \omega_2 \wedge \cdots \wedge \omega_p - a\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_p - b\omega'_1 \wedge \omega_2 \wedge \cdots \wedge \omega_p$$

for all $a, b \in A$ and by $\omega_1 \wedge \cdots \wedge \omega_p$ whenever two of the $\omega_i$ are equal. We call $p$-forms the elements of $\Omega^p_{A/k}$.

As in the case of classical de Rham cohomology recalled in Section 2.2.1, the derivation $d: A \to \Omega^1_{A/k}$ extends in a unique way to $k$-linear maps $d^p: \Omega^p_{A/k} \to \Omega^{p+1}_{A/k}$ satisfying

(a) $d^p \circ d^{p-1} = 0$ for all $p$,

(b) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for all $p$-forms $\alpha$.

Explicitly, one sets

$$d(\omega_1 \wedge \cdots \wedge \omega_p) = \sum_{j=1}^{p} (-1)^{j+1} \omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge d\omega_j \wedge \omega_{j+1} \wedge \cdots \wedge \omega_p.$$

This yields the algebraic de Rham complex

$$\Omega^*_{A/k}: \quad A \longrightarrow \Omega^1_{A/k} \longrightarrow \Omega^2_{A/k} \longrightarrow \cdots \longrightarrow \Omega^d_{A/k}.$$

A $p$-form is said to be closed if it belongs to the kernel of $d^p$ and exact if it is in the image of $d^{p-1}$.

**Definition 2.37.** The algebraic de Rham cohomology of $X = \text{Spec}(A)$ is the cohomology of the algebraic de Rham complex

$$H^*_{\text{dR}}(X) = H^*(\Omega^*_A).$$

In other words, $H^*_{\text{dR}}(X)$ is the quotient of the space of closed $n$-forms on $X$ modulo the subspace of exact $n$-forms.

Both the space of closed forms and the space of exact forms on $X$ have in general infinite dimension. However, we will prove below (Corollary 2.75) that $H^*_{\text{dR}}(X)$ is a finite-dimensional $k$-vector space.

**Example 2.38.** Consider the affine variety $\mathbb{G}_m = \text{Spec} k[t, t^{-1}]$, which is the algebraic analogue of the punctured complex plane from Example 2.4. The de Rham complex reads

$$\begin{align*}
k[t, t^{-1}] & \xrightarrow{d} k[t, t^{-1}] dt \\
t^m & \xrightarrow{\cdot t^{m-1}} dt
\end{align*}$$
and its cohomology is thus given by the kernel and cokernel of \( d \). Note that all \( t^m dt = d(t^{m+1}/(m+1)) \) are in the image of \( d \) except for \( m = -1 \), whence:

\[
H^i_{\text{dR}}(\mathbb{G}_m) = \begin{cases} 
  k & i = 0, \\
  k dt & i = 1, \\
  0 & \text{else.}
\end{cases}
\]

**Example 2.39.** Let \( a, b \in k \) be such that \( 4a^3 + 27b^2 \neq 0 \). Then the polynomial \( f(x) = x^3 + ax + b \) has no double roots, so the equation \( y^2 = f(x) \) defines a smooth affine curve \( X \subseteq k^2 \). We will call \( X \) an affine elliptic curve since its closure in \( \mathbb{P}^2_k \) is an elliptic curve, from which \( X \) is obtained by removing a point. The affine elliptic curve \( X \) is given by the spectrum of the \( k \)-algebra

\[
A = k[x, y]/(y^2 - x^3 - ax - b).
\]

Again, \( H^0_{\text{dR}}(X) = k \) and \( H^i_{\text{dR}}(X) \) vanishes for \( i \geq 2 \). The only interesting cohomology is \( H^1_{\text{dR}}(X) = \text{Coker}(d: A \to \Omega^1_{A/k}) \).

Since \( f \) and \( f' \) are coprime, there exist polynomials \( P, Q \in k[x] \) such that \( Pf + Qf' = 1 \). We consider the 1-form

\[
\omega = Py dx + 2Qdy \in \Omega^1_{A/k}.
\]

Using that the identity \( 2ydy = f'(x)dx \) holds in \( \Omega^1_{A/k} \), one finds

\[
dx = y\omega, \quad dy = f'(x)\omega/2.
\]

(2.40)

Thus, any element of \( \Omega^1_{A/k} \) can be uniquely written as \( (R + Sy)\omega \) for polynomials \( R, S \in k[x] \). By (2.40), all differentials of the form \( Sy\omega \) are exact, so we only need to decide when \( R\omega \) is exact. For this, we compute

\[
d(Ty) = T' y dx + Tdy = (T' f + T f')/2 \omega
\]

for \( T \in k[x] \). Choosing \( T \) with leading term \( \frac{2}{x + 2m}x^m \) for \( m \geq 0 \), one gets \( d(Ty) = (x^{m+2} + \cdots)\omega \); from which it follows that the image of the differential \( d \) consists of elements \( (R + Sy)\omega \) with \( R \in k[x] \) of degree at least two and \( S \in k[x] \) arbitrary. We deduce that

\[
H^1_{\text{dR}}(X) = \langle \omega, x\omega \rangle_k.
\]

### 2.2.4. Algebraic de Rham cohomology of smooth varieties

Let us now turn to the situation where \( X \) is any variety over \( k \), not necessarily affine. Gluing differential forms on affine open subsets, we get a sheaf for the Zariski topology on \( X \).

**Proposition 2.41.** There exists a unique coherent sheaf \( \Omega^1_{X/k} \) on \( X \) whose restriction to every affine open subset \( U \) of \( X \) is the \( \mathcal{O}_X \)-module associated with \( \Omega^1_{\mathcal{O}_X(U)/k} \).
Many of the properties we have discussed in the affine case globalize to arbitrary smooth varieties. In particular, if \( X \) is a smooth variety of dimension \( d \), then the sheaf \( \Omega^1_X \) is locally free sheaf of rank \( d \) and is equipped with the universal \( k \)-derivation \( d: \mathcal{O}_X \to \Omega^1_X \). If \( \Omega^p_X \) denotes the \( p \)-th exterior power of \( \Omega^1_X \), then \( d \) extends to maps \( d^p: \Omega^p_X \to \Omega^{p+1}_X \) satisfying \( d^{p+1} \circ d^p = 0 \).

We denote by \( (\Omega^*_X, d) \) the resulting algebraic de Rham complex

\[
\Omega^*_X: \quad \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots
\]

Note that every term in this complex is a locally free \( \mathcal{O}_X \)-module but the differential \( d \) is not \( \mathcal{O}_X \)-linear, only \( k \)-linear.

**Definition 2.43.** Let \( X \) be a smooth variety over a field \( k \) of characteristic zero. The algebraic de Rham cohomology of \( X \) is the hypercohomology of the de Rham complex:

\[
H^*_\text{dR}(X) = H^*(X, \Omega^*_X).
\]

Recall from Section A.8.2 of the appendix that the hypercohomology of \( \Omega^*_X \) is defined as the cohomology of the complex of global sections of an acyclic resolution of \( \Omega^*_X \), and that there is a spectral sequence

\[
E_1^{p,q} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}_\text{dR}(X)
\]

with differentials \( d_1: E_1^{p,q} \to E_1^{p+1,q} \) induced by \( d: \Omega^p_X \to \Omega^{p+1}_X \). It is called the Frölicher or the Hodge-de Rham spectral sequence.

In practice, a way to compute algebraic de Rham cohomology is to choose a covering of \( X \) by a finite collection of affine open subsets \( U_1, \ldots, U_n \) and to form the double complex

\[
\begin{array}{ccc}
\vdots & \vdots \\
\bigoplus_{i} \Omega^1(U_i) & \longrightarrow & \bigoplus_{i<j} \Omega^1(U_i \cap U_j) \\
\uparrow & & \uparrow \\
\bigoplus_{i} \mathcal{O}(U_i) & \longrightarrow & \bigoplus_{i<j} \mathcal{O}(U_i \cap U_j) \\
\vdots & \vdots \\
\end{array}
\]

where the vertical differentials are the differentials in the algebraic de Rham complex and the horizontal differentials

\[
\begin{array}{c}
\bigoplus_{i_0 < \cdots < i_q} \Omega^p(U_{i_0} \cap \cdots \cap U_{i_q}) \\
\bigoplus_{i_0 < \cdots < i_{q+1}} \Omega^p(U_{i_0} \cap \cdots \cap U_{i_{q+1}})
\end{array}
\]
send a section \( \alpha \in \Omega^p(U_{i_0} \cap \cdots \cap U_{i_q}) \) to the element \( da \) with factors
\[
(d\alpha)_{i_0, \ldots, i_{q+1}} = \sum_{r=0}^{q+1} (-1)^r \alpha_{\hat{i}_0, \ldots, \hat{i}_r, \ldots, \hat{i}_{q+1}} |U_{i_0} \cap \cdots \cap U_{i_{q+1}}.
\]

The algebraic de Rham cohomology of \( X \) is the cohomology of the total complex associated with this double complex.

**Example 2.45.** Let \( X = \mathbb{P}^1_k \) be the projective line over \( k \). We consider the cover by the two affine open subsets
\[
U_0 = \mathbb{P}^1_k \setminus \{0\} = \text{Spec} \, k[t], \quad U_1 = \mathbb{P}^1_k \setminus \{\infty\} = \text{Spec} \, k[s],
\]
whose coordinates are related by \( s = 1/t \) on the intersection \( U_0 \cap U_1 \). Then the only non-zero terms in the above complex are
\[
k[t]dt \oplus k[s]ds \longrightarrow k[t, t^{-1}]dt
\]
\[
d \equiv d \uparrow \quad -d \downarrow
\]
\[
k[t] \oplus k[s] \longrightarrow k[t, t^{-1}].
\]
The horizontal differentials are given by
\[
(f dt, g ds) \mapsto \left( -f(t) - g(1/t)t^{-2} \right) dt, \quad (f, g) \mapsto -f(t) + g(1/t).
\]

Alternatively, we can compute the algebraic de Rham cohomology of \( \mathbb{P}^1_k \) by means of the spectral sequence (2.44). The sheaf of Kähler differentials \( \Omega^1_{\mathbb{P}^1} \) is the line bundle \( \mathcal{O}_{\mathbb{P}^1}(-2) \). By the standard computation of the cohomology of line bundles on \( \mathbb{P}^1 \) [Har77, III, §5], \( \mathcal{O}_{\mathbb{P}^1} \) has only non-vanishing cohomology in degree zero and \( \mathcal{O}_{\mathbb{P}^1}(-2) \) has only non-vanishing cohomology in degree one. Therefore, the spectral sequence reads
\[
\begin{array}{c}
0 \longrightarrow H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \\
H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0
\end{array}
\]
and all differentials vanish already at the first page. This yields isomorphisms
\[
H^n_{\text{dR}}(\mathbb{P}^1) = \begin{cases} 
H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) & n = 0 \\
0 & n = 1 \\
H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) & n = 2
\end{cases}
\]

**Example 2.46.** Let now \( X \) be any smooth connected projective curve over \( k \) and let \( f : X \to \mathbb{P}^1 \) be a non-constant rational function on \( X \). We cover \( X \) by the affine open subsets
\[
U_0 = \setminus f^{-1}(0) \quad U_1 = X \setminus f^{-1}(\infty).
\]
Example 2.47. Let $X$ be a smooth connected projective curve of genus $g$ over $k$. The exact sequence of complexes

$$0 \longrightarrow \Omega_X^1[-1] \longrightarrow \Omega_X^* \longrightarrow \mathcal{O}_X \longrightarrow 0$$

gives rise to the long exact sequence of cohomology groups

$$0 \longrightarrow H^0_{\text{dR}}(X) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \Omega^1_X) \longrightarrow H^1_{\text{dR}}(X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \Omega^1_X) \longrightarrow H^2_{\text{dR}}(X) \longrightarrow 0.$$

Since $X$ is projective and connected, the only global sections of $\mathcal{O}_X$ are the constant functions, that is, $H^0(X, \mathcal{O}_X) = k$. This implies that the first connecting morphism in the above long exact sequence vanishes, as it is given by the differential $d: \mathcal{O}_X(X) \rightarrow \Omega^1_X(X)$. Therefore,

$$H^0_{\text{dR}}(X) = H^0(X, \mathcal{O}_X), \quad H^2_{\text{dR}}(X) = H^1_{\text{dR}}(X, \Omega^1_X)$$

are both one-dimensional, and the first de Rham cohomology group sits in the exact sequence

$$0 \longrightarrow H^0(X, \Omega^1_X) \longrightarrow H^1_{\text{dR}}(X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0.$$

Elements of $H^0(X, \Omega^1_X)$ are called, in the classical literature, differentials of the first kind.

Remark 2.48. When $X$ is affine, there is no need to use hypercohomology. In fact, all coherent sheaves on an affine variety are acyclic. In particular, the complex $\Omega^*_X$ is made of acyclic sheaves, so it is an acyclic resolution of itself. It follows that its hypercohomology agrees with the cohomology of the complex of global sections $(\Omega^*_X(X), d)$. This last complex is called the global de Rham complex. In the affine case, the global de Rham complex is the same as the complex of Kähler differentials

$$\Omega^*_X(X) = \Omega^*_X/k,$$

thus Definitions 2.37 and 2.43 agree for affine varieties.

In general, when $X$ is not affine, the cohomology of the global de Rham complex does not coincide with the algebraic de Rham cohomology. For example, $\Omega^p(X)$ vanishes for $p > n$, and hence so does the cohomology of the global de Rham complex, while a variety will in general have non-trivial cohomology $H^p_{\text{dR}}(X)$ up to degree $2n$. Most varieties in these notes will be affine, so we will often be able to use the global de Rham complex.

Lemma 2.49. Let $X$ be a smooth variety over a field $k$ of characteristic zero, let $K$ be a field extension of $k$, and let $X_K = X \times_{\text{Spec}(k)} \text{Spec}(K)$ denote the extension of scalars. There is a canonical isomorphism

$$H^p_{\text{dR}}(X) \otimes_k K \cong H^p_{\text{dR}}(X_K).$$
The construction of algebraic de Rham cohomology is functorial.

Another important property of Kähler differentials is the compatibility with products. Let $X$ and $Y$ be two varieties over $k$ and let $pr_X$ and $pr_Y$ denote the projections from the product $X \rightarrow Y$ to each of the factors. Then the natural map
\[ pr_X^* \Omega_X^* \otimes pr_Y^* \Omega_Y^* \longrightarrow \Omega_{X \times Y}^* \]
is an isomorphism of sheaves on $X \times Y$. The left-hand side is called the exterior product of $\Omega_X^*$ and $\Omega_Y^*$, and is usually denoted by $\Omega_X^* \boxtimes \Omega_Y^*$, so that there is an isomorphism
\[ \Omega_{X \times Y}^* \simeq \Omega_X^* \boxtimes \Omega_Y^* \]  

(2.50)

2.2.5. Relative de Rham cohomology. There is also a relative version of algebraic de Rham cohomology. For simplicity, we explain the construction only in the affine case. Let $X$ be a smooth affine variety over $k$, and consider a smooth closed subvariety $\iota: Z \hookrightarrow X$, which is hence automatically affine. There is a restriction morphism of complexes $\iota^*: \Omega^*(X) \rightarrow \Omega^*(Z)$. Note that, in contrast to the situation for relative singular homology, the map $\iota^*$ is far from being injective. For example, in the interesting case where $Z$ has smaller dimension than $X$, if $\omega$ is a top degree differential form on $X$, then $\iota^*(\omega) = 0$. Let $\Omega^*(X, Z)$ denote the complex where
\[ \Omega^n(X, Z) = \Omega^n(X) \oplus \Omega^{n-1}(Z) \]
sits in degree $n$ and the differential is given by
\[ d(\alpha, \beta) = (d\alpha, \iota^*(\alpha) - d\beta). \]
There is an exact sequence of complexes
\[ 0 \longrightarrow \Omega^*(X, Z) \longrightarrow \Omega^*(X) \xrightarrow{\iota^*} \Omega^*(Z) \longrightarrow 0 \]  

(2.51)

Remark 2.52. It is instructive to compare this complex to the one used to define relative homology in Section 2.1.2. Mimicking the construction of the cone for cochain complexes, we obtain:
\[ \text{cone}(\iota^*) = \Omega^{n+1}(X) \oplus \Omega^n(Z), \]
with differential
\[ d(\alpha, \beta) = (-d\alpha, -\iota^*(\alpha) + d\beta). \]
Therefore, recalling that the shift $[-1]$ changes the sign of the differential, we see that $\Omega^*(X, Z)$ coincides with the complex $\text{cone}(\iota^*)[-1]$. This last complex is also called the simple of $\iota^*$. The use of the simple or of the cone of a morphism of complexes depends on whether we want that the degree in the obtained complex agrees with the degree in the source complex or in the target complex.
Definition 2.53. For $X$ a smooth affine variety and $Z \hookrightarrow X$ a smooth closed subvariety, the relative de Rham cohomology of the pair $(X, Z)$ is the cohomology of the complex $\Omega^\bullet(X, Z)$:

$$H^n_{dR}(X, Z) = H^n(\Omega^\bullet(X, Z)).$$

Again, a relative de Rham class is represented by a pair of differential forms $(\alpha, \beta)$ such that $\alpha$ is closed and the restriction of $\alpha$ to $Z$ is equal to $d\beta$. However, in general, neither $\alpha$ nor $\beta$ is determined by the other form. Taking the long exact sequence associated with (2.51) one gets

$$\cdots \longrightarrow H^{n-1}_{dR}(Z) \longrightarrow H^n_{dR}(X, Z) \longrightarrow H^n_{dR}(X) \longrightarrow H^n_{dR}(Z) \longrightarrow \cdots$$

Example 2.54. Consider $X = \text{Spec } k[t, t^{-1}]$ and let $Z = \{p, q\}$ be the closed subvariety of $X$ defined by two $k$-points $p$ and $q$. Then $\Omega^\bullet(Z)$ is concentrated in degree zero, $\Omega^0(Z) = k \oplus k$, and the map

$$\iota^* : \Omega^0(X) = k[t, t^{-1}] \to \Omega^0(Z) = k \oplus k$$

is given by evaluating functions at $p$ and $q$, that is, $\iota^*(f) = (f(p), f(q))$. Therefore, the complex $\Omega^\bullet(X, Z)$ reads

\[
d : k[t, t^{-1}] \to k[t, t^{-1}]dt \oplus k \oplus k
\]

\[
f \mapsto (f'(t)dt, f(p), f(q)). \tag{2.55}
\]

The differential $d$ is injective and has image

$$\text{Im}(d) = \langle (0, 1, 1), (nt^{n-1}dt, p^n, q^n) \mid n \in \mathbb{Z} \setminus \{0\}\rangle_k,$$

from which it follows easily that $H^1_{dR}(X, Z)$ is the $k$-vector space generated by the relative differential forms

$$\omega_1 = (0, 1, 0), \quad \omega_2 = (dt/t, 0, 0).$$

Remark 2.56. The de Rham cohomology of affine smooth varieties vanishes above the dimension. If $n = \text{dim } X$, and $Z \subseteq X$ is a closed smooth subvariety of smaller dimension, then a useful part of the long exact sequence of relative cohomology is

$$\cdots \to H^{n-1}_{dR}(Z) \to H^n_{dR}(X, Z) \to H^n_{dR}(X) \to 0. \tag{2.57}$$

2.2.6. The case of normal crossing divisors. In the sequel, we will also need to use relative de Rham cohomology in the case where $Z$ is not smooth, but a simple normal crossing divisor. Using standards tool from homological algebra, the above definition extends to this setting.

Definition 2.58. A divisor $D$ on a smooth algebraic variety $X$ has simple normal crossings if all its irreducible components are smooth and, for each $p \in X$, there exists a local equation of $D$ of the form $x_1 \cdots x_r$ for independent local parameters $x_i \in \mathcal{O}_{X, p}$ and $r \leq \text{dim } X$. 
It follows from the definition that the intersection of \( m \) distinct irreducible components of a simple normal crossing divisor \( D \) is a smooth subvariety of codimension \( m \) in \( X \).

**Construction 2.59.** Let \( X \) be a smooth irreducible affine variety over \( k \) and \( D \) a simple normal crossing divisor, with irreducible components \( D_0, \ldots, D_r \). For simplicity, we assume that all the \( D_i \) are defined over \( k \) as well. For each subset \( I \subseteq \{0, \ldots, r\} \), we set

\[
D_I = \bigcap_{i \in I} D_i.
\]

We define \( D^0 = X \) and, for \( p = 1, \ldots, r+1 \),

\[
D^p = \bigsqcup_{|I| = p} D_I.
\]

Then there is a double complex of \( k \)-vector spaces

\[
K^{p,q} = \Omega^q(D^p),
\]

(2.60)

where the vertical differentials \( d^\text{ver} \) are the differentials \( d \) in the de Rham complex, and the horizontal differentials \( d^\text{hor} \) are alternating sums of restriction maps. More precisely, \( d^\text{hor}: K^{p,q} \to K^{p+1,q} \) is given by

\[
\bigoplus_{|I| = p} \left( \sum_{|J| = p+1} \varepsilon(I,J) d_{J,I} \right),
\]

(2.61)

where \( d_{J,I}: \Omega^q(D_I) \to \Omega^q(D_J) \) denotes the restriction map and the sign \( \varepsilon(I,J) \) is defined as follows: if \( J = \{j_0, \ldots, j_p\} \) with \( j_0 < \ldots < j_p \), and \( I = \{j_0, \ldots, \widehat{j_i}, \ldots, j_p\} \), then \( \varepsilon(I,J) = (-1)^{i} \).

Let \( \Omega^*(X,D) \) denote the total complex associated with \( K^{p,q} \), that is,

\[
\left( \Omega^*(X,D) = \bigoplus_{p+q = *} K^{p,q}, \partial = d^\text{hor} + (-1)^p d^\text{ver} \right).
\]

Note that, thanks to the sign \( (-1)^p \) in front of the vertical differential, the map \( \partial \) satisfies \( \partial^2 = 0 \) and \( \Omega^*(X,D) \) is thus a complex.

**Definition 2.62.** The relative de Rham cohomology \( H^*_{\text{dR}}(X,D) \) is the cohomology of the complex \( \Omega^*(X,D) \).

As for any total complex associated with a double complex, the cohomology can be computed by means of the spectral sequence

\[
E^1_{p,q} = H^q(\Omega^*(D^p)) \implies H^{p+q}_{\text{dR}}(X,D).
\]

(2.63)
Let $n = \dim X$. By definition, a class in the top degree cohomology $H^n(X, D)$ is represented by a tuple

$$(\omega_0, \ldots, \omega_n) \in \bigoplus_{p=0}^{n} \Omega^{n-p}(D^p).$$

What is more, one can always choose $\omega_p = 0$ for $p = 1, \ldots, n$, so that all classes in $H^n(X, D)$ are indeed represented by some $\omega \in \Omega^n(X)$. The key point is that the restriction maps $\Omega^{n-p-1}(D^p) \to \Omega^{n-p-1}(D^{p+1})$ are all surjective [HMS17, Lemma 3.3.20]. We will see in the example below how to use this to find a representative; the general case is analogous.

**Example 2.64.** Let $X = \mathbb{A}^2 = \text{Spec} \, k[x, y]$ and let $D \subset X$ be the union of three lines. After an affine transformation, we may assume without loss of generality that $D$ is the union of the lines

$$D_0 = \{x = 0\}, \quad D_1 = \{y = 0\}, \quad D_2 = \{x + y = 1\}.$$

In this case, the double complex (2.60) is equal to

$$(\Omega^\bullet(\mathbb{A}^2), d) \longrightarrow \bigoplus_{i=0}^{2} (\Omega^\bullet(D_i), d) \longrightarrow \bigoplus_{0 \leq i < j \leq 2} (\Omega^\bullet(D_i \cap D_j), d) \longrightarrow 0.$$

To make all the above terms and maps explicit, we write $D_0 = \text{Spec} \, k[y]$, $D_1 = \text{Spec} \, k[x]$ and we parametrize $D_2 = \text{Spec} \, k[x, y]/(x + y - 1)$ by the

![Figure 3. The triangle D](image-url)
coordinate \( z = x \). Then one gets:

\[
\begin{align*}
  k[x, y]dx \wedge dy & \\
  \quad \downarrow d & \\
  k[x, y]dx \oplus k[x, y]dy & \xrightarrow{c} k[x]dx \oplus k[y]dy \oplus k[z]dz \\
  \quad \downarrow d & \\
  k[x, y] & \xrightarrow{a} k[x] \oplus k[y] \oplus k[z] \\
  \quad \downarrow & \\
  \quad \downarrow & \\
  k & \oplus k \oplus k,
\end{align*}
\]

where \( d \) is the differential, the maps \( a \) and \( b \) are given by

\[
\begin{align*}
  a: & \quad f(x, y) \mapsto (f(0, x), f(y, 0), f(z, 1 - z)), \\
  b: & \quad (f(x), g(y), h(z)) \mapsto (g(0) - f(0), h(0) - f(1), h(1) - g(1)),
\end{align*}
\]

and \( c \) is induced from \( a \) in the obvious way. Therefore, the spectral sequence (2.63) reads

\[
\begin{array}{c}
  0 \\
  0 \\
  k \xrightarrow{a} k \oplus k \oplus k \xrightarrow{b} k \oplus k \oplus k
\end{array}
\]

where the first map sends \( a \) to \((a, a, a)\) and the second one is given by \((a, b, c) \mapsto (b - a, c - a, c - b)\). Since the only non-vanishing cohomology of the bottom complex is in degree two, generated by \((1, 0, 0)\), the second page of the spectral sequence is reduced to \( E_2^{2,0} = k \). It follows that \( H^{i}_{\text{DR}}(A^2, D) \) vanishes for \( i \neq 2 \) and is one-dimensional for \( i = 2 \).

To produce a differential \( \omega \in \Omega^2(A^2) \) representing the cohomology class, we follow the “zig-zag” method, which consists of

- finding \( \omega_1 \in k[x] \oplus k[y] \oplus k[z] \) such that \( b(\omega_1) = (1, 0, 0) \),
- applying \( d^{\text{ver}} \) to get \( \omega_2 = -d\omega_1 \) one row upper,
- choosing \( \omega_3 \in k[x, y]dx \oplus k[x, y]dy \) such that \( c(\omega_3) = -\omega_2 \).

Then, setting \( \omega = -d\omega_3 \), one has

\[
\partial(\omega_1 + \omega_3) = b(\omega_1) - d\omega_1 + c(\omega_3) + d\omega_3 = (1, 0, 0) - \omega,
\]

so that \( \omega \) and \((1, 0, 0)\) are cohomologous.

It is straightforward to check that one can take

\[
\begin{align*}
  \omega_1 &= (y - 1, 0, 0), \\
  \omega_2 &= (-dy, 0, 0), \\
  \omega_3 &= (1 - x)dy - ydx.
\end{align*}
\]
This yields the differential form $\omega = 2dx \wedge dy$ on $\mathbb{A}^2$, which defines a relative cohomology class since it has top degree. In conclusion:

$$H^i_{dR}(X, D) = \begin{cases} (dx \wedge dy)_k & i = 2, \\ 0 & \text{otherwise}. \end{cases}$$

2.2.7. GAGA. Let $X$ be a complex algebraic variety. Serre’s GAGA theorem [Ser56].

**Theorem 2.65 (Serre).** For any projective complex variety $X$ and any algebraic coherent sheaf $\mathcal{F}$ on $X$, the analytification map induces an isomorphism

$$H^i(X, \mathcal{F}) \sim H^i(X^{an}, \mathcal{F}^{an}).$$

**Theorem 2.66 (Poincaré lemma).** Let $M$ be a complex manifold. The inclusion $\mathbb{C} \to \mathcal{O}_M$ of locally constant functions into holomorphic functions induces a quasi-isomorphism $\mathbb{C} \to \Omega^\bullet_M$. Therefore, singular cohomology with complex coefficients is canonically isomorphic to de Rham cohomology:

$$H^\ast(X, \mathbb{C}) \sim H^\ast_{dR}(M).$$

**Proof.** Let $M$ be a complex manifold of dimension $d$. We need to prove that the complex of sheaves

$$0 \to \mathbb{C} \to \mathcal{O}_M \to \Omega^1_M \to \cdots \to \Omega^d_M \to 0$$

is exact, which amounts to showing that the complex of stalks at any point of $M$ is exact. Since $M$ is a complex manifold, every point has an open neighborhood biholomorphic to the polydisc

$$D^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid |z_i| < 1 \text{ for all } i\}$$

and it is enough to show that $0 \to \mathbb{C} \to \mathcal{O}_{D^d} \to \Omega^1_{D^d} \to \cdots$ is exact. Using the isomorphism $\Omega^\ast_{D^d} \simeq (\Omega^\ast_{D})^{\otimes d}$ from (2.50), we are reduced to proving that the complex

$$0 \to \mathbb{C} \to \mathcal{O}(D) \to \mathcal{O}(D)dz \to 0 \quad (2.67)$$

is exact. Holomorphic functions on the disc are given by power series $\sum_{n \geq 0} a_n z^n$ with radius of convergence $\geq 1$ and the differential $d$ maps such
a power series to \((\sum_{n \geq 1} na_n z^{n-1})dz\), which lies in \(\mathcal{O}(D)dz\) since a power series and its derivative have the same radius of convergence. From this it is straightforward to check that the complex \((2.67)\) is exact. \(\square\)

Remark 2.68. The Poincaré lemma fails for the Zariski topology: if \(X\) is an algebraic variety over \(k\), the complex

\[
0 \rightarrow \text{Ker}(d) \rightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_X \rightarrow \cdots \tag{2.69}
\]

of sheaves for the Zariski topology is not exact. For example, in the case where \(X = \mathbb{G}_m = \text{Spec } k[t, t^{-1}]\), a non-empty Zariski open subset of \(X\) is the complement \(U = X \setminus S\) of a finite number of closed points \(S\) and

\[
0 \rightarrow k \rightarrow \mathcal{O}_X(U) \xrightarrow{d} \Omega^1_X(U) \rightarrow 0
\]

always has cohomology in degree two since the class of \(dt/t\) is still non-zero in \(\Omega^1_X(U)/\text{Im}(d)\). In fact, the smaller the Zariski open gets the bigger the cohomology group becomes (see Exercise 2.85).

2.2.8. The sheaf of logarithmic differentials. Recall the notion of simple normal crossing divisor from Definition 2.58. In this paragraph, we explain how to compute algebraic de Rham cohomology of a smooth variety as the hypercohomology of a complex of sheaves on a compactification by a simple normal crossing divisor. The existence of such a compactification is ensured by Hironaka’s theorem on the resolution of singularities.

A resolution of singularities of a variety \(Y\) is a proper birational morphism \(\pi: \tilde{Y} \rightarrow Y\) from a smooth variety \(\tilde{Y}\). Recall that “birational” means that there exists a dense open subset \(U \subset \tilde{Y}\) such that \(\pi^{-1}(U) \rightarrow U\) is an isomorphism; the closed complement \(E \subset \tilde{Y}\) of the largest open subset with this property is called the exceptional locus of \(\pi\).

Theorem 2.70 (Hironaka). Let \(k\) be a field of characteristic zero. Let \(Y\) be a variety over a \(k\) and let \(Z \subset Y\) be a closed subvariety. There exists a resolution of singularities \(\pi: \tilde{Y} \rightarrow Y\) such that

(i) the union of \(\pi^{-1}(Z)\) and the exceptional locus of \(\pi\) is a simple normal crossing divisor,

(ii) \(\pi\) is an isomorphism outside \(Y_{\text{sing}} \cup Z\).

Moreover, such a resolution can be obtained as an iterated blow-up along smooth subvarieties.

We will mainly use the following consequence of Hironaka’s theorem. Start with a smooth variety \(X\) over \(k\) and choose a proper variety \(Y\) over \(k\) containing \(X\) as an open subvariety (for example, if \(X\) is quasi-projective, one can pick as \(Y\) the closure of \(X\) into some projective space on which \(X\) embeds as a locally closed subset; the general case requires Nagata’s compactification theorem). Applied to \(Y\) and \(Z = Y \setminus X\), Hironaka’s theorem
yields a resolution of singularities $\pi: \tilde{Y} \to Y$ that is an isomorphism outside $Z$ and such that $\pi^{-1}(Z)$ is a normal crossing divisor.

**Corollary 2.71.** Let $X$ be a smooth variety over $k$. There exists a smooth proper variety $\tilde{X}$ and an open immersion $j: X \hookrightarrow \tilde{X}$ such that $D = \tilde{X} \setminus X$ is a simple normal crossing divisor.

With these preliminaries out of the way, we now turn to the definition of the complex of logarithmic differentials.

**Definition 2.72 (Deligne).** The complex of sheaves of logarithmic differentials along $D$ is the smallest subcomplex $\Omega^*_X(\log D)$ of $j_* \Omega^*_X$ that is stable under wedge product and contains $\Omega^*_X$ and the logarithmic derivatives $df/f$ of all local sections $f$ of $j_* O_X$ with poles along $D$.

It follows from the definition that $\Omega^1_X(\log D)$ is a locally free $O_X$-module of rank $d = \dim X$. Indeed, if $(z_1, \ldots, z_d)$ are local coordinates such that $D$ is given by $z_1 \cdots z_r = 0$, then $\Omega^1_X(\log D)$ is locally generated by $dz_1/z_1, \ldots, dz_r/z_r, dz_{r+1}, \ldots, dz_d$. Moreover, one has $\Omega^p_X(\log D) = \Lambda^p \Omega^1_X(\log D)$ for all $p \geq 0$.

**Proposition 2.73.** The inclusion of complexes $\Omega^*_X(\log D) \hookrightarrow j_* \Omega^*_X$ is a quasi-isomorphism, and hence

$$H^n(X, \Omega^*_X(\log D)) \simeq H^n(X, j_* \Omega^*_X) \simeq \mathbb{H}^n(X, \Omega^*_X) \simeq H^n_{dR}(X).$$

**Proof.**

**Corollary 2.75.** Let $X$ be a smooth algebraic variety of dimension $d$ over $k$. Algebraic de Rham cohomology $H^n_{dR}(X)$ is a finite-dimensional $k$-vector space. Moreover, $H^n_{dR}(X) = 0$ for all $n > 2d$.

**Proof.** By Proposition 2.73, it suffices to prove that the $k$-vector space $\mathbb{H}^n(\tilde{X}, \Omega^*_X(\log D))$ is finite-dimensional. This cohomology group is the abutment of the spectral sequence

$$E_1^{p, q} = H^q(\tilde{X}, \Omega^p_X(\log D)).$$

Since the logarithmic differentials $\Omega^p_X(\log D)$ are coherent sheaves on the projective variety $\tilde{X}$, all terms $E_1^{p, q}$ of the spectral sequence have finite dimension and vanish unless $0 \leq p \leq d$ and $0 \leq q \leq d$. 

For the proof of the comparison isomorphism, we will also need an analytic variant of Proposition 2.73.

**Proposition 2.76.** Let $M$ be a complex manifold and let $Z \subset M$ be a divisor with normal crossings. The inclusion of complexes $\Omega^*_M(\log Z) \hookrightarrow j_* \Omega^*_M \setminus Z$ is a quasi-isomorphism.
Proof. As in the proof of Proposition 2.73, we will show that
\[
\text{Coker} \left( \Omega^*_M(\log Z) \to j_*\Omega^*_M(Z) \right)
\]
is an exact complex of sheaves on \( M \).
\( \square \)

***

Exercise 2.77. Prove that the axioms (a)-(c) of the definition of the exterior derivative imply that, in local coordinates,
\[
d(fdx_1 \wedge \cdots \wedge dx_p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge dx_p.
\]

Exercise 2.78. Let \( k \) be a field of characteristic zero. Show that \( H^0_{dR}(\mathbb{A}^2_k) = k \) and that all the other cohomology groups vanish.

Exercise 2.79. In Example 2.39 we saw that a basis of the de Rham cohomology of an affine elliptic curve \( X \subseteq \mathbb{A}^2_k \) is given by the classes of the differentials \( \omega \) and \( x\omega \). Let \( \overline{X} \subseteq \mathbb{P}^2_k \) be the projective completion of \( X \), that is, the smooth projective curve obtained by adjoining to \( X \) the point at infinity \( O = [0 : 1 : 0] \). Prove that \( \omega \) extends to a holomorphic differential on \( \overline{X} \), whereas \( x\omega \) has a double pole at \( O \).

Exercise 2.80. We have defined de Rham cohomology for varieties over a field of characteristic zero. Show by means of an example that the same definition gives pathological results in positive characteristic (for instance, the cohomology of \( \mathbb{A}^1 \) has infinite dimension).

Exercise 2.81. Show that the differential \( \omega_1 \) from Example 2.54 is cohomologous to \( \frac{1}{q-p} dt, 0, 0 \). Deduce that \( (dt/t, 0, 0) \) and \( (dt, 0, 0) \) form another basis of the relative cohomology group \( H^1_{dR}(\mathbb{P}^1 \setminus \{0, \infty\}, \{p, q\}) \) and compare it to the previous one.

Exercise 2.82. Let \( A \) be a \( k \)-algebra and let \( \mu: A \otimes_k A \to A \) denote the multiplication map which sends an element \( \sum_i a_i \otimes b_i \) to \( \sum a_i b_i \). Set
\[
I = \text{Ker}(\mu: A \otimes_k A \to A).
\]
The goal of the exercise is to prove that \( \Omega^1_{A/k} \simeq I/I^2 \) as \( A \)-modules.

(a) Show that the map \( a \mapsto 1 \otimes a - a \otimes 1 \) induces a \( k \)-linear derivation \( A \to I/I^2 \), and hence a morphism of \( A \)-modules \( \varphi: \Omega^1_{A/k} \to I/I^2 \) by the universal property (2.34).
(b) Consider the ring $R = A \oplus \Omega^1_{A/k}$, where $A$ acts on $\Omega^1_{A/k}$ through the $A$-module structure and the product of any two elements of $\Omega^1_{A/k}$ is zero. Show that the $k$-bilinear map

$$A \times A \to R \quad (a_1, a_2) \mapsto (a_1 a_2, a_1 da_2).$$

factors through $A \otimes_k A$ and sends $I$ to $\Omega^1_{A/k}$ and $I^2$ to zero. Therefore, it defines a map $\psi: I/I^2 \to \Omega^1_{A/k}$.

c) Prove that $\varphi$ and $\psi$ are inverse of each other.

**Exercise 2.83** (Kähler differentials are not locally free for singular varieties). Set $A = k[x, y]/(xy)$ and $X = \text{Spec}(A)$. By Example 2.35, the module of Kähler differentials $\Omega^1_{X/k}$ has generators $dx$ and $dy$, and one relation $xdy = -ydx$. Let $\omega = xdy$.

a) Show that $\omega \neq 0$ but $x\omega = y\omega = 0$.

b) Let $z \in A$. Show that $xz = yz = 0$ implies $z = 0$. Conclude that $\Omega^1_{X/k}$ is not locally free.

c) Show that $k \cdot \omega$ sits in an exact sequence

$$0 \to k \cdot \omega \to \Omega^1_{X/k} \to k[x]dx \oplus k[y]dy \to 0.$$

and that this exact sequence does not split.

d) Prove that $\Omega^2_X = k \cdot dx \wedge dy$.

**Exercise 2.84.** In this exercise, we show how to compute the algebraic de Rham cohomology of $\mathbb{P}^1$ using global differential forms on an affine variety; this is an instance of Jouanolou’s trick.

a) Let $\Delta \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal and set $X = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$. Prove that $X$ is the affine variety

$$\text{Spec } (k[x, y, z]/(x(x - 1) - yz))$$

and that the projection onto the first factor $\pi: X \to \mathbb{P}^1$ is given in these coordinates by $\pi(x, y, z) = [x : y] = [x - 1 : z]$. Observe that all the fibres of $\pi$ are affine lines.

[Hint: first identify $\mathbb{P}^1 \times \mathbb{P}^1$ with a quadric in $\mathbb{P}^3$ through the Segre embedding].

b) Prove that the complexes $\Omega^*_{\mathbb{P}^1}$ and $R\pi_*\Omega^*_X$ of locally free sheaves on $\mathbb{P}^1$ are quasi-isomorphic. Deduce that the algebraic de Rham cohomologies $H^i_{\text{dR}}(\mathbb{P}^1)$ and $H^i_{\text{dR}}(X)$ are isomorphic.

[Hint: since the morphism $\pi$ is affine, $R\pi_*\Omega^*_X = \pi_*\Omega^*_X$. Then use the Leray spectral sequence].

c) Write down a generator of $H^2_{\text{dR}}(X)$.
Exercise 2.85. A way to rephrase the fact that the Poincaré lemma fails for the Zariski topology, as explained in Remark 2.68, is by saying that, for a smooth connected variety $X$ over $k$, the Zariski sheaves

$$\mathcal{H}_X^q = \frac{\text{Ker}(d: \Omega^q_X \to \Omega^{q+1}_X)}{\text{Im}(d: \Omega^{q-1}_X \to \Omega^q_X)}$$

are not zero in general. Observe that $\mathcal{H}_X^q$ is the sheaf associated with the presheaf $U \rightsquigarrow H^q_{dR}(U)$. As for any hypercohomology of sheaves, there is a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}_X^q) \Rightarrow H^{p+q}_{dR}(X).$$

(a) Prove that the sheaf $\mathcal{H}_X^0$ is flasque, and hence acyclic.

(b) Deduce that the presheaf $U \rightsquigarrow H^1_{dR}(U)$ is already a sheaf on $X$.

2.3. Periods. In this paragraph, we introduce a class of complex numbers called periods. They form a countable subring of $\mathbb{C}$ halfway between algebraic and transcendental numbers: although they tend to be transcendental, they share with algebraic numbers the property that they contain, in some sense, “a finite amount of information”. Moreover, this information has geometric nature. From the modern point of view, periods appear when comparing de Rham and Betti cohomology of algebraic varieties over number fields. We refer to [HMS17] for a detailed exposition of the subject.

2.3.1. Naive periods. The following elementary definition was first written down by Kontsevich and Zagier [KZ01, Section 1.1]:

Definition 2.86. A period is a complex number whose real and imaginary parts can be written as absolutely convergent integrals

$$\int_\sigma f(x_1, \ldots, x_n)dx_1 \cdots dx_n,$$

where the integrand $f$ is a rational function with rational coefficients, i.e. a quotient of two polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$, and the integration domain $\sigma$ is a subset of $\mathbb{R}^n$ defined by a finite union and intersection of subsets of the form $\{g(x_1, \ldots, x_n) \geq 0\}$ with $g$ a rational function with rational coefficients.

Periods form a countable subring of the complex numbers.

One may replace “rational function” with “algebraic function” and “rational coefficients” with “algebraic coefficients” in the above definition, and still obtain the same class of numbers. Standard examples of naive periods include the following:

- All algebraic numbers (see Exercise 2.102).
- The number $\pi = \int_{x^2+y^2 \leq 1} dxdy$.
- Logarithms of rational numbers $\log(q) = \int_1^q \frac{dx}{x}$, where $q \in \mathbb{Q}_{\geq 1}$.
• Elliptic integrals
\[ \int_{1}^{\infty} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \] where \( \lambda \in \mathbb{Q} \setminus \{0,1\} \)
• Multiple zeta values, Feynman integrals, periods of modular forms, some special values of \( L \)-functions, etc.

2.3.2. The comparison isomorphism. Let \( k \) be a subfield of the complex numbers and let \( X \) be a smooth algebraic variety over \( k \). As we have seen, the singular cohomology of \( X(\mathbb{C}) \) is a graded \( \mathbb{Q} \)-vector space and the de Rham cohomology of \( X \) is a graded \( k \)-vector space. Both are related by Grothendieck’s comparison isomorphism.

**Theorem 2.88** (Grothendieck, [Gro66]). Let \( X \) be a smooth variety over a subfield \( k \) of \( \mathbb{C} \). Then there is a canonical isomorphism

\[ \text{comp}_{B, \text{dR}}: H^i_{\text{dR}}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H^i_B(X) \otimes_{\mathbb{Q}} \mathbb{C}. \] (2.89)

When \( X \) is an affine variety, all classes in de Rham cohomology are represented by differential forms. Then the comparison isomorphism is induced by the pairing

\[ H^i_{\text{dR}}(X) \otimes H_i(X(\mathbb{C}), \mathbb{Q}) \xrightarrow{\omega \otimes \sigma \mapsto \int_{\sigma} \omega} \mathbb{C}. \] (2.90)

The fact that (2.90) depends only on the classes of \( \omega \) and \( \sigma \), and is thus well defined, follows from Stokes’ theorem.

**Remark 2.91.** Later on, we will also need the inverse of the comparison isomorphism \( \text{comp}_{B, \text{dR}} \), which will be written as

\[ \text{comp}_{\text{dR}, B}: H^i_B(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^i_{\text{dR}}(X) \otimes_k \mathbb{C}. \]

**Proof of Theorem 2.88.** Let \( X_{\mathbb{C}} \) denote the complex algebraic variety obtained from \( X \) by extension of scalars. Combining the isomorphism

\[ H^*_\text{dR}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H^*_B(X_{\mathbb{C}}) \]

from Lemma 2.49 with the Poincaré lemma

\[ H^*_B(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^*_\text{dR}(X_{\mathbb{C}}^\text{an}), \]

we are reduced to prove that the analytification map

\[ H^*_{\text{dR}}(X_{\mathbb{C}}) \xrightarrow{} H^*_\text{dR}(X_{\mathbb{C}}^\text{an}) \]

from algebraic de Rham cohomology to analytic de Rham cohomology is an isomorphism. Let \( \overline{X} \) be a smooth compactification of \( X \) by a simple normal crossing divisor \( D \). We consider the commutative diagram

\[ \begin{array}{ccc}
H^*_{\text{dR}}(X_{\mathbb{C}}) & \xrightarrow{} & H^*_{\text{dR}}(X_{\mathbb{C}}^\text{an}) \\
\downarrow & & \downarrow \\
\mathbb{H}^*(\overline{X}, \Omega^*_{\overline{X}/(\log D)}) & \xrightarrow{} & \mathbb{H}^*(\overline{X}^\text{an}, \Omega^*_{\overline{X}^\text{an}/(\log D^\text{an})}),
\end{array} \]
where the bottom horizontal map is also given by analytification and the vertical maps are the isomorphisms from Propositions 2.73 and 2.76. It suffices thus to prove that the bottom map is an isomorphism. □

**Idea of the proof.** The strategy to prove the comparison isomorphism is to relate Betti cohomology to an analytic version of de Rham cohomology. Indeed, by “analytification”, the algebraic de Rham complex (2.90) becomes the analytic de Rham complex

\[
\Omega_X^\text{an} : \quad O_X^{\text{an}} \overset{d}{\longrightarrow} \Omega^1_X^{\text{an}} \overset{d}{\longrightarrow} \Omega^2_X^{\text{an}} \overset{d}{\longrightarrow} \cdots
\]

of the analytic complex manifold \(X^{\text{an}}\) associated with the base change \(X \times_k \mathbb{C}\). The hypercohomology of \(\Omega_X^{\text{an}}\) defines the analytic de Rham cohomology groups \(H^i_{\text{dR}}(X^{\text{an}})\) and, again by analytification, we get a canonical morphism of complex vector spaces:

\[
H^i_{\text{dR}}(X) \otimes_k \mathbb{C} \longrightarrow H^i_{\text{dR}}(X^{\text{an}}).
\]

(2.92)

Besides, according to the Poincaré lemma, the complex \(\Omega_X^{\text{an}}\) is a resolution of the constant sheaf \(\mathbb{C}^{\text{an}}\). Since singular cohomology is isomorphic to sheaf cohomology with values in the constant sheaf, we obtain a canonical isomorphism

\[
H^i_B(X) \otimes \mathbb{Q} \mathbb{C} \xrightarrow{\sim} H^i_{\text{dR}}(X^{\text{an}}).
\]

The proof is thus reduced to show that (2.92) is an isomorphism. If we assume \(X\) to be proper, this is a straightforward consequence of Serre’s GAGA theorem, together with the existence of spectral sequences relating algebraic (resp. analytic) de Rham cohomology to the sheaf cohomology \(H^q(X, \mathcal{O}_X^p)\) (resp. \(H^q(X^{\text{an}}, \mathcal{O}_X^{\text{an}})\)). The proof of the general case is more difficult. □

**Remark 2.93.** The theorem does not hold if the smoothness assumption is removed. For instance, if \(X\) is the affine plane curve defined by the equation \(x^5 + y^5 + x^2y^2 = 0\), one can show that \(\dim H^1_{\text{dR}}(X) > \dim H^1_B(X)\) [AK11, Example 4.4]. However, the theorem remains true for singular \(X\) with the “correct” definition of de Rham cohomology [HMS17].

There is also a relative version of the comparison isomorphism:

**Theorem 2.94.** Let \(k\) be a subfield of the complex numbers, \(X\) a smooth variety, and \(Z \subseteq X\) either a smooth closed subvariety or a normal crossing divisor, with everything defined over \(k\). Then there is a canonical isomorphism

\[
H^i_{\text{dR}}(X, Z) \otimes_k \mathbb{C} \xrightarrow{\sim} H^i_B(X, Z) \otimes \mathbb{Q} \mathbb{C}.
\]

(2.95)

**Remark 2.96.** Recall that if \(X\) is affine and \(i : Z \hookrightarrow X\) is a smooth closed subvariety, relative cohomology classes are represented by pairs \((\omega_X, \omega_Z)\) and \((\sigma_X, \sigma_Z)\) satisfying

\[
\partial \sigma_X = -i_* \sigma_Z, \quad i^* \omega_X = d \omega_Z, \quad d \omega_X = 0.
\]
Then the period pairing is given by:
\[
H^i_{\text{dR}}(X, Z) \otimes H^B_i(X, Z) \longrightarrow \mathbb{C} \\
(\omega_X, \omega_Z) \otimes (\sigma_X, \sigma_Z) \longmapsto \int_{\sigma_X} \omega_X + \int_{\sigma_Z} \omega_Z.
\]

2.3.3. **Cohomological periods.** The comparison isomorphism does not respect the rational structures, as it is already clear from the following basic example. In particular, in the case where \( k = \mathbb{Q} \), the vector spaces \( H^i_{\text{dR}}(X) \) and \( H^B_i(X) \) are isomorphic (they have the same dimension), but there is no canonical isomorphism between them!

**Example 2.97.** Let \( X = \mathbb{G}_m = \text{Spec} \mathbb{Q}[t, t^{-1}] \), so the complex points are \( X(\mathbb{C}) = \mathbb{C} \setminus \{0\} \). We know from examples 2.4 and 2.38 that
\[
H^1_{\text{dR}}(X) = \mathbb{Q} \frac{dt}{t}, \quad H_1(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}\sigma,
\]
where \( \sigma \) is the counterclockwise oriented unit circle. Then the comparison isomorphism is given by multiplication by:
\[
\int_\sigma \frac{dt}{t} = 2\pi i.
\]

The fact that the comparison isomorphism does not respect the rational structures gives rise to the *periods*.

**Definition 2.98.** Let \( k \subset \mathbb{C} \) be a number field. Let \( X \) be a smooth variety and \( Z \subset X \) a normal crossing divisor, both defined over \( k \). We call a *period* of the pair \((X, Z)\) any coefficient of a matrix of the isomorphism (2.95) with respect to rational bases of both sides.

It is shown in [HMS17, 11.2] that the naive and the cohomological definitions of periods yield the same subring of \( \mathbb{C} \). However, starting from an integral representation as in (2.87) it is in general not easy to find the pair \((X, Z)\), as we will see when discussing the case of \( \zeta(2) \).

2.3.4. **Examples.**

**Example 2.99.** All algebraic numbers are periods. Indeed, let \( k \) be a number field and consider the zero-dimensional variety \( X = \text{Spec}(k) \), which we regard as defined over \( \mathbb{Q} \). Then \( H^0_{\text{dR}}(X) \) is canonically identified with the \( \mathbb{Q} \)-vector space \( k \). By its very definition, \( X(\mathbb{C}) \) is the set of complex embeddings of \( k \), and hence \( H^0_B(X) = \mathbb{Q}^{\text{Hom}(k, \mathbb{C})} \). If we choose a basis \( a_1, \ldots, a_n \) of \( k \) over \( \mathbb{Q} \) and \( \sigma_1, \ldots, \sigma_n \) denote the complex embeddings of \( k \), the period matrix is \( (\sigma_i(a_j))_{i,j} \).

**Example 2.100.** Let \( X = \mathbb{G}_m, \mathbb{Q} \) and \( Z = \{1, q\} \) for \( q \in \mathbb{Q} \setminus \{0, 1\} \). In Example 2.16 we obtained generators \( \sigma_1 \) and \( \sigma_2 \) of \( H^1_B(X, Z) \) and in Example 2.54 generators \( \omega_1 \) and \( \omega_2 \) of \( H^1_{\text{dR}}(X, Z) \). With respect to these bases the period matrix is
\[
\begin{pmatrix}
\int_{\sigma_2} \omega_1 & \int_{\sigma_2} \omega_2 \\
\int_{\sigma_1} \omega_1 & \int_{\sigma_1} \omega_2
\end{pmatrix} = \begin{pmatrix}
1 & \log(q) \\
0 & 2\pi i
\end{pmatrix},
\]
which shows that logarithms of rational numbers are periods.

2.3.5. **Compatibility with complex conjugation.** We finish this section by stating a result that will be used in Chapter 4. Assume $k \subseteq \mathbb{R}$. Then complex conjugation $c: \mathbb{C} \to \mathbb{C}$ induces a continuous map $X(\mathbb{C}) \to X(\mathbb{C})$, and hence an involution on Betti cohomology $\rho: H^i_{B}(X) \to H^i_{B}(X)$. The functoriality of the comparison isomorphism (2.89) implies:

**Proposition 2.101.** Assume that $k \subseteq \mathbb{R}$. Then the comparison isomorphism (2.89) is equivariant for the action of $\mathbb{Z}/2$ by $id \otimes c$ on the left-hand side (de Rham) and by $\rho \otimes c$ on the right-hand side (Betti).

We illustrate the proposition in the case of $\mathbb{G}_m$ (see Exercise 2.104 below for another instance). We know from Example 2.97 that the comparison isomorphism $\text{comp}_{\partial, \text{dr}}$ sends $dt/t$ to $\sigma^\vee \otimes (2\pi i)$. The differential form being rational, it is invariant under complex conjugation, so $\sigma^\vee \otimes (2\pi i)$ should also be invariant. For this, observe that the image of $\sigma$ by complex conjugation is the clockwise oriented unit circle, whose cohomology class is $-\sigma$. Thus,

$$(\rho \otimes c)(\sigma \otimes (2\pi i)) = -\sigma \otimes (-2\pi i) = \sigma \otimes (2\pi i).$$

***

**Exercise 2.102.** In this exercise, we show that all algebraic numbers are naive periods in the sense of Definition 2.86. For example, the integral representation

$$\sqrt{2} = \int_{x^2 \leq 2}^{x \geq 0} dx$$

shows that $\sqrt{2}$ is a naive period.

(a) Let $P \in \mathbb{Q}[x]$ be an irreducible polynomial and let $\alpha_1, \ldots, \alpha_r$ be its real roots. Generalize the above example to show that all $\alpha_i$ are naive periods.

(b) Using that the real and the imaginary part of a complex algebraic number are real algebraic numbers, deduce that all algebraic numbers are naive periods.

**Exercise 2.103.** Let $X$ be a smooth affine variety of dimension $n$ over a subfield $k$ of the complex numbers. Show that $H^i_B(X) = 0$ for all $i > n$.

**Exercise 2.104.** Let $C \subset \mathbb{A}_Q^2$ be the affine conic given by $x^2 + y^2 = 1$.

(a) Show that the de Rham cohomology group $H^1_\text{dr}(C)$ is generated by the class of the differential form $xdy - ydx$ and that the singular homology $H_1(C(\mathbb{C}), \mathbb{Q})$ is generated by the chain

$$\sigma: [0, 1] \to C(\mathbb{R}), \quad t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$
(b) Prove that the associated period is equal to
\[
\int_{\sigma} xdy - ydx = 2\pi
\]
and check Proposition 2.101 in this case.

(c) Find generators of the singular homology of the conics \( C \) defined by the equations \( x^2 + y^2 = -1 \) and \( x^2 - y^2 = 1 \) and check Proposition 2.101 in these cases as well.

### 2.4. Multiple zeta values as periods.

The previous examples show that algebraic numbers, logarithms of rational numbers, and the ubiquitous \( 2\pi i \) are all periods. From the integral representation (1.109), it follows immediately that multiple zeta values are periods in the sense of Kontsevich and Zagier (Definition 2.86). However, it is not so easy to exhibit the corresponding algebraic varieties. The main goal of this section is to work out the example of \( \zeta(2) \) in detail in order to give an idea of the difficulties involved.

#### 2.4.1. The example of \( \zeta(2) \).

Recall from Example 1.103 that \( \zeta(2) \) admits the integral representation
\[
\zeta(2) = \int_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1}{t_1} \wedge \frac{dt_2}{1 - t_2}.
\]

The integrand is the differential form
\[
\omega = \frac{dt_1}{t_1} \wedge \frac{dt_2}{1 - t_2}
\]
on the affine plane, which is singular along the union of the lines
\[
\ell_0 = \{t_1 = 0\} \quad \text{and} \quad \ell_1 = \{t_2 = 1\}.
\]
Thus, \( \omega \) is a global differential 2-form on \( Y = \mathbb{A}^2 \setminus (\ell_0 \cup \ell_1) \).

The domain of integration is the simplex
\[
\sigma = \{(t_1, t_2) \mid 1 \geq t_1 \geq t_2 \geq 0\} \subset \mathbb{A}^2(\mathbb{C}).
\]
However, if we want to consider the integral (2.105) as a period of \( Y \), relative to some divisor containing the boundary of \( \sigma \), we immediately face the problem that \( \sigma \) is not contained in \( Y \), as the points \( p = (0, 0) \) and \( q = (1, 1) \) belong to \( \sigma \cap (\ell_0 \cup \ell_1) \) (see Figure 5).

A way to remedy this is to perform a geometric construction called *blow-up*, which replaces a point on a variety with a divisor called the *exceptional divisor*. It is a very useful technique in the study of singularities. In our case, we have to blow up the two problematic points \( p \) and \( q \). More precisely, the blow-up of \( \mathbb{A}^2 \) along \( p \) and \( q \) is the closed subvariety \( X \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \) defined by the equations
\[
\begin{align*}
t_1\alpha_1 &= t_2\beta_1, \\
(t_1 - 1)\alpha_2 &= (t_2 - 1)\beta_2,
\end{align*}
\]
where $[\alpha_i : \beta_i]$ are homogeneous coordinates on the two copies of $\mathbb{P}^1$. The projection onto the first factor induces a proper surjective map

$$\pi : X \to \mathbb{A}^2.$$  

It is easy to verify that $\pi^{-1}(p)$ is the projective line

$$E_p = (0, 0) \times \mathbb{P}^1 \times [1 : 1] \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

while $\pi^{-1}(q)$ is the projective line

$$E_q = (1, 1) \times [1 : 1] \times \mathbb{P}^1 \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Moreover, the restriction

$$\pi \mid_{X \setminus (E_p \cup E_q)} : X \setminus (E_p \cup E_q) \to \mathbb{A}^2 \setminus \{p, q\}$$

is an isomorphism. For any closed subset $C \subset \mathbb{A}^2$, the strict transform $\widehat{C}$ of $C$ is the closed subset of $X$ given by

$$\widehat{C} = \pi^{-1}(C \setminus \{p, q\}).$$

In words: we first remove the points $p$ and $q$ if they are in $C$, then we pull-back to $X$ by $\pi$, and finally we take the Zariski closure. The strict transform is contained in the total transform $\pi^{-1}(C)$ but it may be smaller. For instance, the strict transform of $\ell_0$ is the affine line

$$L_0 = \widehat{\ell}_0 = \{((0, t_2), [1 : 0], [1 - t_2 : 1]) \mid t_2 \in \mathbb{A}^1\},$$

while the total transform is $L_0 \cup E_p$. Note that $L_0$ and $E_p$ have only one common point:

$$L_0 \cap E_p = \{((0, 0), [1 : 0], [1 : 1])\}. \quad \text{(2.106)}$$

Similarly, the strict transform of $\ell_1$ is the affine line

$$L_1 = \widehat{\ell}_1 = \{((t_1, 1), [1 : t_1], [0 : 1]) \mid t_1 \in \mathbb{A}^1\},$$

which is disjoint from the exceptional divisor $E_p$, intersects $L_0$ at the point $((0, 1), [1 : 0], [0 : 1])$, and $E_q$ at $((1, 1), [1 : 1], [0 : 1])$. 

![Figure 5. The simplex $\sigma$ and the singular locus $\ell_0 \cup \ell_1$](image-url)
In principle, the pull-back \( \pi^*(\omega) \) of \( \omega \) might have singularities along the total transform of \( \ell_0 \cup \ell_1 \), which would only worsen the initial situation. Fortunately, it is only singular on the strict transform \( L_0 \cup L_1 \). This can be seen using local coordinates in \( X \). For instance, a local chart of \( X \) around the intersection of \( L_0 \) and \( E_p \) is given by the coordinates

\[
t = \frac{\beta_1}{\alpha_1} = \frac{t_1}{t_2}, \quad s = t_2,
\]

in which \( E_p \) and \( L_0 \) have local equations \( s = 0 \) and \( t_1 = 0 \), respectively. Then

\[
\pi^*(\omega) = \frac{d(st)}{st} \wedge \frac{ds}{1-s} = \frac{ds}{s} \wedge \frac{ds}{1-s} + \frac{dt}{t} \wedge \frac{ds}{1-s} = \frac{dt}{t} \wedge \frac{ds}{1-s},
\]

where we have used the Leibniz rule and the fact that \( ds \wedge ds = 0 \). It follows that \( \pi^*(\omega) \) is smooth along \( E_p \). An analogous computation shows that \( \pi^*(\omega) \) has singularities along \( L_1 \) but not along \( E_q \).

The closed points of the exceptional divisor \( E_p \) can be interpreted as lines passing through the point \( p \). This allows us to find the points of \( E_p \) that are contained in \( \hat{\sigma} \):

\[
\hat{\sigma} \cap E_p = \{ ((0, 0), [m : 1], [1 : 1]) \mid 0 \leq m \leq 1 \}.
\]

Combined with (2.106), this implies that \( \hat{\sigma} \cap L_0 = \emptyset \). A similar argument shows that \( \hat{\sigma} \cap L_1 = \emptyset \), so, after passing to the blow-up \( X \), the singular locus of \( \pi^*(\omega) \) and the domain of integration \( \hat{\sigma} \) are disjoint (Figure 6).

\[\text{Figure 6. The strict transform of } \sigma \text{ and the singular locus } L_0 \cup L_1 \text{ of the form } \pi^*(\omega)\]

Write \( L = L_0 \cup L_1 \). The complement \( X \setminus L \) is still an affine variety; in fact, it is the closed subvariety of \( \mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{A}^1 \) defined by

\[
t_1 t = t_2, \\
(t_1 - 1) = (t_2 - 1)s,
\]
where \( t, s \) are the coordinates of the first and the second affine lines. By the previous discussion, \( \pi^*(\omega) \) is an element of \( \Omega^2(X \setminus L) \).

The next issue one needs to deal with is that \( \sigma \) is not a closed chain. Its boundary is contained in the union of the affine lines

\[
m_2 = \{t_1 = t_2\}, \ m_3 = \{t_2 = 0\}, \ m_4 = \{t_1 = 1\},
\]

so we are naturally led to consider the normal crossing divisor

\[
M = \pi^{-1}(m_2 \cup m_3 \cup m_4) = E_p \cup E_q \cup M_2 \cup M_3 \cup M_3 \subset X,
\]

where \( M_i \) denotes the strict transform of \( m_i \). One easily checks that the intersection \( L \cap M \) is reduced to the points \( L_0 \cap E_p \) and \( L_1 \cap E_q \) which we have already computed.

Since \( \hat{\sigma} \) is contained in \( X \setminus L \) and its boundary lies in \( M \), using Remark 2.15 we see that \( \hat{\sigma} \) determines a relative homology class

\[
\hat{\sigma} \in H_2(X \setminus L, M \setminus (L \cap M)).
\]

Besides, the restriction of \( \pi^*(\omega) \) to every irreducible component of \( M \) is zero for dimension reasons, so it defines a relative cohomology class

\[
\pi^*(\omega) \in H^2_{dR}(X \setminus L, M \setminus (L \cap M)).
\]

Pairing these classes through the comparison isomorphism (2.95) yields, as we wanted, the period

\[
\int_{\hat{\sigma}} \pi^*(\omega) = \int_{\pi^*(\hat{\sigma})} \omega = \int_{\sigma} \omega = \zeta(2).
\]

### 2.4.2. Multiple zeta values as periods of the moduli spaces \( \overline{M}_{0,n} \).

For each integer \( n \geq 3 \), let \( M_{0,n} \) be the moduli space of \( n \) ordered distinct points in \( \mathbb{P}^1 \) up to projective equivalence. In other words, two tuples \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) are identified if there exists an element \( g \in \text{PGL}_2 \) such that \( g(x_i) = y_i \) for all \( i \). Since there exists a unique automorphism of \( \mathbb{P}^1 \) sending any given three points to 0, 1 and \( \infty \), we can fix an identification

\[
(x_1, \ldots, x_n) = (0, 1, \infty, t_1, \ldots, t_{n-3})
\]

to get rid of the quotient. This induces an isomorphism

\[
M_{0,n} \simeq (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \{ (t_1, \ldots, t_{n-3}) \mid t_i = t_j \text{ for some } i \neq j \},
\]

which shows that \( M_{0,n} \) is a smooth variety of dimension \( n - 3 \). In particular, \( M_{0,3} \) is reduced to a point and \( M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\} \).

Deligne, Mumford, and Knudsen [Knu83] constructed a smooth compactification \( \overline{M}_{0,n} \) of \( M_{0,n} \) by a normal crossing divisor. The irreducible components of the boundary are in one-to-one correspondence with the partitions of the marked points into subsets of cardinality at least 2. We refer the reader to [KV07] for a nice introduction to these spaces and their compactifications.
Remark 2.107. The blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $(0, 0), (1, 1), \text{ and } (\infty, \infty)$ is isomorphic to the Deligne–Mumford compactification $\overline{M}_{0,5}$ of the moduli space of genus zero curves with 5 marked points. The boundary $\overline{M}_{0,5} \setminus M_{0,5}$ consists of 10 smooth divisors intersecting transversally. The previous constructions shows that $\zeta(2)$ is a period of 

\[ H^2(\overline{M}_{0,5} \setminus A, B \setminus (A \cap B)), \]

where $A$ is the union of 5 irreducible components of the boundary and $B$ consists of the remaining ones.

Although the technical difficulties to transform the integral representation of any multiple zeta value into a period are the same we have encountered for $\zeta(2)$, one needs a more systematic method to deal with all of them. This was accomplished by Goncharov and Manin:

**Theorem 2.108 (Goncharov–Manin [GM04]).** Let $s$ be an admissible multi-index $s$ of weight $n$. There exists two normal crossing divisors $A_s$ and $B$, supported on the boundary of $M_{0,n+3}$ and with no common irreducible components, such that $\zeta(s)$ is a period of 

\[ H^n(\overline{M}_{0,n+3} \setminus A_s, B \setminus (A_s \cap B)). \]  

(2.109)

**Remark 2.110.** A converse to this theorem, due to Brown [Bro09], affirms that, for any choice of boundary divisors $A$ and $B$, all periods of the cohomology groups $H^n(\overline{M}_{0,n+3} \setminus A, B \setminus (A \cap B))$ are $\mathbb{Q}[2\pi i]$-linear combinations of multiple zeta values. This can now be seen as a consequence of Brown’s theorem characterizing the periods of mixed Tate motives over $\mathbb{Z}$ (Corollary 5.105).

In these notes, we will rather follow Deligne and Goncharov [DG05] to show that multiple zeta values are periods associated with the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. A reason to prefer
this approach is that it becomes easier to study the question whether relations between multiple zeta values come from geometry. A third way to see multiple zeta values as periods was proposed by Terasoma in [Ter02].

Exercise 2.111. Show that the boundary of the Deligne–Mumford compactification of $M_{0,n}$ has $2^{n-1} - n - 1$ irreducible components.

2.5. Mixed Hodge structures. Thanks to the comparison isomorphism (2.89), the Betti cohomology of algebraic varieties has richer properties than the singular cohomology of a random topological space. As we will explain in this section, it is endowed with a mixed Hodge structure, which can be thought of as a first approximation to the notion of motive. Usually, the study of a period in the sense of Definition 2.98 begins by understanding the mixed Hodge structure on the cohomology of the pair of varieties from which it arises. This theory was developed by Deligne in the 70s, taking as source of inspiration on the one hand Hodge’s theorem for compact Kähler manifolds and, on the other hand, ℓ-adic cohomology of varieties over finite fields. For a more systematic treatment, we refer the reader to Deligne’s original papers [Del71, Del74] or the monographs [Voi02] and [PS08]. The paper [Dur83] is a user-friendly introduction to the subject.

2.5.1. Pure Hodge structures. Let $M$ be a compact Kähler manifold of dimension $d$, for instance a smooth projective complex variety. For each pair of integers $(p,q)$, let $H^{p,q}(M) \subseteq H^{p+q}(M,\mathbb{C})$ be the subspace of cohomology classes that can be represented by a $C^\infty$-closed differential $(p+q)$-form of type $(p,q)$, i.e. that can be locally written as

$$\sum_{I,J} f_{I,J}(z_1,\ldots,z_d)dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where the sum runs over subsets $I = \{i_1,\ldots,i_p\}$ and $J = \{j_1,\ldots,j_q\}$ of $\{1,\ldots,d\}$, and $f_{I,J}$ are $C^\infty$-functions.

The starting point of Hodge theory is the following theorem:

Theorem 2.112 (Hodge). There is a direct sum decomposition

$$H^n(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}(M).$$

Complex conjugation acts on the right-hand side of (2.113) through the action on the coefficients of the left-hand side, that is,

$$\overline{\sigma \otimes w} = \overline{\sigma} \otimes \overline{w} \quad \text{for all } \sigma \in H^n(M,\mathbb{Q}), \ w \in \mathbb{C}.$$

This action sends $H^{p,q}(M)$ to $H^{q,p}(M)$, a property commonly referred to as Hodge symmetry.
Remark 2.114. Abstractly, what appears in Hodge’s theorem 2.112 is a finite-dimensional $\mathbb{Q}$-vector space $H$, together with a bigrading 

$$H_C = H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

of its complexification satisfying $H^{p,q} = H^{q,p}$. This is called a pure Hodge structure of weight $n$, and the set of pairs $(p, q)$ for which $H^{p,q} \neq 0$ is called the Hodge type. As you will prove in Exercise 2.138, the data of a pure Hodge structure of weight $n$ are equivalent to a decreasing filtration $F^\bullet$ on $H_C$ (the Hodge filtration) such that, for all integers $p$,

$$H_C = F^p H_C \oplus F^{n-p+1} H_C.$$  

(2.115)

This is the definition that one usually finds in textbooks about Hodge theory. However, for the purpose of studying periods it is important to remember that the filtration $F^\bullet$ comes from de Rham cohomology. If $M$ is given by the complex points of an algebraic variety $X$ defined over a subfield $k$ of the complex numbers, then

$$H^n(M, \mathbb{C}) \cong H^n_{\text{dR}}(X) \otimes_k \mathbb{C}$$

and the Hodge filtration is already defined on the $k$-vector space $H^n_{\text{dR}}(X)$. The following definition keeps track of all these elements:

**Definition 2.116.** Let $k$ be a subfield of $\mathbb{C}$. A pure Hodge structure over $k$ is the datum

$$H = (H_B, (H_{\text{dR}}, F^\bullet), \text{comp}_{B,\text{dR}})$$

of a finite-dimensional $\mathbb{Q}$-vector space $H_B$, a finite-dimensional $k$-vector space $H_{\text{dR}}$, together with a decreasing filtration $F^\bullet$, and an isomorphism of complex vector spaces

$$\text{comp}_{B,\text{dR}} : H_{\text{dR}} \otimes_k \mathbb{C} \to H_B \otimes_{\mathbb{Q}} \mathbb{C},$$

such that the induced filtration on $H_C = H_B \otimes_{\mathbb{Q}} \mathbb{C}$, still denoted by $F^\bullet$, satisfies the following: there exists an integer $n$ such that, for all $p$,

$$H_C = F^p H_C \oplus F^{n-p+1} H_C.$$  

(2.117)

We call $n$ the weight of $H$. Abusing language, we will often say that $H_B$ carries a pure Hodge structure.

**Definition 2.118.** A morphism of pure Hodge structures over $k$

$$f : H \to H'$$

is a pair $f = (f_B, f_{\text{dR}})$ consisting of a $\mathbb{Q}$-linear map $f_B : H_B \to H'_B$ and a $k$-linear map $f_{\text{dR}} : H_{\text{dR}} \to H'_{\text{dR}}$ such that $f_{\text{dR}}(F^\bullet H_{\text{dR}}) \subseteq F^\bullet H'_{\text{dR}}$ and that
the following diagram commutes:

$$
\begin{array}{c}
H_{dR} \otimes_k \mathbb{C} \xrightarrow{\text{comp}_{B,dR}} H_B \otimes_{\mathbb{Q}} \mathbb{C} \\
\downarrow f_{dR} \otimes k \text{id} \downarrow \quad \downarrow f_B \otimes q \text{id} \\
H'_{dR} \otimes_k \mathbb{C} \xrightarrow{\text{comp}'_{B,dR}} H'_B \otimes_{\mathbb{Q}} \mathbb{C}.
\end{array}
$$

It follows from this definition that a morphism of Hodge structures of different weight is always zero (Exercise 2.141).

We let $\mathbf{HS}(k)$ denote the category of pure Hodge structures over $k$. If $L$ is another subfield of $\mathbb{C}$ containing $k$, there is an “extension of scalars” functor

$\quad - \otimes_k L: \mathbf{HS}(k) \longrightarrow \mathbf{HS}(L)$

such that $(H \otimes_k L)_B = H_B$ and $(H \otimes_k L)_{dR} = H_{dR} \otimes_k L$ together with the induced filtration and the induced comparison isomorphism via the canonical identification $(H_{dR} \otimes_k L) \otimes_L \mathbb{C} = H_{dR} \otimes_k \mathbb{C}$.

**Example 2.120 (Hodge–Tate structures).** For each integer $n \in \mathbb{Z}$, set $Q(n) = (\mathbb{Q}, (\mathbb{Q}, F^\bullet), \text{comp}_{B,dR})$, where the filtration reads $Q = F^{-n} \mathbb{Q} \supseteq F^{-n+1} \mathbb{Q} = \{0\}$, and the isomorphism $\text{comp}_{B,dR}: \mathbb{C} \to \mathbb{C}$ is given by multiplication by $(2\pi i)^{-n}$. Then $Q(n)$ is a one-dimensional pure Hodge structure of weight $-2n$ over $\mathbb{Q}$. Upon application of the functor (2.119), we obtain a Hodge structure over any subfield $k$ of $\mathbb{C}$ that will be still denoted by $Q(n)$. Note, however, that the special role of $2\pi i$ will be more or less significant depending on the nature of $k$. For example, if $k = \mathbb{C}$, the Hodge structure $Q(n)$ is isomorphic to the one where $\text{comp}_{B,dR}$ is given by the identity, and indeed to any one-dimensional pure Hodge structure of weight $-2n$ (Exercise 2.139).

The Hodge structure $Q(1)$ is known as the Tate Hodge structure. We will call all the $Q(n)$ Hodge–Tate structures. Observe that we already encountered $Q(-1)$. By Example 2.97, it is isomorphic to the triple $H^1(\mathbb{G}_m) = (H^1_B(\mathbb{G}_m), (H^1_{dR}(\mathbb{G}_m), F^\bullet), \text{comp}_{B,dR})$, where $F^\bullet$ is the trivial filtration concentrated in degree 1, and $\text{comp}_{B,dR}$ is Grothendieck’s comparison isomorphism from Theorem 2.88.

Once we have introduced these notions, we can state the following variant of Hodge’s theorem:

**Theorem 2.121.** Let $k$ be a subfield of $\mathbb{C}$ and $X$ a smooth projective variety over $k$. Then the Betti cohomology $H^n_B(X)$ carries a pure Hodge structure of weight $n$ over $k$, functorial for morphisms of algebraic varieties.

More precisely, we consider the triple $H^n(X) = (H^n_B(X), (H^n_{dR}(X), F^\bullet), \text{comp}_{B,dR})$. 
As in the previous example, \( \text{comp}_{\text{B,DR}} \) is the comparison isomorphism of Theorem 2.88. The Hodge filtration \( F^\bullet \) is given by
\[
F^p H^n_{\text{DR}}(X) = \text{Im}(H^n(X, \Omega^\geq_p X) \to H^n(X, \Omega^\bullet_X)),
\]
where \( \Omega^\geq_p X \) stands for the bête truncation of the de Rham complex, namely
\[
\Omega^\geq_p X : 0 \to \cdots 0 \to \Omega^p_X \to \Omega^{p+1}_X \to \cdots.
\]
That the Hodge structure on \( H^n(X) \) is functorial means that, for any morphism \( f : X \to Y \) of smooth projective varieties, the induced map on cohomology \( f^* : H^n(Y) \to H^n(X) \) is a morphism of Hodge structures.

As we have already mentioned, by Exercise 2.141, there are no non-zero morphisms between pure Hodge structures of different weight. However, such maps naturally occur in geometry. For example, if \( Z \hookrightarrow X \) is a smooth closed subvariety of codimension \( c \), then there is a Gysin morphism
\[
H^n(Z) \to H^{n+2c}(X).
\]
In order to turn the Gysin morphism into a morphism of Hodge structures, we introduce Tate twists: given a pure Hodge structure \( H \) of weight \( n \) and another integer \( m \), we denote by \( H(m) \) the pure Hodge structure of weight \( n - 2m \) with the same underlying \( \mathbb{Z} \)-module and \( k \)-vector space, filtration shifted by \( m \) and comparison isomorphism multiplied by \((2\pi i)^{-m}\).

Example 2.122. As Hodge structure, the cohomology of the projective space \( \mathbb{P}^n \) is given by
\[
H^j(\mathbb{P}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q}(-j/2) & 0 \leq j \leq 2n \text{ even}, \\ 0 & \text{else}. \end{cases}
\]

2.5.2. Mixed Hodge structures. Before discussing mixed Hodge structures, we recall some terminology concerning filtrations and morphisms.

Definition 2.123. Let \( k \) be a field and let \( (V, F) \) and \( (V', F) \) be filtered \( k \)-vector spaces. A morphism \( f : V \to V' \) is called filtered if \( f(F^p V) \subseteq F^p V' \) and strict (with respect to \( F \)) if, in addition,
\[
f(F^p V) = F^p V' \cap \text{Im}(f).
\]

Hodge’s Theorem says that the cohomology in degree \( n \) of a smooth projective complex variety carries a pure Hodge structure of weight \( n \). This theorem is no longer true when \( X \) fails to be smooth or projective. For instance, we saw in Example 2.4 that \( H^1(\mathbb{G}_m) \) is one-dimensional, so it cannot carry a pure Hodge structure of weight one. Nevertheless, Deligne proved that the cohomology of any quasi-projective complex variety is an “iterated extension” of pure Hodge structures.
Theorem 2.124 (Deligne). Let $X$ be a quasi-projective variety over the field of complex numbers.

(a) There exists an increasing filtration

$$W_{-1} = 0 \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{2n} = H^n(X),$$

and a decreasing filtration

$$F^0 = H^n(X, \mathbb{C}) \supseteq F^1 \supseteq \cdots \supseteq F^n \supseteq F^{n+1} = 0$$

such that $F^\bullet$ induces a pure Hodge structure of weight $m$ on each graded piece

$$\text{Gr}_W^m H^n(X) = W_m/W_{m-1}.$$

(b) Moreover, if $f : X \to Y$ is a morphism of quasi-projective varieties, the induced map on cohomology $f^* : H^n(Y) \to H^n(X)$ is a filtered morphism with respect to both filtrations, i.e.

$$f^*(W_m H^n(Y)) \subseteq W_m H^n(X),$$

$$f^*(F^p H^n(Y)) \subseteq F^p H^n(X).$$

(c) If $X$ is smooth, then $\text{Gr}_m^W H^n(X) = 0$ for all $m < n$ and, if $X$ is projective, $\text{Gr}_m^W H^n(X) = 0$ for all $m > n$.

Later, in Section 2.7 we will give some ingredients of the proof of the above theorem.

This motivates the following definition:

Definition 2.125. Let $k$ be a subfield of $\mathbb{C}$. A mixed Hodge structure over $k$ is a triple

$$H = ((H_B, W_B^\bullet), (H_{\text{dR}}, F^\bullet, W_{\text{dR}}^\bullet), \text{comp}_{B,\text{dR}})$$

consisting of:

- a finite-dimensional $\mathbb{Q}$-vector space $H_B$, together with an increasing filtration $W_B^\bullet$,
- a finite-dimensional $k$-vector space $H_{\text{dR}}$, together with an increasing filtration $W_{\text{dR}}^\bullet$ and a decreasing filtration $F^\bullet$,
- an isomorphism of complex vector spaces

$$\text{comp}_{B,\text{dR}} : H_{\text{dR}} \otimes_k \mathbb{C} \xrightarrow{\sim} H_B \otimes_{\mathbb{Q}} \mathbb{C}$$

that is filtered with respect to the weight filtration. That is,

$$\text{comp}_{B,\text{dR}}(W_{\text{dR}}^\bullet \otimes_k \mathbb{C}) = W_B^\bullet \otimes_{\mathbb{Q}} \mathbb{C}.$$

We require that these data verify the following: for each integer $m$,

$$\text{Gr}_m^W H = (\text{Gr}_m^W H_B, (\text{Gr}_m^W H_{\text{dR}}, F^\bullet), \text{comp}_{B,\text{dR}})$$

(2.126)

is a pure Hodge structure over $k$ of weight $m$. 
If $H$ is a mixed Hodge structure we will denote $H_C = H_B \otimes \mathbb{Q} \mathbb{C}$ provided with the complex conjugation coming from this rational structure and the Hodge filtration induced by the one in $H_{dR}$ through the comparison isomorphism. Then $H_C$ has a complex conjugate filtration $\overline{F}$.

**Definition 2.127.** A morphism $f: H \to H'$ of mixed Hodge structures over $k$ is a pair $f = (f_B, f_{dR})$ consisting of
- a morphism of $\mathbb{Q}$-vector spaces $f_B: H_B \to H'_B$,
- a morphism of $k$-vector spaces $f_{dR}: H_{dR} \to H'_{dR}$
such that $f_B$ is filtered with respect to the weight filtration, while $f_{dR}$ is filtered with respect to the weight and the Hodge filtrations, and both maps are compatible with the comparison isomorphisms. In other words

$$f_B(W^B H_B) \subseteq W^B H'_B,$$
$$f_{dR}(F^* H_{dR}) \subseteq F^* H'_{dR},$$
$$f_{dR}(W^{dR} H_{dR}) \subseteq W^{dR} H'_{dR},$$
$$f_{dR} \circ \text{comp}_{B, dR} = \text{comp}_{B, dR} \circ (f_B \otimes \text{id}_C).$$

We shall denote by $\text{MHS}(k)$ the category of mixed Hodge structures over $k$. When $k = \mathbb{C}$, we shall simply speak of “mixed Hodge structures” and write $\text{MHS}$ instead of $\text{MHS}(\mathbb{C})$.

**Definition 2.128.** The category $\text{MHS}(k)$ comes naturally with two forgetful functors

$$\omega_B: \text{MHS}(k) \to \text{Vec}_\mathbb{Q},$$
$$\omega_{dR}: \text{MHS}(k) \to \text{Vec}_k$$
sending $H$ to $H_B$ and $H_{dR}$ respectively. These functors are called the Betti fibre functor and the de Rham fibre functor.

**Definition 2.129.** A mixed Hodge structure $H$ over $k$ is called split if there is an isomorphism of mixed Hodge structures

$$H \xrightarrow{\sim} \bigoplus_{m \in \mathbb{Z}} \text{Gr}^W_m H,$$

and therefore $H$ is a direct sum of pure Hodge structures.

As was explained in Remark 2.114, the Hodge filtration of a pure Hodge structure $H$ induces a bigrading of $H_C$ in a natural way. A similar, but more involved, construction applies to mixed Hodge structures as well.

**Proposition 2.130 (Deligne’s splitting).** Let $H$ be a mixed Hodge structure defined over $k$. There is a unique decomposition, called Deligne’s splitting, of $H_C$ into a direct sum

$$H_C = \bigoplus_{p,q} H^{p,q}$$

(2.131)
satisfying the conditions
\[ W_n H_C = \bigoplus_{p+q \leq n} H^{p,q}, \]
\[ F^p H_C = \bigoplus_{p' \geq p} H^{p',q}, \]
\[ \overline{H}^{p,q} \cong H^{q,p} \mod \bigoplus_{r<p,s<q} H^{r,s}. \]

Moreover, this splitting is functorial: given a morphism of mixed Hodge structures \( f: H_1 \to H_2 \), there are induced maps \( f^{p,q}: H_1^{p,q} \to H_2^{p,q} \) compatible with the decomposition (2.131).

**Idea of the proof.** The graded pieces are defined as
\[ H^{p,q} = F^p \cap W_{p+q} \cap \left( F^q \cap W_{p+q} + \sum_{j \geq 2} F^{q-j+1} \cap W_{p+q-j} \right) \]

The proof that this decomposition satisfies the required conditions and is characterized by them can be found in [PS08, Lemma-Definition 3.4].

The functoriality follows from this explicit description. \( \square \)

**Theorem 2.132 (Deligne).** The category \( \text{MHS}(k) \) is abelian.

In [Del71] Deligne proves this result for \( k = \mathbb{C} \), but the proof carries over to the general case.

Deligne’s proof of this theorem is sometimes called “a masterpiece of linear algebra”. The main difficulty comes from the fact that the category of bifiltered vector spaces is not abelian. The key property that makes everything work is that any morphism of mixed Hodge structures is **strict** with respect to the weight and Hodge filtrations. More precisely we have the following lemma that is a consequence of Proposition 2.130.

**Lemma 2.133.** Let \( f: H \to H' \) be a morphism of mixed Hodge structures, then \( f_B \) is strict with respect to the weight filtration and \( f_{dR} \) is strict with respect to the weight and Hodge filtrations.

Another important consequence of Lemma 2.133 is the following:

**Proposition 2.134.** Let \( f \) be a morphism of mixed Hodge structures. Then \( f \) is an isomorphism if and only if either \( \omega_B(f) \) is an isomorphism or \( \omega_{dR}(f) \) is an isomorphism.

**Proof.** Thanks to the comparison isomorphism, \( \omega_B(f) \) is an isomorphism if and only if \( \omega_{dR} \) is one. Thus we only need to prove that \( f \) is an isomorphism if and only if \( \omega_{dR}(f) \) is. In general, a morphism \( g: (V, F) \to (V', F') \) of filtered vector spaces that is an isomorphism of the subjacent vector spaces
is not necessarily an isomorphism because the filtrations, although being compatible they may not match exactly. That is,
\[ g(F^p V) \subset F^p V', \quad g(F^p V) \neq F^p V'. \]
If \( f: H \to H' \) is a morphism of mixed Hodge structures, then the map \( f_{\text{dR}}: (H, W, F) \to (H', W, F) \) is strict with respect to both filtrations. Therefore
\[
\begin{align*}
  f_{\text{dR}}(F^p H_{\text{dR}}) &= F^p H'_{\text{dR}} \cap \text{Im}(f_{\text{dR}}) = F^p H'_{\text{dR}}, \\
  f_{\text{dR}}(W_n H_{\text{dR}}) &= W_n H'_{\text{dR}} \cap \text{Im}(f_{\text{dR}}) = W_n H'_{\text{dR}},
\end{align*}
\]
which implies the result. \( \square \)

### 2.5.3 Mixed Hodge–Tate structures

**Definition 2.135.** A mixed Hodge structure \( H \) over \( k \) is said to be of *Tate type* if \( \text{Gr}^W_{2m+1} H = 0 \) and \( \text{Gr}^W_{2m} H \) is a sum of copies of the pure Hodge–Tate structure \( \mathbb{Q}(-m) \) for all \( m \). Mixed Hodge structures of Tate type are also called mixed Hodge–Tate structures.

We shall denote by \( \text{MHTS}(k) \) the full subcategory of \( \text{MHS}(k) \) consisting of mixed Hodge structures of Tate type over \( k \).

**Remark 2.136.** One can think of mixed Hodge structures as “iterated extensions” of the pure ones. Indeed, given two successive steps of the weight filtration, there is an exact sequence of vector spaces
\[
0 \to W_{m-1} H \to W_m H \to \text{Gr}^W_m H \to 0.
\]
When \( m \) is the highest weight of \( H \) (i.e. \( W_m H = H \)), this exhibits \( H \) as an extension of the pure Hodge structure \( \text{Gr}^W_m H \) by \( W_{m-1} H \), which in turn is an extension of \( \text{Gr}^W_{m-1} H \) by \( W_{m-2} H \), and so on. Then mixed Hodge–Tate structures are those obtained as iterated extensions of the simplest ones, that is, sums of \( \mathbb{Q}(n) \).

In the case of mixed Hodge structures of Tate type, the fibre functor \( f_{\text{dR}} \) does not land just in bifiltered vector spaces but in graded vector spaces.

**Lemma 2.137.** Let \( H \) be a mixed Hodge structure of Tate type. Then \( H_{\text{dR}} \) has a functorial structure of graded \( k \)-vector space given by
\[
H_{\text{dR}} = \bigoplus_p F^p H_{\text{dR}} \cap W_{2p} H_{\text{dR}},
\]
moreover, this bigrading characterizes both the weight and the Hodge filtration.

**Proof.** Exercise 2.143. \( \square \)

***
Exercise 2.138. Prove the claim of Remark 2.114.

[Hint: to get the direct sum decomposition starting from the filtration, define
\( H^{p,q} = F^p H_C \cap F^q H_C \). Conversely, consider \( F^p H_C = \bigoplus_{r \geq p} H^{r,n-r} \).]

Exercise 2.139. Let \( k \) be a subfield of \( \mathbb{C} \). Prove that the set of isomorphism classes of one-dimensional pure Hodge structures over \( k \) is in bijection with \( \mathbb{Z} \times \mathbb{C}^\times / k^\times \).

Exercise 2.140. Let \( H \) be a pure Hodge structure over \( k \) of weight \( n \) and \( p \) an integer. We define the space of \((p,p)\)-classes as
\[
H^{(p,p)} = \begin{cases} \text{comp}_{B,\text{dR}}(F^p H_{\text{dR}}) \cap H_B, & \text{if } n = 2p, \\ \{0\}, & \text{if } n \neq 0. \end{cases}
\]
Show that
\[
\text{Hom}_{\text{MHS}}(k)(\mathbb{Q}(-p), H) = H^{(p,p)},
\]
where \( \mathbb{Q}(-p) \) is the pure Hodge structure of weight \( 2p \) over \( k \) introduced in Example 2.120.

Exercise 2.141. Let \( H \) and \( H' \) be pure Hodge structures over \( k \) of weights \( n \) and \( m \) respectively.

(1) Show that the vector space \( \text{Hom}_{\mathbb{Q}}(H_B, H'_B) \) admits a pure Hodge structure over \( k \) of weight \( m - n \), denoted \( \text{Hom}(H, H') \).

(2) Show that the group of morphisms of Hodge structures between \( H \) and \( H' \) agrees with the subspace \( \text{Hom}(H, H')^{(0,0)} \).

[Hint: given filtered vector spaces \((A, W_\bullet)\) and \((B, W_\bullet)\) with increasing filtrations, one defines an increasing filtration on \( \text{Hom}(A, B) \) as
\[
W_n \text{Hom}(A, B) = \{ f \in \text{Hom}(A, B) \mid f(W_k A) \subset W_{k+n} B \}.
\]
A similar construction is valid for decreasing filtrations.]

(3) Conclude that, if \( n \neq m \), then any morphism of Hodge structures between \( H \) and \( H' \) is zero.

Exercise 2.142. Let \( H \) and \( H' \) be mixed Hodge structures over \( k \). Define a natural mixed Hodge structure on the tensor product \( H \otimes H' \). Show that for any pure Hodge structure \( H \), we have
\[
H(m) = H \otimes \mathbb{Q}(m).
\]

Exercise 2.143. In this exercise we prove Lemma 2.137. Let \( H \) be a mixed Hodge structure of Tate type.

(1) Prove by negative induction over \( n \) that
\[
W_r H_{\text{dR}} = \sum_{2p \leq r} W_{2p} H_{\text{dR}} \cap F^p H_{\text{dR}}.
\]

(2) Show that, for every \( p \in \mathbb{Z} \),
\[
W_{2p-1} H_{\text{dR}} \cap F^p H_{\text{dR}} = \{0\}.
\]
(3) Conclude the proof of Lemma 2.137.

Exercise 2.144. There are two possible ways of inducing $F^\bullet$ on $\text{Gr}_m^W H$. Show that they are equivalent.

Exercise 2.145. Given a morphism $f : H \to H'$ of mixed Hodge structures, prove that the induced maps $f_m : \text{Gr}_m^W H \to \text{Gr}_m H'$ are morphism of pure Hodge structures.

Exercise 2.146. Let $H = (H_{dR}, H_B, \alpha)$ be a triple consisting of

- a finite-dimensional $\mathbb{Q}$-vector space $H_B$, equipped with an increasing filtration $W_2 \cdot H_B$ indexed by even integers,
- a finite-dimensional $\mathbb{Q}$-vector space $H_{dR}$, together with a grading indexed by even integers $H_{dR} = \bigoplus_n (H_{dR})_{2n}$,
- a comparison isomorphism $\alpha : H_{dR} \otimes_{\mathbb{Q}} \mathbb{C} \sim \to H_B \otimes_{\mathbb{Q}} \mathbb{C}$, subject to the condition that $\alpha$ maps $(H_{dR})_{2n} \otimes_{\mathbb{Q}} \mathbb{C}$ to $W_{2n} H_B \otimes_{\mathbb{Q}} \mathbb{C}$, and induces an isomorphism $\alpha_n : (H_{dR})_{2n} \otimes_{\mathbb{Q}} \mathbb{C} \sim \to (W_{2n} H_B/W_{2(n-1)} H_B) \otimes_{\mathbb{Q}} (2\pi i)^n \mathbb{Q}$.

Prove that the category $\text{MHTS}(\mathbb{Q})$ is equivalent to the category whose objects are such triples and whose morphisms are the obvious ones.

2.6. Extensions. We now turn to the description of the extension groups in the category of mixed Hodge structures. Recall that, if no field of definition is explicitly mentioned, by a mixed Hodge structure we mean a mixed Hodge structure over $\mathbb{C}$.

Definition 2.147. Let $A$ and $B$ be two mixed Hodge structures.

1. An extension of $A$ by $B$ is a short exact sequence

$$0 \to B \xrightarrow{\beta} H \xrightarrow{\alpha} A \to 0,$$

where $\alpha$ and $\beta$ are morphisms of mixed Hodge structures. Such an extension is said to be split if there exists a morphism of mixed Hodge structures $s : A \to H$ such that $\alpha \circ s = \text{id}_A$.

2. Two extensions are equivalent if there exists a morphism of mixed Hodge structures $f : H \to H'$ such that the diagram

$$\begin{array}{c}
\begin{array}{c}
0 \\ \| \\
\| \\
\| \\
0 \\
\end{array}
\end{array}
\begin{array}{c}
B \\
\| \\
\| \\
\| \\
B \\
\end{array}
\begin{array}{c}
H \\
\| \\
\| \\
\| \\
H' \\
\end{array}
\begin{array}{c}
A \\
\| \\
\| \\
\| \\
A \\
\end{array}
0
$$
commutes. This defines indeed an equivalence relation (see Exercise 2.156) whose set of equivalence classes will be denoted by

$$\text{Ext}^1_{\text{MHS}}(A, B).$$

2.6.1. Extensions of mixed Hodge structures. We first give Carlson’s formula for the extensions between any two mixed Hodge structures [Car80]. To express it, it is convenient to define a filtration on the space of homomorphisms of filtered vector spaces as in Exercise 2.141. Given two filtered vector spaces \((A, W)\) and \((B, W)\) with increasing filtrations, then \(\text{Hom}(A, B)\) has an induced increasing filtration defined by

$$W_n \text{Hom}(A, B) = \{ f \in \text{Hom}(A, B) | f(W_k A) \subset W_{k+n} B \}.$$ 

A similar construction is valid for decreasing filtrations.

**Theorem 2.148** (Carlson). The extension group of two mixed Hodge structures \(A\) and \(B\) is isomorphic to

$$\text{Ext}^1_{\text{MHS}}(A, B) = \frac{W_0 \text{Hom}_C(A_C, B_C)}{W_0 \cap F^0 \text{Hom}_C(A_C, B_C) + W_0 \text{Hom}_Q(A_B, B_B)}. \quad (2.149)$$

**Proof.** Given an extension of mixed Hodge structures

$$0 \rightarrow B \rightarrow H \rightarrow A \rightarrow 0,$$

we first choose a splitting \(\varphi_1: A_C \rightarrow H_C\) of the underlying complex vector spaces that is compatible with the weight and the Hodge filtration, for example Deligne’s splitting from Proposition 2.130. Choose a second splitting \(\varphi_2: A_C \rightarrow H_C\) compatible with the rational structures \(A_B\) and \(H_B\) and the weight filtration. For every \(a \in A_C\) write \(f(a) \in B_C\) for the unique element whose image in \(H_C\) is \(\varphi_1(a) - \varphi_2(b)\). Clearly, \(f\) is a linear map from \(A\) to \(B\) and respects the weight filtration. Hence \(f \in W_0 \text{Hom}_C(A_C, B_C)\). We have to see what happens if we make different choices of splittings.

Let \(\varphi_2'\) a second choice for the splitting compatible with the weight filtration and the rational structure and let \(f'\) we the corresponding morphism. Then \(f(a) - f'(a) = \varphi_2'(a) - \varphi_2(a)\), so \(f - f'\) respects the rational structure and the weight filtration. In other words,

$$f - f' \in W_0 \text{Hom}_Q(A_B, B_B).$$

Similarly, if \(\varphi_1''\) is a second choice for \(\varphi_1\) and \(f'' \in W_0 \text{Hom}_C(A_C, B_C)\) is the corresponding linear map, then

$$f - f'' \in W_0 \cap F^0 \text{Hom}_C(A_C, B_C).$$

In consequence, the class of \(f\) in the quotient is independent of the choice of splittings. Let now \(H'\) be an equivalent extension. This means that there
is a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & B & \rightarrow & H & \rightarrow & A & \rightarrow & 0 \\
\downarrow & & \downarrow & & \psi & & \downarrow & & \downarrow \\
0 & \rightarrow & B & \rightarrow & H & \rightarrow & A & \rightarrow & 0.
\end{array}
\]

If \(\varphi_1\) and \(\varphi_2\) are choices of splittings for \(H\) then \(\psi \circ \varphi_1 = \psi \circ \varphi_2\) are choices of splittings for \(H'\) that yield the same function \(f\). Therefore we have constructed a map from the left-hand side of (2.149) to the right-hand side.

If the class of \(f\) is the quotient is zero means that we can modify \(\varphi_1\) by an element of \(W_0 \cap F^0 \text{Hom}_C(A_C, B_C)\) to get a new splitting \(\varphi'_1\) and \(\varphi_2\) by an element of \(W_0 \text{Hom}_Q(A_B, B_B)\) to get \(\varphi'_2\) so that \(\varphi'_1 = \varphi_2\). This implies that \(\varphi'_1, \varphi'_2\) defines a splitting of mixed Hodge structures, and the extension was trivial. Therefore the map we have constructed is injective.

To see that it is surjective, we start with a function \(f \in W_0 \text{Hom}_C(A_C, B_C)\). Then we write

\[
(H_B, W) = (A_B, W) \oplus (B_B, W), \quad (H_{dR}, W) = (A_{dR}, W) \oplus (B_{dR}, W),
\]

define the comparison isomorphism in \(H\) and the direct sum of the comparison isomorphisms of \(A\) and \(B\) and define the Hodge filtration on \(H_{dR}\) by

\[
F^p H_{dR} = F^p B_{dR} \oplus \{a + f(a) \mid a \in F^p A_{dR}\}
\]

The fact that \(H = ((H_B, W), (H_{dR}, W, F), \text{comp}_{B, dR})\) (2.150) is a mixed Hodge structure is the content of Exercise 2.157. By construction, the function corresponding to \(H\) is the original function \(f\) thus we see that the map we just constructed is an isomorphism concluding the proof of the theorem.

As a consequence of Carlson’s formula, we next see that the category of mixed Hodge structures has cohomological dimension one, meaning that all higher extension groups vanish. This follows from:

**Theorem 2.151.** For any two mixed Hodge structures \(A, B\), and any integer \(n \geq 2\), we have

\[
\text{Ext}_\text{MHS}^n(A, B) = 0.
\]

**Proof.** In view of Lemma A.17 it is enough to show that, for every mixed Hodge structure \(A\), the functor \(B \mapsto \text{Ext}_\text{MHS}^1(A, B)\) is right exact. Since the functors \(\text{Ext}_\text{MHS}^\bullet(A, -)\) form a cohomological functor, it is enough to show that, if \(B_1 \rightarrow B_2\) is an epimorphism of mixed Hodge structures, then

\[
\text{Ext}_\text{MHS}^1(A, B_1) \rightarrow \text{Ext}_\text{MHS}^1(A, B_2)
\]

is surjective. But this is a direct consequence of Carlson’s formula. \(\square\)
2.6.2. Extensions of Hodge–Tate structures. In the case of Hodge–Tate structures we describe the extensions of mixed Hodge structures defined over $\mathbb{Q}$.

**Theorem 2.152.** Let $m$ and $n$ be two integers. Then

$$\text{Ext}^1_{\text{MHS}(\mathbb{Q})}(\mathbb{Q}(m), \mathbb{Q}(n)) = \begin{cases} \mathbb{C}/(2\pi i)^{n-m}\mathbb{Q} & m < n, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Tensoring the extension by $\mathbb{Q}(-n)$, we can assume without loss of generality that $n = 0$. So let us consider an extension

$$0 \to \mathbb{Q}(0) \overset{\beta}{\longrightarrow} H \overset{\alpha}{\longrightarrow} \mathbb{Q}(m) \to 0.$$ 

Let us first assume that $m > 0$. Then $W_{-2m}H \subseteq H$ is a rank one sub-Hodge structure and the composition $W_{-2m}H \hookrightarrow H \overset{\alpha}{\longrightarrow} \mathbb{Q}(m)$ is an isomorphism. Thus the extension is necessarily split.

For $m = 0$, the weight and the Hodge filtration of $H$ are trivial (the corresponding subobjects are either zero or everything), and hence any section $s_B$ of the map $\alpha_B: H_B \to \mathbb{Q}(0)_B$ induces a morphism of Hodge structures $s: \mathbb{Q}(0) \to H$, so the extension is again split.

Now assume that $m < 0$. The $\mathbb{Q}$-vector space $H_{dR}$ has a canonical splitting

$$H_{dR} = W_0 H_{dR} \oplus F^{-m} H_{dR}.$$ 

Choose a basis $e_0, e_1$ of $H_B$ satisfying $e_0 = \beta(1)$ and $\alpha(e_1) = e$, where 1 is the generator of $\mathbb{Q}(0)_B$ and $e$ is the generator of $\mathbb{Q}(m)_B$. This basis determines uniquely a basis $f_0, f_1$ of $H_{dR}$ by the conditions

$$f_0 \in W_0 H_{dR}, \quad \text{comp}_{B, dR}(f_0) = e_0,$$

$$f_1 \in F^{-m} H_{dR}, \quad \text{comp}_{B, dR}(f_1) \in (2\pi i)^{-m} e_1 + W_{2m} H_B \otimes_{\mathbb{Q}} \mathbb{C}.$$ 

In these bases, the morphism $\text{comp}_{B, dR}$ can be written as

$$\begin{pmatrix} 1 & a \\ 0 & (2\pi i)^{-m} \end{pmatrix}$$

for a complex number $a$ that determines the class of the extension.

We have the right to change the basis $(e_0, e_1)$ by an upper triangular basis with ones in the diagonal and a rational coefficient $b$ in the upper right corner. The basis $(f_0, f_1)$ remains unchanged. In this new basis, the comparison isomorphism will be given by

$$\begin{pmatrix} 1 & a' \\ 0 & (2\pi i)^{-m} \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & (2\pi i)^{-m} \end{pmatrix} = \begin{pmatrix} 1 & a + (2\pi i)^{-m}b \\ 0 & (2\pi i)^{-m} \end{pmatrix}.$$
Thus, two complex numbers $a, a' \in \mathbb{C}$ determine the same extension if and only if $a - a' \in (2\pi i)^{-m}\mathbb{Q}$, from which the result follows.

2.6.3. **Examples.** By Theorem 2.152, the extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ are parametrized by elements in $\mathbb{C}/(2\pi i)^n\mathbb{Q}$. It follows that, for each $n \geq 2$, there is a mixed Hodge structure $\zeta^{MHS}(n)$ sitting in an exact sequence

$$0 \to \mathbb{Q}(n) \to \zeta^{MHS}(n) \to \mathbb{Q}(0) \to 0,$$

whose extension class corresponds to the zeta value $\zeta(n)$. Hence, this extension is split if and only if $\zeta(n) \in (2\pi i)^n\mathbb{Q}$. By Theorem 1.3 and the fact that elements of $(2\pi i)^n\mathbb{Q}$ are purely imaginary for odd $n$, the extension (2.153) is split if and only if $n$ is even. It is a hard problem to construct geometrically these extensions, e.g. as a relative cohomology group.

We now show that, when $n = 1$, all the extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ have geometric origin.

**Example 2.154 (Kummer mixed Hodge structure).** Given a complex number $t \in \mathbb{C}^\times \setminus \{0\}$, consider the relative cohomology

$$H = H^1(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, t\}).$$

The long exact sequence (2.57) gives

$$0 \to H^0(\mathbb{P}^1 \setminus \{0, \infty\}) \to H^0(\{1, t\}) \to H \to H^1(\mathbb{P}^1 \setminus \{0, \infty\}) \to 0.$$ By Example 2.189, one has $H^1(\mathbb{P}^1 \setminus \{0, \infty\}) = \mathbb{Q}(-1)$, and hence

$$0 \to \mathbb{Q}(0) \to H \to \mathbb{Q}(-1) \to 0.$$ The *Kummer mixed Hodge structure* $K^H_t$ is defined to be the dual of $H$, so

$$K^H_t \in \text{Ext}^1_{\text{MHS}(\mathbb{C})}(\mathbb{Q}(0), \mathbb{Q}(1)).$$

For $t = 1$, the Kummer extension is defined as the trivial extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. This yields a map $\mathbb{C}^\times \to \text{Ext}^1_{\text{MHS}(\mathbb{C})}(\mathbb{Q}(0), \mathbb{Q}(1))$.

**Example 2.155.** As another example of how arithmetic information can be encoded through extensions of mixed Hodge structures, let us consider extensions of the first cohomology of a smooth projective curve $C$ by $\mathbb{Q}(-1)$. Then Carlson’s theorem implies that

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(-1), H^1(C)) = \text{Jac}(C)(\mathbb{C}) \otimes_\mathbb{Z} \mathbb{Q}.$$ By Example 2.189, the cohomology of $C \setminus \{p, q\}$ for any pair of points gives such an extension. Through the above isomorphism, the class of the extension is given by the class of the divisor $[p] - [q]$ in $\text{Jac}(C)(\mathbb{C})$. In particular, the extension splits if and only if this divisor is torsion.

\[
\star \star \star
\]
Exercise 2.156. Show that being equivalent in the sense of Definition 2.147 defines an equivalence relation on the set of extensions between mixed Hodge structures.

Exercise 2.157. Let $H$ be the structure given in (2.150). Use that $A$ and $B$ are mixed Hodge structures to prove that, for all $n, p \in \mathbb{Z},$

$$\text{Gr}_n^W H = F^p \text{Gr}_n^W H \oplus \overline{F}^{n-p+1} \text{Gr}_n^W H.$$

Conclude that $H$ is a mixed Hodge structure.

2.7. Examples of mixed Hodge structures. We now explain some ideas behind the construction of mixed Hodge structures on the cohomology of algebraic varieties. The basic tool is the notion of mixed Hodge complex introduced by Deligne in [Del74]. There are several variants of such concept and we will deal here with the one that is relevant for the study of periods.

2.7.1. Mixed Hodge complexes.

Definition 2.158. Let $k$ be a subfield of $\mathbb{C}$. A mixed Hodge complex over $k$ is a 5-tuple

$$A = ((A^*_B, W), (A^*_{\text{DR}}, W, F), (A^*_C, W), \alpha, \beta),$$

where

1. $(A^*_B, W)$ is a bounded below complex of $\mathbb{Q}$-vector spaces with an increasing filtration,

2. $(A^*_{\text{DR}}, W, F)$ is a bounded below complex of $k$-vector spaces with an increasing filtration $W$ and a decreasing filtration $F$,

3. $(A^*_C, W)$ is a bounded below complex of $\mathbb{C}$-vector spaces with an increasing filtration $W$,

4. $\alpha : (A^*_B \otimes_{\mathbb{Q}} \mathbb{C}, W) \to (A^*_C, W)$ is a filtered quasi-isomorphism,

5. $\beta : (A^*_{\text{DR}} \otimes_k \mathbb{C}, W) \to (A^*_C, W)$ is a filtered quasi-isomorphism,

subject to the following two conditions:

1. for every integer $n \in \mathbb{Z}$, the differential induced on the complex $\text{Gr}_n^W A^*_{\text{DR}}$ is strict with respect to the filtration $F$;

2. for all integers $n, m \in \mathbb{Z}$, the triple

$$(H^n(\text{Gr}_n^W A^*_B), (H^n(\text{Gr}_n^W A^*_{\text{DR}}), F), H^n(\alpha)^{-1} \circ H^n(\beta)),$$

is a pure Hodge structure over $k$ of weight $n + m$ in the sense of Definition 2.116.
A morphism of mixed Hodge complexes \( f: A \rightarrow A' \) is a triple
\[
f = (f_B, f_{dR}, f_C)
\]
of morphisms of filtered or bifiltered complexes that commute with the comparison quasi-isomorphisms.

The basic properties of mixed Hodge complexes are summarized in the following result ([Del74, Scholie 8.1.9])

**Proposition 2.159.** Let \( A \) be a mixed Hodge complex over \( k \).

1. For every \( n \), the triple
\[
((H^n(B), W[n]), (H^n(dR), W[n], F), \text{comp}_{B,dR} = H^n(\alpha)^{-1} \circ H^n(\beta))
\]
is a mixed Hodge structure over \( k \).
2. A morphism of mixed Hodge complexes induces a morphism of mixed Hodge structures in cohomology.
3. The spectral sequences associated with the filtered complexes \((A_B, W)\) and \((A_{dR}, W)\) degenerate at the term \( E_2 \).
4. The spectral sequence associated with the filtered complex \((A_{dR}, F)\) degenerates at the term \( E_1 \).

As we will see later it is very useful to combine several mixed Hodge complexes in a single one. This is done using dg-mixed Hodge complexes (differential graded) and the associated total complex.

**Definition 2.160.** A dg-mixed Hodge complex over \( k \) is a 5-tuple
\[
A = ((A^*_{B}, W), (A^*_{dR}, W, F), (A^*_{C}, W), \alpha, \beta),
\]
where

1. \((A^*_{B}, W)\) is a bounded below double complex of \( \mathbb{Q} \)-vector spaces with an increasing filtration,
2. \((A^*_{dR}, W, F)\) is a bounded below double complex of \( k \)-vector spaces with an increasing filtration \( W \) and a decreasing filtration \( F \),
3. \((A^*_{C}, W)\) is a bounded below double complex of \( \mathbb{C} \)-vector spaces with an increasing filtration \( W \),
4. \( \alpha: (A^*_{B} \otimes_{\mathbb{Q}} \mathbb{C}, W) \rightarrow (A^*_{C}, W) \) is a filtered morphism of double complexes,
5. \( \beta: (A^*_{dR} \otimes_{k} \mathbb{C}, W) \rightarrow (A^*_{C}, W) \) is a filtered morphism of double complexes,

such that, for every integer \( p \in \mathbb{Z} \), the 5-tuple
\[
A = ((A^p_{B}, W), (A^p_{dR}, W), (A^p_{C}, W, F), \alpha, \beta),
\]
is a mixed Hodge complex defined over \( k \) in the sense of Definition 2.125.
Let $A$ be a dg-mixed Hodge complex over $k$. We can construct the total complexes $\text{Tot}(A_B)$, $\text{Tot}(A_{dR})$, and $\text{Tot}(A_C)$ as in Definition A.73 from the appendix. On each of them we will denote by $L$ the filtration defined by the second degree and we let $\delta(W, L)$ be the diagonal filtration defined as

$$\delta(W, L)_n A^{p,*}_r = W_{n+p} A^{p,*}_r.$$ (2.161)

**Definition 2.162.** Let

$$A = ((A^*_{B}, W), (A^*_{dR}, W), (A^*_{C}, W, F), \alpha, \beta)$$

be a dg-mixed Hodge complex defined over $k$. Then $\text{Tot}(A)$ is the 5-tuple $((\text{Tot}(A_B)^*, \delta(W, L)), (\text{Tot}(A_{dR})^*, \delta(W, L), F), (\text{Tot}(A_C)^*, \delta(W, L), F), \alpha, \beta)$.

**Proposition 2.163.** If $A$ is a dg-mixed Hodge complex defined over $k$, then $\text{Tot}(A)$ is a mixed Hodge complex defined over $k$.

The need to introduce the diagonal filtration instead of the induced filtration is that the weight filtration in cohomology is not the induced weight filtration, but the deccaled filtration:

$$W_n H^m = \text{Im}(W_{n-m}).$$

Let us see this with an example. Let $A$ be a dg-mixed Hodge complex and $x \in W_r A^{p,q}$ be a cycle. In the cohomology group $H^q(A^{p,*})$, the class of $x$ is an element of weight $r + q$, not $r$. We want all the maps to be compatible with the weight, so in $H^{p+q}(\text{Tot}(A))$, the element $x$ should also have weight $r + q$. This implies that in the complex $\text{Tot}(A)$, the element $x$ should be in the piece $r + q - p - q = r - p$ of the filtration. This is exactly the role of the diagonal filtration:

$$x \in W_r A^{p,q} = \delta(W, L)_{r-p} A^{p,q}.$$

Every time we construct a simple complex from a double complex, it comes equipped with a spectral sequence that relates the cohomology of the total complex with the individual cohomologies of the columns or the rows of the double complex. The added information in the case of dg-mixed Hodge complexes is that this spectral sequence is a spectral sequence of mixed Hodge structures.

**Proposition 2.164.** Let $A$ be a dg-mixed Hodge complex over $k$. There is a spectral sequence

$$E_1^{p,q} = H^q(A^{p,*}) \Rightarrow H^{p+q}(\text{Tot}(A)).$$

Moreover, all the terms $E_r^{p,q}$ carry a mixed Hodge structure and all the maps $d_r$ are morphisms of mixed Hodge structures.

In many cases, this proposition allows one to prove that a spectral sequence degenerates. Indeed, since the differentials $d_r$ are morphisms of mixed Hodge structures, they respect the weight. In particular, whenever two terms have disjoint weights, any map between them is zero.
2.7.2. Smooth projective varieties. Let \( k \) be a subfield of \( \mathbb{C} \) and \( X \) a smooth and projective variety over \( k \). As a warming up, we construct a mixed Hodge complex defined over \( k \) that produces the pure Hodge structure of the cohomology of \( X \) discussed in Theorem 2.121.

The difficulties we have to overcome are twofold. First, algebraic de Rham cohomology of \( X \) is defined as the hypercohomology of the algebraic de Rham complex. Therefore, in order to compute it we need to replace this complex with a complex made out acyclic sheaves. The second is that de Rham cohomology is computed in the algebraic scheme \( X \) with its Zariski topology, while Betti cohomology is computed as a sheaf cohomology in the analytic space \( X(\mathbb{C}) \) with its analytic topology. All the game of Hodge structures is to compare two cohomologies that live in completely different worlds. Luckily, the Godement resolution of Example A.104 has so good properties that solves for us both difficulties.

We start with the de Rham complex \( \Omega^*_{X/k} \), define on it the weight filtration as the trivial filtration
\[
W_{-1}\Omega^*_{X/k} = \{0\}, \quad W_0\Omega^*_{X/k} = \Omega^*_{X/k},
\]
and the Hodge filtration as the bête filtration
\[
F^p\Omega^*_{X/k} = \Omega^{* \geq p}_{X/k}.
\]
For each seaf \( \Omega^p_{X/k} \) we construct the Godement resolution \( C^*(X, \Omega^p_{X/k}) \). Thanks to the functorial properties of the Godement resolution, \( C^*(X, \Omega^*_{X/k}) \) is a double complex (with the appropriate choice of signs) with induced weight and Hodge filtrations. Then the de Rham part of the sought mixed Hodge complex is the complex of global sections of the total complex of that double complex:
\[
(A_{dR}, W, F) = (\Gamma(X, \text{Tot}(C^*(X, \Omega^*_{X/k}))), W, F).
\]

We now look at the complex manifold \( X(\mathbb{C}) \) and let \( \underline{\mathbb{Q}} \) be the constant sheaf on this manifold. Since manifolds are locally contractible, the singular cohomology of \( X(\mathbb{C}) \) with rational coefficients agrees with the sheaf cohomology of \( \underline{\mathbb{Q}} \). Define the weight filtration of \( \underline{\mathbb{Q}} \) as the trivial filtration
\[
W_{-1}\underline{\mathbb{Q}} = \{0\}, \quad W_0\underline{\mathbb{Q}} = \underline{\mathbb{Q}}.
\]
Then the Godement resolution \( C(X(\mathbb{C}), \underline{\mathbb{Q}}) \) has an induced weight filtration and we define the Betti part of the mixed Hodge complex again as the complex of global sections of that complex:
\[
(A_B, W) = (\Gamma(X(\mathbb{C}), C^*(X(\mathbb{C}), \underline{\mathbb{Q}})), W).
\]

Now we need to compare both sides. That is, we need a complex that receives arrows from both complexes, and these arrows are filtered quasi-isomorphisms. To this end we use the complex of holomorphic differential forms \( \Omega^*_{X(\mathbb{C})} \). We introduce again the weight filtration as the trivial filtration,
we apply the Godement resolution to each individual sheaf and take global sections of the total complex of the resulting double complex:

\((A_C, W) = (\Gamma(X(\mathbb{C}), \text{Tot}(\mathcal{C}^*(X(\mathbb{C}), \Omega^*_X(\mathbb{C})))), W)\)

Next we need the comparison maps. The map \(\alpha\) is easy because the complexes involved are both global sections of sheaves living in the same topological space. Since \(\mathbb{Q}\) agrees with the sheaf of locally constant functions on \(X(\mathbb{C})\) and locally constant functions are holomorphic, we deduce a map

\[\mathbb{Q} \to \mathcal{O}_{X(\mathbb{C})} = \Omega^0_{X(\mathbb{C})} \to \Omega^*_X(\mathbb{C}).\]

By the functoriality of the Godement resolution we deduce a map

\[\alpha: A_B \otimes \mathbb{Q} \mathbb{C} \to A_C\]

that, thanks to the Poincaré Lemma is a quasi-isomorphism, and hence a filtered quasi-isomorphism with respect to the weight filtration.

The map \(\beta\) is more complicated as we have to change not only sheaves but also spaces. There is a continuous map between the manifold \(X(\mathbb{C})\) with the analytic topology and the underlying topological space of \(X\) with its Zariski topology. Denote momentarily this map as

\[\psi: X(\mathbb{C}) \to X.\]

Applying Lemma A.105, for each \(p\), we obtain a map

\[\psi^{-1}(\mathcal{C}(X, \Omega^p_{X/k})) \to \mathcal{C}(X(\mathbb{C}), \psi^{-1}\Omega^p_{X/k}).\]

Since an algebraic differential form is always holomorphic, we also have a map of sheaves

\[\psi^{-1}\Omega^p_{X/k} \to \Omega^p_{X(\mathbb{C})}.\]

Taking the Godement resolution of this last map, global sections and total complexes we deduce a map

\[\beta: A_{dR} \otimes_k \mathbb{C} \to A_C\]

that, thanks to the GAGA principle is a quasi-isomorphism, and hence a filtered quasi-isomorphism with respect to the (trivial) weight filtration.

For future reference we wrap the previous complexes in a single symbol.

**Definition 2.165.** Let \(X\) be a smooth projective variety over \(k\). We denote by \(A^H_X\) the mixed Hodge complex constructed in this section.

**Proposition 2.166.** Let \(X\) be a smooth projective variety over \(k\). The mixed Hodge complex \(A^H_X\) induces in the cohomology of \(X\) the Hodge structure of Theorem 2.121. Moreover, the assignment \(X \mapsto A^H_X\) is functorial, so, if \(f: X \to Y\) is a morphism of smooth projective varieties over \(k\), then there is an induced morphism of mixed Hodge complexes \(A^H_f: A^H_Y \to A^H_X\). Moreover the morphism \(A^H_f\) induces the morphism of pure Hodge structures \(f^*: H^*(Y) \to H^*(X)\).
2.7.3. *The smooth case.* Let $X$ be a smooth quasi-projective variety over a subfield $k$ of $\mathbb{C}$. By Theorem 2.88, there is a canonical isomorphism

$$\text{comp}_{B, \text{dR}}: H^n_{\text{dR}}(X) \otimes_k \mathbb{C} \simeq H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C}. \quad (2.167)$$

We want to endow $H^n_B(X)$ with a filtration $W^B_\bullet$ and $H^n_{\text{dR}}(X)$ with two filtrations $W^\text{dR}_\bullet$ and $F^\bullet$ making the triple

$$( (H^n_B(X), W^B_\bullet), (H^n_{\text{dR}}(X), F^\bullet, W^\text{dR}_\bullet), \text{comp}_{B, \text{dR}} )$$

into a mixed Hodge structure over $k$. However, if de Rham cohomology is computed using the complex $\Omega^\bullet_X$ as in Definition 2.43, we face two problems:

(a) One may define a Hodge filtration using the *bête* filtration $\Omega^{\geq p}_X$, but it will not give much information. For example, if $X$ is affine, we saw in Remark 2.48 that $H^n_{\text{dR}}(X)$ is the cohomology of the global de Rham complex, so in this case the definition would yield the trivial filtration $F^n H^n_{\text{dR}}(X) = H^n_{\text{dR}}(X)$.

(b) There is no obvious way to get the weight filtration from $\Omega^\bullet_X$.

To solve these difficulties, we shall instead use the complex of logarithmic differentials, as introduced in Section 2.2.8. In view of Proposition 2.73, the strategy is to define the Hodge and the weight filtrations on the complex $\Omega^\bullet_X \langle \log D \rangle$. The Hodge filtration is given by the *bête* filtration of the complex of logarithmic differentials, that is

$$F^p \Omega^{\geq p}_X \langle \log D \rangle = \Omega^{\geq p}_X \langle \log D \rangle. \quad (2.168)$$

Note that $F^\bullet$ is defined over $k$. The weight filtration is given by the order of poles:

$$W_m \Omega^p_X \langle \log D \rangle = \begin{cases} 0 & m < 0, \\ \Omega^{p-m}_X \langle \log D \rangle & 0 \leq m \leq p, \\ \Omega^p_X \langle \log D \rangle & m \geq p. \end{cases}$$

Once we have a complex of sheaves with two filtrations, in order to produce the de Rham part of a mixed Hodge complex we follow the same strategy used in the smooth projective case. Namely, we define $A'_{\text{dR}}$ as the complex of global sections of the total complex of the Godement resolution of $\Omega^\bullet_X \langle \log D \rangle$ with the induced weight and Hodge filtrations.

$$(A'_{\text{dR}}, W, F) = (\Gamma(X, \text{Tot}(\mathcal{C}^\bullet(X, \Omega^\bullet_X \langle \log D \rangle))), W, F)$$

The weight filtration defined by the order of the poles does not look a priori as a “topological” filtration, so it is not clear how to translate it to the Betti side. The key idea now is to use a different filtration that has a more topological flavour. This is why we marked this complex with a ’ because we want to replace it with a quasi-isomorphic one that is easier to compare with the Betti side.
The canonical filtration on $\Omega^p_X(\log D)$, as defined in Example A.67 from the appendix, is the filtration

$$\tau_{\leq n} \Omega^p_X(\log D) = \begin{cases} 
\Omega^p_X(\log D), & \text{if } p < n \\
\text{Ker } d, & \text{if } p = n, \\
\{0\}, & \text{if } p > n.
\end{cases}$$

In fact, in order to define the weight filtration in cohomology, we do not need logarithmic differentials. Consider the complex of sheaves $j^* \Omega^\ast_X$ on $\overline{X}$. Note that, since $j$ is an affine morphism and the $\Omega^p_X$ are coherent sheaves, all higher direct images vanish. Let $\tau$ denote also the canonical filtration of this complex.

The following result is [Del71, Proposition 3.1.8].

Proposition 2.169. The arrows

$$(\Omega^\ast_X(\log D), W) \leftarrow (\Omega^\ast_X(\log D), \tau) \rightarrow (j^* \Omega^\ast_X, \tau)$$

are filtered quasi-isomorphisms.

In view of this proposition, we define the Godement resolution of $\Omega^\ast_X(\log D)$ with the induced weight and Hodge filtrations:

$$(A_{dR}, W, F) = (\Gamma(\overline{X}, \text{Tot}(C^\ast(X, \Omega^\ast_X(\log D))))), \tau, F).$$

Note that this complex is bifiltered quasi-isomorphic to $A'_{dR}$.

Proposition 2.169 also gives us the idea to define the Betti part of the mixed Hodge complex. Let $\mathbb{Q}$ be the constant sheaf on $X^{an}$, let $C^\ast(X^{an}, \mathbb{Q})$ be its Godement resolution, and let $j: X^{an} \to \overline{X}^{an}$ denote the open immersion of complex manifolds. Since the sheaves composing the complex $C^\ast(X^{an}, \mathbb{Q})$ are flasque, they are acyclic with respect to the functor $j_*$. Therefore, the complex $j_* C^\ast(X^{an}, \mathbb{Q})$ is isomorphic in the derived category of sheaves to $j_* \mathbb{Q}$. Let now $\tau$ denote again the canonical filtration, but this time of the complex $j_* C^\ast(X^{an}, \mathbb{Q})$. Note finally that, since each sheaf $C^p(X^{an}, \mathbb{Q})$ is flasque, the same is true for $j_* C^p(X^{an}, \mathbb{Q})$. Therefore,

$$H^\ast(\Gamma(\overline{X}^{an}, j_* C^\ast(X^{an}, \mathbb{Q}))) = H^\ast(X^{an}, \mathbb{Q}).$$

Therefore, we define the Betti part of the mixed Hodge complex as

$$(A_B, W) = (\Gamma(\overline{X}^{an}, j_* C^\ast(X^{an}, \mathbb{Q})), \tau).$$

As in the case of smooth projective varieties, the comparison between the de Rham and the Betti side is done with holomorphic differential forms on $X$. To them we apply the Godement resolution, the total complex, direct image by $j$, apply the canonical filtration and finally take global sections. So

$$A_C = \Gamma(\overline{X}^{an}, j_* \text{Tot}(C(X^{an}, \Omega^\ast_X))).$$
The comparison maps $\alpha$ and $\beta$ are induced, after going from algebraic forms to holomorphic forms, by the following diagram of sheaves:

$$
\xymatrix{
C^*(\Omega^\infty_X(\log D)) \ar[r] & C^*(j_*\Omega^\infty_X) \ar[r]^-* & j_*C^*(\Omega^\infty_X) \\
 & j_*C^*(X^\infty, \mathbb{Q}) \ar[u]
}
$$

where the arrow marked with $\ast$ follows from Lemma A.105.

**Definition 2.170.** Let $X$ be a smooth projective variety over $k$ and $j : X \to \overline{X}$ a smooth compactification with $D = \overline{X} \setminus X$ a simple normal crossing divisor. We denote by $A^H_X(\log D)$ the mixed Hodge complex constructed in this section.

**Proposition 2.171.** Let $X$ be a smooth variety defined over $k$ and let $\overline{X}$ a smooth compactification with $D = \overline{X} \setminus X$ a simple normal crossing divisor. Then the mixed Hodge complex $A^H_X(\log D)$ induces in the cohomology of $X$ the Hodge structure of Theorem 2.124. Moreover, the assignment

$$X \mapsto A^H_X(\log D)$$

is functorial with respect to pairs of compactifications. Namely, if $f : X \to Y$ is a morphism of smooth varieties over $k$, we can form a commutative diagram

$$
\xymatrix{
X \ar[r]^f \ar[d] & Y \ar[d] \\
\overline{X} \ar[r] & \overline{Y}
}
$$

with, both $D_X = \overline{X} \setminus X$ and $D_Y = \overline{Y} \setminus Y$ simple normal crossing divisors. Then there is a map $f^* : A^H_Y(\log D_Y) \to A^H_X(\log D_X)$ that induces the morphism of mixed Hodge structures $H^*(Y) \to H^*(X)$.

Consider the weight and Hodge filtrations induced on cohomology

$$W^m_{dr}H^n_{dr}(X) = \text{Im}(H^n(X, W^m_{-n}\Omega^\infty_X(\log D)) \to H^n_{dr}(X)), \quad (2.172)$$

$$F^p_{dr}H^n_{dr}(X) = \text{Im}(H^n(X, F^p\Omega^\infty_X(\log D)) \to H^n_{dr}(X)), \quad (2.173)$$

We refer the reader e.g. to [Del71] or [PS08, §4] for a proof that the filtrations we have introduced define a mixed Hodge structure on $H^B_{dr}(X)$.

**Definition 2.174.** We say that a mixed Hodge structure $H$ has weights in a subset $I \subseteq \mathbb{Z}$ if $\text{Gr}^m_W H = 0$ whenever $m \notin I$.

It follows from $(2.172)$ that the cohomology group $H^B_{dr}(X)$ of a smooth variety $X$ has weights in $[n, 2n]$. Moreover, noting that $W_0\Omega^\infty_X(\log D) = \Omega^\infty_X$ and the shift of indices in $(2.172)$, one finds that the first step in the weight filtration is the piece of the cohomology coming from the compactification:

$$W_nH^B_{dr}(X) = \text{Im}(H^B_{dr}(\overline{X}) \to H^B_{dr}(X)).$$
In contrast, when $X$ is projective, the mixed Hodge structure $H^n(X)$ defined in [Del74] has weights in $[0, n]$. The combination of these two statements implies that the cohomology of a smooth projective variety carries a pure Hodge structure.

As we have seen, the definition of de Rham cohomology involves hypercohomology of sheaves; therefore, to compute it concretely, in general we cannot use directly the algebraic de Rham complex but we need a resolution of it, like the Godement resolution. As we have seen in Remark 2.48 for an affine variety $X$, every coherent sheaf is acyclic and we can represent de Rham cohomology with algebraic differentials directly. Nevertheless, the Hodge structure involves a hypercohomology computed on a projective compactification of $X$; therefore, even in the case of affine varieties, in order to compute the Hodge structure we will need an acyclic resolution of the complex of logarithmic differentials, compatible with the weight and the Hodge filtrations.

**Example 2.175.** Let us compute everything for $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, viewed as a variety over $\mathbb{Q}$. As for any smooth curve, there is a canonical smooth compactification, in this case $\overline{X} = \mathbb{P}^1$. Write $D = \{0, 1, \infty\}$ for the divisor at infinity. Recall that $\mathcal{O}_{\mathbb{P}^1}(D)$ stands for the sheaf of rational functions having at most simple poles at $D$ and nowhere else. We have:

$$\Omega^0_{\mathbb{P}^1}(\log D) = \mathcal{O}_{\mathbb{P}^1}, \quad \Omega^1_{\mathbb{P}^1}(\log D) = \mathcal{O}_{\mathbb{P}^1}(D) \otimes \mathcal{O}_{\mathbb{P}^1} \Omega^1_{\mathbb{P}^1}.$$

Since $\Omega^1_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, one sees that $\Omega^1_{\mathbb{P}^1}(\log D) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. By the standard computation of the cohomology of line bundles on $\mathbb{P}^1$ [Har77, III, §5], none of the terms in the complex of logarithmic differentials has higher cohomology. Besides, setting $\omega_0 = \frac{dt}{t}$ and $\omega_1 = \frac{dt}{t-1}$, one has:

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{Q}, \quad H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D)) = \mathbb{Q} \omega_0 \oplus \mathbb{Q} \omega_1$$

(note that these differentials $\omega_0$ and $\omega_1$ have a simple pole at $\infty$ as well). From the spectral sequence (A.108), it follows that

$$H^*_{\text{dR}}(X) = H^*(\mathcal{O}_{\mathbb{P}^1} \xrightarrow{d} \mathcal{O}_{\mathbb{P}^1}(D) \otimes \mathcal{O}_{\mathbb{P}^1} \Omega^1_{\mathbb{P}^1})$$

$$= H^*(\mathbb{Q} \longrightarrow \mathbb{Q} \omega_0 \oplus \mathbb{Q} \omega_1),$$

where the differential in the second complex is the zero map. Thus,

$$H^1_{\text{dR}}(X) = \mathbb{Q} \omega_0 \oplus \mathbb{Q} \omega_1.$$

We now turn to the filtrations. For the Hodge filtration, (2.168) gives

$$H^1_{\text{dR}}(X) = F^0 = F^1 \supseteq F^2 = \{0\}.$$ 

Moreover, the weight filtration on the complex of logarithmic differentials is given by $\Omega_{\mathbb{P}^1}^\bullet = W_0 \subseteq W_1 = \Omega_{\mathbb{P}^1}^\bullet(\log D)$. Since $H^1_{\text{dR}}(\mathbb{P}^1)$ vanishes, we find:

$$\{0\} = W_1 \subseteq W_2 = H^1_{\text{dR}}(X).$$
On the other hand, the first homology group $H_1(X(\mathbb{C}), \mathbb{Q})$ has as a basis the classes of two loops $\sigma_0$ and $\sigma_1$ winding once counterclockwise around the punctures 0 and 1. By Cauchy’s residue theorem, the period matrix reads:

$$\begin{pmatrix}
\int_{\sigma_0} \omega_0 & \int_{\sigma_1} \omega_0 \\
\int_{\sigma_0} \omega_1 & \int_{\sigma_1} \omega_1
\end{pmatrix} = \begin{pmatrix} 2\pi i & 0 \\
0 & 2\pi i \end{pmatrix}.$$

In other words, if $\sigma_0^\vee$ and $\sigma_1^\vee$ are the dual elements in cohomology, the isomorphism $\text{comp}_{\text{B},\text{dR}}$ sends $\omega_0$ to $\sigma_0^\vee \otimes 2\pi i$ and $\omega_1$ to $\sigma_1^\vee \otimes 2\pi i$. Comparing with Example 2.120, one concludes that

$$H_1^1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \simeq \mathbb{Q}(-1)^\oplus 2$$

as mixed Hodge structures over $\mathbb{Q}$.

Observe that all the information in the mixed Hodge structure over $\mathbb{Q}$ of the variety $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ can be read from the complex

$$A^* = A^0 \oplus A^1, \quad A^0 = \mathbb{Q}, \quad A^1 = \mathbb{Q}\omega_0 \oplus \mathbb{Q}\omega_1, \quad (2.176)$$

together with the trivial differential and the filtrations

$$F^0 = A^* \supset F^1 = A^1 \supset F^2 = \{0\},$$

$$W_0 = 0 \subset W_1 \subset W_2 = A^*. \quad (2.177)$$

Note that $A^*$ has an algebra structure given by $\omega_i \wedge \omega_j = 0$, for $i, j \in \{0, 1\}$.

For later reference, we summarize the results of this example in a proposition. We say that a morphism $f: (A^*, W, F) \to (A'^*, W', F')$ between two complexes provided with two filtrations is a bifiltered quasi-isomorphism if $f$ is compatible with the filtrations and the induced maps

$$\text{Gr}_F^p \text{Gr}_W^q A \to \text{Gr}_F'^p \text{Gr}_W'^q A'$$

are quasi-isomorphisms for all $p$ and $n$.

**Proposition 2.178.** Set $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ and let $A^* = A^*_{\text{dR}}$ be the dg-algebra introduced in (2.176) and (2.175). The algebraic de Rham cohomology of $X$ is given by

$$H^*_{\text{dR}}(X) = H^*(A^*).$$

The Hodge and the weight filtration are induced by the filtrations (2.175):

$$F^p H^*_{\text{dR}}(X) = H^*(F^p A^*),$$

$$W_k H^*_{\text{dR}}(X) = H^*(W_k A^*).$$

Moreover, the inclusion of algebras

$$A^* \to E^*_{\mathbb{P}^1(\mathbb{C})}(\log D)$$

induces a bifiltered quasi-isomorphism

$$(A^* \otimes_{\mathbb{Q}} \mathbb{C}, W, F) \to (E^*_{\mathbb{P}^1(\mathbb{C})}(\log D), W, F).$$
2.7.4. Normal crossing divisors. Following the same method we used to define the de Rham cohomology of a normal crossing divisor in Section 2.2.6, we can construct a mixed Hodge complex that endows it with a mixed Hodge structure. In what follows, we discuss the case of a simple normal crossing divisor on a smooth projective variety.

Let $X$ be a smooth variety over $k$ and $D$ a simple normal crossing divisor on $X$. We keep notation from Section 2.2.6, in particular, $D_I$ for the intersection of irreducible components indexed by a subset $I$ and $D^p$ for the disjoint union of all $D_I$ such that $I$ has cardinal $p$. By resolution of singularities (Theorem 2.70), we can choose a smooth compactification $\bar{X}$ of $X$ such that, for each subset $I$, the Zariski closure $\overline{D_I}$ is a smooth compactification of $D_I$ whose complement $E_I = \overline{D_I} \setminus D_I$ is a simple normal crossing divisor.

The mixed Hodge complexes $A^H_{D^p}(\log E_I)$ form a dg-mixed Hodge complex with the same differentials used in loc. cit. Then

$$A^H_B(\log E) = \text{Tot}(A^H_{D^p}(\log E_I)) \quad (2.179)$$

is a mixed Hodge complex that defines a mixed Hodge structure in the cohomology of $D$. In case that $X$ is projective and there is no need to compactify we will denote the complex as $A^H_B$.

Example 2.180. We consider Example 2.83 again. Now, instead of computing the relative cohomology $H^*(X, D)$ we will compute the cohomology $H^*(D)$ with its mixed Hodge structure. The added information is that, by Proposition 2.164, there is a spectral sequence of mixed Hodge structures. Taking into account that the mixed Hodge structure in the $H^0$ of an irreducible smooth variety is always a copy of $\mathbb{Q}(0)$ we obtain that the $E_1$ term of the spectral sequence reads

$$\begin{array}{c}
0 \\
\mathbb{Q}(0) \oplus \mathbb{Q}(0) \oplus \mathbb{Q}(0) \longrightarrow \mathbb{Q}(0) \oplus \mathbb{Q}(0) \oplus \mathbb{Q}(0)
\end{array}$$

where the horizontal map is $(a, b, c) \mapsto (b - a, c - a, c - b)$. From this we easily deduce that

$$H^0(D) = H^1(D) = \mathbb{Q}(0).$$

In fact, using hyper-resolutions, the technique we have used for a normal crossing divisor can be extended to any quasi-projective variety [Del74].

2.7.5. Mixed Hodge structures on relative cohomology. Similarly, one can endow the cohomology with compact support and the relative cohomology with mixed Hodge structures. The basic technique is the following. Let $f: A \to B$ be a morphism of mixed Hodge complexes. Then, changing the sign of the differential of the second complex, we can see $f$ as a dg-mixed Hodge complex. Then

$$\text{Tot}(f) = \text{cone}(f)[-1]$$
is a mixed Hodge complex. We apply this technique to two situations. The cohomology with compact support and the cohomology with support in a subvariety.

Let $X$ be a smooth quasi-projective variety over $k$ and $\overline{X}$ a smooth compactification with $D = \overline{X} \setminus X$ can be computed as the relative cohomology

$$H^*_c(X) = H^*(\overline{X}, D).$$

This cohomology does not depend on the choice of a compactification and sits in a long exact sequence of mixed Hodge structures

$$\cdots \to H^n_c(X) \to H^n(\overline{X}) \to H^n(D) \to \cdots$$

Let $A^H_X$ and $A^H_D$ be the mixed Hodge complexes of sections 2.7.2 and 2.7.4. Then there is a map $f: A^H_X \to A^H_D$ and the mixed Hodge structure of $H^n_c(X)$ is the one induced by Tot$(f)$.

More generally, if $X$ is a smooth quasi-projective variety and $D$ a simple normal crossing divisor of $X$, one can find a compactification as in Section 2.7.4 and the relative cohomology of $(X, D)$ has a mixed Hodge structure induced by Tot$(f)$ where $f: A^H_X(\log \overline{X} \setminus X) \to A^H_D(\log D \setminus D)$.

**Example 2.181.** Let $X$ and $D$ be as in Example 2.64. Then the spectral sequence considered in that example is an exact sequence of mixed Hodge structures that reads

$$\begin{array}{c}
0 \\
0 \\
\mathbb{Q}(0) \to \mathbb{Q}(0) \oplus \mathbb{Q}(0) \oplus \mathbb{Q}(0) \to \mathbb{Q}(0) \oplus \mathbb{Q}(0) \oplus \mathbb{Q}(0) \\
\end{array}$$

from which we derive that

$$H^i(X, D) = \begin{cases} 
\mathbb{Q}(0), & \text{if } i = 2, \\
\{0\}, & \text{otherwise.}
\end{cases}$$

Similarly, let $X$ be a smooth quasi-projective variety, $Z \subset X$ a closed subvariety, and $U = X \setminus Z$ its complement. By resolution of singularities (Theorem 2.70), we can find compactifications $\overline{X}$ and $\overline{U}$ fitting in a commutative square

$$\begin{array}{c}
\begin{array}{c}
U \\
\downarrow \\
\overline{U}
\end{array} \xrightarrow{f} \\
\begin{array}{c}
X \\
\downarrow \\
\overline{X}
\end{array}
\end{array}$$
and such that $D = \overline{X} \setminus X$ and $E = \overline{U} \setminus U$ are simple normal crossing divisors. By functoriality, there is a map of Hodge complexes

$$f: A^H_X(\log D) \to A^H_U(\log E).$$

**Definition 2.182.** The mixed Hodge structure on the cohomology with support is defined as

$$H^*_Z(X) = H^*(X,U) = H^*(\text{Tot}(f)).$$

By construction, $H^*_Z(X)$ sits into a long exact sequence of mixed Hodge structures

$$\cdots \to H^2_Z(X) \to H^n(X) \to H^n(U) \to H^{n+1}_Z(X) \to \cdots$$

For an example, see Exercise 2.199.

2.7.6. **Poincaré duality and the Gysin morphism.** The cup-product in cohomology is also a morphism of mixed Hodge structures.

**Proposition 2.183.** Let $X$ be a quasi-projective variety over $k$ then the cup-product is a morphism of mixed Hodge structures over $k$,

$$H^n(X) \otimes H^m(X) \to H^{n+m}(X).$$

Another useful property is:

**Lemma 2.184.** If $X$ is a smooth irreducible projective variety of dimension $n$ over $k$, then

$$H^{2n}(X) = \mathbb{Q}(-n).$$

**Proof.** By Poincaré duality, we know that $H^{2n}(X,\mathbb{Z}) = \mathbb{Z}$. Let $X \subset \mathbb{P}^N$ be an embedding into a projective space. Then the map

$$H^{2n}(\mathbb{P}^N,\mathbb{Q}) \to H^{2n}(X,\mathbb{Q})$$

that sends the class of a general linear subvariety to its intersection with $X$ is an isomorphism. By Proposition 2.134, the map $H^{2n}(\mathbb{P}^N) \to H^{2n}(X)$ is an isomorphism of mixed Hodge structures. Since $H^{2n}(\mathbb{P}^N) = \mathbb{Q}(-n)$ we obtain the result. $\square$

Putting together Proposition 2.183 and Lemma 2.184 we deduce that Poincaré duality is a morphism of mixed Hodge structures after a twist.

**Proposition 2.185 (Poincaré duality).** Let $X$ be a smooth projective variety of dimension $d$ over $k$, then Poincaré duality gives an isomorphism of mixed Hodge structures

$$H^n(X) \simeq \text{Hom}(H^{2d-n}(X), \mathbb{Q}(-d)).$$

Poincaré duality can be used to define an exceptional functoriality.
Definition 2.186. Let \( Z \) be a smooth closed irreducible subvariety of codimension \( p \) of a smooth variety \( X \) of dimension \( d \). Then, for each \( n \geq 0 \) the Gysin map \( H^n(Z)(-p) \to H^{n+2p}(X) \) is defined as the composition

\[
H^n(Z)(-p) \xrightarrow{\cong} \operatorname{Hom}(H^{2d-2p-n}(Z), \mathbb{Q}(-d+p))(-p) = \operatorname{Hom}(H^{2d-2p-n}(Z), \mathbb{Q}(-d))
\]

\[
\downarrow \quad \operatorname{Hom}(H^{2d-2p-n}(Z), \mathbb{Q}(-d)) \cong H^{n+2p}(X).
\]

The Gysin map has the following very useful property:

Proposition 2.187 (Gysin long exact sequence). Let \( X \) be a smooth variety over \( k \) and \( Z \subseteq X \) a smooth closed subvariety of codimension \( p \). Set \( U = X \setminus Z \). Then, for each \( n \geq 0 \), the Gysin map can be lifted to an isomorphism of mixed Hodge structures \( H^n(Z)(-p) \to H^{n+2p}(X) \). Therefore, there is a long exact sequence of mixed Hodge structures

\[
\cdots \to H^{j-1}(X) \xrightarrow{\alpha} H^{j-1}(U) \xrightarrow{\beta} H^{j-2p}(Z)(-p) \xrightarrow{\gamma} H^j(X) \to \cdots \quad (2.188)
\]

where \( \alpha \) is the usual restriction map and \( \gamma \) is the Gysin map.

As we will see in Exercise 2.202, the smoothness assumption on \( Z \) is necessary.

2.7.7. More examples. We close this section with a few more examples of Hodge structures of mixed Tate type.

Example 2.189 (Smooth open curves). Let \( \overline{C} \) be a smooth projective complex curve and let \( S \subset \overline{C} \) be a non-empty finite subset consisting of \( s \) points. We describe the mixed Hodge structure on the first cohomology group of the open curve \( C = \overline{C} \setminus S \). Since \( S \) is non-empty, the curve \( C \) is affine, and hence \( H^2(C) = 0 \) by Exercise 2.103. Using this vanishing, the Gysin long exact sequence (2.188) reads

\[
0 \to H^1(\overline{C}) \to H^1(C) \to H^0(S)(-1) \to H^2(C) \to 0.
\]

By Lemma 2.184, the last two terms are isomorphic to \( \mathbb{Q}(-1)^{\oplus s} \) and \( \mathbb{Q}(-1) \) and, through this identification, the Gysin map \( \gamma \) is given by the sum of the coordinates. From this we get a short exact sequence

\[
0 \to H^1(\overline{C}) \to H^1(C) \to \mathbb{Q}(-1)^{\oplus (s-1)} \to 0.
\]

The weight filtration is given by

\[
0 = W_0 H^1(C) \subset W_1 H^1(C) = H^1(\overline{C}) \subset W_2 H^1(C) = H^1(C),
\]

so the graded pieces are

\[
\operatorname{Gr}_1^W H^1(C) \simeq H^1(\overline{C}), \quad \operatorname{Gr}_2^W H^1(C) \simeq \mathbb{Q}(-1)^{\oplus (s-1)},
\]
which are indeed pure Hodge structures of weights 1 and 2 respectively. In particular, the mixed Hodge structure $H^1(C)$ is of Tate type if and only if $H^1(C) = 0$, which is equivalent to asking that the curve is a punctured projective line $C = \mathbb{P}^1 \setminus S$.

**Example 2.190 (Moduli spaces $M_{0,n}$).** Let us compute the Hodge structure on the cohomology of the moduli spaces $M_{0,n}$ from paragraph 2.4.2.

**Proposition 2.191.** For each $i \geq 0$, the cohomology group $H^i(M_{0,n})$ carries a pure Hodge–Tate structure of weight $2i$. More precisely,

$$H^i(M_{0,n}) = \mathbb{Q}(-i)^{\oplus b_{i,n}}$$

where the Betti numbers $b_{i,n}$ are given by the generating series

$$\sum_{i \geq 0} b_{i,n} t^i = (1 + 2t)(1 + 3t) \cdots (1 + (n - 2)t).$$

**Proof.** We proceed by induction on $n$. When $n = 3$, the moduli space is reduced to a point, and hence the only non-zero cohomology group is $H^0(M_{0,3}) = \mathbb{Q}(0)$. The case $n = 4$ was settled in Example 2.189, where we saw that the non-trivial cohomology groups are $H^0(M_{0,4}) = \mathbb{Q}(0)$ and $H^1(M_{0,4}) = \mathbb{Q}(-1)^{\oplus 2}$. Let $(0, 1, \infty, t_1, \ldots, t_{n-3})$ denote the coordinates on $M_{0,n}$. For each $n \geq 5$, the map

$$M_{0,n} \rightarrow (M_{0,4} \times M_{0,n-1})$$

$$(0, 1, \infty, t_1, \ldots, t_{n-3}) \mapsto ((0, 1, \infty, t_1), (0, 1, \infty, t_2, \ldots, t_{n-3}))$$

induces an isomorphism between $M_{0,n}$ and the complement of the smooth closed subvariety $Z \subset M_{0,4} \times M_{0,n-1}$ given by

$$Z = \bigsqcup_{i=2}^{n-3} \{t_i = t_1\} \simeq \bigsqcup_{i=2}^{n-3} M_{0,n-1}.$$

We shall compute the cohomology of $M_{0,n} \simeq (M_{0,4} \times M_{0,n-1}) \setminus Z$ by combining the Gysin exact sequence, the Künneth formula and the induction hypothesis. First, the Gysin sequence (2.188) gives

$$\cdots \rightarrow H^{i-2}(Z)(-1) \overset{\alpha}{\rightarrow} H^i(M_{0,4} \times M_{0,n-1}) \rightarrow H^i(M_{0,n}) \rightarrow H^{i-1}(Z)(-1) \overset{\beta}{\rightarrow} H^{i+1}(M_{0,4} \times M_{0,n-1}) \rightarrow \cdots$$

(2.192)

By the Künneth formula and the induction hypothesis, we have

$$H^i(M_{0,4} \times M_{0,n-1}) \cong \bigoplus_{a+b=i} H^a(M_{0,4}) \otimes H^b(M_{0,n-1})$$

$$\cong H^i(M_{0,n-1}) \oplus H^{i-1}(M_{0,n-1})(-1)^{\oplus 2}$$

$$\cong \mathbb{Q}(-1)^{\oplus (b_{i-1,n-1} + 2b_{i-1,n-1})}.$$
It follows that the maps \( \alpha \) and \( \beta \) in (2.192) are morphisms between pure Hodge structures of different weights, and hence identically zero. From this we derive the short exact sequence
\[
0 \to H^i(M_{0,4} \times M_{0,n-1}) \to H^i(M_{0,n}) \to H^{i-1}(M_{0,n-1})(-1)^\oplus(n-4) \to 0.
\]
By (2.193) and the induction hypothesis, the cohomology \( H^i(M_{0,n}) \) is an extension of two pure Hodge–Tate structures of the same weight. Since all such extensions are split by Theorem 2.152, it follows that
\[
H^i(M_{0,n}) = \mathbb{Q}(-i)^{b_i,n} \quad \text{with} \quad b_i,n = b_{i,n-1} + (n-2)b_{i-1,n-1}.
\]
One immediately checks that this recurrence relation amounts to the expression for the Betti numbers given in the statement.

2.7.8. Graph hypersurfaces. Let \( G = (V, E) \) be a finite graph with vertex and edge sets \( V \) and \( E \), respectively. Assume that \( G \) is connected. A subgraph \( T \subseteq G \) is called a spanning tree if \( T \) is a tree (i.e. connected with no loops) and contains all vertices of \( G \). Consider a collection of variables \((x_e)_{e \in E}\) indexed by the edges of \( G \). The first Symanzik polynomial of the graph is defined as
\[
\Psi_G = \sum_{T \subseteq G \atop e \notin T} \prod x_e \in \mathbb{Z}[(x_e)_{e \in E}],
\]
where the sum runs over all spanning trees in \( G \). Let \( n_G \) be the number of edges of \( G \) and \( h_G \) the number of loops. It is easy to see that \( \Psi_G \) is a homogenous polynomial of degree \( h_G \) (Exercise 2.206).

**Definition 2.195.** The graph hypersurface \( X_G \subseteq \mathbb{P}^{n_G-1} \) is the vanishing locus of the polynomial \( \Psi_G \).

Graph hypersurfaces appear in perturbative quantum field theory, a major goal of which is to compute Feynman amplitudes. These are the probabilities that a particle interaction is described by a given graph. The easiest case is when \( n_G = 2h_G \) and \( n_\gamma > 2h_\gamma \) for all non-empty strict subgraphs \( \gamma \subsetneq G \). Then the corresponding Feynmann integral is given, up to a normalization factor, by the convergent integral [BEK06, Prop. 5.2],
\[
I_G = \int_{\sigma} \frac{\Omega}{\psi_G^n},
\]
where we have chosen a numbering of the vertices, so that \( \psi_G \) becomes a polynomial in the variables \( x_0, \ldots, x_{n_G-1} \), the differential form \( \Omega \) is given by
\[
\Omega = \sum_{j=0}^{n_G-1} (-1)^j x_j dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n_G-1},
\]
and one integrates over the real coordinate simplex
\[
\sigma = \{[x_0: \cdots: x_{n_G-1}] \in \mathbb{P}^{n_G-1}(\mathbb{R}) \mid x_i \geq 0 \}.
\]
Note that the condition \( n_G = 2h_G \) implies that the integrand of (2.196), that is written in homogeneous coordinates, is well defined. Setting \( t_i = x_i / x_0 \), the amplitude \( I_G \) can be rewritten as the affine integral

\[
I_G = \int_0^\infty dt_1 \int_0^\infty dt_2 \cdots \int_0^\infty \frac{dt_{n_G-1}}{\psi^2_G(1, t_1, t_2, \ldots, t_{n_G-1})}.
\]

Graphs satisfying the above conditions are called primitive log divergent. Figure 8 gives examples of primitive log divergent graphs and the associated Feynman amplitudes.

![Graphs](image)

\[
6\zeta(3) \quad 20\zeta(5) \quad \frac{45}{4} \zeta(3, 5) + \frac{333}{20} \zeta(5, 3) - \frac{9}{8} \zeta(8)
\]

**Figure 8.** Three examples of primitive log divergent graphs and the corresponding Feynman amplitudes

It was conjectured for some time that the amplitudes \( I_G \) of primitive log divergent graphs were always \( \mathbb{Q} \)-linear combinations of multiple zeta values. This happens to be the case for graphs with \( h_G \leq 6 \). [BS12]

The integrand of (2.196) is a global top-degree differential form \( \omega_G \) on \( \mathbb{P}^{n_G-1} \setminus X_G \), and the boundary of the simplex \( \sigma \) is contained in the union \( D \) of the coordinate hyperplanes \( \{x_i = 0\} \). In general, \( \sigma \) intersects the graph hypersurface \( X_G \), so one is faced with the problem, already encountered in Section 2.4, that the integration cycle does not define an element in the naive relative cohomology group

\[
H^{n_G-1}(\mathbb{P}^{n_G-1} \setminus X_G, D \setminus D \cap X_G).
\]

However, the fact that the coefficients of \( \psi_G \) are positive makes this intersection easy to describe. In fact,

\[
X_G(\mathbb{C}) \cap \sigma = \bigcup_{h_\gamma > 0} L_\gamma(\mathbb{R}_{>0}),
\]
where, if $\gamma$ is a subgraph of $G$, then $L_\gamma$ is the linear subvariety of $\mathbb{P}^{n_G-1}$ of equations $x_e = 0$ for all vertices $e$ of $\gamma$ and

$$L_\gamma(\mathbb{R}_{\geq 0}) = \{[x_e]_{e \in E} \in L_\gamma(\mathbb{R}) \mid x_e \geq 0\}.$$ 

This allowed Bloch, Esnault, and Kreimer to prove the following result in [BEK06, Prop. 7.3]

**Theorem 2.197** (Bloch–Esnault–Kreimer). There exists a tower

$$\pi: P = P_r \longrightarrow \cdots \longrightarrow P_0 = \mathbb{P}^{n_G-1}$$

of blow-ups such that each $P_i$ is obtained by blowing up $P_{i-1}$ along the strict transform of a coordinate linear space $L_i$ and the following conditions hold:

1. The differential $\pi^* \omega_G$ has no poles along the exceptional divisors associated with the blow-ups.
2. The total transform $B$ of $D$ is a normal crossing divisor such that none of the non-empty intersections of its irreducible components is contained in the strict transform $Y$ of $X_G$.
3. The strict transform of $\sigma$ does not meet $Y$.

**Corollary 2.198.** Keeping the notation from the previous theorem, the Feynman amplitude $I_G$ is a period of the mixed Hodge structure

$$H^{n_G-1}(P \setminus Y, B \setminus (B \cap Y)).$$

***

**Exercise 2.199.** Set $X = \mathbb{P}^1_{\mathbb{Q}}$ and let $Z \subset X$ be a closed subvariety consisting of a rational point. Compute the mixed Hodge structure on the cohomology with support $H^*_Z(X)$ introduced in Definition 2.182.

**Exercise 2.200.** Let $X$ be a smooth complex variety, let $Z \subset X$ be a smooth subvariety of codimension $c$, and write $U = X \setminus Z$. Use the Gysin long exact sequence (2.188) to prove that the restriction map $H^i(X) \to H^i(U)$ is an isomorphism for $i < 2c - 1$, and is injective for $i = 2c - 1$.

**Exercise 2.201** (Varieties which admit a compactification by a smooth divisor). Let $U$ be a smooth complex variety. In this exercise, we show that the existence of a smooth compactification by a smooth divisor imposes strong restrictions on the mixed Hodge structure of $U$.

1. Use the Gysin exact sequence (2.188) to show that if $U = X \setminus D$, with $X$ smooth and projective and $D$ smooth, then $H^n(U)$ has only weights in $[n, n + 1]$.

2. Give an example of a smooth surface which does not admit a smooth projective compactification by a smooth divisor.

**Exercise 2.202.** In this exercise, we show that the smoothness assumption in Proposition 2.187 is necessary.
(1) Show that, if $X$ is a smooth projective variety and $Z$ a subvariety, then, for $n > 0$, $H^n_Z(X)$ has weights in $[n - 1, 2n - 2]$. In fact, it can be shown that it has weights in $[n, 2n - 2]$.

(2) Show that, if $X$ is a smooth projective variety and $D$ is a normal crossing divisor, then $H^{n-2}(D)(-1)$ has weights in $[2, n]$. In fact, a similar result holds for any closed subvariety.

(3) As seen in the previous points, the weights of the mixed Hodge structures $H^n_Z(X)$ and $H^{n-2p}(Z)(-p)$ do not need to match, therefore, in general they cannot be isomorphic. But even if the weight match, the map does not need to be an isomorphism. Consider $X = \mathbb{P}^2$ and $D = D_1 \cup D_2$ the union of two coordinate hyperplanes. Then show that

\[
H^n(D) = \begin{cases}
\mathbb{Q}(-1) \oplus \mathbb{Q}(-1), & \text{if } n = 2, \\
\{0\}, & \text{otherwise},
\end{cases}
\]

Therefore, $H^n_D(X) \not\cong H^{n-2}(D)(-1)$.

**Exercise 2.203.** Let $\overline{X}$ be a smooth projective complex variety and $Y_0, Y_1 \subseteq \overline{X}$ two smooth divisors such that $Y_0 \cup Y_1$ has normal crossings. Set $X = \overline{X} \setminus Y_0$ and $Y = Y_1 \setminus (Y_0 \cap Y_1)$. Show that the weight filtration on the relative cohomology group $M = H^n(X,Y)$ is given by

\[
W_{n-2}M = 0, \\
W_{n-1}M = \text{Im}(H^{n-1}(Y_1) \to M), \\
W_nM = \text{Ker}(M \to H^{n-1}(Y_0)(-1)), \\
W_{n+1}M = M.
\]

*[Hint: Consider a diagram of mixed Hodge structures whose rows are Gysin long exact sequences and whose columns are long exact sequences of relative cohomology. Use the fact that $W_m$ is an exact functor and Lemma 2.133.]*

**Exercise 2.204 (The graded pieces of the mixed Hodge structure of a smooth variety).** Let $X$ be a smooth projective variety, $D$ a simple normal crossing divisor an $U = X \setminus D$. Following Construction 2.59, we form

\[
D^0 = X, \quad D^p = \coprod_{i_1, i_2, \ldots, i_p} D_{i_1} \cap \cdots \cap D_{i_p}.
\]

Prove that the weight filtration of $H^n(U)$ is given by

\[
\text{Gr}_m^W H^n(U) = H^{n-m}(- \to H^{m-2}(D^1)(-1) \to H^m(X) \to 0),
\]

(2.205)
where the term $H_n^{n-2p}(D^p)(-p)$ sits in degree $-p$.

**Exercise 2.206.** Prove that the first Symanzik polynomial of a graph, as defined in (2.194), is homogeneous of degree the number of loops in $G$.

**Exercise 2.207 (Deletion-contraction relations).** Let $G$ be a connected graph and $e$ an edge of $G$. We denote by $G\setminus e$ the graph obtained by deleting the edge $e$ and by $G/e$ the graph obtained by contracting the edge $e$. Assume that $G\setminus e$ is still connected and that the two end points of $e$ are different. Show that the following relation holds:

$$\Psi_G = x_e \Psi_{G\setminus e} + \Psi_{G/e}.$$ 

**Exercise 2.208 (The trivial Feynman amplitude).** Consider the graph $G$ with two vertices and two edges connecting them, as in Figure 9. Compute the Feynman amplitude $I_G$ and write down a Hodge structure for which it is a period (no blow-up is needed in this case).

![Figure 9. A simple graph](image)

**2.8. Back to $\zeta(2)$ and irrationality proofs.** We end the chapter by showing that the relative cohomology group attached to $\zeta(2)$ in 2.4.1 is an extension of $\mathbb{Q}(-2)$ by $\mathbb{Q}(0)$. We then discuss the problem of constructing other extensions and a potential application to irrationality proofs.

**2.8.1. The extension associated with $\zeta(2)$.** We prove that the relative cohomology group constructed in paragraph 2.4.1 from the integral representation of $\zeta(2)$ is an extension of $\mathbb{Q}(-2)$ by $\mathbb{Q}(0)$. Recall that we considered the blow-up $X$ of $\mathbb{A}^2$ at the points $p = (0,0)$ and $q = (1,1)$, together with the normal crossing divisors

$$L = L_0 \cup L_1, \quad M = M_0 \cup M_1 \cup M_2 \cup M_3 \cup M_4,$$

where $L_0$ and $L_1$ are the strict transforms of $\{t_1 = 0\}$ and $\{t_2 = 1\}$ (affine lines), $M_0 = E_p$ and $M_1 = E_q$ are the exceptional divisors (projective lines), and $M_2, M_3$ and $M_4$ are the strict transforms of $\{t_1 = t_2\}, \{t_2 = 0\}$ and $\{t_1 = 1\}$ (affine lines again).

---

Many thanks to Clément Dupont and Peter Jossen for their help with this section.
PROPOSITION 2.209. There exists a short exact sequence of mixed Hodge structures

\[ 0 \to \mathbb{Q}(0) \to H^2(X \setminus L, M \setminus (L \cap M)) \to \mathbb{Q}(-2) \to 0. \] (2.210)

PROOF. Let \( X \) be any smooth complex variety, and \( L \) and \( M \) two normal crossing divisors on \( X \) with no common irreducible components and such that \( L \cup M \) has normal crossings as well. By [Dup17, App. A.1], there is a spectral sequence of mixed Hodge structures

\[ E_1^{p,q} = \bigoplus_{j - i = p \atop |I| = i \atop |J| = j} H^{q-2i}(L_I \cap M_J)(-i) \]

\[ \implies \operatorname{gr}^W H^{p+q}(X \setminus L, M \setminus (M \cap L)), \] (2.211)

where the differential \( d_1: E_1^{p,q} \to E_1^{p+1,q} \) is the sum of

1. the restriction maps

\[ H^{q-2i}(L_I \cap M_J)(-i) \to H^{q-2i}(L_I \cap M_{J \cup \{s\}})(-i) \]

induced from the inclusions \( L_I \cap M_{J \cup \{s\}} \hookrightarrow L_I \cap M_J \), multiplied by the signs \( \varepsilon(J,J \cup \{s\}) \);

2. the Gysin morphisms

\[ H^{q-2i}(L_I \cap M_J)(-i) \to H^{q-2i+2}(L_{I \setminus \{r\}} \cap M_J)(-i + 1) \]

associated with the inclusions \( L_I \cap M_J \hookrightarrow L_{I \setminus \{r\}} \cap M_J \), multiplied by the signs \( \varepsilon(I \setminus \{r\}, I) \).

Recall from 2.2.6 that, if \( J = \{j_0, \ldots, j_r\} \) is an index set with \( j_0 < \cdots < j_r \) and \( I \) is obtained from \( I \) by removing \( j_0 \), we set \( \varepsilon(I,J) = (-1)^{j_0} \).

Assume that all the terms \( E_1^{p,q} \) in the spectral sequence carry a pure Hodge structure of weight \( q \). The second page is given by

\[ E_2^{p,q} = \frac{\operatorname{Ker}(d_1: E_1^{p,q} \to E_1^{p+1,q})}{\operatorname{Im}(d_1: E_1^{p-1,q} \to E_1^{p,q})}, \]

together with a differential \( d_2: E_2^{p,q} \to E_2^{p+1,q-1} \). Thus, \( E_2^{p,q} \) has a pure Hodge structure of weight \( q \) as well, which implies \( d_2 = 0 \) since there are no non-trivial morphisms between Hodge structures of different weight. It follows that the spectral sequence degenerates at \( E_2 \) and

\[ E_2^{p,q} = \operatorname{gr}^W H^{p+q}(X \setminus L, M \setminus (M \cap L)). \] (2.212)

Let us now turn to our particular situation. Setting

\[ r = L_0 \cap L_1, \quad s = L_0 \cap E_p, \quad t = L_1 \cap E_q, \quad M_{ij} = M_i \cap M_j, \]

the spectral sequence takes the form of Figure 10. By way of illustration, the piece \( E_1^{1,2} \) is the sum of all possible \( H^{2-2i}(L_i \cap M_J)(-i) \) with \( j = i + 1 \). Then necessarily \( i = 0 \) or \( i = 1 \), and the second case does not appear since there
are no non-empty intersections of one component of $L$ and two components of $M$. For $i = 0$, we get $\bigoplus H^2(M_j) = H^2(E_p) \oplus H^2(E_q)$, taking into account that the remaining components are affine lines. Observe that odd values of $q$ do not need to be considered, since all intersections $L_I \cap M_J$ have only cohomology in even degrees. For the same reason, the assumption that $E_1^{p,q}$ has pure weight $q$ is satisfied in our case.

\[
\begin{array}{cccccc}
H^0(r)(-2) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & H^0(L_0)(-1) & \oplus & H^2(X) & \oplus & H^2(E_p) \\
0 & \oplus & H^0(s)(-1) & \oplus & H^2(E_q) & 0 \\
0 & \oplus & H^0(t)(-1) & \oplus & H^2(E_p) & 0 \\
0 & 0 & H^0(X) & \oplus H^0(M_i) & \oplus H^0(M_{ij}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

**Figure 10.** The first page of the spectral sequence computing the grading $\text{gr}^W H^2(X \setminus L, M \setminus (L \cap M))$

We need to prove that

$$\text{gr}^W H^2(X \setminus L, M \setminus (L \cap M)) = \mathbb{Q}(0) \oplus \mathbb{Q}(-2).$$  \hspace{1cm} (2.213)

The piece $\mathbb{Q}(-2)$ comes from the top-left corner, while $\mathbb{Q}(0)$ arises as the cokernel of the map $\bigoplus H^0(M_i) \to \bigoplus H^0(M_{ij})$, which has rank 4. Indeed, this map is given by

$$(a, b, c, d, e) \mapsto (c - a, d - a, c - b, e - b, e - d).$$

Since the map $H^0(X) \to \bigoplus H^0(M_i)$ sends $a$ to $(a, a, a, a, a)$, the cohomology of the bottom line is concentrated in $E_2^{0,0} = \mathbb{Q}(0)$.

We are thus reduced to show that the complex $E_1^{0,2}$ is exact at the middle term. For this, we first observe that the Gysin maps induce an isomorphism of Hodge structures

$$H^0(E_p)(-1) \oplus H^0(E_q)(-1) \xrightarrow{\sim} H^2(X).$$  \hspace{1cm} (2.214)
This is an instance of the general computation of the Hodge structure of a blow-up, see e.g. [Voi02, 7.3.3]. In the case at hand, it can be seen as follows: the Gysin long exact sequence (2.188) for \( U = X \setminus (E_p \cup E_q) \) reads
\[
\cdots \to H^1(U) \to H^0(E_p)(-1) \oplus H^0(E_q)(-1) \to H^2(X) \to H^2(U) \to \cdots
\]
Since \( U \) is isomorphic, via the blow-up map, to \( \mathbb{A}^2 \setminus \{p, q\} \), the cohomology groups \( H^1(U) \) and \( H^2(U) \) vanish (use Exercise 2.200). It follows that the differential \( d_1 : E_1^{0,2} \to E_1^{1,2} \) in the spectral sequence is given, in suitable bases compatible with the isomorphism (2.214), by
\[
H^2(X) \oplus H^0(s)(-1) \oplus H^0(t)(-1) \to H^2(E_p) \oplus H^2(E_q)
\]
\[
(a, b, c, d) \mapsto (a + c, b + d). \tag{2.215}
\]
To compute the remaining map, one needs to know the cohomology classes \([L_i] \in H^2(X)\). We claim that \([L_0] = -[E_p]\). Indeed, since the total transform of \( \ell_0 \) is the union \( L_0 \cup E_p \), we get
\[
[L_0] + [E_p] = [\pi^{-1}(\ell_0)] = \pi^*[\ell_0] = 0,
\]
where the last equality follows from the fact that \([\ell_0]\) lives in \( H^2(\mathbb{A}^2) = 0\).
Similarly, \([L_1] = -[E_q]\), so that the differential \( d_1 : E_1^{i,-2} \to E_1^{i,2} \) is given by
\[
H^0(L_0)(-1) \oplus H^0(L_1)(-1) \to H^2(X) \oplus H^0(s)(-1) \oplus H^0(t)(-1)
\]
\[
(a, b) \mapsto (-a, -b, a, b). \tag{2.216}
\]
It is now obvious that the middle row of the spectral sequence is exact. Indeed, its whole second page reads
\[
\begin{array}{cccccc}
\mathbb{Q}(-2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{Q}(0).
\end{array}
\]
This concludes the proof of the equality (2.213) and shows, moreover, that the group \( H^i(U \setminus L, M \setminus (L \cap M)) \) vanishes in all degrees \( i \neq 2 \). \hfill \Box

**Remark 2.217.** A byproduct of the proof is that we have canonical identifications (seeExercise 2.221)
\[
g_{\text{tr}}^W H^2(X \setminus L, M \setminus (L \cap M)) = H^2(X \setminus L) = \mathbb{Q}(-2),
\]
\[
g_{\text{tr}}^W H^2(X \setminus L, M \setminus (L \cap M)) = H^2(X, M) = \mathbb{Q}(0). \tag{2.218}
\]
Recall from paragraph 2.4.1 that the differential form \( \pi^*(\omega) \) is an element of \( H^{i, \text{dR}}_d(X \setminus L) \) and the simplex \( \tilde{\sigma} \) belongs to \( H^2(X, M) \). By Theorem 2.152, the class of the extension
\[
[H^2(X \setminus L, M \setminus (L \cap M))] \in \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(-2), \mathbb{Q}(0)) = \mathbb{C}/(2\pi i)^2 \mathbb{Q}
\]
is thus given by \( \int_{\tilde{\sigma}} \pi^*(\omega) = \zeta(2) \). One would like to use this information as follows: imagine that we knew by “pure thought” that all such extensions
given by relative cohomology of varieties defined over \( \mathbb{Q} \) are split. Then \( \zeta(2) \)
would have to vanish in the quotient \( \mathbb{C}/(2\pi i)^2\mathbb{Q} \), which would yield a more
conceptual explanation of why \( \zeta(2) \) is a rational multiple of \( \pi^2 \). To carry
out this program, one needs however to leave the category of mixed Hodge
structures and work with the more abstract notion of mixed Tate motives
which will be introduced in Chapter 4.

2.8.2. Odd zeta extensions. In general, it is a difficult problem to give a
geometric construction of the extension of \( \mathbb{Q}(-n) \) by \( \mathbb{Q}(0) \) whose class in
\( \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(-n), \mathbb{Q}(0)) = \mathbb{C}/(2\pi i)^n\mathbb{Q} \)
is the zeta value \( \zeta(n) \). Besides Proposition 2.209, only the case \( n = 3 \) is
known by work of Brown [Bro16] and Dupont [Dup18].

More precisely, Dupont starts with affine space \( \mathbb{A}^n \) and the hypersurfaces
\( \ell_n = \{x_1 \cdots x_n = 1\} \), \( m_n = \bigcup_{1 \leq i \leq n} \{x_i = 0\} \cup \bigcup_{1 \leq i \leq n} \{x_i = 1\} \).
The divisor \( \ell_n \) is smooth and \( m_n \) is a normal crossing divisor. However, their
union \( \ell_n \cup m_n \) fails to have normal crossings at the point \( p_n = (1, \ldots, 1) \),
where \( n + 1 \) irreducible components intersect. Let \( \pi_n : X_n \to \mathbb{A}^n \)
be the blow-up of \( \mathbb{A}^n \) at \( p_n \) and let \( E_n \) denote the exceptional divisor. We write \( L_n \)
for the strict transform of \( \ell_n \) and \( M_n \) for the union of the strict transform
of \( m_n \) and \( E_n \). We form the relative cohomology:
\[
Z_n = H^n(X_n \setminus L_n, M_n \setminus (L_n \cap M_n)).
\]
Dupont proves that \( Z_n \) fits into an exact sequence of mixed Hodge structures
\[
0 \to \mathbb{Q}(0) \to Z_n \to \mathbb{Q}(-2) \oplus \cdots \oplus \mathbb{Q}(-n)
\]
and that there is a natural isomorphism
\[
Z_n/\mathbb{Q}(0) \xrightarrow{\sim} H^{n-1}(\ell_n, \bigcup_{1 \leq i \leq n} \{x_i = 1\})(-1).
\]
To separate the even and the odd weights, one uses the involution
\[
\tau(x_1, \ldots, x_n) = (x_1^{-1}, \ldots, x_n^{-1}).
\]
Indeed, if \( p : H_n \to H_n/\mathbb{Q}(0) \) denotes the quotient map, one defines \( H_n^{\text{odd}} \)
as \( p^{-1}((H_n/\mathbb{Q}(0))^\tau = 1) \). It then fits into an exact sequence
\[
0 \to \mathbb{Q}(0) \to H_n^{\text{odd}} \bigoplus_{3 \leq 2k+1 \leq n} \mathbb{Q}(-(2k+1)) \to 0
\]
2.8.3. Irrationality proofs. Here is how a typical irrationality proof works. To show that a real number \( \alpha \) is irrational, we proceed in three steps:

1. we construct linear forms

\[
I_n = a_n + b_n \alpha, \quad a_n, b_n \in \mathbb{Q},
\]

such that \( 0 < |I_n| < \varepsilon^n \) for some \( 0 < \varepsilon < 1 \) and \( n \) sufficiently big;

2. if \( r_n \) is the common denominator of \( a_n \) and \( b_n \), then we require that \( r_n < D^n \) for some real number \( D \), again when \( n \) is big enough;

3. \( \varepsilon \) and \( D \) should be related by the inequality \( \varepsilon D < 1 \).

If one succeeds in carrying out these three steps, then \( \alpha \) is irrational. Indeed, assume that \( \alpha = \frac{p}{q} \). Multiplying by \( r_n q \), we get

\[
0 < |r_n a_n q + b_n d_n p| < qr_n \varepsilon^n < q(\varepsilon D)^n,
\]

so the sequence inside the absolute value converges to zero by the assumption that \( \varepsilon D < 1 \). But then, for \( n \) sufficiently big, we would find integers strictly bigger than 0 and smaller than 1, which is of course a contradiction!

Algebraic geometry could be useful in producing the linear forms (2.219). Indeed, assume that we can construct a mixed Hodge structure over \( \mathbb{Q} \) which is an extension of \( \mathbb{Q}(0) \) by \( \mathbb{Q}(n) \) with period matrix

\[
\begin{pmatrix}
1 & \alpha \\
0 & (2\pi i)^n
\end{pmatrix}
\]

with respect to some bases \( \{\omega_0, \omega_1\} \) of \( H_{dR} \) and \( \{\sigma_0, \sigma_1\} \) of \( H_B \). Then, given any \( \omega \in H_{dR} \), there exist rational numbers \( a \) and \( b \) such that \( a\omega_0 + b\omega_1 \), and the integral \( \int_{\sigma_0} \omega \) is equal to \( a + b\alpha \). Typically, \( H \) is given by a relative cohomology group and one considers a sequence \( \omega_n = f^n \omega \) where \( \omega \) is a fixed differential and \( f \) is a function vanishing on the boundary.

**Example 2.220.** Consider the differential form

\[
\omega_{a,b,c} = \frac{(x-1)^a(x-t)^b}{x^{c+1}}dx,
\]

where \( a, b, c \geq 1 \) and \( t \geq 2 \) are integers. Since \( \omega_{a,b,c} \) is only singular along \( x = 0 \) and has top degree, it defines a class in \( H^1_{dR}(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, t\}) \). By Example 2.54, a basis of this relative cohomology group is given by the differentials \( \omega_1 = (0, 1, 0) \) and \( \omega_2 = (\frac{dx}{x}, 0, 0) \), so there exists rational numbers \( A \) and \( B \) such that

\[
\omega_{a,b,c} = A\omega_1 + B\omega_2.
\]
Indeed, elementary manipulations of the complex (2.55) yield the values
\[ A = \sum_{0 \leq i \leq a} \sum_{0 \leq j \leq b} \sum_{i+j \neq c} \binom{a}{i} \binom{b}{j} (-1)^{a-i-j} (t^{b-c+i} - t^{b-j}), \]
\[ B = \sum_{0 \leq i \leq a} \sum_{0 \leq j \leq b} \sum_{i+j = c} \binom{a}{i} \binom{b}{j} (-1)^{a-i-j} t^{b-j}, \]

Note that \( B \) is an integer. In view of Example 2.100, it follows that
\[ \int_1^t \omega_{a,b,c} = A + B \log(t), \]
and choosing the parameters \( a, b, c \) as functions of \( n \) gives a sequence of linear forms in \( 1 \) and \( \log(t) \) as in Step (1).

Let us specialize to the case \( a = b = c = n \) and \( t = 2 \). Then
\[ I_n = \int_1^2 \omega_{n,n,n} = a_n + b_n \log(2), \]
where \( b_n \) is an integer and \( a_n \) is given by the formula
\[ a_n = \sum_{0 \leq i \leq n} \sum_{0 \leq j \leq n} \sum_{i+j \neq n} \frac{\binom{n}{i} \binom{n}{j} (-1)^{a-i-j} (t^{i-n-j})}{i+j-n}. \]

Since the denominators of the summands in \( a_n \) run through \([-n, n]\), one can take \( r_n = \text{lcm}(1, 2, \ldots, n) \). We have:
\[ r_n = \prod_{p \leq n} p^{\left\lfloor \frac{\log n}{\log p} \right\rfloor} < \prod_{p \leq n} p^{\log n / \log p} = n^{\pi(n)}, \]
where \( \pi(n) \) is the number of primes smaller than \( n \). Here is where some deep arithmetic input enters: the prime number theorem asserts that
\[ \pi(n) \sim \frac{n}{\log n} \quad \text{as} \quad n \to +\infty, \]
see e.g. [IK04, Chap. 2]. It follows that \( n^{\pi(n)} \sim e^{(1+\varepsilon)n} \) for all \( \varepsilon > 0 \) and, being generous, \( D = 3 \) works in Step (2).

Next observe that, by the choice of the parameters, \( I_n \) can be written as
\[ I_n = \int_1^2 f^n \frac{dx}{x}, \quad f(x) = \frac{(x-1)(2-x)}{x}. \]
The function \( f \) is strictly positive on the open interval \((1, 2)\) and bounded above by its maximal value \( 3 - 2\sqrt{2} \). Therefore,
\[ 0 < I_n < (3 - 2\sqrt{2})^n \log(2) < (3 - 2\sqrt{2})^n, \]
so \( \varepsilon = 3 - 2\sqrt{2} \) satisfies the assumptions. Luckily, \( \varepsilon D = 0, 5147186 \ldots < 1 \) and, all in all, we have proved that \( \log(2) \) is irrational!

\[ \star \star \star \]

**Exercise 2.221.** Specialize the spectral sequence (2.211) to the cases \( I = \emptyset \) and \( J = \emptyset \). Deduce the identifications (2.218).

**Exercise 2.222.** Let \( L = L_0 \cup L_1 \cup L_2 \) and \( M = M_0 \cup M_1 \cup M_2 \) be two triangles in \( \mathbb{P}^2 \) such that no three lines intersect at a common point. Use the spectral sequence (2.211) to show that

\[ \text{gr} \cdot H^2(\mathbb{P}^2 \setminus L, M \setminus (L \cap M)) = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)^{\oplus 4} \oplus \mathbb{Q}(-2). \]

The question of what happens when the lines are not in general position is studied in great detail in [BGSV90].

**Exercise 2.223 (Irrationality of \( \zeta(3) \)).** The goal of this exercise is to show that \( \zeta(3) \) is irrational following the proof by Beukers [Beu79]. We keep the notation \( r_n = \text{lcm}(1, 2, \ldots, n) \).

(a) Let \( n, m \geq 0 \) be integers and \( \sigma \geq 0 \) a real number. Prove the identity

\[
\int_{[0,1]^2} \frac{x^{n+\sigma}y^{m+\sigma}}{1-xy}dxdy = \begin{cases} 
\frac{1}{n-m} \left( \frac{1}{m+1+\sigma} + \cdots + \frac{1}{n+m} \right) & n > m, \\
\sum_{k=1}^{\infty} \frac{1}{(k+n+\sigma)^2} & n = m.
\end{cases}
\]

(b) Let \( n, m \geq 0 \) be integers. Show that the integral

\[
\int_{[0,1]^2} \frac{-\log xy}{1-xy}x^ny^mdxdy
\]

is a rational number whose denominator divides \( r_n^3 \) if \( n > m \) and that, for \( n = m \), it takes the value

\[
\begin{cases} 
2\zeta(3) & \text{if } n = 0 \\
2 \left( \zeta(3) - 1 - 2^{-3} - \cdots - n^{-3} \right) & \text{if } n > 0.
\end{cases}
\]

[Hint: derivate the formulas of part (a) with respect to \( \sigma \).]

(c) For each integer \( n \geq 1 \), let \( P_n \in \mathbb{Z}[x] \) be the polynomial defined by

\[ n!P_n(x) = \frac{d^n}{dx^n} (x^n(1-x)^n) \]

and consider the integral

\[ I_n = \int_{[0,1]^2} \frac{-\log xy}{1-xy} P_n(x)P_n(y)dxdy. \]
Prove that there exist rational numbers \( a_n, b_n \in \mathbb{Q} \) whose denominators divide \( r_n^3 \) such that

\[
I_n = a_n + b_n \zeta(3).
\]

(d) Prove that the above integral can be rewritten as

\[
I_n = \int_{[0,1]^3} \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}} dxdydz.
\]

[Hint: use the integral representation

\[
-\log xy \quad \frac{1}{1-xy} = \int_0^1 \frac{dw}{1-(1-xy)w}
\]

and the change of variables \( z = (1-w)(1-(1-xy)w)^{-1} \).]

(e) Show that, for all \( 0 \leq x, y, z \leq 1 \), one has

\[
\frac{x(1-x)y(1-y)z(1-z)}{(1-(1-xy)z)} \leq (\sqrt{2} - 1)^4
\]

and deduce that \( 0 < \left| I_n \right| < 2\zeta(3)(\sqrt{2} - 1)^{4n} \).

[Hint: first prove that the maximum occurs for \( x = y \).]

(e) Conclude that \( \zeta(3) \) is irrational.
3. Multiple zeta values and the geometry of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

In this chapter, we start moving towards the goal of upgrading multiple zeta values to their motivic counterparts, which are functions on an algebro-geometric construction associated with the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. To this end, we first look for functions on the space of paths of a differentiable manifold $M$ that are homotopy invariant. These functions are called homotopy functionals. By Stokes’ theorem, examples of homotopy functionals are given by line integrals of closed 1-forms. However, the corresponding functions on the fundamental group always factor through its abelianization and thus cannot detect loops whose homology classes are trivial. Trying to go further, K-T. Chen had the fundamental insight that iterated integrals yield finer invariants, which are in fact sufficient to recover all finite-dimensional unipotent representations of $\pi_1(M)$ and not only the abelian ones. More precisely, his celebrated $\pi_1$-de Rham theorem asserts that the ring of regular functions on the pro-unipotent completion of the fundamental group is isomorphic, as a Hopf algebra, to the zeroth cohomology of the bar complex of any connected model of the algebra of differential forms. This has a number of important consequences, notably the fact due to Hain that the pro-unipotent completion carries a mixed Hodge structure.

In Section 3.1, we review the definition and algebraic properties of iterated integrals. A basic question is when an iterated integral only depends on the homotopy class of a path relative to its endpoints. By relating the parallel transport of connections on the trivial bundle to iterated integrals, we answer the question in length two.

3.1. Iterated integrals and parallel transport. Our presentation follows closely Hain’s survey [Hai87a]. Other nice references are Cartier’s Bourbaki seminar [Car88] and Brown’s notes [Bro13b].

3.1.1. The fundamental groupoid. Let $M$ be a connected differentiable manifold. We say that a continuous function $\gamma: [0, 1] \to M$ is piecewise smooth if there is a partition $0 = a_0 < a_1 < \ldots < a_{n+1} = 1$ of the unit interval such that the restriction of $\gamma$ to each $[a_i, a_{i+1}]$ is smooth, meaning that it can be extended to a smooth function on an open neighborhood of $[a_i, a_{i+1}]$. Similarly, a continuous map $F: [0, 1]^2 \to M$ is said to be piecewise smooth if there exists a finite polyhedral decomposition $[0, 1]^2 = \bigcup_i C_i$ such that all the restrictions $F|_{C_i}$ are smooth, in the sense that they extend to a smooth function on an open neighbourhood of $C_i$.

We call a continuous piecewise smooth map from $[0, 1]$ to $M$ simply a path (see Remark 3.8 below), and denote the space of paths by

$$\mathcal{P}(M) = \{\gamma: [0, 1] \to M \mid \gamma \text{ continuous and piecewise smooth}\}.$$
Given two points $x$ and $y$ in $M$, the subspace of $\mathcal{P}(M)$ consisting of paths from $x$ to $y$ will be denoted by 

$$y\mathcal{P}(M)_x = \{ \gamma \in \mathcal{P}(M) \mid \gamma(0) = x, \gamma(1) = y \}.$$ 

When the endpoints of $\gamma$ agree, we will often call it a *loop*.

**Definition 3.1.** Two paths $\gamma_1, \gamma_2 \in y\mathcal{P}(M)_x$ are said to be *homotopic* if there exists a continuous piecewise smooth function $F: [0,1]^2 \to M$ such that:

$$F(t,0) = \gamma_1(t), \quad F(t,1) = \gamma_2(t), \quad t \in [0,1],$$

$$F(0,s) = x, \quad F(1,s) = y, \quad s \in [0,1]. \quad (3.2)$$

In other words, $F$ is a continuous family of paths $f_s: [0,1] \to M,$

$$t \mapsto f_s(t) = F(t,s),$$

parameterized by $s \in [0,1]$, that interpolates between $\gamma_1$ and $\gamma_2$ while keeping the end points fixed (see Figure 11).

![Figure 11. A homotopy between two paths](image)

It is straightforward to check that “being homotopic” defines an equivalence relation $\sim$ on $y\mathcal{P}(M)_x$. We write

$$\pi_1(M; y, x) = \{ \gamma \in y\mathcal{P}(M)_x \}/\sim$$

for the set of equivalence classes. When the two endpoints agree, we will abbreviate this notation to $\pi_1(M, x)$.

Note that there is a *reversal of paths* operation

$$y\mathcal{P}(M)_x \to x\mathcal{P}(M)_y$$

$$\gamma \mapsto \gamma^{-1}.$$
defined as $\gamma^{-1}(t) = \gamma(1-t)$. Moreover, given a third point $z$ in $M$, we have a composition of paths

$$\gamma_1 \gamma_2 = \begin{cases} 
\gamma_2(2t) & 0 \leq t \leq \frac{1}{2}, \\
\gamma_1(2t-1) & \frac{1}{2} \leq t \leq 1.
\end{cases}$$

(3.3)

Both the reversal and the composition of paths are compatible with the homotopy equivalence relation, and hence induce operations

$$\pi_1(M; y, x) \to \pi_1(M; x, y),$$

(3.4)

$$\pi_1(M; z, y) \times \pi_1(M; y, x) \to \pi_1(M; z, x),$$

(3.5)

which are called “inverse” and “composition” respectively. It is a simple matter of verification to see that (3.5) is associative and that the class of the constant path $\gamma(t) = x$ in $\pi_1(M, x)$ is a neutral element. As such, it will be usually denoted by $1$.

If the endpoints agree, the above operations endow $\pi_1(M, x)$ with the structure of a group: the fundamental group of $M$. In general, when we allow the endpoints to be distinct, we only obtain a groupoid. We recall below the definition, which is in fact tailored to study this example.

**Definition 3.6.** A groupoid $G$ is the data of a set $G_0$ of “objects” and a set $G_1$ of “arrows”, together with the following five operations:

- a source map $s: G_1 \to G_0$;
- a target map $t: G_1 \to G_0$;
- a unit map $u: G_0 \to G_1$ such that $s(u(x)) = t(u(x)) = x$ for all objects $x \in G_0$;
- a composition map $m: G_1 \times G_1 \to G_1$ defined on

$$G_1 \times G_1 \to \{ (f, g) \in G_1 \times G_1 \mid s(f) = t(g) \}
$$

such that $s(m(f, g)) = s(g)$ and $t(m(f, g)) = t(f)$ for all arrows $f, g \in G_1$, and that $u$ is a two-sided unit for $m$. Moreover, the composition is required to be associative.

- an inverse map $i: G_1 \to G_1$ such that, for all arrows $f \in G_1$, $s(i(f)) = t(f)$ and $t(i(f)) = s(f)$ and which is a two-sided inverse for the composition.

Equivalently, a groupoid can be viewed as a small category where all morphisms are isomorphisms (see Exercise 3.37).
Example 3.7 (The fundamental groupoid). The fundamental groupoid of $M$ is the groupoid where $G_0$ is the set of points of $M$ and $G_1$ is the set of homotopy classes of paths in $M$, that is:

$$G_1 = \bigsqcup_{x, y \in M} \pi_1(M; y, x).$$

The source, the target, and the unit are defined in the obvious way, and the inverse and the composition maps are given by (3.4) and (3.5) respectively.

Remark 3.8. When doing homotopy theory on a differentiable manifold, one can choose to work with continuous or piecewise smooth paths. The resulting fundamental group or groupoid is the same in both cases. To make the link with differential forms, it will be most convenient to work with piecewise smooth paths.

3.1.2. Homotopy functionals. We would like to construct functions on the fundamental groupoid of a manifold.

Definition 3.9. A function on $\mathcal{P}(M)$ is called a homotopy functional if the image of every element in $\mathcal{P}(M)$ depends only on its homotopy class, and hence induces a function on $\pi_1(M; y, x)$ for all $x, y \in M$.

The simplest method to construct homotopy functionals is by means of differential forms, as we now recall. Let $k$ be either the real or the complex numbers. We consider the $k$-algebra

$$E^*(M, k) = \bigoplus_{p=0}^{\dim M} E^p(M, k)$$

of smooth $k$-valued differential forms in $M$. Given $\omega \in E^1(M, k)$ and a path $\gamma \in \mathcal{P}(M)$, since $\gamma$ is assumed to be piecewise smooth, we can pullback $\omega$ to the interval $[0, 1]$; the pullback takes the form $\gamma^* \omega = f(t) dt$ for some function $f$. The line integral of $\omega$ along $\gamma$ is defined as

$$\int_\gamma \omega = \int_0^1 \gamma^* \omega = \int_0^1 f(t) dt.$$  (3.10)

This yields a function

$$\int \omega : \mathcal{P}(M) \longrightarrow k, \quad \gamma \longmapsto \int_\gamma \omega.$$

Lemma 3.11. The function $\int \omega$ is a homotopy functional if and only if the 1-form $\omega$ is closed.

Proof. The result follows easily from Stokes’ theorem. First assume that $\omega$ is closed, and that we are given two paths $\gamma_1$ and $\gamma_2$ and a homotopy
$F$ between them. Using the conditions (3.2) in the definition of $F$, we find
\[
\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{[0,1]} \gamma_1^* \omega - \int_{[0,1]} \gamma_2^* \omega = \int_{\partial[0,1]^2} F^* \omega,
\]
where $\partial[0,1]^2$ stands for the boundary of the square $[0,1]^2$. Since $F$ is piecewise smooth, there exists a polyhedral decomposition $[0,1]^2 = \bigcup C_i$ such that $F|_{C_i}$ is smooth. By Stokes’ theorem and the commutativity of $F^*$ with the differential,
\[
\int_{\partial[0,1]^2} F^* \omega = \sum_i \int_{\partial C_i} F^* \omega = \sum_i \int_{C_i} F^*(d\omega) = 0,
\]
thus proving that the line integral is a homotopy functional.

Conversely, assume that the 1-form $\omega$ is not closed. Hence $d\omega \neq 0$, so we can find a smooth map $f : D \to M$ from the unit disc $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ to $M$ such that
\[
\int_{D} f^* d\omega \neq 0.
\]
Consider the paths from $x = f(1,0)$ to $y = f(-1,0)$ given by
\[
\gamma_1(t) = f(\cos(\pi t), \sin(\pi t)), \quad \gamma_2(t) = f(\cos(\pi t), -\sin(\pi t)).
\]
They are homotopic through the homotopy
\[
F(x, y) = f(\cos(\pi x), (1 - 2y) \sin(\pi x)).
\]
On the contrary, another application of Stokes’s theorem gives
\[
\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\partial D} f^* \omega = \int_{D} f^* d\omega \neq 0,
\]
which proves that $\omega$ being closed is a necessary condition as well. □

Line integrals of closed 1-forms produce, however, only a very special kind of homotopy functionals. Indeed, from (3.10) we get the relations
\[
\int_{\gamma_1 \gamma_2} \omega = \int_{\gamma_2} \omega + \int_{\gamma_1} \omega, \quad \int_{\gamma_1^{-1}} \omega = - \int_{\gamma} \omega,
\]
which together imply that, for any pair of loops $\gamma_1, \gamma_2 \in \pi_1(M, x)$, one has:
\[
\int_{\gamma_1^{-1} \gamma_2^{-1} \gamma_1 \gamma_2} \omega = 0. \tag{3.12}
\]

Recall that, given a group $G$, the commutators $[g, h] = g^{-1}h^{-1}gh$ generate a normal subgroup $[G, G]$.

**Definition 3.13.** The *abelianization* of $G$ is the quotient
\[
G^{\text{ab}} = G/[G, G].
\]
It is an abelian group satisfying the universal property that any homomorphism from $G$ to an abelian group factors through $G^{ab}$. In particular, for every closed 1-form $\omega$ the homomorphism

$$\int \omega : \pi_1(M, x) \rightarrow k$$

factors through $\pi_1(M, x)^{ab}$. Now, viewing a loop $\gamma : [0, 1] \rightarrow M$ as a closed singular 1-chain, as defined in Section 2.1, yields a canonical homomorphism

$$h : \pi_1(M, x) \rightarrow H_1(M, \mathbb{Z}),$$

which is often called the Hurewicz map. The following is a basic result from algebraic topology, see e.g. [Hat02, Thm. 2A.1]:

**Theorem 3.14.** The kernel of $h$ consists exactly of the commutator subgroup $[\pi_1(M, x), \pi_1(M, x)]$. Moreover, if $M$ is connected, then $h$ is surjective and thus induces an isomorphism

$$\pi_1(M, x)^{ab} \simeq H_1(M, \mathbb{Z}).$$

Summarizing, line integrals of closed 1-forms always factors through the first homology group of the manifold. Since the fundamental group is a finer invariant, we would like to construct other homotopy functionals which are able to detect the extra information carried by $\pi_1(M, x)$.

### 3.1.3. Iterated integrals

The theory of iterated integrals started with the fundamental observation by K.T. Chen [Che77] that homotopy functionals given by successive integration of 1-forms can detect elements of $\pi_1(M, x)$ whose images by the Hurewicz map are trivial in $H_1(M, \mathbb{Z})$.

**Definition 3.15.** Let $\omega_1, \ldots, \omega_r$ be smooth $k$-valued 1-forms on $M$. The **iterated integral** of $\omega_1, \ldots, \omega_r$ is the function

$$\int \omega_1 \cdots \omega_r : \mathcal{P}(M) \rightarrow k$$

defined as follows:

$$\int_{\gamma} \omega_1 \cdots \omega_r = \int_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} f_1(t_1) \cdots f_r(t_r) dt_1 \cdots dt_r,$$

where $\gamma^* \omega_i = f_i(t) dt$ is the pullback of $\omega_i$ to $[0, 1]$.

More generally, we will call **iterated integral** any function on $\mathcal{P}(M)$ obtained as a $k$-linear combination of (3.16) and the constant function 1, which we viewed as an iterated integral of length 0. We say that an iterated integral has length $\leq s$ if each summand is of the form $\int \omega_1 \cdots \omega_r$ with $r \leq s$.

**Remark 3.17.** Here is an explanation of the term “iterated integral”. Let $S$ be the operator that transforms a 1-form $\eta$ on the interval $[0, 1]$ into the function $S[\eta](t) = \int_0^t \eta$. To obtain the iterated integral we apply $S$ to $\gamma^* \omega_r$, etc.
then multiply the resulting function by $\gamma^*\omega_{r-1}$, apply $S$ again, multiply by $\gamma^*\omega_{r-2}$, etc., and finally evaluate at $t = 1$:

$$\int_\gamma \omega_1 \cdots \omega_r = S[\gamma^*\omega_1 \cdot S[\gamma^*\omega_2 \cdots S[\gamma^*\omega_r]\cdots]](1).$$

Observe that we have already encountered iterated integrals in Chapter 1: the integral representations of multiple zeta values (Theorem 1.108) and polylogarithms (Theorem 1.117) are both examples of iterated integrals.

3.1.4. Basic properties of iterated integrals. The first important property is that iterated integrals are functorial and independent of the parametrization of the path. The proof is left to the reader (see Exercise 3.42).

**Proposition 3.18 (Functoriality).** Let $f: N \to M$ be a smooth map of differentiable manifolds. If $\gamma \in \mathcal{P}(N)$ and $\omega_1, \ldots, \omega_r \in E^1(M,k)$, then

$$\int_\gamma f^*\omega_1 \cdots f^*\omega_r = \int_{f \circ \gamma} \omega_1 \cdots \omega_r.$$

In particular, the iterated integral $\int_\gamma \omega_1 \cdots \omega_r$ does not depend on the choice of parametrization of the path $\gamma$.

We now prove the basic algebraic properties of iterated integrals, which are formulas for the reversal and composition of paths, as well as for the product of two iterated integrals.

**Theorem 3.19.** Let $\omega_1, \ldots, \omega_{r+s}$ be smooth $k$-valued 1-forms on $M$ and let $\gamma, \gamma_1, \gamma_2$ be piecewise smooth paths in $M$ such that $\gamma_2(1) = \gamma_1(0)$. Then the following three equalities hold:

$$\int_\gamma \omega_1 \cdots \omega_r = (-1)^r \int_{\gamma^{-1}} \omega_r \cdots \omega_1, \quad (3.20)$$

$$\int_{\gamma_1 \gamma_2} \omega_1 \cdots \omega_r = \sum_{i=0}^{n} \int_{\gamma_1} \omega_1 \cdots \omega_i \int_{\gamma_2} \omega_{i+1} \cdots \omega_r, \quad (3.21)$$

$$\int_\gamma \omega_1 \cdots \omega_r \int_{\gamma} \omega_{r+1} \cdots \omega_{r+s} = \sum_{\sigma \in \mathcal{W}(r,s)} \int_{\gamma} \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(r+s)}. \quad (3.22)$$

In the last identity, the sum runs over the subset $\mathcal{W}(r,s)$ of the symmetric group $\mathfrak{S}_{r+s}$ consisting of shuffles of type $(r,s)$, as in Definition 1.120.
Proof. The first identity (3.20) is a simple computation using that 
$\gamma^*\omega_i = f_i(t)dt$ implies $(\gamma^{-1})^*\omega_i = -f_i(1-t)dt$, and hence

$$\int_{\gamma^{-1}} \omega_r \cdots \omega_1 = (-1)^r \int_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} f_r(1-t_1) \cdots f_1(1-t_r) dt_1 \cdots dt_r$$

$$= (-1)^r \int_{1 \geq u_r \geq \cdots \geq u_1 \geq 0} f_r(u_r) \cdots f_1(u_1) du_1 \cdots du_r$$

$$= (-1)^r \int_{\gamma} \omega_1 \cdots \omega_r.$$

To pass from the first line to the second we made the change of variables $u_i = 1 - t_{r-i+1}$, whose Jacobian has absolute value 1.

We next prove (3.21). If one writes

$$(\gamma_1 \gamma_2)^* \omega_i = f_i(t)dt, \quad \gamma_1^* \omega_i = g_i(t)dt, \quad \gamma_2^* \omega_i = h_i(t)dt,$$

then the three above functions are related by

$$f_i(t) = \begin{cases} 2h_i(2t) & 0 \leq t \leq \frac{1}{2}, \\ 2g_i(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases} \quad (3.23)$$

We decompose the domain of integration as a union $\Delta^r = \bigcup_{i=0}^r C_i$, where

$$C_i = \{(t_1, \ldots, t_r) \in \mathbb{R}^r \mid 1 \geq t_1 \geq \cdots \geq t_i \geq \frac{1}{2} \geq t_{i+1} \geq \cdots \geq t_r \geq 0\}.$$  

Observe that $C_i \simeq \Delta^i \times \Delta^{r-i}$, as Figure 12 shows in the case $r = 2$.

![Figure 12](image-url)
Now equation (3.21) follows from the computation
\[
\int_{C_i} f_1(t_1) \cdots f_r(t_r) dt_1 \cdots dt_r = \int \frac{f_1(t_1) \cdots f_r(t_r) dt_1 \cdots dt_r}{1 \geq t_1 \geq \cdots \geq t_r \geq 1/2}
\]
\[
= \frac{2^r}{2^{2r}} \int g_1(u_1) \cdots g_i(u_i) h_{i+1}(u_{i+1}) \cdots h_r(u_r) du_1 \cdots du_r
\]
\[
= \int \omega_1 \cdots \omega_i \int \omega_{i+1} \cdots \omega_r,
\]

considering the overlap of the \(C_i\) do not contribute to the integral because they all have codimension at least 1, and hence Lebesgue measure zero. The second equality is obtained by the change of variables
\[
u_j = \begin{cases} 2t_j - 1 & j \leq i, \\ 2t_j & j > i. \end{cases}
\]
The \(2^r\) in the numerator comes from equation (3.23), whereas the \(2^{2r}\) in the denominator is the Jacobian of the change of variables.

Finally, the formula (3.22) is a consequence of the decomposition
\[
\Delta^r \times \Delta^s = \bigcup_{\sigma \in \omega(r,s)} \{ (t_1, \ldots, t_{r+s}) \mid 1 \geq t_{\sigma^{-1}(1)} \geq \cdots \geq t_{\sigma^{-1}(r+s)} \geq 0 \},
\]
that was already used in Proposition 1.123. \(\square\)

### 3.1.5. When are iterated integrals homotopy functionals?

We have seen that iterated integrals do not depend on the parametrization of the path (Proposition 3.18). However, even when all the \(\omega_i\) are closed, they do not always give rise to homotopy functionals, as the example below shows:

**Example 3.24.** Take \(M = \mathbb{R}^2\) with the standard coordinates \(x\) and \(y\). Let \(a, b > 0\) be real numbers and consider the path \(\gamma_{a,b} : [0, 1] \to \mathbb{R}^2\) from \((0, 0)\) to \((1, 1)\) given by \(\gamma_{a,b}(t) = (t^a, t^b)\). Let \(\omega_1 = dx\) and \(\omega_2 = dy\). Then
\[
\gamma_{a,b}^* \omega_1 = at^{a-1} dt, \quad \gamma_{a,b}^* \omega_2 = bt^{b-1} dt,
\]
so one has the iterated integral
\[
\int_{\gamma_{a,b}} \omega_1 \omega_2 = \int_0^1 \left( \int_0^{t_1} \left( \int_0^{t_2} \frac{at^{a-1} dt_2}{bt^{b-1} dt_2} \right) dt_1 \right) = \frac{a}{a + b},
\]
which obviously depends on the choice of \(a\) and \(b\). However, all the paths \(\gamma_{a,b}\) are homotopic to each other!

A natural question is thus when an iterated integral is invariant under homotopy. Theorem 3.148 will give a complete solution to this problem in terms of a construction called the *bar complex*. For the moment, we content
ourselves with a partial answer by linking iterated integrals to connections on trivial bundles through the notion of parallel transport.

3.1.6. *Iterated integrals and connections on trivial bundles.* We continue writing $k$ for either the real or the complex numbers. Let

$$V = k^n \times M$$

be the trivial rank $n$ vector bundle over $M$. Recall that sections of $V$ are functions $x: M \to k^n$. We denote by $C^\infty(V)$ the space of all smooth sections.

**Definition 3.25.** A connection on $V$ is a $k$-linear map

$$\nabla: C^\infty(V) \to C^\infty(V) \otimes_{C^\infty(M)} E^1(M, k)$$

which satisfies the Leibniz rule

$$\nabla(f x) = x \otimes df + f \nabla x$$

for each smooth function $f \in C^\infty(M)$ and each smooth section $x \in C^\infty(V)$.

A connection $\nabla$ canonically extends to a $k$-linear map on $p$-forms, still denoted by $\nabla$, as follows:

$$C^\infty(V) \otimes_{C^\infty(M)} E^p(M, k) \to C^\infty(V) \otimes_{C^\infty(M)} E^{p+1}(M, k)$$

$$x \otimes \eta \mapsto x \otimes d\eta + \nabla(x) \wedge \eta$$

The operator $\nabla^2 = \nabla \circ \nabla$ is called the curvature and one says that the connection $\nabla$ is flat (or integrable) if $\nabla^2$ vanishes.

We call global canonical frame of $V$ the tuple $e = (e_1, \ldots, e_n)$ consisting of the constant functions $e_i: M \to k^n$ with value the $i$-th standard basis vector $(0, \ldots, 1, \ldots, 0)$. By virtue of the Leibniz rule, the connection $\nabla$ is determined by the image of the global canonical frame. Write

$$\nabla e_j = \sum_{i=1}^n e_i \otimes \eta_{ij}, \quad j = 1, \ldots, n$$

with $\eta_{ij} \in E^1(M, k)$. The matrix

$$\omega = (\eta_{ij}) \in E^1(M, k) \otimes_{C^\infty(M)} \text{End}(V) = E^1(M, k) \otimes_k \text{End}(k^n),$$

whose entries are smooth $k$-valued 1-forms on $M$, is called the matrix of the connection in the global canonical frame $e$.

Seeing a section $x: M \to k^n$ as a column vector of smooth functions and invoking the Leibniz rule again, the connection is given by

$$\nabla x = dx + \omega x.$$  

From this one easily computes the curvature:

$$\nabla^2 x = \nabla(dx + \omega x)$$

$$= (d^2 x + d(\omega x) + \omega dx + \omega \wedge \omega x)$$

$$= (d\omega + \omega \wedge \omega) x,$$
where $\omega \wedge \omega$ stands for the product of matrices of 1-forms induced by the usual wedge product. In explicit terms, if $\omega = \sum M_i \eta_i$ with $\eta_i \in E^1(M, k)$ and $M_i \in \text{GL}_n(k)$, then

$$\omega \wedge \omega = \frac{1}{2} \sum_{i,j} [M_i, M_j] \eta_i \wedge \eta_j,$$

where we have used that wedge products anti-commute. The matrix $R = d\omega + \omega \wedge \omega$ is called the curvature matrix of $\nabla$.

Any rank $n$ vector bundle has an associated principal bundle $\text{GL}(V)$ with group $\text{GL}_n(k)$. Since $V$ is trivial the same is true for $\text{GL}(V)$, and hence $\text{GL}(V) \cong \text{GL}_n(k) \times M$. A connection $\nabla$ on $V$ lifts to a connection on $\text{GL}(V)$ that, in this trivialization, is given by the formula

$$\nabla X = dX + \omega X.$$

### 3.1.7. Parallel transport

Given a path $\gamma: [0, 1] \to M$ and a section

$$X: [0, 1] \to \text{GL}_n(k)$$

$$t \mapsto X(t)$$

of $\text{GL}(V)$ along $\gamma$, we say that $X$ is horizontal if

$$\nabla X(t) = 0.$$  \hspace{1cm} (3.26)

Equation (3.26) is equivalent to the condition $dX(t) = -\gamma^*(\omega)X(t)$. If we write $\gamma^*(\omega) = A(t)dt$, then (3.26) becomes the linear differential equation

$$X'(t) + A(t)X(t) = 0.$$

The parallel transport function

$$T: \mathcal{P}(M) \to \text{GL}_n(k)$$

associated with the connection $\nabla$ is defined as follows: if $\gamma: [0, 1] \to M$ is a smooth path, then $T(\gamma) = X(1)$, where $X: [0, 1] \to \text{GL}_n(k)$ is the unique section along the path $\gamma: [0, 1] \to M$ that is horizontal with respect to $\nabla$ and has initial value $X(0) = \text{Id}_n$, the identity $n \times n$ matrix.

**Proposition 3.27.** Let $\gamma, \gamma'$ be smooth paths in $M$ with $\gamma'(1) = \gamma(0)$. Then the following holds:

1. $T(\gamma)$ is independent of the parametrization of $\gamma$.
2. $T(\gamma \gamma') = T(\gamma)T(\gamma')$.

Using Proposition 3.27 we can extend the definition of parallel transport to piecewise smooth paths by reparametrizing them as a finite composition of smooth paths.

We now state the main result which relates connections and homotopy functionals. Recall that the connection $\nabla$ is flat if the associated curvature matrix $R = d\omega + \omega \wedge \omega = 0$ is zero.
Theorem 3.28. The connection $\nabla$ is flat if and only if the parallel transport function is a homotopy functional, in the sense that each component is a homotopy functional.

Proof. See for instance [DK90, Theorem 2.2.1]. \[\square\]

3.1.8. Parallel transport and iterated integrals. Using iterated integrals, one can give the following explicit formula for the parallel transport function:

Proposition 3.29. Let $\nabla = d + \omega$ be a connection on the trivial bundle $k^n \times M \to M$. Then the parallel transport function is given by

$$T(\gamma) = \text{Id}_n - \int_\gamma \omega + \int_\gamma \omega \omega + \ldots,$$

where the products in the integrands are formal products of matrices of 1-forms and the iterated integrals are computed componentwise.

Proof. Let $\gamma^* \omega = A(t)dt$. Then the iterated integrals of formal products of matrices of 1-forms are given by

$$\int_\gamma \underbrace{\omega \omega \cdots \omega}_r = \int_{t_1 \geq t_2 \geq \ldots \geq t_r \geq 0} A(t_1)A(t_2)\cdots A(t_r)dt_1 \cdots dt_r. \quad (3.30)$$

Moreover, the parallel transport function is $T(\gamma) = X(1)$, where $X(t)$ is the unique solution of the differential equation

$$X'(t) + A(t)X(t) = 0 \quad (3.31)$$

with initial condition $X(0) = \text{Id}_n$. Observe that the function $X(t)$ satisfies (3.31) and $X(0) = \text{Id}_n$ if and only if the following integral equation holds

$$X(t) = \text{Id}_n - \int_0^t A(s)X(s)ds. \quad (3.32)$$

We will solve (3.32) by applying the method of Picard-Lindelöf. For this, we define recursively a sequence of approximations to the solution:

$$X_0(t) = \text{Id}_n,$$

$$X_r(t) = \text{Id}_n - \int_0^t A(s)X_{r-1}(s)ds, \quad r \geq 1.$$  

We need to show that the sequence $\{X_r(t)\}$ converges. First we prove by induction that, for all $r \geq 1$, one has:

$$X_r(t) - X_{r-1}(t) = (-1)^r \int_{t \geq s_1 \geq \ldots \geq s_r \geq 0} A(s_1)\cdots A(s_r)ds_1 \cdots ds_r. \quad (3.33)$$

Indeed, by definition

$$X_1(t) - X_0(t) = - \int_0^t A(s)ds,$$
which settles the case $r = 1$. Assume that (3.33) holds for all indices smaller than $r$. By the induction hypothesis

$$X_r(t) - X_{r-1}(t) = -\int_0^t A(s)(X_{r-1}(s) - X_{r-2}(s)) ds$$

$$= -\int_0^t A(s)(-1)^{r-1} \int_{s \geq s_2 \geq \cdots \geq s_r \geq 0} A(s_2) \cdots A(s_r) ds_2 \cdots ds_r ds$$

$$= (-1)^r \int_{t \geq s_1 \geq \cdots \geq s_r \geq 0} A(s_1) \cdots A(s_r) ds_1 \cdots ds_r.$$ 

Using that the volume of the simplex $\Delta^r$ is $1/r!$, we deduce that there exists a constant $K > 0$ such that

$$\int_{t \geq s_1 \geq \cdots \geq s_r \geq 0} A(s_1) \cdots A(s_r) ds_1 \cdots ds_r = O\left(\frac{K^r}{r!}\right).$$

This estimate proves that $\{X_r(t)\}$ is a Cauchy sequence and that its limit is given by the convergent series

$$X_\infty(t) = \sum_{r \geq 0} (-1)^r \int_{t \geq s_1 \geq \cdots \geq s_r \geq 0} A(s_1) \cdots A(s_r) ds_1 \cdots ds_r.$$ 

Clearly, $X_\infty(0) = \text{Id}_n$, and a telescopic argument shows that $X_\infty(t)$ satisfies the differential equation (3.31). Therefore,

$$T(\gamma) = X_\infty(1) = \text{Id}_n - \int_\gamma \omega + \int_\gamma \omega \omega - \cdots \quad \square$$

The entries of the parallel transport matrix involve a priori infinite series, and therefore they are not iterated integrals according to Definition 3.15. On the contrary, if we can assure that the products appearing in the right-hand side of equation (3.30) vanish for $r$ large enough, then all the entries would be finite sums. One can then combine Theorem 3.28 and Proposition 3.29 to give examples of iterated integrals which are homotopy functionals.

**Example 3.34.** A strictly upper triangular matrix $A(t)$ is nilpotent, so there exists $r_0 \geq 1$ such that $A(s_1) \cdots A(s_{r_0}) = 0$. In this case, the parallel transport function reduces to an iterated integral:

$$T = 1 - \int \omega + \cdots + (-1)^{r_0-1} \int_{r_0-1} \omega \omega \cdots \omega.$$ 

For instance, when

$$\omega = \begin{pmatrix} 0 & \omega_1 & \omega_{12} \\ 0 & 0 & \omega_2 \\ 0 & 0 & 0 \end{pmatrix},$$
the parallel transport function is given by

\[ T = \begin{pmatrix} 1 & -\int \omega_1 & \int \omega_1 \omega_2 - \int \omega_{12} \\ 0 & 1 & -\int \omega_2 \\ 0 & 0 & 1 \end{pmatrix} \]

and the curvature of the connection is equal to

\[ d\omega + \omega \wedge \omega = \begin{pmatrix} 0 & d\omega_1 & \omega_1 \wedge \omega_2 + d\omega_{12} \\ 0 & 0 & d\omega_2 \\ 0 & 0 & 0 \end{pmatrix}. \]

Thus, \( \nabla = d + \omega \) is flat if and only if the following two equalities hold

\[ d\omega_1 = d\omega_2 = 0, \quad d\omega_{12} + \omega_1 \wedge \omega_2 = 0. \tag{3.35} \]

It follows that the iterated integral \( \int \omega_1 \omega_2 - \int \omega_{12} \) is a homotopy functional if and only if the conditions (3.35) are satisfied.

More generally, one has the following result [Hai87a, Prop. 3.1]:

**Proposition 3.36.** Let \( \omega, \omega_1, \ldots, \omega_r \) be smooth \( k \)-valued 1-forms on \( M \). Assume that all the \( \omega_i \) are closed. An iterated integral of length two

\[ \sum_{1 \leq i, j \leq r} a_{ij} \int \omega_i \omega_j - \int \omega \]

is a homotopy functional if and only if \( d\omega + \sum_{1 \leq i, j \leq r} a_{ij} \omega_i \wedge \omega_j = 0 \).

We can generalize the previous example to nilpotent flat connections to obtain plenty of iterated integrals that are homotopy functionals. Note that nilpotent flat connections define unipotent local systems, which are nothing else but finite-dimensional unipotent representations of the fundamental group. The next two sections are devoted to detailing this relation.

\[ \star \star \star \]

**Exercise 3.37 (Groupoids as categories).** Let \( C \) be a small category in which all morphisms are isomorphisms. Show that \( C \) yields a groupoid in the sense of Definition 3.6. Conversely, given a groupoid, construct such a category. Note that groups correspond to the case where the set of objects consists of a single element.

**Exercise 3.38 (Integration by parts).** Let \( \omega_1, \ldots, \omega_r \) be smooth \( k \)-valued 1-forms on a differentiable manifold \( M \) and \( f \) a smooth function. Prove that the following three equalities hold for any path \( \gamma \in \mathcal{P}(M) \):

\[ \int_{\gamma} df \omega_1 \cdots \omega_r = (f \circ \gamma)(1) \int_{\gamma} \omega_1 \cdots \omega_r - \int_{\gamma} (f \omega_1) \omega_2 \cdots \omega_r, \tag{3.39} \]
\[ \int_\gamma \omega_1 \cdots \omega_{i-1} df \omega_i \cdots \omega_r = \int_\gamma \omega_1 \cdots (f \omega_i - 1) \omega_i \cdots \omega_r, \quad (3.40) \]
\[ \int_\gamma \omega_1 \cdots \omega_r df = \int_\gamma \omega_1 \cdots \omega_{r-1} (f \omega_r) - (f \circ \gamma)(0) \int_\gamma \omega_1 \cdots \omega_r. \quad (3.41) \]

**Exercise 3.42.** Prove Proposition 3.18.

**Exercise 3.43.** As we have seen in Example 3.24, the iterated integral of the 1-forms \( \omega_1 = dx \) and \( \omega_2 = dy \) on \( \mathbb{R}^2 \) is not a homotopy functional. According to Proposition 3.36, this is explained by the fact that \( \omega_1 \wedge \omega_2 \) does not vanish. Find a 1-form \( \omega_{12} \) such that \( d\omega_{12} + \omega_1 \wedge \omega_2 = 0 \) and check that the value of the iterated integral
\[ \int \omega_1 \omega_2 - \int \omega_{12} : \mathcal{P}(\mathbb{R}^2) \to \mathbb{R} \]
on the paths \( \gamma_{a,b} \) from Example 3.24 is now independent of \( a \) and \( b \).

**Exercise 3.44 (Another proof of (3.20) and (3.21)).** Let \( \omega_1, \ldots, \omega_r \) be 1-forms on a differentiable manifold \( M \). Consider the connection on the rank \((r + 1)\) trivial bundle \( \nabla = d + d\omega \) given by the matrix
\[
\omega = \begin{pmatrix}
0 & \omega_1 & 0 & \cdots & 0 \\
0 & 0 & \omega_2 & \cdots & 0 \\
0 & 0 & 0 & \cdots & \omega_r \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

Show that the parallel transport associated with \( \nabla \) is the \( r + 1 \) by \( r + 1 \) matrix \( T = (T_{ij}) \) with entries
\[
T_{ij} = \begin{cases}
\int \omega_i \cdots \omega_{j-1} & i < j, \\
1 & i = j, \\
0 & i > j.
\end{cases}
\]

Using that \( T(\gamma_1 \gamma_2) = T(\gamma_1)T(\gamma_2) \) by Proposition 3.27, deduce from this computation another proof of formulas (3.20) and (3.21).

**3.2. Affine group schemes, Lie algebras, and Hopf algebras.** In this section, we recall the definition of affine group schemes and of two intimately related algebraic structures: Lie algebras and Hopf algebras. The book [Wat79] is an excellent entry point for readers unfamiliar with these notions. We also recommend [Car07] as an introduction to Hopf algebras. We assume that the reader is familiar with the relation between affine schemes and commutative algebras as in [Har77, Chapter II, Section 2].
Throughout this section, we fix a field \( k \) of characteristic zero (later, in the applications, it will always be equal to \( \mathbb{Q} \)). All undecorated cartesian and tensor products are assumed to be over \( k \). When we want to emphasize that a group is simply a group and does not carry any additional structure (such as a scheme structure or a topology), we will call it “abstract group”.

3.2.1. Affine group schemes. Recall that the category of affine schemes over \( k \) is equivalent to the category of commutative \( k \)-algebras through the contravariant functors

\[ A \mapsto \text{Spec}(A), \quad G \mapsto \mathcal{O}(G), \]

where \( \mathcal{O}(G) \) is the ring of regular functions on \( G \).

**Definition 3.45.** Let \( A \) be a commutative \( k \)-algebra. The corresponding affine \( k \)-scheme \( G = \text{Spec}(A) \) is said to be an affine group scheme if it is endowed with algebraic operations

\[ \mu : G \times G \to G \text{ (product)}, \quad e : \text{Spec}(k) \to G \text{ (unit)}, \quad \iota : G \to G \text{ (inverse)}, \]

satisfying the usual axioms of a group, which are expressed by the commutativity of the following three diagrams:

1. **Associativity:**

\[
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\mu \times \text{Id}} & G \times G \\
\downarrow\text{Id} \times \mu & & \downarrow\mu \\
G \times G & \xrightarrow{\mu} & G.
\end{array}
\]

2. **Unit:**

\[
\begin{array}{ccc}
G \times \text{Spec}(k) & \xrightarrow{\text{Id} \times e} & G \times G \\
\downarrow\text{pr}_1 & & \downarrow\mu \\
G & & \text{Spec}(k) \times G \\
\downarrow\text{pr}_2 & & \downarrow\mu
\end{array}
\]

3. **Inverse:**

\[
\begin{array}{ccc}
G \times G & \xrightarrow{\mu} & G \\
\downarrow\text{Id} \times \iota & & \downarrow\mu \\
G \times G & \xrightarrow{\iota \times \text{Id}} & G \\
\downarrow\iota \times \text{Id} & & \downarrow\mu
\end{array}
\]

where \( \pi \) denotes the structural map of \( G \) as a \( k \)-scheme.
If the algebra $A$ is finitely generated, we say that $G$ is an *algebraic affine group scheme*. We will denote by $\text{AGS}(k)$ the category of affine group schemes over $k$ and by $\text{AAGS}(k)$ the one of algebraic affine group schemes over $k$.

We will see below (Lemma 3.49) that every affine group scheme is in fact a projective limit of algebraic affine group schemes and hence $\text{AGS}(k)$ is equivalent to the category of pro-algebraic affine group Pro($\text{AAGS}$) (Theorem 3.50). See A.5 for a construction of the Pro-category.

A group scheme over $k$ defines a functor from the category of commutative $k$-algebras to the category of abstract groups. Namely, given $G = \text{Spec}(A)$ as in Definition 3.45, one considers the functor:

$$ R \mapsto G(R) = \text{Hom}_{k\text{-alg}}(A, R). $$

The fact that $G$ is a group scheme endows $G(R)$ with a structure of group.

Conversely, we will say that a functor $F$ from commutative $k$-algebras to groups is *representable* if there exist an affine group scheme $G$ and a natural isomorphism of functors between $F$ and $G$.

### 3.2.2. Hopf algebras.

The defining properties of a group scheme can be transferred to the corresponding algebra, yielding the concept of a Hopf algebra. We begin by recalling the definition of algebra, coalgebra, bialgebra, and Hopf algebra. In what follows, we denote by $\tau: H \otimes H \to H \otimes H$ the automorphism that flips the two factors.

**Definition 3.46.** Let $H$ be a $k$-vector space.

1. An *algebra* structure on $H$ is the data of two linear morphisms
   $$ \nabla: H \otimes H \to H \ (\text{product}), $$
   $$ \eta: k \to H \ (\text{unit}), $$

   such that the following diagrams commute:
   
   (a) *Associativity:*
   $$ \begin{array}{c}
   H \otimes H \otimes H 
   \xrightarrow{\nabla \otimes \text{Id}} 
   H \otimes H \\
   \text{Id} \otimes \nabla \\
   \downarrow \\
   H \otimes H 
   \xrightarrow{\nabla} 
   H. 
   \end{array} $$

   (b) *Unit:*
   $$ \begin{array}{c}
   H \otimes k 
   \xrightarrow{\text{Id} \otimes \eta} 
   H \otimes H 
   \xrightarrow{\eta \otimes \text{Id}} 
   k \otimes H, \\
   \cong \\
   \nabla \\
   \cong \\
   \text{Id} \\
   \nabla \\
   \text{Id} \\
   \cong \\
   \text{Id} \\
   \nabla \\
   \text{Id} \\
   \cong \\
   \text{Id} \\
   \nabla \\
   $$

   where the diagonal maps are canonical isomorphisms.
The algebra structure is said to be **commutative** if the following diagram commutes:

\[
\begin{array}{ccc}
H \otimes H & \xrightarrow{\nabla} & H \\
\downarrow^{\tau} & & \downarrow^{\nabla} \\
H \otimes H & \xrightarrow{\nabla} & H.
\end{array}
\]

(2) A **coalgebra** is the dual notion of an algebra. That is, a coalgebra structure on \( H \) is the data of two morphisms

\[
\begin{align*}
\Delta &: H \rightarrow H \otimes H \text{ (coproduct)}, \\
\epsilon &: H \rightarrow k \text{ (counit)},
\end{align*}
\]

such that the following diagrams commute:

(a) **Coassociativity:**

\[
\begin{array}{ccc}
H & \xrightarrow{\Delta} & H \otimes H \\
\downarrow^{\Delta} & & \downarrow^{\Delta \otimes \text{Id}} \\
H \otimes H & \xrightarrow{\Delta \otimes \text{Id}} & H \otimes H \otimes H.
\end{array}
\]

(b) **Counit:**

\[
\begin{array}{ccc}
H \otimes k & \xleftarrow{\text{Id} \otimes \epsilon} & H \otimes H & \xrightarrow{\tau \otimes \text{Id}} & k \otimes H \\
\cong & & \uparrow^{\Delta} & & \cong \\
& & H & & \\
\end{array}
\]

The coalgebra structure is said to be **cocommutative** if the further diagram

\[
\begin{array}{ccc}
H \otimes H & \xrightarrow{\Delta} & H \otimes H \\
\downarrow^{\tau} & & \downarrow^{\Delta} \\
H \otimes H & \xrightarrow{\Delta} & H \otimes H
\end{array}
\]

commutes.

(3) A **bialgebra** structure is a structure of algebra together with a structure of coalgebra that are compatible. This means that the coproduct and the counit are morphisms of algebras and that the product and the unit are morphisms of coalgebras and amounts to the commutativity of the following diagrams:
(a) **Product and coproduct:**

\[
H \otimes H \xrightarrow{\nabla} H \xrightarrow{\Delta} H \otimes H
\]

\[
\Delta \otimes \Delta \downarrow \downarrow \\
H \otimes H \otimes H \otimes H \xrightarrow{\text{Id} \otimes \tau \otimes \text{Id}} H \otimes H \otimes H \otimes H
\]

(b) **Unit and counit:**

\[
k \xrightarrow{\text{Id}} H \xrightarrow{\eta} k
\]

\[
k \xrightarrow{\epsilon} k \xrightarrow{\text{Id}} H \xrightarrow{\epsilon} k
\]

(c) **Unit and coproduct:**

\[
H \xrightarrow{\epsilon \otimes \epsilon} k \otimes k \simeq k
\]

(4) A bialgebra \(H\) is said to be a **Hopf algebra** if it is further equipped with a morphism of algebras

\[
S: H \to H \quad \text{(antipode)}
\]

such that the following diagram commutes:

(e) **Antipode:**

\[
H \otimes H \xrightarrow{S \otimes \text{Id}} H \otimes H \xrightarrow{\nabla} H
\]

\[
\Delta \xrightarrow{S \otimes \text{Id}} H \otimes H \xrightarrow{\epsilon} k \xrightarrow{\eta} H
\]

\[
H \otimes H \xrightarrow{\text{Id} \otimes S} H \otimes H \xrightarrow{\nabla} H
\]

(5) A bialgebra \(H\) is called **commutative** if the product \(\nabla\) is commutative, and **cocommutative** if the coproduct is commutative.

**Remark 3.47.** A bialgebra does not always admit an antipode, see Exercise 3.84 for an example.
Given a commutative Hopf algebra $A$, we can use the algebra structure to define an affine scheme $\text{Spec}(A)$. Then the coproduct, counit and antipode of $A$ give rise to the dual notions of product, unit and inverse on $\text{Spec}(A)$. We immediately obtain the following result:

**Proposition 3.48.** The assignment $A \mapsto \text{Spec}(A)$ is a contravariant equivalence between the category of commutative Hopf $k$-algebras and the category of affine group schemes over $k$. The quasi-inverse equivalence is given by $G \mapsto \mathcal{O}(G)$. Moreover, the group scheme $G$ is commutative if and only if the Hopf algebra $\mathcal{O}(G)$ is cocommutative.

By way of illustration, we show how to use this correspondence to prove the promised result that affine group schemes are pro-algebraic.

**Lemma 3.49.** Every Hopf algebra is a directed union of Hopf subalgebras which are finitely generated $k$-algebras. Therefore, every affine group scheme is a projective limit of algebraic affine group schemes.

**Proof.** Let $H$ be a Hopf algebra and $x \in H$. It suffices to show that $x$ is contained in a finitely generated Hopf subalgebra of $H$. Choose a basis $\{h_i\}$ of $H$ and write $\Delta(x) = \sum_i x_i \otimes h_i$, where only finitely many $x_i$ are non-zero. Let $V \subseteq H$ be the vector subspace spanned by $x$ and the $x_i$. We claim that $\Delta(V) \subseteq V \otimes H$, which amounts of course to saying that $\Delta(x_i) \in V \otimes H$ for all $i$. Indeed, if one writes $\Delta(h_{ij}) = \sum_\ell h_{i\ell} \otimes h_{j\ell}$ with $h_{i\ell} \in k$, then
\[
\sum_i \Delta(x_i) \otimes h_i = (\Delta \otimes \text{Id}) \Delta(x)
= (\text{Id} \otimes \Delta) \Delta(x)
= \sum_{i,j,\ell} x_i \otimes a_{ij\ell} h_j \otimes h_\ell
\]
by the associativity of the coproduct. Comparing the coefficients of $h_\ell$ yields $\Delta(x_i) = \sum_{i,j} x_i \otimes a_{ij\ell} h_j \otimes h_\ell \in V \otimes H$, as we wanted. Now let $\{v_i\}$ be a basis of $V$ and write $\Delta(v_j) = \sum_i v_j \otimes h_{ij}$ with $h_{ij} \in H$. By Exercise 3.83, it follows that $\Delta(h_{ij}) = \sum_\ell h_{i\ell} \otimes h_{j\ell}$, and hence the vector space $U$ generated by $\{v_i\}$ and $\{h_{ij}\}$ satisfies $\Delta(U) \subseteq U \otimes U$. If $W$ is the vector space spanned by $U$ and $S(U)$, then $\Delta(W) \subseteq W \otimes W$ and $S(W) \subseteq W$ using Exercise 3.83 again. Finally, let $A$ be the subalgebra of $H$ generated by $W$. Since $\Delta$ and $S$ are morphisms of algebras, we also have $\Delta(A) \subseteq A \otimes A$ and $S(A) \subseteq A$. It is thus a finitely generated Hopf subalgebra of $H$ containing $x$. \qed

In fact, not only every scheme in groups is pro-algebraic, but the pro-algebraic structure is unique. More precisely, we have the following result.

**Theorem 3.50.** The functor $\text{Pro}(\text{AAGS}(k)) \to \text{AGS}(k)$ given by
\[
(G_d)_{d \in D} \mapsto \lim_{\longleftarrow \atop d \in D} G_d
\]
is an equivalence of categories.
Proof. In view of Lemma 3.49 and Theorem A.55 from the appendix, one only needs to show that $\mathbf{AAGS}(k)$ is the full subcategory of $\mathbf{AGS}(k)$ consisting of co-compact objects. By duality, this amounts to proving that the compact objects in the category of commutative Hopf algebras are those that are finitely generated as algebras.

On one direction, we have to show that a commutative Hopf algebra that is finitely generated as an algebra is a compact object. In other words, using Remark A.50, if $H$ is such a Hopf algebra and $(B_d)_{d \in D}$ is a directed inductive system of Hopf algebras, we have to check that the canonical map
\[
\lim_{d \in D} \text{Hom}(H, B_d) \longrightarrow \text{Hom}(H, \lim_{d \in D} B_d) \tag{3.51}
\]
is an isomorphism. We first prove injectivity. Let $f \in \lim_{d \in D} \text{Hom}(H, B_d)$. Then there is a $d_0 \in D$ and a map $f_{d_0}: H \to B_{d_0}$ that represents $f$. For each $d \geq d_0$, we write $f_d$ for the composition $H \to B_{d_0} \to B_d$. The image of $f$ by the map (3.51) is the composition
\[
H \to B_{d_0} \to \lim_{d \in D} B_d.
\]

Let $a_1, \ldots, a_n$ be a set of generators of $H$. If the image of $f$ under (3.51) is zero, then for each $i = 1, \ldots, n$, there is a $d_i$ such that $f_{d_i}(a_i) = 0$. Taking $d' \geq d_i$, for $i = 1, \ldots, n$, then $f_{d'} = 0$, which implies that $f = 0$. and the map (3.51) is injective.

To prove surjectivity, we use that, since $H$ is a finitely generated $k$-algebra, it is Noetherian. Hence there is an exact sequence of $H$-modules
\[
0 \longrightarrow I \longrightarrow k[x_1, \ldots, x_n] \longrightarrow H \longrightarrow 0,
\]
where the surjection over $H$ is given by $x_i \mapsto a_i$ and $I$ is a finitely generated ideal. Let $f \in \text{Hom}(H, \lim_{d \in D} B_d)$. There is a $d \in D$ such that, for $i = 1, \ldots, n$, the element $f(a_i) \in \text{Im}(B_d)$. Choosing representatives in $B_d$, we construct a map $\tilde{f}_d$ that fits in a commutative diagram
\[
\begin{array}{ccc}
k[x_1, \ldots, x_n] & \longrightarrow & H \\
\tilde{f}_d & & \downarrow f \\
B_d & \longrightarrow & \lim_{d \in D} B_d.
\end{array}
\]
Since $f(I) = 0$ and $I$ is finitely generated, there is a $d' \geq d$ such that the composition $\tilde{f}_{d'}$
\[
k[x_1, \ldots, x_n] \xrightarrow{\tilde{f}_{d'}} B_d \to B_{d'}
\]
satisfies $\tilde{f}_{d'}(I) = \{0\}$. Therefore, we obtain a map $f_{d'}: H \to B_{d'}$, and hence an element in $f'$ in $\lim_{d \in D} \text{Hom}(H, B_d)$. By construction, this element is in the preimage of $f$ and the map (3.51) is surjective.
On the other direction, we have to show that, if \( \text{Hom}(H, -) \) commutes with direct limits, then \( H \) is finitely generated as algebra. As in the proof of Lemma 3.49, we can write
\[
H = \lim_{d \in D} H_d
\]
with \( H_d \) Hopf algebras that are finitely generated as algebras. Consider the identity map
\[
\text{Id}_H \in \text{Hom}(H, H) = \text{Hom}(H, \lim_{d \in D} H_d).
\]
\( \text{Hom}(H, -) \) commutes with direct limits, the map (3.51) is an isomorphism. Therefore, there is a \( d \in D \) and a map \( H \to H_d \) such that the composition \( H \to H_d \to H \) is \( \text{Id}_H \). Hence the map \( H_d \to H \) is surjective and we deduce that \( H \) is finitely generated as an algebra, concluding the proof of the Corollary. \( \square \)

3.2.3. Comodules and Hopf modules.

Definition 3.52. Let \( H \) be a coalgebra over \( k \). A right comodule over \( H \) is a \( k \)-vector space \( V \), together with a coaction
\[
\Delta : V \to V \otimes H
\]
satisfying the following conditions:

1. (associativity) \((\text{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{Id}) \circ \Delta;\)

2. (compatibility with the counit) \((\text{Id} \otimes \epsilon) \circ \Delta = \text{Id} \) once we identify \( V \) with \( V \otimes k.\)

Left comodules are defined in a similar way.

Examples 3.53. The following are examples of comodules.

1. Every coalgebra is a (right and left) comodule over itself.

2. Let \( A \) be a finite-dimensional algebra and \( M \) a finite-dimensional (left or right) \( A \)-module. Then the dual space \( A^\vee \) is a coalgebra and \( M^\vee \) is an \( A^\vee \)-comodule.

3. Let \( V^* = \bigoplus_{n \in \mathbb{Z}} V^n \) be a graded vector space. We consider the \( k \)-vector space
\[
H = \bigoplus_{n \in \mathbb{Z}} ke_n
\]
with the counit \( \epsilon(e_n) = 1 \) for all \( n \) and the coproduct
\[
\Delta(e_n) = e_n \otimes e_n.
\]
Then \( H \) is a coalgebra and \( V^* \) is a right \( H \)-comodule with coaction
\[
\Delta v = v \otimes e_n,
\]
for \( v \in V^n \) (Exercise 3.82). Similarly it can be seen as a left comodule because the Hopf algebra \( H \) is cocommutative.
DEFINITION 3.54. Let $H$ be a commutative Hopf algebra. A left Hopf module is a vector space $V$ that is a module over the algebra structure of $H$ and a left comodule over its coalgebra structure. Moreover, both structures are compatible in the sense that the equality

$$\Delta(hv) = \Delta(h)\Delta(v)$$

holds for all $h \in H$ and for all $v \in V$.

### 3.2.4. Graded Hopf algebras.

DEFINITION 3.55.

1. A bialgebra $H$ is said to be graded if the underlying $k$-vector space has a direct sum decomposition

$$H = \bigoplus_{n \in \mathbb{Z}} H_n$$

compatible with the operations in the sense that, for all $p, q, n \geq 0$,

$$\nabla(H_p \otimes H_q) \subseteq H_{p+q}, \quad \Delta H_n \subseteq \bigoplus_{i+j=n} H_i \otimes H_j$$

If, moreover, $H_n = \{0\}$ for $n < 0$ and $H_0 = k$ we say that $H$ is connected.

2. A graded Hopf algebra is a Hopf algebra such that the underlying bialgebra is graded and the antipode satisfies $SH_n \subseteq H_n$.

One advantage of working with graded connected bialgebras is that they automatically admit a unique antipode turning them into (graded) Hopf algebras (see Exercise 3.85).

### 3.2.5. Examples.

In this paragraph, we give a few examples of affine group schemes and their corresponding Hopf algebras. Of particular interest for the sequel is the Hoffman algebra from Example 3.60.

EXAMPLES 3.56.

1. The trivial group scheme is $\text{Spec}(k)$ with all operations equal to the identity. The corresponding commutative Hopf algebra is $k$ with all operations equal to the identity once we identify $k \otimes k$ with $k$.

2. The multiplicative group $\mathbb{G}_m$. The functor from commutative $k$-algebras to groups given by $R \mapsto R^\times$ is represented by an affine group scheme $\mathbb{G}_m$. The corresponding Hopf algebra is $k[x, x^{-1}]$, together with the coproduct given by

$$\Delta(x) = x \otimes x, \quad \Delta(x^{-1}) = x^{-1} \otimes x^{-1},$$

the counit $\epsilon(x) = \epsilon(x^{-1}) = 1$, and the antipode determined by

$$S(x) = x^{-1} \quad \text{and} \quad S(x^{-1}) = x.$$
(3) The additive group $G_a$. The functor from commutative $k$-algebras to groups given by $R \mapsto (R, +)$ is represented by an affine group scheme $G_a$. The corresponding Hopf algebra is $k[x]$ with

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \epsilon(x) = 0, \quad S(x) = -x.$$ 

(4) The linear group $GL_n$. The functor that, to each commutative $k$-algebra $R$, assigns the group $GL_n(R)$ of invertible $n \times n$ matrices with entries in $R$ is representable by an affine group scheme $GL_n$. The corresponding Hopf algebra is

$$k[t, (x_{ij})_{i,j=1,...,n}]/(t \det(x_{ij}) - 1).$$

Recall that this means that the determinant $\det(x_{ij})$, which is a homogeneous polynomial of degree $n$ in the entries $x_{ij}$, is invertible. Its inverse is the variable $t$. The coproduct is given by

$$\Delta t = t \otimes t, \quad \Delta x_{ij} = \sum_{l=1}^{n} x_{il} \otimes x_{lj}.$$ (3.57)

The counit is the map

$$\epsilon(x_{ij}) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Finally, the antipode can be expressed using Cramer’s rule for the inverse of a matrix in terms of cofactors, that is,

$$S(t) = t^{-1}, \quad S(x_{ij}) = tC_{ji},$$

where $C_{ij}$ is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the $i$-th row and the $j$-th column of $(x_{lm})_{l,m}$. Observe that $C_{ij}$ is a homogeneous polynomial of degree $n - 1$.

(5) Similarly, for every finite-dimensional $k$-vector space $V$, the functor

$$R \mapsto GL(R \otimes V)$$

is representable by an algebraic affine $k$-group scheme $GL(V)$. If $V$ has dimension $n$, the choice of a basis of $V$ induces an isomorphism between $GL(V)$ and $GL_n$.

(6) One needs to be cautious when working with infinite-dimensional vector spaces. In fact, given a $k$-vector space $V$, the functor

$$R \mapsto R \otimes V$$

is representable if and only if $V$ is finite-dimensional [GD71, 9.4.10]. Therefore, the group-valued functor $Aut(V)$ defined by

$$R \mapsto Aut_R(R \otimes V)$$
does not define an affine group scheme when \( V \) has infinite dimension. In fact, the rule (3.57) from Example (4) above does not define a coproduct in the infinite-dimensional case.

Example 3.58. Any Zariski closed subset of \( \text{GL}_n \) that is stable under matrix multiplication and matrix inversion, and contains the identity matrix is also an affine group scheme. This includes all classical algebraic groups such as the special linear group

\[
\text{SL}_n = \text{Spec}(k[(x_{ij})_{i,j=1,...,n}]/(\det(x_{ij}) - 1)).
\]

Example 3.59. Let \( G \) be a finite group. The group algebra

\[
k[G] = \{ \sum_{g \in G} a_g g \mid a_g \in k \}
\]

carries a structure of Hopf algebra. The product is determined by the group structure of \( G \), that is:

\[
\sum_{g \in G} a_g g \cdot \sum_{h \in G} b_h h = \sum_{g,h \in G} a_g b_h g h = \sum_{f \in G} \left( \sum_{g \in G} a_g b_{g^{-1}} f \right) f.
\]

The coproduct is given by \( \Delta g = g \otimes g \), and the antipode by \( S(g) = g^{-1} \). This Hopf algebra is cocommutative but not commutative, unless the group \( G \) is abelian.

Example 3.60. For the purpose of these notes, the main example will be the Hoffman algebra \( \mathcal{H} \) of Section 1.6. Recall that the underlying vector space of \( \mathcal{H} \) is the vector space \( \mathbb{Q}\langle X \rangle \) generated by (non-commutative) words in two letters \( x_0, x_1 \). The Hopf algebra structure is given by

1. **Shuffle product.**

\[
x_{\varepsilon_1} \cdots x_{\varepsilon_r} \shuffle x_{\varepsilon_{r+1}} \cdots x_{\varepsilon_{r+s}} = \sum_{\sigma \in \Omega(r,s)} x_{\varepsilon_{\sigma^{-1}(1)}} \cdots x_{\varepsilon_{\sigma^{-1}(p+q)}}.
\]

2. **Unit.** The map \( \eta: \mathbb{Q} \to \mathcal{H} \) that sends 1 to the empty word.

3. **Deconcatenation coproduct.**

\[
\Delta x_{\varepsilon_1} \cdots x_{\varepsilon_n} = \sum_{j=0}^{n} x_{\varepsilon_1} \cdots x_{\varepsilon_j} \otimes x_{\varepsilon_{j+1}} \cdots x_{\varepsilon_n}.
\]

4. **Counit.** The map \( \epsilon: \mathcal{H} \to \mathbb{Q} \) that sends every non-empty word to 0 and the empty word to 1.

5. **Antipode.**

\[
S(x_{\varepsilon_1} \cdots x_{\varepsilon_n}) = (-1)^n x_{\varepsilon_n} \cdots x_{\varepsilon_1}
\]
For convenience, if $w$ is a word on the letters $x_0$ and $x_1$, we will also use the notation

$$w^* = S(w).$$

(3.61)

Consider the grading of $\mathfrak{H}$ that gives weight $n$ to $x_{\epsilon_1} \cdots x_{\epsilon_n}$. Since all the above operations respect the weight, $\mathfrak{H}$ is a graded Hopf algebra. Moreover it is connected.

3.2.6. The dual of a Hopf algebra. Let $H$ be a Hopf algebra over $k$. If $H$ is a finite-dimensional $k$-vector space, then its dual $H^\vee = \text{Hom}(H, k)$ is again equipped with a Hopf algebra structure, whose product is the dual of the coproduct of $H$, whose coproduct is the dual of the product, and the antipodes of $H$ and $H^\vee$ are dual of each other. In other words, the axioms in Definition 3.46 are self-dual. This uses that the canonical morphism

$$H^\vee \otimes H^\vee \xrightarrow{\sim} (H \otimes H)^\vee$$

(3.62)

is an isomorphism. If $H$ has infinite dimension, the morphism (3.62) fail to be an isomorphism, and hence the dual of the product does not give rise to a coproduct but only to what is called a completed coproduct. Let us explain why. Let $V$ be an infinite-dimensional $k$-vector space and write

$$V = \lim_I V_I,$$

where $I$ runs over the directed set of finite-dimensional subspaces of $V$. Since $\text{Hom}(\cdot, k)$ exchanges inductive and projective limits, the dual of $V$ is

$$V^\vee = \text{Hom}(V, k) = \text{Hom}(\lim_I V_I, k) = \lim_I \text{Hom}(V_I, k) = \lim_I V_I^\vee.$$

Thus, $V^\vee$ has a natural structure of pro-finite-dimensional $k$-vector space.

DEFINITION 3.63. Given a pro-finite-dimensional $k$-vector space

$$W = \lim_I W_I,$$

the completed tensor product with itself is defined as

$$W \hat{\otimes} W = \lim_I (W_I \otimes W_I).$$

Note that the definition requires a structure of pro-finite-dimensional space. When dealing with the dual of an infinite-dimensional vector space we will tacitly assume the previously described structure.

For infinite-dimensional vector spaces, the notion of tensor product is not self-dual. Nevertheless there are canonical arrows

$$V^\vee \otimes V^\vee \longrightarrow (V \otimes V)^\vee \longrightarrow V^\vee \hat{\otimes} V^\vee.$$

That, in general, are not isomorphisms.
Thus, when we dualize the product $A \otimes A \to A$ of an algebra, we only obtain a morphism
\[
A^\vee \longrightarrow (A \otimes A)^\vee \longrightarrow A^\vee \otimes A^\vee
\]
and not necessarily a coproduct $A^\vee \to A^\vee \otimes A^\vee$. A map as in (3.64) is called a \textit{completed coproduct}.

\textbf{Definition 3.65.} A \textit{completed Hopf algebra} $A$ is a pro-finite-dimensional vector space satisfying the analogous properties of a Hopf algebra (Definition 3.46) where all tensor products are replaced by completed tensor products and all the maps are compatible with the pro-finite-dimensional structure. In particular it has a completed coproduct
\[
\Delta: A \longrightarrow A \hat{\otimes} A.
\]
Moreover the algebra product $\nabla: A \otimes A \to A$ factorizes through a \textit{completed product}
\[
A \otimes A \longrightarrow A \hat{\otimes} A \overset{\hat{\nabla}}{\longrightarrow} A.
\]
and the antipode $S$ is compatible with the pro-finite-dimensional structure.

The \textit{dual} of an infinite-dimensional Hopf algebra is a completed Hopf algebra. Typically, we will apply this construction to a connected graded Hopf algebra with finite-dimensional graded pieces. In this case, the notion of completed Hopf algebra can be conveniently written in terms of the topology induced by the augmentation ideal.

\textbf{Example 3.66.} Let $A = k[x]$ be the Hopf algebra of polynomials of Example 3.56 (3). Let $y_m \in A^\vee$ be the element determined by $\langle y_m, x^n \rangle = \delta_{n,m}$. Then

\[
\langle \nabla(y_m \otimes y_n), x^j \rangle = \langle y_m \otimes y_n, \Delta x^j \rangle
\]

\[
= \langle y_m \otimes y_n, (1 \otimes x + x \otimes 1)^j \rangle = \begin{cases} (m+n)! \over n!m!, & \text{if } j = n + m, \\ 0, & \text{otherwise}. \end{cases}
\]

From this equation we deduce that $y_m = y_1^m / m!$ and $A^\vee$ is the algebra of formal series on divided powers. Since we are working over a field of characteristic zero, it is isomorphic to the algebra of formal power series. Thus, writing $y = y_1$, as algebra we have
\[
A^\vee = k[[y]].
\]

One can easily check that the completed coproduct is determined by $\Delta y = 1 \otimes y + y \otimes 1$ and the antipode by $S(y) = -y$. In particular
\[
\Delta y_m = \sum_{j=0}^{m} y_j \otimes y_{m-j}, \quad S(y_m) = -y_m.
\]
The completed coproduct cannot be factored through a true coproduct. Consider the element \( \eta = \sum_{n \geq 0} ny_n \). Then
\[
\Delta \eta = \sum_{n \geq 0} \sum_{j=0}^{n} n y_j \otimes y_{n-j}.
\]
This element does not belong to \( A^\vee \otimes A^\vee \). This can be seen as follows. Any element
\[
\sum_{i,j \geq 0} a_{i,j} y_i \otimes y_j \in A^\vee \otimes A^\vee
\]
 satisfies that the rank of the matrix \((a_{i,j})\) is finite. By contrast, the rank of the matrix \((b_{i,j})\) with \(b_{i,j} = i + j\) is not bounded.

**Example 3.67.** The dual of the Hopf algebra \( \mathcal{H} \) of Example 3.60 is the space \( \mathcal{H}^\vee = \mathbb{Q}(e_0, e_1) \) of series on the non-commutative words in two letters \( e_0, e_1 \). Given a binary sequence \( \alpha \) and an element \( \gamma \in \mathbb{Q}(e_0, e_1) \), the duality is given by the pairing
\[
\langle x_\alpha, \gamma \rangle = \text{coefficient of } e_\alpha \text{ in } \gamma.
\]
This duality and the Hopf algebra structure of \( \mathcal{H} \) endows \( \mathbb{Q}(e_0, e_1) \) with the structures

1. **Concatenation product.** The product \( \Delta^\vee : \mathcal{H}^\vee \otimes \mathcal{H}^\vee \to \mathcal{H}^\vee \) is given by
\[
e_{\varepsilon_1} \cdots e_{\varepsilon_r} \cdot e_{\varepsilon_{r+1}} \cdots e_{\varepsilon_{r+s}} = e_{\varepsilon_1} \cdots e_{\varepsilon_{r+s}}.
\]

2. **Unit.** It is the morphism
\[
\eta^\vee : \mathbb{Q} \to \mathbb{Q}(e_0, e_1)
\]
that sends 1 to the empty word.

3. **Completed coproduct.** It is the unique morphism of algebras
\[
\nabla^\vee : \mathcal{H}^\vee \to \mathcal{H}^\vee \hat{\otimes} \mathcal{H}^\vee
\]
such that
\[
\nabla^\vee e_\varepsilon = 1 \otimes e_\varepsilon + e_\varepsilon \otimes 1, \quad \varepsilon = 0, 1.
\]
This implies that, for any word \( w \) on the alphabet \( \{e_0, e_1\} \),
\[
\nabla^\vee w = \sum_{w_1, w_2} \shuffle (w_1, w_2; w) w_1 \otimes w_2,
\]
where the shuffle index \( \shuffle (w_1, w_2; w) \) was introduced in 1.153.

4. **Counit.** The map
\[
\epsilon^\vee : \mathbb{Q}(e_0, e_1) \to \mathbb{Q}
\]
sending all non-empty words to 0 and the empty word to 1.
(5) **Dual antipode.** It is given by

\[ S^\vee (e_{\varepsilon_1} \cdots e_{\varepsilon_n}) = (-1)^n e_{\varepsilon_n} \cdots e_{\varepsilon_1}. \]

By analogy with (3.61), for a word \( w \) in the letters \( e_0 \) and \( e_1 \), we use the notation

\[ w^* = S^\vee (w). \quad (3.68) \]

### 3.2.7. Lie algebras.

Another important construction attached to a group scheme is its Lie algebra. The definition is modelled after the Lie algebra of a Lie group, which is the tangent space at the origin of the underlying differentiable manifold together with an antisymmetric product that reflects the non-commutativity of the group operation.

**Definition 3.69.** A **Lie algebra** over a field \( k \) is a \( k \)-vector space \( L \) together with a bilinear product

\[ [\cdot, \cdot] : L \otimes L \to L \]

called the **Lie bracket** that satisfies the following two conditions:

1. **Antisymmetry:** \( [a, b] + [b, a] = 0 \) for all \( a, b \in L \).
2. **Jacobi identity:** \([[[a, b], c] + [[b, c], a] + [[c, a], b] = 0 \) for \( a, b, c \in L \).

If, moreover, the underlying vector space has a grading

\[ L = \bigoplus_{n \in \mathbb{Z}} L_n \]

such that \([L_n, L_m] \subseteq L_{n+m}, \) we say that \( L \) is a **graded Lie algebra**.

**Remarks 3.70.**

1. The antisymmetry of the Lie bracket implies that if factors through the exterior product \( L \wedge L \).
2. There is a dual notion to that of Lie algebra called Lie coalgebra.
   We let the reader explore its properties in Exercise 3.88.

To an affine group scheme we can associate a Lie algebra that is the algebraic analogue of the Lie algebra of a Lie group. This Lie algebra can be directly constructed from its Hopf algebra, as we now explain. Let \( G \) be an affine group scheme over \( k \) and let \( A = \mathcal{O}(G) \) be the corresponding commutative Hopf algebra. We keep the notation \((\nabla, \eta, \Delta, \epsilon, S)\) from Definition 3.46.

**Definition 3.71.** The **augmentation ideal** of \( A \) is the kernel of the counit map \( \epsilon : A \to k \). It will be denoted by \( I = \text{Ker}(\epsilon) \).

The augmentation ideal is the maximal ideal of regular functions on \( G \) that vanish at the unit \( e = \eta(1) \). Since \( \epsilon \circ \eta = \text{Id}_k \) there is a a canonical projection \( A \to I \), and therefore a canonical direct sum decomposition \( A = k \oplus I \).
Definition 3.72. The tangent space of the affine group scheme $G$ at the unit element is the $k$-vector space $L = (I/I^2)^\vee$.

To make $L$ into a Lie algebra, we need a bracket $[\cdot, \cdot]: L \wedge L \to L$. We will first define the dual map. For this we observe that the compatibilities of the coproduct with the unit and the counit imply that, if $f \in I$, then
\[ \Delta f = f \otimes 1 - 1 \otimes f \in I \otimes I. \tag{3.73} \]

We now consider the map $I \xrightarrow{\Delta} A \otimes A \to (I/I^2) \otimes (I/I^2) \to (I/I^2) \wedge (I/I^2)$, where the second arrow is induced by the projection $A \to I \to I/I^2$ and the third arrow is the projection from the tensor product to the exterior product. It follows from property (3.73) that the composition of these maps vanishes on $I^2$. Therefore, we obtain a map $d: I/I^2 \to \bigwedge^2(I/I^2)$. \tag{3.74}

By duality, we obtain a map $[\cdot, \cdot]: L \wedge L = (I/I^2)^\vee \wedge (I/I^2)^\vee \to (I/I^2 \wedge I/I^2)^\vee \xrightarrow{d^\vee} (I/I^2)^\vee = L$.

Following Exercise 3.89, the pair $(I/I^2, d)$ is a Lie coalgebra over $k$.

Definition 3.75. The Lie coalgebra associated with the commutative Hopf algebra $A$ is the pair $(I/I^2, d)$. The dual $(L, [\cdot, \cdot])$ is called the Lie algebra of $G$ and denoted $\text{Lie}(G)$.

In practice, to compute the Lie algebra of an affine group scheme $G$, one looks for the elements of $G(k[\varepsilon])$ mapping to the identity in $G(k)$, which is an algebraic characterization of the tangent space at the unit. Here $k[\varepsilon]$ denotes the ring of dual numbers, in which $\varepsilon^2 = 0$.

Examples 3.76.

1. The group $G = \text{GL}_n$ is the open subscheme of the affine space $\mathbb{A}^{n^2}$ defined as the complement of the determinant hypersurface $\{\det = 0\}$. Thus, the tangent space at the origin can be identified with the space $\text{Mat}_n(k)$ of all $n \times n$ matrices over $k$ and the Lie bracket is just the usual commutator of matrices $[A, B] = AB - BA$.

2. The group $G = \text{SL}_n$ is the closed subscheme of $\text{GL}_n$ defined by the equation $\det = 1$. The Lie algebra of $G$ is a subalgebra of $\text{Lie}(\text{GL}_n)$, and hence of $\text{Mat}_n(k)$. To determine it, one needs to find which matrices of the form $1 + \varepsilon M$, with $\varepsilon^2 = 0$, have determinant 1. Since $\det(1 + \varepsilon M) = 1 + \varepsilon \text{Tr}(M)$,
we deduce that $\text{Lie}(\text{SL}_n)$ can be identified with the space of traceless $n$ by $n$ matrices.

3.2.8. The universal enveloping algebra. It is sometimes convenient to replace a Lie algebra with an associative algebra containing the same information. This is the universal enveloping algebra.

**Definition 3.77.** Let $(L, [\cdot, \cdot])$ be a Lie algebra. Its universal enveloping algebra is an associative algebra $U(L)$, together with a universal morphism $\iota_L: L \to U(L)$ such that

$$\iota_L([a, b]) = \iota_L(a)\iota_L(b) - \iota_L(b)\iota_L(a).$$

By “universal” we mean that, if $A$ is another associative algebra with a map $\iota: L \to A$ satisfying $\iota([a, b]) = \iota(a)\iota(b) - \iota(b)\iota(a)$, then there exists a unique map $\varphi: U(L) \to A$ such that $\iota = \varphi \circ \iota_L$.

We next recall the construction of the universal enveloping algebra of a Lie algebra. Let $T(L)$ be the tensor algebra over $L$. That is

$$T(L) = \bigoplus_{n \geq 0} L^\otimes n$$

with the associative product determined by

$$a_1 \otimes \cdots \otimes a_r \cdot a_{r+1} \otimes \cdots \otimes a_{r+s} = a_1 \otimes \cdots \otimes a_{r+s}.$$

If $\dim_k L > 1$ this algebra is non-commutative.

Let $J \subseteq T(L)$ be the two-sided ideal generated by the elements

$$a \otimes b - b \otimes a - [a, b], \quad a, b \in L$$

Then

$$U(L) = T(L)/J,$$

and the map $\iota_L$ is the composition $L \to T(L) \to U(L)$.

By Exercise 3.87 below, the algebra $U(L)$ has a coproduct $\Delta$ determined by the condition

$$\Delta \iota_L(a) = \iota_L(a) \otimes 1 + 1 \otimes \iota_L(a).$$

In fact, $U(L)$ is a Hopf algebra, whose counit $\eta: U(L) \to \mathbb{Q}$ is induced by the zero map $L \to \mathbb{Q}$, and whose antipode is characterized by $S(x) = -x$ for all $x \in L$.

The main structure theorem for universal enveloping algebras is

**Theorem 3.78 (Poincaré–Birkhoff–Witt).** The map $\iota_L$ is injective. In particular, one can recover $L$ from the universal enveloping algebra $U(L)$, together with the coproduct $\Delta$, as the subspace of primitive elements

$$a \in L \iff \Delta a = a \otimes 1 + 1 \otimes a.$$
Let $G$ be an affine group scheme and $L = \text{Lie}(G)$. Being $\mathcal{O}(G)$ a Hopf algebra, its dual $\mathcal{O}(G)^\vee$ has a structure of associative algebra. Since $L = (I/I^2)^\vee$ there is a canonical map

$$\varphi_G : L \to \mathcal{O}(G)^\vee$$

(3.79)

that sends an element $X \in L$ to the composition

$$\mathcal{O}(G) \twoheadrightarrow I/I^2 \xrightarrow{X} k.$$ 

Using Exercise 3.90 and the universal property of the universal enveloping algebra, we obtain a canonical map $U(L) \to \mathcal{O}(G)^\vee$, that, in general, is not an isomorphism.

For the sequel, we also need to introduce the completion of the universal enveloping algebra.

**Definition 3.80.** The completed universal enveloping algebra is the completion $\hat{U}(L)$ of $U(L)$ with respect to the ideal $\text{Ker}(\eta)$, where $\eta$ is the counit of $U(L)$. In other words, writing $J = \text{Ker}(\eta)$, then

$$\hat{U}(L) = \varinjlim U(L)/J^{N+1}.$$ 

**Examples 3.81.**

(1) Let $G = \mathbb{G}_a$ be the additive group over $\mathbb{Q}$. Then its algebra of functions is the polynomial ring $\mathcal{O}(G) = \mathbb{Q}[x]$ and its Lie algebra is the abelian one-dimensional algebra $L = \mathbb{Q}$. Its universal enveloping algebra is the algebra of polynomials $\mathbb{Q}[y]$, while its completed universal enveloping algebra is the algebra of formal power series $\mathbb{Q}[y]$. The map $\mathbb{Q}[y] \to \mathcal{O}(G)^\vee$ sends the divided power $y^n/n!$ to the dual of $x^n$. This map is not an isomorphism, but it can be extended to an isomorphism $\mathbb{Q}[y] \to \mathcal{O}(G)^\vee$.

(2) Let $G = \mathbb{G}_m$ be the multiplicative group over $\mathbb{Q}$. Then its algebra of functions is the ring of Laurent polynomials $\mathcal{O}(G) = \mathbb{Q}[x, x^{-1}]$ and its Lie algebra is again the abelian one-dimensional algebra $L = \mathbb{Q}$ whose universal enveloping algebra is the algebra of polynomials $\mathbb{Q}[y]$, and its completed universal enveloping algebra is the algebra of formal power series $\mathbb{Q}[y]$. The map $\mathbb{Q}[y] \to \mathcal{O}(G)^\vee$ sends $y$ to the linear form $p \mapsto p'(1)$. Hence, identifying $y^n$ with the corresponding element of $\mathcal{O}(G)^\vee$ we have

$$y^n(x^\ell) = \ell^n.$$ 

In this case, the map $\mathbb{Q}[y] \to \mathcal{O}(G)^\vee$ cannot be extended to the completed universal enveloping algebra.

***
Exercises

Exercises 3.82. Prove that the space $H$ of Example 3.53 is a coalgebra and that $V^*$ is an $H$-comodule.

Exercises 3.83. Let $H$ be a Hopf algebra.

(a) Consider a finite-dimensional subvector space $V$ of $H$ satisfying $\Delta(V) \subseteq V \otimes H$. Pick a basis $\{v_i\}$ of $V$ and write $\Delta(v_j) = \sum_i v_i \otimes h_{ij}$. Prove that $\Delta(h_{ij}) = \sum_\ell h_{i\ell} \otimes h_{\ell j}$.

(b) Show that $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$, where $\tau$ is the flip of the factors of $H \otimes H$. Concretely, if $\Delta(h) = \sum_i a_i \otimes b_i$, then $\Delta(S(h)) = \sum_i S(b_i) \otimes S(a_i)$.

Exercises 3.84 (A bialgebra without antipode). Let $H = k[x]$ be the polynomial algebra in one variable. The coproduct $\Delta(x) = x \otimes x$ and the counit $\epsilon(x) = 1$ endow $H$ with the structure of a cocommutative bialgebra. Show that $H$ cannot have an antipode.

Exercises 3.85 (A connected graded bialgebra has an antipode). Let $H$ be a connected graded bialgebra.

(a) Use the commutativity of diagram (2) in Definition 3.46 to prove that the counit $\epsilon: H \to k$ vanishes on $H_n$ for all $n \geq 1$, and hence induces an isomorphism $H_0 \cong k$.

(b) Show that the antipode $S: H \to H$ is the unique algebra morphism such that $S|_{H_0} = \text{Id}$ and, if $x \in H_n$ for $n \geq 1$, $S(x) = -x - \sum \nabla(S(x') \otimes x'''),$

where the sum runs over all elements $x''$ appearing in the coproduct $\Delta(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x'''$.

Exercises 3.86. Let $\mathfrak{H}$ be the Hoffman algebra.

(a) Verify that the operations described in Example 3.60 endow $\mathfrak{H}$ with a Hopf algebra structure.

(b) Recall that $\mathfrak{H}$ is graded by assigning weight $n$ to $x_{\xi_1} \cdots x_{\xi_n}$. Prove by induction on $n$ that the recipe to compute the antipode presented in Exercise 3.85 yields $S(x_{\xi_1} \cdots x_{\xi_n}) = (-1)^n x_{\xi_n} \cdots x_{\xi_1}$.

Exercises 3.87. Consider the coproduct $\Delta: T(L) \to T(L) \otimes T(L)$, which is the unique algebra homomorphism determined by the condition $\Delta a = 1 \otimes a + a \otimes 1$ for $a \in L$. Show that $\Delta J \subseteq J \otimes T(L) + T(L) \otimes J$.

Exercises 3.88 (Lie coalgebras). We introduce Lie coalgebras.
a) Let $L$ be a finite-dimensional Lie algebra over $k$. The dual of the Lie bracket $[,] : L \otimes L \to L$ is a map $d : L^\vee \to L^\vee \otimes L^\vee$. Write down the properties dual to the anti-symmetry and the Jacobi identity from Definition 3.69.

b) Define a Lie coalgebra over $k$ as a $k$-vector space $C$ with a map $d : C \to C \wedge C$ such that $d \circ d = 0$ (when $d$ is appropriately extended to $C \wedge C$). Prove that the dual of a Lie coalgebra, not necessarily of finite dimension, is a Lie algebra.

Exercise 3.89. In this exercise, we show that $I/I^2$ is a Lie coalgebra, and hence $L$ is a Lie algebra.

(a) Check the property (3.73).

(b) Extend $d$ to an operator

$$d : \bigwedge^n(I/I^2) \to \bigwedge^{n+1}(I/I^2)$$

by using the Leibniz rule with appropriate signs. Then show that $d^2 = 0$. This implies that $I/I^2$ is a Lie coalgebra. Deduce from Exercise 3.88 that $L$ is a Lie algebra.

Exercise 3.90. Show that the map $\varphi_G$ in (3.79) satisfies

$$\varphi_G([X,Y]) = \varphi_G(X) \cdot \varphi_G(Y) - \varphi_G(Y) \cdot \varphi_G(X).$$

Exercise 3.91. Show that, in the Hopf algebra $k[x]$ associated with the additive group $G_a$, one has

$$\Delta(x^n) = \sum_{r=0}^n \binom{n}{r} x^r \otimes x^{n-r}.$$ 

Exercise 3.92 (The Hopf algebra of rooted trees). In this exercise, we describe the Hopf algebra of rooted trees introduced by Connes and Kreimer, in connection with the renormalization of quantum field theories [CK98]. Another nice reference is [Foi].

We begin with a couple of definitions. A rooted tree is an oriented finite graph which is connected and simply connected (in other words, a tree), and has a distinguished vertex with no incoming edges called the root. Continuing the metaphor, the vertices with no outcoming edges are called the leaves. A rooted forest is a disjoint union of rooted trees.

Let $H_R$ be the $\mathbb{Q}$-algebra of polynomials in rooted trees, i.e. $H_R$ is the free commutative $\mathbb{Q}$-algebra with unit generated by (isomorphism classes of) rooted trees. The product of two rooted trees can be identified with their disjoint union and the unit is the empty tree $1$. Therefore, as vector space,

$$H_R = \mathbb{Q}[\text{rooted forests}].$$

Let $t$ be a rooted tree. An admissible cut $c$ of $t$ is the choice of a subset of the edges such that any path from the root to the leaves meets at most
one of them. Deleting the edges in \( c \), one gets a rooted forest \( W^c(t) \). Among the connected components of \( W^c(t) \), there is a unique tree \( R^c(t) \) containing the original root. The rooted forest consisting of the remaining components will be denoted by \( P^c(t) \). Two extremes cases of admissible cuts are the empty cut, for which \( R^c(t) = t \) and \( P^c(t) = 1 \), and the total cut, for which \( R^c(t) = 1 \) and \( P^c(t) = V(t) \), the disjoint union of the vertices of \( t \). We shall write \( \text{Adm}_\ast(t) \) for the set of non-total, non-empty admissible cuts of \( t \). We define:

\[
\Delta t = 1 \otimes t + t \otimes 1 + \sum_{c \in \text{Adm}_\ast(t)} P^c(t) \otimes R^c(t).
\]

(3.93)

Since \( \mathcal{H}_R \) is the free algebra in rooted trees, (3.93) extends uniquely to a coproduct \( \Delta : \mathcal{H}_R \rightarrow \mathcal{H}_R \otimes \mathcal{H}_R \). Figure 13 contains an example of an admissible cut and the contribution to the coproduct.

The counit is the map \( \epsilon : \mathcal{H}_R \rightarrow \mathbb{Q} \) which sends the empty tree to 1 and everything else to zero.

1. Prove that \( \Delta \) and \( \epsilon \) satisfy the associativity and counit axioms from Definition 3.46. In other words, \( \mathcal{H}_R \) is a bialgebra.

2. For each integer \( n \geq 0 \), let \( \mathcal{H}_R(n) \subseteq \mathcal{H}_R \) be the vector subspace generated by rooted forests with \( n \) vertices, so that

\[
\mathcal{H}_R = \bigoplus_{n \geq 0} \mathcal{H}_R(n).
\]

Observe that \( \Delta \mathcal{H}_R(n) \subseteq \bigoplus_{i+j=n} \mathcal{H}_R(i) \otimes \mathcal{H}_R(j) \). Since \( \mathcal{H}_R \) is obviously a graded connected algebra, by Exercise 3.85 there is a unique antipode \( S \) turning \( \mathcal{H}_R \) into a Hopf algebra.
(3) Given a rooted tree $t$ and a cut $c$, write $n_c$ for the numbers of cut edges in $c$. Prove that the antipode is given by

$$S(t) = -t - \sum_{c \in \text{Adm}_*(t)} (-1)^{n_c} W_c(t).$$

### 3.3. The pro-unipotent completion of a group.

In this section, we develop some abstract machinery that will be used in the sequel to rephrase the constructions from Section 3.1 in a more conceptual way. There we saw that iterated integrals carry information about the fundamental group of a differentiable manifold. The question we would like to address now is how much of it can be recovered using differential forms. Stated in a vaguer form: what information about the fundamental group is “cohomological”, or even “motivic” if we are dealing with algebraic varieties?

Throughout, $k$ still denotes a field of characteristic zero.

#### 3.3.1. Representations.

We first introduce the notion of representation of an abstract group and of an affine group scheme. In the latter case, one needs to be careful because, as explained in part (6) of Example 3.56, the group-valued functor $\text{Aut}(V)$ is not representable by a scheme when $V$ is an infinite-dimensional vector space.

**Definition 3.94.** Let $\Gamma$ be an abstract group. A $k$-linear representation of $\Gamma$ is a $k$-vector space $V$ together with a group homomorphism $\Gamma \to \text{Aut}_k(V)$.

Let $G$ be an affine group scheme over $k$. A $k$-linear representation of $G$ is a $k$-vector space $V$ together with a natural transformation of group-valued functors $G \to \text{Aut}(V)$. This means that we are given the data of a group homomorphism $G(R) \to \text{Aut}_R(R \otimes V)$ for every $k$-algebra $R$ and, for each morphism of $k$-algebras $R \to R'$, of a commutative diagram

$$
\begin{align*}
G(R) &\longrightarrow \text{Aut}_R(R \otimes V) \\
\downarrow &\quad \downarrow \\
G(R') &\longrightarrow \text{Aut}_{R'}(R' \otimes V).
\end{align*}
$$

Every $k$-linear representation of an affine group scheme determines a representation of the group of $k$-points $G(k)$, but of course not all representations of $G(k)$ arise this way (see Lemma 3.98 and Exercise 3.123 below for an example). Since we will be only working with $k$-linear representations, we will omit the adjective “$k$-linear” and refer to them in what follows simply as “representations”.

In some cases it is more convenient to use the point of view of comodules. For a more detailed proof of the next result see, for instance [Mil17, Remark 4.1].
**Lemma 3.95.** Let $G$ be an affine group scheme over $k$, and let $V$ be a $k$-vector space. There is a natural one-to-one correspondence between linear representations of $G$ on $V$ and right $O(G)$-comodule structures on $V$.

**Proof.** For shorthand we write $A = O(G)$. In fact, the correspondence is in two steps. First each linear representation of $G$ corresponds to a left $A$-comodule structures on the dual vector space $V^\vee$. Then the left $A$-comodule structures on $V^\vee$ correspond to right $A$-comodule structures on $V$.

More precisely, let $\rho: G \to \text{Aut}(V)$ be a representation of $G$. For $\omega \in V^\vee$ we define

$$\Delta_\rho \omega \in A \otimes V^\vee \cong \text{Hom}(V, A)$$

by

$$\langle \Delta_\rho(\omega), v \rangle(g) = \langle \rho(g)(v), \omega \rangle.$$ 

Note that here $\langle \Delta_\rho(\omega), v \rangle$ is meant to be an element of $A = O(G)$ and we determine it by evaluating at elements $g \in G(R)$ for any $k$-algebra $R$. By duality, this defines a right coaction $\Delta_\rho: V \to V \otimes A$, denoted with the same letter, by the rule that, for each $v \in V$ and $\omega \in V^\vee$,

$$\langle \omega, \Delta_\rho(v) \rangle(g) = \langle \Delta_\rho(\omega), v \rangle(g) = \langle \omega, \rho(g)(v) \rangle.$$ 

It is easy to check that, in equation (3.96), the maps $\Delta_\rho$ and $\rho$ determine each other and that $\Delta_\rho$ is a right coaction of $A$ on $V$ if and only if $\rho$ is a representation of $G$.

In the finite dimensional case this can be made more concrete by choosing basis. Let $e_1, \ldots, e_r$ be a basis of $V$. Each linear representation $\rho$ of $G$ on $V$ is a group homomorphism $\rho: G \to \text{Aut}(V)$. After choosing the basis, we can identify $\text{Aut}(V)$ with the space of invertible $r \times r$ matrices. Thus $\rho$ defines an $r \times r$ matrix $(M_{i,j})$ with entries in $A$. The fact that $\rho$ is a group homomorphism is equivalent to

$$M_{i,j}(g \cdot g') = \sum_k M_{i,k}(g)M_{k,j}(g'), \quad M_{i,j}(e) = \delta_{i,j},$$

for $e$ the unit of $G$.

On the other hand, a right coaction $\Delta: V \to V \otimes A$ defines also a matrix $M_{i,j}$ with entries in $A$ by the rule

$$\Delta e_j = \sum_i e_i \otimes M_{i,j}.$$ 

The fact that $\Delta$ is a coaction is again equivalent to (3.97). Both the set of linear representations of $G$ on $V$ and the set of right $A$-coactions on $V$ are thus given by the set of $r \times r$ matrices with entries in $A$ satisfying conditions (3.97).

The first part of the proof of Lemma 3.49 shows the following (see also [DM82, Corollary 2.4]):
**Lemma 3.98.** Every linear representation of an affine group scheme is a directed union of finite-dimensional subrepresentations.

**Remark 3.99.** Recall from Example 3.56 that, if $V$ is a finite dimensional vector space, then the automorphisms of a vector space $V$ form an affine group scheme $\text{GL}(V)$. It turns out that, to give a finite dimensional representation of $G$, is equivalent to give a pair consisting of a $k$-vector space $V$ and a morphism of group schemes $\rho: G \to \text{GL}(V)$. Since we will be mainly interested in finite dimensional representations, this is the point of view that we will use the most.

3.3.2. *The abelianization of the fundamental group.* The obvious piece of information that can be recovered via differential forms is the abelianization of the fundamental group. Indeed, recall from Theorem 3.14 that

$$\pi_1(M, x)^{ab} \cong H_1(M, \mathbb{Z}),$$

so that, passing to the dual, de Rham’s Theorem 2.30 yields an isomorphism

$$H^1_{\text{dR}}(M, \mathbb{R}) \cong \text{Hom}(\pi_1(M, x)^{ab}, \mathbb{R}).$$

Moreover, in the case where $k$ is a subfield of $\mathbb{C}$ and $M = X(\mathbb{C})$ is the set of complex points of a smooth variety $X$ over $k$, we get

$$H^1_{\text{dR}}(X) \otimes \mathbb{C} \cong \text{Hom}(\pi_1(M, x)^{ab}, \mathbb{C}),$$

where the left-hand side stands for algebraic de Rham cohomology (as in Definition 2.43) and has thus a purely algebraic definition.

However, the abelianization of the fundamental group is a very crude invariant that, for example, only knows about abelian representations. We should be able to see much more than just the abelianization of the fundamental group using differential forms. A glimpse of this appeared in Section 3.1 when we saw that iterated integrals are related to nilpotent flat connections, that in turn are related to unipotent representations of the fundamental group. In the next paragraphs, we elaborate on this idea.

3.3.3. *Unipotent and pro-unipotent groups.* Recall from Lemma 3.49 that every affine group scheme is pro-algebraic.

**Definition 3.100.** An affine algebraic group (resp. affine group scheme) $G$ over $k$ is called *unipotent* (resp. *pro-unipotent*) if every non-zero representation $V$ of $G$ has a non-zero fixed vector.

**Remark 3.101.** In view of Lemma 3.98, it is enough to check that every non-zero finite-dimensional representation has a non-zero fixed vector.

**Example 3.102.** Let $\text{Up}_n$ be the functor that associates with each $k$-algebra $R$ the group of $n$ by $n$ upper triangular matrices with 1’s in the diagonal. This functor is represented by an affine group scheme, still denoted by $\text{Up}_n$. The group $\text{Up}_n$ is unipotent. Indeed, let $\rho: \text{Up}_n \to \text{GL}(V)$ be a non-zero finite-dimensional representation. It follows from the definition of
Up_n that \((\rho - 1)^n = 0\). Let \(m\) be an integer such that \((\rho - 1)^m = 0\), but there is an element \(g \in Up_n(k)\) and a vector \(v_1 \in V\) with
\[ v = (\rho(g) - 1)^{m-1}v_1 \neq 0. \]
By construction, \(Up_n \cdot v = v\), showing that \(Up_n\) is unipotent.

For \(n > m\) there is a natural map of affine algebraic groups
\[ Up_n \to Up_m \]
that sends a triangular matrix to its first \(m \times m\) submatrix. Passing to the limit yields the pro-unipotent group
\[ Up_\infty = \varprojlim_n Up_n. \]

For every \(k\)-vector space \(V\) of dimension \(n\), a choice of a basis of \(V\) induces a closed immersion \(Up_n \to GL(V)\).

**Definition 3.103.** Let \(G\) be either an abstract group or an affine group scheme. A finite-dimensional representation \(\rho: G \to GL(V)\) is called *unipotent* if there exists a basis of \(V\) such that \(\rho(G) \subseteq Up_n\).

It follows easily from definitions 3.100 and 3.103 that an affine algebraic group (resp. affine group scheme) \(G\) is unipotent (resp. *pro-unipotent*) if every non-zero finite-dimensional representation \(V\) of \(G\) is unipotent.

**3.3.4. The conilpotency filtration.** We give an alternative characterization of pro-unipotent groups. Let \(G = \text{Spec}(A)\) be an affine group scheme over \(k\). Since \(A\) has a structure of coalgebra coming from the group structure of \(G\), its dual \(A^\vee\) is an algebra, but this time not necessarily commutative. The unit of \(A\) defines an augmentation \(\varepsilon: A^\vee \to k\). Let \(J = \text{Ker}(\varepsilon) \subset A^\vee\) be the augmentation ideal. We denote by \(J^n\) the \(n\)-th power of the ideal \(J\) using the algebra structure of \(A^\vee\). The *conilpotency filtration* is the filtration of \(A\) defined by
\[ 0 \subset C_0 = \text{Ann}_A J \subset C_1 = \text{Ann}_A J^2 \subset \cdots \subset C_i = \text{Ann}_A J^{i+1} \subset \cdots \]
where \(\text{Ann}_A J^{i+1}\) stands for the annhilator of \(J^{i+1}\), that is, the set of elements \(a \in A\) such that \(\langle a, x \rangle = 0\) for all \(x \in J^{i+1} \subset A^\vee\). It is easy to see that \(C_0 = k \cdot 1\), where 1 is the unit of \(A\), and that
\[ \Delta C_i \subset \sum_{a+b=i} C_a \otimes C_b. \]

**Proposition 3.105.** The affine group scheme \(G = \text{Spec}(A)\) is pro-unipotent if and only if the conilpotency filtration of \(A\) is exhaustive, that is:
\[ A = \bigcup_{i=0}^\infty C_i. \]
Proof. Assume that the conilpotency filtration is exhaustive. Let $V$ be a non-zero representation of $G = \text{Spec}(A)$ and denote by $\Delta : V \to V \otimes A$ the corresponding comodule structure given by Lemma 3.95. Since the filtration \( \{C_i\}_{i \geq 0} \) is exhaustive, we deduce that the filtration \( \{V_i\}_{i \geq 0} \) given by
\[
V_i = \{v \in V \mid \Delta v \in V \otimes C_i\}
\]
is also exhaustive. By the axioms of a comodule, if $v \in V_0$, then $\Delta v = v \otimes 1$. Therefore any vector $v \in V_0$ is a fixed vector for the representation. Thus to prove that $G$ is pro-unipotent is enough to show that $V_0$ is non-zero. To this end we show that $V_i = 0$ implies that $V_{i+1} = 0$. So, assume that $V_i = 0$ and let $v \in V_{i+1}$. By (3.104)
\[
(1 \otimes \Delta)\Delta v \in \sum_{a+b=i+1} V \otimes C_a \otimes C_b.
\]
Since $a$ and $b$ cannot be both bigger than $i$, the vector $v$ is sent to zero by the map
\[
V \xrightarrow{\Delta} V \otimes A \xrightarrow{1 \otimes \Delta} V \otimes A \otimes A \xrightarrow{} V \otimes A / C_i \otimes A / C_i.
\]
But, by the associativity property of comodules, this map agrees with the map
\[
V \xrightarrow{\Delta} V \otimes A \xrightarrow{\Delta \otimes 1} V \otimes A \otimes A \xrightarrow{} V \otimes A / C_i \otimes A / C_i.
\]
that is an injection, since $V_i = 0$. Thus $v = 0$, and hence $V_{i+1} = 0$.

Conversely, assume that every non-zero representation of $G$ has a non-zero fixed point. Then every representation $V$ has a filtration $\{V_i\}_{i \geq 0}$ determined by the fact that $V_0$ is the trivial subrepresentation of $V$ and, inductively, $V_{i+1}/V_i$ is the trivial subrepresentation of $V/V_i$. This filtration is exhaustive by Lemma 3.98. The conilpotency filtration agrees with this filtration in the representation given by $A$ itself, thus it is exhaustive. □

3.3.5. The pro-unipotent completion. The central concept of the whole section is the following:

Definition 3.106. Let $\Gamma$ be an abstract group. The pro-unipotent completion $\Gamma^{\text{un}}$ of $\Gamma$ over $k$ is the universal pro-unipotent affine group scheme $G$ over $k$ endowed with a morphism of abstract groups $\Gamma \to G(k)$. More precisely,

- $\Gamma^{\text{un}}$ is a pro-unipotent affine group scheme over $k$ with a morphism $\Gamma \to \Gamma^{\text{un}}(k)$,

- for each pro-unipotent affine group scheme $G$ over $k$ with a morphism $\Gamma \to G(k)$, there is a unique morphism of affine group
schemes $\Gamma^{\text{un}} \to G$ such that the following diagram commutes
\[
\begin{array}{ccc}
\Gamma & \longrightarrow & \Gamma^{\text{un}}(k) \\
\downarrow & & \downarrow \\
G(k) & & G(k)
\end{array}
\]
The pro-unipotent completion of $\Gamma$ over $\mathbb{Q}$ will be called the pro-unipotent completion of $\Gamma$.

Similarly, the pro-algebraic completion $\Gamma^{\text{alg}}$ of $\Gamma$ over $k$ is the universal pro-algebraic affine group scheme $G$ over $k$ endowed with a morphism of abstract groups $\Gamma \to G(k)$.

The pro-unipotent completion is also called the Malcev completion in the literature. As it is always the case with universal objects, when they exist they are unique up to unique isomorphism.

**Remark 3.107.** If the pro-unipotent completion exists, then the groups $\Gamma$ and $\Gamma^{\text{un}}$ have the same finite-dimensional unipotent representations. Therefore, one cannot recover $\Gamma$ by just looking at this kind of representations.

We now present the construction, due to Quillen, of the pro-unipotent completion of a group satisfying a finiteness condition. For the moment, let $\Gamma$ be any abstract group and consider the non-commutative $k$-algebra
\[
k[\Gamma] = \left\{ \sum_{g \in \Gamma} a_g g \mid a_g \in k, a_g = 0 \text{ except for a finite subset} \right\},
\]
with the product structure induced by the group operation of $\Gamma$.

**Definition 3.108.** The augmentation of $k[\Gamma]$ is the algebra morphism
\[
\varepsilon: \quad k[\Gamma] \longrightarrow k \\
\sum_{g \in \Gamma} a_g g \longmapsto \sum_{g \in \Gamma} a_g.
\]
Its kernel $J = \text{Ker}(\varepsilon)$ is called the augmentation ideal:
\[
J = \left\{ \sum_{g \in \Gamma} a_g g \mid \sum_{g \in \Gamma} a_g = 0 \right\}.
\]

The completion of $k[\Gamma]$ with respect to $J$ is the inverse limit
\[
k[\Gamma]^\wedge = \lim_{\leftarrow N} k[\Gamma]/J^{N+1}.
\]
It has a completed coproduct
\[
\nabla^\vee: k[\Gamma]^\wedge \longrightarrow k[\Gamma]^\wedge \otimes k[\Gamma]^\wedge = \lim_{\leftarrow N} k[\Gamma]/J^{N+1} \otimes k[\Gamma]/J^{N+1}
\]
induced by the rule $\nabla^\vee g = g \otimes g$ for all elements $g \in \Gamma$. Moreover, there is an antipode
\[
S^\vee: k[\Gamma]^\wedge \longrightarrow k[\Gamma]^\wedge
\]
determined by \( g \mapsto g^{-1} \). With these operations \( k[\Gamma[^\wedge] \mapsto k[\Gamma[^\wedge]/J_N[^\wedge]+1)^\wedge \)

with the induced structures is a Hopf algebra. The augmentation \( \varepsilon \) extends to an augmentation \( \varepsilon: A[^\wedge] = k[\Gamma[^\wedge] \rightarrow k[^\wedge] \)

introduced in Paragraph 3.3.4. We will also denote by \( J = \text{Ker}(\varepsilon) = Jk[\Gamma[^\wedge] \)

the augmentation ideal of \( k[\Gamma[^\wedge] \).

Let us now assume that \( \Gamma \) satisfies the finiteness condition that \( \Gamma[^\text{ab}] \otimes_\mathbb{Z} k[^\text{fin}] \)

is a finite-dimensional \( k[^\text{vec}] \)-vector space. By Theorem 3.14, this is for instance satisfied when \( \Gamma \) is the fundamental group of a topological space with the homotopy type of a finite CW-complex. In particular, when \( \Gamma \) is the fundamental group of the space of complex points \( X(\mathbb{C}) \) of an algebraic variety \( X \) over \( \mathbb{C} \).

**Lemma 3.109.** If the vector space \( \Gamma[^\text{ab}] \otimes_\mathbb{Z} k[^\text{fin}] \) is finite-dimensional, then all the quotients \( k[\Gamma]/J_N[^\wedge]+1 \) are finite-dimensional as well.

**Proof.** Since \( k[\Gamma] = k \oplus J \), it suffices to prove that \( J/J_N[^\wedge]+1 \) is finite-dimensional for all \( N \geq 0 \). Looking at the filtration

\[
J_N[^\wedge]+1 \subseteq J_N[^\wedge] \subseteq \cdots \subseteq J[^\wedge] \subseteq J,
\]

this amounts to proving that the successive quotients \( J^i/J_i[^\wedge]+1 \) are finite-dimensional for all \( i \geq 1 \). To treat the case \( i = 1 \), we note that the map

\[
\begin{align*}
\Gamma & \rightarrow J^2/J^2 \\
    g & \mapsto (g-1) + J^2
\end{align*}
\]

factors through the abelianization of \( \Gamma \), as can be seen by writing \( gh-1 \) as \( (g-1) + (h-1) + (g-1)(h-1) \). In fact, it induces an isomorphism

\[
\Gamma[^\text{ab}] \otimes_\mathbb{Z} k \xrightarrow{\sim} J^2/J^2
\]

(the inverse is the map that sends the class of a generator \( g-1 \) to the class of \( g \) in \( \Gamma[^\text{ab}] \)). This proves that \( J^i/J_i[^\wedge]+1 \) is finite-dimensional. Taking into account that the multiplication map

\[
(J^i/J_i[^\wedge]+1 \otimes (J^i/J_i[^\wedge]+1)^\wedge \rightarrow J^i/J_i[^\wedge]+1
\]

is surjective for all \( i \geq 1 \), the general result follows.

The following result can be deduced from [Qui69, Appendix A], although the language there is different. A translation into the language of algebraic groups is given in [Hai93, Theorem 3.3]. We sketch the proof.

**Theorem 3.110 (Quillen [Qui69]).** Let \( \Gamma \) be an abstract group such that the vector space \( \Gamma[^\text{ab}] \otimes_\mathbb{Z} k[^\text{fin}] \) has finite dimension. Then the pro-unipotent completion of \( \Gamma \) over \( k \) is the pro-algebraic group \( \text{Spec}(k[\Gamma[^\wedge])[^\wedge]) \).
Proof. As before, write \( A = (k[\Gamma]^\wedge)^\vee \) and \( G = \text{Spec}(A) \). The conilpotency filtration of \( A \) is given by \( \text{Ann}_A J^{N+1} k[\Gamma]^\wedge \). By the definition of \( k[\Gamma]^\wedge \) as a projective limit, it is clear that
\[
\bigcap_{N \geq 0} J^{N+1} k[\Gamma]^\wedge = 0.
\]
Therefore the conilpotency filtration of \( A \) is exhaustive. By Proposition 3.105, we deduce that \( G \) is pro-unipotent.

Let now \( H = \text{Spec}(B) \) be a pro-unipotent group with a group morphism \( \Gamma \to H(k) \). Let \( B^\vee \) be the non-commutative algebra dual to the co-algebra \( B \). There is an inclusion \( H(k) \to B^\vee \) given by evaluating functions at points. The map \( f: \Gamma \to H(k) \) extends to a map \( k[\Gamma] \to B^\vee \) also denoted \( f \). The augmentations of \( k[\Gamma] \) and of \( B^\vee \) are compatible with \( f \). Thus we obtain maps
\[
k[\Gamma]/J^{N+1} \to B^\vee/J^{N+1},
\]
where \( J \) denotes the augmentation ideal in both algebras. Dualizing we obtain maps
\[
\text{Ann}_B J^{N+1} \to \text{Ann}_A J^{N+1} \to A.
\]
Since \( H \) is pro-unipotent, by Proposition 3.105 the conilpotency filtration of \( B \) is exhaustive and we obtain a map \( B \to A \), therefore a map of pro-unipotent groups \( G \to H \). By construction, this is the only map of pro-unipotent groups that preserves the image of \( \Gamma \). Thus \( G \) satisfies the universal property defining \( \Gamma^\text{un} \).

\[ \square \]

3.3.6. Group-like and Lie-like elements. Let \( \Gamma \) be an abstract group such that \( \Gamma^\text{ab} \otimes \mathbb{Z} k \) is finite-dimensional. In this paragraph, we describe the set of rational points \( \Gamma^\text{un}(k) \), the map \( \Gamma \to \Gamma^\text{un}(k) \) and the Lie algebra \( \text{Lie}(\Gamma^\text{un}) \). In the applications we will always be interested in the case \( k = \mathbb{Q} \).

Definition 3.111. An element \( g \in k[\Gamma]^\wedge \) is said to be group-like if it satisfies the conditions \( \varepsilon(g) = 1 \) and \( \nabla^\vee g = g \otimes g \).

The set of group-like elements of \( k[\Gamma]^\wedge \), denoted by \( G(k[\Gamma]^\wedge) \), is a group. Clearly, the image of an element \( g \in \Gamma \) in \( k[\Gamma]^\wedge \) is group-like.

Definition 3.112. An element of \( x \in k[\Gamma]^\wedge \) is called Lie-like if it satisfies the condition \( \nabla^\vee x = 1 \otimes x + x \otimes 1 \).

The set of Lie-like elements of \( k[\Gamma]^\wedge \), denoted by \( L(k[\Gamma]^\wedge) \), is a Lie algebra. The power series \( \exp \) and \( \log \) are bijections, inverses to each other
\[
G(k[\Gamma]^\wedge) \xrightarrow{\log} L(k[\Gamma]^\wedge).
\]

Proposition 3.113. Let \( \Gamma \) be an abstract group such that \( \Gamma^\text{ab} \otimes \mathbb{Z} k \) has finite dimension. Then \( G(k[\Gamma]^\wedge) = \Gamma^\text{un}(k) \), and the natural map \( \Gamma \to k[\Gamma]^\wedge \).
agrees with the structural map $\Gamma \to \Gamma_{un}(k)$. Moreover, the Lie algebra of $\Gamma_{un}$ agrees with $L(k[\Gamma]^\wedge)$.

**Proof.** We just sketch the proof that $G(k[\Gamma]^\wedge) = \Gamma_{un}(k)$. We continue using the notation $A = (k[\Gamma]^\wedge)^\vee$ so that $\Gamma_{un} = \text{Spec}(A)$. By definition the set $\Gamma_{un}(k)$ is in bijection with the set of morphisms $\text{Spec}(k) \to \text{Spec}(A)$, which, in turn is in bijection with the set of algebra homomorphisms $A \to k$. That is, the set of elements $g \in A^\vee = k[\Gamma]^\wedge$ that preserve the unit and the product. For an element $g \in A^\vee$, to preserve the unit is equivalent to $\varepsilon(g) = 1$ and to preserve the product is equivalent to $\nabla^\vee(g) = g \otimes g$. Thus we get bijection $G(k[\Gamma]^\wedge) = \Gamma_{un}(k)$. \hfill \Box

**Example 3.114.** Let us illustrate the above proposition for $\Gamma = \mathbb{Z}$. As we will see in Exercise 3.124, the pro-unipotent completion of $\Gamma$ is the additive group $Ga$ over $k$, so we need to show that group-like elements in $k[[x]]$ are in one-to-one correspondence with $k$. Let $\sum_{n \geq 0} a_n x^n$ be a group-like element. Then $a_0 = 1$ and

$$\nabla^\vee \left( \sum_{n \geq 0} a_n x^n \right) = \left( \sum_{n \geq 0} a_n x^n \right) \otimes \left( \sum_{n \geq 0} a_n x^n \right).$$

Equation (3.115) is thus equivalent to the relation

$$a_k a_m = \binom{k + m}{k} a_{k+m}$$

for all $k, m \geq 0$. In particular, all coefficients are determined by $a_1$ and indeed $a_n = a_1^n / n!$. Hence our element is of the form $\exp(a_1 x)$ and this gives the correspondence.

From the compatibility between the antipode, the product and the completed coproduct we easily deduce the following (Exercise 3.126):

**Lemma 3.116.** If $x$ is a Lie-like element, then $S(x) = -x$, while if $g$ is a group-like element, then it is invertible in the algebra $k[\Gamma]^\wedge$ and satisfies $S(g) = g^{-1}$.

**Example 3.117.** Let $\Gamma$ be the free group on two generators $\gamma_0$ and $\gamma_1$. In this example we compute the pro-unipotent completion of $\Gamma$ over $\mathbb{Q}$. Since $\gamma_0 - 1$ and $\gamma_1 - 1$ belong to the augmentation ideal, we can define

$$\log(\gamma_0) = \log(1 + (\gamma_0 - 1)) = \gamma_0 - 1 - \frac{(\gamma_0 - 1)^2}{2} + \ldots$$

$$\log(\gamma_1) = \log(1 + (\gamma_1 - 1)) = \gamma_1 - 1 - \frac{(\gamma_1 - 1)^2}{2} + \ldots$$
as elements in $\mathbb{Q}[\Gamma]^\wedge$. Recall the algebra $\mathbb{Q}\langle e_0, e_1 \rangle$ from Example 3.67. We define a morphism of algebras $\mathbb{Q}\langle e_0, e_1 \rangle \to \mathbb{Q}[\Gamma]^\wedge$ by sending $e_0$ to $\log(\gamma_0)$ and $e_1$ to $\log(\gamma_1)$. It is easy to verify that this map is an isomorphism compatible with all extra structures (unit, counit, completed coproduct, and antipode) of both completed Hopf algebras. From this it follows that $\Gamma^{\text{un}} = \text{Spec}(\mathcal{H})$, where $\mathcal{H}$ is the Hoffman algebra of Example 3.60.

In particular, we can identify the group of rational points $\Gamma^{\text{un}}(\mathbb{Q})$ with the set of group-like elements of $\mathbb{Q}\langle e_0, e_1 \rangle$, the Lie algebra $\text{Lie}(\Gamma^{\text{un}})$ with the set of Lie-like elements of $\mathbb{Q}\langle e_0, e_1 \rangle$ and the completed universal enveloping algebra of $\text{Lie}(\Gamma^{\text{un}})$ with $\mathbb{Q}\langle e_0, e_1 \rangle$.

3.3.7. The pro-unipotent completion of a torsor. We shall use a variant of Quillen’s construction of the pro-unipotent completion for torsors instead of groups. Let us first recall the definition:

**Definition** 3.118. Let $\Gamma$ be a group. A left torsor under $\Gamma$ is a non-empty set $P$ together with a free and transitive action $\Gamma \times P \to P$.

**Variant** 3.119. We will also use the following variant of the constructions of this section. Let $P$ be a left torsor over $\Gamma$, in other words, a set $P$ on which $\Gamma$ acts freely and transitively on the left. Write $k[P]$ for the $k$-vector space with basis $P$. It has the structure of a left $k[\Gamma]$-module. The completion of $k[P]$ is defined as

$$k[P]^\wedge = \lim_{\leftarrow N} k[P]/J_{N+1}k[P].$$

(3.120)

This is a $k[\Gamma]^\wedge$-module equipped with a completed coproduct

$$\nabla^\vee : k[P]^\wedge \to k[P]^\wedge \otimes k[P]^\wedge,$$

induced by the rule $\nabla^\vee a = a \otimes a$ for any $a \in P$. The completed coproducts of $k[\Gamma]^\wedge$ and $k[P]^\wedge$ are compatible with the module structure in the sense that

$$\nabla^\vee(ga) = \nabla^\vee(g)\nabla^\vee(a).$$

for all $a \in k[P]^\wedge$ and $g \in k[\Gamma]^\wedge$. Dualizing we obtain a commutative algebra $R = (k[P]^\wedge)^\vee$ with a compatible coproduct

$$\Delta : R \to A \otimes R,$$

(3.121)

where $A$ denotes again $A = (k[P]^\wedge)^\vee$. In other words, $R$ is a Hopf module over $A$. The unipotent completion of $P$ is

$$P^{\text{un}} = \text{Spec}(R).$$

The coproduct (3.121) induces an action $\Gamma^{\text{un}} \times P^{\text{un}} \to P^{\text{un}}$ that turns $P^{\text{un}}$ into a left $\Gamma^{\text{un}}$-torsor.

Mutatis mutandis, the same construction can be made for a right torsor $P'$. In this case

$$k[P']^\wedge = \lim_{\leftarrow N} k[P']/k[P']J_{N+1}.$$
Our basic example will be the case when $\Gamma$ is the fundamental group $\pi_1(M,x)$ and $P$ and $P'$ are the torsors of paths $\pi_1(M;x,y)$ and $\pi_1(M;y,x)$ respectively. In this case, there is also an antipode map

$$S: \mathbb{Q}[P]^{\wedge} \to \mathbb{Q}[P']^{\wedge}$$

induced by the rule $S(\gamma) = \gamma^{-1}$ for paths $\gamma \in P$.

Exercise 3.122. Let $V = \bigoplus_{n \in \mathbb{Z}} V^n$ be a graded $k$-vector space. Then there is an induced left action of $\mathbb{G}_m$ on $V$ given by $\lambda \cdot v = \lambda^n v$ on each $v \in V^n$. In fact, giving a $\mathbb{Z}$-grading on $V$ is equivalent to giving an action of $\mathbb{G}_m$.

(1) Prove that the coalgebra $\mathcal{O}(\mathbb{G}_m)$ is isomorphic to the coalgebra $H$ from Example 3.53.

(2) Prove that the coaction of $\mathcal{O}(\mathbb{G}_m)$ on $V$ determined by Lemma 3.95 agrees with the coaction of $H$ from Example 3.53.

Exercise 3.123. Let $G$ be an affine group scheme. We see in this exercise that not every linear representation of the abstract group $G(k)$ has “geometric origin” is an algebraic representation of $G$. For instance, consider the $\mathbb{C}$ vector space $V = K(\mathbb{P}^1_{\mathbb{C}})$ of rational functions on the complex projective line. The group $G(\mathbb{C}) = \text{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$ by Möbius transformations, and hence linearly on $V$.

(1) Let $W \subset V$ be a finite-dimensional vector subspace. Show that the set of poles of the functions belonging to $W$ is finite.

(2) Show that the set of poles that appear in the orbit of the function $t$ is infinite.

(3) Conclude by Lemma 3.98 that the linear representation of $G(\mathbb{C})$ on $V$ does not come from a representation of the algebraic group scheme $G = \text{SL}_2$.

Exercise 3.124. Consider the group $\Gamma = \pi_1(S^1,1) \simeq \mathbb{Z}$. Let $\gamma_0$ be a generator of $\Gamma$ and consider $X_0 = \log(\gamma_0)$ as a power series in $(\gamma_0 - 1) \in J$. Use $\gamma_0$ and $X_0$ to describe explicitly

$$\mathbb{Q}[\pi_1(S^1,1)]/J^{N+1}, \quad \mathbb{Q}[\pi_1(S^1,1)]^{\wedge}, \quad \mathcal{O}(\pi_1(S^1,1)^{\text{un}}),$$

$$\pi_1(S^1,1)^{\text{un}}, \quad \text{Lie}(\pi_1(S^1,1)^{\text{un}}).$$

In particular, deduce that the pro-unipotent completion of $\mathbb{Z}$ is the additive group $\mathbb{G}_a$. Compare this with Exercise 4.23 in the next chapter.

Exercise 3.125. Prove that the pro-unipotent completion of the group $\mathbb{Z}/2\mathbb{Z}$ is the trivial group $\text{Spec}(\mathbb{Q})$. 

3.4. The bar complex and Chen’s $\pi_1$-de Rham theorem. In this section, we make the relation between differential forms and the pro-unipotent completion of the fundamental group of a smooth manifold precise. If one views the latter as the Betti side of a picture, then the de Rham side is given by the cohomology of the bar complex. Both points of view will be related through Chen’s $\pi_1$-de Rham Theorem 3.151.

3.4.1. The reduced bar complex of a connected dg-algebra. We start by recalling the definition of a differential graded algebra.

**Definition 3.127.** Let $k$ be a field of characteristic zero. A differential graded algebra (dg-algebra for short) over $k$ is a graded $k$-vector space

$$A = \bigoplus_{n \in \mathbb{Z}} A^n,$$

together with the following additional structures:

- a multiplication $A^n \otimes A^m \to A^{n+m}$ for all integers $n, m \in \mathbb{Z}$ which makes $A$ into an associative $k$-algebra with unit $1 \in A^0$;

- a differential $d: A \to A$ such that $d(A^n) \subseteq A^{n+1}$ and

$$d(ab) = da \cdot b + (-1)^n a \cdot db, \quad a \in A^n.$$  

We say that $A$ is commutative if, for $a \in A^n$ and $b \in A^m$, the relation $ab = (-1)^{nm} ba$ holds, and connected if $A^n = 0$ for $n < 0$ and $A^0 = k$.

An augmentation of a dg-algebra is a map of dg-algebras $A \to k$, where $k$ is concentrated in degree zero and has trivial differential. It follows immediately from the definitions that a connected dg-algebra has a unique augmentation.

An example to keep in mind, when $k = \mathbb{R}$ or $\mathbb{C}$, is the algebra $E^*(M, k)$ of smooth $k$-valued differential forms on a smooth manifold $M$, together with the wedge product $\wedge$ and the exterior differential $d$ (see Section 2.2.1). A typical augmentation is the evaluation map on an point of $M$. Note that this is not a connected dg-algebra. Similarly, for an arbitrary field $k$, if $X$ is a smooth affine variety over $k$, then $\Omega^*(X)$ is also a dg-algebra.

Since we will apply the general constructions to this setting, in what follows we shall write $\wedge$ for the product in $A$.

**Definition 3.128.** Let $(A^*, \wedge, d)$ be a connected dg-algebra over $k$. Set

$$A^+ = \bigoplus_{n > 0} A^n.$$
The reduced bar complex associated with $A$, denoted by $B^*(A^*)$, is the total tensor algebra of $A^+$:

$$B^*(A^*) = k \oplus A^+ \oplus (A^+ \otimes A^+) \oplus (A^+ \otimes A^+ \otimes A^+) \oplus \ldots$$

An element $x_1 \otimes \cdots \otimes x_n$ for $n \geq 1$ will be denoted by the bar notation

$$[x_1|\cdots|x_n],$$

and the element $1 \in k$ by the empty symbol $[]$.

The reduced bar complex is provided with the following structures:

**Grading:** The degree of an element of $B^*(A^*)$ is given by

$$\deg[x_1|\cdots|x_n] = \sum_{i=1}^{n} \deg(x_i) - n.$$  

**Length filtration:** It is the increasing filtration where $L_mB^*(A^*) \subseteq B^*(A^*)$

is the subspace generated by elements $[x_1|\cdots|x_n]$ with $n \leq m$.

**Differential:** The differential takes into account both the differential and the product structures of $A^*$:

$$d[x_1|\cdots|x_n] = -\sum_{i=1}^{n} (-1)^{\sum_{j=i}^{i-1} \deg(x_j)} [x_1|\cdots|dx_i|\cdots|x_n] + \sum_{i=1}^{n-1} (-1)^{\sum_{j=i}^{i-1} \deg(x_j)} [x_1|\cdots|x_i \land x_{i+1}|\cdots|x_n].$$  \hspace{1cm}  (3.129)

Note that, by the previous convention, $\deg[x_j] = \deg(x_j) - 1$. It is easy to check that $d$ is compatible with the grading and that $d \circ d = 0$. We will write $d = d_I - d_C$, where

$$d_I[x_1|\cdots|x_n] = -\sum_{i=1}^{n} (-1)^{\sum_{j=i}^{i-1} \deg(x_j)} [x_1|\cdots|dx_i|\cdots|x_n]$$  \hspace{1cm}  (3.130)

$$d_C[x_1|\cdots|x_n] = -\sum_{i=1}^{n-1} (-1)^{\sum_{j=i}^{i-1} \deg(x_j)} [x_1|\cdots|x_i \land x_{i+1}|\cdots|x_n].$$  \hspace{1cm}  (3.131)

**Product:** It is the shuffle product

$$\nabla([x_1|\cdots|x_r] \otimes [x_{r+1}|\cdots|x_{r+s}]) = \sum_{\sigma \in \Omega(r,s)} \eta(\sigma)[x_{\sigma^{-1}(1)}|\cdots|x_{\sigma^{-1}(r+s)}],$$  \hspace{1cm}  (3.132)

where $\eta(\sigma)$ is the sign determined by the equation

$$a_1 \land \cdots \land a_{r+s} = \eta(\sigma)a_{\sigma^{-1}(1)} \land \cdots \land a_{\sigma^{-1}(r+s)},$$  \hspace{1cm}  (3.133)
where \( \text{deg}(a_i) = \text{deg}(x_i) - 1 = \text{deg}([x_i]) \). Although \( \eta(\sigma) \) is not determined by \( \sigma \) alone, but also depends on the degrees of the involved elements, this abusive notation is the standard one.

**Coprodut:** The coproduct is the *deconcatenation* coproduct

\[
\Delta[x_1 \cdots |x_n] = \sum_{i=0}^{n} [x_1 | \cdots |x_i] \otimes [x_{i+1} | \cdots |x_n].
\]

**Antipode:** It is given by

\[
S([x_1 | \cdots |x_n]) = (-1)^n \eta(\tau_n) [x_n | \cdots |x_1], \tag{3.134}
\]

where the sign \( \eta(\tau_n) \) is determined by equation (3.133) as before, for the permutation \( \tau_n(i) = n - i \).

**Remark 3.135.** The formula we have written for the differential differs for the classical one that can be found, for instance, in [Tan83, §0.5] by a sign. The reason is the different definition we have for the iterated integral and the fact that we want the bar complex to be compatible with the iterated integrals. To go from one convention of signs to the other we define the operator \( T: B^*(A^*) \to B^*(A^*) \) by

\[
T([x_1 | \cdots |x_n]) = \eta(\tau_n)[x_n | \cdots |x_1] = (-1)^n S([x_1 | \cdots |x_n]),
\]

and \( T_\otimes: B^*(A^*) \otimes B^*(A^*) \to B^*(A^*) \otimes B^*(A^*) \) by

\[
T_\otimes(a \otimes b) = (-1)^{\text{deg}(a) \cdot \text{deg}(b)} T(b) \otimes T(a).
\]

If we denote by \( d', \nabla' \) and \( \Delta' \), the differential, product and coproduct in [Tan83], then one checks that

\[
\nabla = T \circ \nabla' \circ T_\otimes = \nabla',
\]

\[
\Delta = T_\otimes \circ \Delta' \circ T = \Delta'.
\]

That is, \( T \) is an anti-isomorphism of Hopf algebras, and that

\[
d = T \circ d' \circ T.
\]

Thus, the differential here agrees with the differential in [Tan83] twisted by the isomorphism \( T \).

**Lemma 3.136.** Let \((A^*, d, \wedge)\) be a connected commutative dg-algebra. Then the above operations endow \( H^0(B^*(A^*)) \), the zeroth cohomology group of the reduced bar complex, with a commutative Hopf algebra structure.

**Proof.** As stated e.g. in [Tan83, 0.6], the bar construction \( B^*(A^*) \) is a commutative differential graded Hopf Algebra. This means that the product, coproduct and antipode are compatible with the grading and the differential. The latter compatibility is written as

\[
d \circ \nabla = \nabla \circ d_\otimes, \]

\[
\Delta \circ d = d_\otimes \circ \Delta, \]

\[
S \circ d = d \circ S,
\]
where \( d_\otimes \) is the differential induced in \( B^*(A^*) \otimes B^*(A^*) \) that carries the usual sign. All these statements can be checked directly. Once we know that all these operations are compatible with the differential, they are transferred to cohomology cohomology. Since they are compatible with the grading, they induce operations on \( H^0 \).

**Remarks 3.137.**

1. The graded commutativity of the product in \( A^* \) is essential in the previous proof. In fact if the product on \( A^* \) is not graded commutative, it is not true that the shuffle product in \( B^*(A^*) \) is compatible with the differential.

2. The complex \( B^*(A^*) \) is concentrated in positive degrees, so the cohomology we are interested in is simply \( H^0(B^*(A^*)) = \text{Ker}(d: B^0(A^*) \to B^1(A^*)) \).

Note that elements of \( B^0(A^*) \) are \( k \)-linear combinations of \([ \] and \([x_1|\cdots|x_n]\) with \( n \geq 1 \) and \( \deg(x_i) = 1 \) for all \( i = 1, \ldots, n \). Also, observe that, restricted to \( B^0(A^*) \), the differentials are given by the formulas

\[
d_I[x_1|\cdots|x_n] = -\sum_{i=1}^n [x_1|\cdots|dx_i|\cdots|x_n],
\]

\[
d_C[x_1|\cdots|x_n] = -\sum_{i=1}^{n-1} [x_1|\cdots|x_i \wedge x_{i+1}|\cdots|x_n].
\]

3.4.2. **The (non-reduced) bar complex of a dg-algebra.** When considering non-connected dg-algebras, it is convenient to use the (non-reduced) bar complex. This variant will not be needed for the main example of this text \( \mathbb{P}(\mathbb{C})^1 \setminus \{0,1,\infty\} \).

**Definition 3.138.** Let \((A^*, \wedge, d)\) be a dg-algebra over \( k \) and \( \varepsilon_1 \) and \( \varepsilon_2 \) two augmentations (maybe equal). The bar complex associated with \( A, \varepsilon_1, \varepsilon_2 \), denoted by \( B^*(A^*, \varepsilon_2, \varepsilon_1) \), is the total tensor algebra of \( A \):

\[
B^*(A, \varepsilon_2, \varepsilon_1) = k \oplus A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \ldots
\]

As in the case of the reduced bar complex, an element \( x_1 \otimes \cdots \otimes x_n \) for \( n \geq 1 \) will be denoted by the bar notation

\([x_1|\cdots|x_n] \),

and the element \( 1 \in k \) by the empty symbol \([ \] \). The grading of the bar complex is defined in the same way as in the reduced bar complex. The
differential takes into account also the augmentations and is given by

\[
\begin{align*}
    d[x_1|\cdots|x_n] &= -\sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i-1} \deg[x_j]} [x_1|\cdots|dx_i|\cdots|x_n] \\
    &\quad + \sum_{i=1}^{n-1} (-1)^{\sum_{j=1}^{i} \deg[x_j]} [x_1|\cdots|x_i|\wedge x_{i+1}|\cdots|x_n]. \\
    &\quad + \varepsilon_2(x_1)[x_2|\cdots|x_n] + (-1)^{\sum_{j=1}^{n} \deg[x_j]} \varepsilon_1(x_n)[x_1|\cdots|x_{n-1}]. \quad (3.139)
\end{align*}
\]

As before, \( \deg[x_j] = \deg(x_j) - 1 \), and one checks that \( d \) is compatible with the grading and that \( d \circ d = 0 \).

The product is the shuffle product given again by equation (3.132).

If \( \varepsilon_3 \) is a third augmentation (that may agree with the previous ones) there is a coproduct

\[
\Delta : B^*(A,\varepsilon_3,\varepsilon_2) \otimes B^*(A,\varepsilon_2,\varepsilon_1) \to B^*(A,\varepsilon_3,\varepsilon_1)
\]

given by deconcatenation

\[
\Delta[x_1|\cdots|x_n] = \sum_{i=0}^{n} [x_1|\cdots|x_i] \otimes [x_{i+1}|\cdots|x_n].
\]

Finally, the antipode is given again by formula (3.134).

REMARKS 3.140.

1. If \( A \) is a connected dg-algebra and \( \varepsilon \) is the unique augmentation, then the complexes \( B^*(A^*) \) and \( B^*(A^*,\varepsilon,\varepsilon) \) are homotopically equivalent. This is a consequence of Lemma 3.161.

2. If \( (A,\varepsilon_1,\varepsilon_2) \to (A',\varepsilon'_1,\varepsilon'_2) \) is a quasi-isomorphism commuting with the augmentations, then it induces a quasi-isomorphism

\[
B(A,\varepsilon_1,\varepsilon_2) \to B(A',\varepsilon'_1,\varepsilon'_2).
\]

As a consequence of this remark we have

**LEMMA 3.141.** Take \( k = \mathbb{R} \) or \( \mathbb{C} \) and let \( M \) be a connected differentiable manifold. Let \( x,y \in M \) and \( A \subset E^*(M,k) \) a connected dg-algebra such that the inclusion \( A^* \to E^*(M,k) \) is a quasi-isomorphism. Let \( \varepsilon_x,\varepsilon_y \) be the augmentations given by evaluation at the points \( x \) and \( y \) respectively. Then there is a quasi-isomorphism \( B^*(A^*) \to B(E^*(M,k),\varepsilon_y,\varepsilon_x) \). In particular

\[
H^0(B^*(A^*)) = H^0(B(E^*(M,k),\varepsilon_y,\varepsilon_x)).
\]
3.4.3. The reduced bar complex and iterated integrals. Let $M$ be a connected differentiable manifold with the homotopy type of a finite CW complex. Let $E^* (M, \mathbb{C})$ be the differential graded algebra of complex smooth differential forms on $M$. For simplicity of the exposition, we will assume that we have chosen a dg-$\mathbb{C}$-algebra $A^*$ provided with an injective morphism of dg-algebras $\varphi: A^* \to E^* (M, \mathbb{C})$ such that

1. $A^*$ is connected, that is $A^0 = \mathbb{C}$ and $A^n = 0$ for $n < 0$.
2. The induced map in cohomology $\varphi: H^* (A^*) \to H^* (E^* (M, \mathbb{C}))$ is an isomorphism.

And we will use the reduced bar complex of $A^*$. A similar discussion can be made with the bar complex of the whole dg-algebra $E^* (M, \mathbb{C})$. See Exercise 3.157.

The condition of $A^*$ being connected implies that the elements of degree zero of $B^0 (A^*)$ are linear combinations of the form

$$\sum [\eta_1 | \cdots | \eta_r]$$

with $\eta_i \in A^1 \subset E^1 (M)$ one forms. Thus, to any element $x \in B^0 (A^*)$ corresponds an iterated integral

$$[\eta_1 | \cdots | \eta_r] \mapsto \left( \gamma \mapsto \int_\gamma \eta_1 \cdots \eta_r \right).$$

For each pair of points $x, y \in M$, we define a pairing

$$\langle \ , \ \rangle : B^0 (A^*) \otimes \mathbb{Q}[y \mathcal{P} (M)_x] \to \mathbb{C}$$

$$[\eta_1 | \cdots | \eta_r] \otimes \gamma \mapsto \int_\gamma \eta_1 \cdots \eta_r, \quad (3.142)$$

where $y \mathcal{P} (M)_x$ is the set of piecewise smooth paths as in Section 3.1, and $\mathbb{Q}[y \mathcal{P} (M)_x]$ denotes the $\mathbb{Q}$-vector space with basis $y \mathcal{P} (M)_x$.

We can now translate Theorem 3.19 into the language of the bar complex and the pairing (3.142).

**Theorem 3.143.** Let $\gamma, \gamma_1, \gamma_2$ be piecewise smooth paths in $M$ and let $\eta, \eta_1, \eta_2 \in B^0 (A^*)$ be degree zero elements of the reduced bar complex of $A^*$. Then the following three equalities are satisfied:

$$\langle S(\eta), \gamma \rangle = \langle \eta, S(\gamma) \rangle. \quad (3.144)$$

$$\langle \eta, \gamma_1 \gamma_2 \rangle = \langle \Delta \eta, \gamma_1 \otimes \gamma_2 \rangle. \quad (3.145)$$

$$\langle \eta_1 \otimes \eta_2, \nabla^\gamma \rangle = \langle \eta_1 \wedge \eta_2, \gamma \rangle. \quad (3.146)$$

A consequence of the previous theorem is the following result that says that the length filtration of the reduced bar complex is dual to the filtration by the augmentation ideal in the group algebra of paths.
Proposition 3.147. Let $x, y$ be points of $M$. Let $J$ be the augmentation ideal of $\mathbb{Q}[x \mathcal{P}(M)_x]$, $N \geq 0$ an integer and $\gamma \in J^{N+1}\mathbb{Q}[x \mathcal{P}(M)_y]$ or $\gamma \in \mathbb{Q}[y \mathcal{P}(M)_x]J^{N+1}$. If $\eta \in L_N B^0(A^*)$ has length less than or equal to $N$, then $\langle \eta, \gamma \rangle = 0$.

Proof. We only treat the case $\gamma \in J^{N+1}\mathbb{Q}[x \mathcal{P}(M)_y]$ (the other one is completely analogous). The proof proceeds by induction on $N$.

If $N = 0$, every element of $\gamma \in J\mathbb{Q}[x \mathcal{P}(M)_y]$ can be written as

$$\gamma = \sum_{i=1}^r q_i \gamma_i, \quad q_i \in \mathbb{Q}, \quad \sum_{i=1}^r q_i = 0, \quad \gamma_i \in x \mathcal{P}(M)_y.$$

If $\eta \in L_0 B^0(A^*)$, then $\eta = \alpha[ ]$ for $\alpha \in \mathbb{C}$. Since $\langle [ ], \gamma_i \rangle = 1$, for $\gamma_i \in x \mathcal{P}(M)_y$, we deduce the result in the case $N = 0$.

Now fix $N > 0$ and assume that the result holds for all $N' < N$. To prove it for $N$, we may assume that $\gamma = \gamma_1 \gamma_2$ with $\gamma_1 \in J$, $\gamma_2 \in J^{N+1}\mathbb{Q}[x \mathcal{P}(M)_y]$, and $\eta = [\omega_1 | \cdots | \omega_N]$. Then the relation (3.145) yields

$$\langle \eta, \gamma \rangle = \langle \Delta \eta, \gamma_1 \otimes \gamma_2 \rangle = \sum_{i=0}^N \langle [\omega_1 | \cdots | \omega_i], \gamma_1 \rangle \langle [\omega_{i+1} | \cdots | \omega_N], \gamma_2 \rangle$$

$$= \langle [ ], \gamma_1 \rangle \langle [\omega_1 | \cdots | \omega_N], \gamma_2 \rangle + \sum_{i=1}^N \langle [\omega_1 | \cdots | \omega_i], \gamma_1 \rangle \langle [\omega_{i+1} | \cdots | \omega_N], \gamma_2 \rangle.$$

The first summand in the last equality vanishes since $\langle [ ], \gamma_1 \rangle = 0$ and all the factors $\langle [\omega_{i+1} | \cdots | \omega_N], \gamma_2 \rangle$ in the second sum vanishes by the induction hypothesis. Thus, $\langle \eta, \gamma \rangle = 0$, as we wanted to show. \( \square \)

3.4.4. The reduced bar complex and the pro-unipotent completion of the fundamental group. One of the main interests of the reduced bar complex is that it provides us with a criterion to decide whether an iterated integral is a homotopy functional, thus solving the question raised in Section 3.1.

Theorem 3.148. Let $\eta \in B^0(A^*)$. If $d\eta = 0$, then the iterated integral associated with $\eta$ is a homotopy functional.

Proof. Let $x, y \in M$. Consider two homotopic paths $\gamma_1$ and $\gamma_2$ from $x$ to $y$ and let $F$ be a homotopy between them. Recall from Definition 3.1 that $F: [0,1]^2 \to M$ satisfies the conditions

$$F(t,0) = \gamma_1(t), \quad F(t,1) = \gamma_2(t), \quad F(0,s) = x, \quad F(1,s) = y. \quad (3.149)$$
For simplicity, we will assume that $F$ is smooth; the general case follows by taking a polyhedral decomposition, as in the proof of Lemma 3.11. Set

$$F_i : [0,1]^n \times [0,1] \rightarrow M,$$

$$(t_1, \ldots, t_n, s) \mapsto F(t_i, s).$$

The elements of $B^1(A^*)$ are linear combinations of $\nu = [\nu_1 | \cdots | \nu_n]$ with exactly one 2-form among the $\nu_i$’s and the remaining ones being one forms. Given such a $\nu$, with the 2-form in the $i$-th position, we define the integral along $F$ as

$$\int_F \nu = (-1)^i \int_{[0,1] \times \Delta_n} F_1^* \nu_1 \cdots F_n^* \nu_n,$$

where the second integral is oriented by $ds \wedge dt_1 \wedge \cdots \wedge dt_n$. In this integral $\Delta_n$ denotes, as in Notation 1.107, the simplex $\Delta_n = \{(t_1, \ldots, t_n) \mid 1 \geq t_1 \geq \cdots \geq t_n \geq 0\}$.

The definition of the integral along $F$ extends to $B^1(A^*)$ by $\mathbb{C}$-linearity. We claim that

$$\int_{\gamma_2} \omega - \int_{\gamma_1} \omega = \int_F d\omega,$$

and the statement of the theorem will of course be an immediate consequence.

The equality (3.150) is proved by a careful application of Stokes’s theorem. Let $\omega = [\omega_1 | \cdots | \omega_n] \in B^0(A^*)$. First observe that

$$d(F_1^* \omega_1 \wedge \cdots \wedge F_n^* \omega_n) = \sum_{i=1}^n (-1)^{i+1} F_1^* \omega_1 \wedge F_i^* (d\omega_i) \wedge \cdots \wedge F_n^* \omega_n$$

by the properties defining the exterior derivative (see Section 2.2.1) and the commutativity of $d$ and $F_i^*$. Combining this with the definitions of $d_I$ and the integral along $F$, one gets:

$$\int_F d_I \omega = \int_{[0,1] \times \Delta_n} d(F_1^* \omega_1 \wedge \cdots \wedge F_n^* \omega_n).$$

We now apply Stokes’ theorem. Set $\Omega = F_1^* \omega_1 \wedge \cdots \wedge F_n^* \omega_n$.

$$\int_F dI \omega = \int_{\partial([0,1] \times \Delta^n)} F_1^* \omega_1 \wedge \cdots \wedge F_n^* \omega_n$$

$$= \int_{s=1} \Omega - \int_{s=0} \Omega - \int_{t_1=1} \Omega + \sum_{i=1}^{n-1} (-1)^{i+1} \int_{t_i=t_{i+1}} \Omega - (-1)^n \int_{t_n=0} \Omega$$

By the relations satisfied by $F$,

$$\Omega|_{s=1} = \gamma_2 \omega_1 \wedge \cdots \wedge \gamma_2 \omega_n,$$

$$\Omega|_{s=0} = \gamma_1 \omega_1 \wedge \cdots \wedge \gamma_1 \omega_n,$$

$$\Omega|_{t_i=t_{i+1}} = F_i^* \omega_1 \wedge \cdots \wedge F_i^* (\omega_i \wedge \omega_{i+1}) \wedge F_n^* \omega_n.$$
and $\Omega|_{t_1=1}$ (resp. $\Omega|_{t_n=0}$) vanishes since in that case $F_1$ (resp. $F_n$) is a constant function. Besides,

$$\int_F d_C \omega = \sum_{i=1}^{n} (-1)^{i+1}\int_{[0,1] \times \Delta^{n-1}} F_1^* \omega_1 \wedge \cdots \wedge F_i^*(\omega_i \wedge \omega_{i+1}) \wedge \cdots \wedge F_n^* \omega_n.$$ 

Putting everything together yields

$$\int_F d_I \omega = \int_{\gamma_2} \omega - \int_{\gamma_1} \omega + \int_F d_C \omega,$$

which is exactly the content of the claim noting that $d = d_I - d_C$. □

Let $x \in M$ and write $\Gamma = \pi_1(M, x)$. The condition that $M$ has the homotopy type of a finite CW complex implies that $H_1(M)$ is finite dimensional. Thus $\Gamma$ satisfies the hypothesis of Theorem 3.110 and its pro-unipotent completion is given by $\text{Spec}(\mathbb{Q}[\Gamma]^\wedge)$. 

Recall that the zero cohomology group of the reduced bar complex of $A^*$ is just the kernel of the differential map,

$$H^0(B^*(A^*)) = \text{Ker} (d: B^0(A^*) \to B_1(A^*)),$$

which, by Theorem 3.148, consists of homotopy functionals.

Putting together Theorem 3.148 and Proposition 3.147 we obtain a map

$$H^0(L_N B^*(A^*)) \to ((\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)]J^{N+1}) \otimes \mathbb{C})^\vee.$$

**Theorem 3.151 (Chen’s $\pi_1$-de Rham theorem).** For each integer $N \geq 0$ and points $x, y \in M$, the integration map gives an isomorphism

$$H^0(L_N B^*(A^*)) \cong \text{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)]J^{N+1}, \mathbb{C}),$$

and consequently it induces an isomorphism of ind-vector spaces between

$$H^0(B^*(A^*)) = \text{lim}_{N \to \infty} H^0(L_N B^*(A^*))$$

and

$$((\mathbb{C}[\pi_1(M,y,x)]^\wedge)^\vee = \text{lim}_{N \to \infty}((\mathbb{C}[\pi_1(M,y,x)]/\mathbb{C}[\pi_1(M,y,x)]J^{N+1})^\wedge)^\vee.$$ 

In fact, Theorem 3.143 implies that the last isomorphism of Theorem 3.151 is compatible with the Hopf algebra structures on both sides. We will give a proof of this result in the next section.

**Corollary 3.152.** For every point $x \in M$, the iterated integral induces an isomorphism of Hopf algebras

$$H^0(B^*(A^*)) \cong \mathcal{O}(\pi_1(M, x)^{un}) \otimes \mathbb{C}.$$

**Remark 3.153.** The isomorphism of Corollary 3.152 depends on the choice of a base point $x$. 

The case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The main example to which we would like to apply Corollary 3.152 is the manifold $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. This example will be central for the remainder of the book. The fundamental group of $M$ is the free group in two generators. Thus, its pro-unipotent completion is isomorphic to the spectrum of the Hoffman algebra $\mathfrak{H}$ by Example 3.117. We want to recover this fact as a particular case of Chen’s Theorem. For this, we consider the differential forms

$$\omega_0 = \frac{dt}{t}, \quad \omega_1 = \frac{dt}{1-t}. \quad (3.154)$$

Let $A^*_\mathbb{C}$ be the dg-algebra over $\mathbb{C}$ given by

$$A^0_\mathbb{C} = \mathbb{C}, \quad A^1_\mathbb{C} = \mathbb{C}\omega_0 \oplus \mathbb{C}\omega_1, \quad A^2_\mathbb{C} = 0,$$

together with the trivial differential and the obvious multiplication. Thus $A^*_\mathbb{C} = A^* \otimes \mathbb{C}$, where $A$ is the $\mathbb{Q}$ algebra introduced in Example 2.175. In particular, the inclusion $A^*_\mathbb{C} \subset E^*(M, \mathbb{C})$ is a quasi-isomorphism.

Since $d\omega_i = 0$ for $i = 0, 1$ and $\omega_0 \wedge \omega_1 = 0$, formula (3.129) shows that the differential in the reduced bar complex $B^*(A^*)$ is identically zero, and hence

$$H^0(B^*(A^*)) = B^0(A^*).$$

Moreover, there is an isomorphism of Hopf algebras

$$H^0(B^*(A^*)) \longrightarrow \mathfrak{H}, \quad \omega_0 \longmapsto x_0, \quad \omega_1 \longmapsto x_1.$$

That induces an isomorphism of Hopf algebras

$$H^0(B^*(A^*_\mathbb{C})) \longrightarrow \mathfrak{H} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Following Notation 1.153, for a binary sequence $\alpha$, we will denote by $\omega_\alpha$ the element of $H^0(B^*(A^*_\mathbb{C}))$ corresponding to $x_\alpha$.

***

**Exercise 3.155.** Show that the differentials $d_I$ and $d_C$ from equations (3.130) and (3.131) in the definition of the bar complex satisfy

$$d_I^2 = d_C^2 = 0 \quad d_I d_C + d_C d_I = 0.$$ 

Deduce that $d = d_I - d_C$ satisfies $d^2 = 0$ as well.

**Exercise 3.156.** Let $\eta_1, \eta_2$ and $\eta_{12}$ be 1-forms on a differentiable manifold. What conditions should they satisfy for $[\eta_1][\eta_2] - [\eta_{12}]$ to be closed?

**Exercise 3.157.** Let $M$ be a connected differentiable manifold with the homotopy type of a finite CW complex and let $E^*(M, \mathbb{C})$ be the differential graded algebra of complex smooth differential forms on $M$. Consider the projection

$$E^1(M, \mathbb{C}) \longrightarrow E^1(M, \mathbb{C})/dE^0(M, \mathbb{C}).$$
Let \( r \) be a retraction of this projection as complex vector spaces. Show that the subspace
\[
\mathbb{C} \oplus \text{Im}(r) \oplus \bigoplus_{n \geq 2} E^n(M, \mathbb{C}) \subset E^*(M, \mathbb{C})
\]
hits the structure of a complex dg-algebra. It is connected and the inclusion is a quasi-isomorphism.

### 3.5. A geometric description of the pro-unipotent completion of the fundamental group.

We now explain a proof of Chen’s \( \pi_1 \)-de Rham Theorem 3.151. This is not the classical proof that one can find in Hain’s paper [Hai87a, §4], but the strategy we follow will later enable us to exhibit the motivic nature of the pro-unipotent completion of the fundamental group of an algebraic variety. The first step in the proof is to show that the reduced bar complex of the de Rham complex of a differentiable manifold can be seen as the de Rham complex of a cosimplicial manifold. We will be using simplicial techniques and the unfamiliar reader is referred to Section A.7 of the appendix.

#### 3.5.1. The normalized cochain complex and the reduced bar complex.

Let \( M \) be a connected differentiable manifold with the homotopy type of a finite CW complex, and let \( x, y \in M \) be base points.

**Construction 3.158.** We denote by \( yM^\bullet_x \) the cosimplicial manifold with components
\[
yM^n_x = M \times \cdots \times M,
\]
coface maps
\[
\delta^i : yM^n_x \to yM^{n+1}_x, \quad i = 0, \ldots, n + 1,
\]
given by
\[
\delta^i(x_1, \ldots, x_n) = \begin{cases} (y, x_1, \ldots, x_n), & \text{if } i = 0, \\ (x_1, \ldots, x_i, x_i, \ldots, x_n), & \text{if } 0 < i < n + 1, \\ (x_1, \ldots, x_n, x), & \text{if } i = n + 1, \end{cases}
\]
and codegeneracy maps
\[
\sigma^i : yM^{n+1}_x \to yM^n_x, \quad i = 0, \ldots, n,
\]
given by
\[
\sigma^i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_i, x_{i+2}, \ldots, x_{n+1}). \quad (3.159)
\]

As in Section 3.4.3, let \( E^*(M, \mathbb{C}) \) denote the dg-algebra of smooth complex-valued differential forms on \( M \). For simplicity, we will assume that we have chosen a connected differential graded \( \mathbb{C} \)-algebra \( A^*(M) \subseteq E^*(M, \mathbb{C}) \) such that the inclusion is a quasi-isomorphism (see Exercise 3.157). We set
\[
A^*(yM^n_x) = A^*(M) \otimes \cdots \otimes A^*(M).
\]
In particular, \( A^*(yM^n_0) = \mathbb{C} \). The assignment \( \Delta_n \sim A^*(yM^n_x) \) being functorial (Exercise 3.215), these complexes define a simplicial dg-algebra \( A^*(yM^\bullet_x) \).
Remark 3.160. Since $A^*(M)$ is chosen to be connected, the dg-algebra $A^*(y M_x^\bullet)$ does not depend on the base points $x, y \in M$. The reason why keep them in the notation is that, later on, we will build out of $A^*(y M_x^\bullet)$ the de Rham component of a mixed Hodge structure whose Betti component does depend on $x$ and $y$.

Thus $N_s A^*(y M_x^\bullet)$ is a chain complex of cochain complexes. We convert it into a double cochain complex by changing the sign of the chain degree. We denote by $\text{Tot}^n N_s A^*(y M_x^\bullet)$ the associated total complex. Thus

$$\text{Tot}^n N_s A^*(y M_x^\bullet) = \bigoplus_{q-p=n} N_p A^q(y M_x^\bullet).$$

The subcomplexes $\sigma \leq N$ define a filtration of $\text{Tot}^N N_s A^*(y M_x^\bullet)$. Be aware that the index $N$ in the bête filtration refers only to the chain degree and not to the total degree.

Lemma 3.161. The map

$$\psi : B^*(A^*(M)) \to \text{Tot}^N N_s A^*(y M_x^\bullet)$$

$$[\omega_1|\cdots|\omega_n] \mapsto (-1)^{\sum_{i=1}^n (n-i) \deg \omega_i} \omega_1 \otimes \cdots \otimes \omega_n$$

is an isomorphism of complexes that sends the $N$-th step of the length filtration $L_N B^*(A^*(M))$ to $\text{Tot} \sigma \leq N N_s A^*(y M_x^\bullet)$. Similarly, if $\varepsilon$ is the unique augmentation of $A^*(M)$, then the same formula gives us an isomorphism

$$B^*(A^*(M), \varepsilon, \varepsilon) \to \text{Tot} C A^*(y M_x^\bullet).$$

Proof. This lemma is an easy verification. We will only prove the first statement, the second one being analogous. We set

$$A^+ (M) = \bigoplus_{n>0} A^n(M) = A^*(M)/\mathbb{C}.$$

By the shape (3.159) of the codegeneracy maps $\sigma^i$ we deduce that

$$\text{Im}(\sigma_i) = \text{Im}( (\sigma^i)^*) = A^*(M) \otimes \cdots \otimes \mathbb{C} \otimes \cdots \otimes A^*(M)$$

for $i = 0, \ldots, n-1$. Therefore,

$$N_n A^*(y M_x^\bullet) = A^*(y M_x^\bullet)/\sum_{i=0}^{n-1} \text{Im}(\sigma_i) = A^+(M) \otimes \cdots \otimes A^+(M).$$

It follows that the map $\psi$ is an isomorphism of graded vector spaces that respects the filtrations. We next compute the differential in the complex $\text{Tot} \sigma \leq N N_s A^*(y M_x^\bullet)$. Let

$$\omega = \omega_1 \otimes \cdots \otimes \omega_n \in N_n A^m(y M_x^\bullet) \subset \text{Tot}^{m-n} N A^*(y M_x^\bullet).$$
Then $d\omega = d_1 \omega + (-1)^n d_2 \omega$, where $d_1$ is the differential in the normalized complex and $d_2$ is the differential in the de Rham complex. Therefore

$$d\omega = (-1)^n \sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i-1} \deg(\omega_j)} \omega_1 \otimes \cdots \otimes d\omega_i \otimes \cdots \otimes \omega_n \ + \ \sum_{i=1}^{n-1} (-1)^i \omega_1 \otimes \cdots \otimes \omega_i \wedge \omega_{i+1} \otimes \cdots \otimes \omega_n. \quad (3.162)$$

Comparing this formula with the differential in Definition 3.128 one sees that $\psi \circ d = d \circ \psi$. This finishes the proof. $\square$

3.5.2. A Mayer–Vietoris complex of sheaves. The next step is to construct a complex of sheaves that computes certain relative cohomology groups. In fact, we are giving a variant of Construction 2.59 that was used to compute the relative de Rham cohomology in the case of a normal crossings divisor.

As in the previous sections, let $M$ be a connected differentiable manifold which has the homotopy type of a finite CW complex, and let $Y_0, \ldots, Y_k$ be a finite collection of closed subsets of $M$. Write

$$Y = Y_0 \cup \cdots \cup Y_k.$$

**Notation 3.163.** The following notation will be used:

- Recall that $\Delta_k$ stands for the index set $\{0, \ldots, k\}$.
- For each subset $I \subseteq \Delta_k$, we write $Y_I$ for the intersection $\bigcap_{j \in I} Y_j$. We also write $|I|$ for the cardinal of $I$.
- Given a topological space $T$, we denote by $Q_T^x$ the constant sheaf on $T$. If there is a clear inclusion $\iota: T \to M$, by abuse of notation, we will denote also by $Q_T^x$ the extension by zero $\iota_* Q_T^x$. For example, if $x \in M$ is a point, we will write $Q^x$ for the skyscraper sheaf with stalk $Q$ at $x$. For shorthand, in the situation we are considering, we write $Q^I$ for the constant sheaf on $Y_I$ extended by zero to $M$, that is,

$$Q^I = (\iota_I)_* Q_{Y_I},$$

where $\iota_I: Y_I \hookrightarrow M$ is the inclusion. In particular, $Q^\emptyset = Q_M$.
- Given subsets $I \subseteq K \subseteq \Delta_k$, there is an inclusion $Y_K \subseteq Y_I$. We denote by $d_{K,I}: Q_I^x \to Q_K^x$ the corresponding restriction map.
- If $K = \{k_0, \ldots, k_p\}$ with the indices $k_l$ ordered as $k_0 < \cdots < k_p$, and $I = \{k_0, \ldots, k_i, \ldots, k_p\}$, we set

$$\varepsilon(I, K) = (-1)^i$$
as in equation (2.61). We also write
\[
\varepsilon(K) = \prod_{k \in K} (-1)^k.
\] (3.164)

For \(1 \leq p \leq k + 1\), we define a morphism of sheaves
\[
d: \bigoplus_{|I|=p-1} \mathbb{Q}_I \to \bigoplus_{|K|=p} \mathbb{Q}_K, \quad \text{by} \quad d = \bigoplus_{I \subseteq K} \varepsilon(I,K) d_{K,I}.
\]
We define the complex of sheaves \(\tilde{K}(M;Y_0,\ldots,Y_k)\) as
\[
0 \to \bigoplus_{|I|=0} \mathbb{Q}_I \to \bigoplus_{|I|=1} \mathbb{Q}_I \to \cdots \to \bigoplus_{|I|=k} \mathbb{Q}_I \to \bigoplus_{|I|=k+1} \mathbb{Q}_I \to 0,
\] (3.165)
and the complex \(K(M;Y_0,\ldots,Y_k)\) as
\[
0 \to \bigoplus_{|I|=0} \mathbb{Q}_I \to \bigoplus_{|I|=1} \mathbb{Q}_I \to \cdots \to \bigoplus_{|I|=k} \mathbb{Q}_I \to 0.
\] (3.166)
Note that the second complex agrees with the first one except for the last term \(\mathbb{Q}_{\Delta_k}\) that has been deleted.

**Lemma 3.167.** If \(Y\) is locally contractible, then
\[
\mathbb{H}^n(M,\tilde{K}(M;Y_0,\ldots,Y_k)) = H^n(M,Y;\mathbb{Q}),
\]
where the right-hand side is relative singular cohomology.

**Proof.** The sequence of sheaves
\[
0 \to \mathbb{Q}_Y \to \bigoplus_{|I|=1} \mathbb{Q}_I \to \cdots \to \bigoplus_{|I|=p} \mathbb{Q}_I \to \cdots \to \bigoplus_{|I|=k+1} \mathbb{Q}_I \to 0
\]
is a resolution of the constant sheaf \(\mathbb{Q}_Y\) by the finite closed covering given by the \(Y_i\), extended by zero to the whole \(M\) (Exercise 3.216). Therefore, it is exact. Hence the complex \(\tilde{K}(M;Y_0,\ldots,Y_k)\) is quasi-isomorphic to the complex
\[
0 \to \mathbb{Q}_M \to \mathcal{I}_Y \mathbb{Q}_Y \to 0,
\]
where \(\mathcal{I}: Y \to M\) is the inclusion. The result follows then from (2.20). \(\square\)

We now specialize the previous construction to a particular case. Let \(x,y \in M\) be base points, and \(N \geq 0\) an integer. Let \(M^N\) be the \(N\)-fold cartesian product of \(M\). Given a point of \(M^N\), we denote by \(x_1,\ldots,x_N\) its components. Consider the union \(Y = Y_0 \cup \cdots \cup Y_N\) of the closed subspaces \(Y_i \subset M^N\) given by:
\[
Y_0 = \{y = x_1\},
Y_i = \{x_i = x_{i+1}\}, \quad i = 1,\ldots,N-1,
Y_N = \{x_N = x\}.
\]
Sometimes it will be useful to introduce the notation $x_0 = y$ and $x_{N+1} = x$ and write $Y_i = \{ x_i = x_{i+1} \}$ for all $i = 0, \ldots, N$.

Applying the previous construction we define the complexes

\[ y_0 K_x(N) = K(M^N; Y_0, \ldots, Y_N), \]
\[ y_0 \tilde{K}_x(N) = \tilde{K}(M^N; Y_0, \ldots, Y_N). \]

If the base points $x$ and $y$ are different from each other, then $Y_0 \cap \cdots \cap Y_N = \emptyset$ and hence the two complexes agree: $y_0 K_x(N) = \tilde{y}_0 K_x(N)$. By Lemma 3.167, the hypercohomology of $\tilde{y}_0 K_x(N)$ also computes the relative cohomology group:

\[ H^*(M^N, y_0 K_x(N)) = H^*(M^N, Y; \mathbb{Q}), \quad \text{when } x \neq y. \]

In the case where $x = y$, the intersection $Y_0 \cap \cdots \cap Y_N = \{(x, \ldots, x)\}$ consists of a single point and there is a short exact sequence of complexes

\[ 0 \to \mathbb{Q}(x, \ldots, x)[-N - 1] \to x \tilde{K}_x(N) \to x K_x(N) \to 0. \quad (3.168) \]

Note that the leftmost complex has only non-trivial cohomology in degree $N + 1$, where it is isomorphic to $H^0(M^N, \mathbb{Q}(x, \ldots, x)) = \mathbb{Q}$. Thus, taking hypercohomology from (3.168) yields a long exact sequence

\[ 0 \to H^N(M^N, Y; \mathbb{Q}) \to H^N(M^N, y_0 K_x(N)) \xrightarrow{f} \mathbb{Q} \to \cdots \quad (3.169) \]

The map $f$ is surjective because it fits into a commutative diagram

\[ \bigoplus_{|I|=N} H^0(Y_I, \mathbb{Q}) \to H^N(M^N, y_0 K_x(N)) \]

\[ \downarrow \quad f \]

\[ H^0(M^N, \mathbb{Q}(x, \ldots, x)) \]

where the diagonal arrow is surjective. The kernel of $f$ is thus $H^N(M^N, Y; \mathbb{Q})$ and we have a short exact sequence

\[ 0 \to H^N(M^N, Y; \mathbb{Q}) \to H^N(M^N, y_0 K_x(N)) \xrightarrow{f} \mathbb{Q} \to 0. \quad (3.170) \]

3.5.3. An isomorphism of cohomology groups. The next step is to relate the cohomology of the cosimplicial manifold $y_0 M^N_x$ with the cohomology of the sheaf $y_0 K_x(N)$. Since we want an isomorphism on the level of Betti cohomology that is defined over $\mathbb{Q}$ we will use smooth cochains instead of differential forms.

Given a differentiable manifold $M$, we denote by $S^*(M, \mathbb{Q})$ the complex of smooth singular cochains on $X$ with rational coefficients. Recall that the complex $S^*(M, \mathbb{Q})$ computes the Betti cohomology of $M$ and that there is a quasi-isomorphism

\[ E^*(M, \mathbb{C}) \to S^*(M, \mathbb{Q}) \otimes \mathbb{C} \]
given by integration of differential forms over smooth chains, that represents
the comparison isomorphism between de Rham and Betti cohomology in the
differentiable case. As a consequence we have quasi-isomorphisms
\[ A^*(M) \xrightarrow{\sim} S^*(M, \mathbb{Q}) \otimes \mathbb{C}, \quad A^*\left(yM^n_x\right) \xrightarrow{\sim} S^*(M^n, \mathbb{Q}) \otimes \mathbb{C}. \]
Now we write
\[ S^*\cdot = S^*\left(yM^\bullet_x, \mathbb{Q}\right). \]
This is a simplicial object in the category of complex of \(\mathbb{Q}\)-vector spaces and
there is a quasi-isomorphism
\[ \text{Tot} N_* A^*\left(yM^\bullet_x\right) \xrightarrow{\sim} \text{Tot} N_* S^*\cdot \otimes \mathbb{C}. \]
Since \(S^*\cdot\) is a simplicial object in the category of complexes of \(\mathbb{Q}\)-vector
spaces, we can apply to it the functor \(C^\bullet(\Delta N, \cdot)\) defined at the end of Sec-
tion A.7.2. To describe it, for each \(\emptyset \neq I \subset \Delta N\) we denote
\[ yM^I_x = Y_I^c \subset M^N. \]
where \(I^c = \Delta_N \setminus I\). Then
\[ yM^I_x \cong M^{\vert I\vert - 1} \]
To realize this isomorphism we just delete the redundant coordinates. Given
a point \((x_0, \ldots, x_n) \in yM^I_x\) and each \(i \in I^c\) the coordinate \(x_i\) agrees with
the coordinate \(x_{i+1}\). Thus the coordinate \(x_i\) is not needed. Furthermore,
the first coordinate is always equal to \(y\) so it is also redundant. Hence,
if \(I = (i_0, \ldots, i_k)\) we only need to keep the coordinates \(x_{i_1}, \ldots, x_{i_k}\). More
precisely, we denote by \(\iota_I: M^I_x \to M^N\) the composition of the inverse
of the isomorphism (3.172) with the inclusion \(Y_I^c \to M^N\). Then, if \(I = (i_0, \ldots, i_n)\) and
\[ p = (x_{i_1}, \ldots, x_{i_n}) \in yM^I_x, \]
writing \(x_{i_0} = y\) and \(x_{i_{n+1}} = x\), we have
\[ \iota_I(p) = (y_1, \ldots, y_N), \]
where
\[ y_i = x_{\min\{j \in I \cup \{n+1\} \mid j \geq i\}}. \]
For instance, if \(N = 6\) and \(I = \{2, 3, 5\}\), the map \(\iota_I: M^2 \to M^N\) is given by
\[ \iota_I(x_3, x_5) = (y, y, x_3, x_5, x_5, x). \]
If \(K = \{k_0, \ldots, k_n\}\) and \(I = \{k_0, \ldots, \hat{k}_i, \ldots, k_n\}\), there is a face map
\(\delta_{I,K}: yM^I_x \to yM^K_x\) defined by the commutative diagram
\[ \begin{array}{ccc}
yM^I_x & \xrightarrow{\delta_{I,K}} & yM^K_x \\
\| & & \| \\
Y_I^c & \xrightarrow{\iota_I} & Y_K^c.
\end{array} \]
Explicitly,
\[
\delta_{I,K}(x_{k_1}, \ldots, \hat{x}_{k_i}, \ldots, x_{k_n}) = \begin{cases} 
(y, x_{k_2}, \ldots, x_{k_n}), & \text{if } i = 0, \\
(\ldots, x_{k_{i-1}}, x_{k_{i+1}}, x_{k_{i+1}}, \ldots), & \text{if } 0 < i < n, \\
(x_{k_1}, \ldots, x_{k_{n-1}}, x), & \text{if } i = n.
\end{cases}
\]

We now write
\[
S_I^* = S^*(yM^I_x) \simeq S^*(M^{|I|-1}_x) = S_{{|I|-1}}^*.
\]

Then, for each \( p \geq 0 \), we have
\[
C_p(\Delta_N, S^*_\ast) = \bigoplus_{I \subset \Delta_N} S_I^* \bigg|_{|I|=p+1}
\]
with differential \( d: C_p(\Delta_N, S^*_\ast) \to C_{p-1}(\Delta_N, S^*_\ast) \) given by
\[
d = \bigoplus_{I \subset K} \epsilon(I, K) \delta_{I,K}^*.
\]

See equation (A.92).

By Proposition A.93 there is a functorial homotopy equivalence
\[
\phi: \text{Tot} \sigma_{\leq N}S^*_\ast \simeq \text{Tot} C^*(\Delta_N, S^*_\ast).
\]

Explicitly this morphism is given as follows. Let
\[
\omega = (\omega_i)_{0 \leq i \leq N} \in \text{Tot}^* \sigma_{\leq N}S^*_\ast.
\]
Then \( \phi(\omega) = (\omega_I)_{I \subset \Delta_N} \), where
\[
\omega_I = \begin{cases} 
\omega_i, & \text{if } I = \Delta_i, \\
0, & \text{otherwise}.
\end{cases}
\]

Next we compare the cohomology of the complex \( \text{Tot} C^*(\Delta_N, S^*_\ast) \) with that of the complex of sheaves \( yK_x(N) \). To this end we represent the later using also smooth singular cochains.

We define the double complex \( S^{p,q}_Y \) by
\[
S^{p,q}_Y = \bigoplus_{|I|=p} S^q(Y_I, \mathbb{Q}), \quad p \geq 0, \quad 0 \leq p < N,
\]
with horizontal differential
\[
d' : \bigoplus_{|I|=p-1} S^q(Y_I, \mathbb{Q}) \to \bigoplus_{|K|=p} S^q(Y_K, \mathbb{Q}), \quad d' = \bigoplus_{I \subset K} \epsilon(I, K)d_{K,I}.
\]

Let \( \text{Tot}^* S^*_Y \) be the associated total complex. By construction
\[
\mathbb{H}^*(M^N, yK_x(N)[N]) = H^*(\text{Tot}^* S^*_Y[N]).
\]
Lemma 3.175. There is a functorial isomorphism
\[ \text{Tot} C_\ast(\Delta_N, S^*_\ast) \xrightarrow{\simeq} \text{Tot}^\ast S_Y[N] \]
that induces an isomorphism
\[ H^\ast(\text{Tot} C_\ast(\Delta_N, S^*_\ast)) \xrightarrow{\simeq} \mathbb{H}^\ast(M^N, y\mathcal{K}_x(N))[N]. \]

Proof. For every \( \emptyset \neq I \subset \Delta_N \) we have \( yM^I_x = Y_I \). Denote by
\[ f_I : S^\ast(yM^I_x, \mathbb{Q}) \to S^\ast(Y_I, \mathbb{Q}) \]
the identity at the level of smooth singular. The morphisms \( f_I, \emptyset \neq I \subset \Delta_N \) define an isomorphism between the graded \( \mathbb{Q} \)-vector spaces \( \text{Tot} C_\ast(\Delta_N, S^*_\ast) \) and \( \text{Tot}^\ast S_Y[N] \).

But to have an isomorphism of complexes we need to check the compatibility with the differentials. Let \( \emptyset \neq I \subset K \subset \Delta_N \), with \( |K| = |I| + 1 \). The component of the horizontal differential of \( C_\ast(\Delta_N, S^*_\ast) \) between \( S^*_K \) and \( S^*_I \) is \( \varepsilon(I, K)\delta^*_I \), while the component of the horizontal differential in the complex \( S_Y^\ast[N] \) between \( S^*(Y_K) \) and \( S^*(Y_I) \) is \( (-1)^N \varepsilon(K, I) d_{I, K} \). By the commutativity of diagram (3.173), the maps \( \delta^*_I \) and \( d_{I, K} \) agree. Hence we only need to adjust the signs.

Let \( \varepsilon(I) = \prod_{i \in I} (-1)^i \) be the sign introduced in (3.164). It is immediate to check that
\[ \varepsilon(I, K)\varepsilon(K, I) = \varepsilon(I)\varepsilon(K). \]

In consequence, the map
\[ \text{Tot} C^\ast(\Delta_N, S^*_\ast) \to \text{Tot} S_Y^\ast[N] \]
that sends \( S^p(yM^I_x, \mathbb{Q}) \) to \( S^p(Y_I, \mathbb{Q}) \) through the map
\[ (-1)^{|I|} \varepsilon(I)f_I \tag{3.176} \]
is an isomorphism of complexes, thus proving the lemma. The sign \( (-1)^{|I|} \) is needed to take into account that in the complex \( \text{Tot} S_Y^\ast[N] \) one adds the sign \( (-1)^N \) to the differential. \( \square \)

Combining Lemma 3.175 with Lemma 3.161, the homotopy equivalence (3.174) and the fact that, for any differentiable manifold the map \( E^\ast(M, \mathbb{C}) \to S^\ast(M, \mathbb{Q}) \otimes \mathbb{C} \) is a quasi-isomorphism, we deduce the following result.

Corollary 3.177. There is a functorial morphism
\[ L_N B^\ast(A^\ast(M)) \rightarrow \text{Tot}^\ast S_Y \otimes \mathbb{C} \]
that induces an isomorphism
\[ H^0(L_N B^\ast(A^\ast(M))) \simeq \mathbb{H}^N(M^N, y\mathcal{K}_x(N)) \otimes \mathbb{C}. \]
In particular, if \( y \neq x \) we deduce an isomorphism
\[ H^0(L_N B^\ast(A^\ast(M))) \simeq H^N(M^N, Y; \mathbb{Q}) \otimes \mathbb{C}. \]
3.5.4. *Beilinson's Theorem, global version.* As discussed before Theorem 3.151, there is a map

\[
H^0(L_N B^*(A^*(M))) \rightarrow ((\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)])J^{N+1}) \otimes \mathbb{C}^\vee
\]

that we want to prove that is an isomorphism. In view of Corollary 3.177 we deduce a morphism

\[
\mathbb{H}^N(M^N, yK_x(N)) \otimes \mathbb{C} \rightarrow ((\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)])J^{N+1}) \otimes \mathbb{C}^\vee.
\]

We want to see that the later morphism is defined over \( \mathbb{Q} \). To this end we construct directly another morphism

\[
\mathbb{H}^N(M^N, yK_x(N)) \rightarrow ((\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)])J^{N+1}))^\vee.
\]

compatible with the previous one.

Let \( \gamma : [0, 1] \rightarrow M \) be a smooth path such that \( \gamma(0) = x \) and \( \gamma(1) = y \). For each \( 0 \neq I \subset \Delta_N \), we denote by \( \sigma_{\gamma,I}^{N,J} \) the map \( \Delta^{I|J|^{-1}} \rightarrow yM^I_y = Y_{Ic} \simeq M^{[I|J|^{-1}} \) given by

\[
\sigma_{\gamma,I}^{N,J}(t_1, \ldots, t_{|I|^{-1}}) = (\gamma(t_1), \ldots, \gamma(t_{|I|^{-1}})).
\]

If \( |I| = 1 \), then \( Y_{Ic} \) is reduced to a single point and the map \( \sigma_{\gamma,I}^{N,I} \) is constant. Using the maps \( \sigma_{\gamma,I}^{N,J} \) we define a map

\[
\sigma_{\gamma,y}^N : S_Y^*[N] \rightarrow \mathbb{Q},
\]

that sends \( \omega = (\omega_I)_{I \subseteq \Delta_N} \) to

\[
\sigma_{\gamma,y}^N(\omega) = \sum_{I \subseteq \Delta_N} (-1)^{|I|^{-1}} \binom{N(|I|^{-1})}{N} \binom{N(|I|^{-1})}{1} \omega_I (\sigma_{\gamma,I}^{N,I}). \tag{3.178}
\]

Observe the sign relation

\[
(-1)^{|I|^{-1}} \binom{N(|I|^{-1})}{N} = (-1)^{|I|^{-1}} \binom{N(|I|^{-1})}{1}. \tag{3.179}
\]

The reason for the complicated sign in equation (3.178) will be apparent in the proof of the next proposition.

**Proposition 3.180.** The map \( \sigma_{\gamma,y}^N \) for different \( \gamma \) define a morphism

\[
\sigma_y : \mathbb{H}^N(M^N, yK_x(N)) \rightarrow ((\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)])J^{N+1}))^\vee,
\]

such that the diagram

\[
\begin{array}{ccc}
H^0(L_N B^*(A^*(M))) & \rightarrow & ((\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)])J^{N+1}) \otimes \mathbb{C}^\vee \\ \\
\downarrow & & \downarrow \\
\mathbb{H}^N(M^N, yK_x(N)) \otimes \mathbb{C} & \rightarrow & ((\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)])J^{N+1}) \otimes \mathbb{C}^\vee
\end{array}
\]

is commutative.
Proof. We start by proving that, if $\omega$ is exact then $\sigma_{\gamma,y}^N(\omega) = 0$. For this, we compute the boundary of the singular chain $\sigma_{\gamma,y}^N$ and we obtain
\[
\partial\sigma_{\gamma,y}^N = \sum_{I \subset K \atop |K| = |I| + 1} \varepsilon(I, K)\sigma_{\gamma,y}^N.
\] (3.181)

If $\omega = d\eta$ is exact in the complex $S^*_Y[N]$, then
\[
(-1)^N\omega_I = (-1)^{|I|}d\eta_I + \sum_{K \subset I \atop |K| = |I| - 1} \varepsilon(K, I)\eta_K|_{\gamma_I}.
\] (3.182)

Using the sign relation (3.179),
\[
(-1)^{\frac{N(N+1)}{2}}\sigma_{\gamma,y}^N(\omega) = (-1)^N\sum_I (-1)^{\frac{|I|(|I|+1)}{2}}\varepsilon(I^c)\omega_I(\sigma_{\gamma,y}^N)
\]
\[= \sum_I (-1)^{\frac{|I|(|I|+1)}{2}}\varepsilon(I^c)(-1)^{|I|}d\eta_I(\sigma_{\gamma,y}^N)
\]
\[+ \sum_I \sum_{K \subset I \atop |K| = |I| - 1} (-1)^{\frac{|I|(|I|+1)}{2}}\varepsilon(I^c)\varepsilon(K, I)\eta_K(\sigma_{\gamma,y}^N)
\]
\[= \sum_I \sum_{K \supset I \atop |K| = |I| + 1} (-1)^{\frac{|I|(|I|+1)}{2}}\varepsilon(I^c)(-1)^{|I|}\varepsilon(K^c, I^c)\eta_I(\sigma_{\gamma,y}^N)
\]
\[+ \sum_I \sum_{K \supset I \atop |K| = |I| + 1} (-1)^{\frac{|I|(|I|+1)}{2}}\varepsilon(K^c)\varepsilon(I, K)\eta_I(\sigma_{\gamma,y}^N).
\]

In the above computation, the first equality is the definition of $\sigma_{\gamma,y}^N(\omega)$, the second equality is equation (3.182), and in the third equality we apply equation (3.181) to the first term and we interchange the roles of $I$ and $K$ in the second term. The fact that $\sigma_{\gamma,y}^N(\omega) = 0$ follows from the sign identities
\[
\varepsilon(I, K)\varepsilon(K^c, I^c) = \varepsilon(I^c)\varepsilon(K^c), \quad \text{and} \quad (-1)^{\frac{|I|(|I|+1)}{2}}(-1)^{|I|} = -(-1)^{\frac{|K|(|K|+1)}{2}},
\]
for $I \subset K$ with $|K| = |I| + 1$. Therefore, for $\omega$ closed, the value $\sigma_{\gamma,y}^N(\omega)$ only depends on the class of $\omega$ in $\mathbb{H}^N(M^N, y\mathcal{K}_x(\langle N \rangle))$. In consequence we obtain

a pairing
\[
\mathbb{H}^N(M^N, y\mathcal{K}_x(\langle N \rangle)) \otimes \mathbb{Q}[y\mathcal{P}(M)_x] \longrightarrow \mathbb{Q},
\]
where $\mathbb{Q}[y\mathcal{P}(M)_x]$ is as in (3.142).

We next show the above pairing is compatible with the pairing (3.142). Let
\[
[\omega_1] \ldots [\omega_n] \in H^0(L_NB^*(A^*(M))),
\]
since the total degree of this element is zero, every component form $\omega_i \in A^1(M)$. Then the isomorphism
\[
H^0(L_NB^*(A^*(M))) \longrightarrow \mathbb{H}^N(M^N, y\mathcal{K}_x(\langle N \rangle)) \otimes \mathbb{C}
\]
sends \([\omega_1 \ldots | \omega_n]\) to the form \(\omega = (\omega_I)_{I \subseteq \Delta_N}\) with \(\omega_I \in S^n(Y_I, \mathbb{C})\) given by
\[
\omega_I = \begin{cases} 
(-1)^{n(n-1)/2} (-1)^{N(n+1)} \varepsilon(\Delta_n) \omega_1 \boxtimes \cdots \boxtimes \omega_n, & \text{if } I = \Delta_c^n \\
0, & \text{otherwise}
\end{cases}
\]
Then
\[
\sigma_{N, y}^N(\omega) = (-1)^{n(n-1)/2} (-1)^{N(n+1)} \varepsilon(\Delta_n) \omega_\Delta(\sigma_{N, y}^{N, \Delta_n}) = \omega_1 \boxtimes \cdots \boxtimes \omega_n(\sigma_{N, y}^{N, \Delta_n}) = \langle [\omega_1 \ldots | \omega_n], \gamma \rangle, \quad (3.183)
\]
where the last term is the pairing (3.142). So the signs in equation (3.178) are chosen to obtain the sign cancellation in (3.183) and the fact that for \(\omega\) exact, \(\sigma_{N, y}^N(\omega) = 0\).

Once we have established the compatibility with the pairing (3.142), then Theorem 3.148, Proposition 3.147 and Corollary 3.177 imply the proposition.

The following result, due to Beilinson, gives a cohomological interpretation of the finite-dimensional pieces in the pro-unipotent completion of the fundamental group.

**Theorem 3.184.** The map \(\sigma_y\) of Proposition 3.180 is an isomorphism. In particular when \(x \neq y\) there is an isomorphism
\[
\sigma_y : \mathbb{H}^N(M^N, Y; \mathbb{Q}) \to ((\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)])^{N+1})^\vee.
\]

The proof of this result is by induction on \(N\). To put all the pieces in the right position, we need a relative version of the morphism \(\sigma_y\). This is done in the next section.

3.5.5. **Beilinson's theorem, relative version.** We next introduce a relative version of the complex \(\mathcal{K}_x(N)\), where we fix \(x\) but let \(y\) vary. For this, we consider the \((N + 1)\)-fold product
\[
M^{1, N} = M \times M^N = M^{N+1}
\]
regarded as a fibration over \(M\) with fiber \(M^N\). That is to say, we put coordinates \(x_0, \ldots, x_N\) on \(M^{1, N}\) and denote by
\[
\pi: M^{1, N} \to M
\]
the projection over the first factor. We introduce the closed subsets
\[
Z_i = \{x_i = x_{i+1}\} \subseteq M^{1, N}, \quad i = 0, \ldots, N,
\]
where we are still using the convention \(x_{N+1} = x\). For \(y \in M\) and \(i \in \Delta_N\), under the identification \(\pi^{-1}(y) = M^N\), we have
\[
Y_i = Z_i \cap \pi^{-1}(y).
\]
Thus we can see the subsets \(Z_i\) as the family of sets \(Y_i\) for moving \(y\) (in fact, only \(Y_0\) depends on \(y\)). For any subset \(I \subseteq \Delta_N\), we have
\[
Z_I \cap \pi^{-1}(y) = Y_I.
\]
by the above identification. We now define the complexes of sheaves
\[ \mathcal{K}_x(N) = K(M^{1,N}; Z_0, \ldots, Z_N) \]
\[ \tilde{\mathcal{K}}_x(N) = \tilde{K}(M^{1,N}; Z_0, \ldots, Z_N), \]
so that
\[ \mathcal{K}_x(N)|_{\pi^{-1}(y)} = y\mathcal{K}_x(N). \] \hspace{1cm} (3.185)
It is in this sense that \( \mathcal{K}_x(N) \) is a relative version of \( y\mathcal{K}_x(N) \).

The complexes \( \mathcal{K}_x(N) \) and \( y\mathcal{K}_x(N) \) satisfy a recurrence relation that will be useful later. The identity morphism between \( M^{1,(N-1)} \) and \( M^N \) changes the numbering of the components because in the convention we are using, the coordinates of \( M^N \) start with \( x_0 \) while those of \( M^N \) start with \( x_1 \). This identification sends the subset \( Z_i \subset M^{1,(N-1)} \) to the subset \( Y_{i+1} \subset M^N \) for \( i = 0, \ldots, N - 1 \).

Let \( \iota_y : M^{N-1} \to M^N \) be the map
\[ \iota_y(x_1, \ldots, x_{N-1}) = (y, x_1, \ldots, x_{N-1}) \]
For each \( N \geq 1 \), there is an exact sequence of sheaves of complexes
\[ 0 \to (\iota_y)_* y\mathcal{K}_x(N-1)[-1] \to y\mathcal{K}_x(N) \to \mathcal{K}_x(N-1) \to 0. \] \hspace{1cm} (3.186)
To describe this sequence we use the notation that, if \( I = (i_1, \ldots, i_k) \) is a multi-index, then the multi-index \( I + 1 \) is
\[ I + 1 = (i_1 + 1, \ldots, i_k + 1). \]
Then in degree \( 0 \leq k \leq N \), the sequence (3.186) reads
\[ 0 \to \bigoplus_{I \subset \{0, \ldots, N-1\}} \mathbb{Q}_{Y_{(0) \cup (I+1)}} \to \bigoplus_{I \subset \{0, \ldots, N\}} \mathbb{Q}_{Y_I} \to \bigoplus_{I \subset \{1, \ldots, N\}} \mathbb{Q}_{Y_I} \to 0 \]
To identify the rightmost term of this sequence with a piece of \( \tilde{\mathcal{K}}_x(N-1) \) we are using the identification between the sheaf \( \mathbb{Q}_{Z_I} \) on \( M^{1,N-1} \) and the sheaf \( \mathbb{Q}_{Y_{I+1}} \) on \( M^N \). Finally we have to be sure that the map
\[ (\iota_y)_* y\mathcal{K}_x(N-1)[-1] \to y\mathcal{K}_x(N) \]
is compatible with the differential. This amount to the sign relation, for \( I \subset K \), with \( |I| + 1 = |K| \),
\[ -\varepsilon(I, K) = \varepsilon(\{0\} \cup (I + 1), \{0\} \cup (K + 1)). \]
The exact sequence (3.168) induces an exact sequence
\[ 0 \to \mathbb{Q}_{\{x, \ldots, x\}}[-N] \to y\tilde{\mathcal{K}}_x(N-1) \to y\mathcal{K}_x(N-1) \to 0. \] \hspace{1cm} (3.187)
When considering a relative situation, like the family \( \pi : M^{1,N} \to M \), the analogue of the hypercohomology groups of the complex \( y\mathcal{K}_x(N) \) are the higher direct image sheaves \( R^i\pi_*(\mathcal{K}_x(N)) \). As explained in Section A.8.3,
they are defined as the sheaves of vector spaces associated with the presheaves that, to an open subset $U \subseteq M$, assign the vector space

$$\mathbb{H}^i(\pi^{-1}(U), \mathcal{K}_x(N)).$$

To understand them, we shall use the following concrete description of the cohomology. As in the previous sections, let $S^\bullet(T, \mathbb{Q})$ denote the complex of smooth singular cochains on a differentiable manifold $T$. Using the construction (3.166) applied to $M^{1,N}$ and the subsets $Z_0, \ldots, Z_N$, we obtain a double complex

$$0 \to \bigoplus_{|I|=0} S^\bullet(Z_I \cap \pi^{-1}(U), \mathbb{Q}) \to \bigoplus_{|I|=1} S^\bullet(Z_I \cap \pi^{-1}(U), \mathbb{Q}) \to \cdots$$

$$\cdots \to \bigoplus_{|I|=N} S^\bullet(Z_I \cap \pi^{-1}(U), \mathbb{Q}) \to 0 \quad (3.188)$$

which will be denoted by $S^\bullet(Z_\bullet \cap \pi^{-1}(U), \mathbb{Q})$. The associated total complex computes the hypercohomology

$$\mathbb{H}^i(\pi^{-1}(U), \mathcal{K}_x(N)) = H^i(\text{Tot}(S^\bullet(Z_\bullet \cap \pi^{-1}(U), \mathbb{Q}))).$$

**Lemma 3.189.** For every contractible open subset $U$ of $M$ and every point $y \in U$, the inclusion $\pi^{-1}(y) \to \pi^{-1}(U)$ and the identification $\pi^{-1}(y) \simeq M^N$ induce an isomorphism

$$\mathbb{H}^i(\pi^{-1}(U), \mathcal{K}_x(N)) \to \mathbb{H}^i(M^N, y\mathcal{K}_x(N)).$$

**Proof.** For every $I \subseteq \Delta_N$ with $|I| \leq N$, the morphism $\pi|_{Z_I} : Z_I \to M$ is a fibration. Therefore, given any contractible open subset $U \subseteq M$ and any point $y \in U$, the inclusion $Z_I \cap \pi^{-1}(y) \to Z_I \cap \pi^{-1}(U)$ is a homotopy equivalence. The induced morphism of complexes

$$S^\bullet(Z_I \cap \pi^{-1}(U), \mathbb{Q}) \to S^\bullet(Z_I \cap \pi^{-1}(y), \mathbb{Q}) = S^\bullet(Y_I, \mathbb{Q})$$

is a homotopy equivalence as well. The lemma follows from this. $\square$

Thanks to this lemma, the sheaf $R^i\pi_* \mathcal{K}_x(N)$ is a local system on $M$ whose fibre at a point $y$ is given by the hypercohomology group

$$R^i\pi_* \mathcal{K}_x(N)_y = \mathbb{H}^i(M^N, y\mathcal{K}_x(N)).$$

We refer the reader to Section A.8.4 from the appendix for a quick reminder on the different ways to think about local systems. In particular, the sheaf $R^\bullet\pi_* \mathcal{K}_x(N)$ “glues together” the hypercohomology groups $\mathbb{H}^i(M^N, y\mathcal{K}_x(N))$ for all possible base points $y$. The map $f$ in the exact sequence (3.169) yields a morphism from this local system to a skyscraper sheaf

$$R^N\pi_* \mathcal{K}_x(N) \to \underline{\mathcal{O}}_x. \quad (3.190)$$
We have just described some local systems constructed using cohomology. Now we consider some local systems defined using the fundamental groupoid. The point \( x \in M \) continues to be fixed. There are left actions of \( \pi_1(M, x) \) on the \( \mathbb{Q} \)-vector spaces
\[
\mathbb{Q}[\pi_1(M, x)] \quad \text{and} \quad \mathbb{Q}[\pi_1(M, x)]/J^{N+1}
\]
given by path composition. These actions define local systems (see Section A.8.4)
\[
\mathbb{Q}[\pi_1(M; \bullet, x)] \quad \text{and} \quad \mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1}.
\]
The first one may be infinite dimensional, but the second one is always finite dimensional. The fibre at a point \( y \) of the first local system is given by
\[
\mathbb{Q}[\pi_1(M; \bullet, x)]_y = \mathbb{Q}[\pi_1(M; y, x)].
\]
Thus, for every contractible open subset \( U \), the sections of \( \mathbb{Q}[\pi_1(M; \bullet, x)](U) \) are functions
\[
s: U \to \prod_{y \in U} \mathbb{Q}[\pi_1(M; y, x)]
\]
satisfying that, for each pair of points \( y, y' \in U \) and class of paths \( \gamma \in \pi_1(U; y, y') \), the relation
\[
s(y) = \gamma \cdot s(y')
\]
holds. Note that, since \( U \) is assumed to be contractible, \( \pi_1(U; y, y') \) contains a single element. For every point \( y \in U \), there is a canonical identification
\[
\mathbb{Q}[\pi_1(M; \bullet, x)](U) = \mathbb{Q}[\pi_1(M; y, x)].
\]
The description of \( \mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1} \) is similar. The unit of \( \pi_1(M; x) \) induces maps
\[
\mathbb{Q} \to \mathbb{Q}[\pi_1(M; \bullet, x)]_x,
\]
\[
\mathbb{Q} \to \left( \mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1} \right)_x.
\]

We next construct a morphism between the local systems \( R^N \pi_*(\mathcal{K}_x(N)) \) and \( (\mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1})^\vee \). This map is a relative version of the map \( \sigma_y \) of Proposition 3.180.

**Lemma 3.192.** The maps \( \sigma_y \) of Proposition 3.180 for varying \( y \) glue together to a morphism of local systems
\[
\sigma: R^N \pi_*(\mathcal{K}_x(N)) \to (\mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1})^\vee.
\]

**Proof.** We have two local systems and a collection of morphisms between their fibres. To see that they glue together to a morphism of local systems, we need to prove that they are compatible with parallel transport. Assume that we have two local systems \( F \) and \( G \) on \( M \) and for each \( y \in M \) a morphism \( f_y: F_y \to G_y \). To glue all these morphism we have to see that,
given a contractible open subset \( U \subset M \) and points \( y, y' \in U \), then the diagram
\[
\begin{array}{c}
F_y \xrightarrow{\sim} F(U) \xrightarrow{\sim} F_{y'} \\
\downarrow f_y \quad \downarrow f_{y'} \\
G_y \xrightarrow{\sim} G(U) \xrightarrow{\sim} G_{y'}
\end{array}
\]
is commutative. If this is the case, one can show that the \( f_y \) define a morphism of representations of the fundamental group and apply Theorem A.115.

Let \( U \subset M \) be a contractible subset and \( y, y' \in U \) two points. An element of \( (\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)],J^{N+1})_y \) is represented by a linear combination of paths from \( x \) to \( y \) and the parallel transport is given by composition of paths. The fiber of the first local system is
\[
R^N\pi_*(\mathcal{K}_x(N))_y = \mathbb{H}^N(\pi^{-1}(y),y\mathcal{K}_x(N))
\]
and the parallel transport by the composition
\[
\mathbb{H}^N(\pi^{-1}(y),y\mathcal{K}_x(N)) \xrightarrow{\sim} \mathbb{H}^N(\pi^{-1}(U),\mathcal{K}_x(N)) \xrightarrow{\sim} \mathbb{H}^N(\pi^{-1}(y'),y\mathcal{K}_x(N)).
\]
Let \( \omega \in S^N(Z\bullet \cap \pi^{-1}(U),\mathbb{Q}) \) be a closed singular cochain and denote by \( \omega_y \) and \( \omega_{y'} \) the restrictions of \( \omega \) to \( S^*(Z\bullet \cap \pi^{-1}(y),\mathbb{Q}) \) and \( S^*(Z\bullet \cap \pi^{-1}(y'),\mathbb{Q}) \) respectively. Let \( \gamma \in yP(M)_x \) and \( \gamma' \in y'P(U)_y \) be paths. By the previous discussion, to prove the lemma we have to show the equality
\[
\sigma^N_{\gamma,y}(\omega_y) = \sigma^N_{\gamma',y'}(\omega_{y'}).\]
The idea to prove this equality is to construct a singular chain whose boundary is \( \sigma^N_{\gamma,y} - \sigma^N_{\gamma',y'} \).

Recall that any oriented polyhedron \( P \) defines a singular chain after choosing a triangulation. The chains obtained with different triangulations are cohomologous. Any face of \( P \) inherits a orientation from the orientation of \( P \) and a triangulation of \( P \) defines a triangulation of the faces. Fixing a triangulation of \( P \) and identifying \( P \) and its faces with the corresponding singular chains, the equation
\[
\partial P = \sum_{F \text{ face of } P} F.
\]
In this equation, the signs of the boundary of a chain are concealed in the orientation of the faces.

Next we observe that the map \( H : [0,1] \times [0,1] \to M \) defined as
\[
H(s,t) = \gamma'(1 + s)\gamma t/2
\]
satisfies
\[
H(0,t) = \gamma(t), \quad H(1,t) = \gamma'(t), \quad H(s,0) = x, \quad H(s,1) = \gamma'(s).
\]
In the previous section we have identified \( Y_{I^c} \) with \( M^{[I]} \) by deleting the redundant coordinates. In the same way we can identify \( Z_{I^c} \) with \( M^{[I]} \).

With this identification the projection \( Z_{I^c} \to M \) is the projection over the first coordinate.

For each \( \emptyset \neq I \subset \Delta^N \) we denote by

\[
H^{N,I}_{\gamma,\gamma'} : [0,1] \times \Delta^{[I]-1} \to Z_{I^c} \cap \pi^{-1}(U)
\]

the map given by

\[
H^{N,I}_{\gamma,\gamma'}(s,t_1,\ldots,t_{|I|-1}) = (\gamma'(s),\gamma'((1+s)t_1/2),\ldots,\gamma((1+s)t_{|I|-1}/2))
\]

After triangulating \([0,1] \times \Delta^{[I]-1}\) this defines a singular chain in \( Z_{I^c} \cap \pi^{-1}(U) \). Viewed as a chain in \( \pi^{-1}(U) \) satisfies the boundary equation

\[
\partial H^{N,K}_{\gamma,\gamma'} = \sigma^{N,K}_{\gamma,y} - \sigma^{N,K}_{\gamma',y'} - \sum_{I \subset K \atop |K| = |I|+1} \varepsilon(I,K)H^{N,I}_{\gamma,\gamma'}.
\]

For a form \( \eta \in S^{N+1}(Z_x \cap \pi^{-1}(U), \mathbb{Q}) \) we write

\[
H^N_{\gamma,\gamma'}(\eta) = \sum_{I \subseteq \Delta_N} (-1)^{(|I^c|)(|I^c|-2)}/2 (1)^{N(|I^c|)} \varepsilon(I^c)\eta_I(H^{N,I^c}_{\gamma,\gamma'}). \]

Then computing as in the first part of the proof of Proposition 3.180, we have

\[
0 = H^N_{\gamma,\gamma'}(d\omega) = \sigma^N_{\gamma,y}(\omega_y) - \sigma^N_{\gamma',y'}(\omega_{y'}),
\]

proving the lemma.

\( \square \)

The following result, due to Beilinson, is the relative version of Theorem 3.184 and implies it. This gives a cohomological interpretation of the finite-dimensional pieces in the pro-unipotent completion of the fundamental group. There are two proofs of this theorem in the literature [Gon01, §4] and [DG05, §3.3].

**Theorem 3.193 (Beilinson).**

1. The sheaf \( R^i\pi_*(\mathcal{K}_x(N)) \) vanishes for all \( i \leq N - 1 \). In particular,
   \[
   \mathbb{H}^i(M^N, y\mathcal{K}_x(N)) = 0, \quad i \leq N - 1.
   \]

2. The map \( \sigma \) defined in Lemma 3.192 is an isomorphism of local systems
   \[
   \sigma : R^N\pi_*(\mathcal{K}_x(N)) \to (\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,y)])J^{N+1})^\vee.
   \]
   In particular, there are natural isomorphisms
   \[
   \mathbb{H}^N(M^N, y\mathcal{K}_x(N)) \to (\mathbb{Q}[\pi_1(M; y,x)]/\mathbb{Q}[\pi_1(M; y,x)])J^{N+1})^\vee.
   \]
(3) The diagram of sheaves on $M$

$$R^N \pi_*(\mathcal{K}_x(N)) \xrightarrow{\sigma} (\mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]) J^{N+1 \vee} \xrightarrow{\mathbb{Q}_x}$$

where the diagonal arrow is (3.190) and the vertical arrow is induced by the dual of the unit (3.191), is commutative.

PROOF. We first prove statement (3) in the theorem. Since for $y \neq x$ the fibre $(\mathbb{Q}_x)_y = 0$ we only need to check what happens when point $y = x$. Then the statement reduces to the commutativity of the diagram

$$\begin{align*}
\mathbb{H}^N(M^N, x; \mathcal{K}_x(N)) & \xrightarrow{\sigma} (\mathbb{Q}[\pi_1(M, x)]/\mathbb{Q}[\pi_1(M, x)]) J^{N+1 \vee} \\
\mathbb{H}^N(M^N, \mathbb{Q}(x, \ldots, x)[-N]) & \xrightarrow{\mathbb{Q}}
\end{align*}$$

(3.194)

For simplicity, we compute $\mathbb{H}^N(M^N, x; \mathcal{K}_x(N))$ as the cohomology of the complex

$$C^* = \bigoplus_{I \subset \Delta_N, |I| \leq N} \text{Tot}^* (\mathcal{N}S^* (Y_I, \mathbb{Q})),
$$

where $\mathcal{N}S^*$ denotes the normalized complex of smooth cochains. See Section A.7.2 for the definition of the normalized complex associated to a simplicial abelian group. The advantage of this point of view is that the elements of $C^*$ vanish on degenerate chains, simplifying slightly the argument below.

Let $\gamma_x$ be the constant path $x$ in $M$. Since $\gamma_x$ is constant, for $I \subset \Delta_N$, $|I| \leq N$, the singular chain $\sigma_{\gamma_x}^{N, I}$ is supported on $Z_I \cap \pi^{-1}(x) \subset M^{1,N}$. Identifying $\pi^{-1}(x)$ with $M^N$ we can see it as a singular chains on $Y_I$. This chain is degenerate unless $|I| = N$. When $|I| = N$, $\sigma_{\gamma_x}^{N, I}$ is the zero dimensional simplex at the point $(x, \ldots, x)$.

Let $\omega = \sum_I \omega_I \in C^N$ be a closed element. The left vertical map on the diagram followed by the bottom arrow sends $\omega$ to

$$\sum_{|I| = N} \varepsilon(I, \Delta_N) \omega_I(x, \ldots, x).$$

(3.195)

We apply the top arrow followed by the right vertical arrow to $\omega$, using equation (3.178) and taking into account that we are working in the complex of normalized cochains, and we obtain the element

$$\sum_{|I| = N} \varepsilon(I^c) \omega_I(\sigma_{\gamma_x}^{N, I}).$$

(3.196)
The equality between (3.195) and (3.196) follows from the identity
\[ \varepsilon(I^c) = \varepsilon(I, \Delta_N), \text{ for } |I| = N. \]

We now turn to the proof of statements (1) and (2) in the theorem. We proceed by induction on \( N \). The case \( N = 0 \) is obvious. Since we already now that \( \sigma \) is a morphism of sheaves, it is enough to prove the statements fibrewise. Let \( y \in M \). From the exact sequence (3.186), we deduce a long exact sequence
\[ \mathbb{H}^{N-1}(M^{1,N-1}, \mathcal{K}_x(N - 1)) \rightarrow \mathbb{H}^{N-1}(M^{N,1}, \mathcal{K}_x(N)) \rightarrow \mathbb{H}^{N}(M^{1,N-1}, \mathcal{K}_x(N - 1)) \]
and use it to write down the following diagram with exact rows:

\[ \begin{array}{c}
0 \rightarrow (\mathbb{Q}[\pi_1(M; y, x)]/J^N)^\vee \rightarrow (\mathbb{Q}[\pi_1(M; y, x)]/J^{N+1})^\vee \rightarrow (J^N/J^{N+1})^\vee \rightarrow 0.
\end{array} \]

(3.197)

Claim: The left square in the above diagram is commutative.

The first horizontal map sends a closed smooth cochain
\[ \omega = \sum_{I \subseteq \Delta_{N-1}} \omega_I \]
representing a class in \( \mathbb{H}^{N-1}(M^{N,1}, \mathcal{K}_x(N - 1)) \) to the cochain
\[ \iota(\omega) = \sum_{I \subseteq \Delta_{N-1}} \omega_I, \]
where \( \omega_I \) is now seen as a cochain in \( Y_{\{0\} \cup (I+1)} \). Therefore, one has, for every class \( \gamma \in \mathbb{Q}[\pi_1(M; y, x)] \),
\[ \sigma(\omega)(\gamma) = \sum_{I \subseteq \Delta_{N-1}} (-1)^{|I|(I+1)} \frac{|I|(N-1)(N-2)}{2} \varepsilon(I^c) \omega_I(\sigma^{N-1,I}_\gamma), \]
\[ \sigma(\iota(\omega))(\gamma) = \sum_{I \subseteq \Delta_{N-1}} (-1)^{|I|+1} \frac{|I+1|(I+2)(N-1)}{2} \varepsilon((I')^c) \omega_I(\sigma^{N,I'}_\gamma), \]
where \( I' = \{0\} \cup (I+1) \). Since the chains \( \sigma^{N-1,I}_\gamma \) and \( \sigma^{N,\{0\} \cup I}_\gamma \) are equal and one has
\[ (-1)^{|I|(I+1)} \frac{|I|(N-1)(N-2)}{2} = (-1)^{|I+1|} \frac{|I+1|(I+2)(N-1)}{2} = (-1)^{N+|I|} = \varepsilon(I^c) \varepsilon((I')^c) \]
the square commutes. Once we know this, an easy diagram chase shows that there is a map
\[ \sigma : \text{Ker}(g) \to (J^N/J^{N+1})^\vee, \]
slightly abusively still denoted it by \( \sigma \), which completes (3.198) to a commutative diagram.

**Lemma 3.199.**

1. The equality \( \mathbb{H}^i(M^{1,N-1}, \mathcal{K}_x(N-1)) = 0 \) holds for all \( i \leq N - 1 \).

2. The map \( \sigma : \text{Ker}(g) \to (J^N/J^{N+1})^\vee \) is an isomorphism.

**Proof.** As before, let \( \pi : M^{1,N-1} \to M \) be the projection onto the first factor. We shall compute \( \mathbb{H}^i(M^{1,N-1}, \mathcal{K}_x(N-1)) \) using the Leray spectral sequence associated with \( \pi \):

\[ E_2^{p,q} = H^p(M, R^q \pi_* (\mathcal{K}_x(N-1))) \implies \mathbb{H}^{p+q}(M^{1,N-1}, \mathcal{K}_x(N-1)). \]

(3.200)

Taking higher direct images with respect to \( \pi \) from the exact sequence of complexes (3.187) yields isomorphisms

\[ R^i \pi_* (\mathcal{K}_x(N-1)) \cong R^i \pi_* (\mathcal{K}_x(N-1)), \quad i \leq N - 2, \]

and an exact sequence of sheaves

\[ 0 \to R^{N-1} \pi_* (\mathcal{K}_x(N-1)) \to R^{N-1} \pi_* (\mathcal{K}_x(N-1)) \to \mathbb{Q}_x \to 0. \]

(3.201)

The exactness on the right follows, after passing to the fibre at \( x \), from the surjectivity of the map \( f \) in the sequence (3.169).

Now recall that the induction hypothesis in the proof of the theorem is that \( R^i \pi_* (\mathcal{K}_x(N-1)) \) vanishes for all \( i \leq N - 2 \), and hence

\[ R^i \pi_* (\mathcal{K}_x(N-1)) = 0 \text{ for all } i \leq N - 2. \]

Therefore, the Leray spectral sequence (3.200) looks as depicted in Figure 14. There can also be non zero groups above row \( N \). The important point is that all rows strictly below \( N - 1 \) are zero. From this we deduce the equality

\[ \mathbb{H}^i(M^N, \mathcal{K}_x(N-1)) = \begin{cases} 0 & i \leq N - 2 \\ H^0(M, R^{N-1} \pi_* (\mathcal{K}_x(N-1))) & i = N - 1 \end{cases} \]

(3.202)

and a short exact sequence of vector spaces

\[ 0 \to H^1(M, R^{N-1} \pi_* (\mathcal{K}_x(N-1))) \to \mathbb{H}^N(M^N, \mathcal{K}_x(N-1)) \to \]

\[ \to H^0(M, R^N \pi_* (\mathcal{K}_x(N-1))) \to 0. \]

(3.203)

To prove statement (1) in the lemma, it remains to show that

\[ H^0(M, R^{N-1} \pi_* (\mathcal{K}_x(N-1))) = 0. \]

(3.204)
Figure 14. The Leray spectral sequence for $\tilde{\mathcal{K}}_x(N-1)$. 

The long exact sequence of cohomology associated with the short exact sequence of sheaves (3.201) yields

$$0 \rightarrow H^0(R^{N-1}\pi_*(\mathcal{K}_x(N-1))) \rightarrow H^0(R^{N-1}\pi_*(\mathcal{K}_x(N-1))) \xrightarrow{a} \mathbb{Q} \rightarrow H^1(R^{N-1}\pi_*(\mathcal{K}_x(N-1))) \xrightarrow{b} H^1(R^{N-1}\pi_*(\mathcal{K}_x(N-1))) \rightarrow 0.$$  

(3.205)

We shall prove that the map $a$ is an isomorphism. From this fact we obtain equation (3.204) and that the map $b$ is an isomorphism as well. For this we need to compute the cohomology of the sheaf $R^{N-1}\pi_*(\mathcal{K}_x(N-1))$. By the induction hypothesis in the theorem, the map

$$\sigma: R^{N-1}\pi_*(\mathcal{K}_x(N-1)) \rightarrow (\mathbb{Q}[\pi_1(M,\bullet,x)]/J^N)^\vee$$  

(3.206)

is an isomorphism and in particular the sheaf $R^{N-1}\pi_*(\mathcal{K}_x(N-1))$ is a local system on $M$ with fibre

$$R^{N-1}\pi_*(\mathcal{K}_x(N-1))_{x} \cong (\mathbb{Q}[\pi_1(M,x)]/J^N)^\vee.$$  

(3.207)

Setting $\Gamma = \pi_1(M,x)$, the cohomology of $R^{N-1}\pi_*(\mathcal{K}_x(N-1))$ can be computed as the group cohomology of $\Gamma$ acting on (3.207): 

$$H^i(M, R^{N-1}\pi_*(\mathcal{K}_x(N-1))) = H^i(\Gamma,(\mathbb{Q}[\Gamma]/J^N)^\vee).$$

Consider the short exact sequence of $\Gamma$-modules

$$0 \rightarrow (\mathbb{Q}[\Gamma]/J^N)^\vee \rightarrow \mathbb{Q}[\Gamma]^\vee \rightarrow (J^N)^\vee \rightarrow 0.$$  

(3.208)
The $\Gamma$-module $\mathbb{Q}[\Gamma]^{\vee}$ being injective, its cohomology is concentrated in degree zero and there is an exact sequence

$$0 \to H^0(\Gamma, (\mathbb{Q}[\Gamma]/J^N)^{\vee}) \to H^0(\Gamma, \mathbb{Q}[\Gamma]^{\vee}) \to H^0(\Gamma, (J^N)^{\vee}) \to H^1(\Gamma, (\mathbb{Q}[\Gamma]/J^N)^{\vee}) \to 0.$$  

(3.209)

Recall that, if $A$ is a $\Gamma$-module, then $H^0(\Gamma, A)$ is the group of invariants $A^\Gamma$. From this one easily checks:

- The cohomology $H^0(\Gamma, \mathbb{Q}[\Gamma]^{\vee})$ is the one-dimensional $\mathbb{Q}$-vector space generated by the function

$$\sum_{\gamma \in \Gamma} a_\gamma [\gamma] \mapsto \sum_{\gamma \in \Gamma} a_\gamma$$

and the dual of the unit (3.191) induces an isomorphism

$$H^0(\Gamma, \mathbb{Q}[\Gamma]^{\vee}) \sim \mathbb{Q}.$$  

- The cohomology $H^0(\Gamma, (J^N)^{\vee})$ is equal to $(J^N/J^{N+1})^{\vee}$ and the map $H^0(\Gamma, \mathbb{Q}[\Gamma]^{\vee}) \to (J^N/J^{N+1})^{\vee}$

in the long exact sequence (3.209) is the zero map.

Putting together the above facts, the isomorphism (3.207), and the long exact sequence (3.209), we deduce

$$H^0(M, R^{N-1} \pi_*(\mathcal{K}_x(N-1))) \cong \mathbb{Q} \quad \text{(3.210)}$$

$$H^1(M, R^{N-1} \pi_*(\mathcal{K}_x(N-1))) \cong (J^N/J^{N+1})^{\vee}. \quad \text{(3.211)}$$

Besides, the map $a$ in (3.205) agrees with the isomorphism (3.210) by statement (3) in the theorem. From this and (3.202), we derive

$$\mathbb{H}^{N-1}(M^{1,N-1}, \check{\mathcal{K}}_x(N-1)) = H^0(M, R^{N-1} \pi_*(\mathcal{K}_x(N-1))) = 0,$$

thus concluding the proof of statement (1) in the lemma.

We now turn to the proof of (2). Combining the fact that the map $b$ in (3.205) is an isomorphism with (3.211), we get

$$H^1(M, R^{N-1} \pi_*(\check{\mathcal{K}}_x(N-1))) \cong (J^N/J^{N+1})^{\vee}.$$

Besides, by the exact sequence (3.203), there is an inclusion

$$H^1(M, R^{N-1} \pi_*(\check{\mathcal{K}}_x(N-1))) \subseteq \mathbb{H}^N(M^N, \check{\mathcal{K}}_x(N-1)).$$

Claim: This subspace is equal to $\text{Ker}(g)$.  

To prove the claim, we consider the long exact sequence of sheaves obtained by taking higher direct images from (3.186):

\[
\cdots \to R^{N-1} \pi_*(\pi_y)_*(yK_x(N-1)) \xrightarrow{\varphi} R^N \pi_*(yK_x(N)) \to R^N \pi_*(\tilde{K}_x(N-1)) \to R^N \pi_*(\pi_y)_*(yK_x(N-1)) \to \cdots \quad (3.212)
\]

Note that the sheaves \(R^q \pi_*(\pi_y)_*(yK_x(N-1))\) are all skyscraper sheaves supported at the point \(y \in M\), and hence have only cohomology in degree zero. Therefore, in the exact sequence

\[
0 \to H^1(M, R^{N-1} \pi_*(\pi_y)_*(yK_x(N-1))) \to \mathbb{H}^N(M^{N-1}, yK_x(N-1)) \to H^0(M, R^N \pi_*(\pi_y)_*(yK_x(N-1))) \to 0 \quad (3.213)
\]

obtained by applying the Leray spectral sequence to \((\pi_y)_*(yK_x(N-1))\), the leftmost term vanishes and the last but one map is an isomorphism.

Let us introduce the sheaf \(F = \text{Coker}(\varphi)\) and consider the commutative diagram with exact columns

\[
\begin{array}{cccc}
0 & & & \\
\downarrow & & & \downarrow \\
H^1(R^{N-1} \pi_*(\tilde{K}_x(N-1))) & \longrightarrow & 0 & \\
\downarrow & & & \downarrow \\
\mathbb{H}^N(M^{N-1}, yK_x(N-1)) & \xrightarrow{g} & \mathbb{H}^N(M^{N-1}, yK_x(N-1)) & \\
\downarrow & & & \downarrow \\
H^0(F) & \longrightarrow & H^0(R^N \pi_*(\tilde{K}_x(N-1))) & \longrightarrow H^0(R^N \pi_*(\pi_y)_*(yK_x(N-1))) & \\
\downarrow & & & \downarrow \\
0 & & & 0
\end{array}
\]

where the first column is (3.203), the second column is (3.213), and the last row is part of the exact sequence obtained by taking cohomology from (3.212). The above diagram immediately implies that

\[
H^1(M, R^{N-1} \pi_*(\tilde{K}_x(N-1))) \subseteq \text{Ker}(g)
\]

and to prove the reverse inclusion it is enough to show that \(H^0(M, F) = 0\).

To get this vanishing we will show that \(F\) is the extension by zero of a local system on \(M \setminus \{y\}\). We need to distinguish whether the base points \(x\) and \(y\) are distinct or equal.

**Case** \(x \neq y\). Write \(U = M^{1,N-1} \setminus \pi^{-1}(y)\). Since the complex \((\pi_y)_*(yK_x(N-1))\) is supported at \(\pi^{-1}(y)\), one first sees from (3.186) that

\[
yK_x(N)|_U \cong \tilde{K}_x(N-1)|_U
\]
and combining this with (3.187) one obtains a short exact sequence
\[ 0 \to \mathbb{Q}(x_{\ldots}, x)\langle -N \rangle \to y\mathcal{K}_x\langle N \rangle|_U \to \mathcal{K}_x\langle N - 1 \rangle|_U \to 0. \]

In the associated long exact sequence
\[ R^{N - 1}\pi_*(\mathcal{K}_x\langle N - 1 \rangle|_U) \xrightarrow{h} R^N\pi_*\mathbb{Q}(x_{\ldots}, x)\langle -N \rangle \to \]
\[ R^N\pi_*(y\mathcal{K}_x\langle N \rangle|_U) \to R^N\pi_*(\mathcal{K}_x\langle N - 1 \rangle|_U) \to 0, \]
the map \( h \) is surjective, by repeating the argument that yields the surjectivity of the map \( f \) in (3.169). We thus get an isomorphism
\[ R^N\pi_*(y\mathcal{K}_x\langle N \rangle)|_{M\setminus\{y\}} \to R^N\pi_*(\mathcal{K}_x\langle N - 1 \rangle)|_{M\setminus\{y\}}. \]

Since the right-hand side is a local system by Lemma 3.189, the same is true for the left-hand side. Let now \( V \subseteq M \) be a contractible open subset containing \( y \) but not \( x \). Then the restrictions of \( \mathcal{K}_x\langle N - 1 \rangle \) and \( \mathcal{K}_x\langle N - 1 \rangle \) to \( \pi^{-1}(V) \) are isomorphic, so that (3.186) induces a long exact sequence
\[
\cdots \longrightarrow \mathbb{H}^i(\pi^{-1}(V), y\mathcal{K}_x\langle N \rangle) \longrightarrow \mathbb{H}^i(\pi^{-1}(V), \mathcal{K}_x\langle N - 1 \rangle) \longrightarrow \\
\mathbb{H}^{i+1}(\pi^{-1}(y), y\mathcal{K}_x\langle N - 1 \rangle) \longrightarrow \cdots
\]

By Lemma 3.189, the map \( j \) is an isomorphism in all degrees \( i \geq 0 \). This implies in particular the vanishing \( \mathbb{H}^N(\pi^{-1}(V), y\mathcal{K}_x\langle N \rangle) = 0 \) for all contractible open sets \( V \) containing the point \( y \), and hence
\[ R^N\pi_*(y\mathcal{K}_x\langle N \rangle)_y = 0. \]

Finally, since the source of the map
\[ \varphi: R^{N - 1}\pi_*(y\mathcal{K}_x\langle N - 1 \rangle) \to R^N\pi_*(y\mathcal{K}_x\langle N \rangle) \]
is a skyscraper sheaf supported at \( y \), it follows that \( \varphi \) is identically zero. We have thus shown that \( \mathcal{F} = R^N\pi_*(y\mathcal{K}_x\langle N \rangle) \) is the extension by zero of a local system on \( M \setminus \{y\} \).

Case \( x = y \). On \( U = M^{1, N - 1 \setminus \pi^{-1}(x)} \), the exact sequence (3.187) yields an isomorphism
\[ x\mathcal{K}_x\langle N \rangle|_U \cong \mathcal{K}_x\langle N - 1 \rangle|_U, \]
that implies that \( \mathcal{F}|_{M\setminus \{x\}} = R^N\pi_*(x\mathcal{K}_x\langle N \rangle)|_{M\setminus \{x\}} \) is a local system. Let \( V \subseteq M \) be a contractible open subset containing \( x \). In this case it is no longer true that \( y\mathcal{K}_x\langle N \rangle|_{\pi^{-1}(V)} \) has vanishing hypercohomology. Identifying \((x, \ldots, x) \) with \( Y_{\{x, \ldots, x\}} \) there is a map
\[ \mathbb{Q}(x_{\ldots}, x)\langle -N \rangle \to y\mathcal{K}_x\langle N \rangle|_{\pi^{-1}(V)}. \]

Using that \( V \) is contractible, this map induces an isomorphism in hypercohomology
\[ \mathbb{Q} = \mathbb{H}^N(\pi^{-1}(V), \mathbb{Q}(x_{\ldots}, x)\langle -N \rangle) \cong \mathbb{H}^N(\pi^{-1}(V), y\mathcal{K}_x\langle N \rangle). \]
Therefore
\[ R^N \pi_* (yK_x \langle N \rangle)_x = \mathbb{Q} \neq 0. \]
In this case the map
\[ R^{N-1} \pi_*(t_x)_*(\overline{K}_x(N-1))_x \xrightarrow{\iota} R^N \pi_*(K_x(N))_x \]
is surjective and we again deduce that \( \mathcal{F}_x = 0 \). Therefore \( H^0(M, \mathcal{F}) = 0 \),
the map labeled \( e \) is injective and
\[ \text{Ker}(g) = H^1(M, R^{N-1} \pi_*(\overline{K}_x(N-1))) \cong (J^N/J^{N+1})^\vee. \]

To finish the proof of the lemma one needs to check that the above isomorphism is compatible with the map \( \sigma \). We leave this verification to the reader. \( \square \)

We can now finish the proof of Beilinson’s Theorem 3.193.

Recall that statement (1) is the vanishing \( \mathbb{H}^i(M^N, yK_x(N)) = 0 \) in all degrees \( i \leq N - 1 \). By (3.197), this group fits into a long exact sequence
\[ \cdots \to \mathbb{H}^{i-1}(M^{N-1}, yK_x(N-1)) \to \mathbb{H}^i(M^N, yK_x(N)) \to \mathbb{H}^i(M^{1,N-1}, \overline{K}_x(N-1)) \to \cdots \]
For \( i \leq N - 1 \), the leftmost term vanishes by the induction hypothesis and the rightmost term vanishes by the first part of Lemma 3.199, and hence the middle term vanishes as well.

Finally, to prove statement (2) we observe that, thanks to the long exact sequence (3.197), Lemma 3.199, and the induction hypothesis, in the commutative diagram
\[
\begin{array}{cccccc}
0 & \to & \mathbb{H}^{N-1}(M^{N-1}, yK_x(N-1)) & \xrightarrow{\iota} & \mathbb{H}^N(M^N, yK_x(N)) & \to & \text{Ker}(g) & \to & 0 \\
\sigma & & \downarrow & & \sigma & & \downarrow & \sigma & \\
0 & \to & (\mathbb{Q}[\pi_1(M; y, x)]/J^N)^\vee & \to & (\mathbb{Q}[\pi_1(M; y, x)]/J^{N+1})^\vee & \to & (J^N/J^{N+1})^\vee & \to & 0 \\
\end{array}
\]
the rows are exact and the first and third vertical maps are isomorphisms. By the five lemma, the second vertical arrow is also an isomorphism. This concludes the proof. \( \square \)

3.5.6. Proof of Chen’s \( \pi_1 \)-de Rham Theorem. We are now in position to prove Chen’s \( \pi_1 \)-de Rham Theorem using Beilinson’s Theorem 3.193.

**Proof of Theorem 3.151.** If \( N = 0 \), then \( L_0 B^*(A^*(M)) = \mathbb{C} \) given by the constant functions, while
\[ \mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)]J = \mathbb{Q}. \]
Moreover, the map in Theorem 3.151 sends the constant function \( a \in \mathbb{C} \) to the map that sends \( 1 \in \mathbb{Q} \) to \( a \), that is clearly an isomorphism.
Fix now $N > 0$. Applying Lemma 3.161 and Proposition A.93 we obtain a quasi-isomorphism

$$L_N B^*(A^*(M)) \sim \text{Tot}(C_\ast(-, A^*(M_n^\bullet))).$$

For each $n$, the composition

$$A^*(M)^{\otimes n} \otimes \mathbb{C} \longrightarrow E^*(y M_n^\bullet, \mathbb{C}) \longrightarrow S^*(y M_n^\bullet, \mathbb{Q}) \otimes \mathbb{C}$$

is a quasi-isomorphism, functorial in $n$, from which we deduce a quasi-isomorphism

$$L_N B^*(A^*(M)) \otimes \mathbb{C} \sim \text{Tot}(C_\ast(-, S^\bullet_n) \otimes \mathbb{C}).$$

Combining this quasi-isomorphism with Lemma 3.175 and Theorem 3.193 we deduce the isomorphism

$$H^0(L_N B^*(A^*(M)) \otimes \mathbb{C}) \longrightarrow (\mathbb{C}[\pi(M; y, x)]/\mathbb{C}[\pi(M; y, x)]J^{N+1})^\vee.$$

Therefore, we get an isomorphism

$$H^0(B^*(A^*(M)) \otimes \mathbb{C}) = \lim_N H^0(L_N B^*(A^*(M)) \otimes \mathbb{C}) \longrightarrow$$

$$(\lim_N \mathbb{C}[\pi(M; y, x)]/\mathbb{C}[\pi(M; y, x)]J^{N+1})^\vee = (\mathbb{C}[\pi(M; y, x)]^\vee)^\vee,$$

as we wanted to prove. \qed

\section*{Exercise 3.215}

Exercise 3.215. Let $n, m \geq 0$ be integers and let $f: \Delta_n \rightarrow \Delta_m$ be a non-decreasing map. Using the fact that $A^*(M) \subset E^*(M, \mathbb{C})$ is a subalgebra, prove that there exists a unique morphism of dg-algebras $f^*: A^*(y M_n^m) \rightarrow A^*(y M_x^m)$ such that the following diagram commutes

$$
\begin{array}{ccc}
A^*(y M_n^m) & \xrightarrow{f^*} & A^*(y M_x^m) \\
\downarrow & & \downarrow \\
E^*(y M_n^m) & \xrightarrow{f^*} & E^*(y M_x^m)
\end{array}
$$

and making the assignment $\Delta_n \rightsquigarrow y M_n^m$ functorial.

\section*{Exercise 3.216}

Exercise 3.217. Let $M$ be a topological space and $A^*$ a complex of abelian groups. Consider the Godement resolution $C^*$ from Example A.104. Show that there is a natural isomorphism of complexes

$$\text{Tot}^*(\Gamma(M, C^*(A[k]^*))) \longrightarrow \text{Tot}^*(\Gamma(M, C^*(A^*)))\langle k \rangle.$$

\section{3.6. A mixed Hodge structure on the pro-unipotent completion of the fundamental group.}
3.6.1. Construction of the mixed Hodge structure. Hain [Hai87b] and Morgan [Mor78] show that, if $M = X(\mathbb{C})$ is the set of complex points of a smooth algebraic variety, then each of the quotients of the pro-unipotent completion of the fundamental group of $M$ carries a natural mixed Hodge structure. Using the geometric interpretation of such quotients provided by Beilinson’s Theorem 3.193, one can improve this result a little bit, showing that, in fact, if a variety is defined over a subfield $k \subset \mathbb{C}$ we obtain a mixed Hodge structure over $k$. Later we will see that Beilinson’s theorem allow us to upgrade these quotients to motives. For now, the precise statement is the following.

**Theorem 3.218.** Let $k$ be a subfield of $\mathbb{C}$, $X$ a smooth algebraic variety over $k$, $M = X(\mathbb{C})$ the set of complex points of $X$ viewed as a differentiable manifold, and $x, y \in X(k) \subseteq M$ two $k$-rational base points. For each $N \geq 0$, the finite-dimensional $\mathbb{Q}$-vector space $\mathbb{Q}[\pi_1(M; y, x)]/J^{N+1}\mathbb{Q}[\pi_1(M; y, x)]$ carries a mixed Hodge structure over $k$ which is functorial with respect to morphisms of pointed varieties.

Moreover, given integers $N_1 \geq N_2 \geq 0$, the quotient map
\[
\mathbb{Q}[\pi_1(X(\mathbb{C}); y, x)]/J^{N_1+1} \rightarrow \mathbb{Q}[\pi_1(X(\mathbb{C}); y, x)]/J^{N_2+1}
\]
is a morphism of mixed Hodge structures over $k$.

**Proof.** We will prove that, under the assumptions of the statement, the dual vector space
\[
(\mathbb{Q}[\pi_1(M; y, x)]/J^{N+1}\mathbb{Q}[\pi_1(M; y, x)])^\vee
\]
carries a functorial mixed Hodge structure over $k$. By Beilinson’s theorem 3.193, we know that
\[
\mathbb{H}^N(M^N, yK_x(N)) \rightarrow (\mathbb{Q}[\pi(M; y, x)]/J^{N+1}\mathbb{Q}[\pi(M; y, x)])^\vee
\]
and the groups $\mathbb{H}^N(M^N, yK_x(N))$ can be interpreted as certain relative singular cohomology groups of algebraic varieties over $k$, thus can be endowed with a mixed Hodge structure over $k$.

We can also use Lemma 3.175 and Proposition A.93 to identify the groups $\mathbb{H}^N(M^N, yK_x(N))$ with certain singular cohomology groups of a simplicial manifold $yM^*_x$. All the maps involved in $yM^*_x$ are algebraic and defined over $k$, therefore $yM^*_x$ is the simplicial manifold obtained by taking complex points of a simplicial smooth variety over $k$. Using a variant over $k$ of the main construction of [Del74], we endow $\mathbb{H}^N(M^N, yK_x(N))$ with a mixed Hodge structure over $k$.

The claimed functoriality properties follow from the functorial properties of the mixed Hodge structures on the cohomology of simplicial varieties. \[\square\]
We have constructed a pro-mixed Hodge structure on the pro-unipotent completion of the fundamental group by abstract means. Following Hain [Hai87b], Chen’s theorem provides us with a very clear and transparent way to construct such mixed Hodge structure. We now sketch this construction when \( X \) is smooth and defined over \( \mathbb{C} \). We will just show how to define the Hodge and weight filtrations. Consider the dg-algebra \( E^*_{X,\text{an}}(\log D) \) as in Section 2.7.3. It has two augmentations \( \varepsilon_1 \) and \( \varepsilon_2 \) given by evaluating at \( x \) and \( y \) respectively. The Hodge and weight filtrations of \( E^*_{X,\text{an}}(\log D) \) determine the Hodge and weight filtration on \( B^*_{\text{an}}(E^*_{X,\text{an}}(\log D)) \), by saying that, if \( \omega_i \in F_{p_i} \) for \( i = 1, \ldots, r \), then \( [\omega_1| \cdots |\omega_r] \in F_{p_1 + \cdots + p_r} \), while, if \( \omega_i \in W_{n_i} \), then \( [\omega_1| \cdots |\omega_r] \in W_{n_1 + \cdots + n_r + r} \), that is, the Hodge type is the sum of Hodge types, while the weight is the sum of weights plus the length of the element. Then

\[
F^p H^0(B^*(E^*_{X,\text{an}}(\log D))) = \text{Im}(H^0(F^p B^*(E^*_{X,\text{an}}(\log D)))))
\]

\[
W_m H^0(B^*(E^*_{X,\text{an}}(\log D))) = \text{Im}(H^0(W_m B^*(E^*_{X,\text{an}}(\log D))))
\]

3.6.2. The case of \( \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\} \). Let us now specialize to the case where \( X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\} \) and \( M = X(\mathbb{C}) \), as in paragraph 3.4.5 and \( x, y \in X(\mathbb{Q}) \). As we have seen in Example 2.175, we do not need to work with the whole infinite dimensional dg-algebra \( E^*_{X,\text{an}}(\log D) \), but we can work with the smaller \( \mathbb{Q} \)-algebra

\( A = \mathbb{Q} \oplus \mathbb{Q} \omega_0 \oplus \mathbb{Q} \omega_1. \)

In this case both augmentations \( \varepsilon_1 \) and \( \varepsilon_2 \) given by evaluating at \( x \) and \( y \) respectively agree with the trivial augmentation

\[
\varepsilon: \begin{array}{ccc}
A & \rightarrow & \mathbb{Q} \\
1 & \mapsto & 1 \\
\omega_0 & \mapsto & 0 \\
\omega_1 & \mapsto & 0.
\end{array}
\]

(3.219)

This has the added advantage to give us already a mixed Hodge structure over \( \mathbb{Q} \). Since \( A \) is connected we can use the reduced bar complex. Arguing as in paragraph 3.4.5, the Hopf algebra \( H^0(B^*(A)) \) is isomorphic to the Hoffman algebra. In each finite dimensional subspace \( H^0(L_X B^*(A_\mathbb{C})) \), the Hodge filtration is the decreasing filtration determined by

\( [\omega_i] \cdots [\omega_i] \in F^p \)

and the weight filtration is the increasing filtration determined by

\( [\omega_i] \cdots [\omega_i] \in W_{2n}. \)

We now describe an ind-mixed Hodge structure \( \{y^A_{x}^H.N\}_{N \geq 0} \) that corresponds to the algebra of functions over the pro-unipotent completion of
the fundamental group and a dual pro-mixed Hodge structure \( \{ y U_x^{H,N} \}_{N \geq 0} \) that corresponds to the universal enveloping algebra of the Lie algebra of the pro-unipotent completion of the fundamental group.

For the Betti part of \( y A_x^{B,N} \), we write
\[
y A_x^{B,N} = (\mathbb{Q}[\pi(M; y, x)] / J^{N+1}\mathbb{Q}[\pi(M; y, x)])^\vee
\]
with the weight filtration given, for \(-1 \leq k \leq N\), by
\[
W_{2k}(y A_x^{B,N}) = W_{2k+1}(y A_x^{B,N}) = (J^{k+1}\mathbb{Q}[\pi(M; y, x)] / J^{N+1}\mathbb{Q}[\pi(M; y, x)])^\perp.
\]

For the de Rham side, we have
\[
y A_x^{dR,N} = L_N H^0(B^*(A^*))
\]
with the weight filtration given, for \(-1 \leq k \leq N\), by
\[
W_{2k}(y A_x^{dR,N}) = W_{2k+1}(y A_x^{dR,N}) = (L_k H^0(B^*(A^*)))^\perp.
\]
The Hodge filtration is given by defining
\[
F^p(y A_x^{dR,N})
\]
as the subspace generated by words of length \( \ell \) with \( p \leq \ell \leq N \). Note that only the Betti part depends on the points \( x, y \).

By duality we write
\[
y U_x^{B,N} = \mathbb{Q}[\pi(M; y, x)] / J^{N+1}\mathbb{Q}[\pi(M; y, x)]
\]
and
\[
y U_x^{dR,N} = L_N H^0(B^*(A^*))^\vee
\]
with the dual filtrations.

We denote by \( \text{comp}_{dR,B} \) the isomorphism of Theorem 3.151 and by \( \text{comp}_{B,dR} \) its dual. Then the mixed Hodge structures
\[
y A_x^{H,N} = ((y A_x^{B,N}, W), (y A_x^{dR,N}, W, F), \text{comp}_{dR,B}^{-1})
\]
form an inductive system of mixed Hodge structures over \( \mathbb{Q} \) and
\[
y U_x^{H,N} = ((y U_x^{B,N}, W), (y U_x^{dR,N}, W, F), \text{comp}_{B,dR})
\]
form a projective system of mixed Hodge structures over \( \mathbb{Q} \).
3.6.3. Iterated integrals as periods of the fundamental group. We now show that iterated integrals along paths between $x$ and $y$ are periods of the mixed Hodge structure $yA^H_{x}$. We keep the notation $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $M = X(\mathbb{C})$.

Example 3.220. Let $s = (s_1, \ldots, s_n)$ be a positive multi-index of weight $N$ and write $bs(s) = (\varepsilon_1, \ldots, \varepsilon_N)$ for the associated binary sequence. We consider the algebraic differential form on $X^N$ given by
\[
\omega = pr_1^*\omega_{\varepsilon_1} \wedge \cdots \wedge pr_N^*\omega_{\varepsilon_N},
\]
where $\omega_0 = \frac{dt}{t}$ and $\omega_1 = \frac{dt}{1-t}$, as usual, and $pr_i : X^N \to X$ denote the various projections. Since $\omega$ has maximal degree, it defines a class $[(\omega, 0)]$ in the relative de Rham cohomology $H^N_{dR}(X^N, Y)$, where $Y$ is as in Section 3.5.5. From lemmas 3.175 and 3.161 and Proposition A.93, we derive
\[
H^N_{dR}(X^N, Y) = yA^B_{x}.\]

On the other hand, every path $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$, determines a singular simplex
\[
\sigma : \Delta^N \longrightarrow M^N
\]
\[
(t_1, \ldots, t_N) \longmapsto (\gamma(t_1), \ldots, \gamma(t_N)),
\]
where $\Delta^N$ is the simplex of Notation 1.107. Clearly, the support of the chain $\partial \sigma$ is contained in $Y$. Therefore $\sigma$ determines a class $[\sigma]$ in the relative singular homology group $H_N(M^N, Y, \mathbb{Q})$. By Theorem 3.193,
\[
H_N(M^N, Y) = yA^B_{x}.\]

The period associated with these two classes is the iterated integral
\[
\langle [(\omega, 0)], [\sigma] \rangle = \int_{\sigma} \omega = \int_{\gamma} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_N}.
\]
Here we have used two points $x, y \in X(\mathbb{Q})$. In order to obtain multiple zeta values we need to consider the case $x = 0$ and $x = 1$, but these points do not belong to $X(\mathbb{Q})$. For this reason we will need to consider tangential base points in the next section.

***

Exercise 3.221 (The nerve of a category). Let $\mathcal{C}$ be a small category. Let $N(\mathcal{C})_0$ be the set of objects and $N(\mathcal{C})_1$ the set of morphisms. For each $n \geq 2$, define $N(\mathcal{C})_n$ as the set of $n$-tuples of composable morphisms
\[
C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n.
\]
(3.222)

On the one hand, we have maps
\[
\delta_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1} \quad i = 0, \ldots, n,
\]
given by composing at the \(i\)-th object or removing it whenever \(i = 0\) or \(n\).
In other words, \(\delta_i\) sends an \(n\)-tuple as in (3.222) to the \((n-1)\)-tuple
\[
C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i-1} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} C_n.
\]
On the other hand, there are maps
\[
\sigma_i: N(C)_n \to N(C)_{n+1} \quad i = 0, \ldots, n,
\]
obtained by inserting an identity morphism at the \(i\)-th object, that is,
\[
C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} C_i \xrightarrow{\text{Id}} C_i \xrightarrow{f_{i+1} \circ f_i} \cdots \xrightarrow{f_n} C_n.
\]

(a) Prove that \(N(C)_\bullet\), together with the maps \(\delta_i\) as faces and the maps \(\sigma_i\) as degeneracies, has the structure of a simplicial set. This construction is called the \textit{nerve} of the category \(C\).

(b) In particular, identify the simplicial identity which corresponds to the associativity of the composition of morphisms.

**EXERCISE 3.223.** Describe explicitly the mixed Hodge structure on the pro-unipotent completion of the fundamental group of \(G_m\).

---

\section*{3.7. Tangential base points.} In this section, we continue working with the manifold \(M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}\), the differential forms \(\omega_0\) and \(\omega_1\), and the dg-algebra \(A^*_C\) from paragraph 3.4.5. Theorems 1.108 and 1.117 show that multiple zeta values and polylogarithms can be seen as iterated integrals. Nevertheless we face a technical problem. The differential forms \(\omega_0\) and \(\omega_1\) that appear in these theorems have singularities at the points 0, 1 and \(\infty\). Hence they are differential forms on \(M\), but to obtain multiple zeta values we need to integrate along the straight path
\[
dch: \quad [0, 1] \quad \longrightarrow \quad \mathbb{P}^1(\mathbb{C}),
\]
which is not contained in \(M\) because the end points are 0 and 1. Since \(\text{dch}\) is not a path in \(M\), the formulas in theorems 1.108 and 1.117 are not strictly speaking iterated integrals. Thus, to see multiple zeta values and polylogarithms as iterated integrals we have to consider \textit{tangential base points}. As we will see, these are related to the regularization discussed in Section 1.7. Tangential base points will also play an important role later when we consider the algebraic structure of \(\mathbb{P}^1\): the variety \(\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}\) does not contain any smooth integral point, thus we will need tangential base points to have a motivic version of the fundamental group of \(\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}\) defined over \(\mathbb{Z}\).
3.7.1. Paths with tangential base points. For simplicity, we will introduce tangential base points only in the case of $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, the only one we need, but the reader should be aware that the constructions extend easily to any smooth projective curve minus a finite number of points.

**Definition 3.225.** Let $x \in \{0, 1\}$ be either the point zero or the point one in $\mathbb{P}^1(\mathbb{C})$. A **tangential base point** is a pair $(x, v)$, where $v$ is a non-zero tangent vector to $\mathbb{P}^1(\mathbb{C})$ at $x$.

Intuitively, a path has an end point at a tangential base point $(x, v)$ if the end point is $x$ and the tangent vector at the end point is $v$. However, the presence of tangential base points causes a nuisance. On the one hand, in order to be able to compose paths we need to allow tangential points to be reached by the paths at intermediate points. On the other hand, to define homotopy between paths in an easy way it is better to avoid tangential points at intermediate points along the path. To remedy this problem we define two kind of paths, the ones that allow tangential points at intermediate steps (and hence can be composed) and the ones that avoid tangential points. The former will be called cuspidal paths because of the shape we will impose at the tangential points, while the latter will be called clean paths. Then we define a homotopy equivalence of clean paths and a map from the space of cuspidal paths to the space of homotopy classes of clean paths.

**Definition 3.226.** Let $x = (x, v)$ and $y = (y, w)$ be two tangential base points. A **cuspidal path from** $x$ to $y$ is a piecewise smooth map

$$
\gamma : [0, 1] \to M \cup \{0, 1\}
$$

satisfying the following conditions

1. the end points of the path are

$$
\gamma(0) = x, \quad \frac{d\gamma}{dt}(0) = v, \\
\gamma(1) = y, \quad \frac{d\gamma}{dt}(1) = -w;
$$

2. the set $\{t \in (0, 1) \mid \gamma(t) \in \{0, 1\}\}$ is finite. Moreover, if $t_0$ belongs to this set, the left and right tangent vector to $\gamma$ at $t_0$ are non-zero and opposed:

$$
0 \neq \frac{d^+\gamma}{dt}(t_0) = -\frac{d^-\gamma}{dt}(t_0)
$$

This set is called the **set of cusps** of $\gamma$.

When the set of cusps is empty, $\gamma$ is called a **clean path from** $x$ to $y$.

The space of cuspidal paths from $x$ to $y$ is denoted by $\mathcal{P}(M)_x$ while the subspace of clean paths is denoted $\mathcal{P}(M)_{x}^{0}$.
3.7.2. Composition of paths with tangential base points. The composition of paths (3.3) cannot be applied directly to define
\[ P(M)_y \otimes yP(M)_x \rightarrow zP(M)_x \]
for tangential base points \( x, y \) and \( z \) because condition (1) imposes that the derivative of the path at zero and one is a fixed vector, while the parametrization used in (3.3) would multiply this vector by 2. Thus to define the composition of paths we consider the functions
\[ \phi_1(t) = t + 2t^2, \quad \phi_2(t) = 5t - 2 - 2t^2. \]

These functions are smooth and satisfy the properties
\[
\begin{align*}
\phi_1(0) &= 0, & \phi_1(1/2) &= 1, & \phi_1'(0) &= 1, \\
\phi_2(1/2) &= 0, & \phi_2(1) &= 1, & \phi_2'(1) &= 1, \\
\phi_1'(t) &> 0, \ t \in [0, 1/2], & \phi_2'(t) &> 0, \ t \in [1/2, 1], \\
\phi_1'(1/2) &= \phi_2'(1/2).
\end{align*}
\]

In fact, any pair of smooth functions satisfying all the above properties would serve for our purposes.

\[ \gamma_1 \gamma_2(t) = \begin{cases} 
\gamma_2(\phi_1(t)), & 0 \leq t \leq \frac{1}{2}, \\
\gamma_1(\phi_2(t)), & \frac{1}{2} \leq t \leq 1.
\end{cases} \quad (3.227) \]
3.7.3. Homotopy of paths. Let \( \gamma_1, \gamma_2 \in \mathcal{P}(M)^0_{x} \) be two clean paths. A homotopy between \( \gamma_1 \) and \( \gamma_2 \) is a map
\[
F: [0, 1] \times [0, 1] \to M \cup \{0, 1\}
\]
such that
\[
F(t, 0) = \gamma_1(t), \quad F(t, 1) = \gamma_2(t), \quad t \in [0, 1] \\
F(0, s) = x, \quad F(1, s) = y, \quad s \in [0, 1] \\
\frac{\partial F}{\partial t}(0, s) = v, \quad \frac{\partial F}{\partial t}(1, s) = -w, \quad s \in [0, 1] \\
F(t, s) \in M, \quad 0 < t < 1, \quad 0 \leq s \leq 1.
\]
The space \( \pi(M; y, x) \) is the set of homotopy classes of clean paths from \( x \) to \( y \). Similar notation will be used when only one of the base points is tangential.

We next construct a map \( \psi \) from \( \mathcal{P}(M)^0_{x} \) to \( \pi(M; y, x) \). Let \( d(x, y) \) be the standard Euclidean distance in \( \mathbb{C} = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\} \). Let \( \gamma \in \mathcal{P}(M)^0_{x} \). For each \( t_i \) in the set of cusps of \( \gamma \), we can find real numbers \( 0 < \varepsilon_i, \eta_i, \eta'_i \leq \frac{1}{2} \) satisfying the conditions

1. \( t_i \) is the only cusp in the interval \( [t_i - \eta'_i, t_i + \eta_i] \) and \( \gamma \) is smooth in the intervals \( [t_i - \eta'_i, t_i) \) and \( (t_i, t_i + \eta_i] \);
2. the intervals \( [t_i - \eta'_i, t_i + \eta_i] \) are disjoint and do not contain 0 or 1;
3. the image of \( [t_i - \eta'_i, t_i + \eta_i] \) satisfies
\[
d(\gamma(t_i + \eta_i), \gamma(t_i)) = \varepsilon_i, \quad d(\gamma(t), \gamma(t_i)) < \varepsilon_i, \quad \text{for} \quad t_i < t < t_i + \eta_i \\
d(\gamma(t_i - \eta'_i), \gamma(t_i)) = \varepsilon_i, \quad d(\gamma(t), \gamma(t_i)) < \varepsilon_i, \quad \text{for} \quad t_i - \eta'_i < t < t_i;
\]
4. the tangent vector to \( \gamma \) does not change very much
\[
\left\| \frac{d\gamma}{dt}(t) - \frac{d^- \gamma}{dt}(t_i) \right\| \leq \frac{1}{2} \left\| \frac{d^- \gamma}{dt}(t_i) \right\|, \text{for} \quad t \in [t_i - \eta'_i, t_i) \\
\left\| \frac{d\gamma}{dt}(t) - \frac{d^+ \gamma}{dt}(t_i) \right\| \leq \frac{1}{2} \left\| \frac{d^+ \gamma}{dt}(t_i) \right\|, \text{for} \quad t \in (t_i, t_i + \eta_i].
\]
Note that condition (4) implies that the path \( \gamma \) cannot turn around the point \( \gamma(t_i) \) between \( t_i - \eta'_i \) and \( t_i + \eta_i \).

For each cusp \( t_i \) let \( B(\gamma(t_i), \varepsilon_i) \) be the open ball of centre \( \gamma(t_i) \) and radius \( \varepsilon_i \) and let \( r_i: \mathbb{C} \setminus \{\gamma(t_i)\} \to \mathbb{C} \setminus B(\gamma(t_i), \varepsilon_i) \) be the radial retraction. Then we define a new path \( \gamma^o \) defined outside the cusps by
\[
\gamma^o(s) = \begin{cases} 
\gamma(s) & \text{if for all } i, \quad s \notin [t_i - \eta'_i, t_i + \eta_i], \\
r_i(\gamma(s)) & \text{if } s \in [t_i - \eta'_i, t_i + \eta_i], \quad s \neq t_i.
\end{cases}
\]
Condition (2) in the Definition 3.226 implies that \( \gamma^o \) can be extended continuously to the cusps \( t_i \) defining a clean path also denoted \( \gamma^o \). The retraction at a cusp is represented in figure 16.
The following proposition is clear.

**Proposition-Definition 3.229.** The homotopy class of clean paths of $\gamma^\circ$ does not depend on the choice of the numbers $\varepsilon_i, \eta_i, \eta_i'$. The homotopy class of $\gamma^\circ$ in $\pi(M; y, x)$ is denoted by $\psi(\gamma)$.

Using the map $\psi$ we can define a composition of clean paths

$$\pi(M; z, y) \times \pi(M; y, x) \to \pi(M; z, x).$$

**Definition 3.230.** Let $x, y$ and $z$ be base points, tangential or not. Given classes $\gamma_1 \in \pi(M; z, y)$ and $\gamma_2 \in \pi(M; y, x)$, we choose representatives $\tilde{\gamma}_1 \in \mathcal{P}(M)_y^0$ and $\tilde{\gamma}_2 \in \mathcal{P}(M)_x^0$. Then $\tilde{\gamma}_1 \tilde{\gamma}_2 \in \mathcal{P}(M)_x^0$ and we define

$$\gamma_1 \gamma_2 = \psi(\tilde{\gamma}_1 \tilde{\gamma}_2).$$

**Proposition 3.231.** The composition of clean paths given in Definition 3.230 does not depend on the choice of liftings and turns $\pi(M; x, x)$ into a group and $\pi(M; y, x)$ (resp. $\pi(M; x, y)$) into a right (resp. left) $\pi(M; x, x)$-torsor.

The fact that the fundamental groups with different base points are isomorphic can be easily extended to tangential base points. The next proposition is proved like the classical one.

**Proposition 3.232.** Let $x_i, 1 = 1, \ldots, 4$ be any base points of $M$ (tangential or not). Let $\gamma_1 \in \mathcal{P}(M)_x^0$ and $\gamma_2 \in \mathcal{P}(M)_x^0$. Then the following map is an isomorphism:

$$\pi(M; x_3, x_2) \to \pi(M; x_4, x_1)$$

$$\gamma \mapsto \gamma_1 \gamma_2.$$

### 3.7.4. Logarithmic asymptotic developments.

We would like to extend the notion of iterated integral to tangential base points. The main problem is that the integral may diverge, so one needs to regularize it. We start by discussing some preliminaries about asymptotic developments.

**Definition 3.233.** Let $0 < \tau \leq 1$ be a real number and $f: (0, \tau) \to \mathbb{C}$ a continuous function. We say that $f$ admits a logarithmic asymptotic development (of degree less than or equal to $r$) if it can be written as

$$f(t) = f_0(t) + \sum_{k=0}^r a_k \log(t)^k.$$
with \(|f_0(t)| = O(t^{1-\delta})\) for some \(\delta < 1\) and \(a_k \in \mathbb{C}\).

**Lemma 3.234.** Let \(0 < \tau \leq 1\) be a real number and \(f: (0, \tau) \to \mathbb{C}\) a continuous function. If it admits a logarithmic asymptotic development then the development is unique.

**Proof.** Let \(f: (0, \tau) \to \mathbb{C}\) be a continuous function that admits an asymptotic development
\[
f(t) = f_0(t) + \sum_{k=0}^{r} a_k \log(t)^k.
\]
We can recover \(a_r\) as
\[
a_r = \lim_{t \to 0} \frac{f(t) - \log(t)^r}{\log(t)^r}.
\]
Once we know \(a_{s+1}, \ldots, a_r\) we can recover \(a_s\) as
\[
a_s = \lim_{t \to 0} \frac{f(t) - \sum_{k=s+1}^{r} a_k \log(t)^k}{\log(t)^s}.
\]
Finally, \(f_0 = f(t) - \sum_{k=0}^{r} a_k \log(t)^k\). Hence the development is unique. \(\square\)

### 3.7.5. Asymptotic developments of iterated integrals

We now fix the two tangential base points \(0 = (0, 1)\) and \(1 = (1, -1)\), that is, the tangent vector 1 at the point 0 and the tangent vector 1 at the point 1. For instance, the path \(\text{dch}(t) = t\) belongs to \(\mathcal{P}(M)_0\).

Let \(x, y \in \{0, 1\} \cup \mathcal{M}\) be base points (tangential or not), \(\gamma \in y\mathcal{P}(M)_x\) a piecewise smooth clean path, and \((\varepsilon_1, \ldots, \varepsilon_r)\) a binary sequence with \(\varepsilon_1 \in \{0, 1\}\). We consider the iterated integral
\[
\int_{\gamma} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}.
\]
Since the form \(\omega_0\) has a pole at 0 and the form \(\omega_1\) has a pole at 1, this integral may diverge. For instance
\[
\int_{\text{dch}} \omega_0 = \infty.
\]
However, if the form \(\omega_{\varepsilon_1}\) has no pole at the point \(y\) and the form \(\omega_{\varepsilon_r}\) has no pole at the point \(x\), then the above integral is convergent. For instance, if \(\gamma = \text{dch}\), the integral will be convergent when \(\varepsilon_1 = 0\) and \(\varepsilon_r = 1\), that is, when the binary sequence is admissible.

We now describe the regularization process. Let \(\gamma \in y\mathcal{P}(M)_x^0\) be a clean path. For \(0 < \eta < \frac{1}{2}\), we write
\[
\gamma_{\eta}(t) = \gamma(t(1 - \eta) + (1 - t)\eta).
\]
This is a path from \(\gamma(\eta)\) to \(\gamma(1 - \eta)\), and hence completely contained in \(\mathcal{M}\).
Lemma 3.235. Let \((\varepsilon_1, \ldots, \varepsilon_r)\) be a binary sequence. Then the function 
\((0, 1/2) \rightarrow \mathbb{C}\) given by 
\[
\eta \mapsto \int_{\gamma_\eta} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}
\]
admits a logarithmic asymptotic development of degree \(\leq r\).

Proof. We write 
\[
\gamma_{\eta,1}(t) = \gamma(t(1-\eta) + (1-t)/2),
\]
\[
\gamma_{\eta,2}(t) = \gamma(t/2 + (1-t)\eta).
\]
The path \(\gamma_{\eta,2}\) goes from \(\gamma(\eta)\) to \(\gamma(1/2)\) and \(\gamma_{\eta,1}\) is a path from \(\gamma(1/2)\) to \(\gamma(1-\eta)\). Moreover, \(\gamma_\eta = \gamma_{\eta,1}\gamma_{\eta,2}\) (recall that, according to our convention for the composition of paths (3.3), this means that we first walk along \(\gamma_{\eta,2}\), then along \(\gamma_{\eta,1}\)). Using equations (3.20) and (3.21) in Theorem 3.19, it is enough to show that the functions 
\[
\eta \mapsto \int_{\gamma_{\eta,i}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}, \quad i = 1, 2
\]
admit a logarithmic asymptotic development of degree less than or equal to \(r\). Since both cases are analogous, we will only consider \(i = 2\). We prove the existence of a logarithmic asymptotic development by induction on \(r\). The result is clear for \(r = 0\). Let us assume that it holds for a binary sequence of length less than \(r\). If \(\gamma_{\eta,2}^* \omega_{\varepsilon_i} = g_{\varepsilon_i}(t)dt\) and \(\gamma^* \omega_{\varepsilon_i} = h_{\varepsilon_i}(t)dt\), then:
\[
\int_{\gamma_{\eta,2}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \int_{1/2 \geq t_1 \geq \cdots \geq t_r \geq 0} g_{\varepsilon_1}(t_1) \cdots g_{\varepsilon_r}(t_r)dt_1 \cdots dt_r = \int_{1/2 \geq t_1 \geq \cdots \geq t_r \geq \eta} h_{\varepsilon_1}(t_1) \cdots h_{\varepsilon_r}(t_r)dt_1 \cdots dt_r.
\]
Now we compute
\[
I(\eta) = \int_{1/2 \geq t_1 \geq \cdots \geq t_r \geq \eta} h_{\varepsilon_1}(t_1) \cdots h_{\varepsilon_r}(t_r)dt_1 \cdots dt_r = \int_{1/2 \geq t_r \geq \eta} h_{\varepsilon_r}(t_r) \left( \int_{1/2 \geq t_1 \geq \cdots \geq t_{r-1} \geq \eta} h_{\varepsilon_1}(t_1) \cdots h_{\varepsilon_{r-1}}(t_{r-1}) dt_1 \cdots dt_{r-1} \right) dt_r.
\]
By the shape of \(\omega_{\varepsilon_i}\), we deduce that
\[
h_{\varepsilon_r}(t_r) = \alpha/t_r + O(1),
\]
where \(\alpha\) is non-zero if \(\omega_{\varepsilon_i}\) has a pole at the point \(x\) and is zero otherwise. We also apply the induction hypothesis to the inner integral to get
\[
I(\eta) = \int_{1/2 \geq t_r \geq \eta} \left( \frac{\alpha}{t_r} + O(t_r^0) \right) \left( O(t_r^{1-\delta}) + \sum_{k=0}^{r-1} b_k \log(t_r)^k \right) dt_r.
\]
Estimating this integral, we deduce that $I(\eta)$ admits a logarithmic asymptotic development of the sought shape, proving the result.

3.7.6. 

**Regularized iterated integrals.**

**Definition 3.236.** Let $(\varepsilon_1, \ldots, \varepsilon_r)$ be a binary sequence and let $\gamma \in \mathcal{P}(M)^0_x$ be a clean path. Let

$$\int_{\gamma^0} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = a_0.$$

be the logarithmic asymptotic development provided by Lemma 3.235. Then the **regularized iterated integral** along $\gamma$ is defined as

$$\int_{\gamma}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = a_0.$$

**Proposition 3.237.** Let $\gamma \in \mathcal{P}(M)^0_x$ be a cuspidal path and $\gamma^0$ a representative of the class $\psi(\gamma)$ obtained as in (3.228). The **regularized integral**

$$\int_{\gamma^0}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}$$

does not depend on the choice of $\gamma^0$.

**Proof.** Let $\gamma_1^0$ and $\gamma_2^0$ be two choices. Since $\gamma_1^0$ and $\gamma_2^0$ only differ from $\gamma$ in a small neighborhood of the cusps, for small enough $\eta$,

$$\gamma_1^0(\eta) = \gamma_2^0(\eta), \quad \gamma_1^0(1 - \eta) = \gamma_2^0(1 - \eta).$$

Moreover, $\gamma_1^0$ and $\gamma_2^0$ are homotopic. As we saw in paragraph 3.4.5, $H^0(B^*(A^*)) = B^0(A)$, thus all the iterated integrals that can be constructed from $\omega_0$ and $\omega_1$ are homotopy functionals. Therefore

$$\int_{\gamma_1^0}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \int_{\gamma_2^0}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r},$$

from which the result follows. □

**Definition 3.238.** Let $\gamma \in \mathcal{P}(M)^0_x$ be a cuspidal path. Let $\gamma^0$ be a representative of the class $\psi(\gamma)$ obtained as in (3.228). We define

$$\int_{\gamma}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \int_{\gamma^0}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}.$$

Clearly, when the iterated integral is convergent, the value of the regularized integral agrees with the value of the integral.

Regularized iterated integrals share many of the properties of iterated integrals. In particular, Theorem 3.19 can be extended to the new setting.

**Theorem 3.239.** Let $\gamma, \gamma_1, \gamma_2$ be cuspidal in $M$ whose end points are either 0, 1 or belong to $M$ and such that $\gamma_2(1) = \gamma_1(0)$. Let $(\varepsilon_1, \ldots, \varepsilon_{r+s})$ be a binary sequence. Then
\begin{align}
\int_{\gamma}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} &= (-1)^r \int_{\gamma-1}^{\text{reg}} \omega_{\varepsilon_r} \cdots \omega_{\varepsilon_1}. \\
\int_{\gamma_1 \gamma_2}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} &= \sum_{i=0}^{r} \int_{\gamma_1}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_i} \int_{\gamma_2}^{\text{reg}} \omega_{\varepsilon_{i+1}} \cdots \omega_{\varepsilon_r}. \\
\int_{\gamma}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} \int_{\gamma}^{\text{reg}} \omega_{\varepsilon_{r+1}} \cdots \omega_{\varepsilon_{r+s}} &= \sum_{\sigma \in \omega(r,s)} \int_{\gamma}^{\text{reg}} \omega_{\varepsilon_{r-1}(1)} \cdots \omega_{\varepsilon_{r-1}(r+s)}.
\end{align}

**Proof.** We first prove (1). If $\gamma$ is cuspidal and $\gamma^0$ is a clean path in the homotopy class $\psi(\gamma)$ obtained as in (3.228), then $(\gamma^0)^{-1}$ is a clean path in the homotopy class $\psi(\gamma^{-1})$ obtained as in (3.228). Therefore, we can assume that $\gamma$ is a clean path. By construction, $(\gamma^{-1})_\varepsilon = (\gamma_\varepsilon)^{-1}$. By Theorem 3.19 the asymptotic expansions of

\begin{align*}
\int_{\gamma} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}, \text{ and } (-1)^r \int_{\gamma^{-1}}^{\text{reg}} \omega_{\varepsilon_r} \cdots \omega_{\varepsilon_1}
\end{align*}

agree. Thus we have the equality of regularized integrals.

Statement (3) also follows from the corresponding statement in Theorem 3.19.

Statement (2) is slightly more difficult due to the possibility that the joining point is a tangential base point. The proof goes as follows.

Assume that $\gamma_1$ and $\gamma_2$ are clean paths. Let $\gamma = \gamma_1 \gamma_2$ be their composition and $\gamma^0$ a clean path representing $\gamma$ as in (3.228). For sufficiently small $\eta$, the path $(\gamma^0)^{\eta}$ is homotopic to $\gamma_1 \eta \gamma_0 \gamma_2 \eta$, where $\gamma_0 \eta$ denotes the straight path from $\gamma_2 (1 - \eta)$ to $\gamma_1 (\eta)$ (see Figure 17 below). By the usual formula for the composition of paths

\begin{align}
\int_{(\gamma^0)^{\eta}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} &= \sum_{j=0}^{r} \sum_{k=j}^{r} \int_{\gamma_1 \eta}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_j} \int_{\gamma_0 \eta}^{\text{reg}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_k} \int_{\gamma_2 \eta}^{\text{reg}} \omega_{\varepsilon_{k+1}} \cdots \omega_{\varepsilon_r}.
\end{align}

**Lemma 3.241.** One has $\int_{\gamma_0 \eta} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_k} = O(\eta^{k-j}).$

**Proof.** The key point is that we have power series expansions

\begin{align*}
\gamma_2 (1 - \eta) &= \gamma_2 (1) - \gamma_2'(1) \eta + O(\eta^2) \\
\gamma_1 (\eta) &= \gamma_1 (0) + \gamma_1'(0) \eta + O(\eta^2).
\end{align*}
Since \( \gamma_2(1) = \gamma_1(0) \) and \( \gamma'_2(1) = -\gamma'_1(0) \), it follows that
\[
|\gamma_2(1 - \eta) - \gamma_1(\eta)| = O(\eta^2).
\]
Using the equation \( \gamma_{0, \eta} = t\gamma_1(\eta) + (1 - t)\gamma_2(1 - \eta) \), one sees that
\[
\gamma^*_{0, \eta} \frac{dz}{z} = \frac{(\gamma_1(\eta) - \gamma_2(1 - \eta))dt}{t\gamma_2(\eta) + (1 - t)\gamma_2(1 - \eta)}
\]
Since the numerator is \( O(\eta^2) \) and the denominator is \( O(\eta) \), it follows that
\[
\int_{\gamma_{0, \eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_k} = O(\eta^{k-j}),
\]
proving the lemma.

To conclude the proof of the theorem, we observe that, by the lemma, the integral \( \int_{\gamma_{0, \eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_k} \) does not contribute to the constant term in the logarithmic asymptotic development of (3.240) when \( k > j \). Therefore,
\[
\text{const} \int_{(\gamma^0_{0, \eta})} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \sum_{j=0}^{r} \left( \text{const} \int_{\gamma_{1, \eta}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_j} \right) \left( \text{const} \int_{\gamma_{2, \eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_r} \right),
\]
from which the result follows. Here \( \text{const} \) means the constant term \( a_0 \) in the logarithmic asymptotic expansion.

As we did before for “honest” base points, the properties of iterated integrals can be concisely rephrased in terms of the bracket. If \( \gamma \) is a piecewise smooth path and \( \eta \in B^0(\Lambda^*) \), we denote
\[
(\eta, \gamma)_{\text{reg}} = \int_{\gamma} \eta.
\]

**Theorem 3.242.** Let \( \gamma, \gamma_1, \gamma_2 \) be piecewise smooth paths with any base points and let \( \eta, \eta_1, \eta_2 \in B^0(\Lambda^*) \) be elements of the bar complex of \( \Lambda^* \) of degree zero. Then

1. \( (\eta, \gamma)_{\text{reg}} = (S(\eta), \gamma^{-1})_{\text{reg}} \).
\begin{equation}
\langle \eta, \gamma \rangle \equiv \langle \Delta \eta, \gamma_1 \otimes \gamma_2 \rangle \quad (2)
\end{equation}
\begin{equation}
\langle \eta_1, \gamma \rangle \cdot \langle \eta_2, \gamma \rangle = \langle \eta_1 + \eta_2, \gamma \rangle \quad (3)
\end{equation}

3.7.7. **Regularized iterated integrals and regularized zeta values.**

**Example 3.243.** Let us compute an example of a regularized iterated integral in length 3:

\[ \zeta(1, 2) = \int_{\text{reg}} \omega_1 \omega_0 \omega_1. \]

By definition, this is the constant term in the asymptotic logarithmic development of the function

\[ \eta \mapsto - \int_{1-\eta \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1 dt_2 dt_3}{(1-t_1)t_2(1-t_3)}. \]

To be completely precise, according to the above recipe we should have required \( t_3 \geq \eta \) as well. Note, however, that the last form \( \omega_1 \) has no pole at 0, so the two asymptotic logarithmic developments agree.

We first compute the integral following the method of examples 1.103 and 1.105. We obtain

\begin{equation}
\int_{1-\eta \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1 dt_2 dt_3}{(1-t_1)t_2(1-t_3)} = \sum_{m>n>0} \frac{(1-\eta)^m}{n^2 m}. \quad (3.244)
\end{equation}

This power series converges for \( 0 < \eta < 1 \) but diverges for \( \eta = 0 \) and we have to find an asymptotic expansion in \( \log \eta \). To this end, we use the equality

\begin{equation}
\int_{1-\eta \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1 dt_2 dt_3}{(1-t_1)t_2(1-t_3)} = \int_{1-\eta \geq t_2 \geq t_3 \geq 0} \frac{dt_1 dt_2 dt_3}{(1-t_1)t_2(1-t_3)} - 2 \int_{1-\eta \geq t_2 \geq t_1 \geq 0} \frac{dt_1 dt_2 dt_3}{(1-t_1)t_2(1-t_3)}, \quad (3.245)
\end{equation}

which is a simple consequence of the decomposition of the integration domain, together with the fact that the integrand is symmetric in \( t_1 \) and \( t_3 \) (this explains why the last term appears twice). Observe that

\[ \int_{1-\eta \geq t_1 \geq 0} \frac{dt_1}{1-t_1} = \sum_{k \geq 1} \frac{(1-\eta)^k}{k} = - \log(\eta). \]

Combining this with the power series expansions as in Example 1.105, one sees that the right-hand side of (3.245) is equal to

\begin{equation}
- \log(\eta) \sum_{n \geq 1} \frac{(1-\eta)^n}{n^2} - 2 \sum_{m>n \geq 1} \frac{(1-\eta)^m}{m^2 n}. \quad (3.246)
\end{equation}

One can check (Exercise 3.252) directly that this expansion agrees with the right-hand side of (3.244).
To see that the power expansion (3.246) is useful we have to prove that the series appearing in that expansion define continuous functions of $\eta$.

**Lemma 3.247.** The following estimates hold when $\eta$ goes to $0^+$:

$$\sum_{n \geq 1} \frac{(1 - \eta)^n}{n^2} = \zeta(2) + O(\eta \log \eta), \quad (3.248)$$

$$\sum_{m > n \geq 1} \frac{(1 - \eta)^m}{m^2 n} = \zeta(2, 1) + O(\eta \log^2 \eta). \quad (3.249)$$

**Proof.** To prove the estimate (3.248), we need to study

$$\zeta(2) - \sum_{n \geq 0} \frac{(1 - \eta)^n}{n^2} = \sum_{n \geq 0} \frac{1 - (1 - \eta)^n}{n^2}.$$

For $0 < \eta < 1$, we have the inequalities

$$0 < 1 - (1 - \eta)^n < 1, \quad 0 < 1 - (1 - \eta)^n < n\eta.$$

Therefore

$$0 < \sum_{n \geq 1} \frac{1 - (1 - \eta)^n}{n^2} < \sum_{n=1}^{\lfloor \eta \rfloor} \frac{n}{n^2} + \sum_{n > \lfloor \eta \rfloor} \frac{1}{n^2}.$$

Since the first sum is $O(\eta \log \eta)$ and the second is $O(\eta)$, the first estimate follows. The second one is obtained in a similar way. \qed

From Lemma 3.247 we obtain

$$\int_{1-\eta > t_1 > t_2 > t_3 > 0} \frac{dt_1 dt_2 dt_3}{(1 - t_1)t_2(1 - t_3)} = -2\zeta(2, 1) - \zeta(2) \log \eta + O(\eta \log^2 \eta),$$

from which it follows that

$$\zeta(1, 2)^{\text{reg}} = -2\zeta(2, 1).$$

The value of $\zeta(1, 2)^{\text{reg}}$ is equal to the one obtained by shuffle regularization in Example 1.178. This is of course no coincidence, as we now prove:

**Theorem 3.250.** Let $(\varepsilon_1, \ldots, \varepsilon_r)$ be a binary sequence and consider the corresponding word $w = x_{\varepsilon_1} \cdots x_{\varepsilon_r}$. Then:

$$\zeta_w(w) = \int_{\gamma}^{\text{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}.$$

**Proof.** By Proposition 1.173, we need to show that the integral on the right-hand side satisfies the conditions determining $\zeta_w(w)$. Condition (1.174) follows from Theorem 1.108 combined with the observation that, when the binary sequence is admissible, then the regularized integral agrees with the usual integral. Condition (1.175) is checked by a direct computation. Finally condition (1.176) is Theorem 3.239 3. \qed
3.7.8. Chen’s theorem for tangential base points. We finish this section by stating a version of Chen’s theorem with tangential base points. Recall that we are writing \( M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \), and \( A^*_C \) is the dg-algebra of paragraph 3.4.5.

**Theorem 3.251 (Chen’s \( \pi_1 \) theorem for tangential base points).** For each integer \( N \geq 0 \) and each pair of points \( x, y \) (tangential or not), the regularized iterated integrals induce an isomorphism

\[
L_N H^0(B^*(A^*_C)) \overset{\sim}{\rightarrow} \text{Hom}_\mathbb{Q}(\mathbb{Q}[\pi_1(M; y, x)]/J^{N+1}\mathbb{Q}[\pi(M; x)], \mathbb{C}).
\]

Passing to the limit, we deduce an isomorphism between \( H^0(B^0(A^*_C)) \) and the topological dual \( (\mathbb{C}[\pi_1(M; y, x)]^\vee)^\vee \).

**Proof.** We need to show that the pairing between \( L_N H^0(B^*(A^*_C)) \) and \( \pi_1(M; y, x)/J^{N+1} \) is non-degenerate. Since both spaces are finite dimensional, it suffices to prove that there is no non-zero \( \gamma \in \pi_1(M; x, y)/J^{N+1} \) such that \( \langle \omega, \gamma \rangle = 0 \) for all \( \omega \). Indeed, assume that such a \( \gamma \) exists. Choose usual base points \( x' \) and \( y' \) and paths \( \gamma_1, \gamma_2 \) going from \( x \) to \( x' \) and from \( y \) to \( y' \). Then, by Theorem 3.242 (2), for \( \omega \in L_N H^0(B^*(A^*)) \)

\[
\langle \omega, \gamma_1 \gamma_2 \rangle = \sum \langle \omega_1, \gamma_1 \rangle \langle \omega_2, \gamma \rangle \langle \omega_3, \gamma_2 \rangle,
\]

where all the elements \( \omega_1, \omega_2, \omega_3 \) are of length \( \leq N \). Thus \( \langle \omega, \gamma_1 \gamma_2 \rangle = 0 \) for all \( \omega \in L_N H^0(B^*(A^*)) \). By Chen’s Theorem 3.151, \( \gamma_1 \gamma_2 = 0 \) and hence the same is true for \( \gamma \).

\[\square\]

**Exercise 3.252.** By expanding \( \log(\eta) \) as a power series in \( (1 - \eta) \), prove the following equality of functions for \( 0 < \eta < 1 \):

\[
\sum_{m > n > 0} \frac{(1 - \eta)^m}{n^2 m} = -\log(\eta) \sum_{n \geq 0} \frac{(1 - \eta)^n}{n^2} - 2 \sum_{m > n \geq 1} \frac{(1 - \eta)^m}{m^2 n}.
\]

**Exercise 3.253.** Let \( n \geq 2 \) be an integer. Adapt Example 3.243 to compute the regularized iterated integral

\[
\int_{\text{deg}}^{\text{reg}} \omega_1 \omega_0^{n-1} \omega_1
\]

and show that the result coincides with \( \zeta_{\omega}(1, n) \).

3.8. Polylogarithms and their monodromy. In this section, we explain how to make the isomorphism of Chen’s Theorem 3.251 more explicit in the case of \( M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \) by using polylogarithms.
3.8.1. Generators of the fundamental group of $M$. In the previous section, we have introduced the tangential base points 0 and 1. The fundamental group $\pi_1(M,0)$ is generated by the paths $\gamma_0$ and $\gamma_1$ of Figure 18. The space of paths $\pi(M;1,0)$ is generated as a right $\pi_1(M,0)$-module by the straight path $\text{dch}$ also represented in Figure 18.

![Figure 18. Generators](image)

The fundamental group $\pi_1(M,1)$ is generated by the paths

$$
\gamma'_0 = \text{dch} \cdot \gamma_0 \cdot \text{dch}^{-1}, \quad \gamma'_1 = \text{dch} \cdot \gamma_1 \cdot \text{dch}^{-1},
$$

and the space $\pi(M;0,1)$ is generated as a right $\pi_1(M,1)$-module or as a left $\pi_1(M,0)$-module by the path $\text{dch}^{-1}$.

3.8.2. The dual of Chen’s map. We saw in paragraph 3.4.5 that the cohomology in degree zero of the reduced bar complex associated with $A^*_C$ is isomorphic, as a Hopf algebra, to the complex Hoffman algebra $\hat{\mathcal{H}} \otimes \mathbb{C}$. In Example 3.67 we identified the dual $\hat{\mathcal{H}}^\vee$ with the algebra $\mathbb{Q} \langle e_0, e_1 \rangle$. We extend Notation 1.153 as follows.

**Notation** 3.254. If $\alpha$ is a binary sequence, we will denote by $x_\alpha$ the corresponding word in the Hoffman algebra $\hat{\mathcal{H}}$, by $\omega_\alpha$ the differential form $\omega_\alpha$ in $B^0(A^*) \simeq \hat{\mathcal{H}}$ and by $e_\alpha$ the dual element to $x_\alpha$ in $\mathbb{Q} \langle e_0, e_1 \rangle$.

Let $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$, together with two base points $x$ and $y$ (tangential or not). Given a path $\gamma$ from $x$ to $y$ and $\omega \in B^0(A^*_C)$, we define

$$
L_\omega(\gamma) = \int_\gamma^{\text{reg}} \omega \in \mathbb{C}.
$$

Or, in the notation of Theorem 3.242

$$
L_\omega(\gamma) = \langle \omega, \gamma \rangle^{\text{reg}}.
$$

For a binary sequence $\alpha$, we set $L_\alpha(\gamma) = L_{\omega_\alpha}(\gamma)$. Consider the generating series

$$
L(\gamma) = \sum_\alpha L_\alpha(\gamma) e_\alpha \in \mathbb{C} \langle e_0, e_1 \rangle.
$$

Therefore, if $\omega \in B^0(A^*_C) \simeq \hat{\mathcal{H}} \otimes \mathbb{C} \simeq \mathbb{C} \langle e_0, e_1 \rangle^\vee$, we have

$$
L(\gamma)(\omega) = L_\omega(\gamma).
$$
3.8.3. The map $L$ and polylogarithms. Recall that in Definition 1.110 we attached to a positive multi-index $s$ a complex-valued function $L_s$, the polylogarithm, defined on the open unit disc $|z| < 1$. The relation between these notions is explained by the following lemma, whose proof is parallel to that of Theorem 1.117. We left the details to the reader.

Lemma 3.255. Let $z$ be a complex number such that $0 < |z| < 1$, $\gamma$ any path from $0$ to $z$ contained in the unit disc, and $s$ a positive multi-index. Let $bs(s)$ denote the associated binary sequence. Then:

$$L_s(z) = L_{bs(s)}(\gamma).$$

3.8.4. Computation of $L(\gamma_0)$. For any $z \in \mathbb{C} \setminus \{0, 1\}$, any path $\gamma$ from $0$ to $z$ and any binary sequence $\alpha$, $L_\alpha(\gamma)$ is defined. By abuse of notation, we will write $L_\alpha(z)$ and think of it as a multivalued function.

Example 3.256. Let $z \in \mathbb{C} \setminus \{0, 1\}$. Let us show that, for each $n \geq 1$, the following equality of multivalued functions holds:

$$L_0^n(z) = \frac{1}{n!}(\log z)^n$$  \hspace{1cm} (3.257)

Let $\gamma$ be any path from $0$ to $z$. We argue by induction on $n$. First, for $n = 1$, to compute the value

$$L_0(\gamma) = \int_\gamma^{\text{reg}} \frac{dt}{t}$$

one needs to find a logarithmic asymptotic development for

$$\eta \mapsto \int_\eta^{1-\eta} \gamma^* \left(\frac{dt}{\gamma}\right) = \int_{\eta}^{1-\eta} \frac{\gamma'}{\gamma} dt = \log \gamma(1-\eta) - \log \gamma(\eta).$$

Since $\gamma(0) = 0$ and $\gamma'(0) = 1$, one has $\gamma(\eta) = \eta(1 - O(\eta))$ as $\eta$ goes to zero. On the other hand, $\gamma(1 - \eta) = z + O(\eta)$. Thus,

$$\log \gamma(1-\eta) - \log \gamma(\eta) = \log z + O(\eta) - \log \eta$$

and the regularization assigns the value

$$L_0(z) = \log z.$$

Assume now that the identity (3.257) holds for $n - 1$. Since the number of shuffles of type $(1, n - 1)$ is $n$ (cf. Exercise 1.134), relation (3) of Theorem 3.239 gives the result we wanted:

$$nL_0^n(z) = \int_\gamma^{\text{reg}} \omega_0 \int_\gamma^{\text{reg}} \omega_0 \cdots \omega_0 = \frac{1}{(n-1)!}(\log z)^n.$$
Example 3.258. We are now ready to compute $L(\gamma_0)$. Arguing as in Example 3.256, one gets

$$L_{0^n}(\gamma_0) = \frac{1}{n!} (2\pi i)^n.$$ 

If $\alpha$ is a non-empty positive binary sequence, Lemma 3.255 implies that

$$L_{\alpha}(\gamma_0) = 0.$$ 

In fact, it follows from the compatibility with the shuffle product, part (3) of Theorem 3.242, that $L_{\alpha 0^k}(\gamma_0) = 0$ for all $\alpha \neq \emptyset$ and all $k \geq 0$. Summing up, we deduce that

$$L(\gamma_0) = \sum_{\alpha} L_{\alpha}(\gamma_0) e_\alpha = \sum_{n \geq 0} \frac{(2\pi i)^n}{n!} e_0^n = \exp(2\pi i e_0). \tag{3.259}$$

Thanks to the symmetry $z \mapsto 1 - z$, it follows that

$$L(\gamma_1') = \exp(2\pi i e_1). \tag{3.260}$$

3.8.5. $L$ evaluated at $dch$ and the Drinfeld associator.

Example 3.261. Theorem 3.250 implies that, for each binary sequence $\alpha$, the equality $L_{\alpha}(dch) = \zeta_{\omega}(x_\alpha)$ holds. Therefore

$$L(dch) = \sum_{\alpha} \zeta_{\omega}(x_\alpha) e_\alpha. \tag{3.262}$$

We write $\Phi(e_0, e_1)$ for this power series with real coefficients. We also write

$$\Phi_{KZ}(e_0, e_1) = \Phi(e_0, -e_1) = \sum_{\alpha} (-1)^{l(\alpha)} \zeta_{\omega}(x_\alpha) e_\alpha, \tag{3.263}$$

where $l(\alpha)$ is the number of entries equal to 1 in $\alpha$ as in Definition 1.124.

Definition 3.264. The power series $\Phi_{KZ}(e_0, e_1) \in \mathbb{R}[e_0, e_1]$ is called the Drinfeld associator.

3.8.6. Chen’s theorem revisited.

Theorem 3.265. For any two base points $x$ and $y$, the map $L$ can be extended to a continuous $\mathbb{C}$-linear isomorphism

$$L: \mathbb{C}[\pi_1(M; y, x)]^\wedge \longrightarrow \mathbb{C}[e_0, e_1] = \text{Hom}(\tilde{H}, \mathbb{C}).$$

The following properties hold:

1. If $u \in \mathbb{C}[\pi_1(M; y, x)]^\wedge$, then

$$S^\vee(L(u)) = L(S(u)).$$

In particular, if $\gamma \in \pi_1(M; y, x)$ is a path, $S^\vee(L(\gamma)) = L(\gamma^{-1}).$
Given three points $x$, $y$ and $z$, and elements $v \in \mathbb{C}[\pi_1(M; y, x)]^\wedge$, $u \in \mathbb{C}[\pi_1(M; z, y)]^\wedge$, one has
\[
L(uv) = L(u)L(v).
\]

If $u \in \mathbb{C}[\pi_1(M; y, x)]^\wedge$, then
\[
\nabla^\vee(L(u)) = (L \otimes L)(\Delta(u)).
\]

In particular, if $\gamma \in \pi_1(M; y, x)$ is a path, then $L(\gamma)$ is a group-like element.

**Proof.** We first extend $L$ by linearity to $\mathbb{C}[\pi_1(M; y, x)]$. By construction, for any path $\gamma$, the series $L(\gamma)$ starts by one. Therefore, any element in the augmentation ideal of $\mathbb{C}[\pi_1(M; y, x)]$ is sent to an element of the ideal generated by $e_0$ and $e_1$. Thus, it can be extended uniquely to a morphism
\[
L: \mathbb{C}[\pi_1(M; y, x)]^\wedge \longrightarrow \mathbb{C}\langle e_0, e_1 \rangle = \text{Hom}(\mathfrak{H}, \mathbb{C}).
\]

That this yields an isomorphism is simply a reformulation of Theorem 3.251. Clearly, it is enough to check properties (1) to (3) on paths. All of them follow from Theorem 3.239.

We start proving (1) using Theorem 3.242 (1).
\[
L(\gamma^{-1}) = \sum_\alpha \langle \omega_\alpha, \gamma^{-1} \rangle^{\text{reg}} e_\alpha = \sum_\alpha \langle S(\omega_\alpha), \gamma \rangle^{\text{reg}} e_\alpha = \sum_\alpha \langle \omega_\alpha, \gamma \rangle^{\text{reg}} S^\vee(e_\alpha) = S^\vee(L(\gamma)).
\]

We next prove (2) using 3.242 (2)
\[
L(\gamma_1 \gamma_2) = \sum_\alpha \langle \omega_\alpha, \gamma_1 \gamma_2 \rangle^{\text{reg}} e_\alpha = \sum_\alpha \langle \Delta \omega_\alpha, \gamma_1 \otimes \gamma_2 \rangle^{\text{reg}} e_\alpha = \sum_{\alpha', \alpha''} \langle \omega_{\alpha'} \otimes \omega_{\alpha''}, \gamma_1 \otimes \gamma_2 \rangle^{\text{reg}} e_{\alpha'} e_{\alpha''} = L(\gamma_1) L(\gamma_2).
\]

Finally we prove (3) using 3.242 (3).
\[
\nabla^\vee(L(\gamma)) = \sum_\alpha \langle \omega_\alpha, \gamma \rangle^{\text{reg}} \nabla^\vee e_\alpha = \sum_\alpha \langle \omega_\alpha, \gamma \rangle^{\text{reg}} \sum_{\alpha', \alpha''} \mu(\alpha', \alpha'', \alpha) e_{\alpha'} \otimes e_{\alpha''} = \sum_{\alpha', \alpha''} \langle \omega_{\alpha'} \cup \omega_{\alpha''}, \gamma \rangle^{\text{reg}} e_{\alpha'} \otimes e_{\alpha''} = \sum_{\alpha', \alpha''} \langle \omega_{\alpha'}, \gamma \rangle^{\text{reg}} \langle \omega_{\alpha''}, \gamma \rangle^{\text{reg}} e_{\alpha'} \otimes e_{\alpha''} = L(\gamma) \otimes L(\gamma).
\]

This concludes the proof. \qed
EXAMPLE 3.266. From Theorem 3.265 (3) we deduce that $\Phi(e_0, e_1) = L(dch)$ is a group-like element. In particular, it is the exponential of a Lie-like element and its inverse as power series is given by its antipode

$$L(dch^{-1}) = \Phi(e_0, e_1)^{-1} = S^\vee(\Phi(e_0, e_1)). \quad (3.267)$$

From examples 3.258 and 3.261 and the compatibility of $L$ with the composition of paths in 3.265 (2) we can compute $L$ on the remaining generators of $\pi_1(M, 0)$ and $\pi_1(M, 1)$.

$$L(\gamma_1) = \Phi(e_0, e_1)^{-1} \exp(2\pi i e_1) \Phi(e_0, e_1),$$

$$L(\gamma_0) = \Phi(e_0, e_1) \exp(2\pi i e_0) \Phi(e_0, e_1)^{-1}.$$ 

3.8.7. The Knizhnik–Zamolodchikov equation. Theorem 3.265 encodes all the properties of the series $L$, and hence of polylogarithms. The first property is that it satisfies the so-called Knizhnik–Zamolodchikov equation:

**Proposition 3.268.** $L(z)$ satisfies the differential equation

$$\frac{d}{dz} L(z) = \left(\frac{e_0}{z} + \frac{e_1}{1-z}\right) L(z). \quad (3.269)$$

**Proof.** Fix $z \in M$, let $\gamma$ be a path with end point $z$ and let $\gamma_\varepsilon(t) = z + t\varepsilon$. To compute the derivative of $L(z)$ we need to evaluate the limit

$$\lim_{\varepsilon \to 0} \frac{L(\gamma_\varepsilon \gamma) - L(\gamma)}{\varepsilon}.$$ 

By Theorem 3.265 (2)

$$L(\gamma_\varepsilon \gamma) - L(\gamma) = (L(\gamma_\varepsilon) - 1)L(\gamma).$$

Moreover,

$$L(\gamma_\varepsilon) - 1 = \int_{\gamma_\varepsilon}^\text{reg} \omega_0 e_0 + \int_{\gamma_\varepsilon}^\text{reg} \omega_1 e_1 + O(\varepsilon^2).$$

Since

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\gamma_\varepsilon} \omega_0 = \frac{1}{z} \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\gamma_\varepsilon} \omega_1 = \frac{1}{1-z},$$

we conclude

$$\frac{d}{dz} L(z) = \left(\frac{e_0}{z} + \frac{e_1}{1-z}\right) L(z).$$

Finishing the proof. \qed
3.8.8. The monodromy of $L$. The second property we want to derive is an explicit description of the monodromy of $L$ as a multivalued function.

**Theorem 3.270.** Let $z \in M$ and $\gamma$ a path from $0$ to $z$. Then
\[
L(\gamma \cdot \gamma_0) = L(\gamma) \exp(2\pi i e_0),
\]
\[
L(\gamma \cdot \gamma_1) = L(\gamma)\Phi(e_0, e_1)^{-1} \exp(2\pi i e_1)\Phi(e_0, e_1).
\]

**Proof.** Follows immediately from Theorem 3.265 (2) and examples 3.258 and 3.266. \qed

3.8.9. Further properties of the Drinfeld associator. We next derive the basic properties of Drinfeld associator $\Phi_{KZ}$. Let $Ua_4$ be the universal enveloping algebra of the Lie algebra of the pro-unipotent completion of the pure braid group on 4 strings. It is the algebra of power series in letters $t_{i,j}$, $1 \leq i, j \leq 4$ with the relations
\[
t_{i,i} = 0, \quad t_{i,j} = t_{j,i},
\]
\[
[t_{i,j}, t_{i,k} + t_{j,k}] = 0, \quad \text{for } i, j, k \text{ different},
\]
\[
[t_{i,j}, t_{k,l}] = 0, \quad \text{for } i, j, k, l \text{ different}.
\]

**Theorem 3.271** (Drinfeld [Dri90]). The Drinfeld associator satisfies the following relations.

1. **Symmetry relation:** $\Phi_{KZ}(e_0, e_1)\Phi_{KZ}(e_1, e_0) = 1$.

2. **Hexagon relation:** Write $e_\infty = -e_0 - e_1$, then
\[
e^{i\pi e_0}\Phi_{KZ}(e_\infty, e_0)e^{i\pi e_\infty}\Phi_{KZ}(e_1, e_\infty)e^{i\pi e_1}\Phi_{KZ}(e_0, e_1) = 1.
\]

3. **Pentagon relation:** For $t_{i,j} \in Ua_4$ we have
\[
\Phi_{KZ}(t_{1,2}, t_{2,3} + t_{2,4})\Phi_{KZ}(t_{1,3} + t_{2,3}, t_{3,4}) = \Phi_{KZ}(t_{2,3}, t_{3,4})\Phi_{KZ}(t_{1,2} + t_{2,3}, t_{2,4} + t_{3,4})\Phi_{KZ}(t_{1,2}, t_{2,3}).
\]

**Proof.** We start proving (1). Consider the automorphism of $M$ given by $z \mapsto 1 - z$. This automorphism sends the form $\omega_i$ to $-\omega_{1-i}$ for $i = 0, 1$, and hence it sends $e_0$ to $-e_1$ and $e_1$ to $-e_0$. Moreover, it sends $dch$ to $dch^{-1}$. Therefore, we deduce that $L(dch^{-1}) = \Phi(-e_1, -e_0)$. Therefore,
\[
1 = L(dch)L(dch^{-1}) = \Phi(e_0, e_1)\Phi(-e_1, -e_0),
\]
which is equivalent to (1).

To prove (2) we need to introduce more tangential points and paths. Let $0^- = (0, -1)$ be the tangent vector $-1$ at $0$ and $1^- = (1, 1)$ be the tangent vector $1$ at $1$. We consider the point $\infty$ with local coordinate $u = 1/z$ and denote $\infty = (\infty, 1)$ the tangent point $1$ at $\infty$ with respect to this coordinate and $\infty^- = (\infty, -1)$. We denote by $\delta_0 \in \pi(M; 0, 0^-)$ the path that starts in $0^-$, gives half a turn around zero in the counterclockwise direction and ends
in 0. Similarly, \( \delta_1 \in \pi(M; 1^-, 1) \) is the path that starts in 1, gives half a turn in the counterclockwise direction and ends in 1. And \( \delta_\infty \in \pi(M; \infty^-, \infty) \) is the path that starts in \( \infty \), gives half a turn in the counterclockwise direction and ends in \( \infty^- \). Finally, we denote by \( \text{dch}_{\infty,1} \in \pi(M; \infty, 1^-) \) the straight path that starts in \( \infty \) and ends in \( \infty^- \). The straight path that starts in \( 1^- \) and ends in \( \infty^- \) through the real numbers greater than one and by \( \text{dch}_{0,\infty} \in \pi(M; 0^-, \infty^-) \) the straight path that starts in \( \infty^- \) and ends in \( 0^- \) through the negative real numbers. All these paths are represented in Figure 19.

![Figure 19. paths](image)

Clearly, the composition

\[
\delta_0 \cdot \text{dch}_{0,\infty} \cdot \delta_\infty \cdot \text{dch}_{\infty,1} \cdot \delta_1 \cdot \text{dch}
\]

is homotopically equivalent to the trivial path. Therefore, by Theorem 3.265 (2),

\[
L(\delta_0)L(\text{dch}_{0,\infty})L(\delta_\infty)L(\text{dch}_{\infty,1})L(\delta_1)L(\text{dch}) = 1.
\] (3.272)

Arguing as in Example 3.258, we can see that

\[
L(\delta_0) = \exp(\pi i e_0).
\]

We now consider the automorphism of \( M \) given by \( z \mapsto 1/(1-z) \). This map sends \( \delta_0 \) to \( \delta_1 \) and \( \delta_1 \) to \( \delta_\infty \). It also sends \( \text{dch} \) to \( \text{dch}_{\infty,1} \) and \( \text{dch}_{\infty,1} \) to \( \text{dch}_{0,\infty} \).

Moreover the pull back by this isomorphism sends the form \( \omega_0 \) to the form \( \omega_1 \) and the form \( \omega_1 \) to the form \( -\omega_0 - \omega_1 \). Dualizing we deduce that this automorphism sends \( e_0 \) to \( -e_1 \) and \( e_1 \) to \( e_0 - e_1 \). We deduce that

\[
L(\delta_1) = \exp(-\pi i e_1), \quad L(\delta_\infty) = \exp(\pi i (e_1 - e_0)),
\]

\[
L(\text{dch}_{\infty,1}) = \Phi(-e_1, e_0 - e_1), \quad L(\text{dch}_{0,\infty}) = \Phi(e_1 - e_0, -e_0).
\]

Thus equation (3.272) reads

\[
e^{i\pi e_0} \Phi(e_1 - e_0, -e_0)e^{i\pi (e_1 - e_0)} \Phi(-e_1, e_0 - e_1)e^{-i\pi e_1} \Phi(e_0, e_1) = 1,
\]

which is equivalent to

\[
e^{i\pi e_0} \Phi_{KZ}(e_1 - e_0, e_0)e^{i\pi (e_1 - e_0)} \Phi_{KZ}(-e_1, e_1 - e_0)e^{-i\pi e_1} \Phi_{KZ}(e_0, -e_1) = 1.
\]

The hexagon relation is obtained by replacing \( e_1 \) by \( -e_1 \).

The proof of (3) involves considering a path in the moduli space \( M_{0,5} \) which is a complex surface. To write it properly, we would need to discuss
tangential base points and local monodromy in higher dimensions, so we will omit it. See for instance [Had] for an outline.

3.8.10. The associator relations and the extended double shuffle relations. We close this section by quoting

**Theorem 3.273** (Furusho [Fur10], [Fur11]).

a) Let \((\zeta^s(\alpha))_\alpha\) be a collection of real numbers, one for each binary sequence. Denote by \(\zeta^s: \mathcal{S}^0 \to \mathbb{R}\) the map obtained from these numbers by linearity. If the power series
\[
\sum_\alpha (-1)^\alpha \zeta^s(\alpha)e_\alpha
\]
is group-like and satisfies the associator relations of Theorem 3.271, then \((\mathbb{R}, \zeta^s)\) satisfies the extended double shuffle relations (Definition 1.192).

b) Let \(\varphi \in \mathbb{R}\langle e_0, e_1 \rangle\) be a group-like element with the coefficient of \(e_0e_1\) equal to \(-\zeta(2) = -\pi^2/6\). If \(\varphi\) satisfies the pentagon relation 3.271 (3), then it satisfies the symmetry relation 3.271 (1) and the hexagon relation 3.271 (2).

***

**Exercise 3.274.** Compute explicitly the terms up to degree 5 of the Drinfeld associator \(\Phi_{KZ}(e_0, e_1)\). Show that, with the exception of the unit in degree 0, they can be all written as commutators.

**Exercise 3.275.** In this exercise, we show how Theorem 3.270 encodes the monodromy of multiple polylogarithms in one variable. We start with \(\text{Li}_3\), which is the coefficient of \(e_0e_0e_1\) in \(L\). Let \(z \in \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}\) and let \(\gamma\) be a path between 0 and \(z\).

1) Find the coefficient of \(e_0e_0e_1\) in \(L(\gamma \cdot \gamma_0)\) and \(L(\gamma \cdot \gamma_1)\). The obtained expressions give us the monodromy of \(\text{Li}_3\).

2) Compute the monodromy through \(\gamma_0\) and \(\gamma_1\) of the functions
\[
L_0 = 1, \quad L_0, \quad L_1, \quad L_{0001}, \quad L_{01001}.
\]

3.9. The fundamental groupoid of \(\mathbb{P}^1 \setminus \{0,1,\infty\}\). We continue studying the manifold \(M = \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}\), but we view it as the set of complex points of the variety \(X = \mathbb{P}^1(\mathbb{Q}) \setminus \{0,1,\infty\}\) defined over \(\mathbb{Q}\). Recall from Example 2.175 that the dg-algebra \(\hat{A}^*\) computes the algebraic de Rham cohomology of \(X\).
3.9.1. **Summary of structures.** For convenience and to fix notation, we start by summarizing some results of the previous sections.

**Summary 3.276.** Let \( x, y, z \in \{0, 1\} \cup X(\mathbb{Q}) \) be base points, tangential or not. We have at our disposal the following structures.

1. **(Betti side.)** An affine pro-algebraic scheme over \( \mathbb{Q} \)
   \[ y \Pi^B_x = \pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}; y, x)^{\text{un}}, \]
a pro-\( \mathbb{Q} \)-vector space
   \[ y U^B_x = \mathbb{Q}[\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}; y, x)]^\wedge, \]
the subspace of Lie-like elements
   \[ y L^B_x = \{ x \in y U^B_x \mid \nabla^\vee x = 1 \otimes x + x \otimes 1 \}, \]
and an ind-\( \mathbb{Q} \)-algebra
   \[ y A^B_x = \mathcal{O}(y \Pi^B_x) = (y U^B_x)^\vee. \]

2. **(De Rham side.)** An affine pro-algebraic scheme over \( \mathbb{Q} \)
   \[ y \Pi^{dR}_x = \text{Spec}(\mathcal{F}), \]
a pro-\( \mathbb{Q} \)-vector space
   \[ y U^{dR}_x = \mathbb{Q}<e_0, e_1>, \]
the subspace of Lie-like elements
   \[ y L^{dR}_x = \{ x \in y U^{dR}_x \mid \nabla^\vee x = 1 \otimes x + x \otimes 1 \}, \]
and an ind-\( \mathbb{Q} \)-algebra
   \[ y A^{dR}_x = \mathcal{F}. \]

3. **(Comparison.)** Comparison isomorphisms\(^6\)
   \[ \text{comp}_{\Pi, dR, B}^B : y \Pi^B_x \times_{\mathbb{Q}} \mathbb{C} \overset{\sim}{\longrightarrow} y \Pi^{dR}_x \times_{\mathbb{Q}} \mathbb{C}, \]
   \[ \text{comp}_{U, dR, B}^B : y U^B_x \hat{\otimes}_{\mathbb{Q}} \mathbb{C} \overset{\sim}{\longrightarrow} y U^{dR}_x \hat{\otimes}_{\mathbb{Q}} \mathbb{C}, \]
   \[ \text{comp}_{L, dR, B}^B : y L^B_x \hat{\otimes}_{\mathbb{Q}} \mathbb{C} \overset{\sim}{\longrightarrow} y L^{dR}_x \hat{\otimes}_{\mathbb{Q}} \mathbb{C}, \]
   \[ \text{comp}_{A, dR}^B : y A^B_x \otimes_{\mathbb{Q}} \mathbb{C} \overset{\sim}{\longrightarrow} y A^{dR}_x \otimes_{\mathbb{Q}} \mathbb{C}. \]

Observe that the de Rham side on Summary 3.276 is independent of the base points. In fact, there is a canonical de Rham path \( y^{dR}_x \) in \( y \Pi^{dR}_x \) (it is the unit element in the group scheme \( \text{Spec}(\mathcal{F}) \)) and corresponds to the kernel of the counit \( \varepsilon : \mathcal{F} \rightarrow \mathbb{Q} \). Since the pro-algebraic scheme \( y \Pi^{dR}_x \) is independent of the base points, we will suppress them from the notation and we will write \( \Pi^{dR} = \text{Spec} \mathcal{F} \).

\(^6\)Recall that \( y \Pi^B_x \) and \( y \Pi^{dR}_x \) are affine schemes over \( \mathbb{Q} \). Below, the notation \( \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\mathbb{C}) \) is a shorthand for \( \times \text{Spec}(\mathbb{Q}) \text{Spec}(\mathbb{C}) \).
Moreover, for $\pi = B, dR$, the pro-algebraic schemes come together with morphisms
\[ \pi^y_\pi \times \pi^y_\pi \longrightarrow \pi^y_\pi \] (3.277)
induced from the composition of paths on the Betti side and the coproduct of $\pi$ on the de Rham side. These maps turn $\pi^y_\pi$ into a pro-unipotent group scheme and $\pi^y_\pi$ into a right $\pi^y_\pi$ torsor and a left $\pi^y_\pi$ torsor.

Therefore, the pro-$\mathbb{Q}$-vector spaces come equipped with the following structures:

1. A composition of paths
   \[ \Delta^\vee : \pi^y_\pi \otimes \pi^y_\pi \longrightarrow \pi^y_\pi ; \]
2. Units
   \[ \eta^\vee : \mathbb{Q} \longrightarrow \pi^y_\pi ; \]
3. A completed coproduct
   \[ \nabla^\vee : \pi^y_\pi \longrightarrow \pi^y_\pi \otimes \pi^y_\pi ; \]
4. Counits
   \[ \epsilon^\vee : \pi^y_\pi \longrightarrow \mathbb{Q} ; \]
5. A dual antipode
   \[ S^\vee : \pi^y_\pi \longrightarrow \pi^y_\pi . \]

And the ind-algebras $\pi^y_\pi$ come equipped with the dual structures.

All the comparison isomorphisms $\text{comp}_{B, dR}^\pi$ are given by the regularized iterated integrals. For instance
\[ \text{comp}_{dR, B}^U : \pi^y_\pi \longrightarrow \pi^dR_\pi \]
agrees with the map $L$ of Theorem 3.265. Moreover they are compatible with all the structures: The group and torsor structures on $\pi$, the product, unit, coproduct, counit and antipode for $A$ and the completed coproduct, counits, the composition of paths, units and the dual antipode for $U$.

It is immediate to extend the construction of Section 3.6.2 to tangential base points. Therefore the spaces $\pi^y_\pi$ and $\pi^A_x$ come equipped with a weight filtration $W$ and the spaces $\pi^dR_\pi$ and $\pi^A_x$ with a weight filtration $W$ and a Hodge filtration $F$ in such a way that

\[ \pi^A_H = ((\pi^A_\pi, W), (\pi^dR_\pi, W, F), \text{comp}_{B, dR}^\pi) \]
is in ind-$\text{MHS}(\mathbb{Q})$ and
\[ \pi^U_H = ((\pi^U_\pi, W), (\pi^dR_\pi, W, F), \text{comp}_{dR, B}^{-1}) \]
is in pro-$\text{MHS}(\mathbb{Q})$. 
The filtrations in \( yU^1 \) induce a weight filtration on \( yL^B \) and weight and Hodge filtrations on \( yL^dR \), so that
\[
yL^H_x = ((yL^B_x, W), (yL^dR_x, W, F), \text{comp}^{-1}_{dR, B})
\]
is also in \( \text{pro-MHS}(\mathbb{Q}) \).

Moreover, it is easy to check that all the previous structures of \( yA^H \) are morphisms of \( \text{ind-MHS}(\mathbb{Q}) \) and the corresponding structures of \( yU^H \) are morphisms of \( \text{pro-MHS}(\mathbb{Q}) \).

**Variant 3.278.** The same structures are available for other varieties. For instance, everything can be easily generalized to any variety of the form \( X' = \mathbb{P}^1 \setminus S \) for \( S \subset \mathbb{P}^1(\mathbb{Q}) \) a finite set. In this case we will use the notation \( y\Pi(X')_x^B \) for the pro-algebraic scheme in the Betti side and similar notation for the other structures. In the sequel, we will only need the case \( X' = \mathbb{G}_m \).

In this particular case we have
\[
y\Pi(G_m)_x^{dR} = \mathbb{A}^1_x = \mathbb{G}_a
\]
\[
yA(G_m)_x^{dR} = \mathbb{Q}[x_0]
\]
\[
yU(G_m)_x^{dR} = \mathbb{Q}[e_0]
\]
\[
yL(G_m)_x^{dR} = \mathbb{Q}e_0
\]
and the map
\[
\text{comp}^{U}_{dR, B} : yU(G_m)_x^B \longrightarrow yU(G_m)_x^{dR}
\]
will also be denoted by \( L \).

**3.9.2. The fundamental groupoid and the local monodromy.** From now on, we focus our attention on the pro-unipotent group picture \( y\Pi^U_x \). The reader will have no difficulty writing the analogous statements for \( U^?, L^? \) and \( A^? \).

**Definition 3.279.** The diagram consisting of the four schemes \( y\Pi^U_x \), \( x, y \in \{0, 1\} \) with the composition of paths will be called the *tangential fundamental groupoid* of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \). It is represented schematically in Figure 20.

To the tangential fundamental groupoid we want to add the *local monodromy* around 0 and 1.

We start with the local monodromy around 0 in the de Rham side. There is a morphism of Hopf algebras \( \mathfrak{H} \to \mathbb{Q}[x] \) that sends any word containing \( x_1 \) to zero, and \( x_0, \ldots, x_0 \) to \( x^n/n! \). We can see this as a map
\[
0^A^{dR}_0 \longrightarrow 0^A(G_m)_0^{dR}
\]
that induces maps
\[
\mathbb{G}_a = 0\Pi(G_m)_0^{dR} \longrightarrow 0\Pi_0^{dR}, \text{ and } 0U(G_m)_0^{dR} \longrightarrow 0U_0^{dR}.
\]
The local monodromy around 0 in the Betti side is obtained topologically as follows. Let $\Delta^*$ be a small punctured disc around zero in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. The local monodromy is the composition of the inverse of the isomorphism

$$\pi_1(\Delta^*, 0)^{un} \to \pi_1(\mathbb{G}_m, 0)^{un} = \mathbb{G}_a$$

with the natural map

$$\pi_1(\Delta^*, 0)^{un} \to \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, 0)^{un}.$$  

Similarly, the de Rham side of the local monodromy around 1 is induced by the map of Hopf algebras $\mathbb{F} \to \mathbb{Q}[x]$ that sends any word containing $x_0$ to zero, and $x_1.n.x_1$ to $x^n/n!$. While the Betti side is obtained from a small punctured disc around 1.

The local monodromy maps are morphisms of ind-$\text{MHS}(\mathbb{Q})$ in the case of $A$ and morphism of pro-$\text{MHS}(\mathbb{Q})$ in the case of $U$. This means that the pair of maps

$$0A_0^{\text{dR}} \to 0A(\mathbb{G}_m)^{\text{dR}}, \quad 0A_0^{B} \to 0A(\mathbb{G}_m)^{B}$$

is a morphism of ind-$\text{MHS}(\mathbb{Q})$

$$0A_0^{H} \to 0A(\mathbb{G}_m)^{H},$$

while the pair of maps

$$0U(\mathbb{G}_m)^{\text{dR}}_0 \to 0U_0^{\text{dR}}, \quad 0U(\mathbb{G}_m)^{B}_0 \to 0U_0^{B}$$

is a morphism of pro-$\text{MHS}(\mathbb{Q})$

$$0U(\mathbb{G}_m)^{H}_0 \to 0U_0^{H},$$

and the same is true for the local monodromy maps around 1.

**Definition 3.280.** We will denote by $D^{\text{dR}}$ the diagram consisting of the four schemes $y\Pi_x^{\text{dR}}, x, y \in \{0, 1\}$, the morphisms given by the composition of paths, the scheme $\mathbb{G}_a$ and the two local monodromies

$$\mathbb{G}_a \to 0\Pi_0^{\text{dR}}, \quad \mathbb{G}_a \to 1\Pi_1^{\text{dR}}.$$
Similarly, we write $D^\text{dR}_U$, $D^\text{dR}_A$ for the corresponding diagram for the vector spaces $U$ and the algebras $A$. Similarly we will denote $D^\text{B}$ for the corresponding diagrams on the Betti side. Finally we will denote $D^\text{H}_U$ for the pair of diagrams $D^\text{B}_U$ and $D^\text{dR}_U$ together as a diagram of pro-$\text{MHS}(\mathbb{Q})$.

We will see in chapter 4.5 that the diagram $D^\text{H}_U$ is “motivic”.

3.9.3. The automorphisms of $D^\text{dR}$. We denote by $\text{Aut}(D^\text{dR})$ the group of automorphisms of $D^\text{dR}$ in the following sense: to give an element of $\text{Aut}(D^\text{dR})$ amounts to giving an automorphism of pro-algebraic schemes of each $\text{y}_x^{\text{dR}}$ and an automorphism of $\mathbb{G}_a$ that are compatible with the composition of paths (3.277) and the local monodromy maps. The group $\text{Aut}(D^\text{dR})$ is a pro-algebraic group.

We denote by $\text{Aut}^0(D^\text{dR})$ the subgroup of $\text{Aut}(D^\text{dR})$ that acts as the identity on $\mathbb{G}_a$. There is an exact sequence

$$0 \to \text{Aut}^0(D^\text{dR}) \to \text{Aut}(D^\text{dR}) \to \mathbb{G}_m \to 0.$$ 

**Lemma 3.281.** There is an isomorphism of schemes

$$\text{Aut}^0(D^\text{dR}) \to \text{y}_x^{\text{dR}},$$

where $\gamma_f$ is determined by the equation

$$f(1^{\text{dR}}_0) = 1^{\text{dR}}_0 \cdot \gamma_f.$$

**Proof.** Recall that the dual of $\mathcal{F}$, that agrees with the completed universal enveloping algebra of Lie($\Pi^{\text{dR}}_0$), is the algebra $\mathbb{Q}[[e_0, e_1]]$. Let $R$ be a $\mathbb{Q}$-algebra. The elements of $\Pi^{\text{dR}}_0(R)$ are the group-like elements of $R[[e_0, e_1]]$. Moreover we have identities

$$1^{\text{dR}}_1(R) = 1^{\text{dR}}_0\cdot 0^{\text{dR}}_1(R),$$

$$0^{\text{dR}}_1(R) = 0^{\text{dR}}_0(R) \cdot 1^{\text{dR}}_1,$$

$$1^{\text{dR}}_1(R) = 1^{\text{dR}}_0 \cdot 0^{\text{dR}}_0(R) \cdot 0^{\text{dR}}_1.$$ 

(3.282)

Let $f \in \text{Aut}^0(D^\text{dR})(R)$. Since $f$ is the identity in $\mathbb{G}_a$ we deduce that

$$f(\exp(0)) = \exp(0),$$

$$f(1^{\text{dR}}_0 \cdot \exp(1) \cdot 0^{\text{dR}}_1) = 1^{\text{dR}}_0 \cdot \exp(1) \cdot 0^{\text{dR}}_1.$$ 

We also have $f(0^{\text{dR}}_0) = 0^{\text{dR}}_0$ and $0^{\text{dR}}_1 \cdot 1^{\text{dR}}_0 = 0^{\text{dR}}_0$. Therefore the fact that $f$ is compatible with the composition of paths implies that it is determined by the image of $1^{\text{dR}}_0$. We write

$$f(1^{\text{dR}}_0) = 1^{\text{dR}}_0 \cdot \gamma_f$$

for an element $\gamma_f \in 0^{\text{dR}}_0(R) \subset R[[e_0, e_1]]$.

Conversely, let $\gamma \in 0^{\text{dR}}_0(R) = \text{Spec}(\mathcal{F})(R)$. It is a group-like element of the algebra $R[[e_0, e_1]]$. To give an element of $\text{Aut}(0^{\text{dR}}_0(R))$ is equivalent
to give a continuous automorphism of $R\langle e_0, e_1 \rangle$ that is compatible with the completed coproduct and the antipode. We define
\[ f_\gamma(e_0) = e_0, \quad f_\gamma(e_1) = \gamma^{-1} \cdot e_1 \cdot \gamma. \]
This determines a continuous automorphism of $R\langle e_0, e_1 \rangle$. To show that it is compatible with the completed coproduct, it is enough to check it for the generator $e_1$. On the one hand,
\[ f_\gamma(\nabla^\vee(e_1)) = f_\gamma(1 \otimes e_1 + e_1 \otimes 1) = 1 \otimes (\gamma^{-1} \cdot e_1 \cdot \gamma) + (\gamma^{-1} \cdot e_1 \cdot \gamma) \otimes 1. \]
On the other hand, using that $\gamma$ is group-like,
\[ \nabla^\vee(f_\gamma(e_1)) = \nabla^\vee(\gamma^{-1} \cdot e_1 \cdot \gamma) = \gamma^{-1} \otimes \gamma^{-1} \cdot (1 \otimes e_1 + e_1 \otimes 1) \cdot \gamma \otimes \gamma = 1 \otimes (\gamma^{-1} \cdot e_1 \cdot \gamma) + (\gamma^{-1} \cdot e_1 \cdot \gamma) \otimes 1. \]
The fact that $f_\gamma$ is compatible with the dual antipode follows from the fact that, by Lemma 3.116, since $\gamma$ is group-like, then $S^\vee(\gamma) = \gamma^{-1}$.

In consequence $f_\gamma$ determines an element of $\text{Aut}(\mathcal{D}_{\Pi_0^d}(\mathcal{F}))(\mathcal{R})$ that we also denote $f_\gamma$. Writing
\[ f_\gamma(1_{10}^{d_{10}}) = 1_{10}^{d_{10}} \cdot \gamma, \quad f_\gamma(01_1^{d_{11}}) = \gamma^{-1} \cdot 01_1^{d_{11}} \]
and using the identities (3.282), we obtain $\mathcal{R}$-automorphisms of the four schemes $\mathcal{Y}_{x,y} : x, y \in \{0, 1\}$. By construction, these automorphisms are compatible with the composition of paths. Moreover they are compatible with the identity automorphism of $\mathbb{G}_a$ through any of the two local monodromies. Thus we obtain an element $f_\gamma \in \text{Aut}^0(\mathcal{D}_{\Pi}^d)(\mathcal{R})$.

Clearly, the assignments $f \mapsto \gamma_f$ and $\gamma \mapsto f_\gamma$ are inverse to each other, and this concludes the proof of the lemma. \[ \square \]

### 3.9.4. A new product structure.

The isomorphism of schemes of Lemma 3.281 is not a morphism of groups. Therefore, it induces a new group structure on $\text{Spec}(\mathcal{F})$.

**Definition 3.283.** We denote by $(\Pi, \circ)$ the scheme $\Pi = \text{Spec}(\mathcal{F})$ with the product structure induced by the isomorphism of Lemma 3.281.

As schemes $\Pi = \Pi_{\circ}^{d_{\Pi}} = \mathcal{Q}_{\circ}^{d_{\mathcal{Q}}}$ but the product structure is different. Therefore we obtain a new Lie bracket on the Lie algebra of $\Pi$ which is still the set of Lie-like elements of $\mathcal{Q}\langle e_0, e_1 \rangle$ that is called the *Ihara bracket* and a new coproduct on $\mathcal{F} = \mathcal{Q}\langle x_0, x_1 \rangle$ that is called the *Goncharov coproduct*. We now make all these structures explicit.

We start by computing the new product structure of $\Pi$ that we denote by $\circ$. This product is determined by the equation
\[ f_\gamma(f_\mu(1_{10}^{d_{10}})) = 1_{10}^{d_{10}} \cdot (\gamma \circ \mu). \]
For a group-like element $\gamma$, we will denote by $\langle \gamma \rangle_0$ the restriction of $f_\gamma$ to $\Pi^\text{dR}_0$ and also the corresponding continuous automorphism of $\mathbb{Q}\langle e_0, e_1 \rangle$. Recall that it is given by
\[
\langle \gamma \rangle_0(e_0) = e_0, \quad \langle \gamma \rangle_0(e_1) = \gamma^{-1} \cdot e_1 \cdot \gamma.
\]

Since $f_\gamma$ is compatible with the composition of paths, then
\[
f_\gamma(f_\mu(1^\text{dR}_0 \cdot \mu)) = f_\gamma(1^\text{dR}_0 \cdot \mu) = 1^\text{dR}_0 \cdot \gamma \cdot \langle \gamma \rangle_0(\mu),
\]
and hence
\[
\gamma \circ \mu = \gamma \cdot \langle \gamma \rangle_0(\mu).
\]

\[3.284\]

3.9.5. The Ihara bracket. We now compute the new bracket induced in the set of Lie-like elements of $\mathbb{Q}\langle e_0, e_1 \rangle$. Recall the notion of derivation from Definition 2.31. Given a Lie-like element $x \in \mathbb{Q}\langle e_0, e_1 \rangle$, we define a continuous derivation $\partial_x : \mathbb{Q}\langle e_0, e_1 \rangle \to \mathbb{Q}\langle e_0, e_1 \rangle$ as follows:
\[
\partial_x(y) = \frac{d}{dt}\left(\langle \exp(tx) \rangle_0(y)\right)_{|t=0}.
\]
Explicitly, this derivation is determined by
\[
\partial_x e_0 = 0, \quad \partial_x e_1 = -x \cdot e_1 + e_1 \cdot x.
\]

Let now $x$ and $y$ be two Lie-like elements of $\mathbb{Q}\langle e_0, e_1 \rangle$. We denote by $[x, y] = x \cdot y - y \cdot x$ the Lie bracket corresponding to the composition of paths. The Lie bracket induced by $\circ$ will be denoted by $\{x, y\}$. It is determined by
\[
\{x, y\} = \frac{d}{du} \frac{d}{dv} \left(\exp(ux) \circ \exp(vy) \circ \exp(-ux) \circ \exp(-vy)\right)_{|u=0, v=0}.
\]
Explicitly, it is given by
\[
\{x, y\} = [x, y] + \partial_x y - \partial_y x.
\]

\[3.285\]

3.9.6. Goncharov coproduct. Let us now turn to the computation of the new coproduct on the algebra $\mathcal{H} = \mathbb{Q}\langle x_0, x_1 \rangle$.

Following Notation 3.254, if $\alpha$ is a binary sequence, then $x_\alpha \in \mathcal{H}$ is the corresponding word in the alphabet $\{x_0, x_1\}$, while $e_\alpha \in \mathcal{H}$ is the corresponding word in the alphabet $\{e_0, e_1\}$. As a function $x_\alpha \in \mathcal{H} = \mathcal{O}(\Pi)$, the word $x_\alpha$ sends a group-like element of $\mathbb{Q}\langle e_0, e_1 \rangle$ to the coefficient of the word $e_\alpha$.

Recall that, by Lemma 3.116, the dual antipode of a group-like element $\gamma$ is given by $S^\vee(\gamma) = \gamma^{-1}$, while for a word $w = e_{\varepsilon_0} \ldots e_{\varepsilon_n}$ the dual antipode is given in Example 3.67 by
\[
S^\vee(w) = w^* = (-1)^n e_{\varepsilon_n} \ldots e_{\varepsilon_1}.
\]

We deduce that, if $\gamma = \sum_w \gamma_w w$ is a group-like element, then
\[
\gamma^{-1} = \sum_w \gamma_w w^*.
\]

\[3.286\]
The Goncharov coproduct, denoted by $\Delta^\Gamma$, is the coproduct induced in $\mathcal{H}$ by $\circ$ and is determined by the equation
\[
\Delta^\Gamma(x)(\gamma \otimes \mu) = x(\gamma \circ \mu) = x(\gamma \cdot \langle \gamma \rangle_0(\mu)).
\] (3.287)

Note that the product $\circ$ can be defined, for a group-like element $\gamma$ and an arbitrary element $e \in \mathbb{Q}(\langle e_0, e_1 \rangle)$ by
\[
\gamma \circ e = \gamma \cdot \langle \gamma \rangle_0(e).
\] (3.288)

This product is linear in the variable $e$. In particular, for a word $w$ in the alphabet $\{e_0, e_1\}$, the product $\gamma \circ w$ is described as follows:

1. if the word starts with $e_0$ add $\gamma$ at the beginning, while if the word starts with $e_1$ add nothing at the beginning;
2. if the word ends with $e_1$ add $\gamma$ at the end, while if the word ends with $e_0$ add nothing at the end;
3. between $e_0$ and $e_1$ insert $\gamma^{-1}$ and between $e_1$ and $e_0$ insert $\gamma$;
4. between two consecutive occurrences of $e_0$ or two consecutive occurrences of $e_1$ insert nothing.

For instance
\[
\gamma \circ (e_0 e_0 e_1 e_0 e_1 e_1) = \gamma e_0 e_0 \gamma^{-1} e_1 \gamma e_0 \gamma^{-1} e_1 e_1 \gamma.
\]

To give a more compact description of this product we introduce the following notation
\[
1\gamma_0 = \gamma, \ 0\gamma_1 = \gamma^{-1}, \ 0\gamma_0 = 1, \ 1\gamma_1 = 1.
\]

For a binary sequence $\alpha = (\varepsilon_1, \ldots, \varepsilon_n)$, we have
\[
\gamma \circ e_\alpha = 1\gamma_{\varepsilon_1} \cdots e_{\varepsilon_1} \cdots e_{\varepsilon_n} \cdots e_{\varepsilon_1} \cdots e_{\varepsilon_n} \gamma_0.
\] (3.289)

Given the shape (3.289) of the product $\circ$ and the inversion formula (3.286), for any binary sequence $\alpha$, we introduce the following symbols
\[
I(1; \alpha; 0) = x_\alpha, \quad I(0; \alpha; 1) = x^*_\alpha,
\]
\[
I(0; \alpha; 0) = I(1; \alpha; 1) = 1, \quad \text{if } \alpha = \emptyset, \quad I(0; \alpha; 0) = I(1; \alpha; 1) = 0, \quad \text{if } \alpha \neq \emptyset.
\] (3.290)

All of them are elements of $\mathcal{H}$, and hence functions on $\Pi$. Then, for a binary sequence $\alpha$, a group-like element $\gamma \in \Pi(\mathbb{Q})$ and elements $\varepsilon, \varepsilon' \in \{0, 1\}$, we have the duality
\[
x_\alpha(\varepsilon' \gamma \varepsilon) = I(\varepsilon'; \alpha; \varepsilon)(\gamma).
\] (3.291)

Armed with this notation, we can compute Goncharov’s coproduct. Let $\alpha$ be a binary sequence and $\gamma, \mu$ group-like elements of $\mathbb{Q}(\langle e_0, e_1 \rangle)$. Write
\[ \mu = \sum_w \mu_w w. \] Then, by equation (3.287),
\[ (\Delta^\Gamma x_\alpha)(\gamma \otimes \mu) = x_\alpha(\gamma \circ \mu) \]
\[ = x_\alpha(\sum_w \mu_w \gamma \circ w) \]
\[ = x_\alpha\left( \sum_w \mu_w (1 \varepsilon_{\varepsilon_1(w)} \cdot \varepsilon_{\varepsilon_1(w)} \cdots \varepsilon_{\varepsilon_{\mathrm{wt}(w)}(w)} \gamma_0) \right), \]
where \( \mathrm{wt}(w) \) denotes the weight of \( w \) as in Definition 1.124, and \( \varepsilon_i(w) \) is defined to be 0 or 1 depending whether the \( i \)-th letter appearing in \( w \) is \( \varepsilon_0 \) or \( \varepsilon_1 \). Let us write \( \alpha = \varepsilon_1 \cdots \varepsilon_n \) and set \( \varepsilon_0 = 1 \) and \( \varepsilon_{n+1} = 0 \).

We need to compute the coefficient of the word \( \varepsilon_\alpha \) in the above bracketed expression. We will get a contribution for each subword of \( \varepsilon_\alpha \) corresponding to a binary subsequence \( \varepsilon_{i_1} \cdots \varepsilon_{i_k} \) of \( \alpha \). It is easy to see that the coefficient we are looking for is given by:

\[ \sum_{0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n+1} I(\varepsilon_0; \varepsilon_{i_0} \cdots \varepsilon_{i_k}; \varepsilon_{n+1})(\mu) \prod_{p=0}^k I(\varepsilon_{i_{p+1}}; \varepsilon_{i_{p+1}} \cdots \varepsilon_{i_{p+1}}; \varepsilon_{i_{p+1}})(\gamma). \]

The upshot of these computations is the following result, which was first obtained by Goncharov [Gon05, Thm. 1.2].

**Proposition 3.292.** Let \( \varepsilon_0 \cdots \varepsilon_{n+1} \) be a binary sequence. The isomorphism of Lemma 3.281 induces, by transport of structure, the following co-product on the algebra \( \mathfrak{g} \):
\[ \Delta^\Gamma I(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_n; \varepsilon_{n+1}) = \]
\[ \sum_{0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n+1} \prod_{p=0}^k I(\varepsilon_{i_{p+1}}; \varepsilon_{i_{p+1}} \cdots \varepsilon_{i_{p+1}}; \varepsilon_{i_{p+1}}) \otimes I(\varepsilon_0; \varepsilon_{i_1} \cdots \varepsilon_{i_k}; \varepsilon_{n+1}). \]

**Proof.** The case \( \varepsilon_0 = 1 \) and \( \varepsilon_{n+1} = 0 \) was settled above. The other cases follow immediately from (3.290). \( \square \)

**Example 3.293.** If \( n = 1 \),
\[ \Delta^\Gamma I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \]
\[ = I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \otimes I(\varepsilon_0; \varepsilon_2) + I(\varepsilon_0; \varepsilon_1)I(\varepsilon_1; \varepsilon_2) \otimes I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \]
\[ = I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \otimes 1 + 1 \otimes I(\varepsilon_0; \varepsilon_1; \varepsilon_2), \]

since \( I(\varepsilon'; \varepsilon) \) is always equal to 1 regardless of the values of \( \varepsilon \) and \( \varepsilon' \).

**Example 3.294.** If \( n = 2 \), we get contributions from \( k = 0, 1, 2 \). As before, \( k = 0 \) corresponds to the choice of the empty subsequence and gives the value \( I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \otimes 1 \), whereas \( k = 2 \) represents the choice of the whole sequence and contributes with \( 1 \otimes I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \). For \( k = 1 \) we obtain two terms, corresponding to \( i_1 = 1 \) and \( i_1 = 2 \). In both cases, the product
contains only one non-trivial factor ($p = 1$ if $i_1 = 1$ and $p = 0$ if $i_1 = 2$). Putting everything together,

$$
\Delta^\Gamma I(\varepsilon_0; \varepsilon_1 \varepsilon_2; \varepsilon_3) = I(\varepsilon_0; \varepsilon_1 \varepsilon_2; \varepsilon_3) \otimes 1 \\
+ I(\varepsilon_1; \varepsilon_2; \varepsilon_3) \otimes I(\varepsilon_0; \varepsilon_1; \varepsilon_3) \\
+ I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \otimes I(\varepsilon_0; \varepsilon_2; \varepsilon_3) \\
+ 1 \otimes I(\varepsilon_0; \varepsilon_1 \varepsilon_2; \varepsilon_3).
$$

Specializing formula (3.295) to the cases $(1; 1, 0; 0)$ and $(1; 0, 1; 0)$ we get

$$
\Delta^\Gamma (x_0 x_1) = x_0 x_1 \otimes 1 + x_0 \otimes x_1 + x_1 \otimes x_0 + 1 \otimes x_0 x_1, \\
\Delta^\Gamma (x_1 x_0) = x_1 x_0 \otimes 1 + 1 \otimes x_1 x_0.
$$

Just for fun, let us verify the compatibility with shuffle product. On the one hand,

$$
\Delta^\Gamma (x_0 \shuffle x_1) = \Delta^\Gamma (x_0 x_1 + x_1 x_0) \\
= (x_0 x_1 + x_1 x_0) \otimes 1 + 1 \otimes (x_0 x_1 + x_1 x_0) + x_0 \otimes x_1 + x_1 \otimes x_0.
$$

On the other hand,

$$
(\Delta^\Gamma x_1) \shuffle (\Delta^\Gamma x_2) = (1 \otimes x_0 + x_0 \otimes 1) \shuffle (1 \otimes x_1 + x_1 \otimes 1) \\
= 1 \otimes (x_0 \shuffle x_1) + x_0 \otimes x_1 + x_0 \otimes x_1 + (x_0 \shuffle x_1) \otimes 1,
$$

and we see that the expressions are equal.

As the examples show, the formula for Goncharov’s coproduct in Proposition 3.292 contains many trivial factors. Later in Chapter 5 we will give a linearization which is more suitable for computation.

**Exercise 3.296.** Prove formula 3.285.

**Exercise 3.297.** Calculate the number of terms appearing in Goncharov’s coproduct.
4. Mixed Tate motives

The goal of this chapter is to give a precise meaning to the statement that the diagram $\mathcal{D}^{HU}$ of Definition 3.280 has motivic origin. The theory of motives has been a very active area of research in the last decades. This is a rather abstract theory and it is remarkable that, up until today, the only proof that we have for the upper bound of the dimension of the space of multiple zeta values of a given weight uses the theory of motives. A proper treatment of the theory of motives falls outside the scope of this notes. We will use the theory of motives as a black-box and we will limit ourselves to give an idea of its origin and the properties that we will use. The interested reader is referred to the book [And04] and the references therein.

4.1. Tannakian formalism. The link between mixed Tate motives and multiple zeta values is made through the group of symmetries of mixed Tate motives. To make this idea precise we need the formalism of Tannakian categories that we summarize in this section.

The Galois group of a field extension is one of the basic tools in arithmetic and one of its more studied objects. In topology, the fundamental group of a topological space is the analogue of the absolute Galois group of a field and is one of the basic invariants of a topological space. Fueled by the utility of the Galois and fundamental groups it is natural to seek for analogues in other situations. The Tannakian formalism is the basic tool to define analogues of the Galois group in many algebro-geometric situations. The origin of this formalism is Pontryagin duality, according to which a locally compact abelian group is characterized by its character group, and the Tannaka-Krein duality that states that we can recover a compact Lie group from the category of its continuous finite-dimensional real representations. Grothendieck extended the Tannaka-Krein duality to affine algebraic groups. Saavedra-Rivano [SR72] encoded the properties of the category of finite dimensional linear representations of an algebraic group in the concept of a Tannakian category. Thus, by construction, the category of finite dimensional linear representations of an algebraic group is a Tannakian category. The fundamental result of the theory of Tannakian categories is the converse: every Tannakian category is isomorphic to the category of finite dimensional linear representations of a pro-algebraic group.

Note that the formalism of Tannakian categories is tailored to the study of affine group schemes. Thus we will not recover the “true” fundamental group of a topological space nor the Galois group of a field extension with this formalism, but only its so called pro-algebraic envelope.

We will follow the exposition in [DM82] to which the reader is referred for further details. Another nice reference is Chapter 6 of [Sza09]. Through this section we fix a field $k$ (of any characteristic), that will play the role of field of coefficients.
4.1.1. Tensor categories. The definition of Tannakian category gathers the characteristic properties of finite-dimensional $k$-linear representations of affine group schemes. First of all, since morphisms between $k$-linear representations form a vector space, we need the concept of a $k$-linear category.

**Definition 4.1.** A $k$-linear category $\mathcal{C}$ is an additive category such that, for each pair of objects $X,Y \in \text{Ob}(\mathcal{C})$, the group $\text{Hom}_\mathcal{C}(X,Y)$ is a $k$-vector space and the composition maps are bilinear.

The tensor product of two representations is again a representation. Therefore, a Tannakian category should have a tensor product, which is a bilinear functor with some additional properties.

**Definition 4.2.** Let $\mathcal{C}$ be a $k$-linear category, together with a bilinear functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(a) An **associativity constraint** for $(\mathcal{C}, \otimes)$ is a natural transformation

$$\phi = \phi \cdot \cdot : \cdot \otimes (\cdot \otimes \cdot) \longrightarrow (\cdot \otimes \cdot) \otimes \cdot$$

such that the following two conditions hold:

1. For all $X, Y, Z \in \text{Ob}(\mathcal{C})$, the map $\phi_{X,Y,Z}$ is an isomorphism.
2. (Pentagon axiom) For all $X, Y, Z, T \in \text{Ob}(\mathcal{C})$, the following diagram commutes:

\[
\begin{array}{ccc}
X \otimes (Y \otimes (Z \otimes T)) & \xrightarrow{\text{Id} \otimes \phi_{Y,Z,T}} & X \otimes ((Y \otimes Z) \otimes T) \\
\downarrow \phi_{X,Y,Z,T} & & \downarrow \phi_{X,Y,Z,T} \\
(X \otimes (Y \otimes Z)) \otimes T & \xrightarrow{\phi_{X,Y,Z,T} \otimes \text{Id}} & ((X \otimes Y) \otimes Z) \otimes T \\
\end{array}
\]

(b) A **commutativity constraint** is a natural transformation

$$\psi = \psi \cdot \ast : \cdot \otimes \ast \longrightarrow \ast \otimes \cdot$$

such that, for all $X, Y \in \text{Ob}(\mathcal{C})$, the map $\psi_{X,Y}$ is an isomorphism, and the following composition is the identity:

$$\psi_{Y,X} \circ \psi_{X,Y} : X \otimes Y \longrightarrow X \otimes Y.$$

(c) An associativity and a commutativity constrain are said to be **compatible** if, for all objects $X, Y, Z \in \text{Ob}(\mathcal{C})$, the following diagram commutes
(hexagon axiom):

\[
\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\phi_{X,Y,Z}} & (X \otimes Y) \otimes Z \\
\downarrow \phi_{X,Z,Y} & & \downarrow \psi_{X,Y,Z} \\
X \otimes (Z \otimes Y) & \xrightarrow{\psi_{X,Z,Y}} & Z \otimes (X \otimes Y). \\
\end{array}
\]

(d) Finally, we say that a pair \((U,u)\) consisting of an object \(U\) of \(\mathcal{C}\) and an isomorphism \(u: U \to U \otimes U\) is an identity object if the functor \(X \mapsto U \otimes X\) is an equivalence of categories.

We now have all the ingredients to define one of the underlying structures of Tannakian categories.

**Definition 4.3.** A \(k\)-linear tensor category is a tuple \((\mathcal{C}, \otimes, \phi, \psi)\) consisting of a \(k\)-linear category \(\mathcal{C}\), a bilinear functor \(\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}\), and compatible associativity and commutativity constraints \(\phi\) and \(\psi\), such that \(\mathcal{C}\) contains an identity object.

The constraints \(\phi\) and \(\psi\) are usually omitted from the notation and one simply denotes a \(k\)-linear tensor category by \((\mathcal{C}, \otimes)\).

**Remark 4.4.** Two identity objects are canonically isomorphic. From now on, we will fix one and denote it by \((1, e)\).

**Definition 4.5.** An object \(L\) in \(\mathcal{C}\) is called invertible if the functor \(X \mapsto L \otimes X\) is an equivalence of categories.

One easily shows that an object \(L\) is invertible if and only if there exists an object \(L'\) such that \(L \otimes L' \cong 1\). Then \(L'\) is also invertible.

4.1.2. Rigid categories. The set of \(k\)-linear maps between two representations is again a representation and, in particular, a representation on a vector space induces a representation on the dual vector space. Thus a Tannakian category should contain internal Hom’s and duals.

Let \((\mathcal{C}, \otimes)\) be a tensor category and let \(X, Y \in \text{Ob}(\mathcal{C})\). We say that the functor \(T \mapsto \text{Hom}(T \otimes X, Y)\) is representable if there exist an object \(Z \in \text{Ob}(\mathcal{C})\) such that there are functorial isomorphisms

\[
\text{Hom}(T, Z) \to \text{Hom}(T \otimes X, Y) \tag{4.6}
\]

for all \(T \in \text{Ob}(\mathcal{C})\). If this is the case, we denote \(Z\) by \(\text{Hom}(X, Y)\) and we call it internal Hom between the objects \(X\) and \(Y\). Note that any two such objects \(Z\) are related by a unique compatible isomorphism.
Taking $T = \Hom(X, Y)$ in (4.6), the image of the identity $\Id_{\Hom(X, Y)}$ is a morphism which will be denoted by

$$\ev_{X, Y} : \Hom(X, Y) \otimes X \rightarrow Y.$$ 

The dual of an object $X$ is defined as $X^\vee = \Hom(X, 1)$. If $X^\vee$ and $(X^\vee)^\vee$ exist, there is a natural morphism $X \mapsto (X^\vee)^\vee$. We say that $X$ is reflexive if this morphism is an isomorphism.

**Example 4.7.** In the category of groups, $\mathbb{Z}/2$ is not reflexive since its dual is 0. In the category of vector spaces, finite-dimensional vector spaces are reflexive, whereas infinite dimensional ones are not.

**Definition 4.8.** A $k$-linear tensor category is said to be rigid if

1. $\Hom(X, Y)$ exists for all $X, Y \in \Ob(C)$;
2. for all $X_1, X_2, Y_1, Y_2 \in \Ob(C)$, the natural morphism
   $$\Hom(X_1, Y_1) \otimes \Hom(X_2, Y_2) \rightarrow \Hom(X_1 \otimes X_2, Y_1 \otimes Y_2)$$
   is an isomorphism;
3. all objects of $C$ are reflexive.

4.1.3. **Neutral Tannakian categories.** The category of finite-dimensional $k$-linear representations $\Rep_k(G)$ of an algebraic group $G$ over $k$ has other relevant properties. First, it is an abelian category. Second, the one-dimensional representation given by the vector space $k$ with trivial $G$-action is an identity object $\bf 1$ that satisfies $\End(\bf 1) = k$. Finally, the forgetful functor from $\Rep_k(G)$ to the category of finite-dimensional vector spaces $\Vec_k$ that consists in forgetting the action of $G$ is exact, faithful and compatible with the tensor structure on both categories. These will turn out to be all the necessary ingredients to identify the categories of finite-dimensional representations of algebraic groups.

**Definition 4.9.** A neutral Tannakian category over $k$ is a rigid $k$-linear abelian tensor category $C$ such that $\End(\bf 1) = k$ and that there exists an exact faithful $k$-linear tensor functor $\omega : C \rightarrow \Vec_k$. Any such functor is called a fibre functor.

Since we shall never consider non-neutral Tannakian categories in the sequel, we will just refer to them as “Tannakian categories”.

**Examples 4.10.**

1. The category $\Vec_k$ of finite-dimensional vector spaces over $k$, together with the identity functor, is a Tannakian category.
2. Let $\GrVec_k$ be the category of finite-dimensional graded vector spaces over $k$. The objects are finite-dimensional $k$-vector spaces $V$ together with a direct sum decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$, and the
morphism are homogeneous $k$-linear maps. The tensor structure comes from the tensor product of vector spaces, graded by
\[(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j.\]
The forgetful functor $\omega: \text{GrVec}_k \to \text{Vec}_k$ sending $(V, (V_n)_{n \in \mathbb{Z}})$ to $V$ makes $\text{GrVec}_k$ into a Tannakian category.

(3) Let $G$ be any abstract group and $\text{Rep}_k(G)$ the category of finite-dimensional $k$-linear representations of $G$. Let
\[
\omega: \text{Rep}_k(G) \to \text{Vec}_k
\]
be the functor that forgets the action of $G$. Then $\text{Rep}_k(G)$ is a Tannakian category over $k$ and $\omega$ is a fibre functor.

(4) Let $\text{MHS}(\mathbb{Q})$ be the category of mixed Hodge structures over $\mathbb{Q}$ and let $\omega_{\text{B}}$ and $\omega_{\text{dR}}$ the forgetful functors of Definition 2.128. Then $\text{MHS}(\mathbb{Q})$ is a Tannakian category over $\mathbb{Q}$ and both of the functors, $\omega_{\text{B}}$ and $\omega_{\text{dR}}$ are fibre functors.

(5) Let $M$ be a path connected, locally path connected and locally simply connected topological space. The category $\text{Loc}_k(M)$ of local systems of finite-dimensional $k$-vector spaces is a Tannakian category. For each point $x \in M$, the functor
\[
\omega_x: \text{Loc}_k(M) \longrightarrow \text{Vec}_k
\]
that sends a local system $V$ to its fibre at $x$ is a fibre functor.

4.1.4. The fundamental group of a Tannakian category. Fix a Tannakian category $\mathcal{C}$ over $k$ and a fibre functor $\omega$.

**Definition 4.11.** For every $k$-algebra $R$, let $\text{Aut}^{\otimes}(\omega)(R)$ denote the set of families $(\lambda_X)_{X \in \text{Ob}(\mathcal{C})}$ of $R$-linear automorphisms
\[
\lambda_X: \omega(X) \otimes R \longrightarrow \omega(X) \otimes R
\]
such that the following diagrams are commutative:

\[
\begin{align*}
\omega(X_1 \otimes X_2) \otimes R & \xrightarrow{\lambda_{X_1 \otimes X_2}} \omega(X_1 \otimes X_2) \otimes R \\
\omega(X_1) \otimes \omega(X_2) \otimes R & \xrightarrow{\lambda_{X_1} \otimes \lambda_{X_2}} (\omega(X_1) \otimes R) \otimes_R (\omega(X_2) \otimes R),
\end{align*}
\]
\(\omega(1) \otimes R \xrightarrow{\lambda_1} \omega(1) \otimes R\)

\[\begin{array}{ccc}
R & \xrightarrow{\text{Id}} & R \\
\downarrow & & \downarrow \\
\end{array}\]

(2) for every morphism \(\alpha \in \text{Hom}_C(X,Y)\),

\[\begin{array}{ccc}
\omega(X) \otimes R & \xrightarrow{\lambda_X} & \omega(X) \otimes R \\
\downarrow & & \downarrow \\
\omega(Y) \otimes R & \xrightarrow{\lambda_Y} & \omega(Y) \otimes R \\
\end{array}\]

In the above diagrams, all unlabeled tensor products of vector spaces are over \(k\) and the unlabeled arrows are the obvious isomorphisms.

In particular, we define \(\text{Aut}^\otimes(\omega) = \text{Aut}^\otimes(\omega)(k)\). This is the group of \(k\)-linear automorphisms of the functor \(\omega\).

The main theorem of the theory of Tannakian categories is

**Theorem 4.12.** [DM82, Theorem 2.11] Let \(C\) be a Tannakian category over \(k\), together with a fibre functor \(\omega\). Then

1. the functor \(R \mapsto \text{Aut}^\otimes(\omega)(R)\) is representable by an affine group scheme over \(k\) that we denote \(\text{Aut}^\otimes(\omega)\);

2. for every \(X \in \text{Ob}(C)\), the group \(\text{Aut}^\otimes(\omega)\) acts naturally on \(\omega(X)\) and the functor \(C \to \text{Rep}_k(\text{Aut}^\otimes(\omega))\) sending \(X\) to the vector space \(\omega(X)\) with this action of \(\text{Aut}^\otimes(\omega)\) is an equivalence of categories.

**Definition 4.13.** The affine group scheme \(\text{Aut}^\otimes(\omega)\) is called the Tannaka group of \((C,\omega)\). Whenever we want to stress the category we are considering, we will write \(\text{Aut}_C^\otimes(\omega)\).

Given a second fibre functor \(\omega'\), the functor from \(k\)-algebras to sets

\[R \mapsto \text{Isom}^\otimes(\omega, \omega')\]

is representable by an affine scheme which is a right torsor under \(\text{Aut}^\otimes(\omega)\) and a left torsor under \(\text{Aut}^\otimes(\omega')\), see [DM82, Theorem 3.2].

4.1.5. **Matrix coefficients.** Instead of proving Theorem 4.12, we will content ourselves with a description of the Hopf algebra of the Tannaka group using the notion of matrix coefficients from [Del90, §4.7], see also [Bro17] and compare with the notion of framed objects from [BGSV90]. We will assume that all the categories appearing here are small.
Definition 4.14. Let $\mathcal{C}$ be a neutral Tannakian category over $k$, together with two fibre functors $\omega_1$ and $\omega_2$. A matrix coefficient in $(\mathcal{C}, \omega_1, \omega_2)$ is a triple

$$(X, v, f)$$

consisting of an object $X$ of $\mathcal{C}$, an element $v \in \omega_1(X)$, and an element $f \in \omega_2(X)^\vee = \text{Hom}(\omega(X), k)$.

Let $H$ be the $k$-vector space generated by all matrix coefficients, and $V \subseteq H$ the subspace spanned by

1. (bilinearity relations) for every pair of matrix coefficients $(X, v_1, f)$ and $(X, v_2, f)$, and elements $\lambda, \mu \in k$, the relation

$$(X, \lambda v_1 + \mu v_2, f) - \lambda(X, v_1, f) - \mu(X, v_2, f) \in V.$$ 

Similarly, for every pair of matrix coefficients of the form $(X, v, f_1)$ and $(X, v, f_2)$, and elements $\lambda, \mu \in k$, the relation

$$(X, v, \lambda f_1 + \mu f_2) - \lambda(X, v, f_1) - \mu(X, v, f_2) \in V;$$

2. (compatibility relations) for every pair of objects $X, X'$, every morphism $\phi \in \text{Hom}_\mathcal{C}(X, X')$, and $v \in \omega(X)$ and elements $f' \in \omega(X')^\vee$, the relation

$$(X, v, \omega(\phi)^\vee f') - (X', \omega(\phi)v, f') \in V.$$ 

We set $A = H/V$ and write $[X, v, f]$ for the class in $A$ of a matrix coefficient $(X, v, f)$. The vector space $A$ comes with the following structures:

1. Product: The tensor structure of $\mathcal{C}$ induces the product

$$[X, v, f] \cdot [X', v', f'] = [X \otimes X', v \otimes v', f \otimes f'].$$

The associativity and commutativity constraints together with the compatibility relation imply that this product is associative and commutative.

2. Unit: Let $1$ be an identity object. Then $\omega(1) \simeq k$. Choose any $v \in \omega(1) \setminus \{0\}$ and let $f \in \omega(1)^\vee$ be its dual, so that $f(v) = 1$. Then $[1, v, f]$ is a unit for the product. By the bilinearity relations, this class does not depend on the choice of $v$.

3. Counit: The counit is the map $A \to k$ given by $[X, v, f] \mapsto f(v)$.

4. Coproduct: The coproduct is modeled on the Hopf algebra of $\text{GL}_n$ (see Example 3.56). Given an object $X \in \mathcal{C}$, we choose a basis $(e_1, \ldots, e_n)$ of $\omega(X)$. If $(e_1^*, \ldots, e_n^*)$ is the dual basis, then

$$\Delta[X, v, f] = \sum_{j=1}^n [X, v, e_j^*] \otimes [X, e_j, f]. \quad (4.15)$$

One checks that (4.15) does not depend on the choice of the basis.
(5) **Antipode:** Finally, the rigidity of $\mathcal{C}$ allows us to define an antipode. If we identify $\omega(X^\vee)$ with $\omega(X)^\vee$, then

$$S([X, v, f]) = [X^\vee, f, v].$$

It is an easy verification to prove the following:

**Proposition 4.16.** Together with the above structures, $A$ is a commutative Hopf $k$-algebra.

Taking Theorem 4.12 for granted, we can show that $A$ is the Hopf algebra of the Tannaka group $G = \underline{\text{Aut}}^\otimes(\omega)$. More precisely,

**Proposition 4.17.** The map $\varphi: A \to \mathcal{O}(G)$ given by $\varphi([X, v, f])(\lambda) = f(\lambda X(v))$ is an isomorphism of Hopf algebras.

**Proof.** We leave it to the reader to check that $\varphi$ is a morphism of Hopf algebras. By Theorem 4.12, $\mathcal{C}$ is equivalent to the category $\text{Rep}_k(G)$ of finite-dimensional $k$-representations of $G$, and we can identify $\omega$ with the forgetful functor $\text{Rep}_k(G) \to \text{Vec}_k$.

We first prove that $\varphi$ is surjective. Note that there is a left group action of $G$ on $\mathcal{O}(G)$ given by $(\lambda h)(\mu) = h(\mu \lambda)$. By Lemma 3.98, $\mathcal{O}(G)$ is the union of its finite-dimensional subrepresentations. In other words, given $h \in \mathcal{O}(G)$, there exists a finite-dimensional subrepresentation $(V, \rho)$ of $\mathcal{O}(G)$ containing $h$. It determines an object $X$ of $\mathcal{C}$ such that $h$ belongs to $\omega(X) = V$. Let $f \in V^\vee$ be the element given by $f(u) = u(e)$, where $e$ is the unit of $G$ and $u \in V \subseteq \mathcal{O}(G)$. Then, for each element $\lambda \in G$, we have

$$[X, h, f](\lambda) = f(\lambda h) = (\lambda h)(e) = h(e \lambda) = h(\lambda).$$

Therefore, $\varphi([X, h, f]) = h$ and $\varphi$ is surjective.

We next prove injectivity. Assume that $\varphi([X, v, f]) = 0$. We identify $X$ with a finite-dimensional representation $(V, \rho)$ of $G$ such that $v \in V$. Let $V'$ be the simple subrepresentation of $V$ containing $v$. Then $V'$ is generated by elements of the form $\lambda v$ for $\lambda \in G$. Since $\varphi([X, v, f]) = 0$, we deduce that $f|_{V'} = 0$. Let $X'$ be the object of $\mathcal{C}$ corresponding to $(V', \rho)$. By the compatibility relation, the equality

$$[X, v, f] = [X', v, f|_{V'}] = [X', v, 0] = 0$$

holds, thus completing the proof. □
Example 4.18. Let \((\text{GrVec}_k, \omega)\) be the Tannakian category of finite-dimensional graded vector spaces from Example 4.10. It is equivalent to the semisimple category generated by objects \(k_n, n \in \mathbb{Z}\) with

\[
\text{Hom}(k_n, k_m) = \begin{cases} k, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases} \quad k_n \otimes k_m = k_{n+m}, \quad \omega(k_n) \simeq k.
\]

For each \(n\) choose a non-zero element \(u_n \in \omega(k_n)\) and let \(u_n^\vee \in \omega(k_n)^\vee\) be the element defined by \(u_n^\vee(u_n) = 1\). Then every matrix coefficient in \(T\) can be written as a linear combination of the elements

\([k_n, u_n, u_n^\vee], \quad n \in \mathbb{Z}.
\]

Moreover,

\([k_n, u_n, u_n^\vee] \cdot [k_m, u_m, u_m^\vee] = [k_{n+m}, u_{n+m}, u_{n+m}^\vee].
\]

Thus, if we write \(t = [k_1, u_1, u_1^\vee]\), there is an isomorphism of algebras

\(O(\text{Aut}_\otimes(\omega)) = k[t, t^{-1}].
\)

Moreover, the coproduct, the counit and the antipode are given by

\(\Delta t = t \otimes t, \quad \epsilon(t) = \epsilon(t^{-1}) = 1, \quad S(t) = t^{-1}.
\)

From part (2) of Example 3.56, we deduce that \(\text{Aut}_\otimes(\omega) = \mathbb{G}_m\), the multiplicative group. It is a general fact that the presence of a grading is related to an action of \(\mathbb{G}_m\).

Example 4.19. Consider the subgroup of \(\text{GL}_2(\mathbb{R})\) given by:

\(\{(x, y) \in \text{GL}_2(\mathbb{R}) \mid x^2 + y^2 > 0\}.
\)

These are the real points of an affine algebraic group \(S\) over \(\mathbb{R}\) called the Deligne torus. Alternatively, one can define it as

\(S = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m),
\)

where \(\text{Res}_{\mathbb{C}/\mathbb{R}}\) is the Weil restriction functor. This means that, if \(A\) is an \(\mathbb{R}\)-algebra, then \(S(A) = \mathbb{G}_m(A \otimes_{\mathbb{R}} \mathbb{C}) = (A \otimes_{\mathbb{R}} \mathbb{C})^\times\). The category of representations of \(S\) is equivalent to the category of split \(\mathbb{R}\)-mixed Hodge structures.

4.1.6. Tannakian subcategories. Let \(Y\) be an object of a neutral Tannakian category \(C\), we denote by \(\langle Y \rangle\) the full subcategory of \(C\) that contains \(Y\) and is stable by sums, tensor products, dual and subquotients. Then \(\langle Y \rangle\), together with the restriction of any fibre functor \(\omega\) on \(C\) is again a neutral Tannakian category. The action of \(G = \text{Aut}_\otimes(\omega)\) on the vector space \(\omega(Y)\) induces a map \(G \to \text{GL}(\omega(Y))\). The following is shown in the proof of [DM82, Proposition 2.8]

Lemma 4.20. The image \(G^Y \subset \text{GL}(\omega(Y))\) of \(G\) by the above map is a closed subgroup of \(\text{GL}(\omega(Y))\) which agrees with the Tannaka group \(\text{Aut}_\otimes(\omega)\) of the subcategory \(\langle Y \rangle\).
We can order the subcategories of the form $⟨Y⟩$ for $Y$ an object of $C$ by inclusion. With this order they form a directed system. If $⟨Y⟩ ⊂ ⟨Z⟩$, then $Y ∈ ⟨Z⟩$ and $⟨Z⟩ = ⟨Z ⊕ Y⟩$. The projection $Z ⊕ Y → Y$ induces a map

$$\text{Aut}_{⊗}^\otimes (ω) = \text{Aut}_{⊗}^\otimes (ω) \rightarrow \text{Aut}_{⊗}^\otimes (ω).$$

The following lemma exhibits the pro-algebraic nature of $G$.

**Lemma 4.21.** Let $(C, ω)$ be a neutral Tannakian category. Then:

$$\text{Aut}_{⊗}^\otimes (ω) = \text{lim}_{←} \text{Aut}_{⊗}^\otimes (ω) = \text{lim}_{←} G^Y.$$

**Proof.** By Lemma 4.20, there is a surjection $G → G^Y$ for every object $Y$ of $C$. These surjections are compatible with the maps $G^Z = G^Z ⊕ Y → G^Y$ induced by an inclusion $⟨Y⟩ ⊂ ⟨Z⟩$. Therefore, there is a surjection

$$G \rightarrow \text{lim}_{←} G^Y.$$

This map is also injective, because if an element of $G$ is sent to the unit, then it acts trivially on $ω(Y)$ for every object $Y$ and is thus the unit of $G$. □

### 4.1.7. Tannakian categories and the fundamental group.

We next explore what can be recovered from the classical fundamental group of a topological space using the Tannakian formalism. This includes the pro-unipotent completion.

Let $M$ be a path connected, locally path connected and locally simply connected topological space. Let $x_0$ be a point of $M$ and $π_1(M, x_0)$ the fundamental group of $M$ with base point $x_0$. By part (5) of Example 4.10, the category $\text{Loc}_k(M)$ of local systems of finite dimensional $k$-vector spaces over $M$ is a Tannakian category with fibre functor $ω_{x_0}$. Given a local system $V$, the fibre at $x_0$ is a $k$-vector space with an action of $π_1(M, x_0)$. This yields the so-called monodromy representation

$$ρ_V : π_1(M, x_0) → \text{GL}(ω_{x_0}(V)).$$

It follows that $\text{Loc}_k(M)$ is equivalent to the category of finite-dimensional $k$-linear representations of $π_1(M, x_0)$. However, since the fundamental group is not an affine group scheme, it cannot be the Tannaka group of the category $\text{Loc}_k(M)$. In fact, as we will see, the Tannaka group $\text{Aut}_{⊗}^\otimes (ω_{x_0})$ is the pro-algebraic completion of $π_1(M, x_0)$.

For shorthand, write $Γ = π_1(M, x_0)$. Following Lemma 4.21, we can give the following description of the pro-algebraic completion of $Γ$. Let $Y = (V, ρ)$ be a $k$-linear finite-dimensional representation of $Γ$. The group $G^Y$ from Lemma 4.21 is the Zariski closure $ρ(Γ)^Zar$ of the image of $ρ: Γ → \text{GL}(V)$. Let $Y' = (V', ρ')$ be another representation with $⟨Y'⟩ ⊂ ⟨Y⟩$. By Lemma 4.20,
there is a restriction map \( \rho(\Gamma)_{\text{Zar}} \to \rho'(\Gamma)_{\text{Zar}} \). The pro-algebraic completion is the projective limit

\[
\Gamma^{\text{alg}} = \lim_{\langle (V, \rho) \rangle} \rho(\Gamma)_{\text{Zar}},
\]

where the limit is taken with respect to the subcategories \( \langle (V, \rho) \rangle \) ordered by inclusion.

Similarly, we can recover the pro-unipotent completion of \( \Gamma \) using the Tannakian formalism. A local system is called unipotent if its monodromy representation is unipotent (Definition 3.103). The category of unipotent local systems \( \text{ULoc}_k(X) \) on \( X \) is again a Tannakian category and \( \omega_{x_0} \) is again a fibre functor. In this case, the Tannaka group \( \text{Aut}^\otimes(\omega_{x_0}) \) is the pro-unipotent completion of \( \Gamma \). It admits a similar description as the pro-algebraic completion but restricting to finite-dimensional unipotent representations:

\[
\Gamma^{\text{un}} = \lim_{\langle (V, \rho) \rangle} \text{unip.} \rho(\Gamma)_{\text{Zar}},
\]

where the limit again is taken with respect to the subcategories \( \langle (V, \rho) \rangle \) ordered by inclusion.

\[\star\star\star\]

Exercise 4.22. Prove that

\[ [X \oplus Y, u \oplus v, f \oplus g] = [X, u, f] + [Y, v, g]. \]

Exercise 4.23. Consider the unit circle \( S^1 \) as a topological space. Its fundamental group is \( \pi_1(S^1, 1) \simeq \mathbb{Z} \). Prove that the pro-algebraic completion \( \mathbb{Z}^{\text{alg}} \) is infinite-dimensional, while

\[ \mathbb{Z}^{\text{un}} \simeq \mathbb{G}_a, \]

the additive group. For the second part use that to give a unipotent representation of \( \mathbb{Z} \) is equivalent to give a finite-dimensional vector space \( V \) together with a unipotent endomorphism of \( V \) and the fibre functor is just the forgetful functor. Then use the explicit description of the Hopf algebra of the Tannaka group.

Exercise 4.24. Consider the Tannakian category \( \text{Vec}_k \) with the identity as the fibre functor \( \omega \). Prove that \( \text{Aut}^\otimes(\omega) = \text{Spec}(k) \), the trivial group.

Exercise 4.25 (The pro-algebraic completion of a group). Let \( k \) be a field and \( \Gamma \) an abstract group. In this exercise, we present three equivalent constructions of the pro-algebraic completion of \( \Gamma \), which is an affine group scheme \( G = \Gamma^{\text{alg}} \) over \( k \) together with a group morphism \( \Gamma \to G(k) \).

(a) Let \( \mathcal{C} \) be the category of finite-dimensional \( k \)-linear representations of \( \Gamma \). Equipped with the forgetful functor, it is a Tannakian category, and one defines \( G \) as its fundamental group. A \( k \)-point of
$G$ is thus a collection $(\lambda_V)_{V \in \text{Ob}(\mathcal{C})}$ of automorphisms $\lambda_V : V \to V$ satisfying the constraints of Definition 4.11. To each element $\gamma \in \Gamma$ one associates the collection of automorphisms $\lambda^\gamma = (\lambda^\gamma_V)_V$ defined as $\lambda^\gamma_V(v) = \lambda \cdot \gamma$. This yields the map $\Gamma \to G(k)$.

(b) Consider the collection of pairs $(H, \varphi_H)$ consisting of an affine group scheme $H$ over $k$ and a group morphism $\varphi_H : \Gamma \to H(k)$ with Zariski dense image. We define a partial order by setting $(H, \varphi_H) \leq (H', \varphi_{H'})$ whenever there exists a morphism $f : H \to H'$ such that the induced map of $k$-points commutes with $\varphi_H$ and $\varphi_{H'}$ and we define the pro-algebraic completion $G$ as the limit:

$$G = \lim_{\leftarrow} H.$$

(c) The pro-algebraic completion $G$ is an affine group scheme over $k$ with a group morphism $\varphi : \Gamma \to G(k)$ such that, for any affine group scheme $H$ over $k$ and any group morphism $\varphi_H : \Gamma \to H(k)$, there exists a unique morphism $f : G \to H$ such that $f \circ \varphi = \varphi_H$.

Prove that the three constructions give the same pro-algebraic group.

4.2. Voevodsky’s category of motives.

4.2.1. A universal cohomology. Different cohomology theories have been proved useful in the study of algebraic varieties. For instance, as we saw in Chapter 2, to any variety $X$ over a subfield $k$ of $\mathbb{C}$, it is attached the Betti cohomology

$$H^*_B(X) = H^*(X(\mathbb{C}), \mathbb{Q}),$$

which is a finite-dimensional graded $\mathbb{Q}$-vector space. If, in addition, $X$ is smooth, one has also at disposal the de Rham cohomology

$$H^*_\text{dR}(X) = H^*(X, \Omega^*_X),$$

which is now a finite-dimensional graded $k$-vector space. Recall from Theorem 2.88 that both cohomologies are related, after complexification, by the period isomorphism

$$H^*_\text{dR}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H^*_B(X) \otimes_{\mathbb{Q}} \mathbb{C}. \quad (4.26)$$

Another important example is $\ell$-adic cohomology defined, for a variety $X$ over a field $k$ of arbitrary characteristic $p$, a choice of a separable closure $\bar{k}$ of $k$, and a prime number $\ell$ different from $p$, by

$$H^*_\ell(X) = \lim_{\leftarrow} H^*_\ell(X_{\bar{k}}, \mathbb{Z}/\ell^n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$  

When $\bar{k}$ is embeddable into $\mathbb{C}$, Artin proved that there exists a canonical isomorphism

$$H^*_\ell(X) \simeq H^*_B(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \quad (4.27)$$
All the cohomology theories we have mentioned satisfy similar properties, such as homotopy invariance, Poincaré duality, Künneth formulas, Mayer–Vietoris exact sequences etc. A fundamental feature is that the corresponding vector spaces usually come together with extra structures. We have already seen that Betti cohomology can be provided with a mixed Hodge structure, and $\ell$-adic cohomology carries a continuous $\mathbb{Q}_\ell$-linear action of the Galois group $\text{Gal}(\bar{k}/k)$.

The similarities between different cohomology theories, as well as the existence of comparison isomorphisms such as (4.26) or (4.27), led Grothendieck to postulate the existence of a universal cohomology theory which factors all the others: this should be the motive of the variety. Since its introduction by Grothendieck, the theory of motives has inspired a wealth of research but, although we have advanced a lot in our understanding, many fundamental questions remain still unanswered.

Restricting to the case of smooth projective varieties, Grothendieck constructed a category of pure motives over a field $k$ with some of the desired properties. However, in order to prove that it has all of them, he stated a set of conjectures, the standard conjectures, that have proved to be very difficult and seem to be still out of reach. Nevertheless some of the sought properties of the category of pure motives, like the fact that the category of motives modulo numerical equivalence is semi-simple [Jan92], have been proved without the use of the standard conjectures.

The terminology “pure” comes from the fact that for any smooth projective variety, its $n$-th cohomology group always has certain properties that are encoded in the statement “$H^n(X)$ is of pure weight $n$”. For instance, if $X$ is a smooth projective complex variety, the group $H^n_B(X, \mathbb{C})$ has a Hodge decomposition

$$H^n_B(X) \otimes \mathbb{Q} \cong \bigoplus_{p+q=n} H^{p,q}(X).$$

The fact that only factors with $p+q = n$ appear means that its Hodge structure is pure of weight $n$. For varieties over a finite field, the corresponding purity is reflected by the fact that the eigenvalues of the action of Frobenius on étale cohomology have absolute value $q^{n/2}$.

Using resolution of singularities, we can express the cohomology of a singular quasi-projective variety in terms of the cohomology of smooth projective varieties, but in this expression cohomologies of different degrees get mixed. As we have seen in Section 2.5.2 this gives rise to a mixed Hodge structure in the cohomology of $X$. Thus, the motive of a smooth projective variety should be pure while the motive of a singular or quasi-projective variety should be mixed. Since Grothendieck, there has been a great effort to develop a theory of mixed motives.

Abstractly we can think of a cohomology theory in the following way. Fix a field $k$, denote by $\var{Var}_k$ the category of varieties over $k$, and let $\mathcal{A}$ be
an abelian category (or more precisely a Tannakian category). Denote by $\mathcal{D}A$ the derived category of $A$. Then $\mathcal{D}A$ is a triangulated category provided with a $t$-structure (see Section A.9 for a definition) that allows us to recover $A$ from $\mathcal{D}A$. A cohomology theory (with values in $A$) is a contravariant functor

$$H: \text{Var}_k \rightarrow \mathcal{D}A$$

satisfying certain properties. We can recover the “cohomology groups” of $X$ from $H(X)$ using the $t$-structure:

$$H^n(X) = t_{\leq n} t_{\geq n} H(X) \in A.$$ 

Voevodsky was able to define a triangulated category $\mathbf{DM}_{gm}(k)$, which is a candidate for the derived category of mixed motives over $k$. The main missing piece is a suitable “motivic” $t$-structure. Recently, Beilinson [Bei12] showed that, when $k$ has characteristic zero, the existence of such motivic $t$-structure implies the standard conjectures. Conversely, Hanamura proved in [Han99] that, over any field $k$, the conjunction of the standard conjectures and conjectures by Murre and Beilinson-Soulé implies the existence of the motivic $t$-structure. Thus we are back to Grothendieck insight that to have a full theory of motives we need to prove the standard conjectures.

4.2.2. The derived category of mixed motives. Let $k$ be a field. In what follows, we give a sketch of Voevodsky’s construction of a derived category of mixed motives over $k$ with rational coefficients, which will be denoted by $\mathbf{DM}(k) = \mathbf{DM}_{gm}(k)_\mathbb{Q}$.

For more details we refer the reader to the original paper [Voe00], the lecture notes [MVW06] or part II of the introductory book [And04].

We start with the category $\text{Sm}(k)$ of smooth varieties over $k$. This category is not additive, for it does not make sense to “sum” two morphisms of schemes. The first step of the construction will be to enlarge the set of morphisms through the notion of finite correspondence.

4.2.3. First step: the category of finite correspondences.

**Definition 4.28.** Let $X$ and $Y$ be objects of $\text{Sm}(k)$. A finite correspondence from $X$ to $Y$ is a $\mathbb{Z}$-linear combination of integral closed subschemes $W \subseteq X \times Y$ such that the projection $W \to X$ is finite and surjective over a connected component of $X$.

Finite correspondences form an abelian subgroup of the group of algebraic cycles $Z^{\dim Y}(X \times Y)$, which will be denoted by $c(X, Y)$.

**Example 4.29.** Given any morphism of schemes $f: X \to Y$, the graph $\Gamma_f \subseteq X \times Y$ is a finite correspondence. In general, we can think of finite correspondences as multivalued maps on a connected component of $X$. 


Given $X, Y, Z \in \text{Sm}(k)$, we will denote by $p_{XY}, p_{XZ}$ and $p_{YZ}$ the projections from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$ respectively.

**Lemma 4.30.** Let $X, Y, Z$ be objects in $\text{Sm}(k)$. Consider finite correspondences $W \in c(X, Y)$ and $W' \in c(Y, Z)$. Then the cycles $p_{XY}^*(W)$ and $p_{YZ}^*(W')$ intersect properly on $X \times Y \times Z$. Moreover, the projection of the cycle $p_{XZ}(p_{XY}^* \alpha \cdot p_{YZ}^* \beta)$ is finite over $X$ and surjective over a connected component.

Thanks to the above lemma, we can define the composition

$$\circ : c(X, Y) \times c(Y, Z) \to c(X, Z)$$

by

$$\alpha \circ \beta = p_{XZ}(p_{XY}^* \alpha \cdot p_{YZ}^* \beta). \quad (4.31)$$

The category $\text{SmCor}(k)$ has the same objects as $\text{Sm}(k)$, but the morphisms are given by finite correspondences with $\mathbb{Q}$-coefficients:

$$\text{Hom}_{\text{SmCor}(k)}(X, Y) = c(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$  

There is a functor $\text{Sm}(k) \to \text{SmCor}(k)$ that is the identity on objects and sends a map $f : X \to Y$ to its graph $\Gamma_f$. By Exercise 4.52, the composition of maps is compatible with the composition (4.31) of finite correspondences. We denote by $[X]$ the image in $\text{SmCor}(k)$ of a smooth variety $X$.

The direct sum in $\text{SmCor}(k)$ is given by the disjoint union of varieties. This category is also equipped with the tensor product

$$[X] \otimes [Y] = [X \times_k Y].$$

4.2.4. **Second step: A triangulated category with homotopy invariance and Mayer–Vietoris.** The second step is similar to the construction of the derived category of an abelian category. We start with the category

$$C^b(\text{SmCor}(k))$$

of bounded chain complexes in $\text{SmCor}(k)$. The objects are diagrams

$$\cdots \to [X_n] \xrightarrow{\partial_n} [X_{n-1}] \to \cdots,$$

where $X_i$ is in $\text{Sm}(k)$ and $\partial_n \in c(X_n, X_{n-1}) \otimes \mathbb{Q}$ are finite correspondences such that $\partial_{n-1} \circ \partial_n = 0$. Then we define the homotopy category $K^b(\text{SmCor}(k))$ as the one having the same objects as $C^b(\text{SmCor}(k))$, and morphisms given by homotopy classes of morphisms of complexes.

Two examples of objects of $K^b(\text{SmCor}(k))$ are:

1. (homotopy complex) for any $X$ in $\text{Sm}(k)$, the complex

$$[X \times \mathbb{A}^1] \xrightarrow{pr} [X]$$

placed in degrees 1 and 0.
(2) (Mayer–Vietoris complex) for any $X$ in $\mathbf{Sm}(k)$ and any open cover $X = U \cup V$, the complex

$$[U \cap V] \xrightarrow{i_{U \cap V,U} + i_{U \cap V,V}} [U] \oplus [V] \xrightarrow{i_{U,X} - i_{V,X}} [X],$$

where $[U \cap V]$ sits in degree 2, and the arrows $i_{U,X}, i_{V,X}, i_{U \cap V,U}$ and $i_{U \cap V,V}$ are the obvious inclusions.

We want to force the homotopy invariance and the Mayer–Vietoris property, which mean that the above two complexes are acyclic. To this end, we take the quotient of $K^b(\text{SmCor}(k))$ by the thick triangulated subcategory generated by all homotopy and Mayer–Vietoris complexes. It has the structure of a triangulated category.

4.2.5. Third step: The pseudo-abelian envelope. The next step is to take the pseudo-abelian envelope of the quotient obtained in the previous step. The resulting category is denoted by $DM^{\text{eff}}_{gm}(k)$.

Recall the construction of the pseudo-abelian envelope

**Definition 4.32.** Let $\mathcal{C}$ be an additive category. The pseudo-abelian envelope of $\mathcal{C}$ is the category with

- **objects:** $(X, p)$ where $X$ is an object of $\mathcal{C}$ and $p \in \text{Hom}_\mathcal{C}(X, X)$ is an idempotent, that is, $p^2 = p$.

- **morphisms:** $\text{Hom}((X, p), (Y, q)) \subseteq \text{Hom}_\mathcal{C}(X, Y)$ is the subgroup of those $f$ such that $f = q \circ f \circ p$.

There is a fully faithful functor $\mathcal{C} \rightarrow \mathcal{C}_{pa}$ sending $X$ to $(X, \text{id})$. Passing to the pseudo-abelian envelope allows us to consider the kernel of each idempotent $p: X \rightarrow X$ as a subobject of $X$. This will be crucial when we want to talk about “pieces of the cohomology”.

**Remark 4.33.** By a result of Balmer and Schlichting [BS01], the pseudo-abelian envelope of a triangulated category remains triangulated. Thus, $DM^{\text{eff}}_{gm}(k)$ is still a triangulated category.

We have a functor $M: \mathbf{Sm}(k) \rightarrow DM^{\text{eff}}_{gm}(k)$ sending $X$ to $[X]$, regarded as a complex concentrated in degree zero. The category $DM^{\text{eff}}_{gm}(k)$ is also equipped with a tensor product that is characterized by the property

$$M(X) \otimes M(Y) = M(X \times Y).$$

The unit object is the motive of the base field, which will be denoted by $Q(0) = M(\text{Spec}(k))$.

Note also that there is a functor

$$C^b(\text{SmCor}(k)_{pa}) \rightarrow DM^{\text{eff}}_{gm}(k)$$

from the category of bounded complexes in the pseudo-abelian envelope of $\text{SmCor}(k)$ to the category of effective motives $DM^{\text{eff}}_{gm}(k)$. 

(4.34)
4.2.6. **Fourth step: inversion of the Tate motive.** Given \( X \) in \( \text{Sm}(k) \), let \( X \to \text{Spec}(k) \) denote the structural morphism. We can think of it as a complex sitting in degrees 0 and \(-1\):

\[
[X] \longrightarrow [\text{Spec } k].
\]

**Definition 4.36.** The reduced motive of \( X \) is the object \( \widetilde{M}(X) \) of \( DM_{\text{eff}}^\text{gm}(k) \) determined by the complex (4.35).

When \( X \) has a \( k \)-rational point, there is a direct sum decomposition (see exercise 4.53)

\[
M(X) = \mathbb{Q}(0) \oplus \widetilde{M}(X).
\]

**Definition 4.37.** The Tate motive \( \mathbb{Q}(1) \) is \( \widetilde{M}(\mathbb{P}^1_k)[-2] \). For \( n \geq 0 \), one defines \( \mathbb{Q}(n) \) as \( \mathbb{Q}(1)^{\otimes n} \).

The last step of the construction of \( DM(k) \), necessary to obtain a rigid tensor category, is to formally invert the motive \( \mathbb{Q}(1) \). By this we mean the following: an object of the new category \( DM(k) \) is a pair \( (M, m) \), where \( M \) is an object of \( DM_{\text{eff}}^\text{gm}(k) \) and \( m \in \mathbb{Z} \). Morphisms are given by

\[
\text{Hom}_{DM(k)}((M, m), (N, n)) = \lim_{\longrightarrow} \text{Hom}_{DM_{\text{eff}}^\text{gm}(k)}(M \otimes \mathbb{Q}(m+r), N \otimes \mathbb{Q}(n+r)).
\]

The resulting category has the following property:

**Theorem 4.38 (Voevodsky).** *The category \( DM(k) \) is a rigid tensor \( \mathbb{Q} \)-linear triangulated category.*

**Proof.** See \([MVW06, \text{Theorem } 20.17}\). \( \square \)

4.2.7. **Properties of \( DM(k) \).** All the usual machinery to compute the homology of algebraic varieties is still available in the derived category of motives:

1. (Künneth): \( M(X \times Y) = M(X) \otimes M(Y) \).
2. (\( \mathbb{A}^1 \)-homotopy invariance): \( M(X \times \mathbb{A}^1) = M(X) \).
3. (Mayer–Vietoris): For \( X = U \cup V \) as before, there is a distinguished triangle
   \[
   M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1].
   \]
4. (Gysin) If \( Z \subset X \) is a smooth closed subscheme of codimension \( c \) of a smooth scheme \( X \), then there is a distinguished triangle
   \[
   M(X \setminus Z) \to M(X) \to M(Z)(c)[2c] \to M(X \setminus Z)[1].
   \]
(5) (Blow-ups) Let $Z \subseteq X$ be a smooth closed subscheme of a smooth scheme, $\text{Bl}_Z X$ the blow-up of $X$ along $Z$, and $E$ the exceptional divisor. Then there is a distinguished triangle

$$M(E) \to M(\text{Bl}_Z X) \oplus M(Z) \to M(X) \to M(E)[1].$$

Moreover, if $Z$ has codimension $c$ in $Z$, the triangle yields a canonical isomorphism

$$M(\text{Bl}_Z X) = M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)(i)[2i].$$

(6) (Duality) There is a duality $A \mapsto A^\vee$ that, for $X$ smooth and projective of dimension $d$, satisfies

$$M(X)^\vee = M(X)(-d)[-2d].$$

(7) (Adjunction) The duality and tensor product are related by the adjunction formulas

$$\text{Hom}(A \otimes B^\vee, C) = \text{Hom}(A, C \otimes B),$$
$$\text{Hom}(A \otimes B, C) = \text{Hom}(B, A^\vee \otimes C).$$

Remark 4.39. We observe that the functor from $\text{Sm}(k)$ to $\text{DM}(k)$ is covariant, thus it is a “homological” functor in contrast to the contravariant functor chosen by Grothendieck for pure motives that was cohomological.

Example 4.40. Let us use some of these properties to show that

$$M(\mathbb{P}^n) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2] \oplus \cdots \oplus \mathbb{Q}(n)[2n].$$

This should be compared with Example 2.122, where the cohomology of $\mathbb{P}^n$ is computed, but noting that $M(\mathbb{P}^n)$ is to be seen as the homology of $\mathbb{P}^n$.

We proceed by induction on $n$, the case $n = 1$ being reduced to the definition of $\mathbb{Q}(1)$. For $n \geq 2$, the standard closed immersion $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ satisfies $\mathbb{P}^n \setminus \mathbb{P}^{n-1} = \mathbb{A}^n$. By the Gysin property, we have the distinguished triangle

$$M(\mathbb{A}^n) \to M(\mathbb{P}^n) \to M(\mathbb{P}^{n-1})(1)[2] \to M(\mathbb{A}^n)[1]. \quad (4.41)$$

Note that $M(\mathbb{A}^n) = \mathbb{Q}(0)$, as one can prove by repeatedly applying the $\mathbb{A}^1$-homotopy property. Moreover, the composition

$$M(\mathbb{A}^n) \to M(\mathbb{P}^n) \to M(\text{Spec}(k))$$

is the identity $\mathbb{Q}(0) \to \mathbb{Q}(0)$. Thus, the triangle (4.41) is split and $M(\mathbb{P}^n) = \mathbb{Q}(0) \oplus M(\mathbb{P}^{n-1})(1)[2]$. The result follows by induction hypothesis.

Remark 4.42. To understand the different roles of the twist and the shift, it is instructive to compare the reduced motives of $\mathbb{P}^1$ and $\mathbb{G}_m$. In the first case, we have $\tilde{M}(\mathbb{P}^1) = \mathbb{Q}(1)[2]$. For the second case, one can use the...
Mayer–Vietoris triangle for the open covering $\mathbb{P}^1 = U \cup V$, with $U = \mathbb{P}^1 \setminus \{0\}$. One gets an exact triangle

$$M(\mathbb{G}_m) \longrightarrow \mathbb{Q}(0) \oplus \mathbb{Q}(0) \longrightarrow \mathbb{Q}(1)[2] \longrightarrow M(\mathbb{G}_m)[1],$$

from which it follows that $M(\mathbb{G}_m) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1]$, thus $\hat{M}(\mathbb{G}_m) = \mathbb{Q}(1)[1]$. This can be compared with the fact that, for any cohomology theory, $H^1(\mathbb{G}_m)$ and $H^2(\mathbb{P}^1)$ are isomorphic, but they lie in different degree. In particular, the Hodge structure $H^2(\mathbb{P}^1)$ is pure of weight 2 and Hodge type $(1, 1)$. The same is true for $H^1(\mathbb{G}_m)$, but, since this last group lies in degree one, we consider it as a mixed Hodge structure.

4.2.8. Motivic cohomology. Voevodsky also computed some morphism groups in the category $\text{DM}(k)$. In particular, he defined:

**Definition 4.43.** The motivic cohomology of $X$ is

$$H^n_M(X, \mathbb{Q}(p)) = \text{Hom}_{\text{DM}(k)}(M(X), \mathbb{Q}(p)[n]).$$

Using Bloch’s formula relating higher Chow groups and $K$-theory he proves ([Voe02], [Blo86], [Lev94])

**Theorem 4.44.** Given a smooth variety $X$, there is an isomorphism

$$H^n_M(X, \mathbb{Q}(p)) = (\mathbb{K}_{2n-p}(X) \otimes \mathbb{Q})^{(p)},$$

where $\mathbb{K}_*(X)$ denotes Quillen’s $K$-theory of $X$ and the index $(p)$ means the eigenspace for the Adams operations.

4.2.9. The normalization of a cosimplicial scheme. To every variety $X$, not necessarily smooth, it is attached a motive $M(X)$ in Voevodsky’s category. Using tools from homological algebra, one can construct more general motives, for instance the motive of a cosimplicial variety.

Recall that in Section A.7.2 we defined the normalized complex associated with a cosimplicial object in an abelian category. It turns out that it is enough to work in a pseudo-abelian category.

**Lemma 4.45.** Let $X^\bullet$ be a cosimplicial object in $\text{Sm}(k)$. Given integers $m > n \geq 0$, the following endomorphism in $\text{SmCor}(k)$ is idempotent:

$$p_n = (1 - \delta^0 \sigma^0)(1 - \delta^1 \sigma^1) \cdot \cdots \cdot (1 - \delta^n \sigma^n) : [X^m] \rightarrow [X^m].$$

**Proof.** We argue by induction on $n$. For $n = 0$, note that the relation $\sigma^0 \delta^0 = \text{Id}$ implies that $\delta^0 \sigma^0$ is an idempotent, and hence the same holds for $1 - \delta^0 \sigma^0$. Let us now assume that $p_{n-1}$ is idempotent. We next observe that for $i = 0, \ldots, n-1$, the face $\sigma^n$ commutes with $\delta^i \sigma^i$. Indeed, by relations (c) and (b) in (A.81),

$$\sigma^n(\delta^i \sigma^i) = \delta^i \sigma^{n-i-1} \sigma^i = (\delta^i \sigma^i) \sigma^n.$$
Moreover, relation (d) in (A.81) implies \( \sigma^n(1 - \delta^n \sigma^n) = 0 \). These two equations together imply
\[
\sigma^n(1 - \delta^0 \sigma^0) \cdots (1 - \delta^n \sigma^n) = 0. \tag{4.46}
\]
We now compute, using equation (4.46), and the induction hypothesis,
\[
p^2_n = (1 - \delta^0 \sigma^0) \cdots (1 - \delta^{n-1} \sigma^{n-1})(1 - \delta^n \sigma^n) \]
\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad = p^2_{n-1}(1 - \delta^n \sigma^n) = p_{n-1}(1 - \delta^n \sigma^n) = p_n,
\]
as we wanted to show. \( \square \)

Since \( p_n \) is idempotent, \( \text{Im}(p_n) \) is an object of the pseudo-abelian envelope of \( \text{SmCor}(k) \). By convention, we write \( p_{-1} = \text{Id} \).

**Definition 4.47.** Let \( X^\bullet \) be a cosimplicial object in \( \text{Sm}(k) \). The **normalization** of \( X^\bullet \) is the complex in \( \text{SmCor}(k)_{pa} \) given by
\[
\mathcal{N}(X^\bullet)^n = \text{Im}(p_{n-1} : [X^n] \to [X^n]),
\]
together with the differential
\[
d = \sum_{i=0}^{n+1} (-1)^i \delta^i : \mathcal{N}(X^\bullet)^n \to \mathcal{N}(X^\bullet)^{n+1}.
\]

In general, the complex \( \mathcal{N}(X^\bullet) \) is not bounded. To obtain a bounded complex, we consider the \( \text{bête} \) truncation \( \sigma_{\leq N} \mathcal{N}(X^\bullet) \), that is,
\[
\sigma_{\leq N} \mathcal{N}(X^\bullet)^n = \begin{cases}
\mathcal{N}(X^\bullet)^n & n \leq N, \\
0 & n > N.
\end{cases}
\]
This is now an element of \( C^b(\text{SmCor}(k)_{pa}) \). For each \( N \geq 0 \), applying the functor (4.34), we obtain a motive
\[
[\sigma_{\leq N} \mathcal{N}(X^\bullet)].
\]
Clearly, given integers \( M \geq N \geq 0 \), there is a morphism of complexes
\[
\sigma_{\leq M} \mathcal{N}(X^\bullet) \to \sigma_{\leq N} \mathcal{N}(X^\bullet).
\]
The system \( ([\sigma_{\leq N} \mathcal{N}(X^\bullet)])_{N \geq 0} \) is a pro-object in \( \text{DM}(k) \).

**Remark 4.48.** The advantage of using Lemma 4.45 is that it provides us with an explicit idempotent cutting out the normalized complex from the cochain complex. However, we could have also constructed it directly by abstract means, as we now explain.\(^7\) Recall that a category is said to be \textit{preadditive} if the morphism sets are abelian groups and the composition of maps is bilinear. Given a preadditive category \( \mathcal{A} \), let \( \text{Ab}(\mathcal{A}) \) denote the

\(^7\)We thank J. Ayoub for pointing this argument to us.
category of presheaves of abelian groups on $\mathcal{A}$, by which we simply mean additive contravariant functors from $\mathcal{A}$ to the category $\textbf{Ab}$ of abelian groups. Then $\textbf{Ab}(\mathcal{A})$ is an abelian category, and the Yoneda lemma ensures that the natural functor

$$h: \mathcal{A} \rightarrow \textbf{Ab}(\mathcal{A})$$

which sends $X$ to $\text{Hom}(\cdot, X)$ is fully faithful. Assume now that $\mathcal{A}$ is pseudo-abelian. If $Y'$ is a direct factor of an object of the form $h(X)$, then projecting to the complement one gets an idempotent $p$ of $h(X)$ such that $Y' = \text{Ker}(p)$. By fully-faithfulness, we can see $p$ as an idempotent of $X$, and the object $Y = \text{Ker}(p)$ in $\mathcal{A}$, determined up to unique isomorphism, satisfies $h(Y') = Y'$.

4.2.10. Hodge realization. From now on, we assume that $k$ has characteristic zero and comes with an embedding $k \hookrightarrow \mathbb{C}$. We end this section recalling the existence of the Hodge realization functor.

**Theorem 4.49.** There is a covariant functor of $\mathbb{Q}$-linear rigid tensor triangulated categories

$$R^H: \text{DM}(k) \rightarrow D^b(\text{MHS}(k)).$$

The proof of this theorem is sketched in [DG05, §1.5]. The main difficulty is the covariance of the de Rham complex for finite correspondences. A more detailed version of the argument is exposed in [Bou09].

We now give a sketch of the construction of the Hodge realization functor in the case of the motive $[\sigma_{\leq N}\mathcal{N}(X^\bullet)]$ from the previous section. Let $X^\bullet$ be a cosimplicial object in $\text{Sm}(k)$. Assume that there is an embedding $j^\bullet: X^\bullet \rightarrow \overline{X}^\bullet$ of cosimplicial smooth varieties over $k$ such that $\overline{X}^n$ is a smooth projective varieties and $D^n = \overline{X}^n \setminus X^n$ is a simple normal crossing divisor for all $n$. The Hodge realization of $[\sigma_{\leq N}\mathcal{N}(X^\bullet)]$ is constructed as follows.

1. **Betti part** $R^B$. For each $n$, let $\mathcal{C}^*(X^n(\mathbb{C}), \mathbb{Q})$ be the Godement canonical flasque resolution of the constant sheaf $\mathbb{Q}$ on the complex manifold $X^n(\mathbb{C})$ and let $j_n^\ast \mathcal{C}^*(X^n(\mathbb{C}), \mathbb{Q})$ be the complex of sheaves on $\overline{X}^n$ obtained by taking the direct image by the inclusion $j_n: X_n \hookrightarrow \overline{X}_n$. We equip this complex with the canonical
increasing filtration (see Example A.67)

\[ W_m j_n \mathcal{C}^k(X^n(\mathbb{C}), \mathbb{Q}) = \begin{cases} 
  j_n \mathcal{C}^k(X^n(\mathbb{C}), \mathbb{Q}), & \text{if } k < m, \\
  \text{Ker } d, & \text{if } k = m, \\
  0, & \text{if } k > m. 
\end{cases} \]

We construct filtered acyclic resolutions \((K_{B,n}^*, W)\) of the complex \((j_n \mathcal{C}(X^n(\mathbb{C}), \mathbb{Q}), W)\) in a functorial way. For instance using again the Godement canonical flasque resolution, this time on \(X^n\). Taking now global sections we obtain a filtered simplicial complex \((\Gamma(X^\bullet, K_{B,n}^*), W)\). Finally, taking the normalization, the truncation and the total complex of the resulting double complex we obtain a filtered complex

\[(\text{Tot } \sigma \leq_N \mathcal{N}(X^\bullet, K_{B,n}^*), W).\]

Finally, since we want the realization functor to be covariant, so we write

\[ (R^B(\sigma \leq_N \mathcal{N}X^\bullet), W) = (\text{Tot } \sigma \leq_N \mathcal{N}(X^\bullet, K_{B,n}^*)^\vee, W). \quad (4.50) \]

Here it is important to note that the normalization of simplicial and cosimplicial objects are dual of each other.

(2) de Rham part \(R^{dR}\). For each \(n\), let \(\Omega^*_{X^n}(\log D^n)\) be the de Rham complex of algebraic forms on \(X^n\) with logarithmic poles along \(D^n\). This complex has a decreasing Hodge filtration \(F\) that counts the number of differentials and an increasing weight filtration \(W\) that counts the number of poles of a differential form. We construct an acyclic bifiltered resolution \((K_{dR}^*, F, W)\) again in a functorial way. We now repeat the process done in the Betti case: we take global sections, the normalization and truncation on the simplicial direction, the total complex and the dual to obtain a bifiltered complex

\[ (R^{dR}(\sigma \leq_N \mathcal{N}X^\bullet), F, W) = ((\text{Tot } \sigma \leq_N \mathcal{N}(X^\bullet, K_{dR}^*))^\vee, F, W). \quad (4.51) \]

(3) The comparison isomorphism. Going to the cosimplicial complex manifold \(X^\bullet(\mathbb{C})\), we can construct a bifiltered complex

\[ (R^{dR}(\sigma \leq_N \mathcal{N}X^\bullet(\mathbb{C})), F, W), \]

that is the analogue of the complex \((R^{dR}(\sigma \leq_N \mathcal{N}X^\bullet), F, W)\) but using holomorphic forms. Then the maps

\[ (R^B(\sigma \leq_N \mathcal{N}X^\bullet), W) \otimes_{\mathbb{Q}} \mathbb{C} \quad \longrightarrow \quad (R^{dR}(\sigma \leq_N \mathcal{N}X^\bullet(\mathbb{C})), W) \]

\[ (R^{dR}(\sigma \leq_N \mathcal{N}X^\bullet), W) \otimes_{\mathbb{Q}} \mathbb{C} \]
are filtered quasi-isomorphisms giving the comparison isomorphism.

***

**Exercise 4.52.** Prove that the composition of the finite correspondences given by the graphs of two morphisms of algebraic varieties $f: X \to Y$ and $g: Y \to Z$, as defined in (4.31), is the graph of $g \circ f: X \to Z$.

**Exercise 4.53.** Let $X$ be a smooth variety over $k$, together with a rational point $x: \text{Spec}(k) \to X$. Consider the composition $p: X \to \text{Spec}(k) \xrightarrow{x} X$.

1. Show that $p$ is a projector and that the class of $(X, 1 - p)$ agrees with the reduced motive $\tilde{M}(X)$ from Definition 4.36. Thus there is a decomposition $M(X) = \mathbb{Q}(0) \oplus \tilde{M}(X)$.

2. Show that the reduced motive $\tilde{M}(X)$ is independent of the choice of the rational point $x$.

4.3. **Mixed Tate motives.** As was mentioned in the previous section, it is not known how to construct a motivic $t$-structure yielding the desired abelian category of mixed motives. However, when $k$ is number field, one can extract from $\text{DM}(k)$ an abelian category of mixed Tate motives with similar properties to mixed Hodge Tate structures. The keystone is Borel’s computation of the $K$-theory of number fields.

4.3.1. **The derived category of mixed Tate motives.** The motives $\mathbb{Q}(n)$ are the simplest non-trivial objects of the category $\text{DM}(k)$. It is thus reasonable to figure out what can be built starting from them.

**Definition 4.54.** The derived category of mixed Tate motives over $k$ is the smallest triangulated subcategory $\text{DMT}(k)$ of $\text{DM}(k)$ containing the objects $\mathbb{Q}(n)$, for all $n \in \mathbb{Z}$, and stable under extensions.

Recall that the latter condition means that if $A \to B \to C \to A[1]$ is a distinguished triangle in $\text{DM}(k)$ and two objects among $A, B, C$ belong to $\text{DMT}(k)$, then so does the third.

Thanks to the comparison between motivic cohomology and $K$-theory (Theorem 4.44), the extension groups of simple objects in the category $\text{DMT}(k)$ are given by

$$\text{Ext}^i(\mathbb{Q}(l), \mathbb{Q}(m)) = \text{Ext}^i(\mathbb{Q}(0), \mathbb{Q}(m - l))$$

$$= \text{Hom}_{\text{DM}(k)}(M(\text{Spec}(k)), \mathbb{Q}(m - l)[i])$$

$$= (K_{2(m-l)-i}(k) \otimes \mathbb{Q})^{(m-l)}.$$
The $K$-theory groups of general fields are still largely unknown, but, when $k$ is a number field, Borel computed their ranks:

**Theorem 4.55 (Borel, [Bor74]).** Let $k$ be a number field with $r_1$ (resp. $2r_2$) real (resp. complex) embeddings. Then:

$$ (K_2(m-l)-i(k) \otimes \mathbb{Q})^{(m-l)} = \begin{cases} 
\mathbb{Q}, & \text{if } i = 0, \ m - l = 0, \\
\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } i = 1, \ m - l = 1, \\
\mathbb{Q}^{r_1+r_2}, & \text{if } i = 1, \ m - l \geq 3 \text{ odd,} \\
\mathbb{Q}^{r_2}, & \text{if } i = 1, \ m - l \geq 2 \text{ even,} \\
0, & \text{otherwise.}
\end{cases} $$

The important information we should get from this is

1. the only non-zero groups $\text{Ext}^i$ occur for $i = 0, 1$;
2. $\text{Ext}^0(\mathbb{Q}(l), \mathbb{Q}(m)) = \text{Hom}(\mathbb{Q}(l), \mathbb{Q}(m)) = 0$ unless $m = l$, for which it is equal to $\mathbb{Q}$;
3. if $\text{Ext}^1(\mathbb{Q}(l), \mathbb{Q}(m)) \neq 0$, then $m > l$;
4. the only infinite-dimensional group is $\text{Ext}^1(\mathbb{Q}(l), \mathbb{Q}(l+1))$.

In particular, when $k = \mathbb{Q}$, we have $r_1 = 1$ and $r_2 = 0$, so

$$ \text{Ext}^1_{\text{DMT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = \begin{cases} 
\mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } n = 1, \\
\mathbb{Q} & \text{if } n \geq 3 \text{ odd} \\
0 & \text{otherwise.}
\end{cases} $$

This and the fact that $\text{Ext}^i_{\text{DMT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ for $i \geq 2$ will determine the structure of the category of mixed Tate motives over $\mathbb{Q}$.

**Example 4.56 (Kummer motives).** Since

$$ \text{Ext}^1_{\text{DMT}(k)}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q}, $$

there are plenty of non-trivial extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. They are all rational linear combinations of Kummer motives. For each $t \in \mathbb{Q}^\times \setminus \{1\}$, consider the complex $K_t$ in $\text{SmCor}(k)$ given by

$$ \text{Spec}(k) \oplus \text{Spec}(k) \xrightarrow{f_t} \mathbb{P}^1_k \setminus \{0, \infty\}, $$

where $\text{Spec}(k) \oplus \text{Spec}(k) = \{*,_1, *,_2\}$ sits in degree 0 and the finite correspondence $f_t$ is defined by the cycle $[(*, t)] - [(*, 1)]$.

The class of $K_t$ in $\text{DM}(k)$ belongs to $\text{DMT}(k)$ and the degrees are chosen so that it belongs to $\text{MT}(k)$. The Kummer motive $K^{\text{Mat}}_t$ is the class of $K_t$ in $\text{MT}(k)$. For $t = 1$ we write $K^{\text{Mat}}_1$ for the trivial extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. The Hodge realization of the Kummer motive is the Kummer mixed Hodge structure of Example 2.154.
Another well understood case is the K-theory of finite fields, which was completely computed by Quillen in [Qui72, Thm. 8], shortly after he introduced the definition of higher algebraic K-theory:

**Theorem 4.57** (Quillen, [Qui72]). Let \( \mathbb{F}_q \) be the finite field with \( q \) elements. Then:

\[
K_i(\mathbb{F}_q) = \begin{cases} 
\mathbb{Z} & i = 0, \\
\mathbb{Z}/(q^n - 1) & i = 2n - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

**Conjecture 4.58** (Beilinson–Soulé). If \( k \) is a field, then \( K_n(k)^{(r)}_\mathbb{Q} \) vanishes for all \( n > 2r \).

An immediate corollary of Borel and Quillen’s theorems is:

**Corollary 4.59.** The Beilinson-Soulé conjecture holds when \( k \) is either a number field or a finite field.

### 4.3.2. A t-structure on the derived category of mixed Tate motives (after Levine).

For each pair of integers \( a \) and \( b \), let us denote by \( \mathcal{T}[a,b] \) the strictly full triangulated subcategory of \( \text{DMT}(k) \) generated by the objects \( \mathbb{Q}(n) \) for \( a \leq -2n \leq b \). We denote \( \mathcal{T}[a,a] \) simply by \( \mathcal{T}_a \), and we extend the definition to cover the cases \( a = -\infty \) or \( b = \infty \) as well. In particular, \( \mathcal{T}_{(-\infty,\infty)} = \text{DMT}(k) \).

**Lemma 4.60.** Let \( a \leq b \leq c \) be integers (the cases \( a = -\infty \) and \( c = \infty \) are also allowed). Then \( (\mathcal{T}[a,b], \mathcal{T}[b,c]) \) is a t-structure on \( \mathcal{T}[a,c] \).

In particular, for each integer \( b \), the pair \( (\mathcal{T}_{(-\infty,b]}, \mathcal{T}_{[b+1,\infty)}) \) provides a t-structure on \( \text{DMT}(k) \). Let us emphasize that this is not the t-structure we are looking for, since its heart is reduced to zero. However, it will allow us to define a weight structure.

The truncation functors for the t-structure \( (\mathcal{T}_{(-\infty,b]}, \mathcal{T}_{[b+1,\infty)}) \) on \( \text{DMT}(k) \) will be denoted by

\[
W_\leq b : \text{DMT}(k) \to \mathcal{T}_{(-\infty,b]}
\]

\[
W^{>b} : \text{DMT}(k) \to \mathcal{T}_{[b+1,\infty)}.
\]

The reason for the subindex or superindex is that one will give an increasing filtration whereas the other will give a decreasing filtration.

Let \( W^{>b} \) denote \( W^{>b-1} \) and define

\[
\text{Gr}^W_b(M) = W^{>b}W_\leq b(M).
\]

For each even integer \( a \), let \( \mathcal{T}_{[a,0]} \) (resp. \( \mathcal{T}_{a,0}^{>0} \)) be the full subcategory of \( \mathcal{T}_a \) generated by \( \mathbb{Q}(-a/2)[n] \) for \( n \leq 0 \) (resp. \( n > 0 \)). Finally, let \( \mathcal{T}_{[a,b]}^{<0} \) (resp. \( \mathcal{T}_{a,b}^{\geq 0} \)) be the full subcategory of \( \mathcal{T}_{[a,b]} \) generated by the objects \( M \) such that \( \text{Gr}_{c}^W(M) \) belongs to \( \mathcal{T}_{c}^{<0} \) (resp. \( \mathcal{T}_{c}^{\geq 0} \)) for all \( a \leq c \leq b \).
Theorem 4.61 (Levine). Assume that the field $k$ satisfies the Beilinson-
Soulé conjecture. Then the pair of strictly full subcategories
\[(\mathcal{T}_{\leq 0}(\mathcal{D}_{\mathbb{C}}), \mathcal{T}_{\geq 0}(\mathcal{D}_{\mathbb{C}}))\]
forms a non-degenerate $t$-structure on $\mathcal{M}(k)$.

Definition 4.62. The category $\mathcal{M}(k)$ of mixed Tate motives over $k$ is
the heart of the above $t$-structure.

The category $\mathcal{M}(k)$ has the following properties:

1. It is a neutral Tannakian category generated under extensions by the
   objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.

2. Each object $M$ of $\mathcal{M}(k)$ has an increasing weight filtration $W_\bullet M$
such that
   \[Gr^{W_{2n}}_n M \simeq \mathbb{Q}(n)^\oplus k_n, \quad Gr^{W_{2n+1}}_n = 0\]
   for some natural numbers $k_n$.

3. A fibre functor is given by
   \[\omega(M) = \bigoplus_n \text{Hom}(\mathbb{Q}(n), Gr^{W}_{2n} M).\] (4.63)

Moreover, Wildeshaus [Wil09, Théorème 1.3] proved that there exists
a canonical equivalence of categories
\[F: D^b(\mathcal{M}(k)) \rightarrow \mathcal{M}(k).\] (4.64)

The functor $F$ is $t$-exact, induces the identity on the heart $\mathcal{M}(k)$, and
has the property that the composition with the cohomology functor $H^0$
associated with the $t$-structure as in (A.132) coincides with the canonical
cohomology functor $D^b(\mathcal{M}(k)) \rightarrow \mathcal{M}(k)$. In view of Remark A.130, the
main difficulty does not lie in proving that the two categories are equivalent
but in constructing a functor between them.

4.3.3. Examples. If the motive of a variety $X$ is of mixed Tate type, i.e.
belongs to $\mathcal{M}(k)$, then decomposing $M(X)$ (or rather its dual) by means
of Levine's $t$-structure we obtain the cohomology motives
\[h^i(X) = t_{\leq 0}t_{\geq 0}(M(X)^\vee[i]) \in \mathcal{M}(k).\]
Thus we can isolate the different cohomological degrees, something we do
not know how to do for general motives.

Example 4.65. By Example 4.40, the motive of the projective space
$M(\mathbb{P}_k^n)$ is of mixed Tate type and one has
\[h^i(\mathbb{P}_k^n) = \begin{cases} \mathbb{Q}(-m) & i = 2m, \ 0 \leq m \leq n \\ 0 & \text{otherwise.} \end{cases}\]
Using properties of $\text{DMT}(k)$ such as the homotopy invariance or the long exact sequence of a closed immersion, we can show that certain motives are mixed Tate. For instance, if a variety $X$ possesses a stratification such that the motive of each locally closed stratum is mixed Tate, then the whole $M(X)$ is a mixed Tate motive.

Example 4.66. Let $n \geq 3$ be an integer and consider the moduli space $M_{0,n}$ of distinct $n$-points in $\mathbb{P}^1$. It is a smooth variety of dimension $n - 3$ which is defined over $\mathbb{Q}$. Since any three points can be sent to $0, 1, \infty$ by a projective transformation, one has $M_{0,3} = \text{Spec}(\mathbb{Q})$ and $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In general, $M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \text{diagonals}$. We will write elements of $M_{0,n}$ as tuples $(0, 1, \infty, x_4, \ldots, x_n)$.

Let us show by induction that $M(M_{0,n})$ belongs to $\text{DMT}(\mathbb{Q})$. The result is clear for $n = 3$ and 4. For $n \geq 5$, we can decompose $M_{0,n}$ as follows:

$M_{0,n} \simeq (M_{0,4} \times M_{0,n-1}) \setminus \bigcup_{i=5}^{n} \{x_i = x_4\}$.

By the Künneth formula and the induction hypothesis, the motive of $X = M_{0,4} \times M_{0,n-1}$ belongs to $\text{DMT}(\mathbb{Q})$. The same is true for the motive of $Z = \bigsqcup_{i=5}^{n} \{x_i = x_4\}$. Now the Gysin triangle reads

$M(M_{0,n}) \to M(X) \to M(Z)(1)[2] \to M(M_{0,n})[1]$.

and since $M(X)$ and $M(Z)(1)[2]$ belong to $\text{DMT}(\mathbb{Q})$, so does $M(M_{0,n})$.

Example 4.67. Let $L = L_0 \cup \cdots \cup L_n$ and $M = M_0 \cup \cdots \cup M_n$ be hyperplanes in the projective space $\mathbb{P}^n$. Assume that they are in general position, meaning that the divisor $L \cup M$ has normal crossings. Then the following motive belongs to $\text{MT}(k)$:

$H^2(\mathbb{P}^n \setminus L, M \setminus (M \cap L))$.

4.3.4. Realizations. Recall that in Definition 2.135 we introduced a category $\text{MHTS}(\mathbb{Q})$ of mixed Hodge Tate structures over $\mathbb{Q}$. Then the functor $R^H$ of Theorem 4.49 restricts to a functor

$\text{DMT}(\mathbb{Q}) \to D^b(\text{MHTS}(\mathbb{Q}))$.

As explained in Example A.126, the category appearing on the right-hand side has a canonical $t$-structure. We have also defined a $t$-structure on $\text{DMT}(\mathbb{Q})$. Since it is motivic, any realization functor is $t$-exact in the sense of Definition A.123, and hence restricts to a functor on the hearts. Specializing to $R^H$, we obtain a functor from $\text{MT}(\mathbb{Q})$ to $\text{MHS}(\mathbb{Q})$. Taking into account that the Hodge realization of a mixed Tate motive is a mixed Hodge Tate structure, we actually get a functor

$R^H : \text{MT}(\mathbb{Q}) \to \text{MHTS}(\mathbb{Q})$ \hfill (4.68)

which respects the weight filtrations.
It is important to note that the category $\text{MHTS}(\mathbb{Q})$ is much bigger than $\text{MT}(\mathbb{Q})$. For instance compare the set of extensions of $\mathbb{Q}(m)$ and $\mathbb{Q}(n)$ in the category $\text{MHTS}(\mathbb{Q})$ given by Theorem 2.152, that is uncountable, with the set of extensions in $\text{MT}(\mathbb{Q})$ given by Theorem 4.55, that is countable. Thus it is important to know which mixed Hodge structures come from geometry. This leads to the precise meaning to the word “motivic” when speaking about a mixed Hodge Tate structure:

**Definition 4.69.** We say that a mixed Hodge Tate structure over $\mathbb{Q}$ is **motivic** if it lies in the essential image of the functor $R^H$. The same definition applies to pro-mixed Hodge Tate structures. More generally, we say that a diagram of pro-mixed Hodge Tate structures is motivic if it is isomorphic to the image by the functor $R^H$ of a diagram of pro-mixed Tate motives.

Even if $\text{MHTS}(\mathbb{Q})$ is much bigger than $\text{MT}(\mathbb{Q})$, the realization functor between them is fully faithful and stable by subobjects. This is a very useful result to prove that many mixed Hodge structures have motivic origin. We should mention that to determine whether the Hodge realization functor from the hypothetical category of mixed motives is fully faithful (i.e. bijective on Hom sets) would be a extremely difficult problem. For instance, if one restricts to the category of pure motives it amounts to the Hodge conjecture. That we can do it for $\text{MT}(\mathbb{Q})$ relies again on Borel’s results about the $K$-theory of number fields.

**Proposition 4.70 (Deligne–Goncharov).** The realization functor (4.68) is fully faithful and its essential image is stable under subobjects.

**Proof.** The key point of the argument is that the realization functor $R^H$ determines injections

$$\text{Ext}^1_{\text{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) \to \text{Ext}^1_{\text{MHS}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n))$$

(4.71)

into the extension groups which were computed in Theorem 2.152. For $n = 1$, this follows from the injectivity of

$$\log |\cdot| : \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{C}/2\pi i \mathbb{Q}.$$  

For $n > 1$, the injectivity follows by interpreting $\text{Ext}^1_{\text{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n))$ as a part of the motivic cohomology of $\text{Spec}(\mathbb{Q})$, which can be computed using $K$-theory:

$$\text{Ext}^1_{\text{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = H^1_{\mathcal{M}}(\text{Spec}(\mathbb{Q}), \mathbb{Q}(n)) = K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q},$$

then interpreting $\text{Ext}^1_{\text{MHS}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n))$ as Deligne cohomology groups:

$$\text{Ext}^1_{\text{MHS}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = H^{1}_D(\text{Spec}(\mathbb{Q}), \mathbb{Q}(n)).$$

Under this interpretation, the realization map (4.71) should correspond to the Borel regulator map mentioned in Digression 1.14, which is known to be injective by the work of Borel.
Consider now the fibre functors $\omega_{dR}$ on $\text{MHS}(\mathbb{Q})$ (Definition 2.128) and $\omega$ on $\text{MT}(\mathbb{Q})$ (4.63). These fibre functors are compatible and induce maps at the level of Tannaka groups
\[
G^H_{\omega_{dR}} = \text{Aut}_{\text{MHTS}(\mathbb{Q})}^\otimes(\omega_{dR}) \to \text{Aut}_{\text{MT}(\mathbb{Q})}^\otimes(\omega) = G_\omega.
\]
(4.72)
By the Tannakian dictionary, the functor $R^H$ is fully faithful if and only if the morphism (4.72) is surjective.

To show this we argue as follows: both $G^H_{\omega_{dR}}$ and $G_\omega$ can be written as the semidirect product of $\mathbb{G}_m$ and a pro-unipotent group.
\[
G^H_{\omega_{dR}} = U^H_{\omega_{dR}} \rtimes \mathbb{G}_m, \quad G_\omega = U_\omega \rtimes \mathbb{G}_m.
\]
Then the injectivity of (4.71) implies the surjectivity of (4.72) (see the proof of Theorem 4.104 for the precise relationship between the Ext groups and the Lie algebra of $U_\omega$).

\[\Box\]

**Example 4.73.** Let $n > 0$ be an even integer and $H$ a mixed Hodge structure over $\mathbb{Q}$ that is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$. If this extension is non-trivial then it is not motivic over $\mathbb{Q}$, in the sense that it cannot be the Hodge realization of a motive over $\mathbb{Q}$. Indeed, assume that there is a mixed Tate motive over $\mathbb{Q}$ whose Hodge realization is $H$. Since the realization functor is fully faithful, from the exact sequence
\[
0 \to \mathbb{Q}(n) \to H \to \mathbb{Q}(0) \to 0
\]
corresponds an exact sequence of mixed Tate motives
\[
0 \to \mathbb{Q}(n) \to M \to \mathbb{Q}(0) \to 0.
\]
Since $\text{Ext}^1_{\text{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ this extension is split. Hence the sequence of mixed Hodge structures is also split.

Of course, there exist motivic non-trivial extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ defined over non-totally real number fields.

\[\star\star\star\]

**Exercise 4.74.** Prove that the pair of subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$ of Example A.126 forms indeed a $t$-structure.

**Exercise 4.75.** Let $\text{Gr}(d, n)$ be the Grassmanian scheme of $d$-planes in $k^n$. Show that the motive of $\text{Gr}(d, n)$ belongs to $\text{DMT}(k)$.

**4.4. Mixed Tate motives over $\mathbb{Z}$.** From now on, we specialize further to the case $k = \mathbb{Q}$. The category $\text{MT}(\mathbb{Q})$ is still too big for our purposes since the extension group
\[
\text{Ext}^1_{\text{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(1)) \simeq \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \bigoplus_{p \text{ prime}} \mathbb{Q}
\]
is infinite-dimensional. To remedy this, Goncharov [Gon01, §3] introduced a subcategory of “mixed Tate motives over $\mathbb{Z}$”.

**4.4.1. Definition and basic properties.**

**Definition 4.76.** A motive $M$ in $\text{MT}(\mathbb{Q})$ is said to be everywhere unramified if, given any integer $n$, there is no subquotient $E$ of $M$ which fits into a non-split extension $0 \to \mathbb{Q}(n+1) \to E \to \mathbb{Q}(n) \to 0$. The full subcategory $\text{MT}(\mathbb{Z})$ of $\text{MT}(\mathbb{Q})$ consisting of everywhere unramified motives is called the category of mixed Tate motives over $\mathbb{Z}$.

To a motive $M$ over $\mathbb{Q}$ and a prime number $\ell$, we can associate the $\ell$-adic realization of $M$. For instance, to the motive corresponding to a smooth variety $X$ over $\mathbb{Q}$ we associate the dual of the $\ell$-adic cohomology $H^\ast_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. The $\ell$-adic realization is a $\mathbb{Q}_\ell$-vector space, together with a continuous action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $p$ be a prime number distinct from $\ell$.

The choice of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and a field embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_p$ allows one to see the Galois group $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as a subgroup of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By restriction, we obtain a representation of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Recall that the Galois group of the maximal unramified extension $\mathbb{Q}_p \subset \mathbb{Q}_p^{ur} \subset \overline{\mathbb{Q}}_p$ is isomorphic to $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. The inertia subgroup $I_p$ is defined by

$$1 \to I_p \to \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1.$$ 

**Definition 4.77.** Let $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}(V)$ be an $\ell$-adic representation, and $p$ a prime number distinct from $\ell$. We say that $\rho$ is unramified at $p$ if its restriction to the inertia subgroup $I_p \subseteq \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is trivial.

We have at our disposal the following criterion to decide whether a mixed Tate motive over $\mathbb{Q}$ belongs to $\text{MT}(\mathbb{Z})$.

**Proposition 4.78** (Deligne–Goncharov). A mixed Tate motive $M$ over $\mathbb{Q}$ belongs to $\text{MT}(\mathbb{Z})$ if and only if, for each prime number $p$, there exists a prime $\ell \neq p$ such that the $\ell$-adic realization $\omega_\ell(M)$ is unramified at $p$.

**Proof.** See [DG05, Prop. 1.8].

**Example 4.79.** Let $K^\text{Mot}_t$ be the Kummer motive associated with an element $t \in \mathbb{Q}^\times$ as in Example 4.56. For each prime $\ell$, the $\ell$-adic realization of $K^\text{Mot}_t$ is the extension

$$0 \to \mathbb{Q}_\ell(1) \to K^\ell_t \xrightarrow{f} \mathbb{Q}(0) \to 0$$

corresponding to the $\mathbb{Q}_\ell(1)$-torsor given by the projective limit of $\ell^n$-th roots of unity of $t$. This is unramified everywhere if and only if $t \in \mathbb{Z}^\times$. Thus, taking into account that $\mathbb{Z}^\times \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, the only Kummer motive that belongs to $\text{MT}(\mathbb{Z})$ is the trivial one $K^\text{Mot}_1$. This solves the problem of the extension groups being infinite-dimensional.
The main properties of the category $\text{MT}(\mathbb{Z})$ are summarized in the following theorem

**Theorem 4.80.**

1. $\text{MT}(\mathbb{Z})$ is a Tannakian category generated by the objects $\mathbb{Q}(n)$ for all integers $n \in \mathbb{Z}$.

2. Each object $M$ of $\text{MT}(\mathbb{Z})$ has a canonical increasing weight filtration $W$ indexed by even integers, and such that
   $$\text{Gr}^W_{2n} M \cong \mathbb{Q}(-n)^{\oplus k_n}$$
   for some integers $k_n \geq 0$.

3. The extension groups in the category $\text{MT}(\mathbb{Z})$ are given by
   $$\text{Ext}^i_{\text{MT}(\mathbb{Z})}(\mathbb{Q}(l), \mathbb{Q}(m)) = \begin{cases} 
   \mathbb{Q}, & \text{if } i = 0, \ m - l = 0, \\
   \mathbb{Q}, & \text{if } i = 1, \ m - l \geq 3 \ \text{odd}, \\
   0, & \text{otherwise.}
   \end{cases}$$
   Hence all of them are finite-dimensional.

Since $\text{MT}(\mathbb{Z}) \subset \text{MT}(\mathbb{Q})$ is stable under subobjects, we immediately deduce from Proposition 4.70:

**Corollary 4.81.** The realization functor
$$R: \text{MT}(\mathbb{Z}) \rightarrow \text{MHTS}(\mathbb{Q})$$
is fully faithful with essential image stable under subobjects.

### 4.4.2. Fibre functors.

In this section, we introduce various fibre functors on the category $\text{MT}(\mathbb{Z})$ and compute the corresponding Tannaka groups. The first one is defined using the weight structure on $\text{MT}(\mathbb{Z})$ given by part (2) of Theorem 4.80. For each motive $M$ in $\text{MT}(\mathbb{Z})$ and each integer $n \in \mathbb{Z}$, we write
$$\omega_n(M) = \text{Hom}_{\text{MT}(\mathbb{Z})}(\mathbb{Q}(n), \text{Gr}^W_{2n} M)$$
and define a fibre functor $\omega: \text{MT}(\mathbb{Z}) \rightarrow \text{Vec}_\mathbb{Q}$ by
$$\omega(M) = \bigoplus_n \omega_n(M). \quad (4.82)$$

Observe that $\omega$ factors through the category of graded $\mathbb{Q}$-vector spaces.

From the Hodge realization of a motive we obtain two fibre functors. The de Rham fibre functor, denoted by $\omega_{\text{dR}}$, is the de Rham part of the Hodge structure. For a motive $M \in \text{MT}(\mathbb{Z})$, the vector space $\omega_{\text{dR}}(M)$ comes equipped with two filtrations, the decreasing Hodge filtration $F$, and the increasing weight filtration $W$. Since $(\omega_{\text{dR}}(M), F, W)$ is part of a mixed Tate Hodge structure, these filtrations are opposed in the sense that, if we write
$$\omega_{\text{dR}}(M)^n = F^{-n} \omega_{\text{dR}}(M) \cap W_{-2n} \omega_{\text{dR}}(M),$$
\[ \omega_{\text{dR}}(M) = \bigoplus_n \omega_{\text{dR}}(M)^n, \]
\[ F^{-p}\omega_{\text{dR}}(M) = \bigoplus_{m \leq p} \omega_{\text{dR}}(M)^m, \]
\[ W_{-2n}\omega_{\text{dR}}(M) = \bigoplus_{m \geq n} \omega_{\text{dR}}(M)^m. \]

Thus the de Rham fibre functor \( \omega_{\text{dR}} \) also factors through the category of graded vector spaces.

**Lemma 4.83.** The de Rham fibre functor \( \omega_{\text{dR}} \) is canonically isomorphic to the fibre functor \( \omega \).

There is also a Betti fibre functor \( \omega_B \) given by the Betti part of the Hodge realization. The rational vector space \( \omega_B \) is provided with a weight filtration \( W \), but not a Hodge filtration. Note that \( \omega_B \) does not factor canonically through the category of graded vector spaces.

Finally there is a comparison isomorphism
\[ \text{comp}_{\text{B,dR}}: \omega_{\text{dR}} \otimes Q C \longrightarrow \omega_B \otimes Q C. \] (4.84)

**Example 4.85.** In this example we compute explicitly the de Rham and Betti realizations of \( Q(1) \) and the comparison isomorphism. First we need a variety whose motive contains \( Q(1) \). Let
\[ X = \mathbb{P}_Q^1 \setminus \{0, \infty\} = \mathbb{A}_Q^1 \setminus \{0\} = \mathbb{G}_{m,Q} = \text{Spec}(Q[x, x^{-1}]). \]
Recall from Remark 4.42 that \( M(X) = Q(0) \oplus Q(1)[1] \), and hence
\[ t_0(M(X)[-i]) = \begin{cases} Q(i), & \text{if } i = 0, 1, \\ 0, & \text{otherwise.} \end{cases} \]

We already have a nice compactification \( X \subset \mathbb{P}_Q^1 \). We can write down explicitly the complex of differential forms on \( \mathbb{P}_Q^1 \) with logarithmic poles along \( \{0, \infty\} \). The sheaf \( \Omega^0_{\mathbb{P}_Q^1}(\log\{0, \infty\}) \) is \( \mathcal{O}_{\mathbb{P}_Q^1} \), the sheaf of rational functions on \( \mathbb{P}_Q^1 \). The sheaf \( \Omega^1_{\mathbb{P}_Q^1}(\log\{0, \infty\}) \) is the \( \mathcal{O}_{\mathbb{P}_Q^1} \)-module generated by the differential form \( \frac{dx}{x} = -\frac{dx}{x-1} \). Thus, as a sheaf, is isomorphic to \( \mathcal{O}_{\mathbb{P}_Q^1} \). Since
\[ H^i(\mathbb{P}_Q^1, \mathcal{O}_{\mathbb{P}_Q^1}) = 0, \quad \text{for } i > 0, \]
there is no need to search for a resolution of the complex \( \Omega^*_{\mathbb{P}_Q^1}(\log\{0, \infty\}) \) and we can use directly the complex of global sections to compute de Rham cohomology. We have
\[ \Gamma(\mathbb{P}_Q^1, \Omega^0_{\mathbb{P}_Q^1}(\log\{0, \infty\})) = Q[x, x^{-1}], \]
\[ \Gamma(\mathbb{P}_Q^1, \Omega^1_{\mathbb{P}_Q^1}(\log\{0, \infty\})) = \frac{Q[x, x^{-1}]}{x}. \]
The differential map is given by $dx^n = nx^{n-1}$. Hence

$$H^0_{dR}(X) = \mathbb{Q}, \quad H^1_{dR}(X) = \mathbb{Q} \frac{dx}{x}.$$ 

Therefore

$$\omega_{dR}(\mathbb{Q}(1)) = \left( \mathbb{Q} \frac{dx}{x} \right)^\vee.$$ 

Thus $\omega_{dR}(\mathbb{Q}(1))$ is a one dimensional vector space and we have identified a canonical generator $(dx/x)^\vee$.

The Betti realization is given by the singular homology of the space of complex points. Thus

$$\omega_{B}(\mathbb{Q}(1)) = H_1(\mathbb{C} \setminus \{0\}, \mathbb{Q})$$

This is again a rational vector space of dimension 1. A generator of it is given by the unit circle traveled in the counterclockwise direction, that we denote $\gamma$.

The comparison isomorphism is obtained from the integration of differential forms along singular chains. Since

$$\int_{\gamma} \frac{dx}{x} = 2\pi i$$

we deduce that $\text{comp}_{dR,B}(\gamma) = (dx/x)^\vee \otimes (2\pi i)$.

### 4.4.3. Tannaka groups of $\mathbf{MT}(\mathbb{Z})$.

We now turn to the description of the affine group schemes associated with the various fibre functors on the category of mixed Tate motives over $\mathbb{Z}$.

**Notation 4.86.** The following notation will be used throughout:

$$G_{dR} = \text{Aut}^\otimes(\omega) = \text{Aut}^\otimes(\omega_{dR}),$$

$$G_{B} = \text{Aut}^\otimes(\omega_{B}),$$

$$P_{B, dR} = \text{Isom}^\otimes(\omega_{dR}, \omega_{B}),$$

$$P_{dR, B} = \text{Isom}^\otimes(\omega_{B}, \omega_{dR}).$$

Observe that both $P_{B, dR}$ and $P_{dR, B}$ are $G_{dR}$-torsors and $\text{comp}_{B, dR}$ (resp. $\text{comp}_{dR, B}$) is a complex point of $P_{B, dR}$ (resp. $P_{dR, B}$).

In what follows, we will use the subscript $dR/B$ for properties which are common to $G_{dR}$ and $G_{B}$.

**Lemma 4.91.** The groups $G_{dR/B}$ fit into an exact sequence

$$1 \rightarrow U_{dR/B} \rightarrow G_{dR/B} \rightarrow \mathbb{G}_m \rightarrow 1,$$  \hspace{1cm} (4.92)

where $U_{dR/B}$ is a pro-unipotent group.
Proof. Recall that the category $\text{MT}(\mathbb{Z})$ contains the object $\mathbb{Q}(1)$. Since $\omega_{\text{dR}/\text{B}}(\mathbb{Q}(1))$ is a one-dimensional $\mathbb{Q}$-vector space, we obtain a morphism

$$t_{\text{dR}/\text{B}}: G_{\text{dR}/\text{B}} \to \text{GL}(\omega_{\text{dR}/\text{B}}(\mathbb{Q}(1))) = \mathbb{G}_m.$$  

(4.93)

We define $U_{\text{dR}/\text{B}}$ as the kernel of this morphism.

Since the action of $G_{\text{dR}/\text{B}}$ is compatible with the tensor product, an element $g \in G_{\text{dR}/\text{B}}$ acts on $\omega_{\text{dR}/\text{B}}(\mathbb{Q}(n))$ as $t_{\text{dR}/\text{B}}(g)^n$. Since the weight filtration is a filtration in the category of motives, $G_{\text{dR}/\text{B}}$ respects the weight filtration. This means that, if $g \in G_{\text{dR}/\text{B}}$ and $X \in \text{Ob}(\text{MT}(\mathbb{Z}))$, the action of $g$ in $\omega_{\text{dR}/\text{B}}(X)$ sends $W_n \omega_{\text{dR}/\text{B}}(X) = \omega_{\text{dR}/\text{B}}(W_n X)$ to $W_n \omega_{\text{dR}/\text{B}}(X)$.

Therefore, it acts on $\text{Gr}^W W_n \omega_{\text{dR}/\text{B}}(X)$. Since $\text{Gr}^W W_n \omega_{\text{dR}/\text{B}}(X)$ is a sum of copies of $\omega_{\text{dR}/\text{B}}(\mathbb{Q}(n))$, $g$ acts on $\text{Gr}^W W_n \omega_{\text{dR}/\text{B}}(X)$ as $t_{\text{dR}/\text{B}}(g)^n$ and the action of an element $u \in U_{\text{dR}/\text{B}}$ on the same space is trivial. This implies that $U_{\text{dR}/\text{B}}$ is a pro-unipotent group, that is, an inverse limit of unipotent affine algebraic groups.

At this level, an advantage of using the de Rham fibre functor $\omega = \omega_{\text{dR}}$ instead of the Betti one $\omega_{\text{B}}$ is that the exact sequence (4.92) admits a canonical splitting $\tau: \mathbb{G}_m \to G_{\text{dR}}$. Indeed:

**Lemma 4.94.** One has

$$G_{\text{dR}} = U_{\text{dR}} \rtimes \mathbb{G}_m.$$

**Proof.** We use the fact that $\omega = \omega_{\text{dR}}$ factors through the category of graded vector spaces. Given $t \in \mathbb{G}_m$, let $\tau(t) \in G_{\text{dR}}$ denote the element that acts as multiplication by $t^n$ on $\omega_n$. This defines a section $\tau: \mathbb{G}_m \to G_{\text{dR}}$ of $t_{\text{dR}}$. Hence $G_{\text{dR}}$ is a semidirect product. \hfill $\Box$

**Corollary 4.95.** Any $G_{\text{dR}}$-torsor is trivial.

**Proof.** We assume that the reader is familiar with the vanishing of the Galois cohomology groups

$$H^1(\mathbb{Q}, \mathbb{G}_m) = H^1(\mathbb{Q}, \mathbb{G}_a) = 0$$

(see for instance [Wat79, 18.2] or [Ser94, Chap. II, §1.2, Prop. 1]). It follows that, for any unipotent group $U$ or any group $G$ that is an extension of $\mathbb{G}_m$ by $U$, the Galois cohomology groups are also trivial

$$H^1(\mathbb{Q}, U) = H^1(\mathbb{Q}, G) = 0.$$

Now, the group $G_{\text{dR}}$ can be written as

$$G_{\text{dR}} = \lim_{\leftarrow} G_{\text{dR}}^N,$$

where each $G_{\text{dR}}^N$ is an extension of $\mathbb{G}_m$ by a unipotent group and all the transition maps are surjective. By Mittag–Leffler we deduce that

$$H^1(\mathbb{Q}, G_{\text{dR}}) = \lim_{\leftarrow} H^1(\mathbb{Q}, G_{\text{dR}}^N) = 0,$$
which implies that any $G_{dR}$-torsor defined over $\mathbb{Q}$ is trivial.

The corollary has the important following consequence, which will be exploited in the next chapter.

**Proposition 4.96.** There exists an element $a \in G_{dR}(\mathbb{C})$ such that, for all motives $M$ of $\mathrm{MT}(\mathbb{Z})$, one has

$$\omega_B(M) = (\comp_{B,dR} \circ a)(\omega_{dR}(M)). \quad (4.97)$$

Moreover, $a$ can be chosen of the form $a = u_0 \cdot \tau(2\pi i)$ with $u_0 \in U_{dR}(\mathbb{R})$.

**Proof.** We follow [Del89, §8.10]. Recall from (4.89) that

$$P_{B,dR} = \mathrm{Isom}^Q(\omega_{dR}, \omega_B)$$

is a $G_{dR}$-torsor with a complex point $\comp_{B,dR} \in P_{B,dR}(\mathbb{C})$. In particular, $P_{B,dR}$ is non-empty. This implies that $P_{B,dR}$ has a $\overline{\mathbb{Q}}$-rational point, and hence it is a trivial torsor over $\overline{\mathbb{Q}}$. By Corollary 4.95, the torsor has to be trivial already over $\mathbb{Q}$, which implies the existence of a rational point, that is an isomorphism of fibre functors $\alpha : \omega_{dR} \rightarrow \omega_B$. Define

$$a = \comp_{dR,B} \circ \alpha. \quad (4.98)$$

By construction, $a$ is an element of $G_{dR}(\mathbb{C})$ and $\comp_{B,dR} \circ a = \alpha$, from which (4.97) follows. Note also that any other element of $G_{dR}(\mathbb{C})$ satisfying this property is of the form $a\gamma$ with $\gamma \in G_{dR}(\mathbb{Q})$.

Let us now turn to the second assertion, that $a$ can be chosen of the form $u_0 \cdot \tau(2\pi i)$ with $u_0 \in U_{dR}(\mathbb{R})$. This uses in a crucial way the compatibility between the comparison isomorphism and complex conjugation explained in Proposition 2.101. Interpreted in our context, it says that the following diagram of fibre functors is commutative:

\[
\begin{array}{ccccccccc}
\omega_{dR} & \overset{\alpha}{\longrightarrow} & \omega_B & \overset{\comp}{\longrightarrow} & \omega_{B} \otimes \mathbb{C} & \overset{\comp}{\longrightarrow} & \omega_{dR} \otimes \mathbb{C} \\
\rho \downarrow & & \rho \otimes c & & \Id \otimes c \downarrow & & \rho \otimes c \\
\omega_{dR} & \overset{a}{\longrightarrow} & \omega_B & \overset{\comp}{\longrightarrow} & \omega_{B} \otimes \mathbb{C} & \overset{\comp}{\longrightarrow} & \omega_{dR} \otimes \mathbb{C},
\end{array}
\]

where $\rho$ is the map induced from complex conjugation on the topological space and $c$ is complex conjugation on the coefficients. Note that $\rho$ is a rational point of $G_B$. The complex conjugate of $a$ is $\bar{a} = \Id \otimes \sigma \circ a$. Define $x = a^{-1}\bar{a}$. By the commutativity of the diagram, $x = a^{-1}\rho a$. Thus $x \in G_{dR}(\mathbb{Q})$ and has order two.

Let us apply (4.98) to the motive $\mathbb{Q}(1)$. Since

$$\comp_{dR,B} : \omega_B(\mathbb{Q}(1)) \rightarrow \omega_{dR}(\mathbb{Q}(1))$$
is multiplication by $2\pi i$ by Example 4.85 and $\alpha(\mathbb{Q}(1))$ is an invertible map of one-dimensional $\mathbb{Q}$-vector spaces, it follows that $t_{dR}(a) \in \mathbb{G}_m$ lies in $2\pi i \mathbb{Q}^\times$. Thus, up to replacing $a$ by $a\gamma$ with $\gamma \in G_{dR}(\mathbb{Q})$, we can assume that

$$a^{-1}a = \tau(-1). \quad (4.99)$$

Any other element satisfying both (4.97) and (4.99) is of the form $a\gamma$ for some $\gamma \in G_{dR}(\mathbb{Q})$ such that $\gamma^{-1}\tau(-1)\gamma = \tau(-1)$. In particular, any $\gamma \in \tau(\mathbb{Q}^\times)$ works. Therefore, replacing $a$ by $a\gamma$ with $\gamma \in G_{dR}(\mathbb{Q})$, we can assume that $a^{-1}a = \tau(-1)$.

(4.99)

Thus, up to replacing $a$ by $a\gamma$ with $\gamma \in G_{dR}(\mathbb{Q})$, we can assume that $a^{-1}a = \tau(-1)$.

It remains to show that $u_0 \in U_{dR}(\mathbb{R})$. By (4.99),

$$\tau(2\pi i)^{-1}u_0^{-1}u_0^{-1} = \tau(-1)$$

and writing $\tau(-1) = \tau(2\pi i)^{-1}\tau(-2\pi i)$ one gets $u_0 = u_0$. □

4.4.4. *The period map and the period conjecture.* Recall from the previous sections that $P_{B, dR}$ denotes the scheme of tensor isomorphisms between $\omega_B$ and $\omega_{dR}$, which has the structure of a pro-algebraic variety over $\mathbb{Q}$. The ring of regular functions $\mathcal{O}(P_{dR, B})$ forms an ind-object in the category of $\mathbb{Q}$-algebras of finite type.

**Definition 4.100.** The period map is the ring morphism

$$\text{per}: \mathcal{O}(P_{dR, B}) \to \mathbb{C}$$

given by evaluation at the point $\text{comp}_{dR, B}$:

$$\text{per}(f) = f(\text{comp}_{dR, B}).$$

Similarly, evaluation at the point $\text{comp}_{B, dR}$ yields a period map

$$\mathcal{O}(P_{B, dR}) \to \mathbb{C}.$$ 

The following is a variant of Grothendieck’s period conjecture for the category of mixed Tate motives over $\mathbb{Z}$ (cf. also [And04, 25.2]).

**Conjecture 4.102 (Grothendieck).** The point $\text{comp}_{dR, B}$ is generic.

To give a meaning to the word “generic”, observe that, as in Lemma 4.21, $P_{B, dR}$ can be written as the projective system of torsors $P^Y_{B, dR}$ for mixed Tate motives $Y$. Then, by “generic” we mean that, for every quotient $P_{B, dR} \to P^Y_{B, dR}$ the image $\text{comp}_{B, dR}^Y$ of the point $\text{comp}_{B, dR}$ in $P^Y_{B, dR}$ is not contained in any proper subvariety defined over $\mathbb{Q}$. Therefore $\text{comp}_{B, dR}$ is generic if and only if, for every mixed Tate motive, the period map

$$\text{per} = \text{ev}_{\text{comp}_{B, dR}^Y}: \mathcal{O}(P^Y_{B, dR}) \to \mathbb{C}$$

is injective. Moreover, if $\text{comp}_{B, dR}$ is generic, then the transcendence degree of the residue field of $\text{comp}_{B, dR}^Y$ is equal to the dimension of $P^Y_{B, dR}$.

From the previous discussion, we see that Grothendieck’s period conjecture for mixed Tate motives is equivalent to the following:
Conjecture 4.103. The period map \((4.101)\) is injective.

4.4.5. Lie algebras. Let \(u_{dR}\) be the Lie algebra of \(U_{dR}\). The decomposition \(G_{dR} = U_{dR} \rtimes \mathbb{G}_m\) from Lemma 4.94 yields an action of \(\mathbb{G}_m\) on \(u_{dR}\) which is compatible with the Lie algebra structure in that \(t \cdot [a, b] = [t \cdot a, t \cdot b]\) for all \(t \in \mathbb{G}_m\) and all \(a, b \in u_{dR}\). Let \(u^0_{dR} \subseteq u_{dR}\) be the subspace where \(t \in \mathbb{G}_m\) acts as multiplication by \(t^n\). Then

\[ [u^n_{dR}, u^m_{dR}] \subseteq u^{n+m}_{dR} \]

and therefore we get a graded Lie algebra

\[ u^\text{gr}_{dR} = \bigoplus_{n \in \mathbb{Z}} u^n_{dR}. \]

The fibre functor \(\omega_{dR}\) induces an equivalence of categories between finite-dimensional graded vector spaces together with an action of \(u^\text{gr}_{dR}\) compatible with the gradings and the category \(\mathcal{MT}(\mathbb{Z})\).

The main result of this section is the following:

Theorem 4.104. The graded Lie algebra \(u^\text{gr}_{dR}\) is free with one generator in each positive odd degree \(n \geq 3\).

The theorem will be a consequence of Lemma 4.107 below. Since we have not found a suitable reference, we include a proof of it.

Recall that a finite-dimensional Lie algebra \(\mathfrak{L}\) is said to be nilpotent if there exists an integer \(n\) such that \([a, [a, \ldots, [a, b] \ldots]]\) for all \(a, b \in \mathfrak{L}\). This definition admits several generalizations to infinite-dimensional Lie algebras. The one that will be useful for us is the following:

Definition 4.105. A Lie algebra \(\mathfrak{L}\) is called quasi-nilpotent if

\[ \bigcap_{n} [\mathfrak{L}, [\mathfrak{L}, \ldots, [\mathfrak{L}, \mathfrak{L}] \ldots] = 0. \]

Examples 4.106. Any nilpotent Lie algebra is quasi-nilpotent. A pro-nilpotent Lie algebra is quasi-nilpotent. The graded Lie algebra associated with a pro-nilpotent graded Lie algebra is also quasi-nilpotent. Any subalgebra of a quasi-nilpotent Lie algebra is quasi-nilpotent.

Lemma 4.107. Let \(\mathfrak{L} = \bigoplus_n \mathfrak{L}_n\) be a quasi-nilpotent graded Lie algebra over \(\mathbb{Q}\) with \(H_1(\mathfrak{L}, \mathbb{Q})\) concentrated in positive degrees and \(H_2(\mathfrak{L}, \mathbb{Q}) = 0\). Then \(\mathfrak{L}\) is isomorphic to the free algebra generated by \(H_1(\mathfrak{L}, \mathbb{Q})\).

Proof. We use the Koszul complex of \(\mathfrak{L}\) to compute its homology

\[ \ldots \rightarrow \mathfrak{L} \wedge \mathfrak{L} \wedge \mathfrak{L} \rightarrow \mathfrak{L} \wedge \mathfrak{L} \xrightarrow{[\cdot, \cdot]} \mathfrak{L} \xrightarrow{0} \mathbb{Q}, \]

where the last map in the complex is the zero map and the previous to the last is given by the Lie bracket. From this complex we derive the well known identity

\[ H_1(\mathfrak{L}, \mathbb{Q}) = \mathfrak{L}/[\mathfrak{L}, \mathfrak{L}]. \]
The map $\mathcal{L} \to H_1(\mathcal{L}, \mathbb{Q})$ is homogeneous and surjective, thus we can choose a homogeneous lifting $H_1(\mathcal{L}, \mathbb{Q}) \to \mathcal{L}$. In general, this lifting is non-canonical. Let $\mathfrak{F}$ be the free Lie algebra generated by $H_1(\mathcal{L}, \mathbb{Q})$. It is a graded algebra. By the universal property of free Lie algebras, the chosen lifting defines a graded map $\mathfrak{F} \to \mathcal{L}$. We want to show that this map is an isomorphism.

Let $F_n$ denote the increasing filtration of $\mathcal{L}$ and $\mathfrak{F}$ given by the degree:

$$F_n \mathcal{L} = \bigoplus_{n' \leq n} \mathcal{L}_{n'}, \quad F_n \mathfrak{F} = \bigoplus_{n' \leq n} \mathfrak{F}_{n'}.$$  

We prove by induction on $n \geq 0$ that the map $F_n \mathfrak{F} \to F_n \mathcal{L}$ is surjective. By construction, $F_0 \mathfrak{F} = 0$. Since $\mathcal{L}$ is graded, we deduce that $F_0 \mathcal{L}$ is a Lie subalgebra. Since $\mathcal{L}$ is quasi-nilpotent, the same is true for $F_0 \mathcal{L}$. Since $H_1(\mathcal{L}, \mathbb{Q})$ is concentrated in positive degrees, $F_0 \mathcal{L}$ is also perfect: $F_0 \mathcal{L} = [F_0 \mathcal{L}, F_0 \mathcal{L}]$. This implies that $F_0 \mathcal{L} = \{0\}$ so we get the case $n = 0$ in the induction process.

We assume now that $F_{n'} \mathfrak{F} \to F_{n'} \mathcal{L}$ is surjective for all $n' < n$. Since we can write $F_n \mathcal{L}/F_{n-1} \mathcal{L} = H_1(\mathcal{L}, \mathbb{Q})_n + [\mathcal{L}, \mathcal{L}]_n$ the definition of $\mathfrak{F}$, the fact that $F_0 \mathcal{L} = 0$ and the induction hypothesis imply that the map $F_n \mathfrak{F} \to F_n \mathcal{L}$ is surjective. Since $\mathcal{L}$ is graded,

$$\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n = \bigcup_{n \geq 0} F_n \mathcal{L},$$  

and we conclude the surjectivity of $\mathfrak{F} \to \mathcal{L}$.

Let now $\mathfrak{k} \subset \mathfrak{F}$ denote the kernel of the map $\mathfrak{F} \to \mathcal{L}$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{F} \wedge \mathfrak{k} & \longrightarrow & \mathfrak{k} \\
\downarrow & & \downarrow \\
\mathfrak{F} \wedge \mathfrak{F} \wedge \mathfrak{F} & \longrightarrow & \mathfrak{F} \wedge \mathfrak{F} & \longrightarrow & \mathfrak{F} & \longrightarrow & H_1(\mathcal{L}, \mathbb{Q}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{L} \wedge \mathcal{L} \wedge \mathcal{L} & \longrightarrow & \mathcal{L} \wedge \mathcal{L} & \longrightarrow & \mathcal{L} & \longrightarrow & H_1(\mathcal{L}, \mathbb{Q}),
\end{array}
$$

where $\mathfrak{F} \wedge \mathfrak{k}$ is the image of $\mathfrak{F} \otimes \mathfrak{k}$ in $\mathfrak{F} \wedge \mathfrak{F}$. The long vertical sequences and the upper long horizontal sequence are exact by definition. The lower long sequence is exact because $H_2(\mathcal{L}, \mathbb{Q}) = 0$. From this we deduce

$$\mathfrak{k} \subset [\mathfrak{k}, \mathfrak{F}].$$

Since $\mathfrak{F}$ is also quasi-nilpotent we conclude that $\mathfrak{k} = 0$, thus showing the injectivity of the map $\mathfrak{F} \to \mathcal{L}$. $\square$

**Proof of Theorem 4.104.** We start by computing the Lie algebra cohomology of $u^{\text{gr}}_{\text{dR}}$. To this end, let $\text{Rep}_{\mathbb{Q}}^\infty(U_{\text{dR}})$ (respectively $\text{Rep}_{\mathbb{Q}}^\infty(G_{\text{dR}})$)
denote the category of continuous $\mathbb{Q}$-linear representations of $U_{\text{dR}}$ (respectively $G_{\text{dR}}$), not necessarily of finite dimension. We have a fully faithful functor

$$\text{MT}(\mathbb{Z}) = \text{Rep}_\mathbb{Q}(G_{\text{dR}}) \rightarrow \text{Rep}_\mathbb{Q}^\infty(G_{\text{dR}}).$$

In particular, there are representations $\mathbb{Q}(n)$ of $G_{\text{dR}}$ on which $G_{\text{dR}}$ acts through its quotient $\mathbb{G}_m$. Then

$$H^i(u_{\text{dR}}^\text{gr}, \mathbb{Q}) = \text{Ext}^i_{\text{Rep}_\mathbb{Q}^\infty(U_{\text{dR}})}(\mathbb{Q}, \mathbb{Q}),$$

where $\mathbb{Q}$ is viewed as the trivial representation of $U_{\text{dR}}$.

In order to compute the groups $\text{Ext}^i_{\text{Rep}_\mathbb{Q}^\infty(U_{\text{dR}})}(\mathbb{Q}, \mathbb{Q})$ we will use the theory of induction and restriction of representations. From the inclusion $U_{\text{dR}} \rightarrow G_{\text{dR}}$ we have a functor from the category of representations of $G_{\text{dR}}$ to the category of representations of $U_{\text{dR}}$ that consist simply in restricting the group that act. This functor is denoted $\text{Res}^G_{U_{\text{dR}}}$.

This functor admits a left adjoint denoted $\text{Ind}^G_{U_{\text{dR}}}$.

The properties we need are the computations

$$\text{Res}^G_{U_{\text{dR}}}(\mathbb{Q}) = \mathbb{Q}, \quad \text{and} \quad \text{Ind}^G_{U_{\text{dR}}}(\mathbb{Q}) = \prod_{n \in \mathbb{Z}} \mathbb{Q}(n),$$

and the adjoint property. Then

$$\text{Ext}^i_{\text{Rep}_\mathbb{Q}^\infty(U_{\text{dR}})}(\mathbb{Q}, \mathbb{Q}) = \text{Ext}^i_{\text{Rep}_\mathbb{Q}^\infty(U_{\text{dR}})}(\mathbb{Q}, \text{Res}^G_{U_{\text{dR}}}(\mathbb{Q})) = \text{Ext}^i_{\text{Rep}_\mathbb{Q}^\infty(G_{\text{dR}})}(\text{Ind}^G_{U_{\text{dR}}}(\mathbb{Q}), \mathbb{Q})$$

$$= \text{Ext}^i_{\text{Rep}_\mathbb{Q}^\infty(G_{\text{dR}})}(\prod_{n \in \mathbb{Z}} \mathbb{Q}(n), \mathbb{Q})$$

$$= \bigoplus_{n \in \mathbb{Z}} \text{Ext}^i_{\text{Rep}_\mathbb{Q}^\infty(G_{\text{dR}})}(\mathbb{Q}(n), \mathbb{Q}).$$

It follows from part (3) of Theorem 4.80 that

$$H^1(u_{\text{dR}}^\text{gr}, \mathbb{Q}) = \bigoplus_{n < -3} \text{Ext}^1_{\text{MT}(\mathbb{Z})}(\mathbb{Q}(n), \mathbb{Q}(0)),$$

$$H^2(u_{\text{dR}}^\text{gr}, \mathbb{Q}) = 0,$$

where each summand $\text{Ext}^1_{\text{MT}(\mathbb{Z})}(\mathbb{Q}(n), \mathbb{Q}(0))$ is one-dimensional and sits in odd degree $n < -3$. Going to homology we deduce that

$$H_1(u_{\text{dR}}^\text{gr}, \mathbb{Q}) = \bigoplus_{n \geq 3} \mathbb{Q},$$

$$H_2(u_{\text{dR}}^\text{gr}, \mathbb{Q}) = 0.$$

To prove the theorem we only need to show that $u_{\text{dR}}^\text{gr}$ satisfies the hypothesis of Lemma 4.107. By definition, it is a graded Lie algebra. Since $U_{\text{dR}}$ is pro-unipotent, we deduce that $u_{\text{dR}}$ is pro-nilpotent, and hence $u_{\text{dR}}^\text{gr}$
is quasi-nilpotent. The other assumptions of the lemma are nothing but conditions \((4.108)\) and \((4.109)\) above.

\[\square\]

**Remarks 4.110.**

1. Following [DG05] and [Del13], the grading on \(u^{\text{gr}}_{\text{dR}}\) that we consider is the one coming from the action of \(\mathbb{G}_m\), where \(t\) acts as \(t\) on \(Q(1)\). This is why we obtain a positively graded Lie algebra in contrast with [And04] or [Bro12] that have a negatively graded Lie algebra.

2. Consider the abelianization
\[
(u^{\text{gr}}_{\text{dR}})^{ab} = u^{\text{gr}}_{\text{dR}}/[u^{\text{gr}}_{\text{dR}}, u^{\text{gr}}_{\text{dR}}],
\]
which is a graded vector space. The proof of Theorem 4.104 yields a canonical identification
\[
(u^{\text{gr}}_{\text{dR}})^{ab}_n = (\text{Ext}^1_{\text{MT}}(\mathbb{Z}, Q(n)))^\vee.
\]
Moreover, \(u^{\text{gr}}_{\text{dR}}\) is isomorphic to the free Lie algebra generated by \((u^{\text{gr}}_{\text{dR}})^{ab}\). Nevertheless, there is no canonical lifting from \((u^{\text{gr}}_{\text{dR}})^{ab}\) to \(u^{\text{gr}}_{\text{dR}}\), and hence no canonical isomorphism between \(u^{\text{gr}}_{\text{dR}}\) and the free Lie algebra generated by \((u^{\text{gr}}_{\text{dR}})^{ab}\).

3. Note also that \(u^{\text{dR}}\) and \(u^{\text{gr}}_{\text{dR}}\) are not isomorphic. In fact, \(u^{\text{dR}}\) is the completion of \(u^{\text{gr}}_{\text{dR}}\) with respect to the grading, which implies that \(u^{\text{dR}}\) is not a free Lie algebra.

4.4.6. *The Hilbert–Poincaré series.* From Theorem 4.104, we deduce that the universal enveloping algebra \(U(u^{\text{gr}}_{\text{dR}})\) of \(u^{\text{gr}}_{\text{dR}}\) is the free associative graded algebra with one generator in each odd degree \(n \geq 3\). The algebra of regular functions \(O(U^{\text{dR}})\) is also graded and is the dual of the completed universal enveloping algebra \(\hat{U}(u^{\text{gr}}_{\text{dR}})\) in the graded sense.

For simplicity we will consider the grading by the codegree in \(O(U^{\text{dR}})\) that is the opposite of the one induced by the grading of \(u^{\text{gr}}_{\text{dR}}\). Thus it is also positively graded. We can compute its Hilbert-Poincaré series

\[
H_{O(U^{\text{dR}})}(t) = \frac{1}{1 - t^2 - t^2 - t^5 - \ldots} = \frac{1 - t^2}{1 - t^2 - t^5} \quad (4.111)
\]

from the dimension of the graded pieces of \(U(u^{\text{gr}}_{\text{dR}})\).

Let us now, somehow artificially, introduce the algebra

\[
\mathcal{H}^{\text{MT}} = O(U^{\text{dR}}) \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \quad (4.112)
\]

where \(f_2\) is in degree 2. From \((4.111)\) we immediately deduce:
Lemma 4.113. The Hilbert-Poincaré series of $\mathcal{H}^{\text{MT}}$ is given by

$$H_{\mathcal{H}^{\text{MT}}}(t) = \frac{1}{1 - t^2 - t^3} = \sum_{k \geq 0} d_k t^k,$$

where the integers $d_k$ are the same as in Zagier’s Conjecture 1.71.

Following Deligne, Goncharov, and Terasoma, in order to prove the upper bound $\dim \mathbb{Z}_k \leq d_k$ of Theorem 1.95, we will construct in Chapter 5 a $\mathbb{Q}$-algebra $\mathcal{H}$, which injects into $\mathcal{H}^{\text{MT}}$, and comes together with a surjective graded map $\mathcal{H} \to \bigoplus \mathbb{Z}_k$. This will imply immediately the bound. The reason we have changed the grading of $O(U_{\text{dr}})$ is precisely to make this map compatible with the degree. We have already seen that multiple zeta values appear as periods of the pro-unipotent completion of the fundamental group of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$. The motivic interpretation will give the link between $\mathcal{H}$ and $\bigoplus \mathbb{Z}_k$.

⋆⋆⋆

Exercise 4.114. Find examples which show that all the hypothesis in Lemma 4.107 are needed.

4.5. The motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. We continue considering the algebraic variety

$$X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$$

over $\mathbb{Q}$ and the complex manifold

$$M = X(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}.$$ 

As in Section 3.9, we set:

0 = the tangential base point $(0,1)$, i.e. the tangent vector 1 at 0,
1 = the tangential base point $(1,-1)$, i.e. the tangent vector $-1$ at 1.

Let $\mathbf{x}, \mathbf{y} \in X(\mathbb{Q}) \cup \{0,1\}$ be rational or tangential base points. The aim of this section is to explain that the pro-unipotent completion of the torsor of paths from $\mathbf{x}$ to $\mathbf{y}$, as well as the extra structures given by composition of paths and local monodromy, are motivic in the sense of Definition 4.69. In fact, we want to add to Summary 3.276 a motivic side whose Betti and de Rham realizations give the Betti and de Rham sides of that summary. To exhibit the motivic nature of the group schemes and torsors in that summary, it seems necessary to use the language of algebraic geometry over a Tannakian category [Del89, §6]. In order to avoid this language, we will only consider the motivic analogues of $U^\bullet_\bullet$ and $L^\bullet_\bullet$. 
4.5.1. The pro-mixed Tate motive $yU^\text{Mot}_x$. We start with the case of two rational base points $x, y \in X(\mathbb{Q}) \subseteq M$. Recall the cosimplicial manifold $yM^\bullet_x$ from Construction 3.158. As we already used in Section 3.6.1, when endowing the fundamental group with a mixed Hodge structure over $\mathbb{Q}$, all the maps involved in $yM^\bullet_x$ are algebraic and, the points $x, y$ being rational, defined over $\mathbb{Q}$. We will denote by $yX^\bullet_x$ the corresponding cosimplicial object in the category $\text{Sm}(\mathbb{Q})$.

As explained in Section 4.2.9, to $yX^\bullet_x$ one associates a family of motives
\[
\{[\sigma \leq N]_yX^\bullet_x\}_{N \geq 0}.
\]
By construction, given integers $M \geq N \geq 0$, there is a morphism
\[
\sigma \leq M [\sigma \leq N]_yX^\bullet_x \rightarrow \sigma \leq N [\sigma \leq N]_yX^\bullet_x
\]
making $\{[\sigma \leq N]_yX^\bullet_x\}_{N \geq 0}$ into a projective system of motives.

**Lemma 4.115.** The object $[\sigma \leq N]_yX^\bullet_x$ belongs to $\text{DMT}(\mathbb{Q})$.

**Proof.** Exercise 4.143. $\square$

We can therefore consider its cohomology with respect to the $t$-structure of $\text{DMT}(\mathbb{Q})$.

**Definition 4.116.** For each $N \geq 0$, we define a mixed Tate motive
\[
yU^\text{Mot, Mot}_x N = H_0([\sigma \leq N]_yX^\bullet_x) \in \text{MT}(\mathbb{Q}).
\]
As $N$ varies, these motives fit into a pro-mixed Tate motive $yU^\text{Mot}_x$.

We also consider the constant cosimplicial variety $\text{Spec}(\mathbb{Q})^\bullet$ given by $\text{Spec}(\mathbb{Q})$ in all degrees, with coface and codegeneracy maps all equal to the identity. Applying the previous construction to $\text{Spec}(\mathbb{Q})^\bullet$, one easily finds (Exercise 4.144) that, for all $N \geq 0$,
\[
H_0([\sigma \leq N]_y\text{Spec}(\mathbb{Q})^\bullet) = \mathbb{Q}(0).
\]

4.5.2. The structures of $yU^\text{Mot}_x$. We next introduce some extra structures carried by $yU^\text{Mot}_x$: the unit and counit, the completed coproduct, the composition of paths and the antipode. The idea is to give a geometric analogue of the constructions in the reduced bar complex of a connected dg-algebra (see Definition 3.128), in such a way that they are compatible with the isomorphism from Lemma 3.161.

We start with the unit and counit. Each point $x \in X(\mathbb{Q})$ determines a morphism of cosimplicial varieties
\[
\eta^y_x : \text{Spec}(\mathbb{Q})^\bullet \rightarrow xX^\bullet_x
\]
which sends $\text{Spec}(\mathbb{Q})^n = \text{Spec}(\mathbb{Q})$ to the point $(x, \ldots, x) \in xX^n_x$. Besides, we have for each pair of points $x, y \in X(\mathbb{Q})$ a map of cosimplicial varieties
\[
e^y : yX^\bullet_x \rightarrow \text{Spec}(\mathbb{Q})^\bullet
\]
given by the structural map in all degrees. These induce morphisms
\[
\eta^\vee: \mathbb{Q}(0) \to \mathcal{U}^\text{Mot}_x,
\]
\[
\epsilon^\vee: \mathcal{U}^\text{Mot}_x \to \mathbb{Q}(0),
\]
which are called \textit{unit} and \textit{counit} respectively.

**Remark 4.119.** To understand the notation we will use in the following constructions, recall from 4.2.3 that the direct sum in in the category \textit{SmCor}(\mathbb{Q}) corresponds to the disjoint union of varieties, whereas the tensor product is given by the cartesian product of varieties. Note also that the description we will give of morphisms should be understood as correspondences. For instance, the map for the antipode below is the cycle in \(X^n \times X^n\) given by \((-1)^{n+1/2}\Gamma\), where \(\Gamma\) is the graph of the map \((x_1, \ldots, x_n) \mapsto (x_n, \ldots, x_1)\).

For any two rational points \(x, y \in X(\mathbb{Q})\), consider the unbounded complex \(C^*(y, X^*_x)\) in the category \textit{SmCor}(\mathbb{Q}) given by
\[
C^n(y, X^*_x) = y^*X^n_x,
\]

We consider the morphism
\[
\big[ X \big]^\otimes_n \to \bigoplus_{p+q=n} \big[ X \big]^\otimes_p \otimes \big[ X \big]^\otimes_q
\]
in \textit{SmCor}(\mathbb{Q}) that sends the point \((x_1, \ldots, x_n)\) to
\[
\sum_{p+q=n} \sum_{\sigma \in \text{id}(p,q)} (-1)^\sigma(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) \otimes (x_{\sigma(p+1)}, \ldots, x_{\sigma(n)}), \quad (4.120)
\]
where \((-1)^\sigma\) is the sign of the permutation \(\sigma\).

**Remark 4.121.** Notice that what appears in this product is the permutation \(\sigma\) instead of \(\sigma^{-1}\) as in Proposition 1.151 or Definition 3.128. This is due to the contravariant nature of differential forms.

One can check that this map induces a morphism of complexes
\[
\nabla^\vee: C^*(y, X^*_x) \to C^*(y, X^*_x) \otimes C^*(y, X^*_x).
\]
Now, for points \(x, y, z \in X(\mathbb{Q})\), and integers \(p, q \geq 0\), we consider the map
\[
\big[ X \big]^\otimes_p \otimes \big[ X \big]^\otimes_q \to \big[ X \big]^\otimes(p+q)
\]
given by
\[
(x_1, \ldots, x_p) \otimes (y_1, \ldots, y_q) \mapsto (x_1, \ldots, x_p, y_1, \ldots, y_q). \quad (4.122)
\]
Varying $p,q$ we obtain a morphism of complexes
\[ \Delta^\vee : C^\ast(z X_y^\bullet) \otimes C^\ast(y X_x^\bullet) \to C^\ast(z X_x^\bullet). \]
Finally, the correspondence $[X]^{\otimes n} \to [X]^{\otimes n}$ given by
\[ (x_1, \ldots, x_n) \mapsto (-1)^{\frac{n(n+1)}{2}} (x_n, \ldots, x_1) \] (4.123)
defines a morphism of complexes, called the dual antipode,
\[ S^\vee : C^\ast(y X_x^\bullet) \to C^\ast(z X_y^\bullet). \]

The next step is to induce morphisms at the level of the normalized complexes $\mathcal{N}(y X_x^\bullet)$. For this, one needs to check that the chain morphisms commute with the projector $p_n$ of Lemma 4.45 and take care of the truncations.

**Lemma 4.124.** Let $N, M \geq 0$ be integers.

1. If $N \geq 2M$, the map $\nabla^\vee$ induces a morphism of complexes
\[ \nabla^\vee : \sigma_{\leq N} \mathcal{N}(y X_x^\bullet) \to \sigma_{\leq M} \mathcal{N}(y X_x^\bullet) \otimes \sigma_{\leq M} \mathcal{N}(y X_x^\bullet). \]
2. If $N \geq M$, the map $\Delta^\vee$ induces a morphism of complexes
\[ \Delta^\vee : \sigma_{\leq N} \mathcal{N}(z X_y^\bullet) \otimes \sigma_{\leq N} \mathcal{N}(y X_x^\bullet) \to \sigma_{\leq M} \mathcal{N}(z X_y^\bullet). \]
3. If $N \geq M$, the map $S^\vee$ induces a morphism of complexes
\[ S^\vee : \sigma_{\leq N} \mathcal{N}(y X_x^\bullet) \to \sigma_{\leq M} \mathcal{N}(z X_y^\bullet). \]

Moreover, when $N$ and $M$ vary within the above constraints, the three morphisms yield maps of projective systems.

As a consequence of Lemma 4.124 we obtain the following result.

**Proposition 4.125.** Given any three points $x, y, z \in X(\mathbb{Q})$, there are morphisms of pro-mixed Tate motives

1. a composition of paths
\[ \Delta^\vee : \underline{\chi}_y^{\text{Mot}} \otimes \underline{\chi}_y^{\text{Mot}} \to \underline{\chi}_y^{\text{Mot}}; \]
2. a unit
\[ \eta^\vee : \mathbb{Q}(0) \to \chi_x^{\text{Mot}}; \]
3. a completed coproduct
\[ \nabla^\vee : y \underline{\chi}_x^{\text{Mot}} \to y \underline{\chi}_x^{\text{Mot}} \otimes y \underline{\chi}_x^{\text{Mot}}; \]
4. a counit
\[ \epsilon^\vee : y \underline{\chi}_x^{\text{Mot}} \to \mathbb{Q}(0); \]
5. a dual antipode
\[ S^\vee : y \underline{\chi}_x^{\text{Mot}} \to y \underline{\chi}_x^{\text{Mot}}. \]
4.5.3. The motivic nature of the fundamental groupoid of $\mathbb{P}_Q^1 \setminus \{0, 1, \infty\}$.

**Theorem 4.126** (Deligne–Goncharov [DG05]). For $x, y \in X(\mathbb{Q})$, the Hodge realization of $y U_x^{\text{Mot}}$ agrees with the pro-mixed Hodge structure $y U_x^H$ described in Summary 3.276:

$$R^H(y U_x^{\text{Mot}}) = y U_x^H.$$  

Moreover, $R^H$ is compatible with the composition of paths, the unit, the completed coproduct, the counit and the dual antipode. In particular, the diagram $U_x^H$ for $* \mapsto x$ varying in rational base points, is motivic.

**Proof.** Let $A^*$ be the differential graded algebra given in Example 2.175. Recall that it is given by

$$A^0 = \mathbb{Q}, \quad A^1 = \mathbb{Q} \omega_0 \oplus \mathbb{Q} \omega_1$$

with zero differential. The product in this algebra satisfies $\omega_0 \wedge \omega_1 = 0$. The Hodge filtration is given by

$$F^0 = A^* \supset F^1 = A^1 \supset F^2 = 0$$

and the weight filtration by

$$W_{-1} = 0 \subset W_0 = A^0 \subset W_1 = A^*.$$  

As we have seen in Proposition 2.178, the differential graded algebra $A^*$ allows us to compute the de Rham cohomology of $\mathbb{P}_Q^1 \setminus \{0, 1, \infty\}$ with its weight and Hodge filtration. We have seen also in Section 3.6.2 that it can be used to compute the de Rham side of $U_x^H$.

We will now use this algebra to compute the de Rham side of $R^H(y U_x^{\text{Mot}})$.

Consider the variety $(\mathbb{P}_Q^1)^n$ and the divisor $D_n$ consisting of all points with one coordinate equal to 0, 1 or $\infty$. This is a simple normal crossing divisor. Then, for every pair of rational points $x, y \in X(\mathbb{Q})$, the $n$-th component of the cosimplicial scheme $y X^n_x$ is given by

$$y X^n_x = (\mathbb{P}_Q^1)^n \setminus D_n.$$  

Let $(E^{*}_{\mathbb{P}_Q^1(\mathbb{C})^n}(\log D_n), F, W)$ be the de Rham algebra of complex-valued smooth differential forms on $(\mathbb{P}_Q^1(\mathbb{C}))^n$ with logarithmic poles along $D_n$ with its Hodge and weight filtration (see Section 2.7.3). We now denote

$$A^*_x(y X^n_x) = A^* \otimes^{\mathbb{Q}} \cdots \otimes A^*.$$  

The Hodge and weight filtrations of $A^*$ induce Hodge and weight filtrations on $A^*(y X^n_x)$. For all rational points $x, y$ and integer $n \geq 0$, there is an inclusion

$$A^*(y X^n_x) \hookrightarrow E^{*}_{\mathbb{P}_Q^1(\mathbb{C})^n}(\log D_n)$$

given by

$$1 \otimes \cdots \otimes \omega_{\epsilon_i} \otimes \cdots \otimes 1 \mapsto \omega_{\epsilon_i}(t_i),$$
where $\varepsilon_i = 0$, 1, the 1-form $\omega_i$ is in the position $i$ and $t_i$ is the $i$-th coordinate of $A^n \subset (\mathbb{P}^1)^n$. From the fact that

$$A^* \otimes \mathbb{C} \to E^*_n(\log D_n)$$

is a bifiltered quasi-isomorphism (see the end of Example 2.175) we deduce that the map

$$A^*(y X^n_x) \otimes \mathbb{C} \to E^*_n(\log D_n)$$

is also a bifiltered quasi-isomorphism. Thus $A^*(y X^n_x)$ determine the Hodge and weight filtration of the de Rham cohomology of $y X^n_x$, even with its $\mathbb{Q}$-structure. The important point to note now, that is easy to check, is that the previous inclusions are functorial with respect to any morphism involved in the structures of $y X^n_x$. More precisely

**Lemma 4.127.** The family of inclusions

$$A^*(y X^n_x) \hookrightarrow E^*_n(\log D_n), \quad \text{(4.128)}$$

for $x, y \in X(\mathbb{Q})$ and $n \geq 0$ is functorial with respect to

1. the coface and codegeneracy maps of the cosimplicial schemes $y X^\bullet_x$;
2. the maps (4.117) and (4.118), where we identify $\text{Spec}(\mathbb{Q})$ with $y X^0_x$ through the structure map of $\mathbb{Q}$-schemes;
3. the maps (4.120), (4.122) and (4.123) that will induce the product, the coproduct and the antipode.

Moreover, each map in the family is a filtered quasi-isomorphism.

**Proof.** The fact that each map in the family is a quasi-isomorphism has already been discussed. To be precise of the meaning of functoriality in this lemma we spell out the case of a coface, being all the other maps treated in a similar way. Consider the coface

$$\delta^0: y X^n_x \to y X^{n+1}_x$$

given by $\delta^0(x_1, \ldots, x_n) = (y, x_1, \ldots, x_n)$. Then there is a diagram

$$\begin{array}{ccc}
A^*(y X^{n+1}_x) & \to & E^*_n(\log D_{n+1}) \\
\downarrow & & \downarrow^{(\delta^0)^*} \\
A^*(y X^n_x) & \to & E^*_n(\log D_n).
\end{array}$$

The statement of the lemma means that there is a unique morphism, also denoted by $(\delta^0)^*$,

$$A^*(y X^{n+1}_x) \to A^*(y X^n_x)$$

completing the diagram to a commutative square. By the fact that the horizontal arrows are injective the unicity is clear and we have to show the existence. The needed map is obviously given by

$$(\delta^0)^*(a_1 \otimes \cdots \otimes a_{n+1}) = \varepsilon(a_1)a_2 \otimes \cdots \otimes a_{n+1},$$
where $\varepsilon$ is the augmentation of $A^*$ given by (3.219). All the remaining maps are defined in a similar way. The compatibility of all the morphisms with the composition of maps is just a consequence of the injectivity of the morphisms (4.128).

The main consequence of Lemma 4.127 is that to compute the de Rham realization functor as explained in section 4.2.10 we can use the algebras $A^*(y^{X^n}_x)$ and we deduce that

$$R^\text{dR}(y^{U^{\text{Mot}}}_x) = \lim_{\rightarrow N} H_0(\text{Tot} \sigma \leq N A^*(y^{M^\bullet}_x)).$$

By Lemma 3.161, there is a canonical isomorphism

$$\text{Tot} A^*_\bullet \sim \rightarrow B^*(A^*).$$

Taking the truncation, the cohomological functor $H^0$ and the inductive limit we deduce that

$$R^\text{dR}(y^{U^{\text{Mot}}}_x) = y^\text{dR}.$$ 

The next step is to check the compatibility with the structures on both sides. This is the content of next lemma.

**Lemma 4.129.** The morphism $\psi$ of Lemma 3.161 is compatible with the shuffle product, the coproduct and the antipode.

**Proof.** Since the different structures do not depend on the rational points $x, y$ we omit them from the notation. We begin by proving the compatibility with the shuffle product. For non-negative integers $p, q, r, s$, the map (4.120) induces a map

$$\nabla: A^r(X^p) \otimes A^s(X^q) \rightarrow A^{r+s}(X^{p+q})$$

given by

$$\nabla((\omega_1(x_1) \wedge \cdots \wedge \omega_p(x_p)) \otimes (\omega_{p+1}(x_{p+1}) \wedge \cdots \wedge \omega_{p+q}(x_{p+q}))) = \sum_{\sigma \in \omega(p,q)} (-1)^{\sigma} (-1)^{ps} \omega_1(x_{\sigma(1)}) \wedge \cdots \wedge \omega_{p+q}(x_{\sigma(p+q)}).$$

The sign $(-1)^{\sigma}$ comes from the definition of the map (4.120), while the sign $(-1)^{ps}$ comes from the fact that we have to swap the simplicial degree $p$ with the differential degree $s$. We now compute

$$\nabla(\psi([\omega_1| \cdots |\omega_p]) \otimes \psi([\omega_{p+1}| \cdots |\omega_{p+q}])) = \sum_{\sigma \in \omega(p,q)} (-1)^{\Sigma_{i=1}^{p+q} i \deg(\omega_i)} (-1)^{\sigma} \omega_1(x_{\sigma(1)}) \wedge \cdots \wedge \omega_{p+q}(x_{\sigma(p+q)}).$$

In this equality we have used that

$$\sum_{i=1}^{p} i \deg(\omega_i) + \sum_{j=1}^{q} j \deg(\omega_{p+j}) + p \sum_{j=1}^{q} \deg(\omega_{p+j}) = \sum_{i=1}^{p+q} i \deg(\omega_i) \quad (4.130)$$
We also compute
\[
\psi(\nabla([\omega_1|\cdots|\omega_p] \otimes [\omega_{p+1}|\cdots|\omega_{p+q}])) = \sum_{\sigma \in \omega(p,q)} \eta(\sigma)(-1)^{\sum_{i=1}^{p+q} i \deg(\omega_{\sigma^{-1}(i)})} \omega_{\sigma^{-1}(1)}(x_1) \wedge \cdots \wedge \omega_{\sigma^{-1}(p+q)}(x_{p+q}),
\]
where \(\eta(\sigma)\) is the sign determined by equation (3.133). In order to see that the signs in both expressions agree we introduce formal variables \(a_1\ldots a_{p+q}\) of degree \(-1\), and put \(a = a_1 \wedge \cdots \wedge a_{p+q}\). Then, on the one hand,
\[
\eta(\sigma)(-1)^{\sum_{i=1}^{p+q} i \deg(\omega_{\sigma^{-1}(i)})} \omega_{\sigma^{-1}(1)}(x_1) \wedge \cdots \wedge \omega_{\sigma^{-1}(p+q)}(x_{p+q}) \wedge a
\]
\[
= \eta(\sigma) a_1 \wedge \omega_{\sigma^{-1}(1)}(x_1) \wedge \cdots \wedge a_{p+q} \wedge \omega_{\sigma^{-1}(p+q)}(x_{p+q})
\]
\[
= a_{\sigma(1)} \wedge \omega_1(x_{\sigma(1)}) \wedge \cdots \wedge a_{\sigma(p+q)} \wedge \omega_{p+q}(x_{\sigma(p+q)}),
\]
while, on the other hand,
\[
(-1)^{\sum_{i=1}^{p+q} i \deg(\omega_i)} (-1)^p \omega_1(x_{\sigma(1)}) \wedge \cdots \wedge \omega_{p+q}(x_{\sigma(p+q)}) \wedge a
\]
\[
= (-1)^{\sum_{i=1}^{p+q} i \deg(\omega_i)} \omega_1(x_{\sigma(1)}) \wedge \cdots \wedge \omega_{p+q}(x_{\sigma(p+q)}) \wedge a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(p+q)}
\]
\[
= a_{\sigma(1)} \wedge \omega_1(x_{\sigma(1)}) \wedge \cdots \wedge a_{\sigma(p+q)} \wedge \omega_{p+q}(x_{\sigma(p+q)}),
\]
proving the compatibility with the shuffle product.

We next prove the compatibility with the coproduct. The maps (4.122) induce morphisms
\[
\Delta: A^r(X^n) \longrightarrow \bigoplus_{r+s=t} \bigoplus_{p+q=n} A^r(X^p) \otimes A^s(X^q)
\]
given by
\[
\Delta(\omega_1(x_1) \wedge \cdots \wedge \omega_n(x_n))
\]
\[
= \sum_{p=0}^{n} (-1)^p \sum_{i=p+1}^{n} \deg(\omega_i) \omega_1(x_1) \wedge \cdots \wedge \omega_p(x_p) \otimes \omega_{p+1}(x_{p+1}) \wedge \cdots \wedge \omega_n(x_n),
\]
where the sign comes again from the fact that we are swapping a simplicial degree with a differential degree. Then \(\Delta \circ \psi = \psi \circ \Delta\) is easily checked using equation (4.130).

Finally, the map (4.123) induces a morphisms
\[
S: A^r(X^n) \longrightarrow A^r(X^n)
\]
given by
\[
S(\omega_1(x_1) \wedge \cdots \wedge \omega_n(x_n)) = (-1)^{n(n+1)/2} \omega_1(x_n) \wedge \cdots \wedge \omega_n(x_1).
\]
The proof of the compatibility of the antipode \(S\) with the map \(\psi\) follows the same method as the previous compatibilities. \(\square\)
As a consequence of this lemma we know that the de Rham realization $R^{dR}(yU^\text{Mot}_x)$ agrees with $yU^B_x$ including all the structures.

To conclude, the fact that

$$R^B(yU^\text{Mot}_x) = yU^B_x$$

follows from Theorem 3.193, Lemma 3.175, Proposition A.93 and the description of the Betti realization functor in Section 4.2.10. □

4.5.4. The case of tangential base points. We next have to consider the case of tangential base points and prove that the space of paths with tangential base points is also motivic.

We start with the particular case of $\mathbb{G}_m = \mathbb{P}_Q^1 \setminus \{0, \infty\}$ and the tangential base point $0 = (0,1)$. Recall that in Variant 3.278 we have stated that the method used to study $\mathbb{P}_Q^1 \setminus \{0,1, \infty\}$ can be used to study $\mathbb{G}_m$. In this case the dg-algebra we use is $A(\mathbb{G}_m) = Q \oplus Q \omega_0$ and we obtain that $yU(\mathbb{G}_m)^{dR} = Q[e_0]$

**Proposition 4.131.** There is an isomorphism

$$0U(\mathbb{G}_m)_0^H \sim \alpha^1U(\mathbb{G}_m)_1^H.$$  
Moreover, if $x \in \mathbb{G}_m(Q)$, then there is an isomorphism

$$0U(\mathbb{G}_m)_x^H \sim \alpha^1U(\mathbb{G}_m)_x^H.$$  

**Proof.** We only prove the second statement. The proof of the first one is similar. We define the de Rham component of the sought isomorphism as the identity. Clearly it is compatible with the Hodge and the weight filtrations. This is justified because as was the case of $\mathbb{P}_Q^1 \setminus \{0,1, \infty\}$, the de Rham side is independent of the base points.

We have introduced the straight path $dch$ between 0 and 1, given by $dch(t) = t$ for $t \in [0,1]$. We define the Betti part of the isomorphism as the map induced by the composition of paths which sends a path $\gamma \in \pi_1(\mathbb{G}_m; 0, x)$ to the path $dch \cdot \gamma \in \pi_1(\mathbb{G}_m; 1, x)$. We need to prove that both isomorphisms are compatible with the comparison isomorphism. The comparison map $\text{comp} = \text{comp}_{dR,B}$ is given by the iterated integral map

$$\text{comp}(\gamma) = \sum_{n \geq 0} e_0^n \int_{\gamma} \omega_0 \cdot \omega_0 \cdots \omega_0$$

and satisfies $\text{comp}(\gamma \cdot \gamma') = \text{comp}(\gamma) \text{comp}(\gamma')$. Thus we only need to check that $\text{comp}(dch) = 1$. This last equality follows by taking the limit $z \to 1$ in Example 3.256.

That the Betti part of the isomorphism is compatible with the weight filtration is now a consequence of the fact that the de Rham side is. □

From the proposition we immediately deduce:
Corollary 4.132. The pro-mixed Hodge structures \( \mathfrak{o}U(G_m)_x^H \) and \( \mathfrak{o}U(G_m)_0^H \) are motivic (i.e. they are in the essential image of \( R^H \)).

The next lemma describes the structure of \( \mathfrak{o}U(G_m)_0^H \).

Lemma 4.133. The pro-mixed Hodge structure \( \mathfrak{o}U(G_m)_0^H \) is split and agrees with \( \prod_{n \geq 0} \mathbb{Q}(n) \).

In particular, \( \mathfrak{o}\mathcal{L}(G_m)_0^H = \mathbb{Q}(1) \).

Proof. Let \( f_n \) and \( b_n \) be generators of \( \mathbb{Q}(n)_{dR} \) and \( \mathbb{Q}(n)_B \) respectively; they satisfy \( \text{comp}(b_n) = (2\pi i)^n f_n \). Let \( \gamma_0 \) be the generator of \( \pi_1(G_m,0) \) introduced in Section 3.8.1. By Example 3.258, we know that \( \text{comp}_{dR,B}(\gamma_0) = \exp(2\pi i e_0) \). Consider the power series

\[
\log(\gamma_0) = \log(1 + (\gamma_0 - 1)) \in \mathbb{Q}[\pi_1(G_m,0)]^\wedge.
\]

For each \( n \), we define a map

\[
\varphi_n: \mathbb{Q}(n) \to \mathfrak{o}U(G_m)_0^H
\]

which sends \( f_n \) to \( e_0^n \in \mathbb{Q}\langle e_0 \rangle \) and \( b_n \) to \( \log(\gamma_0)^n \in \mathbb{Q}[\pi_1(G_m,0)]^\wedge \). This map is compatible with the comparison isomorphism:

\[
\text{comp}_{dR,B}(\varphi_n(b_n)) = \text{comp}_{dR,B}(\log(\gamma_0)^n) = (2\pi i)^n e_0^n = \varphi_n(\text{comp}_{dR,B}(b_n)).
\]

Moreover, taking into account that

\[
\log(\gamma_0)^n \in J^n \mathbb{Q}[\pi_1(G_m,0)]^\wedge = W_{-2n} \mathbb{Q}[\pi_1(G_m,0)]^\wedge
\]

and \( e_0^n \in F^{-n} \cap W_{-2n} \mathbb{Q}\langle e_0 \rangle \), the map (4.134) is a morphism of mixed Hodge structures. The maps \( \varphi_n \) induce the sought isomorphism of pro-mixed Hodge structures. The second statement follows immediately from the first one.

We next reduce the question of showing that the mixed Hodge structure of the universal enveloping algebra is motivic to the question that the one of the Lie algebra is motivic.

Lemma 4.135. Let \( x \) and \( y \) be two base points of \( M \) (tangential or not). Then the pro-mixed Hodge structure \( yU_x^H \) is motivic if and only if the structure \( y\mathcal{L}_x^H \) is.

Proof. Since \( y\mathcal{L}_x^H \) is a sub-mixed Hodge structure of \( yU_x^H \), by Proposition 4.70, if \( yU_x^H \) is motivic, then \( y\mathcal{L}_x^H \) is also motivic.

Conversely, assume that \( y\mathcal{L}_x^H \) is motivic. Recall that \( y\mathcal{L}_x^H \) is an inverse limit

\[
y\mathcal{L}_x^H = \lim_{N} y\mathcal{L}_x^H / (y\mathcal{L}_x^H)_{\geq N+1}.
\]
By Proposition 4.70, each quotient in this limit is motivic. Since
\[ y^AH_x = \lim_{N \to \infty} \text{Sym}^*(y^L^H_x/(y^L^H_x)_{\geq N+1})^\vee, \]
we deduce that \( y^AH_x \) is also motivic. By duality, we conclude that \( y^U^H_x \) is also motivic. \( \square \)

Now let \( x \in X(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q}) \setminus \{0,1,\infty\} \) be a rational point and \( 0 \) the tangential base point \((0,1)\). By Lemma 4.135, to show that \( 0^L^H_x \) is motivic, it is enough to show that \( 0^L^H_x \) is. To show that \( 0^L^H_x \) is motivic, we will embed it in a mixed Hodge structure that we know is motivic. Once this is proved, that \( 1^U^H_x \) is motivic follows from the symmetry of \( X \) that sends \( x \) to \( 1-x \).

Let \( f: X \to \mathbb{G}_m \) be the natural inclusion. Then \( f \) induces a morphism of mixed Hodge structures
\[ \varphi_1: 0^L^H_x \to 0^L^H_x. \]  \hspace{1cm} (4.136)
The map \( f \) also induces a local monodromy map
\[ f^*: 0^U^H_{\mathbb{G}_m} \to 0^U^H_0. \]

Consider the composition of morphisms of mixed Hodge structures
\[ 0^U^H_x \otimes 0^U^H_{\mathbb{G}_m} \xrightarrow{\nabla^\vee \otimes \text{Id}} 0^U^H_x \otimes 0^U^H_x \otimes 0^U^H_{\mathbb{G}_m} \]
\[ \xrightarrow{S^\vee \otimes \text{Id} \otimes \text{Id}} x^U^H_0 \otimes x^U^H_x \otimes 0^U^H_{\mathbb{G}_m} \to x^U^H_x, \]
where the last morphism is induced by the composition of paths
\[ \gamma_1 \otimes \gamma_2 \otimes \gamma_3 \mapsto \gamma_1 \cdot f^*(\gamma_3) \cdot \gamma_2. \]

Restricting to Lie type elements we obtain a map
\[ 0^L^H_x \otimes 0^L^H_{\mathbb{G}_m} \to x^L^H_x. \]  \hspace{1cm} (4.137)

Now the identification \( 0^L^H_{\mathbb{G}_m} = \mathbb{Q}(1) \) yields a morphism of pro-mixed Hodge structures
\[ \varphi_2: 0^L^H_x \to x^L^H_x(-1). \]  \hspace{1cm} (4.138)

**Lemma 4.139.** The following morphism of pro-mixed Hodge structures is injective:
\[ \varphi_1 + \varphi_2: 0^L^H_x \to 0^L^H_x \oplus x^L^H_x(-1). \]

**Proof.** It is enough to check the injectivity on the de Rham side. Let \( \mathcal{L} \) be the free Lie algebra with generators \( e_0 \) and \( e_1 \) on degree \(-1\). Let \( \hat{\mathcal{L}} \) be the completion of \( \mathcal{L} \) with respect to this grading. Then we have \( 0^L^dR_x = \hat{\mathcal{L}} \) and \( 0^L^H_{\mathbb{G}_m} = \mathbb{Q} e_0 \). Clearly, the map \( \varphi_1 \) is the projection to the \( e_0 \) component. By construction, the map \(\varphi_2: \hat{\mathcal{L}} \to \hat{\mathcal{L}} \) is given by \( a \otimes e_0 \mapsto [e_0, a] \). Therefore, the map \( f^*: \hat{\mathcal{L}} \to \hat{\mathcal{L}} \) is given by \( a \mapsto [e_0, a] \). Denote by \( \varphi_2': \mathcal{L} \to \mathcal{L} \) the map given by the same formula. By [Reu93, Theorem 2.10] the kernel of the map \( \varphi_2' \) is \( \mathbb{Q} e_0 \). It is an easy exercise on inverse limits to show that this
implies that the kernel of \( \varphi_2 \) is also \( \mathbb{Q}e_0 \). Since \( \varphi_1 \) does not vanish on the kernel of \( \varphi_2 \) we deduce the lemma.

Combining Proposition 4.131 and Theorem 4.126 we know that the pro-mixed Hodge structure \( _0\mathcal{L}(\mathbb{G}_m)_x^H \oplus \mathcal{L}_x^H(-1) \) is motivic. By Proposition 4.70, we deduce that \( _0\mathcal{L}_x^H \) is motivic and by Lemma 4.135 that \( _0U^H_x \) is motivic.

We now have to consider the case of two tangential base points. Let \( x, y \in \{0, 1\} \) two tangential base points of \( X \). Let \( z \in X(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\} \) be a rational point. The composition of paths gives us a surjection

\[ yU^H_z \otimes zU^H_x \rightarrow yU^H_x. \]

Since we already know that the structures on the left-hand side are motivic, we deduce that \( yU^H_x \) is also motivic. Once we know that, for all \( x, y \in \{0, 1\} \), the mixed Hodge structure \( yU^H_x \) is motivic, the realization functor \( R^H \) being fully faithful, any morphism among them is also motivic. Therefore, the composition of paths, the completed coproduct, the antipode, the unit and the counit, and the local monodromy maps are motivic.

4.5.5. The main theorem and some consequences. From the previous discussion we deduce

**Theorem 4.140** (Deligne–Goncharov [DG05]). For each pair of tangential base points \( x, y \in \{0, 1\} \) of \( X \), there is a pro-mixed Tate motive \( yU^\text{Mot}_x \) whose Hodge realization is isomorphic to \( yU^H_x \). By the fully faithfulness of the realization functor, \( yU^\text{Mot}_x \) is unique up to unique isomorphism. Moreover, the unit and the counit, the composition of paths, the completed coproduct, the antipode, and the local monodromy maps are motivic.

In fact we can do even more

**Theorem 4.141** (Deligne–Goncharov [DG05]). For each pair of tangential base points \( x, y \in \{0, 1\} \), the pro-mixed Tate motive \( U^\text{Mot}_x \) is a pro-object in the category \( \text{MT}(\mathbb{Z}) \). The motive \( _0U(\mathbb{G}_m)_0^\text{Mot} \) belongs to \( \text{MT}(\mathbb{Z}) \).

**Proof.** The proof of this theorem relies on showing that the \( \ell \)-adic realizations of these motives are unramified (see [DG05, Proposition 4.17]) and using Proposition 4.78.

**Corollary 4.142.** The diagram \( D^H_U \) of definition 3.280 is motivic and defined over \( \mathbb{Z} \).

The importance of this result is that it connects a very abstract and non-explicit group \( G_{\text{dr}} = \text{Aut}^\otimes(\omega_{\text{dr}}) \), but with known structure (see sections 4.4.4 and 4.4.5), with a very concrete combinatorial group \( \text{Aut}(D^H_U) \) (see section 3.9.3). The group \( G_{\text{dr}} \) is the group of symmetries of the category \( \text{MT}(\mathbb{Z}) \). Therefore it acts on every motive defined over \( \mathbb{Z} \) or even in any
diagram of motives defined over \( \mathbb{Z} \). Since the diagram \( D_{dR}^{dR} \) is motivic, the group \( G_{dR} \) acts on it and we obtain a group homomorphism

\[ G_{dR} \longrightarrow \text{Aut}(D_{dR}^{dR}). \]

The subgroup \( U_{dR} \subset G_{dR} \) acts trivially on the motive \( \mathbb{Q}(1) \), which implies that its image acts trivially on \( _0\mathcal{L}(\mathbb{G}_m)^{dR} \), and hence on \( _0U(\mathbb{G}_m)^H \). Therefore the image of \( U_{dR} \) is contained in \( \text{Aut}^0(D_{dR}) \) and we obtain a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & U_{dR} & \longrightarrow & G_{dR} & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Aut}^0(D_{dR}) & \longrightarrow & \text{Aut}(D_{dR}) & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0
\end{array}
\]

The next chapter will be mainly devoted to extract consequences of this diagram.

\[ \star \star \star \]

**Exercise 4.143.** Use that \([X]\) belongs to \( \text{DMT}(\mathbb{Q}) \) and the fact that \( \text{DMT}(\mathbb{Q}) \) is closed under products and extensions to prove by induction that \([\sigma_{\leq N} N_{\mathcal{X}} X^\bullet]\) belongs to \( \text{DMT}(\mathbb{Q}) \).

**Exercise 4.144.** Show that the complex \( \mathcal{N} \text{Spec}(\mathbb{Q})^\bullet \) is isomorphic in \( C(\text{SmCor}(\mathbb{Q})_{pa}) \) to the complex \( \text{Spec}(\mathbb{Q}) \) concentrated in degree zero. Conclude that \( H_0(\sigma_{\leq N} \mathcal{N} \text{Spec}(\mathbb{Q})^\bullet) = \mathbb{Q}(0) \) for all \( N \geq 0 \).
5. Motivic multiple zeta values
(after Brown, Deligne, and Goncharov)

In this final chapter, we pull together all the techniques developed so far to prove theorems A and B from the preface. An important part of the strategy will be to upgrade multiple zeta values to motivic multiple zeta values, which are elements of a Hopf algebra. At the end of the chapter, we will state some remarkable consequences of both theorems, among which are the fact that periods of mixed Tate motives over \(\mathbb{Z}\) are \(\mathbb{Q}\)-linear combinations of multiple zeta values, and that Zagier’s conjecture implies the algebraic independence of \(\pi, \zeta(3), \zeta(5), \ldots\).

5.1. The upper bound. We now have all that we need to prove Theorem A, the upper bound for the dimension of the \(\mathbb{Q}\)-vector space generated by multiple zeta values of a given weight.

5.1.1. Setting. Recall the construction of the Tannakian category \(\text{MT}(\mathbb{Z})\) of mixed Tate motives over \(\mathbb{Z}\). The simple objects are the Tate motives \(\mathbb{Q}(n)\), for all \(n \in \mathbb{Z}\), and the structure is determined by the extension groups

\[
\text{Ext}^1_{\text{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \begin{cases} 
\mathbb{Q} & \text{if } n \geq 3 \text{ odd} \\
0 & \text{otherwise}
\end{cases} \quad (5.1)
\]

and the vanishing of higher extensions. The fibre functor

\(\omega: \text{MT}(\mathbb{Z}) \to \text{Vec}_\mathbb{Q}\)

from (4.82) makes \(\text{MT}(\mathbb{Z})\) into a Tannakian category, and hence equivalent to the category of representations of the pro-algebraic \(\mathbb{Q}\)-group

\(G_{\text{dR}} = \text{Aut}^\otimes(\omega)\).

We have already determined the structure of \(G_{\text{dR}}\) using the computation of the extension groups. It is a semidirect product

\(G_{\text{dR}} \cong U_{\text{dR}} \rtimes \mathbb{G}_m\), \quad (5.2)

where \(U_{\text{dR}}\) is a pro-unipotent algebraic group over \(\mathbb{Q}\). The Lie algebra \(u_{\text{dR}}\) of \(U_{\text{dR}}\) is (non-canonically) isomorphic to the completion of the free Lie algebra with one generator \(\sigma_{2n+1}\) in each degree \(- (2n + 1)\) for all \(n \geq 1\). Therefore, the graded Lie algebra \(u_{\text{dR}}^{\text{gr}}\) is a free Lie algebra.

Besides, in Section 3.9 we introduced the algebraic groups of symmetries of the de Rham fundamental groupoid of \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\), which were denoted by \(\text{Aut}^0(D_{\text{dR}})\) and \(\text{Aut}(D_{\text{dR}})\). Moreover we showed in Lemma 3.281 that there is an isomorphism of \(\mathbb{Q}\)-schemes

\(\text{Aut}^0(D_{\text{dR}}) \simeq \Pi_0^{\text{dR}}\).

This led us to define an algebraic group \((\Pi, \circ)\) with underlying scheme \(\Pi_0^{\text{dR}}\) and the multiplication induced by the above isomorphism (Definition 3.283).
The group \((\Pi, \circ)\) acts on \(1\Pi_0^{\text{dR}}\) and the map \(v \mapsto v(1)\) is an isomorphism. Thus, \(1\Pi_0^{\text{dR}}\) is a trivial torsor under \((\Pi, \circ)\).

Theorem 4.140 implies that the diagram \(D^{\text{dR}}\) is the de Rham realization of a diagram of mixed Tate motives over \(\mathbb{Z}\). Therefore we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & U_{\text{dR}} & \longrightarrow & G_{\text{dR}} & \longrightarrow & G_m & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Pi & \longrightarrow & \text{Aut}(D^{\text{dR}}) & \longrightarrow & G_m & \longrightarrow & 0.
\end{array}
\] (5.3)

and we denote the first vertical arrow by

\[I: U_{\text{dR}} \longrightarrow \Pi.\] (5.4)

In particular, \(G_{\text{dR}}\) acts on the pro-scheme \(1\Pi_0^{\text{dR}}\).

We introduce the notation

\[A^{\text{MT}} = O(U_{\text{dR}}), \quad A = O(I(U_{\text{dR}})).\] (5.5)

Note that there is an injective morphism of algebras \(A \hookrightarrow A^{\text{MT}}\).

In (4.112) we introduced

\[H^{\text{MT}} = A^{\text{MT}} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2].\]

It is a Hopf module over \(A^{\text{MT}}\), with \(f_2\) in degree two, and its Hilbert-Poincaré series is given by

\[H^{\text{MT}}(t) = \sum_{k \geq 0} d_k t^k.\]

5.1.2. The algebra of motivic multiple zeta values. From now on we will let \(\text{dch}^{\text{dR}}\) denote the image by

\[\text{comp}_{\text{dR}, B}: 1\Pi_0^B(\mathbb{C}) \longrightarrow 1\Pi_0^{\text{dR}}(\mathbb{C})\]

of the straight path \(\text{dch} \in 1\Pi_0^B(\mathbb{Q})\) from \(0\) to \(1\). This is nothing other than what was previously denoted in (3.262) by:

\[L(\text{dch}) = \sum_{\alpha} \zeta_{\omega}(x_{\alpha}) e_{\alpha}.\]

In particular, \(\text{dch}^{\text{dR}}\), which was a priori only a complex point of \(1\Pi_0^{\text{dR}}\), lives actually in \(1\Pi_0^{\text{dR}}(\mathbb{R})\), since all (regularized) multiple zeta values \(\zeta_{\omega}(x_{\alpha})\) are real numbers.

Recall that the affine ring of \(1\Pi_0^{\text{dR}}\) is

\[O(1\Pi_0^{\text{dR}}) = \mathbb{Q}(x_0, x_1) = \mathcal{O}.\]

Evaluating an element \(f \in O(1\Pi_0^{\text{dR}})\) at the point \(\text{dch}^{\text{dR}}\) yields a map

\[\text{dch}: O(1\Pi_0^{\text{dR}}) \longrightarrow \mathbb{R}.\]
given by \( \text{dch}(f) = f(\text{dch}^{\text{dR}}) \). For a word \( w \) in the alphabet \( x_0, x_1 \), we get \( w(\text{dch}^{\text{dR}}) = \zeta_w(w) \). Thus, by Corollary 1.177, we obtain a surjective map from \( O(1\Pi^{\text{dR}}_0) \) to the algebra \( Z \) of multiple zeta values.

**Remark 5.6.** This map is very far from being injective, as all relations between multiple zeta values belong to the kernel. As a result, the algebra \( Q\langle x_0, x_1 \rangle \), which has the advantage of being elementary, is too big for the purpose of proving Theorem A. The algebra \( O(G_{\text{dR}}) \) looks more promising but it is still too big. In fact \( O(G_m) = Q[x, x^{-1}] \) with \( x \) in degree 1. Using the splitting of Lemma 4.94 we derive

\[
O(G_{\text{dR}}) \simeq A^{\text{MT}} \otimes_Q Q[x, x^{-1}].
\]

The presence of \( x^{-1} \), that has degree \(-1\), implies that the dimension of each graded piece of \( O(G_{\text{dR}}) \) is infinite, therefore this algebra is also not useful for our purposes. Identifying \( f_2 \) with \( x^2/24 \) we obtain an injective map \( H^{\text{MT}} \to O(G_{\text{dR}}) \). The strategy to prove Theorem A is to prove that the evaluation map “dch” factors through \( H^{\text{MT}} \). This can be done in an ad hoc way or we can use a nice geometric interpretation due to Brown.

Following Brown [Bro12, §2.3], we define a closed subvariety \( Y \subseteq 1\Pi^{\text{dR}}_0 \) as the Zariski closure of the orbit of \( \text{dch}^{\text{dR}} \), that is:

\[
Y = \overline{G_{\text{dR}} \cdot \text{dch}^{\text{dR}}}. \tag{5.7}
\]

**Lemma 5.8.** The subvariety \( Y \) is defined over \( Q \).

**Proof.** To see that \( Y \) is defined over \( Q \) we give another interpretation of it. Recall that \( P_{\text{dR,B}} \) is the torsor of isomorphisms between the fibre functors \( \omega_B \) and \( \omega_{\text{dR}} \). Thus there is an action

\[
P_{\text{dR,B}} \times 1\Pi^{\text{dR}}_0 \to 1\Pi^{\text{dR}}_0.
\]

The point \( \text{dch} \in 1\Pi^{B}(Q) \) induces a map \( \text{dch}: P_{\text{dR,B}} \to 1\Pi^{\text{dR}}_0 \). This map is \( G_{\text{dR}} \)-equivariant and sends \( \text{comp}_{\text{dR,B}} \) to \( \text{dch}^{\text{dR}} \). Thus, \( Y \) is the Zariski closure of the image of the map \( \text{dch} \). The point \( \text{dch} \) being rational, we deduce that \( Y \) is defined over \( Q \). \( \square \)

We consider the \( Q \)-algebra

\[
H = O(Y).
\]

The action of \( G_m \) on \( Y \) induces a grading of \( H \). Since \( Y \) contains \( \text{dch}^{\text{dR}} \), the map “dch” factors through \( H \) giving a map

\[
\text{per} : H \to \mathbb{R}. \tag{5.9}
\]

By Corollary 1.177 the image of “per” is \( Z \). Moreover, since the action of \( G_m \) on \( Y \) is compatible with its action on \( 1\Pi^{\text{dR}}_0 \) and the grading that this action induces on \( O(1\Pi^{\text{dR}}_0) \) agrees with the natural grading of \( Q[x_0, x_1] \), we deduce that the image of \( H_k \) is \( Z_k \).
Definition 5.10. \( \mathcal{H} \) is called the *algebra of motivic multiple zeta values* and the map “per” the *period map*.

The map (5.9) is called the period map is because it is compatible with the period map from Definition 4.100. In fact, since
\[
dch(\text{comp}_{\text{dR}, B}) = dch^{\text{dR}},
\]
there is a commutative diagram
\[
\begin{array}{ccc}
\mathcal{O}(\mathcal{Y}) & \xrightarrow{\text{dch}^*} & \mathcal{O}(\mathcal{P}_{\text{dR}, B}) \\
& \text{per} \searrow & \text{per} \\
& & \mathbb{C}.
\end{array}
\]

Remark 5.12. We can interpret \( \mathcal{H} \) as follows. Let \( \mathcal{I} \subset \mathbb{Q}(x_0, x_1) \) be the ideal of functions vanishing on \( dch \). This is the ideal of rational relations among multiple zeta values. The ideal of \( \mathcal{Y} \), denoted by \( \mathcal{J}^{MT} \), is the ideal of motivic relations between multiple zeta values, that is, those explained by geometry. The fact that \( \mathcal{J}^{MT} \subseteq \mathcal{I} \) will imply the upper bound of the dimension of the space of multiple zeta values, while Zagier’s conjecture will be equivalent to \( \mathcal{J}^{MT} = \mathcal{I} \), that is, that every rational relation among multiple zeta values comes from geometry.

Now, the strategy to prove Theorem A is to get the inequality \( \dim \mathcal{H}_k \leq d_k \) from an injection \( \mathcal{H} \to \mathcal{H}^{MT} \). This injection will come from the study of the geometry of \( \mathcal{Y} \). As a consequence of Theorem B to be proved later, we will see that, in fact, the equality \( \dim \mathcal{H}_k = d_k \) holds and the algebras \( \mathcal{H} \) and \( \mathcal{H}^{MT} \) are isomorphic.

5.1.3. The structure of \( \mathcal{Y} \). Recall from Proposition 4.96 that there exists an element
\[
a = u_0 \cdot \tau(2\pi i) \in U_{\text{dR}}(\mathbb{R})\tau(2\pi i) \subset G_{\text{dR}}(\mathbb{C})
\]
such that \( \omega_B(M) = (\text{comp}_{B, \text{dR}} \circ a)(\omega_{\text{dR}}(M)) \) for all \( M \) in \( \text{MT}(\mathbb{Z}) \).

Lemma 5.14. There exists \( \gamma \in \Pi(\mathbb{Q}) \) such that
\[
dch^{\text{dR}} = (I(u_0) \circ \tau(2\pi i))(1_{10}).
\]
Moreover, for any \( \gamma \) satisfying (5.15), one has \( \tau(-1)(\gamma) = \gamma \).

Proof. By Proposition 4.96 there is a \( \gamma' \in 1\Pi_{0}^{\text{dR}}(\mathbb{Q}) \) such that \( dch^{\text{dR}} = a(\gamma') \). Let \( \gamma \in \Pi(\mathbb{Q}) \) such that \( \gamma' = \gamma(1_{10}) \). Then
\[
dch^{\text{dR}} = (u_0 \cdot \tau(2\pi i))(\gamma')
= I(u_0)(\tau(2\pi i)(\gamma(1_{10})))
= I(u_0) \circ \tau(2\pi i)(\gamma)(1_{10}).
\]

The second assertion follows from the fact that both \( dch \) and \( u_0 \) are real, and hence so is \( \tau(2\pi i)(\gamma) \). Writing \( \gamma = \sum c_w w \) in \( \mathbb{Q}[e_0, e_1] \), it follows that \( c_w = 0 \) for \( w \) of odd degree, thus \( \tau(-1)(\gamma) = \gamma \). \( \square \)
Let us write \( \gamma = \sum c_w w \) in \( \mathbb{Q}\langle e_0, e_1 \rangle \). Since \( \tau(-1)(\gamma) = \gamma \) by the previous lemma, only monomials \( w \) of even degree contribute. It follows that the map \( G_m \rightarrow \Pi \) defined by \( t \mapsto \tau(t)(\gamma) \) depends only on \( t^2 \). Indeed, if one defines
\[
\rho(t) = \sum t^{\deg(w)\over 2} c_w w 
\]
(5.16) one has \( \tau(t)(\gamma) = \rho(t^2) \). Observe that \( \rho \) extends to \( \mathbb{A}^1 \) with \( \rho(0) = 1 \).

**Theorem 5.17.** The morphism of schemes
\[
\psi: I(U_{dR}) \times \mathbb{A}^1 \longrightarrow \Pi \\
(u, t) \longmapsto u \circ \rho(t)
\]
induces an isomorphism \( I(U_{dR}) \times \mathbb{A}^1 \simeq \mathcal{Y} \) given by \( (u, t) \mapsto \psi(u, t)(1^0_1) \).

**Proof.** Recall that the graded Lie algebra \( u_{gr, dR} \) is positively graded and is zero in degree \( < 3 \) by Theorem 4.104. Thus any element \( u \in I(U_{dR}) \) can be written as
\[
u = 1 + \sum_{\deg(w) \geq 3} e_w w.
\]
The coefficients of the monomial \( e_0 e_1 \) in \( \rho(t) \) and \( u \circ \rho(t) \) agree. Let us compute the former. Recall that
\[
dch_{dR} = 1 - \zeta(2)e_0 e_1 + \text{higher degree}.
\]
Since \( dch_{dR} = (u \circ \tau(2\pi i)(\gamma))(1^0_1) \) by Lemma 5.14, one has \( (2\pi i)^2 c_{e_0 e_1} = -\zeta(2) \), which yields the value \( c_{e_0 e_1} = \frac{1}{24} \) by Euler’s formula. The coefficient of \( e_0 e_1 \) in \( \rho(t) \) is thus equal to \( \frac{t}{24} \).

This leads naturally to consider the maps
\[
c: \Pi \longrightarrow \mathbb{A}^1 \\
x \longmapsto 24 \cdot \text{coefficient of } e_0 e_1 \text{ in } x.
\]
\[
\varphi: \Pi \longrightarrow \Pi \\
x \longmapsto x \circ \rho(c(x))^{-1}.
\]
By the previous discussion, we have \( c(\psi(u, t)) = t \) and \( \varphi(\psi(u, t)) = u \). In particular, the morphism \( \psi \) is injective.

Observe that \( x \in \Pi \) is in the image of \( \psi \) if and only if \( \varphi(x) \) belongs to \( I(U_{dR}) \). Therefore, \( \text{Im } \psi = \varphi^{-1}(I(U_{dR})) \). Since \( I(U_{dR}) \) is closed in \( \Pi \), the same holds for \( \text{Im } \psi \). By Lemma 5.14, \( (\text{Im } \psi)_{1^0_1} \) contains \( G_{dR} \cdot \text{dch}_{dR} \) as an open dense subset, so it has to be equal to its closure \( \mathcal{Y} \). Write \( \mathcal{Y}' \) for the preimage of \( \mathcal{Y} \) in \( \Pi \). To conclude, we note that the map \( \mathcal{Y}' \rightarrow I(U_{dR}) \times \mathbb{A}^1 \) given by \( x \mapsto (\varphi(x), c(x)) \) is an inverse of \( \psi \). \( \square \)

**Corollary 5.18.** There is an isomorphism of graded algebras
\[
\mathcal{H} \simeq \mathbb{A} \otimes_{\mathbb{Q}} \mathbb{Q}[t],
\]
where \( t \) sits in degree two. This isomorphism induces an injection \( \mathcal{H} \hookrightarrow \mathcal{H}^{MT} \) that sends \( t \) to \( 24f_2 \).
Proof. We need to show that the map $\psi$ from the previous theorem is $G_m$-equivariant provided that one makes $\lambda \in G_m$ act on $A^1$ by $t \mapsto \lambda^2 t$. On the one hand, formula (5.16) gives $\rho(\lambda^2 t) = \tau(\lambda)(\rho(t))$. On the other hand, using Proposition 4.94, we get

$$\tau(\lambda)(u \circ \rho(t)) = \tau(\lambda)(u) \circ \tau(\lambda)(\rho(t)),$$

from which the result follows. \qed

5.1.4. Proof of Theorem A. Since the map (5.9) is surjective and respects the weight, it suffices to show that $\dim H_k \leq d_k$ for each $k \geq 2$. But Corollary 5.18 and Lemma 4.113 yield

$$\dim H_k \leq \dim(H^{MT})_k = d_k.$$

5.2. Motivic multiple zeta values and the motivic coaction. In this section, we define some elements of the algebra $\mathcal{H}$ which will be called motivic multiple zeta values. Thanks to the existence of the coproduct, we can find many relations between them, which will translate into relations for the usual numbers.

5.2.1. The structure of $A^{MT}$. From the fact that $\text{Lie}(U_{dR})$ is isomorphic to the completion of the free Lie algebra with one generator in each odd degree $\leq -3$, we know that $A^{MT}$ is non-canonically isomorphic to the graded Hopf algebra

$$U' = \mathbb{Q}\langle f_3, f_5, f_7, \ldots \rangle$$

whose underlying space is the set of non-commutative words in symbols $f_{2i+1}, i \geq 1$ in degree $2i+1$, whose product is the shuffle product and whose coproduct is the deconcatenation coproduct

$$\Delta(f_{i_1} f_{i_2} \cdots f_{i_r}) = \sum_{k=0}^r f_{i_1} \cdots f_{i_k} \otimes f_{i_{k+1}} \cdots f_{i_r}. \quad (5.19)$$

We introduce the commutative graded algebra

$$U = U' \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \quad (5.20)$$

with $f_2$ in degree 2. There is a coaction

$$\Delta: U \longrightarrow U' \otimes_{\mathbb{Q}} U \quad (5.21)$$

obtained by declaring $\Delta f_2 = 1 \otimes f_2$, that turns $U$ into an $U'$-comodule. Clearly $H^{MT}$ is non-canonically isomorphic to $U$.

For later use, it will also be convenient to introduce the elements $f_{2n}$ for $n \geq 2$. They are defined as $f_{2n} = b_n f_2^n$, where $b_n$ is the rational number satisfying $\zeta(2n) = b_n \zeta(2)^n$ by Euler’s Theorem 1.3.

The Hopf algebra $U'$ and its comodule $U$ are useful for explicit computations. Later we will fix a convenient isomorphism

$$\phi: H^{MT} \rightarrow U \quad (5.22)$$
satisfying certain normalization requirements. But for the moment we denote by \( \phi \) any such isomorphism.

For compatibility with the theory of multiple zeta values, the grading in \( \mathcal{U}, \mathcal{U}', \mathcal{H} \) and the other algebras will be called the weight.

We first present the computational tools we will use at the level of \( \mathcal{U}' \). As in Definition 3.75, the Lie coalgebra associated with \( \mathcal{U}' \) is

\[
L = \mathcal{U}_0' / (\mathcal{U}_0')^2.
\]

Since there is a canonical decomposition \( \mathcal{U}' = \mathbb{Q} \oplus \mathcal{U}_0' \), we have a projection \( q: \mathcal{U}' \to L \). The Lie coalgebra \( L \) inherits a grading from \( \mathcal{U}' \). Let \( L_N \) be the subspace of weight \( N \) and \( p_N: L \to L_N \) the projection. We define a map

\[
D_{2r+1}: \mathcal{U} \longrightarrow L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U}
\]

as the composition

\[
\mathcal{U} \xrightarrow{\Delta} \mathcal{U}' \otimes_{\mathcal{U}} \mathcal{U} \xrightarrow{q \otimes \text{Id}} L \otimes_{\mathcal{U}} \mathcal{U} \xrightarrow{p_{2r+1} \otimes \text{Id}} L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U},
\]

where \( \Delta \) is the extended coproduct (5.21). We will see in Exercise 5.46 that the maps \( D_{2r+1} \) are derivations. We put

\[
D_{<N} = \bigoplus_{3 \leq 2r+1 < N} D_{2r+1}.
\]

**Lemma 5.25.** Let \( N \geq 2 \) be an integer. Then:

\[
(\ker D_{<N}) \cap \mathcal{U}_N = \mathbb{Q} f_N.
\]

**Proof.** We first show that \( f_N \in \ker D_{<N} \). When \( N \) is even, we already have \( \Delta f_N - 1 \otimes f_N = 0 \). If \( N \) is odd and \( 2r+1 < N \), then

\[
D_{2r+1} f_N = p_{2r+1}(q(f_N)) \otimes 1 = 0.
\]

Thus \( f_N \in \ker D_{<N} \). Conversely, let \( \xi \in \mathcal{U}_N \). Such an element can be uniquely written as

\[
\xi = \alpha f_N + \sum_{3 \leq 2r+1 < N} f_{2r+1} v_r
\]

with \( v_r \in \mathcal{U}_{N-2r-1} \) and \( \alpha \in \mathbb{Q} \). Using the explicit expression of the coproduct \( \Delta \), we see that

\[
D_{2r+1} \xi = f_{2r+1} \otimes v_r + \text{other terms},
\]

where none of the monomials of \( \mathcal{U}_{2r+1}' \) which appear in the extra terms is \( f_{2r+1} \). Thus, if \( D_{2r+1} \xi = 0 \) we deduce that \( v_r = 0 \). \( \square \)
5.2.2. **Motivic multiple zeta values.** Recall that, in formula (3.290) at the end of chapter 3, we introduced, for each binary sequence \( \alpha \), a function on \( \Pi \) which we denoted by
\[
I(1; \alpha; 0) = x_\alpha.
\]
We now let \( I^m(1; \alpha; 0) \) denote the restriction of this function to \( \mathcal{Y} \), that is, the projection to the quotient
\[
I^m(1; \alpha; 0) \in \mathcal{H} = \mathcal{O}(\Pi)/\mathcal{J}^{MT}.
\]
Following formulas (3.290), for later use, we denote
\[
I^m(0; \alpha; 1) = x^*_\alpha |_{\mathcal{Y}}.
\]
and
\[
I^m(0; \alpha; 0) = I^m(1; \alpha; 1) = \begin{cases} 1 & \alpha = \emptyset, \\ 0 & \alpha \neq \emptyset. \end{cases} \tag{5.26}
\]
The symbols \( I^m \) we have introduced are called **motivic iterated integrals**.

We now list some useful properties of the motivic iterated integrals.

**Lemma 5.27.**

(1) If \( N \geq 1 \), then \( I^m(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_N; \varepsilon_{N+1}) = 0 \) when \( \varepsilon_1 = \cdots = \varepsilon_N \).

(2) Reflection formula
\[
I^m(1; \varepsilon_1 \cdots \varepsilon_N; 0) = (-1)^N I^m(0; \varepsilon_N \cdots \varepsilon_1; 1)
= I^m(1; 1 - \varepsilon_N \cdots 1 - \varepsilon_1; 0)
\]

**Proof.** Property (2) follows from Theorem 3.239 (1) and the symmetry of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) given by \( z \mapsto 1 - z \) that reverses the path \( dch \).

We prove property (1). Since \( I^m(0; \alpha; 0) = I^m(1; \alpha; 1) = 0 \) for a non-empty binary sequence \( \alpha \) and, by (2), \( I^m(0; \varepsilon(\mathbb{N}); 1) = (-1)^N I^m(1; \varepsilon(\mathbb{N}); 0) \), it suffices to show that \( I^m(1; \varepsilon(\mathbb{N}); 0) = 0 \). For this, we use the identity
\[
I^m(1; \varepsilon(\mathbb{N}); 0) = \frac{1}{N!} I^m(1; \varepsilon; 0)^N
\]
and the fact that \( I^m(1; \varepsilon; 0) = 0 \) since the algebra \( \mathcal{H} \) has no elements in degree one. \( \square \)

**Definition 5.28.** For a positive multi-index \( s = (s_1, \ldots, s_r) \), the associated **motivic multiple zeta value** is the element of \( \mathcal{H} \) given by
\[
\zeta^m(s) = I^m(1; 0^{(s_1-1)} 1 \cdots 0^{(s_r-1)} 1; 0).
\]

The binary sequence \((0^{(s_1-1)}, 1, \cdots, 0^{(s_r-1)}, 1)\) is called the binary sequence associated with \( s \) and is denoted in Definition 1.124 as \( bs(s) \).

The period map \( \text{per} : \mathcal{H} \rightarrow \mathbb{R} \) from Definition 5.10 satisfies
\[
\text{per}(\zeta^m(s)) = \zeta_\omega(s).
\]
Remark 5.29. Due to the different convention on the definition of multiple zeta values and of iterated integrals there is a discrepancy between the symbols used here and the symbols used in [Bro12], that we summarize here. We denote $\zeta_B^m(s_1, \ldots, s_r)$, $I_B^m(\varepsilon_0 : \varepsilon_1, \ldots, \varepsilon_n : \varepsilon_{n+1})$ and $\zeta_B(s_1, \ldots, s_r)$ the motivic multiple zeta values, motivic iterated integrals and multiple zeta values used in [Bro12]. Then

$$
\zeta_B^m(s_1, \ldots, s_r) = \zeta^m(s_r, \ldots, s_1) \quad \quad I_B^m(\varepsilon_0 : \varepsilon_1, \ldots, \varepsilon_n : \varepsilon_{n+1}) = I^m(\varepsilon_{n+1} : \varepsilon_n, \ldots, \varepsilon_1 : \varepsilon_0) \quad \quad \zeta_B(s_1, \ldots, s_r) = \zeta(s_r, \ldots, s_1).
$$

The map per is the same in [Bro12] and in this book because it is the evaluation morphism at a point. The translation from motivic multiple zeta values in [Bro12] is given by

$$
\zeta_B^m(s_1, \ldots, s_r) = I_B^m(0 : 10^{(s_1-1)} \ldots 10^{(s_r-1)} : 1),
$$

while here is given by

$$
\zeta^m(s_1, \ldots, s_r) = I^m(1 : 0^{(s_1-1)} 1 \ldots 0^{(s_r-1)} 1 : 0).
$$

Both equations are compatible via the change of notation.

If $s$ is admissible, then $\zeta^m(s) \neq 0$. In particular, $\zeta^m(2) \neq 0$. In fact $\zeta^m(2)$ is the function on $\mathcal{V}$ that sends an element $g$ of $\mathcal{V}(\mathbb{Q}) \subset \mathbb{Q}[[e_0, e_1]]$ to its coefficient on $e_0e_1$. It follows that $\zeta^m(2)$ is sent to $\frac{t}{24}$ under the isomorphism $\mathcal{H} \to A \otimes \mathbb{Q}[t]$ of Corollary 5.18. Therefore it is sent to $f_2$ under the injection $\mathcal{H} \to \mathcal{H}^{\text{MT}}$ of the same corollary.

Remark 5.30. The fact that $\zeta^m(2)$ is not zero is an important difference between Brown’s and Goncharov’s approaches to motivic multiple zeta values. Recall that $U_{\text{dR}} \subset G_{\text{dR}}$ and that we had elements $1_{\text{dR}} \in \Pi(\mathbb{Q})$ and $\text{dch}_{\text{dR}} \in \Pi(\mathbb{C})$. Goncharov works with the orbit of $1_{\text{dR}}$ under $U_{\text{dR}}$:

$$
\mathcal{X} = U_{\text{dR}} \cdot 1_{\text{dR}} \subset \Pi
$$

which is isomorphic, as a variety, to $I(U_{\text{dR}})$. Hence $\mathcal{O}(\mathcal{X}) \simeq A$. However, Brown works with the variety $\mathcal{V}$ defined as the closure of the orbit of $\text{dch}_{\text{dR}}$ under $G_{\text{dR}}$

$$
\mathcal{V} = G_{\text{dR}} \cdot \text{dch}_{\text{dR}} \simeq I(U_{\text{dR}}) \times A^1.
$$

Since the leading term of $\text{dch}_{\text{dR}}$ is $1_{\text{dR}}$ we deduce that

$$
\lim_{t \to 0} \tau(t)\text{dch}_{\text{dR}} = 1_{\text{dR}}.
$$

Therefore $\mathcal{X} \subset \mathcal{V}$ and the inclusion from $\mathcal{X}$ to $\mathcal{V}$ is given by $x \mapsto (x, 0)$. In other words, the inclusion $\mathcal{X} \hookrightarrow \mathcal{V}$ is given by the algebra morphism

$$
\pi: \mathcal{H} \twoheadrightarrow \mathcal{H}/\zeta^m(2)\mathcal{H} = A.
$$

(5.31)
5.2.3. *The motivic coaction.* Goncharov’s coproduct from Proposition 3.292 induces a coaction
\[ \Delta : \mathcal{H} \rightarrow A \otimes \mathcal{H} \]
given by the formula
\[ \Delta I^m(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_N; \varepsilon_{N+1}) = \sum_{0=i_0 < i_1 < \cdots < i_k < i_{k+1}=N+1} \pi \left( \prod_{p=0}^{k} I^m(\varepsilon_{i_p}; \varepsilon_{i_p+1} \cdots \varepsilon_{i_{p+1}-1}; \varepsilon_{i_{p+1}}) \right) \otimes I^m(\varepsilon_0; \varepsilon_{i_1} \cdots \varepsilon_{i_k}; \varepsilon_{N+1}), \] (5.33)
where \( \pi \) denotes the projection (5.31).

**Lemma 5.34.** For all \( N \geq 2 \),
\[ \Delta \zeta^m(N) = 1 \otimes \zeta^m(N) + \pi(\zeta^m(N)) \otimes 1. \]

**Proof.** By Definition 5.28, we have \( \zeta^m(N) = I^m(1; 0^{(N-1)}1; 0) \). Using part (1) of Lemma 5.27, we see that the only non-vanishing terms in the coproduct formula (5.33) correspond to the partitions
\[ k = 0, \ i_0 = 0, \ i_1 = N + 1, \ k = N, \ i_j = j, \ j = 0, \ldots, N + 1. \]
The first partition yields the term \( \pi(\zeta^m(N)) \otimes 1 \), while the second one gives \( 1 \otimes \zeta^m(N) \), thus proving the result. \( \square \)

The formula (5.33) is rather complicated, so we will use an infinitesimal version of it, which is the analogue of the derivations \( D_{2r+1} \) for the algebra of motivic multiple zeta values \( \mathcal{H} \). For this, we consider the Lie coalgebra
\[ \mathcal{L} = A_{>0}/(A_{>0})^2, \]
which inherits a grading from \( A \). Let \( \mathcal{L}_N \) be the subspace of degree \( N \) and \( p_N : \mathcal{L} \rightarrow \mathcal{L}_N \) the projection. We still have a projection \( q : A \rightarrow \mathcal{L} \).

**Definition 5.35.** We define a map
\[ D_{2r+1} : \mathcal{H} \rightarrow \mathcal{L} \otimes \mathcal{H} \]
as the composition
\[ \mathcal{H} \xrightarrow{\Delta - \text{Id}} A_{>0} \otimes \mathcal{H} \xrightarrow{q \text{Id}} \mathcal{L} \otimes \mathcal{H} \xrightarrow{p_{2r+1} \text{Id}} \mathcal{L} \otimes D_{2r+1} \otimes \mathcal{H}. \]
We put
\[ D_{<N} = \bigoplus_{3 \leq 2r+1 < N} D_{2r+1}. \] (5.37)

For any isomorphism \( \phi : A^{MT} \rightarrow U' \) of Hopf algebras, we extend it to an algebra isomorphism, \( \phi : \mathcal{H}^{MT} \rightarrow U \) by sending \( f_2 \) to \( f_2 \). It is also an isomorphism of comodules. By abuse of notation, we will denote by \( \phi \) the
restriction of \( \phi \) to \( \mathcal{H} \) and the map \( \mathcal{L} \to L \) induced by \( A \hookrightarrow A^{MT} \xrightarrow{\phi} \mathcal{U}' \).

Then the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\phi} & \mathcal{U} \\
\downarrow D_{2r+1} & & \downarrow D_{2r+1} \\
\mathcal{L}_{2r+1} \otimes \mathcal{H} & \xrightarrow{\phi \otimes \phi} & L_{2r+1} \otimes \mathcal{U}.
\end{array}
\] (5.38)

**Lemma 5.39.** There exists an isomorphism \( \phi: \mathcal{H}^{MT} \to \mathcal{U} \) as before that sends \( \zeta_m(N) \) to \( f_N \) for all \( N \geq 2 \).

**Proof.** We start with any such isomorphism \( \phi \). By construction \( \phi \) sends \( f_2 \) to \( f_2 \). By the discussion before Remark 5.30, \( \phi \) sends \( \zeta_m(2) \) to \( f_2 \).

By Lemma 5.34 we deduce that \( D_{<N} \zeta_m(N) = 0 \). By the commutativity of the diagram (5.38) we deduce that \( D_{<N} \phi(\zeta^m(N)) = 0 \). By Lemma 5.25, \( \phi(\zeta^m(N)) = \alpha_N f_N \) for \( \alpha_N \in \mathbb{Q}^\times \). For \( N = 2r \) even,

\[
\phi(\zeta^m(2r)) = \alpha_{2r} f_{2r} = \alpha_{2r} b_r f^r_2 = \phi(\alpha_{2r} b_r \zeta^m(2)^r).
\]

By the injectivity of \( \phi \) we deduce that \( \zeta^m(2r) = \alpha_{2r} b_r \zeta^m(2)^r \). Taking the period map we see that \( \alpha_{2r} = 1 \).

By the structure of \( \mathcal{U}' \), for any family of non-zero rational numbers \( \alpha_{2r+1}, r \geq 1 \), there is an automorphism of \( \mathcal{U}' \) that sends \( f_{2r+1} \) to \( \alpha_{2r+1}^{-1} f_{2r+1} \). Therefore we can normalize \( \phi \) as we want. \( \square \)

As a byproduct of the proof, we have seen that

\[
\zeta^m(2r) = b_r \zeta^m(2)^r. \tag{5.40}
\]

The following projection will appear in the explicit description of the operators \( D_n \).

**Definition 5.41.** For each \( n > 1 \), we denote \( \varpi_n = p_n \circ q \circ \pi: \mathcal{H} \to \mathcal{L}_n \).

The projection \( \varpi_n \) kills \( \zeta^m(2) \), all products and all motivic multiple zeta values of weight different from \( n \).

**Proposition 5.42.** For \( n < N \) odd, the action of \( D_n \) is given by

\[
D_n I^m(\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_N, \varepsilon_{N+1}) = \sum_{p=0}^{N-n} \varpi_n \left( I^m(\varepsilon_p; \varepsilon_{p+1}, \ldots, \varepsilon_{p+n}; \varepsilon_{p+n+1}) \right) \otimes I^m(\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_p, \varepsilon_{p+n+1}, \ldots, \varepsilon_N; \varepsilon_{N+1}). \tag{5.43}
\]

**Proof.** The projection \( q \) kills all decomposable elements of \( A_{>0} \) and the projection \( p_n \) kills all the elements of degree different from \( n \). Taking
into account that
\[ I^m(\varepsilon; \alpha; \varepsilon') = 1, \text{ if } \alpha = \emptyset, \text{ while} \]
\[ I^m(\varepsilon; \alpha; \varepsilon') \in \mathcal{A}_{> 0}, \text{ if } \alpha \neq \emptyset, \]
it follows that in the sum that runs over partitions
\[ 0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = N + 1 \]
only the terms having exactly one gap of length \( n \) survive. This gives the desired formula. \( \Box \)

5.2.4. \textit{The kernel of } \( D_{< N} \).

\textbf{Theorem 5.44.} Let \( N \geq 2 \) be an integer. Then
\[ \ker D_{< N} \cap \mathcal{H}_N = \mathbb{Q} \zeta_m(N). \]

\textbf{Proof.} Choosing a normalized isomorphism \( \phi \) as in Lemma 5.39, the result follows from the combination of Lemma 5.25 and the commutativity of the diagram (5.38). \( \Box \)

The theorem has the following useful corollary:

\textbf{Corollary 5.45.} Let \( N \geq 2 \) be an integer and \( a^m \) an element of \( \mathcal{H}_N \). Assume that \( D_{< N}(a^m) = 0 \) and \( \text{per}(a^m) = \alpha \zeta(N) \) for some rational number \( \alpha \). Then \( a^m = \alpha \zeta^m(N) \) in \( \mathcal{H} \).

\textbf{Proof.} Since \( a^m \in \ker D_{< N} \cap \mathcal{H}_N \), Theorem 5.44 gives the existence of a rational number \( \beta \) such that \( a^m = \beta \zeta^m(N) \). Applying the period map, one gets \( \beta \zeta(N) = \text{per}(a^m) = \alpha \zeta(N) \), and hence \( \beta = \alpha \).

The importance of this corollary is that it allows one to lift relations between classical multiple zeta values to their motivic counterparts.

\textbf{Exercise 5.46.} Show that the maps \( D_{2r+1}: \mathcal{U} \to L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U} \) from (5.23) are derivations, that is, they satisfy
\[ D_{2r+1}(\xi_1 \xi_2) = (1 \otimes \xi_1)D_{2r+1}(\xi_2) + (1 \otimes \xi_2)D_{2r+1}(\xi_1) \]
for all \( \xi_1, \xi_2 \in \mathcal{U} \). The same holds for the maps \( D_{2r+1}: \mathcal{H} \to L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{H} \) introduced in Definition 5.35.

\textbf{Exercise 5.47} (Linear independence of \( \zeta^m(2, 3) \) and \( \zeta^m(3, 2) \)). The goal of this exercise is to prove the linear independence of the motivic multiple zeta values \( \zeta^m(2, 3) \) and \( \zeta^m(3, 2) \) by exploiting the derivation \( D_3 \). Since \( \mathcal{H}_5 \) has dimension at most \( d_5 = 2 \) by Theorem A, it will follow that they form a basis. This is the first non-trivial case of Brown’s theorem.

(a) Prove that \( I^m(1; 010; 0) = -2 \zeta^m(3) \) and \( I^m(0; 100; 1) = -\zeta^m(3) \).
(b) Use the general formula (5.43) for the action of the derivation $D_3$ and the identities from part (a) of the exercise to compute
\[
D_3\zeta^m(2, 3) = -2\varpi_3(\zeta^m(3)) \otimes \zeta^m(2), \\
D_3\zeta^m(3, 2) = 3\varpi_3(\zeta^m(3)) \otimes \zeta^m(2).
\]

(c) Now assume that $\zeta^m(2, 3) = \lambda \zeta^m(3, 2)$ for some rational number $\lambda$. By part (b), then one necessarily has $\lambda = -2/3$. Upon application of the period map, argue that this leads to a contradiction.

**Exercise 5.48 (Brown’s proof in weight 5).** The trick from the previous exercise does not generalize to higher weight. Here we present an alternative argument which can be seen as a toy case of Brown’s proof.

(a) Prove the equality $D_3(\zeta^m(3) \otimes \zeta^m(2)) = \varpi_3(\zeta^m(3)) \otimes \zeta^m(2)$. Together with the computations in Exercise 5.47 and Theorem 5.44, this implies that there exist rational numbers $\alpha, \beta \in \mathbb{Q}$ such that
\[
\zeta^m(2, 3) + 2\zeta^m(3)\zeta^m(2) = \alpha\zeta^m(5), \\
\zeta^m(3, 2) - 3\zeta^m(3)\zeta^m(2) = \beta\zeta^m(5).
\]

(b) By virtue of Corollary 5.45, the stuffle product and the first identity in (1.68), deduce that $\alpha = 9/2$ and $\beta = -11/2$.

(c) Let $\text{gr}_1^F H_{5,3}^2 \subset H_5$ be the subspace spanned by $\zeta^m(2, 3)$ and $\zeta^m(3, 2)$ (the reason for this notation will become apparent later). We define a linear map $(f, g): \text{gr}_1^F H_{5,3}^2 \to \mathbb{Q}^2$ by requiring
\[
D_3(a) = f(a)\varpi_3(\zeta^m(3)) \otimes \zeta^m(2), \\
D_5(a) = g(a)\varpi_5(\zeta^m(5)) \otimes 1
\]
for all $a \in \text{gr}_1^F H_{5,3}^2$. Use parts (a) and (b) to show that this map has rank two, and hence $\zeta^m(2, 3)$ and $\zeta^m(3, 2)$ form a basis of $H_5$.

5.3. Two families of motivic multiple zeta values and Zagier’s theorem.

5.3.1. Certain relations among motivic multiple zeta values.

**Lemma 5.49.** For each $n \geq 1$, the following equality holds:
\[
\zeta^m(2^n) = \frac{6^n}{(2n+1)!}\zeta^m(2)^n.
\]

**Proof.** Recall that
\[
\zeta^m(2^n) = t^m(1; 01\ldots01; 0).
\]
Then $D_{2r+1}\zeta^m(2^n) = 0$ for all $3 \leq 2r + 1 < 2n$, because in formula (5.43) every sequence of the form $e_p, \ldots, e_{p+2r+2}$ will start and end with the same
value. By (5.26) the corresponding motivic iterated integral is zero. Hence 
$\zeta_m(2^{(n)}) \in \text{Ker } D_{<2n}$. By Theorem 5.44 and equation (5.40), we deduce that 
$\zeta_m(2^{(n)})$ is a rational multiple of $\zeta_m(2^n)$. To get the precise multiple we use 
the period map and Example 1.29. □

In order to simplify notation, we write
$$
\zeta^m_1(s) = I^m(1; 0^{(s_1-1)}1 \cdots 0^{(s_r-1)}10; 0).
$$

**Lemma 5.50.** For $n \geq 1$ the following equalities hold:

$$
\zeta^m_1(2^{(n)}) = -2 \sum_{i=0}^{n-1} \zeta^m(2^{(i)}32^{(n-i-1)}), \quad (5.51)
$$

$$
\zeta^m_1(2^{(n)}) = 2 \sum_{i=1}^{n} (-1)^i \zeta^m(2i+1) \zeta^m(2^{(n-i)}). \quad (5.52)
$$

**Proof.** Recall from (5.26) that $I^m(1; 0) = 0$. Since the multiplication in $H$ is given by the shuffle product, we have

$$
0 = I^m(1; 01^n; 1) = \zeta^m_1(2^{(n)}) + 2 \sum_{i=0}^{n-1} \zeta^m_1(2^{(i)}32^{(n-i-1)}),
$$

from which the identity (5.51) follows.

To prove equation (5.52) we first show the equality of multiple zeta values

$$
- \sum_{i=0}^{n-1} \zeta(2^{(i)}32^{(n-i-1)}) = \sum_{i=1}^{n} (-1)^i \zeta(2i+1) \zeta(2^{(n-i)})
$$

(5.53)

using the stuffle product. Indeed, by Exercise 1.46, we have

$$
\zeta(3) \zeta(2^{(n-1)}) = \sum_{i=0}^{n-1} \zeta(2^{(i)}32^{(n-1-i)}) + \sum_{i=0}^{n-2} \zeta(2^{(i)}52^{(n-2-i)})
$$

$$
\zeta(5) \zeta(2^{(n-2)}) = \sum_{i=0}^{n-2} \zeta(2^{(i)}52^{(n-2-i)}) + \sum_{i=0}^{n-3} \zeta(2^{(i)}72^{(n-3-i)})
$$

$$
\vdots
$$

$$
\zeta(2n-1) \zeta(2) = \zeta(2n-1, 2) + \zeta(2, 2n-1) + \zeta(2n+1)
$$

$$
\zeta(2n+1) = \zeta(2n+1).
$$

Taking the alternate sum of these equalities we obtain equation (5.53).

We now prove equation (5.52) by induction on $n$. The case $n = 1$ is contained in Exercise 5.47. By Exercise 5.77, for $3 \leq 2r+1 < 2n$,

$$
D_{2r+1} \zeta^m_1(2^{(n)}) = \varpi_{2r+1}(\zeta^m_1(2^{(r)})) \otimes \zeta^m(2^{n-r}), \quad (5.54)
$$
by induction hypothesis and the fact that \( \varpi_{2r+1} \) kills products
\[
D_{2r+1} \psi_{1}^{m}(2^{(n)}) = 2(-1)^{r} \varpi_{2r+1}(\psi^{m}(2r + 1)) \otimes \psi^{m}(2^{n-r}).
\]
Moreover, using the fact that \( D_{2r+1} \) is a derivation,
\[
D_{2r+1}(\psi^{m}(2r + 1)\psi^{m}(2^{n-r})) = \varpi_{2r+1}(\psi^{m}(2r + 1)) \otimes \psi^{m}(2^{n-r}) \quad (5.55)
\]
and for \( r \neq i \)
\[
D_{2r+1}(\psi^{m}(2i + 1)\psi^{m}(2^{n-i})) = 0. \quad (5.56)
\]
Therefore, if \( \Theta \) is the difference of the two terms of equation (5.52), then for \( 3 \leq 2r + 1 < 2n \),
\[
D_{2r+1}\Theta = 0.
\]
Hence, by Theorem 5.44, \( \Theta \) is a multiple of \( \psi^{m}(2n + 1) \), and formula (5.52) follows from Corollary 5.45 and equations (5.53) and (5.51).

Given two integers \( r \) and \( s \), we let \( \mathbb{I}(r \geq s) \) denote the indicator function
\[
\mathbb{I}(r \geq s) = \begin{cases} 1 & r \geq s, \\ 0 & \text{else}. \end{cases}
\]

**Lemma 5.57.** Let \( a, b \geq 0 \) be integers. For each \( 1 \leq r \leq a + b \), one has
\[
D_{2r+1}\psi^{m}(2^{r})32^{(a)} = \varpi_{2r+1}(\xi^{r}_{a,b}) \otimes \psi^{m}(2^{a+b+1-r}),
\]
where the element \( \xi^{r}_{a,b} \in \mathcal{H} \) is given by
\[
\xi^{r}_{a,b} = \sum_{\begin{smallmatrix} \alpha \leq a \\ \beta \leq b \\ \alpha + \beta = r-1 \end{smallmatrix}} \psi^{m}(2^{\beta})32^{(a)} - \sum_{\begin{smallmatrix} \alpha \leq a \\ \beta \leq b \\ \alpha + \beta = r-1 \end{smallmatrix}} \psi^{m}(2^{(\alpha)})32^{(\beta)} + \left( \mathbb{I}(b \geq r) - \mathbb{I}(a \geq r) \right)\psi^{m}(2^{(r)}). \quad (5.58)
\]

**Proof.** To prove the result it is enough to check which non-zero terms appear in formula (5.43). These terms are given by consecutive subsequences of \( 2r + 1 \) entries and can be of the following types:

1. subsequences containing 001 and starting with 1, these contribute to the first sum;
2. subsequences containing 001 and starting with 0, after applying the reflection formula of Lemma 5.27, these subsequences contribute to the second sum;
3. when \( b \geq r \) there is exactly one sequence ending with 00 this gives the term \( \mathbb{I}(b \geq r)\psi^{m}(2^{(r)}) \);
4. when \( a \geq r \) there is exactly one sequence starting with 00. After applying the reflection formula we obtain the term \( -\mathbb{I}(a \geq r)\psi^{m}(2^{(r)}) \).

Using equation (5.26) is easy to check that all the other subsequences do not contribute to the result.
Proposition 5.59. Given $a, b \geq 0$, write $n = a + b + 1$. There exists a unique $n$-tuple of rational numbers $(\alpha_{a,b}^r)_{r=1,\ldots,n}$ such that

$$\zeta^m(2^b \cdot 32^a) = \sum_{r=1}^{n} \alpha_{a,b}^r \zeta^m (2r+1) \zeta^m (2^{n-r}).$$  \hspace{1cm} (5.60)

Proof. The proof proceeds by induction on $n$. Assume that the result holds for all integers smaller than $n$. In particular, all the numbers $\alpha_{a,b}^r$ are defined for $a + b + 1 < n$. Consider $a, b$ such that $a + b + 1 = n$. We compute $D_{2r+1} \zeta^m (2^b \cdot 32^a)$ for all $r < n$. Let $\xi_{a,b}^r$ be the term that appears in Lemma 5.57. By induction hypothesis and equation (5.52) we can write

$$\xi_{a,b}^r = \alpha_{a,b}^r \zeta^m (2r+1) \mod \text{products}$$

for a rational number $\alpha_{a,b}^r$. Therefore

$$\varpi_{2r+1} (\xi_{a,b}^r) = \alpha_{a,b}^r \varpi_{2r+1} (\zeta^m (2r+1)),$$

from which it follows that

$$D_{2r+1} \zeta^m (2^b \cdot 32^a) = \alpha_{a,b}^r \varpi_{2r+1} (\zeta^m (2r+1)) \otimes \zeta^m (2^{n-r}).$$  \hspace{1cm} (5.62)

Using equations (5.55) and (5.56) we deduce that both sides of the equation to be proved have the same image under $D_{2n+1}$. By Theorem 5.44, they differ by a rational multiple of $\zeta^m (2n+1)$ and one defines $\alpha_{a,b}^n$ in such a way that this difference is zero. \hfill $\Box$

We have here a remarkable example of both the strength and the limits of the motivic formalism. Applying the period map (5.9), the motivic identity (5.60) implies that the same holds for usual multiple zeta values, something which would have been difficult to predict working only with numbers, where the coaction is invisible. However, the motivic formalism alone does not allow us to compute the precise value of the constants $\alpha_{a,b}^r$. For this one needs to prove the corresponding identity of numbers first, then show that it is motivic. The first task was accomplished by Zagier in [Zag12].

5.3.2. Zagier’s theorem. Define, for each $a, b, r \geq 0$, rational numbers

$$A_{a,b}^r = \left(\frac{2r}{2a+2}\right), \quad B_{a,b}^r = (1 - 2^{-2r}) \left(\frac{2r}{2b+1}\right).$$  \hspace{1cm} (5.63)

As in the previous paragraph, we set $n = a + b + 1$.

Theorem 5.64 (Zagier, [Zag12]). The following equality holds:

$$\zeta (2^b \cdot 32^a) = 2 \sum_{r=1}^{n} (-1)^r (A_{a,b}^r - B_{a,b}^r) \zeta (2r+1) \zeta (2^{n-r}).$$  \hspace{1cm} (5.65)

Remark 5.66. The original proof of Zagier’s theorem has been simplified in [Li13]. It is also worth mentioning that Terasoma [Ter13] showed that the relation (5.65) holds for any associator.
5.3.3. **Lifting Zagier’s theorem to a motivic identity.** The first non-trivial case of Zagier’s theorem are the identities

\[
\zeta(2, 3) = -2\zeta(3)\zeta(2) + \frac{9}{2}\zeta(5),
\]
\[
\zeta(3, 2) = 3\zeta(3)\zeta(2) - \frac{11}{4}\zeta(5).
\]

In Exercise 5.48 we show that they lift to motivic equalities.

**Theorem 5.67.** For \(a, b \geq 0\) and \(1 \leq r \leq a + b + 1\), the numbers \(a_{a,b}^r\) from the statement of Proposition 5.59 are equal to

\[
a_{a,b}^r = (-1)^r 2 \left( A_{a,b}^r - B_{a,b}^r \right). \tag{5.68}
\]

In other words, writing \(n = a + b + 1\), the following identity of motivic multiple zeta values holds:

\[
\zeta^m(2^{(b)}32^{(a)}) = 2 \sum_{r=1}^{n} (-1)^r \left( A_{a,b}^r - B_{a,b}^r \right) \zeta^m(2r + 1) \zeta^m(2^{(n-r)}). \tag{5.69}
\]

**Proof.** We first note that, for any \(a, b \geq 0\) and \(1 \leq r \leq a + b + 1\), the following identities are satisfied:

\[
A_{a,b}^r = \sum_{\substack{a \leq a' \leq b \\ \beta \leq \beta' \leq b' \leq b+1}} A_{a',\beta'}^{r-1} - \sum_{\substack{a \leq a' \leq b' \leq b+1 \\ \alpha \leq \alpha' < r-1}} A_{a',\beta'}^{r-1} + \mathbb{I}(b \geq r) - \mathbb{I}(a \geq r), \tag{5.70}
\]

\[
B_{a,b}^r = \sum_{\substack{a \leq a' \leq b' \leq b+1 \\ \alpha \leq \alpha' < r-1}} B_{a',\beta'}^{r-1} - \sum_{\substack{a \leq a' \leq b' \leq b+1 \\ \alpha \leq \alpha' < r-1}} B_{a',\beta'}^{r-1}. \tag{5.71}
\]

This can be proved using that \(A_{a,b}^r\) does not depend on \(b\), that \(B_{a,b}^r\) does not depend on \(a\), and the symmetries \(A_{a,b}^{\alpha+\beta+1} = A_{b-1,a+1}^{\alpha+\beta+1}\) and \(B_{a,b}^{\alpha+\beta+1} = B_{b,a}^{\alpha+\beta+1}\). For instance, the second equality is clear because by symmetry each term of the second sum cancels one term of the first sum; the only remaining term in the first sum is \(B_{r-1-b,b}^r\) that agrees with \(B_{a,b}^r\) because it is independent of \(a\). To prove the first equality we may distinguish different cases according to whether \(a\) and \(b\) are bigger than or equal to \(r\) or not. For instance if \(a < r\) and \(b \geq r\), the term \(A_{a,b}^r\) is different from zero. In this case both sums range from \((\alpha, \beta) = (a, r - 1 - a)\) to \((0, r - 1)\). By the symmetry of the \(A\)’s all terms cancel except \(A_{a,r-1-a}^r\) from the first sum, that agrees with \(A_{a,b}^r\) and \(-A_{a,r-1}^r = -1\) that cancels with \(\mathbb{I}(b \geq r)\). The remaining cases are similar.

We now prove the result by induction on \(n = a + b + 1\). So we assume that equation (5.68) is true for all \(a', b'\) with \(a' + b' < n - 1\) and all \(1 \leq r' \leq a' + b' + 1\) and fix \(a, b\) with \(a + b + 1 = n\). We compute \(D_{2r+1} \zeta^m(2^{(b)}32^{(a)})\) in two ways and compare the results. The first way is equation (5.62), while the second is to apply Lemma 5.57, then use Lemma 5.50 to get rid of the terms \(\zeta_1^m(2^{(r)})\)
and apply equation (5.61) to the terms \( \varpi_{2r+1}(\zeta^m(2^{(2\{B\})}) \)). Comparing both results we obtain

\[
\alpha^r_{a,b} = \sum_{a \leq \alpha, \beta \leq b} \alpha_{a,\beta}^r - \sum_{\alpha \leq a, \beta < b} \alpha_{\beta,\alpha}^r + 2(-1)^r \left( \mathbb{I}(b \geq r) - \mathbb{I}(a \geq r) \right).
\]

By the induction hypothesis and equations (5.70) and (5.71), we deduce the equality (5.68) for \( 1 \leq r \leq a + b \).

To treat the remaining case \( r = a + b + 1 \), let \( \Theta \) be the difference between the left-hand side and the right-hand sides of equation (5.69); it is a motivic zeta value of weight \( 2a + 2b + 3 \). The identities we already proved and equation (5.60) yield \( D_{< 2a + 2b + 3}(\Theta) = 0 \). By Zagier’s Theorem 5.64, we obtain \( \text{per}(\Theta) = 0 \). Finally, Corollary 5.45 implies \( \Theta = 0 \), thus proving the result.

### 5.3.4. The coefficients \( c_s \)

Among the coefficients \( \alpha^r_{a,b} \), the leading one \( \alpha^a_{a,b} \) will play a special role, so we single it out.

**Definition 5.72.** Let \( s = 2^{(b)}32^{(a)} \) be an admissible multi-index with only one entry equal to 3 and all the remaining entries equal to 2. We set

\[
c_s = \alpha^{a+b+1}_{a,b}.
\]

We will also write

\[
c_{12^{(n)}} = 2(-1)^n.
\]

**Lemma** 5.50 and Proposition 5.59 are then rephrased as follows:

**Corollary 5.73.** For positive integers \( n, a, b \) with \( n = a + b + 1 \), the following equalities hold:

1. \( \varpi_{2n+1}(\zeta^m(2^{(n)})) = c_{12^{(n)}} \zeta^m(2n + 1) \),
2. \( \varpi_{2n+1}(\zeta^m(2^{(b)}32^{(a)})) = c_{2^{(b)}32^{(a)}} \zeta^m(2n + 1) \).

Moreover,

\[
c_{12^{(n)}} = -2 \sum_{i=0}^{n-1} c_{2^{(1)}32^{(n-i-1)}}.
\]

Recall that, given a prime number \( p \), the \( p \)-adic valuation of a non-zero rational number \( x \) is the only integer \( v_p(x) \) such that \( x \) can be written in the form \( x = p^{v_p(x)} \frac{a}{b} \) with \( a \) and \( b \) relatively prime to \( p \). We set \( v_p(0) = \infty \).

As a consequence of Theorem 5.67, the coefficients \( c_w \) have the following 2-adic properties.

**Lemma 5.75.** Let \( w \) a word of the form \( w = 2^{(b)}32^{(a)} \) and denote by \( w^* \) the word written in reverse order, i.e. \( w^* = 2^{(a)}32^{(b)} \). Then

1. \( c_w \in \mathbb{Z} \left\lceil \frac{1}{2} \right\rceil \),
(2) \( c_w - c_{w^*} \) is an even integer,

(3) \( v_2(c_2(n-1)3) = v_2(c_{32(n-1)}) \leq v_2(c_w) \leq 0 \).

**Proof.** Set \( n = a + b + 1 \). Recall the formula from Theorem 5.67:

\[
c_w = (-1)^n 2 \left( A^n_{a,b} - B^n_{a,b} \right).
\]

Since \( A^n_{a,b} \) is an integer and \( B^n_{a,b} \) belongs to \( \mathbb{Z}[\frac{1}{2}] \), the first claim follows.

Property (2) is obtained from the symmetry \( B^n_{a,b} = B^n_{b,a} \). Indeed,

\[
c_w - c_{w^*} = (-1)^n 2[A^n_{a,b} - A^n_{b,a}] \in 2\mathbb{Z}.
\]

To prove (3), we first observe that \( v_2((2n)! < 2n \), and hence

\[
v_2(2^{-2n} \binom{2n}{2b+1}) < 0.
\]

Using the triangle inequality, it follows that

\[
v_2(c_w) = v_2(2 \cdot 2^{-2n} \binom{2n}{2b+1}) = 1 + v_2(2^{-2n} \binom{2n}{2b+1}) \leq 0.
\]

For the remaining inequality, we write

\[
\binom{2n}{2b+1} = \frac{2n}{2b+1} \binom{2n-1}{2b}.
\]

Therefore,

\[
v_2(c_w) = 2 - 2n + v_2(n) + v_2(\binom{2n-1}{2b}).
\]

Since \( v_2(\binom{2n-1}{2b}) \geq 0 \), the right-hand side attains its minimum for \( b = n - 1 \) and \( b = 0 \), which correspond to the cases \( w = 2^{(n-1)3} \) and \( w = 32^{(n-1)} \). □

***

**Exercise 5.76.** Show that one may replace the multiple zeta value \( \zeta(2^{(n-r)}) \) with either \( \zeta(2n - 2r) \) or \( \zeta(2)^{n-r} \) in the right-hand side of Zagier’s theorem 5.64 without losing the rationality of the coefficients \( \alpha_{a,b}^{n,r} \).

**Exercise 5.77.** Prove equation (5.54).

### 5.4. The subspaces \( \mathcal{H}^{2,3} \).

**Definition 5.78.** We denote by \( \tilde{\mathcal{H}}^{2,3} \subset \mathcal{O}(\Pi) \) the subspace generated by the functions \( f(1; \alpha; 0) \), where \( \alpha \) is the binary sequence associated with an admissible multi-index containing only 2 and 3 as entries, and by \( \mathcal{H}^{2,3} \subset \mathcal{H} \) the image of \( \tilde{\mathcal{H}}^{2,3} \) under the restriction map

\[
\text{res}: \mathcal{O}(\Pi) \rightarrow \mathcal{H}.
\]
Clearly, $\mathcal{H}^{2,3}$ is the $\mathbb{Q}$-vector space spanned by the motivic multiple zeta values $\zeta^m(s_1, \ldots, s_r)$ with $s_i \in \{2, 3\}$.

From now on, we identify the set of words in the alphabet $\{2, 3\}$ with the set of admissible multi-indices with only 2 and 3 as entries.

We filter $\tilde{\mathcal{H}}^{2,3}$ by the number of entries equal to 3 in the admissible multi-index. Precisely, for each integer $\ell \geq 0$, consider

$$F_{\ell} \tilde{\mathcal{H}}^{2,3} = \langle I(1; bs(s); 0) \mid s \text{ contains } \leq \ell \text{ entries equal to 3} \rangle_\mathbb{Q}.$$ 

This defines an increasing level filtration

$$0 \subseteq F_{0} \tilde{\mathcal{H}}^{2,3} \subseteq F_{1} \tilde{\mathcal{H}}^{2,3} \subseteq \cdots$$

By restriction, we deduce an increasing filtration on $\mathcal{H}^{2,3}$ with

$$F_{\ell} \mathcal{H}^{2,3} = \langle \zeta^m(s_1, \ldots, s_r) \in \mathcal{H}^{2,3} \mid \text{number of } s_i = 3 \leq \ell \rangle_\mathbb{Q}.$$ 

The associated graded pieces $\text{gr}_{\ell} F \mathcal{H}^{2,3}$ are the $\mathbb{Q}$-linear spans of motivic multiple zeta values with exactly $\ell$ entries equal to 3. In particular,

$$\text{gr}_{0} F \mathcal{H}^{2,3} = \langle \zeta^m(2^n) \mid n \geq 1 \rangle_\mathbb{Q},$$

$$\text{gr}_{1} F \mathcal{H}^{2,3} = \langle \zeta^m(2^b) 32(a) \mid a, b \geq 0 \rangle_\mathbb{Q}.$$ 

Note that these are precisely the two families of motivic multiple zeta values that we studied in the previous section.

**Remark 5.79.** The $\mathbb{Q}$-vector space $\text{gr}_{\ell} F \tilde{\mathcal{H}}^{2,3}_N$ is non-empty if and only if the weight $N$ and the level $\ell$ have the same parity. When this is the case, writing $N = 2m + 3\ell$, the dimensions are given by

$$\dim_\mathbb{Q} \text{gr}_{\ell} F \tilde{\mathcal{H}}^{2,3}_N = \binom{m + \ell}{\ell}.$$ 

**5.4.1. The level lowering operator.** Recall that in Section 3.9.6 we introduced Goncharov’s coproduct as a morphism

$$\Delta : \mathcal{O}(\Pi) \longrightarrow \mathcal{O}(\Pi) \otimes_\mathbb{Q} \mathcal{O}(\Pi). \quad (5.80)$$

From this we obtained the motivic coaction (5.32)

$$\Delta : \mathcal{H} \longrightarrow \mathcal{A} \otimes_\mathbb{Q} \mathcal{H}$$

that we have been using in the last pages. In what follows, we will also use an intermediate version

$$\Delta : \mathcal{O}(\Pi) \longrightarrow \mathcal{A} \otimes \mathcal{O}(\Pi)$$

which is simply obtained from (5.80) via the projection $\mathcal{O}(\Pi) \to \mathcal{A}$ (recall that this corresponds to restricting a function on $\Pi$ to the subvariety $\mathcal{X}$ of Remark 5.30). This is nothing else but the coaction associated with the action of $I(U_{dR})$ on $\Pi$. As in Definition 5.35, there are maps

$$D_{2r+1} : \mathcal{O}(\Pi) \longrightarrow \mathcal{L}_{2r+1} \otimes \mathcal{O}(\Pi).$$
Following the proof of Proposition 5.42 we see that, for all odd integers \( n < N \), the analogue of (5.43) also holds:

\[
D_n I(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_N; \varepsilon_{N+1}) = 
\sum_{p=0}^{N-n} \varpi_n \left( I_m(\varepsilon_p; \varepsilon_{p+1} \cdots \varepsilon_{p+n}; \varepsilon_{p+n+1}) \right) 
\otimes I(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_p, \varepsilon_p+1, \cdots, \varepsilon_N; \varepsilon_{N+1}).
\]

(5.81)

We now study how the filtered subspace \( \tilde{H}^{2,3} \subset O(\Pi) \) behaves with respect to the coproduct and the infinitesimal coaction.

**Lemma 5.82.** The subspace \( \tilde{H}^{2,3} \) is stable under the coaction:

\[
\Delta: \tilde{H}^{2,3} \rightarrow A \otimes_{Q} \tilde{H}^{2,3}.
\]

**Proof.** Let \( I(1; \alpha; 0) \) be an element of \( \tilde{H}^{2,3} \). Then \( \alpha \) is a binary sequence obtained by successive concatenation of the subsequences 01 and 001. From the explicit formula for the coproduct (5.33) and the fact that the iterated integrals \( I(\varepsilon; \alpha'; \varepsilon') \) vanish when \( \varepsilon = \varepsilon' \) and \( \alpha' \neq \emptyset \), we deduce that each non-trivial term appearing in \( \Delta I(1; \alpha; 0) \) has, in the right-hand side of the coaction, a factor of the form \( I(1; \beta; 0) \), where \( \beta \) is again a concatenation of the subsequences 01 and 001. \( \square \)

**Remark 5.83.** In [Del13, §6.3], the above result is rephrased by saying that the subspace \( \tilde{H}^{2,3} \) is “motivic”, thus invariant under the action of \( U_{dR} \).

From this we immediately deduce:

**Corollary 5.84.** For each \( r \geq 1 \), the derivation \( D_{2r+1} \) restricts to a map

\[
D_{2r+1}: \tilde{H}^{2,3} \rightarrow L_{2r+1} \otimes_{Q} \tilde{H}^{2,3}.
\]

In fact, more is true:

**Lemma 5.85.** The derivations \( D_{2r+1} \) are compatible with the level filtration, in the sense that:

\[
D_{2r+1}: F_{\ell} \tilde{H}^{2,3} \rightarrow L_{2r+1} \otimes_{Q} F_{\ell-1} \tilde{H}^{2,3}.
\]

**Proof.** Given a word \( s \) in the alphabet \( \{2, 3\} \) of level \( \ell \), then \( bs(s) \) contains exactly \( \ell \) subsequences 00. Any subsequence of odd length of \( (1; bs(s); 0) \) that begins and ends with the same symbol will be killed by \( I_m \) and will not contribute to \( D_{2r+1} \). Otherwise it must contain at least a subsequence 00. Thus the complementary quotient sequence will contain at most \( \ell-1 \) subsequences 00. Hence will have level at most \( \ell-1 \). \( \square \)

The above lemma yields a map

\[
\text{gr}_{\ell} F D_{2r+1}: \text{gr}_{\ell} F \tilde{H}^{2,3} \rightarrow L_{2r+1} \otimes \text{gr}_{\ell-1} F \tilde{H}^{2,3}.
\]

(5.86)
Lemma 5.87. For all $r, \ell \geq 1$, one has

$$\text{gr}^F_\ell D_{2r+1} (\text{gr}^F_\ell \widetilde{H}^{2,3}_N) \subseteq \mathbb{Q} \varpi_{2r+1} (\zeta^m (2r+1)) \otimes \text{gr}^F_{\ell-1} \widetilde{H}^{2,3}.$$  

**Proof.** Let $s$ be a word in the alphabet $\{2, 3\}$ of level $\ell$, and let $I^m(1; bs(s); 0)$ be the corresponding motivic iterated integral. From the definition of $D_{2r+1}$, we have

$$\text{gr}^F_\ell D_{2r+1} (\zeta^m(s)) = \sum_{\gamma} \varpi_{2r+1} (I^m(\gamma)) \otimes \zeta^m(s_{\gamma}),$$  

where the sum runs over all subsequences $\gamma$ of $(1; bs(s); 0)$ of length $2r+1$, and $s_{\gamma}$ is obtained by removing the internal part of $\gamma$.

If $\gamma$ contains more than one subsequence $00$, then $s_{\gamma}$ has level $< \ell - 1$, and hence does not contribute. If $\gamma$ begins and ends in the same symbol, then $I^m(\gamma)$ is zero. One checks that $I^m(\gamma)$ can be of four remaining types:

1. $I^m(1; 01 \ldots 01; 0) = \zeta^m (2\{\beta\}, 32\{\alpha\}),$
2. $I^m(0; 10 \ldots 10; 1) = -\zeta^m (2\{\beta\}, 32\{\alpha\}),$
3. $I^m(1; 01 \ldots 10; 0) = \zeta^m (2^r),$
4. $I^m(0; 01 \ldots 10; 1) = -\zeta^m (2^r).$

By Corollary 5.73, in all cases one has $\varpi_{2r+1}(I^m(\gamma)) \in \mathbb{Q} \zeta^m (2r+1)$. \hfill \qedsymbol

Remark 5.89. Lemma 5.87 says that the map (5.86) factors through the one-dimensional subspace $[(\text{ud}_R)^{ab}_{2r+1}]^\vee$ of $L_{2r+1}$.

The above lemma justifies the following definition:

**Definition 5.90.** For all $N, \ell \geq 1$, the level lowering operator $\widetilde{\partial}_{N, \ell}$ is the $\mathbb{Q}$-linear map

$$\widetilde{\partial}_{N, \ell}: \text{gr}^F_\ell \widetilde{H}^{2,3}_N \longrightarrow \bigoplus_{3 \leq r+1 \leq N} \text{gr}^F_{\ell-1} \widetilde{H}^{2,3}_{N-2r-1}$$  

obtained by first applying

$$\bigoplus_{3 \leq r+1 \leq N} \text{gr}^F_{\ell} D_{2r+1} |_{\text{gr}^F_\ell \widetilde{H}^{2,3}_N}$$

and then sending $\varpi_{2r+1}(\zeta^m (2r+1))$ to 1.

5.4.2. A pair of bases. We next describe bases of the source and the target of the map (5.91). For $\ell \geq 1$ and $N \geq 3$, we define

$B_{N, \ell} = \text{set of words in the alphabet } \{2, 3\} \text{ of weight } N \text{ and level } \ell.$

$B^\prime_{N, \ell} = \text{set of words in the alphabet } \{2, 3\} \text{ of weight } \leq N - 3 \text{ and level } \ell - 1$

(this includes the empty word if $\ell = 1$).
Clearly, $B_{N,\ell}$ gives a basis $B_{N,\ell}$ of $\operatorname{gr}^F_{\ell} \tilde{H}^2_{N-1}$, while $B'_{N,\ell}$ determines a basis $B'_{N,\ell}$ of $\bigoplus_{3 \leq 2r+1 \leq N} \operatorname{gr}^F_{\ell-1} \tilde{H}^2_{N-2r-1}$. Write $N = 3\ell + 2m$, so $m$ is the number of 2s in an element of $B_{N,\ell}$. Then

$$|B_{N,\ell}| = \left(\ell + m\right) = \sum_{m'=0}^{m} \left(\ell - 1 + m'\right) = |B'_{N,\ell}|.$$

We provide $B_{N,\ell}$ with the lexicographic order for the ordering $2 < 3$ and $B'_{N,\ell}$ with the order $s \leq s'$ if and only if $\operatorname{wt}(s) < \operatorname{wt}(s')$ or $\operatorname{wt}(s) = \operatorname{wt}(s')$ and $s$ is smaller than or equal to $s'$ in the lexicographic order.

**Lemma 5.92.** The map $B'_{N,\ell} \rightarrow B_{N,\ell}$ that sends an element $s \in B'_{N,\ell}$ to $2^{2r-1}3s \in B_{N,\ell}$, where $2r = N - 1 - \operatorname{wt}(s)$ is an order-preserving bijection.

**Proof.** Denote by $\upsilon$ the map in the statement. If $\operatorname{wt}(s) < \operatorname{wt}(s')$, then $r > r'$, and hence $\upsilon(s) = 2^{2r-1}3s < 2^{2r'-1}3s' = \upsilon(s')$. If $\operatorname{wt}(s) = \operatorname{wt}(s')$ but $s$ is smaller than $s'$ in the lexicographic order, then $\upsilon(s) = 2^{2r-1}3s < 2^{2r-1}3s' = \upsilon(s')$.

Therefore, $\upsilon$ is injective and order-preserving. Since the sets $B'_{N,\ell}$ and $B_{N,\ell}$ have the same cardinality, $\upsilon$ is a bijection. □

### 5.5. Brown’s theorem.

5.5.1. **Statement.** The goal of this section is to prove the following result, which directly implies Theorem B:

**Theorem 5.93 (Brown).** The set of elements

$$\{\zeta^m(s_1, \ldots, s_r) \mid s_i \in \{2, 3\}\}$$

forms a basis of the $\mathbb{Q}$-vector space of motivic multiple zeta values.

Before going into the proof, let us mention the immediate corollary:

**Corollary 5.94 (Theorem B).** Every multiple zeta value is a $\mathbb{Q}$-linear combination of MZVs with only 2 and 3 as entries.

**Proof.** Apply the period map (5.9). □

**Remarks 5.95.**

(1) Unfortunately, the proof does not give an algorithm to compute such a linear combination.

(2) The missing information to deduce that such multiple zeta values furnish a basis, as it is conjectured, is to know that all relations among multiple zeta values have motivic origin.
5.5.2. **Strategy of the proof.** The key point to prove Theorem 5.93 is the following lemma.

**Lemma 5.96.** For all \( N, \ell \geq 1 \), the level lowering operator \( \tilde{\partial}_{N, \ell} \) is an isomorphism of \( \mathbb{Q} \)-vector spaces.

We show how to deduce Theorem 5.93 from Lemma 5.96. This amounts to proving the following:

**Lemma 5.97.** The map \( \tilde{H}^{2,3} \to H^{2,3} \) is an isomorphism.

**Proof.** We first prove by induction on the level that, for every weight \( N \) and level \( \ell \), the restriction map \( \text{gr}_F^{\ell} \tilde{H}^{2,3} \to \text{gr}_F^{\ell} H^{2,3} \) is an isomorphism.

The initial step is \( \ell = 0 \). If \( N = 2r \) is even, the space \( \text{gr}_F^0 \tilde{H}^{2,3} \) is one-dimensional generated by the symbol \( I(1; bs(2\{r\}); 0) \) while the space \( \text{gr}_F^0 H^{2,3} \) is generated by \( \zeta(2\{r\}) \neq 0 \). Thus the restriction map

\[
\text{gr}_F^0 \tilde{H}^{2,3} \to \text{gr}_F^0 H^{2,3}
\]

(5.98)

is an isomorphism. If \( N \) is odd, then both spaces are zero and therefore the map (5.98) is also an isomorphism.

We now consider the commutative diagram

\[
\begin{array}{c}
\text{gr}_F^{\ell} \tilde{H}^{2,3} \\
\downarrow \\
\text{gr}_F^{\ell} H^{2,3}
\end{array}
\begin{array}{c}
\tilde{\partial}_{N, \ell} \\
\oplus_{3 \leq 2r+1 \leq N} \text{gr}_F^{\ell-1} \tilde{H}^{2,3}_{N-2r-1}
\end{array}
\begin{array}{c}
\oplus_{3 \leq 2r+1 \leq N} \text{gr}_F^{\ell-1} H^{2,3}_{N-2r-1}
\end{array}
\]  

(5.99)

By definition, the left vertical arrow is an epimorphism. By the induction hypothesis, the right vertical map is an isomorphism and by Lemma 5.96 the upper horizontal map is injective. We conclude that the left vertical arrow is an isomorphism.

Once we now that all the restriction maps \( \text{gr}_F^{\ell} \tilde{H}^{2,3} \to \text{gr}_F^{\ell} H^{2,3} \) are isomorphisms, we deduce that the restriction map \( \tilde{H}^{2,3} \to H^{2,3} \) is an isomorphism by using the fact that the filtration \( F \) is bounded below and the five lemma. Finally, since the weight is a grading in both \( \tilde{H}^{2,3} \) and \( H^{2,3} \) we obtain that the map \( \tilde{H}^{2,3} \to H^{2,3} \) is an isomorphism.

\[\square\]

5.5.3. **Proof of Lemma 5.96.** The proof is based on the study of the 2-adic valuation of the coefficients of the matrix of \( \tilde{\partial}_{N, \ell} \) with respect to the bases introduced in Section 5.4.2. We shall use the following lemma:

**Lemma 5.100.** Let \( A = (a_{ij})_{i,j} \) be a square matrix of size \( n \) with rational coefficients. Assume that there exists a prime number \( p \) such that the following conditions hold:

(a) \( v_p(a_{ij}) \geq 1 \) for all \( i > j \),
Then $A$ is invertible.

**Proof.** Consider the matrix $A'$ obtained by multiplying the $i$-th row of $A$ by $p^{-v_p(a_{ii})}$. By condition (b), the coefficients of $A'$ are $p$-integral, so we can reduce modulo $p$. Since we still have $v_p(a_{ij}') \geq 1$ for $i > j$ but now $v_p(a_{ii}') = 0$, the reduction is upper triangular with non-zero elements in the diagonal. It follows that the determinant of $A'$, and hence the determinant of $A$, is non-zero.

We next see that, up to terms with even coefficients, the map $\tilde{\partial}_{N,\ell}$ acts by deconcatenation.

**Theorem 5.101.** Let $s$ be a word in the alphabet $\{2, 3\}$ of weight $N$ and level $\ell$. Then

$$\tilde{\partial}_{N,\ell}I(1; bs(s); 0) = \sum_{\substack{u=uv \\
\deg_3 u = 1}} c_u I(1; bs(v); 0) + \text{terms with } 2\mathbb{Z} \text{ coefficients},$$

where $\deg_3 u$ is the numbers of 3 in the word $u$.

**Proof.** Following the proof of Lemma 5.87, there are four types of terms in $\tilde{\partial}_{N,\ell}I(1; bs(s); 0))$. We start with (3) and (4). Since $c_{12(n)} = 2(-1)^n$, these terms contribute with even coefficients. Besides, almost all terms of types (1) and (2) can be grouped in pairs. Choose four positions as follows

\[ I(\ldots 01a \ldots 10b \ldots 01 \ldots) \]

that is, $a$ and $b$ (resp. $c$ and $d$) are consecutive, $a$ (resp. $d$) contains a 0 and $b$ (resp. $c$) contains a 1. Combining Lemma 5.27 (2) Lemma 5.75 (2), the sum of the contributions of the subsequences $ac$ and $bd$ has again coefficients in $2\mathbb{Z}$. The only terms that cannot be paired this way are the leftmost subsequences appearing in the sum of the statement.

**Corollary 5.102.** With respect to the bases $B_{N,\ell}$ and $B'_{N,\ell}$, ordered as in paragraph 5.4.2, the matrix $M_{N,\ell}$ of the operator $\tilde{\partial}_{N,\ell}$ satisfies the assumptions of Lemma 5.100 for the prime $p = 2$.

**Proof.** Let $v$ be an admissible multi-index with only 2 and 3 as entries, of weight $\leq N - 3$ and level $\ell - 1$. Put $2r = N - 1 - \text{wt}(v)$ and $s = 2^{(r-1)}3v$. Then $s$ is the multi-index corresponding to $v$ under the order-preserving bijection from Lemma 5.92. Consider any admissible multi-index with only 2 and 3 as entries, of weight $N$ and level $\ell$ that can be written as $uv$ with $\deg_3 u = 1$. If $s \neq uv$, then the number on $2s$ before the first 3 in $u$ is smaller than $r - 1$. Hence $uv > s$. By Theorem 5.101, this implies that any term in $M_{N,\ell}$ that is not an even integer is above the diagonal. Moreover, by the same theorem and the statement (3) of Lemma 5.75, the coefficient
of $\nu$ in $\bar{\partial}_{N,\ell}$s sitting at the diagonal of $M_{N,\ell}$ has 2-adic valuation smaller than or equal to zero and it realizes the minimum of this valuation within its row. Therefore, the assumptions of Lemma 5.100 are satisfied.\hfill \Box

Clearly, Lemma 5.96 is a consequence of Corollary 5.102 and Lemma 5.100, thus finishing the proof of Theorem 5.93.

5.5.4. Some consequences of Brown’s theorem. We conclude this chapter with some corollaries of Brown’s theorem.

**Corollary 5.103.** The map $U_{\text{dR}} \rightarrow I(U_{\text{dR}})$ is a group isomorphism.

**Proof.** Recall from (5.5) that $A^{\text{MT}} = \mathcal{O}(U_{\text{dR}})$ and $A = \mathcal{O}(I(U_{\text{dR}}))$. We want to show that the injective map $A \hookrightarrow A^{\text{MT}}$ induced by $U_{\text{dR}} \rightarrow I(U_{\text{dR}})$ is surjective. In Corollary 5.18 we proved that this map extends to an injection $\mathcal{H} \hookrightarrow \mathcal{H}^{\text{MT}}$ compatible with the gradings on both sides. Brown’s theorem implies that the dimension of the graded pieces of $\mathcal{H}$ agree with those of $\mathcal{H}^{\text{MT}}$, and hence the algebras are isomorphic.\hfill \Box

Let $\text{MT}'(\mathbb{Z})$ be the full Tannakian subcategory of $\text{MT}(\mathbb{Z})$ generated by the objects $xU_{\text{y Mot},N}$ for $N \geq 0$ and $x, y \in \{0, 1\}$ and let $\omega'_{\text{dR}}$ be the fibre functor $\omega_{\text{dR}}$ restricted to $\text{MT}'(\mathbb{Z})$.

**Corollary 5.104.** The quotient $\text{Aut}_{\text{MT}(\mathbb{Z})}^\otimes(\omega_{\text{dR}}) \rightarrow \text{Aut}_{\text{MT}'(\mathbb{Z})}^\otimes(\omega'_{\text{dR}})$ is an isomorphism of affine group schemes. It follows that the inclusion $\text{MT}'(\mathbb{Z}) \rightarrow \text{MT}(\mathbb{Z})$ is an equivalence of Tannakian categories, so that every mixed Tate motive over $\mathbb{Z}$ is a subquotient of a tensor construction on one of the finite-dimensional pieces of the motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

**Proof.** The Tannaka group $\text{Aut}_{\text{MT}(\mathbb{Z})}^\otimes(\omega_{\text{dR}})$ is $I(U_{\text{dR}}) \times \mathbb{G}_m$. Thus the fact that the morphism of Tannaka groups is an isomorphism follows from Corollary 5.103. As a consequence, both $\text{MT}(\mathbb{Z})$ and $\text{MT}'(\mathbb{Z})$ are equivalent to the category of finite dimensional representations of $G_{\text{dR}}$.\hfill \Box

**Corollary 5.105.** The periods of every mixed Tate motive over $\mathbb{Z}$ are linear combinations with $\mathbb{Q}[\frac{1}{2\pi i}]$ coefficients of multiple zeta values. In other words, the ring of periods of mixed Tate motives over $\mathbb{Z}$ is $\mathbb{Z}[\frac{1}{2\pi i}]$.

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
U_{\text{dR}} \times \mathbb{G}_m & \xrightarrow{f_1} & G_{\text{dR}} & \xrightarrow{f_2} & P_{\text{dR,B}} \\
\downarrow g_1 & & & & \downarrow g_2 \\
U_{\text{dR}} \times \mathbb{A}^1 & \xrightarrow{f_3} & \mathcal{Y}
\end{array}
\]
where
\[
\begin{align*}
f_1(u, s) &= u \cdot \tau(s), \quad g_1(u, s) = (u, s^2) \\
f_2(g) &= g \cdot a^{-1} \cdot \text{comp}_{\text{dR}, B} \\
g_2(p) &= p \cdot \text{dch}, \quad f_3(u, t) = \psi(u, t)(1_{1_0}),
\end{align*}
\]
where \(a\) is defined in Proposition 4.96. The commutativity of the above diagram follows from the definition of \(\psi\) in Theorem 5.17. The upper horizontal arrows are clearly isomorphisms and the lower horizontal arrow is an isomorphism by Corollary 5.103.

By (5.13), \(f_1(u_0, 2\pi i) = a\). Clearly
\[
f_2(a) = \text{comp}_{\text{dR}, B}, \quad g_2(\text{comp}_{\text{dR}, B}) = \text{dch}^{\text{dR}}, \quad g_1(u_0, 2\pi i) = (u_0, (2\pi i)^2).
\]
By the commutativity of the diagram \(f_3(u_0, (2\pi i)^2) = \text{dch}^{\text{dR}}\). All the morphisms on the diagram are defined over \(\mathbb{Q}\).

The algebra of periods of \(\text{MT}(\mathbb{Z})\) is
\[
\text{ev}_{\text{comp}_{\text{dR}, B}}(\mathcal{O}(P_{\text{dR}, B})) = \text{ev}_{(u_0, 2\pi i)}(\mathcal{O}(U_{\text{dR}} \times \mathbb{G}_m)).
\]
The algebra of multiple zeta values is
\[
\text{ev}_{\text{dch}^{\text{dR}}}(\mathcal{O}(Y)) = \text{ev}_{(u_0, (2\pi i)^2)}(\mathcal{O}(U_{\text{dR}} \times \mathbb{A}^1)).
\]
Since \(\mathcal{O}(U_{\text{dR}} \times \mathbb{G}_m) = \mathcal{O}(U_{\text{dR}} \times \mathbb{A}^1)[s^{-1}]\) and \(s(u_0, 2\pi i) = 2\pi i\), the result follows.

**Corollary 5.106.** Zagier’s conjecture 1.71 implies that the numbers \(\pi, \zeta(3), \zeta(5), \ldots\) are algebraically independent over \(\mathbb{Q}\).

**Proof.** The key ingredient is a structure theorem for Hopf algebras due to Milnor and Moore [MM65]:

**Theorem 5.107 (Milnor–Moore).** Let \(k\) be a field of characteristic zero and \(A = \bigoplus_{n \geq 0} A_n\) a graded connected commutative Hopf algebra over \(k\) such that \(A_n\) is finite-dimensional for all \(n\). Then \(A\) is the symmetric algebra
\[
A = \text{Sym}[A_{>0}/(A_{>0})^2].
\]

We will use the theorem through the following straightforward corollary: if \(x_1, x_2, \ldots\) are elements of \(A_{>0}\) whose classes in the quotient \(A_{>0}/(A_{>0})^2\) are linearly independent, then \(x_1, x_2, \ldots\) are algebraically independent.

Let us apply this corollary to the Hopf algebra \(A = \mathcal{O}(U_{\text{dR}})\) and the motivic zeta values \(\zeta^m(3), \zeta^m(5), \ldots\). These elements lie in different degrees and their images in the quotient \(\mathcal{L} = A_{>0}/(A_{>0})^2\) are non-zero, so they are linearly independent. By the corollary, \(\zeta^m(3), \zeta^m(5), \ldots\) are algebraically independent in \(\mathcal{A}\). Since \(\mathcal{H} = \mathcal{A}[\zeta^m(2)]\), we deduce that the motivic zeta values \(\zeta^m(2), \zeta^m(3), \zeta^m(5), \ldots\) are algebraically independent in \(\mathcal{H}\). Now, if one assumes Zagier’s conjecture, the period map \(\text{per}: \mathcal{H} \to \mathcal{Z}\) is an isomorphism. Since \(\text{per}(\zeta^m(n)) = \zeta(n)\) and \(\zeta(2) = \frac{\pi^2}{6}\), it follows that the numbers \(\pi, \zeta(3), \zeta(5), \ldots\) are algebraically independent over \(\mathbb{Q}\). \(\square\)
Corollary 5.108. Zagier’s conjecture 1.71 is equivalent to Grothendieck’s period conjecture for mixed Tate motives 4.103.

Proof. Zagier’s conjecture is equivalent to the injectivity of the map per: \( \mathcal{H} \rightarrow \mathbb{C} \). Since \( \mathcal{O}(P_{dR,B}) = \mathcal{H}[s^{-1}] \) with \( s^2 = -24\zeta^m(2) \), this is equivalent to the injectivity of the period map per: \( \mathcal{O}(P_{dR,B}) \rightarrow \mathbb{C} \), which is precisely the content of Conjecture 4.103. \( \square \)
Appendix A. Some results from homological algebra

For the convenience of the reader, in this appendix we gather a number of notions and results from category theory, homological algebra, and sheaf theory that are used through the main text. We assume that the reader is familiar with the notion of category and of functor between two categories. Recall that a category is called small if its objects and morphisms form a set. All the categories we will work with will be essentially small in the sense that they are equivalent to a small one. To avoid deep set-theoretic considerations, we will always replace such categories with small equivalent ones.

A.1. Abelian categories.

Definition A.1. A category $\mathcal{C}$ is called additive if the following conditions are satisfied:

1. For all objects $X, Y \in \text{Ob}(\mathcal{C})$, the set $\text{Hom}_{\mathcal{C}}(X, Y)$ has the structure of an abelian group. In particular, there is a zero morphism 0 between any two objects of $\mathcal{C}$. We denote by $+$ the group law and call it addition. It is further required that composition of morphisms and addition satisfy the distributivity relation:
   \[
   f \circ (g + h) = f \circ g + f \circ h, \quad (g + h) \circ f = g \circ f + h \circ g.
   \]

2. There exists a zero object, that is, there is an object $0 \in \text{Ob}(\mathcal{C})$ such that, for every $X \in \text{Ob}(\mathcal{C})$, there is a unique morphism $0 \to X$ and a unique morphism $X \to 0$. In other words, 0 is an initial and final object of the category. The zero morphism from (1) is nothing but the composition $X \to 0 \to Y$.

3. Given two objects $X, Y \in \text{Ob}(\mathcal{C})$, there exists a direct sum object i.e. an object $X \oplus Y \in \text{Ob}(\mathcal{C})$ and morphisms
   \[
   X \longrightarrow X \oplus Y \leftarrow Y
   \]
   such that, for each object $Z \in \text{Ob}(\mathcal{C})$ equipped with morphisms $X \to Z$ and $Y \to Z$, there exists a unique morphism $X \oplus Y \to Z$ making the following diagram commutative:
   \[
   \begin{array}{ccc}
   X & \longrightarrow & X \oplus Y \\
   \downarrow & & \downarrow \\
   Z & \longrightarrow & Y.
   \end{array}
   \]

A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is called additive if, for any two objects $X, Y \in \text{Ob}(\mathcal{C})$, the morphism
   \[
   F(X) \oplus F(Y) \longrightarrow F(X \oplus Y)
   \]
obtained from property (3) above for the objects $F(X), F(Y)$, and $F(X \oplus Y)$ of $\mathcal{D}$ is an isomorphism.
DEFINITION A.2. Let \( f: X \to Y \) be a morphism in an additive category \( C \). A kernel of \( f \) is a pair consisting of an object \( \text{Ker}(f) \) of \( C \) and a morphism \( \iota: \text{Ker}(f) \to X \) such that \( f \circ \iota = 0 \) and that the following universal property holds: for every morphism \( g: Z \to X \) with \( f \circ g = 0 \), there exists a unique morphism \( \varphi: Z \to \text{Ker}(f) \) making the diagram below commutative:

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
\downarrow{\varphi} & & \downarrow{\iota} \\
\text{Ker}(f) & & \\
\end{array}
\]

Such an object may or may not exist (Exercise A.13). Whenever it does, \( \text{Ker}(f) \) is not unique but unique up to a unique isomorphism, as is the case for all solutions of universal problems. In practice, we will identify all possible solutions through the unique isomorphisms and pretend that \( \text{Ker}(f) \) is unique. By abuse of language, the symbol \( \text{Ker}(f) \) will denote at the same time the object and the morphism \( \text{Ker}(f) \to X \).

DEFINITION A.3. Let \( f: X \to Y \) be a morphism in an additive category \( C \). A cokernel of \( f \) is a pair consisting of an object \( \text{Coker}(f) \) of \( C \) and a morphism \( p: Y \to \text{Coker}(f) \) satisfying \( p \circ f = 0 \) and that such that \((\text{Coker}(f), p)\) is universal among pairs with this property.

Similarly, the cokernel of a map is the dual notion to the kernel (Exercise A.14).

Whenever they exist, the image and the coimage of a morphism \( f \) in an additive category are defined as

\[
\text{Im}(f) = \text{Ker}(\text{Coker}(f)), \quad \text{Coim}(f) = \text{Coker}(\text{Ker}(f)).
\]

If \( \text{Im}(f) \) and \( \text{Coim}(f) \) exist, then there is a canonical morphism (Exercise A.15)

\[
\text{Coim}(f) \to \text{Im}(f).
\]

DEFINITION A.4. An abelian category is an additive category \( C \) satisfying the following two conditions:

1. Every morphism \( f \) in \( C \) has a kernel and a cokernel. Therefore, every morphism has an image and a coimage.
2. For every morphism \( f \), the canonical map \( \text{Coim}(f) \to \text{Im}(f) \) is an isomorphism.

See Exercise A.75 for a typical example of an additive category that has all kernels and cokernels but is not abelian since condition (2) above fails.

The following result allows us to work in any small abelian category as if it were a category of modules over a ring. For a proof, see [Wei94, Theorem 1.6.1].
THEOREM A.5 (Freyd–Mitchell). Let $\mathcal{C}$ be a small abelian category. There exists a ring $R$ and an exact and fully faithful functor from $\mathcal{C}$ to the category of $R$-modules.

DEFINITION A.6. Let $\mathcal{A}$ be an additive category.

1. A cochain complex $A = (A^\ast, d)$ is a collection of objects $A^n$ of $\mathcal{A}$, one for each $n \in \mathbb{Z}$, and morphisms

\[ \cdots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \cdots \]

called differentials such that, for all $n \in \mathbb{Z}$, we have

\[ d^n \circ d^{n-1} = 0. \quad (A.7) \]

2. A morphism of complexes $f : (A^\ast, d) \to (B^\ast, d)$ is a collection of maps $f^n : A^n \to B^n$ commuting with the differential, that is,

\[ f^n \circ d^n = d^n \circ f^n - 1. \]

We picture it as follows:

\[
\begin{array}{ccc}
\cdots & A^{n-1} & d^{n-1} & A^n & d^n & A^{n+1} & \cdots \\
\downarrow{f^{n-1}} & f^n & \downarrow{f_{n+1}} \\
\cdots & B^{n-1} & d^{n-1} & B^n & d^n & B^{n+1} & \cdots 
\end{array}
\]

3. A cochain complex is called bounded if there exists an integer $M$ such that $A^n = 0$ for all $|n| \geq M$. Similarly, one defines the notion of bounded below and bounded above cochain complex.

4. The notion of chain complex in an additive category is the dual one, that is, the differential $d$ lowers the degree. Therefore, a chain complex $(A_\ast, d)$ is a collection of objects $A_n$, one for each integer $n \in \mathbb{Z}$, and of morphisms

\[ \cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \cdots \]

called differentials such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Since we will mainly use cochain complexes through these notes, we will often abbreviate and simply call them “complexes”. Sometimes it is convenient to think of a complex as a graded object

\[ A^\ast = \bigoplus_{n \in \mathbb{Z}} A^n \]

together with an operator $d : A^\ast \to A^\ast$ which is homogeneous of degree one, i.e. $d(A^n) = A^{n+1}$. Observe that for this to make sense we need either that the abelian category admits infinite sums or that the complex is bounded. With this convention, condition (A.7) simply reads $d \circ d = 0$. 
Remark A.8. If $\mathcal{A}$ is an abelian category, condition (A.7) is equivalent to imposing
\[ \text{Im}(d^{n-1}) \subseteq \text{Ker}(d^n) \]
for all integers $n \in \mathbb{Z}$.

Definition A.9. Let $\mathcal{A}$ be an abelian category and let $A = (A^*, d)$ be a complex in $\mathcal{A}$. The cohomology of $A$ is the graded object
\[ H^*(A) = \bigoplus_{n \in \mathbb{Z}} H^n(A), \quad H^n(A) = \text{Ker}(d^n) / \text{Im}(d^{n-1}). \]
In other words, one sets $H^*(A) = \text{Ker}(d)/\text{Im}(d)$ with the graduation induced by that of $A$ and the fact that $d$ is homogeneous.

A complex $A = (A^*, d)$ is called an exact sequence if its cohomology vanishes, i.e. $H^*(A) = 0$ or, equivalently, if $\text{Im}(d) = \text{Ker}(d)$.

If $\mathcal{A}$ is an abelian category, the category of complexes of $\mathcal{A}$, denoted by $\mathcal{C}(\mathcal{A})$, the category of bounded below complexes, denoted by $\mathcal{C}^-(\mathcal{A})$, and the category of bounded complexes, denoted by $\mathcal{C}^b(\mathcal{A})$ are abelian categories in turn. The cohomology functors $H^n: \mathcal{C}(\mathcal{A}) \to \mathcal{A}$, $n \in \mathbb{Z}$, satisfy the property that, for every short exact sequence of complexes
\[ 0 \to A \to B \to C \to 0, \] (A.10)
there are maps $\partial^n: H^n(C) \to H^{n+1}(A)$ such that the sequence
\[ \cdots \to H^n(A) \to H^n(B) \to H^n(C) \xrightarrow{\partial^n} H^{n+1}(A) \to H^{n+1}(B) \to \cdots \]
is exact. This property leads to the definition of cohomological functors and is the inspiration for the definition of derived functors.

Definition A.11. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A cohomological $\delta$-functor between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $F^n: \mathcal{A} \to \mathcal{B}$ indexed by the integers $n \in \mathbb{Z}$ and, for each short exact sequence like (A.10) in $\mathcal{A}$, maps $\partial^n: F^n(C) \to F^{n+1}(A)$ called connection morphisms, such that

1. the sequence
\[ \cdots \to F^n(A) \to F^n(B) \to F^n(C) \xrightarrow{\partial^n} F^{n+1}(A) \to F^{n+1}(B) \to \cdots \]
is exact;
2. for every morphism of short exact sequences
\[ \begin{array}{cccccc}
0 & \to & A & \to & B & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & A' & \to & B' & \to & C' & \to & 0,
\end{array} \]
the diagram
\[
\begin{array}{c}
F^n(C) \xrightarrow{\partial^n} F^{n+1}(A) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
F^n(C') \xrightarrow{\partial^n} F^{n+1}(A')
\end{array}
\]

is commutative.

In Definition A.21 below, we will introduce the notions of \(\delta\)-functor and cohomological functor in the setting of triangulated categories (see also Exercise A.32 for the relation between the three concepts).

We also recall the definition of exact functors.

**Definition A.12.** Let \(A\) and \(B\) be abelian categories and \(F : A \to B\) a covariant additive functor. We say that \(F\) is **exact** if, for every exact sequence \(0 \to A \to B \to C \to 0\), then the sequence
\[
0 \to F(A) \to F(B) \to F(C) \to 0
\]
is exact. \(F\) is said to be **right exact** if for all exact sequences as before, only
\[
F(A) \to F(B) \to F(C) \to 0
\]
is exact, while is called **left exact** if
\[
0 \to F(A) \to F(B) \to F(C)
\]
is exact. There are analogous definitions for contravariant functors.

**Exercise A.13.** Let \(C\) be the category whose objects are pairs \((V, W)\) of vector spaces of the same dimension and whose morphisms are pairs of linear maps. Show that \(C\) is an additive category but there are morphisms in \(C\) not having a kernel. This is a toy example for the category of vector bundles on a topological space, in this case consisting of two points, which is typically not abelian by contrast with the category of coherent sheaves.

**Exercise A.14.** Given an additive category \(C\), let \(C^{\text{op}}\) denote the **opposite category**, which has the same objects as \(C\) but reversed morphisms \(\text{Hom}_{C^{\text{op}}}(X, Y) = \text{Hom}_C(Y, X)\) for all objects \(X\) and \(Y\); it is an additive category as well. Show that a cokernel for \(f : X \to Y\) in \(C\) is a kernel for the corresponding morphism from \(Y\) to \(X\) in \(C^{\text{op}}\), and vice versa.

**Exercise A.15.** Let \(C\) be an additive category and let \(f\) be a morphism in \(C\). Assume that the image \(\text{Im}(f)\) and the coimage \(\text{Coim}(f)\) both exist. Show that there is a canonical morphism
\[
\text{Coim}(f) \to \text{Im}(f).
\]
A.2. Yoneda extensions.

Definition A.16. Let $\mathcal{A}$ be an abelian category, let $A$ and $B$ be objects of $\mathcal{A}$, and let $n \geq 1$ an integer. An extension of degree $n$ of $A$ by $B$ is an exact sequence

$$E: 0 \to B \overset{\iota}{\to} C_{n-1} \to \cdots \to C_0 \overset{\pi}{\to} A \to 0.$$  

Given two extensions of the same degree $E$ and $E'$, if there exists a commutative diagram

$$
\begin{array}{ccccc}
E: & 0 & \to & B & \to & C_{n-1} & \to & \cdots & \to & C_0 & \to & A & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E': & 0 & \to & B & \to & C'_{n-1} & \to & \cdots & \to & C'_0 & \to & A & \to & 0.
\end{array}
$$

we say that $E$ is equivalent to $E'$.

Note that, for $n = 1$, this notion agrees with the one presented in Definition 2.147 in the particular case of mixed Hodge structures. In contrast to that case, the existence of the above diagram does not define an equivalence relation for $n \geq 2$ since it is not symmetric. We consider instead the equivalence relation generated by such relations, which means that we force symmetry by adding the reverse equivalences and then transitivity for the mix of the old and new relations. We denote by $\text{Ext}_A^n(A,B)$ the set of equivalence classes of degree $n$ extensions of $A$ by $B$.

Let $E$ and $E'$ be two degree $n$ extensions of $A$ by $B$. Assume first that $n > 1$. Let $C_i$ denote the objects appearing in the first extension and $C'_i$ the ones appearing in the second one. We denote by $C''_0$ the pull-back of $C_0$ and $C'_0$ over $A$, this means that

$$C''_0 = \text{Ker}(C_0 \oplus C'_0 \overset{\phi}{\to} A),$$

with $\phi(c,c') = \pi'(c') - \pi(c)$. If $(c,c') \in C''_0$, then by construction $\pi(c) = \pi'(c')$, therefore there is a well defined map $\pi'': C''_0 \to A$ given by

$$\pi''(c,c') = \pi(c) = \pi'(c').$$

Dually, we denote by $C''_{n-1}$ the push-out of $C_{n-1}$ and $C'_{n-1}$ under $B$. This means that

$$C''_{n-1} = \text{Coker}(B \overset{\psi}{\to} C_{n-1} \oplus C'_{n-1}),$$

where $\psi(b) = (\iota(b), -\iota'(b))$. Both maps $b \mapsto (\iota(b),0)$ and $b \mapsto (0,\iota'(b))$ from $B$ to $C_{n-1} \oplus C'_{n-1}$ induce the same map to $C''_{n-1}$ that we denote $\iota''$.

Then the Baer sum of $E$ and $E'$ is the degree $n$ extension given by

$$0 \to B \to C''_{n-1} \to C_{n-2} \oplus C''_{n-2} \to \cdots \to C_1 \oplus C'_1 \to C''_0 \to A \to 0.$$
When \( n = 1 \), since \( n - 1 = 0 \), we have to combine the construction of \( C''_0 \) and \( C''_{n-1} \) in a single construction. Let

\[
0 \to B \to E \to A \to 0,
\]

\[
0 \to B \to E' \to A \to 0
\]

be two extensions of degree 1. Write

\[
E'' = \text{Coker}(B \xrightarrow{\psi} \text{Ker}(E \oplus E' \xrightarrow{\phi} A)) = \text{Ker}(\text{Coker}(B \xrightarrow{\psi} E \oplus E') \xrightarrow{\phi} A).
\]

As before we have induced maps \( B \to E'' \) and \( E'' \to A \). The Baer sum of \( E \) and \( E' \) is the extension

\[
0 \to B \to E'' \to A \to 0.
\]

With the Baer sum the equivalence class sets \( \text{Ext}^n(A, B) \) form a group (Exercise A.18).

Formation of extensions is functorial. This means that, given objects \( A \), \( B \), and \( B' \), a morphism \( f : B \to B' \); and an extension

\[
E : 0 \to B \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \xrightarrow{d_0} A \to 0,
\]

one constructs an extension \( \text{Ext}^n(A, f)(E) \in \text{Ext}^n(A, B') \) as follows.

First define \( C'_{n-1} \) as the pushout of \( C_{n-1} \) and \( B' \) under \( B \). Then there are maps \( f_n : C_{n-1} \to C'_{n-1} \) and \( d'_n : B' \to C'_{n-1} \). For notational convenience write \( C_n = B \), \( C'_n = B' \) and \( f_n = f \). Assume that we have defined the groups \( C'n, \ldots, Ck_1 \) and the maps \( f_i \), \( i = n, \ldots, k \) and \( d_i' \), \( i = n, \ldots, k + 1 \), where \( 1 \leq k \leq n - 1 \). Then the group \( C'_{k-1} \) is defined as

\[
C'_{k-1} = \text{Coker}(\psi : C'_{k+1} \oplus C_k \to C'_k \oplus C_{k-1}),
\]

where the map \( \psi \) is given by

\[
\psi(c', c) = (d'_{k+1}(c') + f_k(c), -d_k(c)).
\]

There are induced maps \( f_k : C_{k-1} \to C'_{k-1} \) and \( d'_k : C_k \to C'_{k-1} \). Using the universal property of the cokernel, one proves that there is a map \( d'_0 : C'_0 \to A \). The resulting sequence

\[
0 \to B' \xrightarrow{d'_n} C'_{n-1} \xrightarrow{d'_{n-1}} \cdots \xrightarrow{d'_1} C'_0 \xrightarrow{d'_0} A' \to 0,
\]

is exact and defines the extension \( E' = \text{Ext}^n(A, f)(E) \in \text{Ext}^n(A, B') \).

Let \( 0 \to B_1 \to B_2 \to B_3 \to 0 \) be a short exact sequence. There is a map \( \partial^0 : \text{Hom}(A, B_3) \to \text{Ext}^1(A, B_1) \) sending a morphism \( f \) to the sequence \( 0 \to B_1 \to E \to A \to 0 \), where \( E \) is the pull-back of \( B_2 \) and \( A \) over \( B_3 \). There are also maps \( \partial^n : \text{Ext}^n(A, B_3) \to \text{Ext}^{n+1}(A, B_1) \) that send the extension

\[
0 \to B_3 \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \xrightarrow{d_0} A \to 0,
\]

to the extension

\[
0 \to B_1 \xrightarrow{d_n} B_2 \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_{n-1} \xrightarrow{d_0} A \to 0.
\]
We write $\text{Ext}^0(A, B) = \text{Hom}(A, B)$ and $\text{Ext}^n(A, B) = \{0\}$ for $n < 0$. In Exercise A.19 you will prove that the extension groups together with the connection morphisms defined above form a cohomological $\delta$-functor (Definition A.11).

**Lemma A.17.** Let $A$ be an abelian category. Assume that there exists an integer $n_0 \geq 0$ such that, for all objects $A$ in $A$, the functor $\text{Ext}^{n_0}(A, -)$ is right exact. Then $\text{Ext}^n(A, B) = 0$ for all objects $A, B$ and all integers $n > n_0$.

**Proof.** It is enough to prove that, if the functor $\text{Ext}^n(A, -)$ is right exact for all objects $A$, then $\text{Ext}^{n+1}(A, B) = 0$ for all objects $A, B$, because the functor zero is right exact and we proceed by induction. We start with the case $n = 0$. Assume that $\text{Ext}^0(A, -) = \text{Hom}(A, -)$ is right exact for all $A$ and let $0 \to B \to E \xrightarrow{\pi} A \to 0$ an extension. Since $\text{Hom}(A, -)$ is right exact, there exist a map $f: A \to E$ such that $\pi \circ f = \text{Id}_A$. This means that the extension is split and we conclude that $\text{Ext}^1(A, B) = 0$.

Let now $n \geq 1$ and assume that $\text{Ext}^n(A, -)$ is right exact. Let

$$E: \quad 0 \to B \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} C_0 \xrightarrow{d_0} A \to 0,$$

be an extension in $\text{Ext}^{n+1}(A, B)$. Let $C = \text{Coker}(d_{n+1})$ and consider the exact sequence

$$0 \to B \to C_n \to C \to 0$$

and the extension

$$E': \quad 0 \to C \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \xrightarrow{d_0} A \to 0,$$

then $E' \in \text{Ext}^n(A, C)$ and $E = \partial^n(E')$. Since $\text{Ext}^n(A, -)$ is right exact, necessarily $\partial^n(E') = 0$. Therefore $E = 0$. \qed

**Exercise A.18.** Show that the Baer sum is well defined on equivalence classes and hence defines a group structure on $\text{Ext}^n(A, B)$, $n \in \mathbb{Z}$.

**Exercise A.19.** Let $A$ be an abelian category. Prove that, for every object $A$ of $A$, the functors $\text{Ext}^n(A, -)$ together with the connection morphisms $\partial^n$ defined in the text form a cohomological $\delta$-functor.

**A.3. Triangulated and derived categories.** A triangulated category is a machine to produce long exact sequences.

**Definition A.20 (Verdier).** A *triangulated category* $\mathcal{T}$ is an additive category, together with the following extra data:
a) a self-equivalence of categories

\[ [1]: \mathcal{T} \rightarrow \mathcal{T} \]

\[ \xymatrix{ X \ar[r]^{	ext{Id}} & X[1] } \]

We denote by \( f[1] \) the image of a morphism \( f \). Once the self equivalence \([1]\) is given, we shall call triangles all sequences of the form

\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]. \]

A morphism of triangles is a commutative diagram

\[ \xymatrix{ X \ar[r]^u \ar[d] & Y \ar[r]^v \ar[d] & Z \ar[r]^w \ar[d] & X[1] \ar[d] \\
X' \ar[r]^{u'} & Y' \ar[r]^{v'} & Z' \ar[r]^{w'} & X'[1]. } \]

We will use the convention that an arrow decorated with \([1]\) like \( A \xrightarrow{[1]} B \) means a map \( A \rightarrow B[1] \).

b) A class of triangles called distinguished triangles.

These data are required to satisfy the following axioms:

**(T1)** a) For any \( X \in \text{Ob}(\mathcal{T}) \), the triangle

\[ X \xrightarrow{\text{Id}} X \xrightarrow{0} X[1] \]

is distinguished.

b) Any triangle isomorphic to a distinguished one is distinguished.

c) Any morphism \( X \xrightarrow{u} Y \) can be completed to a distinguished triangle

\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]. \]

**(T2)** The triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \) is distinguished if and only if the triangle \( Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1] \) is distinguished.

**(T3)** Given two distinguished triangles

\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1], \quad X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1], \]

and morphisms \( f: X \rightarrow X' \) and \( g: Y \rightarrow Y' \) such that \( g \circ u = u' \circ f \), there exists \( h: Z \rightarrow Z' \) (not necessarily unique) such that

\[ \xymatrix{ X \ar[r]^u \ar[d]_f & Y \ar[r]^v \ar[d]_g & Z \ar[r]^w \ar[d]_h & X[1] \ar[d]_{f[1]} \\
X' \ar[r]^{u'} & Y' \ar[r]^{v'} & Z' \ar[r]^{w'} & X'[1]. } \]

is a morphism of triangles.
(T4) Given a diagram of solid arrows

\[
\begin{array}{c}
\text{Y}' \\
\downarrow \downarrow \\
\text{X}' \\
\downarrow \downarrow \\
\text{Z}' \\
\downarrow \downarrow \\
\text{X} \\
\text{Y} \\
\end{array}
\]

if the three triangles

\[
\begin{align*}
X & \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k} X[1] \\
Y & \xrightarrow{v} Z \xrightarrow{j} X' \xrightarrow{i} Y[1] \\
X & \xrightarrow{v \circ u} Z \xrightarrow{m} Y \xrightarrow{n} X[1]
\end{align*}
\]

are distinguished, then there exist dashed arrows \( f \) and \( g \) as in the triangle

\[
\begin{array}{c}
\text{Z}' \\
\downarrow \downarrow \\
\text{Y}' \\
\downarrow \downarrow \\
\text{X}' \\
\downarrow \downarrow \\
\text{Z} \\
\end{array}
\]

is distinguished and the following commutation relations hold:

\[
\begin{align*}
k &= n \circ f, & \ell &= g \circ m, \\
m \circ v &= f \circ j, & u[1] \circ n &= i \circ g.
\end{align*}
\]

**Definition A.21.** Let \( \mathcal{T} \) and \( \mathcal{T}' \) be triangulated categories. A triangulated functor \( F: \mathcal{T} \to \mathcal{T}' \) is an additive functor that respects the shift \( F(C[1]) = F(C)[1] \) and that sends distinguished triangles to distinguished triangles.

Let \( \mathcal{A} \) be an abelian category. A cohomological functor \( H: \mathcal{T} \to \mathcal{A} \) is an additive functor such that, for each distinguished triangle

\[
\begin{align*}
X & \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1],
\end{align*}
\]
the sequence

\[ \cdots \rightarrow H(A[n]) \xrightarrow{H(u)} H(B[n]) \xrightarrow{H(v)} H(C[n]) \xrightarrow{H(w)} H(A[n+1]) \rightarrow \cdots \]

is exact.

The first example of a triangulated category is the category of bounded below complexes up to homotopy. We recall here the construction. Let \( \mathcal{A} \) be an abelian category and \( \mathcal{C}^+(\mathcal{A}) \) the category of bounded below cochain complexes on \( \mathcal{A} \). Complexes are denoted by \( A^\bullet \) or, if we want to emphasize the differential, by \( (A^\bullet, d) \). For a complex \( A^\bullet \), the condition of being bounded below means that there is an \( n_0 \in \mathbb{Z} \) such that \( A^n = \{0\} \) for all \( n \leq n_0 \).

Given a complex \( (A^\bullet, d) \), its \textit{shift} \( A^{\bullet+1} \) is defined as

\[ A^{n+1} = A^n, \quad d_{n+1} = -d_n. \] (A.22)

A morphism of complexes \( f : A^\bullet \to B^\bullet \) induces a morphism of cohomology groups

\[ H(f) : H^\bullet(A^\bullet) \to H^\bullet(B^\bullet). \]

A morphism \( f \) is called a \textit{quasi-isomorphism} whenever \( H(f) \) is an isomorphism.

An important construction in the category \( \mathcal{C}^+(\mathcal{A}) \) is the \textit{cone} of a morphism of complexes. Let \( f \) be a morphism of complexes as before, then the cone of \( f \) is the complex

\[ \text{cone}(f)^n = A^{n+1} \oplus B^n, \quad d(a, b) = (-da, db - f(a)). \]

The cone is provided with two morphisms of complexes

\[ B \to \text{cone}(f), \quad b \mapsto (0, -b) \]
\[ \text{cone}(f) \to A[1], \quad (a, b) \mapsto a, \]

that induce a long exact sequence of cohomology groups

\[ \cdots \rightarrow H^n(A^\bullet) \xrightarrow{H(f)} H^n(B^\bullet) \rightarrow H^n(\text{cone}(f)) \rightarrow H^{n+1}(A^\bullet) \rightarrow \cdots \] (A.23)

Given two morphisms of complexes \( f, g : A^\bullet \to B^\bullet \), a \textit{homotopy} between them is a collection of maps \( s^n : A^n \to B^{n-1} \) such that

\[ f^n - g^n = d^{n-1} \circ s^n + s^{n+1} \circ d^n. \]

If such a homotopy exists, we say that \( f \) and \( g \) are homotopically equivalent. Two homotopically equivalent morphisms induce the same morphism in cohomology groups.

\textsc{Definition A.24.} The \textit{homotopy category} \( \mathcal{K}^+(\mathcal{A}) \) is the category whose objects are the same as those of \( \mathcal{C}^+(\mathcal{A}) \) but whose morphisms are equivalence classes with respect to the homotopy equivalence of morphisms in \( \mathcal{C}^+(\mathcal{A}) \).
The shift functor \( [1]: \mathcal{K}^+(A) \to \mathcal{K}^+(A) \) is the one defined in \((A.22)\). A distinguished triangle is a triangle isomorphic to a triangle of the form

\[
A^* \xrightarrow{f} B^* \to \text{cone}(f) \to A[1].
\]

The category \( \mathcal{K}^+(A) \) is a triangulated category.

**Example A.25.** Let \( A \) be an abelian category. The exactness of the sequence \((A.23)\) shows that the functor \( H: \mathcal{K}^+(A) \to A \) given by \( A^* \mapsto H^0(A^*) \) is a cohomological functor.

**Example A.26.** Let \( F: A \to B \) be an additive functor. The induced functor \( F: \mathcal{K}^+(A) \to \mathcal{K}^+(B) \) given by \( F((A^*, d))^n = F(A)^n \) with differential \( F(d) \) is a \( \delta \)-functor. See Exercise A.31.

The main example of a triangulated category is the (bounded) derived category of an abelian category. We quickly recall the main structures associated with the definition of derived categories.

The construction of the derived category is done in two steps. The first is the construction of the homotopy category \( \mathcal{K}^+(A) \) that we have already sketched. In the second step, one constructs \( \mathcal{D}^+(A) \) by inverting the quasi-isomorphisms. That is, the objects of \( \mathcal{D}^+(A) \) are the same as the objects of \( \mathcal{K}^+(A) \) (which are the same as the ones of \( \mathcal{C}^+(A) \)), while the morphism on \( \mathcal{D}^+(A) \) between two objects \( A^* \) and \( B^* \) are equivalence classes of diagrams of the form

\[
\begin{array}{ccc}
A^* & \xrightarrow{\sim} & C^* \\
\downarrow & & \downarrow \\
B^* & \xrightarrow{\sim} & A^*
\end{array}
\]

where the arrow to the left is a quasi-isomorphism. The diagrams

\[
\begin{array}{ccc}
A^* & \xrightarrow{\sim} & C_1^* \\
\downarrow & & \downarrow \\
B^* & \xrightarrow{\sim} & A^*
\end{array} \quad \text{and} \quad \begin{array}{ccc}
C_2^* & \xrightarrow{\sim} & B^* \\
\downarrow & & \downarrow \\
A^* & \xrightarrow{\sim} & B^*
\end{array}
\]

are equivalent if there is a third diagram of the same type and morphisms \( C_3^* \to C_1^* \) and \( C_3^* \to C_2^* \) such that

\[
\begin{array}{ccc}
A^* & \xrightarrow{\sim} & C_1^* \\
\downarrow & & \downarrow \\
C_3^* & \xrightarrow{\sim} & B^*
\end{array} \quad \text{and} \quad \begin{array}{ccc}
C_2^* & \xrightarrow{\sim} & B^* \\
\downarrow & & \downarrow \\
A^* & \xrightarrow{\sim} & C_2^*
\end{array}
\]

commutes in \( \mathcal{K}^+(A) \). This means that all the triangles in the diagram \((A.27)\) are commutative up to homotopy but they are not necessarily commutative.
To have such a simple description of the morphisms is the main reason to define the derived category in two steps. One can invert directly the quasi-isomorphisms in $C^+(A)$, but then morphisms will be chains of the form

$$
\begin{array}{cccc}
A^* & \rightarrow & C_1^* & \rightarrow & \cdots & \rightarrow & C_k^* & \rightarrow & B^*
\end{array}
$$

where all the arrows in the left direction are quasi-isomorphisms.

The category $D^+(A)$ is a triangulated category, where the self equivalence $[1]$ is defined by the shift, while the class of distinguished triangles are those triangles that are isomorphic (in $D^+(A)$) to one of the form

$$A^* \xrightarrow{f} B^* \rightarrow \text{cone}(f) \rightarrow A[1]^*.$$

**Definition A.28.** Let $A$ be an abelian category. Then the (bounded below) derived category is the category we just described. The localization functor is the functor $Q: K^+(A) \rightarrow D^+(A)$ that is the identity at the level of objects.

**Remark A.29.** There is also a definition of the bounded homotopy category $K^b(A)$ and bounded derived category $D^b(A)$ that is obtained replacing bounded below complexes with bounded complexes in all the above discussion.

We now discuss Verdier localization. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{E}$ be a triangulated subcategory of $\mathcal{T}$.

**Proposition A.30.** There exists a triangulated category $\mathcal{T}/\mathcal{E}$ which is universal for the following two properties:

a) There exists a canonical triangulated functor $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{E}$ which is the identity on objects.

b) Every object $X$ of $\mathcal{E}$ is isomorphic to the zero object $0$ in $\mathcal{T}/\mathcal{E}$.

***

**Exercise A.31.** Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories.

1) If $(A^*,d)$ is a bounded below complex in $\mathcal{A}$, then $(F(A^*),F(d))$ is a bounded below complex in $\mathcal{B}$. Observe that, if $f$ and $g$ are homotopically equivalent morphisms in $\mathcal{A}$, then $F(f)$ and $F(g)$ are also homotopically equivalent in $\mathcal{B}$. Therefore, $F$ induces a functor $F: K^+(\mathcal{A}) \rightarrow K^+(\mathcal{B})$. 
(2) Prove that the induced functor $F$ is compatible with the shift functor and with the formation of the cone. Conclude that $F$ is a $\delta$-functor.

Exercise A.32. As we have seen in Example A.25, taking cohomology in degree zero is the basic example of a cohomological functor. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories.

1. Show that the cohomological functor $H^0$ from $\mathcal{K}^+ (\mathcal{B})$ to $\mathcal{B}$ descends to a cohomological functor $\mathcal{D}^+ (\mathcal{B}) \to \mathcal{B}$.

2. Let $F: \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ be a $\delta$-functor and define the functors $F^n: \mathcal{A} \to \mathcal{B}$ as the composition

\[ \mathcal{A} \to \mathcal{D}^+(\mathcal{A}) \xrightarrow{F} \mathcal{D}^+(\mathcal{B}) \xrightarrow{[n]} \mathcal{D}^+(\mathcal{B}) \xrightarrow{H^0} \mathcal{B}, \]

where the first map sends an object in $\mathcal{A}$ to this object seen as a complex concentrated in degree zero. Show that the functors $F^n$ with the appropriate connection morphisms form a cohomological $\delta$-functor.

3. Let $F: \mathcal{K}^+(\mathcal{A}) \to \mathcal{K}^+(\mathcal{B})$ be a $\delta$-functor and define the functors $F^n: \mathcal{A} \to \mathcal{B}$ as the composition

\[ \mathcal{A} \to \mathcal{K}^+(\mathcal{A}) \xrightarrow{F} \mathcal{K}^+(\mathcal{B}) \xrightarrow{[n]} \mathcal{D}^+(\mathcal{B}) \xrightarrow{H^0} \mathcal{B}. \]

Show with an example that, in general, the functors $F^n$ do not form a cohomological $\delta$-functor.

A.4. Derived functors. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor. We know from Example A.26 that $F$ induces a $\delta$-functor, also denoted by $F$, between the homotopy categories $\mathcal{K}^+(\mathcal{A})$ and $\mathcal{K}^+(\mathcal{B})$. If $F$ is exact, then this extension sends quasi-isomorphisms to quasi-isomorphisms and hence yields a functor between the derived categories $\mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$. By contrast, if $F$ is not exact, then it does not extend naively to the derived categories. Whenever it exists, the derived functor is the best approximation to $F$ between the derived categories. Recall that $Q: \mathcal{K}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A})$ denotes the localisation functor from Definition A.28.

Definition A.33. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor. The \textit{(total) right derived functor} of $F$ is a $\delta$-functor

\[ RF: \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B}) \]

together with a natural transformation

\[ \xi: Q \circ F \to RF \circ Q \]

of functors from $\mathcal{K}^+(\mathcal{A})$ to $\mathcal{D}^+(\mathcal{B})$ that is universal. This means that, for any $\delta$-functor $G: \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$, equipped with a natural transformation
\( \zeta : Q \circ F \to G \circ Q \), there exists a unique natural transformation \( \eta : RF \to G \) such that

\[ \zeta = (\eta \circ Q) \circ \xi. \]

We give more details on what this definition means. Recall that the functor \( Q \) is the identity on objects. So we will denote \( Q(A) \) just by \( A \). The natural transformation \( \xi \) provides, for every object \( A \) of \( K^+ \), a morphism \( \xi(A) : F(A) \to RF(A) \) such that, for every morphism \( f \in \text{Hom}_{K}(A, B) \), the diagram

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\downarrow{\xi(A)} & & \downarrow{\xi(B)} \\
RF(A) & \xrightarrow{RF(f)} & RF(B)
\end{array}
\]

is commutative in \( D^+(B) \). If \( G \) is another functor as in the definition, provided with a natural transformation \( \zeta \), then for every object \( A \) of \( K^+ \), we also have a morphism

\[ \zeta(A) : F(A) \to G(A) \]

such that, for every morphism \( f \in \text{Hom}_{K}(A, B) \), the diagram

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\downarrow{\zeta(A)} & & \downarrow{\zeta(B)} \\
G(A) & \xrightarrow{G(f)} & G(B)
\end{array}
\]

commutes, then the universality of \( RF \) means that we should have morphisms

\[ \eta(A) : RF(A) \to G(A), \quad A \in \text{Ob}(D^+(A)) = \text{Ob}(K^+). \]

with \( \eta(A) \circ \xi(A) = \zeta(A) \). This \( \eta \) is a morphism of functors in the sense that for every \( f \) as before we have \( G(f) \circ \eta(A) = \eta(B) \circ RF(f) \).

**Remark A.34.** There is also the notion of bounded derived functor \( RF : D^b(A) \to D^b(B) \).

From the total right derived functor we define the cohomological derived functors.

**Definition A.35.** Let \( F : A \to B \) be an additive functor between abelian categories. Denote also by \( F : K^+ \to K^+(B) \) the extension to the homotopy category. If the total derived functor \( RF \) exists, then the cohomological derived functors \( R^nF, n \in \mathbb{Z} \) are defined as the composition

\[ A \to D^+(A) \xrightarrow{RF} D^+(B) \xrightarrow{[n]} D^+(B) \xrightarrow{H^0} B. \]
The standard situation where one can show the existence of the right derived functor is when $F : A \to B$ is a left exact additive functor and the category $A$ has enough injectives.

**Definition A.36.** Let $A$ be an abelian category. An object $I \in \text{Ob}(A)$ is called *injective* if, for every exact sequence $0 \to A \to A'$ and every morphism $A \to I$, there is a morphism $A' \to I$ making the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & A \\
& \searrow & | \\
& & I
\end{array}
$$

commutative. In other words, for every monomorphism $A \to A'$ the map $\text{Hom}(A', I) \to \text{Hom}(A, I)$ is surjective.

Injective objects satisfy the following properties:

**Lemma A.37.** Let $A$ be an abelian category and $I$ and injective object.

1. The functor $\text{Hom}(-, I)$ is exact.
2. Every short exact sequence $0 \to I \to E \to A \to 0$ is split.
3. For every object $A \in \text{Ob}(A)$ and every integer $n > 0$, the Yoneda extension group $\text{Ext}^n(A, I)$ vanishes.

**Proof.** The first statement is just a reformulation of the definition because $\text{Hom}(-, I)$ is always left exact. Let now $0 \to I \to E \to A \to 0$ be a short exact sequence. By the definition of injective object, there is a morphism $E \to I$ making the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & I \\
& \nearrow & | \\
& & E
\end{array}
$$

commutative. This exactly means that the sequence is split.

The third statement follows from the first one and Lemma A.17 or from the second statement using the connection morphism.

Intuitively, on an abelian category that contains only injective objects any additive functor would be exact, because any short exact sequence would be a direct sum and an additive functor preserves direct sums. So, the idea to correct the lack of exactness of a functor is to replace any object by an injective one.

**Definition A.38.** Let $A$ be an abelian category. We say that $A$ has *enough injectives* if, for every object $A$ of $A$, there is an injective object $I$ and a monomorphism $A \to I$. 
If an abelian category $\mathcal{A}$ has enough injectives and $A^*$ is a bounded below cochain complex, then there exists a bounded below complex $I^*$ of injective objects and a quasi-isomorphism $\psi: I^* \to K^*$, $\varphi: J^* \to K^*$ such that $\psi \circ f$ and $\varphi \circ g$ are homotopically equivalent (see for instance [Wei94, §2.3] for the needed ingredients). Any such complex $I^*$ is called an injective resolution of $A^*$.

**Proposition A.39.** Let $\mathcal{A}, \mathcal{B}$ be a abelian categories and $F$ a left exact functor from $\mathcal{A}$ to $\mathcal{B}$. Assume that $\mathcal{A}$ has enough injectives. Then the functor $RF: D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined by

$$RF(A^*) = F(I^*),$$

where $A^*$ is an object of $C^+(\mathcal{A})$ and $I^*$ is an injective resolution of $A^*$ satisfies the universal property of Definition A.33 and hence is the derived functor of $F$. Note that $RF(A^*)$ is well defined up to a unique isomorphisms. The cohomological derived functors $R^n F: \mathcal{A} \to \mathcal{B}$ are given by

$$R^n F(A) = H^n(RF(A)),$$

where $A$ denotes at the same time an object of $\mathcal{A}$ and a complex in $C^+(\mathcal{A})$ concentrated in degree zero.

**Lemma A.40.** Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $F: \mathcal{A} \to \mathcal{B}$ a left exact functor. Assume that $\mathcal{A}$ has enough injectives.

1. For every object $A \in \mathcal{A}$, the equation $F(A) = R^0 F(A)$ holds.
2. If $I$ is an injective object. Then, for all $n > 0$, the condition $R^n F(I) = 0$ holds.

**Proof.** Exercise A.44. □

**Example A.41.** Let $\mathcal{A}$ be an abelian category with enough injectives. Then the Yoneda extension groups can be also constructed as the higher derived functors of the functor Hom. More precisely, given an object $A \in \text{Ob}(\mathcal{A})$ the functor $\text{Hom}(A, -)$ is left exact and, for any other object $B$, there are functorial isomorphisms

$$\text{Ext}^n(A, B) \cong R^n \text{Hom}(A, -)(B).$$

For ease of notation we write $R^n \text{Hom}(A, -)(B) = R^n \text{Hom}(A, B)$. This is justified because

$$R^n \text{Hom}(A, -)(B) = R^n \text{Hom}(-, B)(A),$$

so the ambiguity is harmless. We prove the claim. We already know that $\text{Ext}^0(A, B) = \text{Hom}(A, B) = R^0 \text{Hom}(A, B)$. So we have our functorial isomorphism in degree zero. Assume by induction that there are functorial isomorphism $\text{Ext}^m(A, B) \cong R^m \text{Hom}(A, B)$ for all $m < n$. Let

$$0 \to B \to I \to C \to 0$$
be a short exact sequence with $I$ injective. By induction we have a commutative diagram

$$
\begin{array}{cccccc}
\text{Ext}^{n-1}(A, I) & \longrightarrow & \text{Ext}^{n-1}(A, C) & \longrightarrow & \text{Ext}^{n}(A, B) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & & & \\
R^{n-1}\text{Hom}(A, I) & \longrightarrow & R^{n-1}\text{Hom}(A, C) & \longrightarrow & R^{n}\text{Hom}(A, B) & \longrightarrow & 0
\end{array}
$$

with exact rows. Since all the vertical arrows are isomorphisms, there is a unique isomorphism $\text{Ext}^{n}(A, B) \cong R^{n}\text{Hom}(A, B)$ making the diagram commutative. It is easy to show that this isomorphism is independent of the chosen exact sequence and is functorial.

In the previous example we have interpreted the Yoneda extension groups as the derived functors of the Hom functor. But there is another interpretation as the Hom functor in the derived category.

**Proposition A.42.** Let $A$ be an abelian category. Let $A$ and $B$ be objects of $A$. We see $A$ and $B$ as objects in $D^+(A)$ concentrated in degree zero. Then there are functorial isomorphisms

$$\text{Ext}^{n}(A, B) \cong \text{Hom}_{D^+(A)}(A, B[n]).$$

**Proof.** We start by constructing the map. Let

$$E: 0 \longrightarrow B \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \xrightarrow{\pi} A \longrightarrow 0.$$

be an extension. Let $C^*$ be the complex obtained from $E$ by deleting $A$ and putting $C_i$ in degree $-i$. Thus $B$ sits in degree $-n$. The map $C_0 \rightarrow A$ induces a morphisms of complexes $C^* \rightarrow A$. By the exactness of $E$ this morphism is a quasi-isomorphism. The identity in $B$ defines also a morphism of complexes $C^* \rightarrow B[n]$. To the extension $E$ it corresponds the morphism in the derived category

$$\begin{array}{ccc}
A & \xrightarrow{\cong} & C^* \\
\downarrow & & \downarrow \cong \\
B[n] & & 
\end{array}
$$

(A.43)

Conversely, assume there is a morphism in $\text{Hom}_{D^+(A)}(A, B[n])$ represented by a diagram like (A.43) with $(C^*, d)$ a complex quasi-isomorphic to $A$. We write

$$\begin{align*}
C'_0 &= \text{Ker}(d^0: C^0 \rightarrow C^1), \\
C'_i &= C^{-i}, \quad i = 1, \ldots, n - 1 \\
B' &= C^{-n}/\text{Im}(d^{-n-1}).
\end{align*}$$

We obtain an extension $E' \in \text{Ext}^{n}(A, B')$ and a map $B' \rightarrow B$. By the functoriality of Yoneda extensions we deduce an extension $E \in \text{Ext}^{n}(A, B)$.

The following facts are left as an exercise
(1) Two equivalent extensions give rise to the same morphism in the derived category.

(2) Two representations of the same morphism in the derived category define the same class of extensions.

(3) The two constructions are inverse of each other.

\[\square\]

***

Exercise A.44. Prove Lemma A.40.

Exercise A.45. Fill the details in the proof of Proposition A.42.

**A.5. Ind and pro-objects in a category.** Inductive and projective limits are important operations in category theory. Nevertheless in many interesting categories such limits may not exist. This is the case of the category of mixed Hodge structures. To remedy this situation, given a category \(\mathcal{C}\), one can define categories \(\text{Ind}(\mathcal{C})\) and \(\text{Pro}(\mathcal{C})\) of inductive and projective systems in \(\mathcal{C}\), where inductive or projective limits in \(\mathcal{C}\) exist. We refer the reader to [KS06] for more details on \(\text{Ind}(\mathcal{C})\) and \(\text{Pro}(\mathcal{C})\).

A filtered category is a generalization of the concept of directed set.

**Definition A.46.** A filtered category \(\mathcal{D}\) is a category satisfying the following properties

(1) There exists at least one object in \(\mathcal{D}\).

(2) For every two objects \(a, b \in \text{Ob}(\mathcal{D})\) there exists an object \(c\) and maps \(a \to c\) and \(b \to c\).

(3) For every two maps \(f_1, f_2 : a \to b\) with the same source and target, there exists a map \(g : b \to c\) such that \(g \circ f_1 = g \circ f_2\).

**Example A.47.** A directed set \(I\) gives rise to a filtered category by stipulating that \(\text{Hom}(a, b)\) contains one element if \(a \leq b\) and is empty otherwise.

**Definition A.48.** Let \(\mathcal{C}\) be a category and \(\mathcal{D}\) a small filtered category.

(1) An inductive system \(X\) in \(\mathcal{C}\) indexed by \(\mathcal{D}\) is a functor from \(X : \mathcal{D} \to \mathcal{C}\).

(2) Let \(X\) and \(Y\) be inductive systems indexed by \(\mathcal{D}\) and \(\mathcal{E}\) respectively. A morphism \(f\) from \(X\) to \(Y\) is the data of a functor map \(f_\sharp : \mathcal{D} \to \mathcal{E}\) and a natural transformation from \(X\) to \(Y \circ f_\sharp\).

(3) A projective system \(X\) in \(\mathcal{C}\) indexed by \(\mathcal{D}\) is a functor \(C : \mathcal{D}^{\text{op}} \to \mathcal{C}\).

(4) If \(X\) is a projective system indexed by \(\mathcal{D}\) and \(Y\) is a projective system indexed by \(\mathcal{E}\), then a morphism \(f\) between \(X\) and \(Y\) is a
functor $f_\sharp: D \to E$ and a natural transformation between $X$ and $Y \circ f_\sharp^\op$.

A inductive system will be called a directed system if the index category is the category associated with a directed set as in Example A.47.

**Definition A.49.** Let $\mathcal{C}$ be a category, $D$ a small filtered category and $X = (X_d)_{d \in D}$ an inductive system indexed by $D$. An inductive limit of this system is a universal solution to the problem: find an object $X_0$ in $\mathcal{C}$ together with a morphism of inductive systems $X \to X_0$. Here $X_0$ denotes at the same time the object $X_0$ and the constant inductive system $1 \mapsto X_0$. If $X_0$ is such a universal solution, it is written as

$$X_0 = \lim_{d \in D} X_d.$$  

An inductive limit whose index category is a direct set is called a direct limit.

Let now $X = (X_d)_{d \in D}$ be a projective system. A projective limit of this system in a universal solution to the problem: find an object $X_0$ in $\mathcal{C}$ together with a morphism of projective systems $X_0 \to X$. If $X_0$ is such universal solution it is written as

$$X_0 = \lim_{d \in D} X_d.$$  

A projective limit whose index category is a direct set is called a direct limit.

Inductive limits are also called colimits, while projective limits are also called just limits.

**Remark A.50.** In many cases it is important to know if a functor respects inductive limits. For this it is enough to check if the functor respects direct limits. Similarly, a functor that respects inverse limits, also respects projective limits with respect to any small filtered category.

**Definition A.51.** Let $\mathcal{C}$ be any category. The ind-category of $\mathcal{C}$ is the universal category that “contains” $\mathcal{C}$ and is closed under inductive limits. More precisely, is a category $\text{Ind}(\mathcal{C})$ $\text{Ind}(\mathcal{C})$ closed under inductive limits, together with a functor $\mathcal{C} \to \text{Ind}(\mathcal{C})$ such that, for any category $\mathcal{A}$ closed under inductive limits, with a functor $\mathcal{C} \to \mathcal{A}$, there exists a unique functor $\text{Ind}(\mathcal{C}) \to \mathcal{A}$ preserving inductive limits and making the triangle

$$\begin{array}{ccc}
\mathcal{C} & \to & \mathcal{A} \\
\downarrow & & \uparrow \\
\text{Ind}(\mathcal{C}) & \to & \mathcal{A}
\end{array}$$

commutative.

The pro-category of $\mathcal{C}$ is a category $\text{Pro}(\mathcal{C})$ closed under projective limits together with a functor $\mathcal{C} \to \text{Pro}(\mathcal{C})$ such that, for any category $\mathcal{A}$, closed
under projective limits, with a functor \( \mathcal{C} \to \mathcal{A} \), there exists a unique functor \( \mathcal{A} \to \text{Pro}(\mathcal{C}) \) preserving projective limits and making the triangle

\[
\begin{array}{ccc}
\mathcal{C} & \to & \mathcal{A} \\
\downarrow & & \downarrow \\
& \text{Pro}(\mathcal{C}) & \\
\end{array}
\]

commutative.

We next give a construction of the categories \( \text{Ind}(\mathcal{C}) \) and \( \text{Pro}(\mathcal{C}) \).

**Proposition A.52.** Let \( \mathcal{C} \) be a category.

1. Then the category \( \text{Ind}(\mathcal{C}) \) is the category of whose objects are inductive systems in \( \mathcal{C} \) and whose morphisms are given, for \( X = (X_d)_{d \in D} \) and \( Y = (Y_e)_{e \in E} \), by

\[
\text{Hom}_{\text{Ind}(\mathcal{C})}(X, Y) = \lim_{\leftarrow d \in D} \lim_{\rightarrow e \in E} \text{Hom}_\mathcal{C}(X_d, Y_e).
\]

2. The category \( \text{Pro}(\mathcal{C}) \) is the category of whose objects are projective systems in \( \mathcal{C} \) and whose morphisms are given, for \( X = (X_d)_{d \in D} \) and \( Y = (Y_e)_{e \in E} \), by

\[
\text{Hom}_{\text{Ind}(\mathcal{C})}(X, Y) = \lim_{\rightarrow e \in E} \lim_{\leftarrow d \in D} \text{Hom}_\mathcal{C}(X_d, Y_e).
\]

**Remark A.53.** As you will prove in exercise A.62, a morphism of inductive systems in \( \mathcal{C} \) induces a morphism in \( \text{Ind}(\mathcal{C}) \) and, if they exist, a morphism between the corresponding inductive limits. Moreover, if two morphisms induce the same morphism in \( \text{Ind}(\mathcal{C}) \), they also induce the same morphism between inductive limits. Similar result is true for projective systems.

One has to be careful with the fact that the objects in \( \text{Ind}(\mathcal{C}) \) are “formal” limits and not “true” limits in the sense that, if \( \mathcal{C} \) already is closed under inductive limits, then the functor \( \mathcal{C} \to \text{Ind}(\mathcal{C}) \) does not preserve, in general, inductive limits (see Exercise A.61). In this case, to understand the difference between \( \mathcal{C} \) and \( \text{Ind}(\mathcal{C}) \) is convenient to introduce the notion of compact objects.

**Definition A.54.** Let \( \mathcal{C} \) be a category that admits inductive limits. An objects \( X \in \text{Ob}(\mathcal{C}) \) is called **compact** if for every inductive system \( (Y_d)_{d \in D} \) the canonical map

\[
\lim_{d \in D} \text{Hom}(X, Y_d) \to \text{Hom}(X, \lim_{d \in D} Y_d)
\]

is an isomorphism.

Dually, if \( \mathcal{C} \) is a category that admits projective limits, then an object \( X \in \mathcal{C} \) is called **co-compact** if the canonical map

\[
\lim_{d \in D} \text{Hom}(Y_d, X) \to \text{Hom}(\lim_{d \in D} Y_d, X)
\]

is an isomorphism.
is an isomorphism.

The interest of the notion of compact object is the following result.

**Theorem A.55.** Let $\mathcal{C}$ be a category that admits inductive limits and $\mathcal{C}^\text{cpt}$ the full subcategory of compact objects. If every object of $\mathcal{C}$ is an inductive limit of compact objects, then the composition

$$\text{Ind}(\mathcal{C}^\text{cpt}) \rightarrow \text{Ind}(\mathcal{C}) \rightarrow \mathcal{C}$$

is an equivalence of categories. Dually, let $\mathcal{C}$ be a category that admits projective limits and $\mathcal{C}^\text{ccpt}$ the full subcategory of co-compact objects. If every object of $\mathcal{C}$ is a projective limit of co-compact objects, then the composition

$$\text{Pro}(\mathcal{C}^\text{ccpt}) \rightarrow \text{Pro}(\mathcal{C}) \rightarrow \mathcal{C}$$

is an equivalence of categories.

**Example A.56.** In the category of vector spaces the compact objects are the finite dimensional vector spaces. Also the co-compact objects are the finite dimensional spaces.

**A.5.1. Derived functors via ind-objects.** One of the uses of the ind-category is that it allows us to construct derived functors even when the original category does not have enough injectives (see for instance [Wil00]).

We state the construction for the bounded below derived category, but similar constructions can also be made in the bounded case.

Let $\mathcal{A}$ be a small abelian category. We denote by

$$+: \mathcal{K}^+(\mathcal{A}) \rightarrow \text{Ind}(\mathcal{K}^+(\mathcal{A}))$$

the functor that associates to a complex $C$ the complex

$$C^+ = \lim_{C \rightarrow D} D$$

where the limit is taken with respect to all the quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$. By the definition, it follows that, if $E \xrightarrow{f} C$ is a quasi-isomorphism, then the induced map

$$f^+: E^+ \rightarrow D^+$$

is an isomorphism. Let now $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories with $\mathcal{A}$ small. Then $F$ extends to a functor

$$F: \text{Ind}(\mathcal{K}^+(\mathcal{A})) \rightarrow \text{Ind}(\mathcal{K}^+(\mathcal{B})).$$

Since the functor $+$ inverts quasi-isomorphism, the composition

$$\mathcal{K}^+(\mathcal{A}) \xrightarrow{+} \text{Ind}(\mathcal{K}^+(\mathcal{A})) \xrightarrow{F} \text{Ind}(\mathcal{K}^+(\mathcal{B})) \rightarrow \text{Ind}(\mathcal{D}^+(\mathcal{B}))$$
factors uniquely through $D^+(A)$ defining a functor

$$RF: D^+(A) \to \text{Ind}(D^+(B)).$$

**Definition A.57** (Deligne [Del73, Définition 1.2.1. (iii)]). $F$ is called *right derivable* (in the bounded below derived category) if $RF$ factors trough $D^+(B)$. In this case the functor

$$RF: D^+(A) \to D^+(B)$$

is called the *total right derived functor*.

The following lemma justifies calling it the total right derived functor.

**Lemma A.58** ([Del73, p. 23]). Let $A$ and $B$ be abelian categories with $A$ small. If the total right derived functor $RF$ exists then it satisfies the universal property of Definition A.33.

There is also a criterion for when the total right derived functor exist.

**Proposition A.59** ([Del73, Proposition 1.2.2. (ii)]). $F$ is right derivable (in the bounded below derived category) if and only if

$$RF(A) \in \text{Ind}(D^+(B))$$

lies in $D^+(B)$ for any object $A \in A$.

**Example A.60.** The abelian category $\text{MHS}$ does not have enough injectives. The main reason for this is that, if $H$ is a mixed Hodge structure, then $H_B$ and $H_{dR}$ are finite dimensional vector spaces. Nevertheless, in Exercise A.64 we will see that the functor $\text{Hom}_{\text{MHS}}(k)(H, -)$ is right derivable.

***

**Exercise A.61.** In this exercise, we illustrate the fact that, when $C$ admits inductive limits, then $C$ and $\text{Ind}(C)$ are not equivalent. Let $\text{QVec}_Q$ be the category of arbitrary $Q$-vector spaces, non necessarily of finite dimension. Let $V$ be a vector space with a countable basis and write it as

$$V = \lim_{\longrightarrow} W_n$$

with $W_n$ a vector space of dimension $n$. Write also $V_n = V$ for all $n \in \mathbb{N}$. Thus

$$V = \lim_{\longrightarrow} V_n.$$  

Show that the natural map in $\text{Ind}(\text{QVec}_Q)$

$$(W_n)_{n \in \mathbb{N}} \to (V_n)_{n \in \mathbb{N}}$$

is not an isomorphism. Conclude that the functor

$$\text{Ind}(\text{QVec}_Q) \to \text{QVec}_Q$$

is not an equivalence of categories.
Exercise A.62. Let $\mathcal{C}$ be a category.

(1) Let $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$ be inductive systems and $f$ a morphism of inductive systems. Assume that the inductive limits $X$ and $Y$ of the inductive systems exist. Then $f$ induces a morphism, also denoted $f$ between $X$ and $Y$. Prove that, if $g$ is an equivalent morphism, then the morphisms induced by $f$ and $g$ agree.

(2) Let $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$ be projective systems and $f$ a morphism of projective systems. Assume that the projective limits, $X$ and $Y$, of the projective systems exists. Then $f$ induces a morphism, also denoted $f$ between $X$ and $Y$. Prove that, if $g$ is an equivalent morphism, then the morphisms induced by $f$ and $g$ agree.

Exercise A.63. Let $\textbf{Vec}_k$ be the category of finite-dimensional vector spaces over $k$.

(1) Prove that, if $V$ is an ind-vector space, then its dual $V^\vee$ is a pro-vector space.

(2) If $f: V \to W$ is a morphism of ind-vector spaces, show that it induces a morphism $f^\vee$ of pro-vector spaces.

(3) Show that $\text{Ind}(\textbf{Vec}_k)$ is equivalent to the category of arbitrary vector spaces $\textbf{QVec}_k$.

Exercise A.64. Let $k$ be a subfield of $\mathbb{C}$ and let $H$ be a mixed Hodge structure over $k$. Show that the functor $\text{Hom}_{\text{MHS}(k)}(H, -): \text{MHS}(k) \to \text{Ab}$ is right derivable and that there is an equality $R^i \text{Hom}_{\text{MHS}(k)}(H, -) = \text{Ext}^i_{\text{MHS}(k)}(H, -)$.


A.6.1. Basic definitions.

Definition A.65. Let $\mathcal{A}$ be an abelian category and let $V$ be an object of $\mathcal{A}$. A decreasing filtration on $V$ is a collection of subobjects

$$\{0\} \hookrightarrow \ldots \hookrightarrow F^{p+1}V \hookrightarrow F^pV \hookleftarrow F^{p-1}V \hookrightarrow \ldots \hookrightarrow V.$$ We will use the notation $F^{\infty}V = \{0\}$ and $F^{-\infty}V = V$. A decreasing filtration is called finite if there are integers $p_1$ and $p_2$ such that $F^{p_1}V = \{0\}$ and $F^{p_2}V = V$.

Remark A.66. There is the analogue notion of increasing filtration that will be denoted with the index as a subscript:

$$\{0\} \hookrightarrow \ldots \hookrightarrow F_{p-1}V \hookrightarrow F_pV \subset F_{p+1}V \hookrightarrow \ldots \hookrightarrow V.$$ Starting from a decreasing filtration $F$, we can define an increasing filtration by the rule

$$F_pV = F^{-p}V.$$
We leave to the reader the task of translating all the notions of this subsection from decreasing to increasing filtrations.

Let \((V,F)\) be a filtered object of \(\mathcal{A}\). The associated graded object is

\[ \text{Gr}_F^* V = \bigoplus_{p \in \mathbb{Z}} \text{Gr}^p_F V, \quad \text{Gr}^p_F V = F^p V / F^{p+1} V. \]

If \(F\) is a decreasing filtration on an object \(V\) and \(n \in \mathbb{Z}\), the shifted filtration \(F[n]\) is defined by

\[ F[n]^p V = F^{n+p} V. \]

Let \((V,F)\) and \((V',F')\) be filtered objects of \(\mathcal{A}\) with decreasing filtrations. A morphism \(f : V \to V'\) is called filtered if \(f(F^p V) \subset F^p V'\) for all \(p \in \mathbb{Z}\) and strict (with respect to the filtration \(F\)) if, in addition,

\[ f(F^p V) = F^p V' \cap \text{Im}(f). \]

Let \(A = (A^*,d)\) be a cochain complex in \(C^+ (\mathcal{A})\). A filtration \(F\) on \(A\) is the data of filtrations \(F^n\) on each \(A^n\) that are compatible with the differential in that, for any two integers \(n\) and \(p\), one has

\[ d(F^n A^n) \subset F^n A^{n+1}. \]

A filtered complex is called strict if the differential is strict with respect to the filtration. A filtered complex is called biregular if, for every \(n \in \mathbb{Z}\), the filtration \(F^n\) on the object \(A^n\) is finite.

**Example A.67.** Given a cochain complex \(A^*\), the following filtrations are widely used:

1. The (decreasing) bête filtration \(\sigma_{\geq p}\) is given by

\[ \sigma_{\geq p} A^n = \begin{cases} \{0\}, & \text{if } n < p \\ A^n, & \text{if } n \geq p. \end{cases} \]

2. The decreasing canonical filtration \(\tau_{\geq p}\) is given by

\[ \tau_{\geq p} A^n = \begin{cases} \{0\}, & \text{if } n < p, \\ A^n / \text{Im}(d), & \text{if } n = p, \\ A^n, & \text{if } n > p. \end{cases} \]

3. The increasing canonical filtration \(\tau_{\leq p}\) is given by

\[ \tau_{\leq p} A^n = \begin{cases} A^n, & \text{if } n < p, \\ \text{Ker}(d), & \text{if } n = p, \\ \{0\}, & \text{if } n > p. \end{cases} \]
The canonical filtrations satisfy
\[ H^n(\tau^{\geq p} A) = \begin{cases} H^n(A), & \text{if } n \geq p, \\ \{0\}, & \text{if } n > p, \end{cases} \]
\[ H^n(\tau_{\leq p} A) = \begin{cases} \{0\}, & \text{if } n < p, \\ H^n(A), & \text{if } n \geq p. \end{cases} \]

A.6.2. Spectral sequences. Given a filtered complex, we can use the filtration to construct successive approximations to its cohomology. This can be seen as a generalization of the long exact sequence arising from a short exact sequence of complexes and will be achieved through spectral sequences.

Let \((A^*, F)\) be a complex together with a biregular decreasing filtration. The filtration \(F\) induces a filtration in cohomology as follows:
\[ F^p H^n(A) = \text{Im}(F^p A^n \cap ZA^n), \]
where \(ZA^n = \ker(d: A^n \to A^{n+1})\) is the subgroup of cycles. Hence we have an associated graded vector space \(\text{Gr}_F H^*(A)\). The spectral sequence will allow us to recover the graded object \(\text{Gr}_F H^*(A)\) rather than the total cohomology space \(H^*(A)\).

For simplicity of the exposition, we will assume in the sequel that the category \(A\) is the category of modules over a ring so we can use elements in the discussion. The basic idea of spectral sequences is that we approximate \(H^*(A)\) by first computing the cohomology of \(\text{Gr}_F^p A\). That is, the first page of the spectral sequence is
\[ E_1^{p,q} = H^{p+q}(\text{Gr}_F^p A) = \frac{\{x \in F^p A^{p+q} \mid dx \in F^{p+1} A^{p+q+1}\}}{d(F^p A^{p+q-1}) + F^{p+1} A^{p+q}}. \]
In approximating \(\text{Gr}_F^p H^{p+q}(A)\) by \(E_1^{p,q}\) we are making two errors. First, we are taking the elements \(x \in F^p A^{p+q}\) such that \(dx \in F^{p+1} A^{p+q+1}\), while the elements in \(\text{Gr}_F^p H^{p+q}(A)\) should be be represented by \(x \in F^p A^{p+q}\) with \(dx = 0\). The second error is that we are taking the quotient by elements of the form \(d(F^p A^{p+q-1})\), while we should be taking the quotient by all boundaries. So \(\text{Gr}_F^p H^{p+q}(A)\) should be a certain subquotient of \(E_1^{p,q}\). In fact, the differential \(d\) induces a map
\[ d_1: E_1^{p,q} \to E_1^{p+1,q}, \]
see Exercise A.76. Then the second page of the spectral sequence
\[ E_2^{p,q} = \frac{\ker(d_1: E_1^{p,q} \to E_1^{p+1,q})}{\text{Im}(d_1: E_1^{p-1,q} \to E_1^{p,q})} \]
is a better approximation to \(\text{Gr}_F^p H^{p+q}(A)\) than the first page. In fact,
\[ E_2^{p,q} = \frac{\{x \in F^p A^{p+q} \mid dx \in F^{p+2} A^{p+q+1}\}}{dF^{p-1} A^{p+q-1} \cap F^p A^{p+q} + F^{p+1} A^{p+q} \cap d^{-1} F^{p+2} A^{p+q+1}}. \]
So now the errors we are committing are that $dx \in F^{p+2}A^{p+q+1}$ instead of being zero and that we are taking the quotient by $dF^{p-1}$ instead of taking the quotient by all the boundaries. Something has improved, because in general $F^{p+2}A^{p+q+1}$ is smaller than $F^{p+1}A^{p+q+1}$ so we are closer to the condition of $dx = 0$ and $dF^{p-1}$ is bigger than $dF^p$, so we are closer to taking the quotient by all the boundaries.

This process can be iterated. There is a differential $d^2: E^{p,q}_2 \to E^{p+2,q-1}_2$, the third page $E^{*,*}_3$ is defined as the cohomology of this differential, and so on. Explicitly, for every $r > 0$, we write

$$E^{p,q}_r = \left\{ x \in F^pA^{p+q} \mid dx \in F^{p+r}A^{p+q+1} \right\}.$$  

(A.68)

This definition makes sense for $r = \infty$ and we obtain

$$E^{p,q}_\infty = \frac{\left\{ x \in F^pA^{p+q} \mid dx = 0 \right\}}{dA^{p+q-1} \cap F^pA^{p+q} + F^{p+1}A^{p+q} \cap d^{-1}F^{p+r}A^{p+q+1}} = \text{Gr}_F^p H^{p+q}(A).$$

Again, the map $[x] \mapsto [dx]$ defines a differential

$$d_r: E^{p,q}_r \to E^{p+r,q-r+1}_r$$

and

$$E^{p,q}_{r+1} = \frac{\text{Ker}(d_r: E^{p,q}_r \to E^{p+r,q-r+1}_r)}{\text{Im}(d_r: E^{p+r,q+r-1}_r \to E^{p,q}_r)}.$$  

(A.69)

This is the outcome of Exercise A.77.

If the filtration $F$ is biregular, then for every $p, q \in \mathbb{Z}$ there is an integer $r_0 \geq 1$ such that $E^{p,q}_{r_0} = E^{p,q}_\infty$ for all $r \geq r_0$. Therefore, we can recover $\text{Gr}_F^r H(A)$ from the spectral sequence.

**Definition A.70.** Let $(A, F)$ be a filtered complex. We will denote by $E^{p,q}_r$ the elements defined by equation (A.68) and the collection of pages $E^{*,*}_r$ and morphisms $d_r$ will be called the spectral sequence associated with the filtration $F$. For any $r \geq 1$ we use the notation

$$E^{p,q}_r \Longrightarrow H^{p+q}(A)$$

to indicate a spectral sequence that converges to the cohomology of $A$. But recall that we only recover the graded vector space.

If there is an $r \geq 1$ such that, for all $p, q$, the equality $E^{p,q}_{r_0} = E^{p,q}_\infty$ holds, we say that the spectral sequence degenerates at the term $E_{r_0}$ or that it degenerates at the $r_0$-th page.

If we want to stress the filtration, because the original complex may have more that one filtration, then we will denote the spectral sequence as $F E^{p,q}_r$.

**A.6.3. Degeneracy criteria.** In this section we discuss several criteria for the degeneration of an spectral sequence.

**Proposition A.71.** Let $(A, F)$ be a filtered complex.
If there is an integer \( r \geq 1 \) such that the page \( E_r^*,* \) is reduced to one row, that is
\[
E_r^{p,q} = \{0\} \quad \text{for all } q \neq q_0,
\]
then the spectral sequence degenerates at the page \( E_r^*,* \) and there is an identity
\[
H^{p+q_0}(A) = E_r^{p,q_0}.
\]
In this case, we recover the cohomology and not the associated graded space.

Assume that there is an integer \( r_0 \geq 1 \) such that the page \( E_{r_0}^*,* \) is reduced to two rows, that is,
\[
E_{r_0}^{p,q} = \{0\} \quad \text{for all } q \neq q_0, q_1 \text{ with } q_0 < q_1.
\]
Write \( r = q_1 - q_0 + 1 \). Then the spectral sequence degenerates at the page \( E_{\max(r_0,r+1)}^*,* \). If \( r_0 \leq r \), then there is a long exact sequence
\[
\cdots \to H^{p+q_1}(A) \to E_{r_0}^{p,q_1} \xrightarrow{d_r} E_{r_0}^{p+r,q_1-r+1} \to H^{p+q_1+1}(A) \to E_{r_0+1,q_1} \to \cdots
\]

**Proof.** Exercise A.78.

**Proposition A.72.** Let \((A,F)\) be a filtered complex. If \( d \) is strict with respect to the filtration \( F \) then the spectral sequence \( E_{r,*}^* \) degenerates at the term \( E_1 \). In this case, the following holds:
\[
F_p H^n(A) = H^n(F_p A), \quad Gr_F^p H^n(A) = H^n(Gr_F^p A).
\]

**Proof.** Consider the diagram
\[
\begin{array}{ccc}
\{x \in F_p A^n | dx = 0\} & \xrightarrow{f} & E_1^{p,n-p} \\
F^{p,r+n,F} A^n & \xrightarrow{g} & E_{\infty}^{p,n-p}
\end{array}
\]
If \( x \in F_p A^n \) satisfies that \( dx \in F^{p+1} A^{n+1} \), then by the strictness of \( d \), there is an element \( y \in F^{p+1} A^n \) such that \( dx = dy \). This implies that the map \( f \) in the above diagram is an isomorphism.

If \( x \in F_p A^n \) satisfies that \( x \in \text{Im} d \), again by strictness, there is an element \( z \in F^{p+1} A^{n-1} \) such that \( x = dz \). This shows that the map \( g \) in the above diagram is an isomorphism.

Since the maps \( f \) and \( g \) are isomorphisms, we deduce that the spectral sequence degenerates at \( E_1 \). In particular, this implies that
\[
H^n(Gr_F^p A) = E_1^{p,n-p} = E_{\infty}^{p,n-p} = Gr_F^p H^n(A).
\]
The equality \( H^n(F_p A) = F_p H^n(A) \) follows from
\[
dF_p A^{n-1} = dA^{n-1} \cap F_p A^n
\]
that is true by the strictness of $d$. □

A.6.4. The case of a dg-complex. A differential graded complex (dg-complex for short) is a collection of complexes $(A^p, d^\text{ver})$ and morphisms of complexes $d^\text{hor}: A^p, * \to A^{p+1, *}$ satisfying $(d^\text{hor})^2 = 0$. Since the maps $d^\text{hor}$ and $d^\text{ver}$ commute, we have to change some signs to define the total complex.

A dg-complex $A = (A^*, d^\text{hor}, d^\text{ver})$ is called bounded (respectively bounded below) if there exist integers $p_0$ and $q_0$ such that $A^{p,q} = \{0\}$ whenever $|p| > p_0$ or $|q| > q_0$ (respectively $p < p_0$ or $q < q_0$).

**Definition A.73.** Let $A = (A^*, d^\text{hor}, d^\text{ver})$ be a bounded below dg-complex. Its total complex $\text{Tot}(A)$ is the complex

$$\text{Tot}(A)^n = \bigoplus_{p+q=n} A^{p,q}$$

with differential $d: \text{Tot}(A)^n \to \text{Tot}(A)^{n+1}$ given, in the piece $A^{p,q}$ by

$$d = d^\text{hor} + (-1)^p d^\text{ver}.$$

The total complex of a dg-complex comes always equipped with the horizontal bête filtration

$$\sigma^{\geq p} \text{Tot}(A)^n = \bigoplus_{p' \geq p} A^{p', n-p'}.$$

The corresponding spectral sequence has $E_1$ term

$$E_1^{p,q} = H^q(A^p, *) \implies H^{p+q}(A). \quad (A.74)$$

***

**Exercise A.75.** Let $k$ be a field and let $\mathbf{FVec}(k)$ be the category of filtered $k$-vector spaces together with filtered morphisms. This is an additive category.

1. Show that every morphism in $\mathbf{FVec}(k)$ has a kernel and a cokernel. More precisely, the kernel of an arrow $f$ agrees with the kernel computed in $\mathbf{Vec}(k)$ together with the induced filtration as a subobject. Similarly, the cokernel agrees with the one computed in $\mathbf{Vec}(k)$, together with the induced filtration as a quotient.

2. Let $f: (V, F) \to (W, F)$ be a morphism of filtered vector spaces. Show that the arrow $f$ is strict with respect to $F$ if and only if the canonical map $\text{Coim}(f) \to \text{Im}(f)$ is an isomorphism.

3. Conclude that $\mathbf{FVec}(k)$ is not an abelian category.

**Exercise A.76.** Show that the map $E_1^{p,q} \to E_1^{p+1,q}$ given by $[x] \mapsto dx$ is well defined.
EXERCISE A.77. Show that, for each \( r \geq 1 \), the map \( E^{p,q}_r \to E^{p+r,q-r+1}_r \) given by \([x] \mapsto dx\) is well defined and prove equation (A.69).

EXERCISE A.78. Prove Proposition A.71.

EXERCISE A.79. In this exercise we introduce the decaled filtration. Let \((A,F)\) be a filtered complex the decaled filtration \(\text{Dec}(F)\) is defined as

\[
\text{Dec}(F)^pA^n = \{ x \in F^{p+n}A^n \mid dx \in F^{p+n+1}A^{n+1} \}.
\]

(1) Prove that \((A,\text{Dec}(F))\) is a filtered complex.

(2) Prove that there are isomorphisms compatible with the differentials

\[
\text{Dec}(F)^pE_{r}^{p,n-p} \to F^{p,n-p}_{r+1}.
\]

A.7. Simplicial techniques. A very useful tool in homological algebra is that of simplicial objects. A good reference for simplicial techniques is the book \([GJ09]\).

A.7.1. Simplicial and cosimplicial objects. We first review the definition of simplicial and cosimplicial objects in a category. Let \(\Delta\) denote the category with objects the finite ordered sets

\(\Delta_n = \{0, \ldots, n\}, \quad n \geq 0,\)

and morphisms the non-decreasing maps between the various \(\Delta_n\). Any morphism in \(\Delta\) can be written as a composition of faces \(\delta^i: \Delta_n \to \Delta_{n+1}\), for \(i = 0, \ldots, n + 1\), and degeneracies \(\sigma^i: \Delta_{n+1} \to \Delta_n\), for \(i = 0, \ldots, n\), which are defined as follows:

\[
\delta^i(j) = \begin{cases} 
  j & \text{if } j < i, \\
  j + 1 & \text{if } j \geq i,
\end{cases} \quad \sigma^i(j) = \begin{cases} 
  j & \text{if } j \leq i, \\
  j - 1 & \text{if } j > i.
\end{cases}
\]

In other words, the face \(\delta^i\) is the map that skips \(i\), while the degeneracy \(\sigma^i\) is the map that repeats \(i\).

**Definition A.80.** Let \(\mathcal{C}\) be a category. A simplicial (resp. cosimplicial) object in \(\mathcal{C}\) is a functor \(\Delta^{\text{op}} \to \mathcal{C}\) (resp. \(\Delta \to \mathcal{C}\)).

Using the above characterization of morphisms in \(\Delta\), simplicial and cosimplicial objects admit a very concrete description. For instance, a cosimplicial object \(X^\bullet\) becomes a collection \((X^n)_{n \geq 0}\) of objects of \(\mathcal{C}\), each \(X^n\) being the image of \(\Delta_n\) through the functor \(\Delta \to \mathcal{C}\), together with morphisms

\[
\delta^i: X^n \to X^{n+1}, \quad i = 0, \ldots, n + 1 \\
\sigma^i: X^{n+1} \to X^n, \quad i = 0, \ldots, n,
\]
satisfying the commutativity relations

\[(a)\quad \delta^i \delta^j = \delta^j \delta^{i-1}, \quad \text{for } i < j,\]
\[(b)\quad \sigma^j \sigma^i = \sigma^i \sigma^{j+1}, \quad \text{for } i \leq j,\]
\[(c)\quad \sigma^j \delta^i = \delta^i \sigma^{j-1}, \quad \text{for } i < j,\]
\[(d)\quad \sigma^j \delta^i = \text{Id}, \quad \text{for } i = j, j + 1,\]
\[(e)\quad \delta^i \sigma^i = \delta^{i-1} \sigma^i, \quad \text{for } i > j + 1.\]

The maps \(\delta^i\) and \(\sigma^i\) are again called faces and degeneracies, and one usually represents these data by a diagram of the form

\[
\begin{array}{ccccccc}
X^0 & \overset{\delta_0}{\longrightarrow} & X^1 & \overset{\delta_1}{\longrightarrow} & \cdots \\
\end{array}
\]

The description of a simplicial object is the dual one. It is thus given by a collection of objects \((X_n)_{n \geq 0}\), together with morphisms

\[
\delta_i : X_{n+1} \to X_n, \quad i = 0, \ldots, n + 1
\]
\[
\sigma_i : X_n \to X_{n+1}, \quad i = 0, \ldots, n
\]
satisfying the commutativity relations dual to (A.81), that is:

\[(a)\quad \delta_i \delta_j = \delta_j \delta_{i-1}, \quad \text{for } i < j,\]
\[(b)\quad \sigma_i \sigma_j = \sigma_j \sigma_{i+1}, \quad \text{for } i < j,\]
\[(c)\quad \delta_i \sigma_j = \sigma_{j-1} \delta_i, \quad \text{for } i < j,\]
\[(d)\quad \delta_i \sigma_j = \text{Id}, \quad \text{for } i = j, j + 1,\]
\[(e)\quad \delta_i \sigma_j = \sigma_j \delta_{i-1}, \quad \text{for } i > j + 1.\]

Remark A.83. The category \(\Delta\) is equivalent to the category of totally ordered non-empty finite sets, denoted by \(\text{FOS}\). Therefore, we can also view a simplicial object \(X_\bullet\) as a functor \(\text{FOS}^{\text{op}} \to C\) by sending such a set \(I\) to the object \(X_I = X_{|I|-1}\), where \(|I|\) denotes the cardinal of \(I\).

Example A.84. Recall from Notation 1.107 that the symbol \(\Delta^n\) was already used for the topological simplex

\[
\Delta^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 1 \geq t_1 \geq \cdots \geq t_n \geq n\}.
\]

As \(n\) varies, the simplices \(\Delta^n\) form a cosimplicial object \(\Delta^\bullet\) in the category of topological spaces. The faces are given by

\[
\delta^i(t_1, \ldots, t_n) = \begin{cases}
(1,t_1,\ldots,t_n), & \text{if } i = 0, \\
(t_1,\ldots,t_i,t_{i+1},\ldots,t_n), & \text{if } i = 1,\ldots,n, \\
(t_1,\ldots,t_n,0), & \text{if } i = n+1,
\end{cases}
\]

and the degeneracies by

\[
\sigma^i(t_1, \ldots, t_{n+1}) = (t_1, \ldots, \hat{t}_{i+1}, \ldots, t_{n+1}), \quad i = 0 \ldots, n,
\]

where the symbol \(\hat{t}_{i+1}\) means that the coordinate \(t_{i+1}\) is omitted.
What we called the standard simplex in Section 2.1, that is,
\[ \Delta_{st}^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, \ t_i \geq 0, \ i = 0, \ldots, n \} \]
gives a more symmetric representation of the same cosimplicial topological space (see Exercise A.95). The face maps \( \delta^i : \Delta_{st}^n \to \Delta_{st}^{n+1} \) are now defined as
\[ \delta^i(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n), \quad i = 0, \ldots, n + 1, \]
and the degeneracy maps \( \sigma^i : \Delta_{st}^{n+1} \to \Delta_{st}^n \) as
\[ \sigma^i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1}), \quad i = 0, \ldots, n \]

**A.7.2.** Simplicial abelian groups and chain complexes. Simplicial and cosimplicial objects in an abelian category are closely related to chain and cochain complexes, as introduced in Section A.1. In this paragraph, we review some constructions making this relation precise. To begin with, we associate a chain complex with a simplicial object in an abelian category.

**Definition A.85.** Given a simplicial object \( X_* \) in an abelian category, the **associated chain complex** \( CX_* \) with
\[ CX_n = X_n, \quad \partial_n = \sum_{i=0}^{n} (-1)^i \delta_i : CX_n \to CX_{n-1}. \]
When there is no need to emphasize the degree, we will denote the differential of this complex simply by \( \partial \).

We can also consider a simpler complex called the normalized complex.

**Definition A.86.** Given a simplicial object \( X_* \) in an abelian category, the associated **normalized chain complex** \( N X_* \) with
\[ N X_n = \bigcap_{i=0}^{n-1} \text{Ker} \delta_i, \quad \partial_n = (-1)^n \delta_n. \]
The **group of degenerate elements** is
\[ DX_n = \sum_{i=0}^{n-1} \sigma_i(X_{n-1}). \]

We ask the reader to check in Exercise A.96 that \( CX_* \) and \( N X_* \) are indeed complexes and that degenerate elements form a subcomplex of \( CX_* \).

The chain complex \( CX_* \) and the normalized chain complex \( N X_* \) have the same homological properties.

**Theorem A.87.** If \( X_* \) is a simplicial object in an abelian category, then the composition
\[ N X_* \to CX_* \to CX_* / DX_* \]
is an isomorphism. Moreover, the inclusion \( N X_* \to CX_* \) is a quasi-isomorphism.
Proof. We define the subspaces
\[ N_k X_n = \bigcap_{j=0}^{\min(k,n-1)} \ker \delta_j, \quad D_k X_n = \sum_{i=0}^{\min(k,n-1)} \sigma_i (X_{n-1}). \]
Using the simplicial identities is easy to show that \((N_k X_\ast, \partial)\) and \((D_k X_\ast, \partial)\) are subcomplexes of \((CX_\ast, \partial)\).

We prove by induction on \(k\) that, for all \(n \geq 0\), the composition
\[ N_k X_n \to CX_n \to CX_n / D_k X_n \quad \text{(A.88)} \]
is an isomorphism and that the inclusion
\[ N_k X_\ast \to N_{k-1} X_\ast \]
is a quasi-isomorphism.

Since for \(k = -1\), we have \(N_0 X_n = CX_n\) and \(D_0 X_n = \{0\}\), the above statements are true for \(k = -1\). Since, for \(k \geq n - 1\), \(N_k X_n = N X_n\), if we prove the above statements for all \(k\) we deduce the theorem.

We assume that both statements are true for \(j < k\) and we prove them for \(k\). We first prove that the composition (A.88) is an isomorphism. For \(n \leq k\) this follows directly by the induction hypothesis. Let \(x \in N_{k-1} X_n\) with \(n \geq k + 1\). Then by the simplicial identities,
\[ x - \sigma_k \delta_k x \in N_k X_n. \]

Using the induction hypothesis we conclude that \(CX_n = N_k X_n + D_k X_n\) and the composition (A.88) is surjective.

Using again the simplicial identities, it follows that
\[ \sigma_k (N_{k-1} X_n) \subset N_{k-1} X_n, \quad \sigma_k (D_{k-1} X_n) \subset D_{k-1} X_n. \]
In particular \(\sigma_k\) induces a map \(\sigma_k : X_n / D_{k-1} X_n \to X_n / D_{k-1} X_n\), by induction hypothesis there is a commutative diagram
\[
\begin{array}{ccc}
N_{k-1} X_n & \cong & X_n / D_{k-1} X_n \\
\downarrow \sigma_k & & \downarrow \sigma_k \\
N_{k-1} X_n & \cong & X_n / D_{k-1} X_n.
\end{array}
\]
There is a canonical isomorphism
\[
\frac{X_n}{D_k X_n} = \frac{X_n}{D_{k-1} X_n + \sigma_k (X_{n-1})} \cong \frac{X_n / D_{k-1} X_n}{\sigma_k (X_n / D_{k-1} X_n)}.
\]
From this isomorphism and the previous commutative diagram we deduce that, if \(x \in N_{k-1} X_n\) belongs to \(D_k X_n\) then, \(x = \sigma_k y\) with \(y \in N_{k-1} X_n\).

Nor let \(x \in N_k X_n \cap D_k X_n\). By the previous discussion \(x = \sigma_k y\) with \(y \in N_{k-1} X_n\). Since \(x \in \ker \delta_k\), then
\[ 0 = \delta_k x = \delta_k \sigma_k y = y. \]
So \( y = 0 \) and, in consequence \( x = 0 \). This shows that \( \mathcal{N}_k X_n \cap D_k X_n = \{0\} \) and that the composition (A.88) is injective. Thus is an isomorphism.

Now we prove that the inclusion \( \mathcal{N}_k X_* \to \mathcal{N}_{k-1} X_* \) is a quasi-isomorphism. In fact we will prove more. We prove that it is a homotopy equivalence. Denote by \( \iota_k : \mathcal{N}_k X_* \to \mathcal{N}_{k-1} X_* \) the inclusion and by \( \pi_k \) the composition
\[
\mathcal{N}_{k-1} X_* \to X_*/D_{k-1} X_* \cong \mathcal{N}_k X_* .
\]
One checks that, for \( x \in \mathcal{N}_{k-1} X_n \),
\[
\pi_k \circ \iota_k (x) = x = \text{Id}(x) \\
\iota_k \circ \pi_k (x) = x, \quad \text{if } n \leq k, \\
\iota_k \circ \pi_k (x) = x - \sigma_k \delta_k(x) \quad \text{if } n > k.
\]
We need to show that \( \iota_k \circ \pi_k \) is homotopy equivalent to the identity. Let \( s : \mathcal{N}_{k-1} X_* \to \mathcal{N}_{k-1} X_* \) be the map that sends \( x \in \mathcal{N}_{k-1} X_n \) to
\[
s(x) = \begin{cases} 
0, & \text{if } n < k; \\
(-1)^k \sigma_k(n), & \text{if } n \geq k.
\end{cases}
\]
Using the simplicial identities it follows that
\[
x - \iota_k \circ \pi_k (x) = (\partial s + s \partial)(x)
\]
showing that \( \mathcal{N}_k X_* \) and \( \mathcal{N}_{k-1} X_* \) are homotopy equivalent and finishing the proof of the theorem. \( \square \)

Example A.89. Let \( M \) be a topological space. As in Section 2.1, we denote by \( C_n(M) = C_n(M, \mathbb{Z}) \) the free abelian group generated by all continuous maps \( \sigma : \Delta^n_{st} \to M \). The cosimplicial structure of \( \Delta^\bullet_{st} \) induces a simplicial group structure on \( C^\bullet(M) \). We will denote by \( C_*(M) \) the associated chain complex and by \( \tilde{C}_*(M) \) the normalized chain complex. As explained in Section 2.1, the complex \( (C_*(M), \partial_*) \) computes singular homology of \( M \). The same is true for the complex \( (\tilde{C}_*(M), \partial_*) \).

Similarly, if \( M \) is a differentiable manifold, we will denote by \( S_*(M) \) the chain complex of smooth singular chains and by \( \tilde{S}_*(M) \) the normalized complex of smooth singular chains.

Dualizing the construction of the chain complex we obtain the definition of the associated cochain complex.

Definition A.90. Let \( X^\bullet \) be a cosimplicial object in an abelian category, the associated cochain complex is the complex \( CX^\bullet \) with
\[
CX^n = X^n, \quad d = \sum_{i=0}^{n+1} (-1)^i \delta^i : CX^n \to CX^{n+1},
\]
and the normalized cochain complex is
\[
\mathcal{C}X^n = X^n / \sum_{i=0}^{n-1} \text{Im } \delta^i \cong \bigcap_{i=0}^{n-1} \text{Ker } \sigma^i, \quad d = \sum_{i=0}^{n+1} (-1)^i \delta^i.
\]
The statements that $X^n / \sum_{i=0}^{n-1} \text{Im} \delta^i$ and $\bigcap_{i=0}^{n-1} \text{Ker} \sigma^i$ are isomorphic and that the inclusion $\mathcal{N}X^* \to CX^*$ is a quasi isomorphism are dual to Theorem A.87 and are proved in a similar way.

**Example A.91.** Let $M$ be a topological space. For any coefficient ring $R$, the groups

$$C^n(M, R) = \text{Hom}(C_n(M), R)$$

form a cosimplicial abelian group. We will also denote by $C^*(M, R)$ the associated cochain complex and by $\widetilde{C}^*(M, R)$ the normalized cochain complex. Similarly, when $M$ is a differentiable manifold, $S^*(M, R)$ and $\widetilde{S}^*(M, R)$ will denote the cochain complex of smooth singular cochains and the corresponding normalized complex. The four complexes in this example compute the singular cohomology of $M$ with coefficients in $R$.

In the course of the proof of Beilinson’s Theorem 3.193, one needs to associate with a cosimplicial manifold a variant of the normalized cochain complex from Definition A.90. In fact, it is homotopically equivalent to a truncation of the normalized one. For each $N \geq 0$ and each simplicial object $X_\bullet$ in an abelian category, we introduce a new complex $C_*(\Delta_N, X_\bullet)$. For each $0 \neq I \subset \Delta_N$, using the convention of Remark A.83, we have the object $X_I = X_{|I|-1}$. If $K = \{k_0, \ldots, k_p\}$ with the indices $k_l$ in increasing order, and $I = \{k_0, \ldots, \hat{k}_i, \ldots, k_p\}$, we set the sign $\varepsilon(I, K) = (-1)^i$, which is the same appearing in Notation 3.163. We also denote

$$d_{I,K} = \delta_i: X_K \to X_I.$$  

For each $p \geq 0$, we define

$$C_p(\Delta_N, X_\bullet) = \bigoplus_{I \subset \Delta_N, |I| = p+1} X_I$$  

with differential $d: C_p(\Delta_N, X_\bullet) \to C_{p-1}(\Delta_N, X_\bullet)$ given by

$$d = \bigoplus_{I \subset K} \varepsilon(I, K) d_{I,K}.$$  

For a chain complex $C_*$, let $\sigma_{\leq N}$ denote the \textit{bête} filtration

$$\sigma_{\leq N} C_n = \begin{cases} C_n, & \text{if } n \leq N, \\ 0, & \text{if } n > N. \end{cases}$$

This filtration is the dual of the bête filtration in a cochain complex given in Example A.67.

For a conceptual proof of the following result, see [DG05, Proposition 3.10]. We give an elementary proof similar to the proof of Theorem A.87.

**Proposition A.93.** Given a simplicial object $X_\bullet$ in an abelian category, the complexes $C_*(\Delta_N, X_\bullet)$ and $\sigma_{\leq N} \mathcal{N}X_*$ are functorially homotopically equivalent.
Proof. Let \( \phi: \sigma_{\leq N} \mathcal{N} X_\ast \to C_\ast(\Delta_N, X_\ast) \) be the map sending the object \( \mathcal{N} X_\ast \) to \( X_{\Delta_n} \subset C_n(\Delta_N, X_\ast) \). This map is a morphism of complexes because, for each subset \( J \subset \Delta_n \) with \( |J| = n \), the restriction to \( \mathcal{N} X_\ast \) of the map \( d_{\Delta_n,J} \) vanishes unless \( J = \Delta_{n-1} \). The map \( \phi \) is injective and we identify \( \sigma_{\leq N} \mathcal{N} X_\ast \) with its image inside \( C_\ast(\Delta_N, X_\ast) \).

We consider the decreasing filtration \( F^\ast \) on the complex \( C_\ast(\Delta_N, X_\ast) \) defined, for \( I = \{i_0, \ldots, i_k, \ldots \} \) in increasing order, by

\[
F^p X_I = \left\{ \bigcap_{j=0}^{\min(p,|I|-1)} \ker \delta_j, \quad \text{if } i_j = j \text{ for } j = 0, \ldots, \min(p,|I|-1), \right. \\
\left. \{0\}, \quad \text{otherwise.} \right.
\]

The simplicial identities readily show that this filtration is by subcomplexes. Moreover

\[
F^{-1}C_\ast(\Delta_N, X_\ast) = C_\ast(\Delta_N, X_\ast),
\]

\[
F^N C_\ast(\Delta_N, X_\ast) = \sigma_{\leq N} \mathcal{N} X_\ast.
\]

Therefore, it is enough to show, for \( p \geq 0 \), that each inclusion

\[
F^p C_\ast(\Delta_N, X_\ast) \hookrightarrow F^{p-1}C_\ast(\Delta_N, X_\ast),
\]

is a homotopy equivalence.

For \( p \geq 0 \), let \( I \subset \Delta_N \) be a subset of the form

\[
I = \{0, 1, \ldots, p-1, i_p, \ldots, i_n\}, \quad p < i_p < i_{p+1} < \cdots < i_n. \tag{A.94}
\]

Write \( I_p = \{0, 1, \ldots, p-1, i_p, \ldots, i_n\} \). There is an increasing map \( \sigma^p: I_p \to I \) that sends \( p \) to \( i_p \) and the other elements to themselves. Being \( X_\ast \) a simplicial object, there is a map \( \sigma^p: X_I \to X_{I_p} \).

For \( p \geq 0 \) we define the degree one map

\[
s^p: F^{p-1}C_\ast(\Delta_N, X_\ast) \longrightarrow F^{p-1}C_\ast(\Delta_N, X_\ast)
\]

that, in the component \( X_I \) is equal to \( (-1)^p \sigma_p: X_I \to X_{I_p} \) if \( I \) has the shape (A.94) and is zero otherwise.

We look at the map

\[
\psi^p: F^{p-1}C_\ast(\Delta_N, X_\ast) \longrightarrow F^{p-1}C_\ast(\Delta_N, X_\ast)
\]
given as \( \psi^p = \text{Id} - (ds^p + s^p d) \). The simplicial identities imply that,

1. if \( x \in F^p C_\ast(\Delta_N, X_\ast) \), then \( \psi^p(x) = x \);
2. for \( x \in F^{p-1} C_\ast(\Delta_N, X_\ast) \), then \( \psi^p(x) \in F^p C_\ast(\Delta_N, X_\ast) \).

Thus \( \psi^p \) induces a morphism of complexes

\[
\psi^p': F^{p-1}C_\ast(\Delta_N, X_\ast) \longrightarrow F^p C_\ast(\Delta_N, X_\ast)
\]

that, combined with the inclusion in the opposite direction, yields a homotopy equivalence.
Writing $\psi = \psi_{N-1} \circ \cdots \circ \psi_0$ we obtain a homotopy inverse of the map $\phi$. Since we have only used the simplicial maps, it is clear that the resulting homotopy equivalence is functorial. 

\[\square\]

**Exercise A.95.** Construct homeomorphisms $\Delta^n \to \Delta^n_{st}$ for each $n \geq 0$ that commute with the face and degeneracy maps from Example A.84.

**Exercise A.96.** Let $X_\bullet$ be a simplicial object $X_\bullet$ in an abelian category. Use the simplicial identities (A.82) to prove the following statements. 

(1) The composition $\partial_n \circ \partial_{n+1} = 0$. Therefore, $(CX_\bullet, \partial)$ is a complex. 

(2) Show that $\partial_n$ sends $\mathcal{N}X_n$ to $\mathcal{N}X_{n-1}$ and, when restricted to $\mathcal{N}X_n$, agrees with $(-1)^n \delta_n$. 

(3) Show that $\partial_n$ sends $DX_n$ to $DX_{n-1}$.

### A.8. Sheaf cohomology

In this section, we give a brief summary of sheaf cohomology. More details can be found e.g. in [Har77, Chapter II] or in [KS06].

**Definition A.97.** Let $M$ be a topological space. A sheaf of abelian groups $F$ on $M$ is an assignment that, to each open subset $U \subset M$, associates an abelian group $F(U)$ satisfying the following properties:

1. If $U \subset V$ is the inclusion of an open subset, then there is a restriction map $\rho_{U,V}: F(V) \to F(U)$. Most often, one uses the notation 
   \[ t|_U = \rho_{U,V}(t). \]
   This restriction satisfies $\rho_{U,U} = \text{Id}$. 
2. If $U \subset V \subset W$ are inclusions of open subsets, then 
   \[ \rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}. \]
3. If $U = \bigcup_{i \in I} U_i$ is an open covering and $t \in F(U)$ is a section such that $t|_{U_i} = 0$ for all $i \in I$, then $t = 0$. 
4. If $U = \bigcup_{i \in I} U_i$ is an open covering and $t_i \in F(U_i)$ are such that 
   \[ t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j} \]
   for all $i, j \in I$, then there exists $t \in F(U)$ such that $t|_{U_i} = t_i$. 

An assignment as before that only satisfies (1) and (2) is called a presheaf.

**Remarks A.98.**
(1) Properties (1) and (2) are summarized as follows: let $\text{Op}(M)$ be the category whose objects are the open subsets of $M$ and whose morphisms $\text{Hom}(U,V)$ are a singleton if $U \subset V$ and empty otherwise; then $F$ is a contravariant functor from $\text{Op}(M)$ to the category $\text{Ab}$ of abelian groups. Property (3) is the locality property, while (4) is the gluing property. From now on, we convene that, unless otherwise indicated, the word “sheaf” means “sheaf of abelian groups” and we denote by $\text{ShAb}(M)$ the category.

(2) It follows from the definition that, if $F$ is a sheaf, then $F(\emptyset) = \{0\}$.

(3) To define a sheaf $F$, it suffices to specify the sections of $F$ on a basis of the topology of the underlying topological space. For example, if $M$ is a differentiable manifold, it is enough to give the value on contractible open subsets (Exercice A.118).

The category $\text{ShAb}(M)$ is abelian, so it makes sense to talk about kernels, cokernels, and images; of complexes of sheaves and of the cohomology of a complex of sheaves; also the notion of exact sequence of sheaves is well defined and the usual definition of injective object in homological algebra also applies to sheaves.

In many cases, it is easier to describe a presheaf than to describe a sheaf, as we need to fulfill less conditions. Fortunately, there is a canonical procedure to go from presheaves to sheaves. For the proof of the following proposition, see [Har77, Proposition-Definition II 1.2].

**Proposition A.99.** Given a presheaf $F$ there is a sheaf $F^+$ and a morphism of presheaves $\theta: F \to F^+$ that is universal. This means that, for every morphisms of presheaves $f: F \to G$ with $G$ a sheaf, there is a unique map of sheaves $\phi: F^+ \to G$ such that $f = \phi \circ \theta$.

**Examples A.100.** Let $M$ be a topological space.

(1) Let $A$ be an abelian group. The constant presheaf $A^o$ is the presheaf that sends any open subset $U$ to $A^o(U) = U$ with all restriction maps $\rho_{U,V}$ equal to the identity. This is not a sheaf. The constant sheaf $\underline{A}$ is the associated sheaf, which agrees with the sheaf of locally constant functions with values in $A$. When we want to emphasise the topological space on which $A$ lives, we shall write $\underline{A}_M$. We say that a sheaf is constant if it is of the form $\underline{A}$ for some abelian group $A$.

(2) Let $A$ be an abelian group and let $x \in M$ be a point. The skyscraper sheaf $A_x$ is the sheaf given by

$$A_x(U) = \begin{cases} A & x \in U, \\ 0 & x \notin U. \end{cases}$$
Given a sheaf $F$ and an open subset $U \subseteq M$, the elements of $F(U)$ are called \textit{sections} of $F$ over $U$. The group $F(M)$ is also denoted by $\Gamma(M, F)$ and its elements are called \textit{global sections}. The assignment $F \mapsto \Gamma(M, F)$ gives rise to the global section functor $\Gamma : \text{ShAb}(M) \to \text{Ab}$. This functor is not exact, but only left exact: if $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence of sheaves, then the sequence of abelian groups $0 \to \Gamma(M, F_1) \to \Gamma(M, F_2) \to \Gamma(M, F_3)$ is exact, but the rightmost map does not need to be surjective. This observation is the starting point of the definition of sheaf cohomology. In the language of derived categories (see Section A.4), \textit{sheaf cohomology} is the derived functor of the functor of global sections.

**Definition A.101.** Let $M$ be a topological space and $F$ a sheaf on $M$. Then the cohomology groups of $M$ are the derived functors of the global sections functor:

$$H^n(M, F) = R^n\Gamma(M, F).$$

**Example A.102.** It is a classical result that, if $M$ is a locally contractible paracompact topological space, then for any abelian group of coefficients $R$, singular cohomology (Section 2.1) agrees with sheaf cohomology of the constant sheaf $R$, that is:

$$H^*(M, R) = H^*(M, R).$$

The preprint [Sel16] contains a proof of this comparison result without the paracompactness assumption.

The category $\text{ShAb}(M)$ has enough injectives. This means that, for each sheaf $F$, there exists a long exact sequence

$$0 \to F \to I^0 \to I^1 \to I^2 \to \ldots$$

where all the $I^i$ are injective sheaves. Such an exact sequence is called an \textit{injective resolution}. Taking global sections yields a complex

$$\Gamma(M, I^0) \to \Gamma(M, I^1) \to \Gamma(M, I^2) \to \cdots$$

and the sheaf cohomology groups of $F$ are defined as

$$H^n(M, F) = H^n(\Gamma(M, I^*)).$$

The resulting groups do not depend on the choice of the injective resolution.

For theoretical purposes, injective resolutions are very nice, but it is not easy to write them down explicitly and it is useful to have more concrete ways to compute cohomology.

**Definition A.103.** Let $F$ be a sheaf on $X$.

1. We say that $F$ is \textit{flasque} (or \textit{flabby}) if the restriction maps $F(V) \to F(U)$ are surjective for all open subsets $U \subset V \subset X$. 


(2) We say that $F$ is acyclic if $H^i(X, F) = 0$ in all degrees $i > 0$.

An acyclic resolution of a sheaf $F$ is an exact sequence

$$0 \rightarrow F \rightarrow A^0 \rightarrow A^1 \rightarrow A^2 \rightarrow \ldots$$

where all the $A^i$ are acyclic sheaves. The sheaf cohomology of $F$ can be computed using any acyclic resolution, that is

$$H^n(M, F) = H^n(\Gamma(M, A^*)) .$$

**Example A.104.** Every sheaf $F$ has a canonical acyclic resolution called the *Godement resolution*. To introduce it we recall the notion of stalk. If $x$ is a point of $M$, then the stalk of $F$ at $x$ is the direct limit

$$F_x = \lim_{\rightarrow} F(U) ,$$

where $U$ is an open subset of $M$. An element of $F_x$ is a class of pairs $(U, t)$ where $U \subset M$ is an open subset and $t \in F(U)$ is a section, with respect to the equivalence generated by the relations

$$(U, t) \sim (V, t|_V) , \quad \text{for } x \in V \subset U .$$

The Godement resolution is then constructed as follows: one first assigns to each open subset $U$ of $M$ the product

$$C^0(F)(U) = \prod_{x \in U} F_x$$

of the stalks of $F$ at all points in $U$. Together with the obvious restriction maps, one obtains a sheaf $C^0(F)$ on $M$. Moreover, the natural morphism of sheaves $F \rightarrow C^0(F)$ is injective. Then one defines

$$C^1(F) = C^0(C^0(F)/F) .$$

Iterating this process yields the Godement resolution

$$C^*(M, F): \quad C^0(F) \rightarrow C^1(F) \rightarrow C^2(F) \rightarrow \ldots$$

An important property of the Godement resolution is that it gives a functorial way of choosing an acyclic resolution: if $F \rightarrow G$ is a morphism of sheaves, then there is a morphism of complexes $C^*(M, F) \rightarrow C^*(M, G)$ compatible with composition of morphisms.

The Godement resolution satisfies the following property:

**Lemma A.105.** Let $f: X \rightarrow Y$ be a continuous map of topological spaces and let $F$ be a sheaf on $Y$. There is a natural morphism of complexes of sheaves

$$f^{-1}C^*(Y, F) \rightarrow C^*(X, f^{-1}F) .$$

**Proof.** See [Hub95, Proposition 5.2.1]. \qed
A.8.2. Hypercohomology. Let us now consider a complex $F^*$ of sheaves of abelian groups

$$F^*: \ldots \to F^{n-1} \xrightarrow{d} F^n \xrightarrow{d} F^{n+1} \to \ldots.$$  

For the cohomology of $F^*$ we can mean two things. The first is to take the cohomology in the abelian category $\text{ShAb}$, in this case the cohomology objects will be sheaves. The second is to consider the derived functor of the functor of global sections $\Gamma$. In this second case the cohomology objects will be abelian groups. In order to distinguish between these two possibilities the second is classically called the hypercohomology of the complex.

A resolution of the complex $F^*$ is a complex $D^*$ together with a quasi-isomorphism $F^* \to D^*$. If all the sheaves $D^n$ are injective (resp. acyclic), we say that $D^*$ is an injective (resp. acyclic) resolution.

**Definition A.106.** The hypercohomology of $F^*$ is the cohomology of the complex of global sections of any acyclic resolution $D^*$ of $F^*$:

$$\mathbb{H}^n(M, F^*) = H^n(\Gamma(M, D^*)).$$

**Example A.107.** Since Godement’s canonical resolution from Example A.104 is functorial, we can use it to construct a resolution of any bounded below complex of sheaves. Let

$$\ldots \to F^{n-1} \xrightarrow{d} F^n \xrightarrow{d} F^{n+1} \to \ldots$$

be a complex of sheaves such that there exists an $n_0 \in \mathbb{Z}$ such that $F^n = 0$ for all $n \leqslant n_0$. For each $n$, let $C^*(M, F^n)$ be Godement’s canonical resolution. By functoriality, there are commuting maps

$$d^{\text{hor}} : C^m(M, F^n) \to \text{Hom}(C^m, C^{m+1}(M, F^n)).$$

The total complex of $C^*(M, F^*)$ is the complex $\text{Tot}^*(C(M, F))$ with

$$\text{Tot}^n(C(M, F)) = \bigoplus_{p+q=n} C^q(M, F^p)$$

and differential $d$ given, for each $x \in C^q(M, F^p)$, by

$$dx = d^{\text{hor}} x + (-1)^p d^{\text{ver}} x.$$

There is a quasi-isomorphism $F^* \to \text{Tot}^*(C(M, F))$ of complexes of sheaves that makes $\text{Tot}^*(C(M, F))$ into an acyclic resolution of $F^*$. Thus

$$\mathbb{H}^n(M, F^*) = H^n(\Gamma(M, \text{Tot}^*(C(M, F)))).$$

Since the hypercohomology of a sheaf is computed as the cohomology of a dg-complex it comes equipped with the spectral sequence (A.74). The first page of this spectral sequence is

$$E_1^{p,q} = H^q(M, F^p) \Rightarrow \mathbb{H}^{p+q}(M, F^*).$$
A.8.3. **Higher direct images.** Higher direct images can be seen as a relative version of cohomology groups.

**Definition A.109.** Let \( f: M \to N \) be a continuous map between topological spaces and let \( F \) be a sheaf on \( M \). The **direct image** sheaf \( f_* F \) is the sheaf on \( N \) whose sections on an open subset \( U \subset N \) are given by

\[
f_* F(U) = F(f^{-1}(U)).
\]

(A.110)

In fact, the direct image construction (A.110) defines a functor

\[ f_*: \text{ShAb}(M) \to \text{ShAb}(N). \]

The functor \( f_* \) is left exact and the category \( \text{ShAb}(M) \) has enough injectives, so the general constructions of Section A.4 apply to this setting, thus giving rise to a total right derived functor \( Rf_* \) and to cohomological right derived functors \( R^n f_* \) for each \( n \in \mathbb{N} \).

**Example A.111.** The cohomology groups of a sheaf \( F \) on \( M \) are a particular case of higher direct images. Indeed, let \( M \) be a topological space and let \( F \) be a sheaf on \( M \). Let \( * \) denote the topological space consisting of a single point and let \( \pi: M \to * \) be the unique map with target \( * \). Then, we can identify \( \text{ShAb}(*) \) with \( \text{Ab} \) and, under this identification,

\[
f_* F = \Gamma(M, F), \quad R^n f_* F = H^n(M, F).
\]

The stalks of the higher direct image sheaves are related with the cohomology of a small neighbourhood of the fibre.

**Proposition A.112.** Let \( f: X \to Y \) be a continuous map of topological spaces, \( F \) a sheaf on \( X \), and \( y \in Y \) a point. The stalk of the higher direct image sheaf \( R^n f_* F \) at \( y \) is given by

\[
(R^n f_* F)_y = \lim_{y \in U} H^n(f^{-1}(U), F),
\]

where the limit runs over all open sets \( U \) of \( Y \) containing \( y \).

Given a continuous map \( f: X \to Y \), the **Leray spectral sequence** allows us to compute the cohomology of a sheaf \( F \) on \( X \) in terms of the cohomology of its higher direct images sheaves on \( Y \).

**Proposition A.113.** Let \( f: X \to Y \) be a continuous map and let \( F \) be a sheaf on \( X \). There is spectral sequence, called the **Leray spectral sequence**, with second page

\[
E_2^{p,q} = H^p(Y, R^q f_* F) \implies H^{p+q}(X, F).
\]

A.8.4. **Local systems as representations of the fundamental group.** In this section, we discuss a class of sheaves on a topological space called **local systems** and we prove that the category they form is equivalent to the representations of the fundamental group.
Definition A.114. A sheaf \( F \) on a topological space \( M \) is said to be \textit{locally constant} if there exists an open cover \( \{U_i\}_{i \in I} \) of \( M \) such that all the restrictions \( F|_{U_i} \) are constant sheaves as in (1) of Examples A.100. A locally constant sheaf is also called a \textit{local system}.

Theorem A.115. Let \( M \) be a Hausdorff, second countable, connected, locally compact and locally contractible topological space, \( x_0 \in M \) a point and \( V \) a vector space. Then there is an equivalence of categories between the category of local systems \( F \) on \( M \) with fibre \( F_{x_0} = V \) and representations of \( \pi_1(X, x_0) \) on \( V \).

Proof. We first show how to construct a local system from a representation of the fundamental group. For every point \( x \in M \) we choose once and for all a path \( \alpha_x \in \pi_1(M, x_0) \). Thus \( \alpha_x : [0,1] \) is a piecewise smooth path with \( \alpha_x(0) = x_0 \) and \( \alpha_x(1) = x \). Let \( \rho : \pi_1(M, x_0) \to GL(V) \) be a representation. We define a sheaf \( F \) by describing its sections. For every open set \( U \subset M \), let \( F(U) \) be the vector space of all functions \( f : U \to V \) satisfying that, for every pair of points \( x, y \in U \) and path \( \gamma \in \gamma_x \), the relation
\[
[\alpha_y^{-1} \cdot \gamma \cdot \alpha_x]f(x) = f(y)
\]
holds. The fact that \( F \) is a sheaf is left as Exercise A.120. We now show that \( F \) is locally constant. Since \( M \) is assumed to be locally contractible we can cover \( M \) with contractible open subsets. Let \( U \) be one of these subsets. We show that \( F(U) \simeq V \). Choose a point \( x \in U \) and let \( \varphi_x : F(U) \to V \) be the map \( f \mapsto f(x) \). The map \( \varphi_x \) is injective. Indeed, since \( U \) is contractible, it is in particular simply-connected. Therefore, for each \( y \) there is a path \( \gamma \in \gamma_y \), so, if \( \varphi_x(f) = f(x) = 0 \), then
\[
f(y) = [\alpha_y^{-1} \cdot \gamma \cdot \alpha_x]f(x) = 0,
\]
thus \( f = 0 \). We now show that \( \varphi_x \) is surjective. Let \( v \in V \), for every point \( y \) choose a path \( \gamma_y \in \gamma_y \). Since \( U \) is contractible, it is in particular simply-connected. Therefore if \( \gamma'_y \) is another choice, then \( [\gamma_y] = [\gamma'_y] \in \pi(M; y, x) \). Define \( f_v \) by the rule
\[
f_v(y) = [\alpha_y^{-1} \cdot \gamma_y \cdot \alpha_x]v.
\]
By the previous discussion, this is independent of the choice of \( \gamma_y \) and is easily seen to be a section of \( F(U) \). Moreover \( f_v(x) = v \), showing that \( \varphi_x \) is surjective and hence an isomorphism. The same argument can be repeated to show that for each connected open subset \( U' \subset U \) the restriction \( F(U) \to F(U') \) is an isomorphism showing that \( F|_U \) is isomorphic to the constant sheaf.

The next step is to produce a representation of \( \pi_1(M, x_0) \) starting with a locally constant sheaf with fibre \( F_{x_0} = V \). Let \( F \) be such a sheaf. An open subset \( U \subset M \) will be called good (for this sheaf) if \( U \) is connected and \( F|_U \) is isomorphic to the constant sheaf \( V \) on \( U \). Since \( F \) is locally constant, \( M \) can be covered by good open subsets. Let \( \gamma \in \gamma_x \) be a path between
two points. Using that $[0, 1]$ is compact, we can choose a finite set of points $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ and good open sets $U_j$, $j = 0, \ldots, k$ such that $\gamma([t_j, t_{j+1}]) \subset U_j$. For each $j = 0, \ldots, k$ there is an isomorphism

$$\rho_j: F_\gamma(t_j) \to F_\gamma(t_{j+1})$$

given as the composition

$$F_\gamma(t_j) \xrightarrow{\sim} F(U_j) \xrightarrow{\sim} F_\gamma(t_j).$$

We denote by $\rho_\gamma: F_x \to F_y$ the composition

$$\rho_\gamma = \rho_k \circ \cdots \circ \rho_0.$$

The fact that $\rho_\gamma$ is independent of the choices follows from two properties:

1. the isomorphism $\rho_j$ does not depend on the choice of the good open set $U'$ containing $\gamma(t_j)$ and $\gamma(t_{j+1})$;
2. the composition $\rho_\gamma$ does not change if we add a point $t'_j$ with $t_j < t'_j < t_{j+1}$.

The next step is to show that $\rho_\gamma$ only depends on the homotopy class $[\gamma] \in \pi_1(M; y, x)$. Let $\gamma$ and $\gamma'$ be two paths and $H$ a homotopy between them. Then we can find points

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1, \quad 0 = s_0 < s_1 < \cdots < s_{\ell} < s_{\ell+1} = 1$$

and good open sets $U_{i,j}$ such that $H([t_i, t_{i+1}] \times [s_j, s_{j+1}]) \subset U_{i,j}$. Then one checks that the square

$$\begin{array}{ccc}
F_H(t_i, s_j) & \xrightarrow{\sim} & F_H(t_i, s_{j+1}) \\
\downarrow \sim & & \downarrow \sim \\
F_H(t_{i+1}, s_j) & \xrightarrow{\sim} & F_H(t_{i+1}, s_{j+1})
\end{array}$$

is commutative. From this it follows that $\rho_{\gamma'} = \rho_\gamma$. The map $\rho_\gamma$ is called the parallel transport along $\gamma$.

As a consequence of this construction there is a representation

$$\rho: \pi_1(X, x_0) \to \text{GL}(V)$$

given by $[\gamma] \mapsto \rho_\gamma$.

We leave to the reader to check that the constructions we have described are inverses of each other and are functorial.

**Remark A.116.** It is clear from the proof of Theorem A.115 that the hypothesis are an overkill, but are satisfied in the examples we are interested in. For instance, instead of asking the topological space to be locally contractible it is enough to ask it to be locally path-connected and semi-locally simply-connected. The second condition means that every point has a neighborhood such that every loop in the neighborhood is null-homotopic in the whole space.
Remark A.117. The relation between the categories of local systems and that of representations of the fundamental group implies a relation between the cohomology of a local system as a sheaf and group cohomology of its fiber. More precisely, let $M, x_0$ and $V$ as in Theorem A.115. Write $\Gamma = \pi_1(M, x_0)$ for the fundamental group of $M$ let $F$ be a local system on $M$ with fiber $V$. This turns $V$ into a $\Gamma$-module. Then, for all $i \geq 0$,

$$H^i(M, F) = H^i(\Gamma, V),$$

where the left-hand side is sheaf cohomology and the right-hand side is group cohomology.

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Exercise A.118. Let $M$ be a topological space and let $\mathcal{B}$ be a basis of its topology, that is, a collection of open subsets that cover $M$ and such that, for any two basis elements $U_1, U_2 \in \mathcal{B}$ and any point $x \in U_1 \cap U_2$, there exists a basis element $U_3 \in \mathcal{B}$ such that $x \in U_3 \subset U_1 \cap U_2$. Show that an assignement $U \mapsto F(U)$, for each $U \in \mathcal{B}$, satisfying the four conditions in Definition A.97 determines uniquely a sheaf $F$ on $M$.

Exercise A.119. Let $M$ be a topological space and let $\iota: N \hookrightarrow M$ be the inclusion of a closed subspace. Show that, given a sheaf $F$ on $N$, the direct image sheaf $\iota_* F$ from Definition A.109 is the extension by zero, that is, the sheaf on $N$ with stalks $F_x$ if $x \in N$ and 0 otherwise. Then prove that $\iota_*$ is an exact functor, so that the higher direct image sheaves vanish.

Exercise A.120. Let $M$ be a connected locally contractible topological space and let $\rho$ be a representation of the fundamental group. Show that the presheaf $F$ constructed in the proof of Theorem A.115 is a sheaf.

A.9. $t$-structures. There are many natural situations where one is able to construct a triangulated category but would like to obtain an abelian category instead. In their work on perverse sheaves [BBD82], Beilinson, Bernstein and Deligne introduced the notion of $t$-structure as a way of extracting an abelian category from a triangulated category. This is how the abelian category of mixed Tate motives over a number field is constructed in Section 4.3.

Definition A.121 (Beilinson–Bernstein–Deligne). Let $\mathcal{T}$ be a triangulated category. A $t$-structure on $\mathcal{T}$ is a pair of strictly full (that is, full and closed under isomorphism) subcategories

$$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$$

such that, defining for each integer $n$

$$\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n], \quad \mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n],$$

the following three conditions are satisfied:
(1) One has $\mathcal{T}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}$.

(2) (Orthogonality) If $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$, then $\text{Hom}_\mathcal{T}(X, Y) = 0$.

(3) Each object $X$ of $\mathcal{T}$ fits into a distinguished triangle
\[ Y \to X \to Z \to Y[1] \] (A.122)
with $Y \in \mathcal{T}^{\leq 0}$ and $Z \in \mathcal{T}^{\geq 1}$.

We say that the $t$-structure is non-degenerate if, moreover, the intersections $\bigcap_{n \in \mathbb{Z}} \mathcal{T}^{\leq n}$ and $\bigcap_{n \in \mathbb{Z}} \mathcal{T}^{\geq n}$ are reduced to zero.

**Definition A.123.** The heart of a $t$-structure on $\mathcal{T}$ is the full subcategory $\mathcal{T}^0 = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$.

A functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ between triangulated categories equipped with $t$-structures is said to be $t$-exact whenever $F(\mathcal{T}_1^{\leq 0}) \subseteq \mathcal{T}_2^{\leq 0}$ and $F(\mathcal{T}_1^{\geq 0}) \subseteq \mathcal{T}_2^{\geq 0}$. It restricts thus to a functor between the hearts.

Note that the objects $Y$ and $Z$ in the triangle (A.122) are not a priori required to be unique. However, this follows from the other axioms:

**Lemma A.124.**

(1) The inclusion of $\mathcal{T}^{\leq n}$ into $\mathcal{T}$ admits a right adjoint $t_{\leq n} : \mathcal{T} \to \mathcal{T}^{\leq n}$ and the inclusion $\mathcal{T}^{\geq n}$ into $\mathcal{T}$ admits a left adjoint $t_{\geq n} : \mathcal{T} \to \mathcal{T}^{\geq n}$.

(2) For each object $X$ in $\mathcal{T}$, there exists a unique morphism
\[ w \in \text{Hom}_\mathcal{T}(t_{\geq 1}X, t_{\leq 0}X[1]) \]
such that the following is a distinguished triangle:
\[ t_{\leq 0}X \to X \to t_{\geq 1}X \xrightarrow{w} t_{\leq 0}X[1]. \]
Up to unique isomorphism, this triangle is the only one satisfying the condition (3) in Definition A.121.

Moreover, if $a \leq b$, there is a unique isomorphism
\[ t_{\geq a}t_{\leq b}X \xrightarrow{\sim} t_{\leq b}t_{\geq a}X. \] (A.125)

The standard example of $t$-structure is the following:

**Example A.126.** Let $\mathcal{A}$ be an abelian category. Recall from Section A.3 that the bounded below derived category $D^+(\mathcal{A})$ is a triangulated category. It comes together with a canonical $t$-structure which measures how far a complex is from having cohomology concentrated in degree zero. Precisely, for each integer $n$, one considers the full subcategories
\[ \mathcal{T}^{\leq n} = \{ C^\bullet \in D^b(\mathcal{A}) \mid H^m(C^\bullet) = 0 \text{ for all } m > n \}, \]
\[ \mathcal{T}^{\geq n} = \{ C^\bullet \in D^b(\mathcal{A}) \mid H^m(C^\bullet) = 0 \text{ for all } m < n \}. \]
It is easy to check that the pair \((T^{\leq 0}, T^{\leq 0})\) satisfies the axioms (1) to (3) from Definition A.121. Moreover, this \(t\)-structure is non-degenerate.

The functors \(t_{\leq n}\) and \(t_{\geq n}\) are given by the canonical filtrations of Example A.67.

\[
t_{\leq n}C = \tau_{\leq n}C, \quad t_{\geq n}C = \tau_{\geq n}C.
\]

It follows that

\[
t_{\leq 0}t_{\geq 0}(C[n]) = H^n(C^\bullet).
\]

Viewing an object of \(A\) as a complex concentrated in degree zero, one gets an equivalence between \(A\) and the heart of \(D^b(A)\).

For more general triangulated categories, the following theorem makes it possible to extract an abelian category [BBD82, Thm 1.3.6]. Recall that an abelian subcategory \(A\) of a triangulated category \(T\) is said to be admissible whenever short exact sequences in \(A\) are exactly those sequences

\[
0 \to B \xrightarrow{u} C \xrightarrow{v} A \to 0
\]

such that there exists a distinguished triangle

\[
B \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B[1]. \quad (A.127)
\]

Remark A.128. The extension to a distinguished triangle is not unique, unless \(A\) is a full subcategory, that is, \(\text{Hom}_A(X,Y) = \text{Hom}_T(X,Y)\) for all objects \(X,Y \in A\). Indeed, it follows from axiom (T3) in the definition of triangulated categories that, given two extensions as in (A.127), the identity maps \(B \to B\) and \(C \to C\) can be completed to a morphism of triangles

\[
\begin{array}{ccc}
B & \xrightarrow{u} & C \\
\downarrow & & \downarrow \\
A & \xrightarrow{w} & B[1]
\end{array}
\begin{array}{ccc}
B & \xrightarrow{u} & C \\
\downarrow & & \downarrow h \\
A & \xrightarrow{w'} & B[1]
\end{array}
\]

in \(T\). In particular, \(w = w' \circ h\) and uniqueness amounts to proving that \(h\) is the identity. Since \(A\) is a full subcategory, \(h: A \to A\) is a morphism in \(A\) such that \(h \circ v = v\), and the surjectivity of \(v\) implies \(h = \text{Id}_A\).

The following theorem is proved in [BBD82, Thm. 1.3.6]:

Theorem A.129 (Beilinson–Bernstein–Deligne). The heart of a \(t\)-structure on a triangulated category is a full admissible abelian subcategory.

Remark A.130. It is not true, however, that \(T\) is equivalent, as triangulated category, to the derived category of the heart of a \(t\)-structure. Usually, one does not even have a functor \(D^b(T^0) \to T\) (see Exercise A.135).

Definition A.131. Let \(n\) be an integer. The \(n\)-th cohomology of \(X \in T\) with respect to the \(t\)-structure is the following object of the heart:

\[
h^n(X) = t_{\leq 0}t_{\geq 0}(X[n]) \in T^0. \quad (A.132)
\]
This yields cohomological functors $h^\bullet: T \to T^0$, in the sense that it maps distinguished triangles $X \to Y \to Z \to X[1]$ to long exact sequences

$$\cdots \to h^n(X) \to h^n(Y) \to h^n(Z) \to h^{n+1}(X) \to \cdots$$

A.9.1. Extensions. Recall that, in Proposition A.42 we have seen that the extension groups in an abelian category can be interpreted as morphism groups in the derived category. Up to some extent this can be generalized to abelian subcategory of a triangulated category.

Consider a full admissible abelian subcategory $A$ of a triangulated category $T$. The definition of the map $\text{Ext}^n(A, B) \to \text{Hom}_{\mathcal{T}}(A, B[n])$ can be adapted to the triangulated category $T$.

Let $0 \to B \to C \to A \to 0$ be an extension in $A$. By Remark A.128, it extends to a unique distinguished triangle $B \to C \to A \to B[1]$, yielding a map $w: A \to B[1]$. Moreover, the same argument shows that two equivalent extensions give rise to the same $w$. We thus obtain a homomorphism

$$\varphi_1: \text{Ext}^1_A(A, B) \to \text{Hom}_T(A, B[1]).$$

More generally, breaking a degree $n$ extension

$$0 \to B \to C_{n-1} \to \cdots \to C_0 \to A \to 0$$

into several short exact sequences gives a morphism $A \to B[n]$ which only depends on the equivalence class of the extension. For instance, if $n = 2$, one associates to $0 \to B \to C_1 \xrightarrow{a} C_0 \xrightarrow{b} A \to 0$ the short exact sequences

$$0 \to B \to C_1 \to \text{Im}(a) \to 0$$

$$0 \to \text{Ker}(b) \to C_0 \to A \to 0.$$

Setting $D = \text{Im}(a) = \text{Ker}(b)$ and applying $\varphi_1$ to the rows of the above diagram, we get maps $\alpha: D \to B[1]$ and $\beta: A \to D[1]$. Then we form $\alpha[1] \circ \beta: A \to B[2]$.

**Proposition A.133.** Let $A$ be a full admissible abelian subcategory of a triangulated category $T$. Assume that $A$ is stable under extension. Then

$$\varphi_n: \text{Ext}^n_A(A, B) \to \text{Hom}_T(A, B[n]).$$

is an isomorphism for $n = 1$ and an injection for $n = 2$.

**Proof.** See [Lev93, Prop. 1.6].

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Exercise A.134. Show that the distinguished triangle (A.122) in the definition of $t$-structure is uniquely determined by $X$ up to a unique isomorphism. Thus, it makes sense to write $Y = X^{< 0}$ and $Z = X^{> 1}$. Moreover, the assignments $X \mapsto X^{< 0}$ and $X \mapsto X^{> 1}$ determine functors $t_{< 0}$ and $t_{> 0}$.

Exercise A.135 (A $t$-structure such that the derived category of the heart is not equivalent to the original triangulated category). Let $X$ be a connected finite CW-complex and let $\text{Sh}(X)$ be the abelian category of sheaves of $\mathbb{Q}$-vector spaces on $X$. Consider the full subcategory $T \subseteq \text{D}^{b}(\text{Sh}(X))$ consisting of complexes of sheaves $C$ such that all the cohomology sheaves $H^i(C)$ are constant. Then $T$ inherits a structure of triangulated category. We define

$$T^{< 0} = \{C \mid H^i(C) = 0 \text{ for } i > 0\},$$

$$T^{> 0} = \{C \mid H^i(C) = 0 \text{ for } i < 0\}.$$

1. Show that the pair $(T^{< 0}, T^{> 0})$ forms a $t$-structure on $T$, whose heart is equivalent to the category $\text{Vec}_{\mathbb{Q}}$ of finite-dimensional $\mathbb{Q}$-vector spaces.

2. Let $\mathbb{Q}_X$ be the constant sheaf on $X$. Show that

$$\text{Hom}_{T}(\mathbb{Q}_X, \mathbb{Q}_X[2]) = H^2(X, \mathbb{Q}).$$

However, using the fact that $\text{D}^{b}(T^0)$ is equivalent to the category $\text{D}^{b}(\text{Vec}_{\mathbb{Q}})$, we have

$$\text{Hom}_{\text{D}^{b}(T^0)}(\mathbb{Q}_X, \mathbb{Q}_X[2]) = 0.$$

Deduce that, as long as $H^2(X, \mathbb{Q}) \neq 0$, the triangulated category $T$ is not equivalent to the derived category of the heart.

A.10. Algebraic $K$-theory of a ring. We refer the reader to [Wei13] and [Sri96] for more details.

Definition A.136. Let $\mathcal{A}$ be an abelian category. The Grothendieck group $K_0(\mathcal{A})$ is the quotient of the free abelian group on the isomorphism classes $[A]$ of objects of $\mathcal{A}$ by the subgroup generated by $[A] - [B] - [C]$ for all exact sequences $0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$.

Given a ring $R$, let $\text{P}(R)$ be the category of finitely generated projective $R$-modules. We define $K_0(R)$ as the Grothendieck group of $\text{P}(R)$. If $R$ is a Dedekind domain, one can show that

$$K_0(R) = \mathbb{Z} \oplus \text{Cl}(R).$$

For instance, if $R$ is the ring of integers $\mathcal{O}_k$ of a number field $k$, then $K_0(\mathcal{O}_k)$ is $\mathbb{Z} \oplus \text{Cl}(k)$. In particular, $K_0(\mathbb{Z}) = \mathbb{Z}$. 

A.10.1. The infinite general linear group. Let $R$ be a commutative ring with unit. The groups $\text{GL}_n(R)$ of invertible $n$ by $n$ matrices with entries in $R$ form an inductive system with respect to the maps
\[ \varphi_{n,m}: \text{GL}_n(R) \to \text{GL}_m(R) \]
defined, for integers $n \leq m$, by
\[ \varphi_{n,m}(A) = \begin{pmatrix} A & 0 \\ 0 & \text{Id}_{m-n} \end{pmatrix}. \]

**Definition A.137.** The infinite general linear group is the limit
\[ \text{GL}(R) = \lim_{\to} \text{GL}_n(R), \]
while the first $K$-group is the group
\[ K_1(R) = \text{GL}(R)^{\text{ab}} = \text{GL}(R)/[\text{GL}(R), \text{GL}(R)]. \]
The determinant of a matrix provides a group homomorphism
\[ \det: K_1(R) \to R^\times. \]

$K_2$ of a ring was defined by Bass and Milnor. In the case of a field it admits the following elementary description
\[ K_2(F) = F^\times \otimes F^\times / \langle x \otimes (1 - x) \mid x \neq 0, 1 \rangle \]

A.10.2. The classifying space of a group. Let $G$ be an abstract group. We turn $G$ into a topological space by endowing it with the discrete topology. Let $EG$ be any contractible space together with a proper free action of $G$.

**Definition A.138.** The classifying space of $G$ is the quotient
\[ BG = EG/G. \]

Since $EG$ is a fibration over $BG$ with fiber $G$, there is a long exact sequence of homotopy groups
\[ \cdots \to \pi_1(EG) \to \pi_1(BG) \to \pi_0(G) \to \pi_0(EG) \to \pi_0(BG) \to 0. \]
Recalling that $EG$ is contractible, we have $\pi_i(EG) = 0$ for all $i$. Moreover, $\pi_0(G) = G$ and the higher homotopy groups vanish. Therefore,
\[ \pi_n(BG) = \begin{cases} G & n = 1, \\ 0 & \text{otherwise}. \end{cases} \]
Such a space is called a $K(G,1)$-space. One can prove that a $K(G,1)$-space is unique up to homotopy, so the homotopy type of $BG$ does not depend on the choice of the contractible space where $G$ acts freely.

A concrete way to construct the classifying space is to start with
\[ E\cdot G: \quad G \xrightarrow{\cdot} G \times G \xrightarrow{\cdot} G \times G \times G \cdots, \]
which is the simplicial set having:
The geometric realization $E_nG = G^{n+1}$ for all $n \geq 0$,

- faces $\delta_i: E_{n+1}G \to E_nG$ ($i = 0, \ldots, n+1$) given by
  \[ \delta_i(g_0, \ldots, g_{n+1}) = (g_0, \ldots, \widehat{g_i}, \ldots, g_{n+2}) \]

- degeneracies $\sigma_i: E_nG \to E_{n+1}G$ ($i = 0, \ldots, n$) given by
  \[ \sigma_i(g_0, \ldots, g_n) = (g_0, \ldots, g_i, g_i, g_{i+1}, \ldots, g_n) \]

The geometric realization $EG = |E_nG|$ is then a contractible topological space. The group $G$ acts on $E_nG$ by diagonal multiplication and this action is free. The quotient is a simplicial set $B_nG$: \[
A.139 \quad \{e\} \xrightarrow{G} \xrightarrow{G \times G \cdots},
\]
where $e$ stands for the neutral element of $G$. The faces and degeneracies now involve the group operation, namely:

- faces $\delta_i: B_{n+1}G \to B_nG$ ($i = 0, \ldots, n+1$) given by
  \[
  \delta_i(g_1, \ldots, g_{n+1}) = \begin{cases}
  (g_2, \ldots, g_{n+1}) & i = 0 \\
  (g_1, \ldots, g_ig_{i+1}, \ldots, g_{n+1}) & i = 1, \ldots, n \\
  (g_1, \ldots, g_n) & i = n + 1
  \end{cases}
  \]

- degeneracies $\sigma_i: B_nG \to B_{n+1}G$ ($i = 0, \ldots, n$) given by
  \[ \sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, e, g_i+1, \ldots, g_n) \]

The quotient map $E_nG \to B_nG$ is given by
\[
A.140 \quad (g_0, \cdots, g_n) \mapsto (g_0g_1^{-1}, g_1g_2^{-1}, \cdots, g_{n-1}g_n^{-1}).
\]
We left to the reader the verification that (A.140) commutes with all faces and degeneracies.

A.10.3. Quillen's construction. In view of the definitions of $K_1(R)$ and $K_2(R)$ already available, it was understood that the higher $K$-theory of a ring will have something to do with the higher homotopy of some space associated with $\text{GL}(R)$. However, the classifying space was not enough because all its higher homotopy groups vanish. To remedy this, Quillen introduced the \textit{plus construction} which assigns to a connected CW-complex $X$ another connected CW-complex $X^+$ such that

1. $X^+$ is a connected $H$-space, that is, a connected topological space together with a multiplication $X^+ \times X^+ \to X^+$ which satisfies the group axioms up to homotopy,
2. the fundamental group is abelianized: $\pi_1(X^+) = \pi_1(X)^{ab}$,
3. the homology groups do not change: $H_n(X, \mathbb{Z}) \simeq H_n(X^+, \mathbb{Z})$.

Moreover, $X^+$ is an $H$-space. In fact, (a) can be achieved by adding 2-cells to $X$ and then it suffices to add 3-cells to reverse the change of the homology.
Definition A.141 (Quillen, [Qui73]). Let $n \geq 1$ be an integer. The $n$-th $K$-theory group of the ring $R$ is given by

$$K_n(R) = \pi_n(BGL(R)^+).$$

The homology of $BG$ with constant coefficients is the group homology of $G$.

A.11. Deligne–Beilinson cohomology. In this section, we attach to a smooth projective complex variety $X$ a collection of real vector spaces called Deligne–Beilinson cohomology. See [EV88] for more details.

Let $X$ be a smooth projective complex variety and $\Lambda$ a subring of $\mathbb{R}$ (typically $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ itself). For each integer $n \geq 0$, set $\Lambda(n) = (2\pi i)^n \Lambda \subseteq \mathbb{C}$.

We consider the complex of sheaves

$$\Lambda(n)_{\mathcal{D}}: \Lambda(n) \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \cdots \longrightarrow \Omega^{n-1}_X,$$

where $\Lambda(n)$ is viewed as a constant sheaf sitting in degree 0 and $\Omega^p_X$ is placed in degree $p + 1$ for all $p = 0, \ldots, n - 1$.

Definition A.143. The Deligne–Beilinson cohomology of $X$ is the hypercohomology of this complex

$$H^*_\mathcal{D}(X, \Lambda(n)) = \mathbb{H}^*(X, \Lambda(p)_{\mathcal{D}}).$$

Remark A.144. Recall that the Hodge filtration is given by the $bête$ truncation

$$F^n\Omega^*_X: \quad 0 \rightarrow \Omega^n_X \longrightarrow \Omega^{n+1}_X \longrightarrow \cdots$$

From this it follows that the complex $\Lambda(n)_{\mathcal{D}}$ is quasi-isomorphic to

$$\operatorname{cone}(\Lambda(n) \oplus F^n\Omega^*_X \longrightarrow \Omega^*_X)[-1].$$

In view of the remark, the Deligne–Beilinson cohomology fits into a long exact sequence

$$\cdots \rightarrow H^i(X, \mathbb{C}) \longrightarrow H^i_{\mathcal{D}}(X, \Lambda(n)) \longrightarrow H^i(X, \Lambda(n)) \oplus F^n H^i(X, \mathbb{C}) \rightarrow \cdots$$

A.11.1. The Deligne–Beilinson cohomology of $BGL(\mathbb{C})$.

Example A.145. Let $X = \text{Spec}(\mathbb{C})$. Then:

$$H^n_{\mathcal{D}}(X, \mathbb{R}(p)) = \begin{cases} \mathbb{R} & n = p = 0, \\ \mathbb{R}(p-1) & n = 1, p > 0 \end{cases}$$
A.12. The Beilinson regulator.

Definition A.146. Let \( n \geq 1 \) be an integer. The Beilinson regulator on the \( K \)-theory group \( K_{2n-1}(\mathbb{C}) \) is the map 
\[
K_{2n-1}(\mathbb{C}) \rightarrow \mathbb{R}(n-1)
\]
obtained by composition of the Hurewicz morphism 
\[
\text{Hur}: K_{2n-1}(\mathbb{C}) \rightarrow H_{2n-1}(B \text{GL}(\mathbb{C})^\delta, \mathbb{R})
\]
and the evaluation at the Chern character 
\[
\text{ev}^*(\text{ch}): H_{2n-1}(B \text{GL}(\mathbb{C})^\delta, \mathbb{R}) \rightarrow \mathbb{R}(n-1).
\]

The first step in the construction of the Beilinson regulator is to use the Hurewicz morphism, which maps the \( n \)-th homotopy group of any topological space to its \( n \)-th homology group. Since Quillen’s plus construction does not change the homology, in this way we get 
\[
K_n(A) = \pi_n(B \text{GL}(A)^+) \xrightarrow{\text{Hur}} H_n(B \text{GL}(A)^+, \mathbb{Z}) = H_n(B \text{GL}(A), \mathbb{Z}).
\]

A.13. Compatibility of the regulator with the Hodge realization.

Theorem A.147. Let \( k \) be a number field. For each complex embedding \( \sigma: k \rightarrow \mathbb{C} \), the following diagram commutes:
\[
\begin{array}{cccc}
\text{Ext}^1_{\text{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) & \xrightarrow{R_{\mu}} & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \mathbb{Q}(n)) & \\
\downarrow & & \downarrow \sim & \\
\text{Hom}_{\text{DM}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)[1]) & & & \mathbb{C}/(2\pi i)^n \mathbb{Q} \\
\downarrow & & & \downarrow R \\
K_{2n-1}(k)_{\mathbb{Q}} & \xrightarrow{\sigma} & K_{2n-1}(\mathbb{C})_{\mathbb{Q}} &
\end{array}
\]

say very carefully what every map is

\[\text{"sûrement vrai" [DG05, p.9]}\]

[Hub00]

***

Exercise A.148. Given a group \( G \), let \( \mathcal{C} \) be the category consisting of one object with endomorphism group \( G \). Show that the simplicial set \( B_* G \) from (A.139) coincides with the nerve of the category \( \mathcal{C} \), as introduced in Exercise 3.221.
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