Multiple zeta values: from numbers to motives

José Ignacio Burgos Gil and Javier Fresán

With contributions by Ulf Kühn

Contents

Preface

Multiple zeta values (MZVs for short) are real numbers of the form

$$\zeta(s_1, s_2, \ldots, s_\ell) = \sum_{n_1 > n_2 > \cdots > n_\ell \geqslant 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}, \tag{0.1}$$

where all the exponents s_i are integers greater than or equal to 1, and we make the assumption $s_1 \geqslant 2$ to ensure that the series converges. For $\ell = 1$, these are nothing but the values at integers $s \geqslant 2$ of the Riemann zeta function

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}.$$

Euler proved in 1735 that, when s is even, $\zeta(s)$ is a rational multiple of π^s. Thanks to Lindemann's proof of the transcendence of π a century and a half later, it follows that all the numbers $\zeta(2), \zeta(4), \ldots$ are transcendental. For example,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \text{etc.}$$

The values of the Riemann zeta function at odd integers are much more mysterious. Indeed, a folklore conjecture asserts that they are all "new" transcendental numbers:

Transcendence conjecture. *The numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.*

Being algebraically independent over $\mathbb{Q}$ means that, for each integer $k \geqslant 0$, there exists no non-zero polynomial $P \in \mathbb{Q}[x_0, \ldots, x_k]$ such that

$$P(\pi, \zeta(3), \ldots, \zeta(2k+1)) = 0,$$

and in particular that the numbers $\zeta(3), \zeta(5), \ldots$ are transcendental. This conjecture seems completely out of reach: at the time of writing, the best we know is that $\zeta(3)$ is irrational (Apéry, 1978) and that there are infinitely many irrational numbers among the remaining values at odd integers (Ball and Rivoal, 2001). Neither the transcendence of $\zeta(3)$, let alone its algebraic independence with π, nor the irrationality of $\zeta(5)$ have been proved!

The case $\ell = 2$ was also considered by Euler, back in his 1776 article *Meditationes circa singulare serierum genus* ("Meditations about a singular type of series") [**Eul76**]. In an attempt to find a closed formula for $\zeta(3)$, he looked for linear relations with integer coefficients among the numbers π^3, $\pi^2 \log 2$, and $(\log 2)^3$. This led him to the discovery of remarkable identities involving double zeta values, the simplest being $\zeta(3) = \zeta(2, 1)$.

After more than two centuries of oblivion, multiple zeta values were independently rediscovered in the 1990s by Hoffman and Zagier. It was soon realized that these numbers appear in a wealth of different contexts, including Witten's zeta functions, Kontsevich's deformation quantization, Vassiliev knot invariants, and the theory of mixed Tate motives. Most of these topics share a physics flavour and, roughly at the same time, the physicists Broadhurst and Kreimer found that a lot of Feynman amplitudes in quantum field theory can be expressed as linear combinations of multiple zeta values. The next two decades saw extensive work by a host of mathematicians, including Brown, Cartier, Deligne, Drinfeld, Écalle, Goncharov, Hain, Hoffman, Kontsevich, Terasoma, Zagier, and many others. Although major progress was made, fundamental questions remain open and multiple zeta values are still nowadays an active and rapidly moving field of research

The product of two multiple zeta values is a linear combination, with integral coefficients, of multiple zeta values. For instance, the identity

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

was already known to Euler. Said differently, the $\mathbb{Q}$-subvector space $\mathcal{Z} \subseteq \mathbb{R}$ spanned by all multiple zeta values is an algebra. Contrary to the algebra generated by Riemann zeta values, which according to the transcendence conjecture should simply be a polynomial algebra in $\zeta(2), \zeta(3), \zeta(5) \dots$, multiple zeta values satisfy a plethora of relations that endow $\mathcal{Z}$ with a rich combinatorial structure. One can argue that the main goal of the theory is to understand all linear relations among these numbers.

To make this more precise, we attach to each multiple zeta value $\zeta(s_1, \dots, s_\ell)$ the integer $s_1 + \cdots + s_\ell$, which is called the *weight*. Let $\mathcal{Z}_k \subseteq \mathcal{Z}$ be the vector subspace generated by multiple zeta values of weight k, with the convention that $\mathcal{Z}_0 = \mathbb{Q}$ and $\mathcal{Z}_1 = \{0\}$. Based on a mix of numerical evidence and pure thought, Zagier conjectured "after many discussions with Drinfeld, Kontsevich, and Goncharov" that there is a direct sum decomposition

$$\mathcal{Z} = \bigoplus_{k \geqslant 0} \mathcal{Z}_k,$$

and that the dimension of each graded piece is given by a Fibonacci-like sequence

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k. \tag{0.2}$$

Precisely, $(d_k)_{k\geqslant 0}$ is defined by the initial terms $d_0 = d_2 = 1$ and $d_1 = 0$, and the recurrence relation $d_k = d_{k-2} + d_{k-3}$ for all $k \geqslant 3$, so that the generating series is

$$\sum_{k\geqslant 0} d_k t^k = \frac{1}{1 - t^2 - t^3}. \tag{0.3}$$

This would imply that the dimension of $\mathcal{Z}_k$ grows like a constant multiple of r^k, where $r = 1.3247\dots$ is the real root of $x^3 - x - 1$, which is much smaller than the number 2^{k-2} of multi-indices $(s_1, \dots, s_\ell)$ of weight k for which (0.1) converges.

Plan. The goal of these notes is to give a reasonably self-contained proof of the following results towards Zagier's conjecture:

Theorem A (Deligne–Goncharov [**DG05**], Terasoma [**Ter02**]). *The integers d_k are upper bounds for the dimensions of $\mathcal{Z}_k$:*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leqslant d_k.$$

Theorem B (Brown [**Bro12**]). *Each multiple zeta value can be written as a $\mathbb{Q}$-linear combination of multiple zeta values with only 2s and 3s as exponents. That is, the following family generates the $\mathbb{Q}$-vector space $\mathcal{Z}$:*

$$\{\zeta(s_1, \dots, s_\ell) \mid s_i \in \{2, 3\}\}. \tag{0.4}$$

In fact, Hoffman conjectured that (0.4) forms a *basis* of $\mathcal{Z}$. By a simple counting argument, equality (0.2) would follow from this. Theorem B addresses the "algebraic" part of this conjecture, which suffices to deduce Theorem A. It is also worth mentioning that, taking these results for granted, the algebraic independence of the numbers $\pi, \zeta(3), \zeta(5), \dots$ is a consequence of Zagier's conjecture. In a sense, we have "linearized" the transcendence conjecture. On the negative side, let us emphasize that, despite the progress made thus far, we still do not know a single k for which the dimension of $\mathcal{Z}_k$ is bigger than one!

Surprisingly enough, the proofs of these easy-to-state theorems use the machinery of motives. Kontsevich noticed that multiple zeta values of weight k admit a representation as iterated integrals

$$\zeta(s_1,\ldots,s_\ell)=\int_{\Delta^k}\omega_0(t_1)\cdots\omega_0(t_{s_1-1})\,\omega_1(t_{s_1})\,\omega_0(t_{s_1+1})\cdots\omega_1(t_k), \tag{0.5}$$

where $\omega_0(t)=\mathrm{d}t/t$ and $\omega_1(t)=\mathrm{d}t/(1-t)$ are differential forms on $\mathbb{P}^1\setminus\{0,1,\infty\}$, and the integration domain is the simplex

$$\Delta^k=\{(t_1,\ldots,t_k)\in[0,1]^k \mid 1\geqslant t_1\geqslant t_2\geqslant\cdots\geqslant t_k\geqslant 0\}.$$

This integral representation exhibits multiple zeta values as periods of algebraic varieties. In the words of Deligne [**Del13**, p. 3], "whereas the notion of infinite sum is unfamiliar (*étrangère*) to algebraic geometry, the study of integrals of algebraic quantities is one of its sources." Thanks to the identity (0.5), "algebraic geometry, and more precisely the theory of mixed Tate motives, is useful for the study of multiple zeta values"

Usually, the philosophy of motives represents a powerful tool to predict all algebraic relations between periods. However, when it comes to proving them, one is confronted with the problem that even the first step in this program—getting a category of motives with all the desired properties—remains conjectural. In contrast, for mixed Tate motives over a number field, there is an unconditional theory that relies ultimately on Borel's deep results about the K-theory of number fields. This gives good control over the group governing the symmetries of multiple zeta values. Using this group, one can construct a pro-algebraic variety, together with an action of $\mathbb{G}_m$, in such a way that the Hilbert–Poincaré series of its graded algebra of functions $\mathcal{H}$ coincides with (0.3). The *raison d'être* of this construction is the existence of a surjective map $\mathcal{H}\to\mathcal{Z}$ compatible with the weight; we shall refer to elements of $\mathcal{H}$ as "motivic multiple zeta values". The existence of such a map immediately implies Theorem A. To prove Theorem B, one exploits the motivic coaction, a new structure of $\mathcal{H}$, invisible at the level of numbers, that allows one to get relations among motivic multiple zeta values in a systematic way. A variant of the Grothendieck period conjecture asserts that the algebras $\mathcal{H}$ and $\mathcal{Z}$ are isomorphic, from which Zagier's conjecture would follow.

Outline. Let us now give a more detailed description of the contents of each chapter. The word cloud on the next page should also give a quick idea of the main concepts involved.

Chapter 1 lays out what could be called the "minimal theory" of multiple zeta values. We first define them as infinite series and prove that the product of two multiple zeta values is a linear combination of multiple zeta values by decomposing the indexation domain. This so-called *stuffle* product makes $\mathcal{Z}$ into a $\mathbb{Q}$-algebra, conjecturally graded by the weight. We discuss Zagier's conjecture for the dimension of the graded pieces, as well as refinements due to Hoffmann, and Broadhurst and Kreimer. That progress has been made towards these conjectures relies very much on the existence of the integral representation (0.5). We prove that the decomposition of the product of two simplices yields a new algebra structure on $\mathcal{Z}$, the *shuffle* product. Comparing the stuffle and the shuffle product, one gets many relations among multiple zeta values but not all of them. As we explain in the

last section of the chapter, to conjecturally describe the full algebraic structure, one needs to introduce a regularization process that assigns a finite value to the divergent series $\zeta(1, s_2, \ldots, s_\ell)$.

The goal of Chapter 2 is to show that multiple zeta values are periods of algebraic varieties. To begin with, we briefly recall the definition of singular cohomology of a differential manifold and de Rham's theorem, which says that it can be computed using analytic differential forms. Grothendieck's breakthrough was to realize that, if we are dealing with algebraic varieties, then algebraic differential forms suffice; this gives rise to algebraic de Rham cohomology and the period isomorphism. After introducing these concepts, we give a first interpretation of multiple zeta values as periods of the moduli spaces $\overline{M}_{0,n}$ of stable genus zero curves due to Goncharov and Manin. We then move to mixed Hodge structures (a first approximation to the notion of motive), discuss a number of examples, and compute the extension groups of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$. We end the chapter with a discussion of the problem of finding a geometric construction of these extensions, as well as a potential application to irrationality proofs following Brown.

Chapter 3 introduces iterated integrals, a second way to interpret multiple zeta values as periods. We first present the basic definitions and tackle the question of which iterated integrals are homotopy invariant. We then recall the notions of affine group scheme and Hopf and Lie algebras, which will be extensively used in the sequel. We define the pro-unipotent completion of a group and we construct it, under some finiteness assumptions, following work of Quillen. One of the main results of the chapter is Chen's π_1-de Rham theorem, which roughly says that functions on the pro-unipotent completion of the fundamental group of a differential manifold M are given by homotopy invariant iterated integrals. A consequence, due to Hain, is that when M underlies an algebraic variety, this pro-unipotent completion carries a mixed Hodge structure. The general formalism being settled, we specialize everything to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Multiple zeta values are iterated integrals along the straight path from 0 to 1. Since the endpoints do not belong to the space, this forces us to work with tangential base points. The last section contains a detailed analysis of all the structures carried by the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, including Goncharov's coproduct that will be a fundamental tool in the proof of Brown's theorem.

In Chapter 4, we study the category of mixed Tate motives over $\mathbb{Z}$. The first section contains a crash course in the tannakian formalism. We then sketch a construction of Voevodsky's triangulated category of mixed motives over a field k. It is unknown how to extract an abelian category with good properties from it. However, it was observed by Levine that, when k is a number field, Borel's computation of the rational K-theory of k enables one to extract an abelian category of mixed Tate motives over k, which is moreover tannakian. Even for $k = \mathbb{Q}$, this category is too large for the purposes of studying multiple zeta values. To remedy this, one defines the subcategory of mixed Tate motives over $\mathbb{Z}$. We determine the structure of its Tannaka group and show, after Deligne and Goncharov, that it contains a pro-object whose Hodge realization is the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Finally, in Chapter 5 we pull everything together to prove the main results. In the first section, we construct the graded algebra $\mathcal{H}$ of motivic multiple zeta values and a surjective map $\mathcal{H} \to \mathcal{Z}$ compatible with the grading. Using the structure of the Tannaka group of the category of mixed Tate motives over $\mathbb{Z}$, we show that the graded piece $\mathcal{H}_k$ is of dimension d_k. The existence of the above map then implies the upper bound for the dimensions of $\mathcal{Z}_k$ in Theorem A. We then present the proof of Theorem B, following closely Brown's original paper, and a few consequences. Namely, we explain how to deduce the fact that all periods of mixed Tate motives over $\mathbb{Z}$ are polynomials expressions in $1/2\pi i$ and multiple zeta values, as well as the fact that Zagier's conjecture implies the algebraic independence of $\pi, \zeta(3), \zeta(5), \ldots$.

The book is supplemented by an appendix were we give an introduction to some of the notions and techniques from homological algebra (abelian categories, triangulated categories and t-structures, derived functors, filtrations and spectral sequences, sheaf cohomology...) that are used in the main body.

Warning. Before continuing, we should warn the reader that there are two competing conventions for multiple zeta values in the literature, sometimes in the same paper! Other authors, including Brown, define $\zeta(s_1, \ldots, s_\ell)$, for integers $s_i \geqslant 1$ and $s_\ell \geqslant 2$, as the sum

$$\sum_{1 \leqslant n_1 < n_2 < \cdots < n_\ell} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.$$

In fact, one needs to fix conventions for the order of composition of paths, the definition of iterated integrals, and the expression of multiple zeta values as iterated integrals. Things get simpler if they are compatible. We have chosen those conventions for which the monodromy of a local system is a group morphism.

Prerequisites. The difficulty of the exposition increases as the notes progress. In Chapter 1, besides a couple of digressions, the emphasis is mainly on combinatorial aspects and very little background is required. From Chapter 2 on, we assume some familiarity with algebraic varieties, the language of schemes and cohomology of sheaves, at the level of any introductory book, for instance Hartshorne's [**Har77**].

Chapter 3 contains a crash course on algebraic groups and Lie and Hopf algebras, with an emphasis on unipotent groups and nilpotent Lie algebras, which will play an important role in the sequel. Finally, in Chapter 4 we freely use basic notions from category theory and homological algebra, most of which are gathered in the appendix for the convenience of the reader. We have done our best to present all the materials in the most clear and accessible way, but occasionally we were unable to prevent the text from being sketchy. Unfortunately, Borel's theorem about the K-theory of number fields is used as a black box.

Notation and conventions.

- By an *algebraic variety* over some field k, we mean a reduced separated scheme of finite type over k.
- Given a set S and functions $f\colon S \to \mathbb{C}$ and $g\colon S \to \mathbb{R}_{\geqslant 0}$, the notation $f = O(g)$ means that there exists a real number $C \geqslant 0$ such that the inequality $|f(x)| \leqslant Cg(x)$ holds for all $x \in S$.
- The word *positive* means strictly bigger than 0 and the word *negative* strictly smaller than 0, so *non-negative* means bigger than or equal to 0.
- We denote by $\lfloor x \rfloor$ the *floor* of a real number x. That is, $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. Similarly, $\lceil x \rceil$ denotes the *ceiling* of x, that is, the smallest integer greater than or equal to x.
- If R is a ring and S is a set, then $\langle S \rangle_R$ denotes the R module generated by S. If S is an "abstract" set, then $\langle S \rangle_R$ is a free R-module, while if M is an R-module and $S \subset M$, then $\langle S \rangle_R$ means the submodule of M generated by S. For instance, $\langle x, y \rangle_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space of dimension 2. When the ring R is understood, the subindex will usually be omitted.
- If R is a commutative ring and S is a set, then $R[S]$ denotes the commutative associative R-algebra generated by S, and $R\langle S \rangle$ denotes the associative R-algebra generated by S. As before, these may have two different meanings depending on the nature of S.
- If R is a commutative ring and S is a set, then $R[\![S]\!]$ denotes the completion of $R[S]$ with respect to the ideal generated by S, and $R\langle\!\langle S \rangle\!\rangle$ the completion of $R\langle S \rangle$, again with respect to the ideal generated by S.

Acknowledgments. These notes are the outgrowth of a series of lectures given by the first author at the CMI summer school *Periods and Motives: Feynman Amplitudes in the 21st Century*, and by the second author at the MPIM Bonn and the University of Freiburg. The starting point of the first chapter were notes taken by Michael Lalla of a course on multiple zeta values given by Ulf Kühn at the Universität Hamburg. We would like to thank all the participants of these events for their feedback during and after the lectures, their many questions, and their encouragement to pursue this project. Parts of the book were written while the authors were enjoying the hospitality of many institutions, including the ÉNS Paris, the ICMAT Madrid, the MPIM Bonn, the University of Kyoto, the ETH Zürich, the École polytechnique, and the IMPAN at Warsaw.

The first author thanks the Clay Mathematics Institute for making the summer school possible. Moreover, he wants to acknowledge support from the MINECO grants MTM2013-42135-P, MTM2016-79400-P, PID2019-108936GB-C21 and PID2022-142024NB-I00 during the preparation of these notes.

The second author was first introduced to the subject at the master course *Multizêtas et groupe fondamental* given by Francis Brown in Jussieu during the spring of 2012. Later that summer he participated at the Alpbach workshop *Multiple zeta values*, organized by Joseph Ayoub and Sergey Gorchinsky. Finally, he wants to acknowledge support by the SNSF grants 200021-150099 and 200020-162928 and by the grant "Périodes en Géométrie Arithmétique et Motivique" ANR-18-CE40-0017 from Agence National de la Recherche during the preparation of these notes.

We would like to thank the following people who generously reviewed parts of these notes and answered our questions: Giuseppe Ancona, Joseph Ayoub, Olivier Benoist, Spencer Bloch, Francis Brown, Antoine Chambert-Loir, Ishai Dan-Cohen, Frédéric Déglise, Brad Drew, Clément Dupont, Kurusch Ebrahimi-Fard, Benjamin Enriquez, Riccardo Ferrario, Sergey Galkin, Herbert Gangl, Claire Glanois, Ryotaro Harada, Annette Huber, Peter Jossen, Masanobu Kaneko, Shane Kelly, Jean Lannes, Marco Maculan, Nils Matthes, Simon Pepin Lehalleur, Corentin Perret, Jérôme Poineau, Alexander Saad, Masha Vlasenko, Michel Waldschmidt, Jörg Wildeshaus, Don Zagier, and Federico Zerbini.

We warmly thank the referees for their very careful and friendly reading of previous versions of the manuscript.

1. Classical theory of multiple zeta values (by J.I. Burgos Gil, J. Fresán, and U. Kühn)

In this chapter, we introduce multiple zeta values and begin to study their basic properties. These are the real numbers

$$\zeta(s_1,\ldots,s_\ell)=\sum_{n_1>n_2>\cdots>n_\ell\geqslant 1}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_\ell^{s_\ell}}$$

associated with tuples of integers $\boldsymbol{s}=(s_1,\ldots,s_l)$ satisfying $s_i\geqslant 1$ and $s_1\geqslant 2$, so that the series converges. The sum of the exponents $w=s_1+\cdots+s_\ell$ is called the weight and ℓ is referred to as the length. Of great importance is that multiple zeta values cannot only be written as infinite series as above, but also as integrals

$$\zeta(s_1,\ldots,s_\ell)=\int_{1\geqslant t_1\geqslant\cdots\geqslant t_w\geqslant 0}\frac{\mathrm{d}t_1\cdots\mathrm{d}t_w}{t_1\cdots t_{s_1-1}(1-t_{s_1})t_{s_1+1}\cdots(1-t_w)}.$$

These representations give two different ways of writing the product of $\zeta(\boldsymbol{s})$ and $\zeta(\boldsymbol{s}')$ as a linear combination with integral coefficients of multiple zeta values or, in more algebraic terms, of showing that the $\mathbb{Q}$-vector space $\mathcal{Z}\subseteq\mathbb{R}$ generated by multiple zeta values has an algebra structure. From the series representation one obtains the stuffle product, whereas the integral representation gives the shuffle product. Comparing these two products yields many relations among multiple zeta values. Not all of them, however, can be obtained by this method: since a product of multiple zeta values has weight at least 4, Euler's identity $\zeta(3)=\zeta(2,1)$ does not arise in this manner. A way to accommodate this and other relations is to introduce a regularization process that assigns a finite value to the divergent series corresponding to multi-indices with $s_1=1$. There will be, in fact, two kinds of regularizations, modelled on the stuffle and the shuffle product. Conjecturally, all relations among multiple zeta values come from comparing them. A few good references for the material of this chapter are the survey articles by Cartier [**Car02**], Waldschmidt [**Wal12**], and Zudilin [**Zud03**], as well as Chapter 3 of Zhao's book [**Zha16**].

1.1. Riemann zeta values.

The Riemann zeta function is one of the most famous objects in mathematics. One often hears that it encodes all arithmetic properties of prime numbers: our task is to extract them!

DEFINITION 1.1. The *Riemann zeta function* is defined, on the half-plane of complex numbers s with $\mathrm{Re}(s)>1$, by the absolutely convergent series

$$\zeta(s)=\sum_{n\geqslant 1}\frac{1}{n^s}. \tag{1.2}$$

It admits a meromorphic continuation to the whole complex plane with a single pole at $s=1$ (see, for example, [**Tit86**, Chap. II], where no less than seven different methods to obtain this continuation are explained). The Riemann zeta function still keeps many mysteries. The most impenetrable of them is undoubtedly the Riemann hypothesis (the conjecture that all the non-trivial zeros of $\zeta(s)$ lie in the line $\mathrm{Re}(s)=1/2$), which has many far-reaching consequences for the study of the distribution of prime numbers in analytic number theory. The aim of this book is to glimpse at other aspects of this function, namely the question:

which numbers do we get when evaluating $\zeta(s)$ at integers s?

1.1.1. *Even zeta values.* The story in fact began 120 years before Riemann's article [**Rie59**], with Euler's solution to the so-called *Basel problem*, which asked for the computation of the special value

$$\zeta(2) = \sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

For the prehistory of the Riemann zeta function, we refer the reader to Weil's beautiful account [**Wei89**]. In fact, Euler proved much more than this.

THEOREM 1.3 (Euler, 1735). *The values of the zeta function at even positive integers are given by the formula*

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k} \qquad (k \geqslant 1). \tag{1.4}$$

Here B_{2k} are rational numbers, called *Bernoulli numbers* and defined by the power series identity

$$\frac{t}{e^t - 1} = 1 + \sum_{k \geqslant 1} B_k \frac{t^k}{k!}. \tag{1.5}$$

Note that the function

$$f(t) = \frac{t}{e^t - 1} + \frac{1}{2} t = \frac{t(1 + e^t)}{2(e^t - 1)}$$

is even, *i.e.* satisfies $f(t) = f(-t)$. Hence, the Bernoulli numbers satisfy $B_1 = -1/2$ and $B_k = 0$ for all odd integers $k \geqslant 3$. The first few are easily computed:

k	2	4	6	8	10	12
B_k	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$

PROOF OF THEOREM 1.3. The key ingredient is an identity for the cotangent function, also due to Euler (see Exercise 1.17): for $x \in \mathbb{C} \setminus \mathbb{Z}$, the equality

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n \geqslant 1} \frac{2x}{x^2 - n^2} \tag{1.6}$$

holds. For $0 < |x| < 1$, we can expand the quotient inside the summation sign as a geometric series. Since the resulting double series is absolutely convergent, we can then interchange the order of summation to obtain

$$\pi \cot(\pi x) = \frac{1}{x} - 2 \sum_{k \geqslant 1} \zeta(2k) x^{2k-1}. \tag{1.7}$$

Besides, the identities

$$\frac{1}{e^t - 1} = \frac{e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \quad \text{and} \quad -\frac{1}{e^{-t} - 1} = \frac{e^{\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}},$$

along with the definition (1.5) of Bernoulli numbers and the fact that they vanish for odd $k \geqslant 3$, imply the equality

$$\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{2}{t} + 2 \sum_{k \geqslant 1} \frac{B_{2k} t^{2k-1}}{(2k)!}.$$

Therefore, formula (1.6) can be rewritten as

$$\pi\cot(\pi x) = \pi i \frac{e^{\frac{2\pi i x}{2}} + e^{-\frac{2\pi i x}{2}}}{e^{\frac{2\pi i x}{2}} - e^{-\frac{2\pi i x}{2}}} = \frac{1}{x} + \sum_{k\geqslant 1} \frac{(2\pi i)^{2k} B_{2k}}{(2k)!} x^{2k-1}, \tag{1.8}$$

and we conclude by identifying the coefficients in (1.7) and (1.8) term by term. □

REMARK 1.9.

i) Euler's formula (1.4) implies the equality

$$\mathbb{Q}[\zeta(2), \zeta(4), \ldots] = \mathbb{Q}[\pi^2]$$

of subrings of the ring of real numbers.

ii) The Riemann zeta function satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.10}$$

where Γ is the gamma function; see [**Tit86**, (2.1.13)]. Using that $\Gamma(s)$ has a simple pole of residue $(-1)^k/k!$ at all non-positive integers $s=-k$ and this functional equation, we derive from Euler's formula the values of the Riemann zeta function at negative integers:

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1} \qquad (k \geqslant 1).$$

In particular, $\zeta(s)$ vanishes at $s=-2k$ for all $k \geqslant 1$; these are the "trivial zeros". One can also compute the value $\zeta(0) = -1/2$ on noting that $\zeta(s)$ has a simple pole of residue 1 at $s=1$.

1.1.2. *Odd zeta values.* By contrast, despite the many efforts of the mathematical community, nobody has been able to give closed-form expressions for the values of the Riemann zeta function at odd positive integers $s = 3, 5, \ldots$ in terms of previously known numbers like π. This led to the following conjecture:

CONJECTURE 1.11 (Transcendence conjecture). *The numbers*

$$\pi, \zeta(3), \zeta(5), \ldots$$

are algebraically independent over $\mathbb{Q}$. *That is, for each integer* $k \geqslant 0$ *and each non-zero polynomial* $P \in \mathbb{Q}[x_0, \ldots, x_k]$, *one has* $P(\pi, \zeta(3), \ldots, \zeta(2k+1)) \neq 0$.

This conjecture seems completely out of reach of the current techniques in transcendence theory. The transcendence of π was proved by Lindemann in his 1882 paper [**Lin82**]. Combined with Euler's formula (1.4), it implies that the numbers $\zeta(2k)$ are transcendental for all $k \geqslant 1$. But, when it comes to the values at odd integers, we do not even know whether $\zeta(3)$ is transcendental, not to speak of its algebraic independence with π, or if $\zeta(5)$ is irrational. The few known results, at the moment of writing, are summarized below. The Bourbaki seminar [**Fis04**] contains an excellent survey of the developments prior to 2004.

- Apéry proved the irrationality of $\zeta(3)$ in 1978; see [**Apé79**] for a short announcement and [**vdP79**] for a more detailed account. Different proofs by Beukers [**Beu79, Beu87**], Nesterenko [**Nes96**], Sorokin [**Sor98**], and Prévost [**Pré96**], among others, are now available, but none of them seems to generalize in any way to other odd zeta values such as $\zeta(5)$.

- Rivoal [Riv00] and Ball and Rivoal [BR01] proved the inequality

$$\dim_{\mathbb{Q}}\langle 1, \zeta(3), \zeta(5), \ldots, \zeta(n)\rangle \geqslant \frac{1}{3}\log(n)$$

  for all odd integers $n \geqslant 3$; see also the exposition in [Col03]. In particular, infinitely many odd zeta values $\zeta(2k+1)$ are irrational. A proof "by elementary means" of this corollary was recently given by Sprang [Spr18] building on ideas of Zudilin [Zud18]; see also [FSZ19] and [Fis21]

- Zudilin [Zud01] proved that out of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ at least one is irrational.

Brown has suggested in [Bro16] a common geometric framework for these irrationality proofs. The approach is based on the study of periods of the moduli spaces $M_{0,n}$ of curves of genus zero with n marked points (see Section 2.9.3).

Digression 1.12. Despite their "simplicity", special values of the Riemann zeta function are linked to much interesting mathematics. For instance, K-groups and regulators explain why the values at even integers are easier to understand than those at odd integers. The material on the Dedekind zeta function and the class number formula that we mention in what follows is covered, for example, in Neukirch's book [Neu99, Chap. VII, §5].

Let F be a number field and $\mathcal{O}_F$ its ring of integers. The *Dedekind zeta function* of F is defined, on the half-plane $\mathrm{Re}(s) > 1$, by the absolutely convergent series

$$\zeta_F(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s},$$

where $\mathfrak{a}$ runs through all non-zero ideals of $\mathcal{O}_F$ and $N(\mathfrak{a})$ denotes the cardinal of the finite field $\mathcal{O}_F/\mathfrak{a}$. In particular, $\zeta_{\mathbb{Q}}$ agrees with the Riemann zeta function (1.2).

The Dedekind zeta function extends to a meromorphic function on the complex plane, with a simple pole at $s = 1$. Its residue is given by the *class number formula*

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F R_F}{w_F\sqrt{|d_F|}},$$

where r_1 (resp. r_2) denotes the number of real (resp. pairs of conjugate complex) embeddings of F, h_F is the class number, w_F is the number of roots of unity contained in F, and d_F stands for the discriminant.

The remaining term R_F is defined using the *Dirichlet regulator* map

$$\begin{aligned} \rho\colon\ \mathcal{O}_F^\times &\longrightarrow \quad \mathbb{R}^{r_1+r_2} \\ u \ &\longmapsto (\log ||u||_v)_v. \end{aligned}$$

Here v runs over all archimedean places of F, and we write

$$||u||_v = \begin{cases} |\sigma(u)|, & \text{if } v = \sigma \text{ is a real place,} \\ |\sigma(u)|^2, & \text{if } v = \{\sigma, \overline{\sigma}\} \text{ is a complex place.} \end{cases}$$

The product formula $\prod_v ||u||_v = 1$ implies that ρ lands in the hyperplane of points whose coordinates sum to zero. In fact, Dirichlet showed that the image of ρ is a lattice in $\mathbb{R}^{r_1+r_2-1}$, that is, a subgroup of the form $\mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_{r_1+r_2-1}$ for linearly independent vectors $v_1, \ldots, v_{r_1+r_2-1}$ (*Dirichlet's unit theorem*). By definition, the *covolume* of such a lattice is the Lebesgue measure of the set

$$\{x_1v_1 + \cdots + x_{r_1+r_2-1}v_{r_1+r_2-1} \mid x_i \in \mathbb{R},\ 0 \leqslant x_i \leqslant 1\}.$$

The covolume of the lattice $\rho(\mathcal{O}_F^\times)$ is a real number R_F, abusively called Dirichlet regulator as well.

Borel generalized this picture to other values of the Dedekind zeta function. The role of the units $\mathcal{O}_F^\times$ is played by the higher K-groups $K_n(\mathcal{O}_F)$, certain finitely generated abelian groups, whose definition is rather involved, that carry a lot of information about the "hidden" arithmetic of F. Borel computed the rank of these groups and defined, for each $n \geqslant 2$, the *Borel regulator* map

$$\rho_n \colon K_{2n-1}(\mathcal{O}_F) \longrightarrow \mathbb{R}^{d_n}, \quad d_n = \begin{cases} r_1 + r_2, & \text{if } n \text{ is odd,} \\ r_2, & \text{if } n \text{ is even.} \end{cases}$$

Its image is again a lattice, whose covolume is a real number R_n also called Borel regulator. Letting $\zeta_F^*(1-n)$ denote the first non-vanishing coefficient in the Taylor expansion of the Dedekind zeta function at $s = 1-n$, he proved the relation

$$\zeta_F^*(1-n) \sim_{\mathbb{Q}^\times} R_n.$$

(The notation $\sim_{\mathbb{Q}^\times}$ means that the left-hand side and the right-hand side agree up to a non-zero rational number.) The Dedekind zeta function satisfies a functional equation similar to (1.10), from which it follows that $\zeta_F(n)$ is, up to some easy factor involving the discriminant of F and powers of π, a rational multiple of R_n:

$$\zeta_F(n) \sim_{\mathbb{Q}^\times} \frac{\pi^{n(r_1+2r_2-d_n)}}{\sqrt{|d_F|}} R_n.$$

When $F = \mathbb{Q}$ is the field of rational numbers, the K-group $K_{2n-1}(\mathbb{Z})$ has rank 1 if $n \geqslant 3$ is odd, and 0 otherwise (see Section 4.3). Therefore, $R_n = 1$ holds for even n. Thus, $\zeta(n)$ is a rational multiple of π^n for even n, while it involves the "mysterious" Borel regulator for odd n. This result will play a pivotal role in the motivic approach to multiple zeta values. For more details, we refer the reader to the original papers by Borel [**Bor74**] and [**Bor77**], the first author's monograph [**BG02**], or Soulé's short survey [**Sou10**].

1.1.3. *Double zeta values.* In order to investigate possible relations among zeta values, Euler looked at the algebraic structure of these numbers. The product of two Riemann zeta values gives rise to a new kind of interesting sum:

$$\begin{aligned} \zeta(s_1)\cdot\zeta(s_2) &= \left(\sum_{n_1\geqslant 1}\frac{1}{n_1^{s_1}}\right)\cdot\left(\sum_{n_2\geqslant 1}\frac{1}{n_2^{s_2}}\right) \\ &= \sum_{n_1,n_2\geqslant 1}\frac{1}{n_1^{s_1}n_2^{s_2}} \\ &= \sum_{n_1>n_2\geqslant 1}\frac{1}{n_1^{s_1}n_2^{s_2}} + \sum_{n_2>n_1\geqslant 1}\frac{1}{n_2^{s_2}n_1^{s_1}} + \sum_{n=n_1=n_2\geqslant 1}\frac{1}{n^{s_1+s_2}}. \end{aligned} \tag{1.13}$$

The first two terms in the last line are called *double zeta values* and admit the various representations

$$\begin{aligned}
\zeta(s_1, s_2) &= \sum_{n_1 > n_2 \geqslant 1} \frac{1}{n_1^{s_1} n_2^{s_2}} \\
&= \sum_{n \geqslant 2} \frac{1}{n^{s_1}} \left(1 + \frac{1}{2^{s_2}} + \cdots + \frac{1}{(n-1)^{s_2}}\right) \qquad (1.14) \\
&= \sum_{m,n \geqslant 1} \frac{1}{(n+m)^{s_1} n^{s_2}}.
\end{aligned}$$

With this notation, equation (1.13) can be rewritten as

$$\underbrace{\zeta(s_1) \cdot \zeta(s_2)}_{\text{product of zeta values}} = \underbrace{\zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)}_{\text{sum of zeta and double zeta values}}. \qquad (1.15)$$

This identity already appears in Euler's work [**Eul76**, p. 144] under the name of "prima methodus".

Example 1.16. Taking $s_1 = s_2 = 2$, we get $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$, and hence

$$\zeta(2,2) = \frac{\pi^4}{120}$$

by Euler's formula (1.4). Similarly, one finds that the double zeta value $\zeta(2k, 2k)$ is a rational multiple of π^{4k} for all $k \geqslant 1$.

As we have seen, products of two Riemann zeta values are linear combinations of zeta and double zeta values. To handle products of more factors, multiple zeta values of higher length are needed. These new numbers satisfy many linear relations, and one can argue that the main goal of the theory is to fully understand them.

⋆⋆⋆

Exercise 1.17. Prove that the logarithmic derivative of Euler's product expansion for the sine function

$$\frac{\sin \pi z}{\pi z} = \prod_{n \geqslant 1} \left(1 - \frac{x^2}{n^2}\right)$$

yields the identity

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n \geqslant 1} \frac{2x}{x^2 - n^2} \qquad (x \in \mathbb{C} \backslash \mathbb{Z}),$$

and deduce formula (1.7) in the proof of Theorem 1.3.

Exercise 1.18. Prove that the Taylor expansion of the logarithm of the gamma function at $z = 0$ is given by

$$\log \Gamma(1 - z) = \gamma z + \sum_{n \geqslant 2} \zeta(n) \frac{z^n}{n},$$

where γ is the Euler constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right).$$

[Hint: use Weierstrass's factorization formula for the gamma function.]

Exercise 1.19 (Tornheim sums). Given integers $a, b, c \geqslant 0$, the series

$$S(a,b,c) = \sum_{m,n\geqslant 1}\frac{1}{m^a n^b (m+n)^c},$$

is called a *Tornheim sum*, in reference to the article [Tor50].

i) Prove that $S(a,b,c)$ converges if and only if the following inequalities hold:
$$a+c>1, \quad b+c>1, \quad a+b+c>2.$$
ii) Show that the following Pascal triangle-like recurrence holds:
$$S(a,b,c) = S(a-1,b,c+1) + S(a,b-1,c+1).$$
iii) Deduce that $S(a,b,c)$ is a linear combination with integral coefficients of double zeta values. For example, the identity $S(1,1,1) = 2\zeta(2,1)$ holds.
iv) Prove by direct computation the equality
$$S(1,1,1) = \zeta(2,1) + \zeta(3),$$
and deduce Euler's identity $\zeta(3) = \zeta(2,1)$. [Hint: use the equality
$$\frac{1}{mn(m+n)} = \frac{1}{m^2}\left(\frac{1}{n} - \frac{1}{m+n}\right)$$
to transform the sum over n into a telescoping series.]

1.2. Definition of multiple zeta values. It is now time to introduce the main character of this book. We start with some terminology.

1.2.1. *Multi-indices and multiple zeta values.*

Definition 1.20. A *multi-index*

$$\boldsymbol{s} = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$$

is called *positive* if $s_i \geqslant 1$ for all $i = 1, \dots, \ell$, and *admissible* if it is positive and, in addition, satisfies $s_1 \geqslant 2$. The *weight* of $\boldsymbol{s}$ is the sum $s_1 + \cdots + s_\ell$, and ℓ is called its *length* (it is also called *depth* in the literature). By convention, the empty multi-index ($\ell = 0$) will also be considered to be admissible of weight and length both equal to 0. We will call s_i an *entry* of the multi-index s.

Lemma 1.21. *Let $\boldsymbol{s} = (s_1, s_2, \dots, s_\ell)$ be a non-empty admissible multi-index. Then the following series converges:*

$$\zeta(\boldsymbol{s}) = \zeta(s_1, s_2, \dots, s_\ell) = \sum_{n_1 > n_2 > \cdots > n_\ell \geqslant 1}\frac{1}{n_1^{s_1} n_2^{s_2}\cdots n_\ell^{s_\ell}}.$$

Proof. In view of the inequality

$$\zeta(\boldsymbol{s}) \leqslant \zeta(2, \underbrace{1, \dots, 1}_{\ell-1}),$$

it suffices to show that $\zeta(2,1,\ldots,1)$ converges. Using the estimate

$$\sum_{k=1}^{n}\frac{1}{k}\leqslant 1+\log(n),$$

which is obtained by comparison with the integral $\int_1^n \mathrm{d}x/x$, one gets:

$$\begin{aligned}\zeta(2,1,\ldots,1) &= \sum_{n_1>n_2>\cdots>n_\ell\geqslant 1}\frac{1}{n_1^2 n_2\cdots n_\ell}\\ &\leqslant \sum_{n\geqslant 1}\frac{1}{n^2}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^{\ell-1} \qquad (1.22)\\ &\leqslant \sum_{n\geqslant 1}\frac{(1+\log(n))^{\ell-1}}{n^2}.\end{aligned}$$

The last series converges, as can be seen as follows: from the limit

$$\lim_{n\to+\infty}\frac{\log(1+\log(n))}{\log(n)}=0,$$

we deduce that there is an integer n_0 such that the inequality $(1+\log(n))^{\ell-1}<\sqrt{n}$ holds for all $n\geqslant n_0$. The tail of the last series in (1.22) is thus bounded by the convergent series $\sum_{n\geqslant n_0} n^{-3/2}$, so it is itself convergent. □

Definition 1.23. The *multiple zeta value* associated with an admissible multi-index $\boldsymbol{s}=(s_1,\ldots,s_\ell)$ is the real number

$$\zeta(\boldsymbol{s})=\sum_{n_1>n_2>\cdots>n_\ell\geqslant 1}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_\ell^{s_\ell}}.$$

We shall adopt the convention $\zeta(\emptyset)=1$.

Remark 1.24. Abusing notation, we will sometimes write

$$\mathrm{wt}(\zeta(\boldsymbol{s}))=\mathrm{wt}(\boldsymbol{s})=s_1+\cdots+s_\ell, \qquad (1.25)$$

$$\ell(\zeta(\boldsymbol{s}))=\ell(\boldsymbol{s})=\ell. \qquad (1.26)$$

In particular, $\mathrm{wt}(1)=\ell(1)=0$ for the empty multi-index. Strictly speaking, only the weight and the length of the multi-index $\boldsymbol{s}$, as opposed to the multiple zeta value $\zeta(\boldsymbol{s})$, are well defined, since there are equalities $\zeta(\boldsymbol{s})=\zeta(\boldsymbol{s}')$ for different multi-indices. Conjecturally, when such an equality holds $\boldsymbol{s}$ and $\boldsymbol{s}'$ have the same weight, and hence (1.25) makes sense. By contrast, the length is only well defined for multi-indices, as Euler's relation $\zeta(2,1)=\zeta(3)$ already shows that the same value can be represented by multi-indices of different lengths (see Exercise 1.19 or Corollary 1.56 below).

Example 1.27. Let $2^{\{n\}}=(2,\ldots,2)$ denote the admissible multi-index of length n whose entries are all equal to 2. We compute the value of $\zeta(2^{\{n\}})$ using the method of *generating series* and Euler's product expansion for the sine function

$$\frac{\sin\pi x}{\pi x}=\prod_{n\geqslant 1}\left(1-\frac{x^2}{n^2}\right). \qquad (1.28)$$

Plugging the definition of $\zeta(2^{\{n\}})$ into the power series below, we get:

$$\begin{aligned}\sum_{n\geqslant 0}\zeta(2^{\{n\}})(-x^2)^n &= \sum_{n\geqslant 0}\sum_{m_1>\cdots>m_n\geqslant 1}\left(-\frac{x^2}{m_1^2}\right)\cdots\left(-\frac{x^2}{m_n^2}\right)\\ &= \prod_{m\geqslant 1}\left(1-\frac{x^2}{m^2}\right)\\ &= \sum_{n\geqslant 0}(-1)^n\frac{\pi^{2n}}{(2n+1)!}x^{2n}.\end{aligned}$$

The second equality above comes from the elementary observation that, in the power series expansion of the infinite product, the terms of degree $2n$ correspond bijectively to choices of n integers satisfying $m_1 > m_2 > \cdots > m_n \geqslant 1$. The third equality is the combination of (1.28) and the power series expansion of the sine function. Now, identification of the coefficients yields

$$\zeta(2^{\{n\}}) = \frac{\pi^{2n}}{(2n+1)!}. \tag{1.29}$$

Note the particular case $\zeta(2,2) = \pi^4/120$ from Example 1.16.

1.2.2. *The algebra of multiple zeta values.*

Definition 1.30. We will write $\mathcal{Z}$ for the $\mathbb{Q}$-subvector space of $\mathbb{R}$ generated by all multiple zeta values:

$$\mathcal{Z} = \langle 1, \zeta(2), \zeta(3), \zeta(2,1), \zeta(4), \dots\rangle_{\mathbb{Q}}.$$

Given integers $k, \ell \geqslant 0$, we also consider the following subvector spaces of $\mathcal{Z}$:

$$\begin{aligned}\mathcal{Z}_k &= \langle \zeta(\boldsymbol{s}) \mid \mathrm{wt}(\boldsymbol{s}) = k\,\rangle_{\mathbb{Q}},\\ F_\ell\mathcal{Z} &= \langle \zeta(\boldsymbol{s}) \mid l(\boldsymbol{s}) \leqslant \ell\,\rangle_{\mathbb{Q}},\\ F_\ell\mathcal{Z}_k &= \langle \zeta(\boldsymbol{s}) \mid \mathrm{wt}(\boldsymbol{s}) = k,\ \ell(\boldsymbol{s}) \leqslant \ell\,\rangle_{\mathbb{Q}}.\end{aligned}$$

In particular, the equalities $\mathcal{Z}_0 = \mathbb{Q}$ and $\mathcal{Z}_1 = \{0\}$ hold.

Remark 1.31. The subspaces $F_\ell\mathcal{Z}$ define an increasing filtration of $\mathcal{Z}$:

$$\mathbb{Q} = F_0\mathcal{Z} \subseteq F_1\mathcal{Z} \subseteq F_2\mathcal{Z} \subseteq \cdots.$$

There is an obvious inclusion

$$F_\ell\mathcal{Z}_k \subseteq F_\ell\mathcal{Z} \cap \mathcal{Z}_k$$

that is expected to be an equality (Exercise 1.101), but this is not yet known.

The identity (1.15) is the first indication that the $\mathbb{Q}$-vector space $\mathcal{Z}$ has the richer structure of an *algebra*. Recall that this simply means that $\mathcal{Z}$ is equipped with a bilinear multiplication map $\mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$.

Theorem 1.32. *The multiplication of real numbers induces an associative commutative algebra structure on $\mathcal{Z}$ that is compatible with the weight and the length filtration in that there is an inclusion*

$$F_{\ell_1}\mathcal{Z}_{k_1} \cdot F_{\ell_2}\mathcal{Z}_{k_2} \subseteq F_{\ell_1+\ell_2}\mathcal{Z}_{k_1+k_2}$$

for all non-negative integers ℓ_1, ℓ_2, k_1, k_2.

The theorem says, in particular, that every product of multiple zeta values can be written as a linear combination of multiple zeta values, hence the following:

Corollary 1.33. *Every polynomial relation among Riemann zeta values $\zeta(k)$ gives rise to a linear relation among multiple zeta values.*

Thus, finding algebraic relations among zeta values amounts to finding linear relations among multiple zeta values; this is a first interpretation of what we meant by "*linearizing* the transcendence conjecture" in the preface.

1.2.3. *Proof of Theorem 1.32.* The result will directly follow from Lemmas 1.38 and 1.39 below. Before stating them, we need to introduce the notion of *stuffle multiplicities* of multi-indices.

Construction 1.34. Given positive multi-indices

$$\boldsymbol{s} = (s_1, s_2, \ldots, s_\ell), \qquad \boldsymbol{s}' = (s'_1, s'_2, \ldots, s'_{\ell'}),$$

consider the set of all $2 \times \ell''$-matrices, for integers ℓ'' from $\max(\ell, \ell')$ to $\ell + \ell'$, satisfying the following properties:

i) the entries of the first row are the numbers s_i, for $1 \leqslant i \leqslant \ell$, in this order, plus some interlaced zeros;
ii) the entries of the second row are the numbers s'_i, for $1 \leqslant i \leqslant \ell'$, in this order, plus some interlaced zeros;
iii) no column has two zeros.

Each such matrix defines a new positive multi-index $\boldsymbol{s}'' = (s''_1, \ldots, s''_{\ell''})$ by adding the two entries of each column.

An equivalent construction will be given in Exercise 1.46.

Example 1.35. For the multi-indices $\boldsymbol{s} = (2, 1, 1)$ and $\boldsymbol{s}' = (2, 3)$, two possible choices of such a matrix are

$$\begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 3 & 0 \end{pmatrix},$$

from which we get the multi-index $\boldsymbol{s}'' = (2, 2, 4, 1)$, and

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \end{pmatrix},$$

which gives $\boldsymbol{s}'' = (4, 1, 4)$. Observe that the length of the resulting $\boldsymbol{s}''$ varies.

Definition 1.36. Let $\boldsymbol{s}$, $\boldsymbol{s}'$, and $\boldsymbol{s}''$ be positive multi-indices. The *stuffle multiplicity* $\mathrm{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'')$ is the number of times that the multi-index $\boldsymbol{s}''$ appears as an outcome of the previous construction with inputs $\boldsymbol{s}$ and $\boldsymbol{s}'$.

By definition, the stuffle multiplicity is a non-negative integer.

Example 1.37. In the easy case $\boldsymbol{s} = (2)$ and $\boldsymbol{s}' = (2)$, all possible matrices are

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

from which one gets multi-indices $(4),(2,2)$ and $(2,2)$. Hence, in this example the stuffle multiplicity is equal to

$$\operatorname{st}(\boldsymbol{s},\boldsymbol{s}';\boldsymbol{s}'')=\begin{cases}1, & \text{if } \boldsymbol{s}''=(4),\\ 2, & \text{if } \boldsymbol{s}''=(2,2),\\ 0, & \text{otherwise.}\end{cases}$$

From conditions i), ii), and iii) in Construction 1.34, we immediately deduce the following properties of the stuffle multiplicity:

Lemma 1.38. *Let $\boldsymbol{s}$, $\boldsymbol{s}'$, and $\boldsymbol{s}''$ be positive multi-indices satisfying the condition* $\operatorname{st}(\boldsymbol{s},\boldsymbol{s}';\boldsymbol{s}'')>0$. *Then the following holds:*

i) $\operatorname{wt}(\boldsymbol{s}'')=\operatorname{wt}(\boldsymbol{s})+\operatorname{wt}(\boldsymbol{s}')$;
ii) $\ell(\boldsymbol{s}'')\leqslant\ell(\boldsymbol{s})+\ell(\boldsymbol{s}')$;
iii) *if $\boldsymbol{s}$ and $\boldsymbol{s}'$ are admissible, then so is $\boldsymbol{s}''$.*

The main reason to introduce the stuffle multiplicity is the following result, which together with the previous lemma implies Theorem 1.32.

Lemma 1.39. *Let $\boldsymbol{s}=(s_1,s_2,\dots,s_\ell)$ and $\boldsymbol{s}'=(s'_1,s'_2,\dots,s'_{\ell'})$ be admissible multi-indices. The following equality holds:*

$$\zeta(\boldsymbol{s})\cdot\zeta(\boldsymbol{s}')=\sum_{\boldsymbol{s}''}\operatorname{st}(\boldsymbol{s},\boldsymbol{s}';\boldsymbol{s}'')\zeta(\boldsymbol{s}'').$$

Proof. Multiplying the series

$$\zeta(\boldsymbol{s})=\sum_{n_1>\cdots>n_\ell\geqslant1}\frac{1}{n_1^{s_1}\cdots n_\ell^{s_\ell}}\quad\text{and}\quad\zeta(\boldsymbol{s}')=\sum_{m_1>\cdots>m_{\ell'}\geqslant1}\frac{1}{m_1^{s'_1}\cdots m_{\ell'}^{s'_{\ell'}}},$$

one gets the equality

$$\zeta(\boldsymbol{s})\zeta(\boldsymbol{s}')=\sum_{\substack{n_1>\cdots>n_\ell\geqslant1\\ m_1>\cdots>m_{\ell'}\geqslant1}}\frac{1}{n_1^{s_1}\cdots n_\ell^{s_\ell}m_1^{s'_1}\cdots m_{\ell'}^{s'_{\ell'}}}.\tag{1.40}$$

We now decompose the sum (1.40) according to the possible orderings of the terms of the sequence $n_1,\dots n_\ell,m_1,\dots m_{\ell'}$. For instance, if $\ell=\ell'=1$, we distinguish the three cases $n_1>m_1$, $n_1=m_1$ and $n_1<m_1$, which results in the decomposition

$$\sum_{\substack{n_1\geqslant1\\ m_1\geqslant1}}\frac{1}{n_1^{s_1}m_1^{s'_1}}=\sum_{n_1>m_1\geqslant1}\frac{1}{n_1^{s_1}m_1^{s'_1}}+\sum_{n_1\geqslant1}\frac{1}{n_1^{s_1+s'_1}}+\sum_{m_1>n_1\geqslant1}\frac{1}{n_1^{s_1}m_1^{s'_1}},$$

which was already obtained in formula (1.15). By construction, the number of times that a given sum

$$\zeta(\boldsymbol{s}'')=\sum_{k_1>\cdots>k_{\ell''}\geqslant1}\frac{1}{k_1^{s''_1}\cdots k_{\ell''}^{s''_{\ell''}}}$$

appears in this process is precisely the stuffle multiplicity $\operatorname{st}(\boldsymbol{s},\boldsymbol{s}';\boldsymbol{s}'')$. $\square$

EXAMPLE 1.41. Let a, b, c be integers satisfying $a, c \geqslant 2$ and $b \geqslant 1$. We decompose the product $\zeta(a,b)\zeta(c)$ as:

$$\begin{aligned}
\zeta(a,b)\zeta(c) &= \sum_{\substack{n_1>n_2\geqslant 1\\ m\geqslant 1}} \frac{1}{n_1^a n_2^b m^c} \\
&= \sum_{m>n_1>n_2\geqslant 1} \frac{1}{m^c n_1^a n_2^b} + \sum_{m=n_1>n_2\geqslant 1} \frac{1}{n_1^{a+c} n_2^b} + \sum_{n_1>m>n_2\geqslant 1} \frac{1}{n_1^a m^c n_2^b} \\
&\qquad + \sum_{n_1>m=n_2\geqslant 1} \frac{1}{n_1^a n_2^{b+c}} + \sum_{n_1>n_2>m\geqslant 1} \frac{1}{n_1^a n_2^b m^c} \\
&= \zeta(c,a,b) + \zeta(a+c,b) + \zeta(a,c,b) + \zeta(a,b+c) + \zeta(a,b,c).
\end{aligned}$$

More examples will be presented in the next sections.

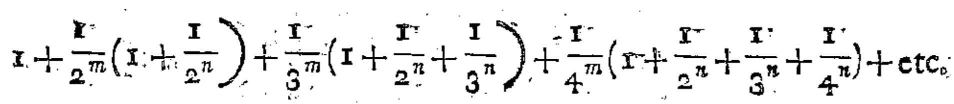
$\star\star\star$

EXERCISE 1.42. Let $k \geqslant 2$ be an integer. Prove that there are 2^{k-1} positive multi-indices of weight k and that 2^{k-2} among them are admissible.

EXERCISE 1.43. It would have been possible, as Euler did in length two (see Figure 1 below), to define multiple zeta values as

$$\zeta^\star(s_1, s_2, \ldots, s_\ell) = \sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_\ell\geqslant 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.$$

Find the relation between $\zeta(s_1, s_2, \ldots, s_\ell)$ and $\zeta^\star(s_1, s_2, \ldots, s_\ell)$.



$$1 + \frac{1}{2^m}\left(1 + \frac{1}{2^n}\right) + \frac{1}{3^m}\left(1 + \frac{1}{2^n} + \frac{1}{3^n}\right) + \frac{1}{4^m}\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n}\right) + \text{etc.}$$

FIGURE 1. Euler's definition of double zeta values in [**Eul76**].

EXERCISE 1.44. Given an integer $s \geqslant 2$, we let $s^{\{n\}} = (s, \ldots, s)$ denote the admissible multi-index of length n with all entries equal to s.

i) Adapt the argument from Example 1.27 to prove the equality

$$\sum_{n\geqslant 0} \zeta(s^{\{n\}}) x^n = \exp\left(\sum_{k\geqslant 1} (-1)^{k-1} \frac{\zeta(sk)}{k} x^k\right).$$

ii) Deduce that the multiple zeta value $\zeta(s^{\{n\}})$ belongs to the subring

$$\mathbb{Q}[\zeta(s), \zeta(2s), \zeta(3s), \ldots] \subset \mathbb{R}.$$

More precisely, consider an infinite collection of weighted variables $(x_k)_{k\geqslant 1}$, where x_k is given weight sk. Then, for each $n \geqslant 1$, there exists a polynomial with rational coefficients $P_n(x_1, \ldots, x_n)$, homogeneous of weight sn, such that the equality

$$\zeta(s^{\{n\}}) = P_n(\zeta(s), \zeta(2s), \ldots, \zeta(ns))$$

holds. Combined with this, Euler's formula (1.4) implies that the multiple zeta value $\zeta(s^{\{n\}})$ is a rational multiple of π^{ns} for even s.

iii) Some explicit formulas:

$$\zeta(4^{\{n\}}) = \frac{(2\pi)^{4n}}{2^{2n-1}(4n+2)!}, \qquad \zeta(6^{\{n\}}) = \frac{6(2\pi)^{6n}}{(6n+3)!},$$
$$\zeta(8^{\{n\}}) = \frac{(2\pi)^{8n}}{2^{2n-2}(8n+4)!}\Big[(\sqrt{2}+1)^{4n+2} + (\sqrt{2}-1)^{4n+2}\Big].$$

Despite its appearance, the last factor is rational since it is invariant under the substitution $\sqrt{2} \mapsto -\sqrt{2}$.

EXERCISE 1.45. Use the stuffle multiplicities to prove the equality

$$\zeta(2k+1)\zeta(2^{\{n-k\}}) = \sum_{i=0}^{n-k} \zeta(2^{\{i\}}, 2k+1, 2^{\{n-k-i\}}) + \sum_{i=0}^{n-k-1} \zeta(2^{\{i\}}, 2k+3, 2^{\{n-k-1-i\}})$$

for all integers $n, k \geqslant 1$.

EXERCISE 1.46. Let $\mathrm{st}(\ell, \ell'; r)$ denote the set of surjective maps

$$\sigma\colon \{1, 2, \ldots, \ell+\ell'\} \longrightarrow \{1, 2, \ldots, \ell+\ell'-r\}$$

satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(\ell)$ and $\sigma(\ell+1) < \cdots < \sigma(\ell+\ell')$.

i) Determine the cardinality of $\mathrm{st}(\ell, \ell'; r)$ and show how to get from σ a matrix satisfying the three conditions in Construction 1.34.

ii) Prove the identity

$$\sum_{\boldsymbol{s}''} \mathrm{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'')\zeta(\boldsymbol{s}'') = \sum_{r=0}^{\min(\ell,\ell')} \sum_{\sigma \in \mathrm{st}(\ell,\ell';r)} \zeta(s''(\sigma)_1, \ldots, s''(\sigma)_{\ell+\ell'-r}),$$

where $\ell = \ell(\boldsymbol{s})$, $\ell' = \ell(\boldsymbol{s}')$ and $s''(\sigma)$ is the multi-index with

$$s''(\sigma)_k = \begin{cases} s_i, & \text{if } \sigma^{-1}(k) = \{i\} \text{ for } i \leqslant \ell, \\ s'_j, & \text{if } \sigma^{-1}(k) = \{\ell+j\}, \\ s_i + s'_j, & \text{if } \sigma^{-1}(k) = \{i, \ell+j\}. \end{cases}$$

(By definition of $\mathrm{st}(\ell, \ell'; r)$, all possibilities for $\sigma^{-1}(k)$ are covered.)

1.3. Relations among double zeta values. We now undertake the task of finding linear relations among multiple zeta values by elementary methods. Historically, one of the first techniques consisted in reordering multiple sums by means of a partial fraction decomposition. In what follows, we show how this yields linear relations among double zeta values.

1.3.1. *Partial fraction decompositions.* For integers a, b with $b \geqslant 0$, we shall use the standard convention for binomial coefficients:

$$\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!}. \tag{1.47}$$

In particular, $\binom{a}{0} = 1$ holds for all a and, if $b > a \geqslant 0$, then $\binom{a}{b} = 0$ holds.

LEMMA 1.48. *Let $i, j \geqslant 1$ be integers. The following equality of rational functions holds:*

$$\frac{1}{x^i y^j} = \sum_{r=1}^{i+j-1} \left[\frac{\binom{r-1}{i-1}}{(x+y)^r y^{i+j-r}} + \frac{\binom{r-1}{j-1}}{(x+y)^r x^{i+j-r}} \right]. \tag{1.49}$$

PROOF. We proceed by induction on i and j. The proof in the case $i = j = 1$ reduces to a simple checking. Assume that (1.49) holds for a given pair (i, j). Differentiating with respect to x, we find that $1/x^{i+1}y^j$ is equal to

$$\frac{1}{i} \sum_{r=1}^{i+j-1} \left[\frac{r\binom{r-1}{i-1}}{(x+y)^{r+1} y^{i+j-r}} + \frac{r\binom{r-1}{j-1}}{(x+y)^{r+1} x^{i+j-r}} + \frac{(i+j-r)\binom{r-1}{j-1}}{(x+y)^r x^{i+j+1-r}} \right] =$$
$$\frac{1}{i} \sum_{r=2}^{i+j} \left[\frac{(r-1)\binom{r-2}{i-1}}{(x+y)^r y^{i+1+j-r}} + \frac{(r-1)\binom{r-2}{j-1}}{(x+y)^r x^{i+1+j-r}} \right] + \frac{1}{i} \sum_{r=1}^{i+j-1} \frac{(i+j-r)\binom{r-1}{j-1}}{(x+y)^r x^{i+j+1-r}}.$$

Thanks to the identities

$$(r-1)\binom{r-2}{i-1} = i\binom{r-1}{i} \quad \text{and} \quad (r-1)\binom{r-2}{j-1} = (r-j)\binom{r-1}{j-1},$$

and taking the convention (1.47) into account, the previous expression becomes

$$\sum_{r=1}^{i+j} \left[\frac{\binom{r-1}{i}}{(x+y)^r y^{i+1+j-r}} + \frac{\binom{r-1}{j-1}}{(x+y)^r x^{i+1+j-r}} \right],$$

which agrees with the right-hand side of (1.49) for $(i+1, j)$. The induction step from (i, j) to $(i, j+1)$ is completely symmetric. □

COROLLARY 1.50. *Let $p, q \geqslant 1$ be integers. For any non-zero complex number a, the following equality of rational functions holds:*

$$\frac{1}{u^p(u-a)^q} = (-1)^q \sum_{k=0}^{p-1} \frac{\binom{q+k-1}{q-1}}{u^{p-k} a^{q+k}} + \sum_{k=0}^{q-1} (-1)^k \frac{\binom{p+k-1}{p-1}}{a^{p+k}(u-a)^{q-k}}. \tag{1.51}$$

PROOF. Take $y = u$ and $x = a - u$ in equation (1.49). To transform the resulting expression into (1.51), observe that the binomial coefficient $\binom{r-1}{q-1}$ vanishes unless $q \leqslant r \leqslant p + q - 1$, and hence the indexes r that actually contribute to the sum can all be written as $r = q + k$ for some $k = 0, \ldots, p-1$. The same holds for the binomial coefficient $\binom{r-1}{p-1}$. □

1.3.2. *Applications.* A straightforward consequence of the partial fraction decomposition from Lemma 1.48 is the *shuffle relation*

$$\zeta(j)\zeta(k-j) = \sum_{r=2}^{k-1} \left[\binom{r-1}{j-1} + \binom{r-1}{k-j-1} \right] \zeta(r, k-r) \tag{1.52}$$

for any $k \geqslant 4$ and $2 \leqslant j \leqslant k-2$. Replacing the product in the left-hand side of (1.52) with the stuffle formula (1.15) we get the linear identity

$$\zeta(j, k-j) + \zeta(k-j, j) + \zeta(k) = \sum_{r=2}^{k-1} \left[\binom{r-1}{j-1} + \binom{r-1}{k-j-1} \right] \zeta(r, k-r), \tag{1.53}$$

which is called a *double shuffle relation*. The reason for these names will become apparent in Section 1.5.

A more sophisticated application of partial fraction decompositions gives the following result, essentially what Euler calls "tertia methodus" in [**Eul76**]. We refer the reader to [**Har18**] for a modern exposition of his techniques.

THEOREM 1.54 (Euler, 1776). *Given integers $p \geqslant 2$ and $q \geqslant 1$, the following equality holds:*

$$\begin{aligned}\zeta(p,q) = \sum_{k=0}^{q-2}(-1)^k\tbinom{p+k-1}{p-1}\zeta(q-k)\zeta(p+k) \\ + (-1)^q\sum_{k=0}^{p-2}\tbinom{q+k-1}{q-1}\zeta(p-k,q+k) \\ + (-1)^{q-1}\tbinom{p+q-2}{p-1}\big[\zeta(p+q)+\zeta(p+q-1,1)\big].\end{aligned}$$

REMARK 1.55. The assumptions $p \geqslant 2$ and $q \geqslant 1$ ensure that all the terms in the above formula are convergent series. Euler also allowed the case $p = 1$. Then the sum contains divergent terms such as $\zeta(1)$ or $\zeta(1,1)$ that need to be regularized; see [**Har18**] for a rigorous treatment of Euler's method.

Making $q = 1$ we immediately get:

COROLLARY 1.56 (Euler's sum formula). *If $s \geqslant 3$, then*

$$\zeta(s) = \sum_{j=1}^{s-2}\zeta(s-j,j). \tag{1.57}$$

In particular, the equality $\zeta(3) = \zeta(2,1)$ holds.

PROOF OF THEOREM 1.54. We borrow the argument from Nielsen's book; see [**Nie65**, Chap. III, § 18, p. 48]. Recall the equality

$$\zeta(p,q) = \sum_{n>m\geqslant 1}\frac{1}{n^p m^q} = \sum_{n\geqslant 2}\left(\sum_{a=1}^{n-1}\frac{1}{n^p(n-a)^q}\right) \tag{1.58}$$

from (1.14). Applying the partial fraction decomposition from Corollary 1.50 to each summand in the right-hand side of this equality and separating the terms coming from $k = p-1$ and $k = q-1$ yields

$$\begin{aligned}\sum_{a=1}^{n-1}\frac{1}{n^p(n-a)^q} =& (-1)^q\sum_{k=0}^{p-2}\sum_{a=1}^{n-1}\frac{\binom{q+k-1}{q-1}}{n^{p-k}a^{q+k}} \\ &+ \sum_{k=0}^{q-2}\sum_{a=1}^{n-1}(-1)^k\frac{\binom{p+k-1}{p-1}}{a^{p+k}(n-a)^{q-k}} \\ &+ (-1)^q\tbinom{q+p-2}{p-1}\sum_{a=1}^{n-1}\left[\frac{1}{na^{p+q-1}} - \frac{1}{a^{p+q-1}(n-a)}\right].\end{aligned}$$

The sum over n of the first two terms in the above expression converges, whereas the sum of each individual summand of the third term diverges. We will show below that the sum over n of the third term is also convergent.

Using identity (1.58), the sum over n of the first term can be written as

$$\sum_{n\geqslant 2}(-1)^q\sum_{k=0}^{p-2}\sum_{a=1}^{n-1}\frac{\binom{q+k-1}{q-1}}{n^{p-k}a^{q+k}} = (-1)^q\sum_{k=0}^{p-2}\tbinom{q+k-1}{q-1}\zeta(p-k,q+k).$$

We next observe the equality

$$\zeta(p)\zeta(q) = \sum_{n\geqslant 2}\sum_{a=1}^{n-1}\frac{1}{(n-a)^p a^q},$$

which implies that the sum over n of the second term is equal to

$$\sum_{n\geqslant 2}\sum_{k=0}^{q-2}\sum_{a=1}^{n-1}\frac{(-1)^k\binom{p+k-1}{p-1}}{a^{p+k}(n-a)^{q-k}} = \sum_{k=0}^{q-2}(-1)^k\binom{p+k-1}{p-1}\zeta(q-k)\zeta(p+k).$$

For the last term, we use the identity

$$\sum_{a=1}^{n-1}\frac{1}{a^{p+q-1}(n-a)} =$$
$$\sum_{a=1}^{a<n/2}\frac{1}{(n-a)a^{p+q-1}} + \sum_{a>n/2}^{n-1}\frac{1}{a^{p+q-1}(n-a)} + \begin{cases}\frac{1}{\left(\frac{n}{2}\right)^{q+p}}, & \text{if } n \text{ is even,}\\ 0, & \text{if } n \text{ is odd.}\end{cases}$$

We note the equalities

$$\sum_{\substack{n\geqslant 2\\ n \text{ even}}}\frac{1}{\left(\frac{n}{2}\right)^{q+p}} = \zeta(p+q) \quad\text{and}\quad \sum_{n\geqslant 2}\sum_{a>\frac{n}{2}}^{n-1}\frac{1}{a^{p+q-1}(n-a)} = \zeta(p+q-1,1).$$

We finally estimate the remaining term. For $N \geqslant 3$, one has:

$$\sum_{n=2}^{N}\left(\sum_{a=1}^{n-1}\frac{1}{n(n-a)^{p+q-1}} - \sum_{a=1}^{a<\frac{n}{2}}\frac{1}{(n-a)a^{p+q-1}}\right) = \sum_{n>\frac{N+1}{2}}^{N}\sum_{a=N-n+1}^{n-1}\frac{1}{na^{p+q-1}}.$$

Using the assumption $p+q-1 \geqslant 2$, one sees that the last term converges to zero as N goes to ∞. The theorem results from summing up all the computations. $\square$

Corollary 1.59 (Nielsen). *For each $n \geqslant 2$, the following equalities hold:*

$$\sum_{r=1}^{n-1}\zeta(2r, 2n-2r) = \frac{3}{4}\zeta(2n),$$
$$\sum_{r=1}^{n-1}\zeta(2r+1, 2n-2r-1) = \frac{1}{4}\zeta(2n).$$

Proof. Following [**Nie65**, Chap. III, § 19, p. 49], we use the identity

$$\sum_{r=2}^{p-1}\zeta(r)\zeta(p-r+1) = p\zeta(p+1) - 2\zeta(p,1), \tag{1.60}$$

which results from the decomposition (1.15) of the product of two zeta values and Euler's sum formula (1.57). This will be combined with the equality

$$(2n-2)\left[\zeta(2n)+\zeta(2n-1,1)\right] = \sum_{k=0}^{2n-4}(-1)^k(k+1)\zeta(k+2)\zeta(2n-k-2),$$

which is obtained by applying Theorem 1.54 to $p = 2$ and $q = 2n-2$. Note that the term $\zeta(k+2)\zeta(2n-k-2)$ on the right-hand side is invariant under the substitution $k \mapsto 2n-k-4$ and that it appears with multiplicity $(-1)^k(2n-2)$. Therefore,

$$\begin{aligned} 2\left[\zeta(2n) + \zeta(2n-1,1)\right] &= \sum_{k=0}^{2n-4} (-1)^k \zeta(k+2)\zeta(2n-k-2) \\ &= \sum_{r=1}^{n-1} \zeta(2r)\zeta(2n-2r) - \sum_{r=1}^{n-2} \zeta(2r+1)\zeta(2n-2r-1). \end{aligned} \tag{1.61}$$

Summing and subtracting equations (1.61) and (1.60) for $p = 2n-1$ yields the recursion formulas

$$\sum_{r=1}^{n-1} \zeta(2r)\zeta(2n-2r) = \frac{2n+1}{2}\zeta(2n) \qquad (n \geqslant 2),$$
$$\sum_{r=1}^{n-2} \zeta(2r+1)\zeta(2n-2r-1) = \frac{2n-3}{2}\zeta(2n) - 2\zeta(2n-1,1) \qquad (n \geqslant 3).$$

The statement is proved by replacing the products of zeta values in the left-hand sides with their expression (1.15) as sums of double zeta values. □

Remark 1.62. The previous corollary was rediscovered by Gangl, Kaneko, and Zagier; see [**GKZ06**, Thm. 1] and Exercise 1.66.

1.3.3. *Relations in low weight.* We now show how to use the above results to get linear relations among multiple zeta values of low weight.

Corollary 1.63. *The following relations hold in $\mathcal{Z}$:*

i) *in weight 3:*
$$\zeta(3) = \zeta(2,1);$$
ii) *in weight 4:*
$$\begin{aligned} \zeta(4) &= 4\zeta(3,1), \\ \zeta(2,2) &= 3\zeta(3,1); \end{aligned}$$
iii) *in weight 5:*
$$\begin{aligned} \zeta(5) &= -4\zeta(4,1) + 2\zeta(2,3), \\ \zeta(3,2) &= -5\zeta(4,1) + \zeta(2,3); \end{aligned}$$
iv) *in weight 6:*
$$\begin{aligned} \zeta(6) &= 4\zeta(5,1) + 4\zeta(3,3), \\ \zeta(2,4) &= \frac{13}{3}\zeta(5,1) + \frac{7}{3}\zeta(3,3), \\ \zeta(4,2) &= -\frac{4}{3}\zeta(5,1) + \frac{2}{3}\zeta(3,3). \end{aligned}$$

Proof. All the relations follow from Theorem 1.54 together with the decomposition (1.15). We have already seen that the equality $\zeta(3) = \zeta(2,1)$ is the first instance of Euler's sum formula (1.57).

Let us now derive the two relations in weight 4. On the one hand, Theorem 1.54 applied to $p = q = 2$ gives $\zeta(2)^2 = 2\zeta(4) + 2\zeta(3,1)$. Combining this with the identity $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$ from Example 1.16, we obtain

$$\zeta(4) + 2\zeta(3,1) = 2\zeta(2,2).$$

On the other hand, by Euler's sum formula, $\zeta(4) = \zeta(3,1) + \zeta(2,2)$, hence the equalities $\zeta(4) = 4\zeta(3,1)$ and $\zeta(2,2) = 3\zeta(3,1)$.

The remaining identities are left as an exercise. □

1.3.4. *An upper bound for the dimension of $F_2\mathcal{Z}_k$.* Putting all the identities of this section together, one gets upper bounds for the dimension of the $\mathbb{Q}$-vector space generated by zeta and double zeta values of a given weight. However, as we will see in the next section, these bounds are not expected to be optimal in general (see Remark 1.97).

PROPOSITION 1.64. *For each $k \geqslant 4$, the $\mathbb{Q}$-vector space $F_2\mathcal{Z}_k$ spanned by zeta and double zeta values of weight k is of dimension*

$$\dim_{\mathbb{Q}} F_2\mathcal{Z}_k \leqslant \left\lceil \frac{k-2}{2} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

PROOF. The space $F_2\mathcal{Z}_k$ is generated by the $k-1$ elements $\zeta(k)$ and $\zeta(j, k-j)$ for $j = 2, \ldots, k-1$. Recall from Corollary 1.56 that they satisfy Euler's sum formula

$$\zeta(2, k-2) + \cdots + \zeta(k-1, 1) - \zeta(k) = 0,$$

as well as the double shuffle relations (1.53)

$$\begin{aligned}\zeta(j, k-j) + {} & \zeta(k-j, j) + \zeta(k) \\ & = \sum_{r=2}^{k-1} \left[\binom{r-1}{j-1} + \binom{r-1}{k-j-1} \right] \zeta(r, k-r) \qquad (j = 2, \ldots, k-2).\end{aligned}$$

Since the latter are invariant under the substitution $j \mapsto k-j$, it suffices to consider the equations for $j \leqslant k-j$, that is, $j \leqslant \lfloor \frac{k}{2} \rfloor$.

One gets one equation from Euler's sum formula, and $\lfloor \frac{k}{2} \rfloor - 1$ equations from the double shuffle relations. We claim that these $\lfloor \frac{k}{2} \rfloor$ equations are linearly independent. As $k-1-\lfloor \frac{k}{2} \rfloor = \lceil \frac{k-2}{2} \rceil$, this implies the statement. Indeed, by the convention (1.47), the double shuffle relations take the form

$$\sum_{r=j+1}^{k-1} a_r \zeta(r, k-r) - \zeta(k) = 0 \qquad (j = 2, \ldots, k-2),$$

for positive integers a_r. The matrix of relations is thus upper triangular with non-zero entries in the diagonal, and hence invertible. □

⋆ ⋆ ⋆

EXERCISE 1.65. Derive the remaining relations of Corollary 1.63.

EXERCISE 1.66 (Gangl–Kaneko–Zagier). Define the generating function of double zeta values of weight k as the formal power series

$$T_k(X,Y) = \sum_{\substack{r+s=k\\ r,s\geqslant 1}} \zeta(r,s) X^{r-1} Y^{s-1}\,.$$

i) Use the double shuffle relation (1.53) to prove the functional equation
$$\begin{aligned} T_k(X+Y,Y) &+ T_k(X+Y,X) \\ &= T_k(X,Y) + T_k(Y,X) + \zeta(k)\frac{X^{k-1}-Y^{k-1}}{X-Y} \end{aligned}$$
for all integers $k \geqslant 3$.

ii) Give an alternative proof of Corollary 1.59 using the above functional equation for $(X,Y) = (1,0)$ and $(1,-1)$.

1.4. The Zagier and the Broadhurst–Kreimer conjectures. As we saw in the previous section, there are many linear relations among multiple zeta values. To get an intuition of what the structure of the algebra $\mathcal{Z}$ might be, one can start by performing numerical experiments.

1.4.1. *Numerical experiments.* The first step is to use clever techniques to accelerate the convergence of the infinite series defining multiple zeta values. With these techniques, one can compute them with very high precision (for instance, 800 significant digits) in a reasonable amount of time; see [**Bro96**, §4] for a description of such techniques, as well as [**BBV10**] for the state of the art some years ago. Then we can apply lattice algorithms such as the LLL algorithm or the PSLQ algorithm to find linear relations with integer coefficients among the computed multiple zeta values. At a given precision, we will find many spurious relations (as we are only working with rational approximations), but we can easily tell true relations from spurious ones. The true relations should have small coefficients compared to the inverse of the precision that was used. Moreover, the true relations will survive after doubling the precision, say from 100 to 200 significant digits.

After extensive experimentation by many mathematicians, no non-trivial linear relations among multiple zeta values of different weight have been found: all known relations are homogeneous. Moreover, we can write a table with the "experimental" dimension of each vector space $\mathcal{Z}_k$. Below, k is the weight, d_k^{exp} is the apparent dimension of $\mathcal{Z}_k$ given by the experiments, and 2^{k-2} is the number of admissible multi-indices of weight k (Exercise 1.42), that is, the dimension $\mathcal{Z}_k$ would have had if there were no $\mathbb{Q}$-linear relations at all.

k	2	3	4	5	6	7	8	9	10	11	12	13
2^{k-2}	1	2	4	8	16	32	64	128	256	512	1024	2048
d_k^{exp}	1	1	1	2	2	3	4	5	7	9	12	16

TABLE 1.1. Experimental dimension

Of course, the experiments are not conclusive. There may exist linear relations with "big" coefficients that we have not yet found; then the dimension of $\mathcal{Z}_k$ would be smaller than d_k^{exp}. In fact, there is not even a single value of k for which the dimension of $\mathcal{Z}_k$ is known to be strictly bigger than 1.

Many of the relations obtained experimentally can be proved theoretically. For instance, Euler's sum formula (1.57) gives

$$\zeta(3) = \zeta(2,1),$$

the expected relation in weight 3. In weight 4, there are four admissible multi-indices but $d_4^{\mathrm{exp}} = 1$; we thus need to find three independent relations. Indeed, from Corollary 1.63 and Example 1.140 below, we get

$$\zeta(3,1) = \frac{1}{4}\zeta(4), \quad \zeta(2,2) = \frac{3}{4}\zeta(4), \quad \zeta(2,1,1) = \zeta(4).$$

In weight 5, we expect six relations. In fact, by Corollary 1.63 and Exercise 1.147 below, there are linear relations

$$\begin{aligned} \zeta(5) &= \frac{4}{5}\zeta(3,2) + \frac{6}{5}\zeta(2,3), & \zeta(4,1) &= -\frac{1}{5}\zeta(3,2) + \frac{1}{5}\zeta(2,3), \\ \zeta(5) &= \zeta(2,1,1,1), & \zeta(4,1) &= \zeta(3,1,1), \\ \zeta(2,1,2) &= \zeta(2,3), & \zeta(2,2,1) &= \zeta(3,2). \end{aligned} \tag{1.67}$$

However, given the lack of a theoretical proof, it is conceivable that experimental relations survive up to the number of significant digits that we have used but fail with higher precision.

1.4.2. *Does the weight define a grading?* The fact that all known relations among multiple zeta values are homogeneous led to the following:

Conjecture 1.68. *The subspaces $\mathcal{Z}_k \subseteq \mathcal{Z}$ are in direct sum:*

$$\mathcal{Z} = \bigoplus_{k \geqslant 0} \mathcal{Z}_k.$$

As we already know the inclusion $\mathcal{Z}_{k_1} \cdot \mathcal{Z}_{k_2} \subseteq \mathcal{Z}_{k_1+k_2}$ from Theorem 1.32, this conjecture will be reformulated below as the statement that the weight defines a *grading* on the $\mathbb{Q}$-algebra $\mathcal{Z}$.

Remark 1.69. Assuming Conjecture 1.68, we immediately deduce that all multiple zeta values of positive weight are transcendental numbers. Indeed, let $\boldsymbol{s}$ be an admissible multi-index of weight $w > 0$. If $\zeta(\boldsymbol{s})$ were algebraic, it would satisfy a polynomial equation of the form $\sum_{k=0}^{d} a_k \zeta(\boldsymbol{s})^k = 0$, where the a_k are rational numbers. But then one would have

$$a_d \zeta(\boldsymbol{s})^d \in \mathcal{Z}_{wd} \cap \bigoplus_{d' < d} \mathcal{Z}_{wd'},$$

and hence $a_d = 0$ since subspaces of different weights intersect only at 0.

1.4.3. *Zagier's conjecture.* The conjectural dimension of the $\mathbb{Q}$-vector spaces $\mathcal{Z}_k$ is given by a Fibonacci-like sequence of integers, namely the sequence $\{d_k\}_{k \geqslant 0}$ recursively defined by the conditions

$$\begin{gathered} d_0 = 1, \ d_1 = 0, \ d_2 = 1, \\ d_k = d_{k-2} + d_{k-3}. \end{gathered} \tag{1.70}$$

These numbers fit together into the generating series

$$\sum_{k \geqslant 0} d_k t^k = \frac{1}{1 - t^2 - t^3}.$$

Indeed, $1-t^2-t^3$ is invertible in $\mathbb{Q}[\![t]\!]$, and the coefficients a_k of the inverse power series must satisfy the relations (1.70) for the equality $(1-t^2-t^3)\sum a_k t^k = 1$ to hold; they are thus equal to d_k.

There is an overwhelming amount of numerical evidence for the following conjecture, stated by Zagier in [**Zag94**, p. 509] "after many discussions with Drinfel'd, Kontsevich and Goncharov".

Conjecture 1.71 (Zagier). *The equality* $\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$ *holds.*

Hoffman [**Hof97**, Conj. C, p. 493] proposed a refinement of Zagier's conjecture, in which not only the dimension of $\mathcal{Z}_k$ but also a particular $\mathbb{Q}$-basis is postulated. This conjecture is based on the observation that the equality

$$d_k = |\{\text{multi-indices of weight } k \text{ with entries 2 and 3}\}| \tag{1.72}$$

holds because the numbers on the right-hand side also verify conditions (1.70).

Conjecture 1.73 (Hoffman). *For each integer $k \geqslant 2$, the multiple zeta values $\zeta(s_1, \ldots, s_\ell)$ of weight k with $s_i \in \{2,3\}$ form a $\mathbb{Q}$-basis of $\mathcal{Z}_k$.*

This would imply the following representations of the spaces $\mathcal{Z}_k$:

$$\begin{aligned}
\mathcal{Z}_2 &= \mathbb{Q}\zeta(2),\\
\mathcal{Z}_3 &= \mathbb{Q}\zeta(3),\\
\mathcal{Z}_4 &= \mathbb{Q}\zeta(2,2),\\
\mathcal{Z}_5 &= \mathbb{Q}\zeta(2,3) \oplus \mathbb{Q}\zeta(3,2),\\
\mathcal{Z}_6 &= \mathbb{Q}\zeta(2,2,2) \oplus \mathbb{Q}\zeta(3,3),\\
\mathcal{Z}_7 &= \mathbb{Q}\zeta(2,2,3) \oplus \mathbb{Q}\zeta(2,3,2) \oplus \mathbb{Q}\zeta(3,2,2).
\end{aligned}$$

Remark 1.74.

i) By the relations (1.67), we see that $\mathcal{Z}_5$ is generated by $\zeta(2,3)$ and $\zeta(3,2)$. Thus, the first step towards the Zagier and the Hoffman conjectures would be to prove that these numbers are $\mathbb{Q}$-linearly independent.

ii) Having the right number of elements does not mean finding a basis. For instance, one could have thought that the elements

$$\zeta(2n_1+1, \ldots, 2n_r+1)\zeta(2)^k,$$

for $r \geqslant 0$, $k \geqslant 0$, and $n_i \geqslant 1$, form a basis of $\mathcal{Z}$, since their number in a given weight agrees with the conjectural dimension (Exercise 1.103). But Gangl, Kaneko, and Zagier [**GKZ06**, p. 74] discovered the relation

$$28\zeta(9,3) + 150\zeta(7,5) + 168\zeta(5,7) = \frac{5197}{691}\zeta(12),$$

which disproves such an expectation (note that $\zeta(12)$ is a rational multiple of $\zeta(2)^6$ thanks to Euler's formula (1.3)).

1.4.4. *Algebra generators of multiple zeta values.* In the remainder of this section, the word $\mathbb{Q}$-algebra (without any further qualifier) is tacitly understood to mean an associative commutative algebra with unit, *i.e.* a $\mathbb{Q}$-vector space A endowed with a bilinear multiplication $A \times A \to A$ that is associative, commutative, and has a neutral element 1. A morphism of $\mathbb{Q}$-algebras is a linear map $f\colon A \to B$ that sends 1 to 1 and preserves multiplication.

DEFINITION 1.75. A *graded* $\mathbb{Q}$-algebra is a $\mathbb{Q}$-algebra A, together with a direct sum decomposition (called *grading*)

$$A = \bigoplus_{k \in \mathbb{Z}} A_k$$

into $\mathbb{Q}$-vector subspaces A_k of A satisfying $A_k \cdot A_{k'} \subseteq A_{k+k'}$. Note that the unit of the algebra then belongs necessarily to A_0, hence a map $\eta\colon \mathbb{Q} \to A_0$.

A graded $\mathbb{Q}$-algebra is said to be *connected* if $A_k = 0$ for all $k < 0$ and η is an isomorphism. Moreover, A is said to be *free* if it is isomorphic to a polynomial algebra $\mathbb{Q}[X_1, \ldots, X_n, \ldots]$ with X_i homogeneous of some degree; the X_i are then called *free algebra generators* of A.

DEFINITION 1.76. Let $A = \bigoplus_{k \in \mathbb{Z}} A_k$ be a graded $\mathbb{Q}$-algebra such that all A_k are finite-dimensional. The *Hilbert–Poincaré series* of A is defined as

$$H_A(t) = \sum_{k \in \mathbb{Z}} (\dim_{\mathbb{Q}} A_k) t^k.$$

If A is connected, then its Hilbert–Poincaré series has only non-negative degrees and the constant coefficient is equal to 1. Moreover, the number of free algebra generators in a given degree is well-defined, *i.e.* does not depend on the choice of an isomorphism with a particular polynomial algebra (Exercise 1.108).

LEMMA 1.77. *Let A be a connected graded free $\mathbb{Q}$-algebra such that all A_k are finite-dimensional, and let D_k denote the number of free algebra generators in degree k. Then the Hilbert–Poincaré series of A is equal to*

$$H_A(t) = \prod_{k \geqslant 1} (1 - t^k)^{-D_k}. \tag{1.78}$$

PROOF. Let $X_{1,1}, \ldots, X_{1,D_1}, \ldots, X_{\ell,1}, \ldots, X_{\ell,D_\ell}, \ldots$ be a set of homogeneous free algebra generators of A, with $X_{i,j}$ of degree $i \geqslant 1$. It suffices to observe that the coefficient of t^k in the power series expansion of the product (1.78) agrees with the number of monomials of degree k in the variables $X_{i,j}$, and hence with the dimension of A_k since we are dealing with a free algebra. □

We now explain how to compute the number of algebra generators in terms of the logarithm of the Hilbert–Poincaré series. Let us keep the assumptions on A from Lemma 1.77, and write

$$\log H_A(t) = \sum_{n \geqslant 1} c_n t^n. \tag{1.79}$$

Recall that the Möbius function μ takes the value 1 (resp. -1) on square-free integers with an even (resp. odd) number of prime factors, and 0 on integers with squared prime factors. In particular, $\mu(1) = 1$. The *Möbius inversion formula* is the statement that, if two sequences of complex numbers $(a_n)_{n \geqslant 1}$ and $(b_n)_{n \geqslant 1}$ are related by the equality $a_n = \sum_{d|n} b_d$ for all $n \geqslant 1$, then

$$b_n = \sum_{d|n} \mu(d) a_{n/d}.$$

LEMMA 1.80. *Let A be as in Lemma 1.77, let D_k denote the number of free algebra generators in degree k, and let c_n be the coefficient of t^n in $\log H_A(t)$ as in (1.79). Then the following equality holds:*

$$D_k = \sum_{d|k} \frac{\mu(d)}{d} c_{k/d}. \tag{1.81}$$

PROOF. Taking the logarithm of the identity (1.78) and using the formal power series expansion $-\log(1-x) = \sum_{n\geqslant 1} \frac{x^n}{n}$, one gets

$$\log H_A(t) = -\sum_{k\geqslant 1} D_k \log(1-t^k) = \sum_{k\geqslant 1} D_k \sum_{d\geqslant 1} \frac{t^{kd}}{d} = \sum_{n\geqslant 1}\left(\sum_{d|n} \frac{D_{n/d}}{d}\right) t^n.$$

Comparison of coefficients then yields

$$c_n = \sum_{d|n} \frac{D_{n/d}}{d} = \frac{1}{n}\sum_{d|n} dD_d,$$

and the equality (1.81) follows from the Möbius inversion formula applied to the sequences $a_n = nc_n$ and $b_n = nD_n$. □

Let us specialize the above discussion to the algebra $\mathcal{Z}$ of multiple zeta values. According to Zagier's conjecture, its Hilbert–Poincaré series is given by

$$H_{\mathcal{Z}}(t) = \frac{1}{1-t^2-t^3}.$$

CONJECTURE 1.82. *$\mathcal{Z}$ is a connected graded free $\mathbb{Q}$-algebra.*

Assuming this and Zagier's conjecture, we would like to compute the number D_k of free algebra generators in weight k. For this, we define a sequence of integers $(P_d)_{d\geqslant 1}$ by the equality

$$\sum_{d\geqslant 1} P_d t^d = \sum_{d\geqslant 1} dc_d t^d = t\frac{\mathrm{d}}{\mathrm{d}t}\log H_{\mathcal{Z}}(t) = \frac{2t^2+3t^3}{1-t^2-t^3}.$$

Equivalently, it is the sequence uniquely determined by the conditions

$$P_1 = 0,\ P_2 = 2,\ P_3 = 3,$$
$$P_d = P_{d-2} + P_{d-3}$$

for all $d \geqslant 4$. Therefore, Lemma 1.80 gives

$$D_k = \frac{1}{k}\sum_{d|k} \mu(k/d) P_d.$$

The first values of P_k and D_k are given in Table 1.2 below.

k	1	2	3	4	5	6	7	8	9	10	11	12	13
P_k	0	2	3	2	5	5	7	10	12	17	22	29	39
D_k	0	1	1	0	1	0	1	1	1	1	2	2	3

TABLE 1.2. Conjectural values of D_k for the algebra $\mathcal{Z}$

Recall that Hoffman's conjecture 1.73 predicts that the set of multiple zeta values $\zeta(s_1, \ldots, s_\ell)$ with all entries $s_i \in \{2,3\}$ is a graded $\mathbb{Q}$-basis of $\mathcal{Z}$. It is then only natural to try to extract from these elements a set of algebra generators; this is done through the theory of Lyndon words.

DEFINITION 1.83. Let X be the alphabet $\{a, b\}$, and let X^* be the set of words in X. We endow X^* with the lexicographic order for which $a < b$. A *Lyndon word* is a non-empty word $w \in X^*$ such that, for each non-trivial decomposition $w = uv$, the inequality $w < v$ holds.

For example, ab is a Lyndon word because $ab < b$, but none of the words aa, ba, bb is Lyndon.

Every word in the alphabet $\{2,3\}$ can be seen as an admissible multi-index with entries $s_i \in \{2,3\}$, and hence defines a multiple zeta value.

CONJECTURE 1.84. *The free $\mathbb{Q}$-algebra generated by Lyndon words on the alphabet $\{2,3\}$ with the order $2 < 3$ is isomorphic to $\mathcal{Z}$. The isomorphism is given by sending a Lyndon word to the corresponding multiple zeta value.*

Assuming that the conjecture holds, the free algebra generators in weights up to 13 are listed in Table 1.3 below.

weight	generators	weight	generators
2	$\zeta(2)$	8	$\zeta(2,3,3)$
3	$\zeta(3)$	9	$\zeta(2,2,2,3)$
4	$\emptyset$	10	$\zeta(2,2,3,3)$
5	$\zeta(2,3)$	11	$\zeta(2,2,2,2,3), \zeta(2,3,3,3)$
6	$\emptyset$	12	$\zeta(2,2,2,3,3), \zeta(2,2,3,2,3)$
7	$\zeta(2,2,3)$	13	$\zeta(2,2,2,2,2,3), \zeta(2,2,3,3,3),$ $\zeta(2,3,2,3,3)$

TABLE 1.3. First Lyndon words on the alphabet $\{2,3\}$

For example, the other conjectural basis element in weight 5, namely the multiple zeta value $\zeta(3,2)$, can be written as the polynomial expression

$$\zeta(3,2) = \frac{5}{9}\zeta(2)\zeta(3) - \frac{11}{9}\zeta(2,3).$$

See Exercise 1.104 for weight 6.

1.4.5. *The Broadhurst–Kreimer conjecture.* So far, we have only taken the weight of multiple zeta values into account. A difficulty to add the length to the picture is that it is only expected to induce a filtration and not a grading, as it is already clear from the existence of relations among multiple zeta values associated with multi-indices of different length such as $\zeta(3) = \zeta(2,1)$.

DEFINITION 1.85.

i) A *filtration* on a $\mathbb{Q}$-algebra A is an increasing sequence of vector subspaces

$$\cdots \subseteq F_{\ell-1}A \subseteq F_\ell A \subseteq F_{\ell+1}A \subseteq \ldots$$

indexed by $\ell \in \mathbb{Z}$ and satisfying

$$F_\ell A \cdot F_{\ell'}A \subseteq F_{\ell+\ell'}A.$$

A filtration is called *separated* if $\bigcap_\ell F_\ell A = 0$ and *exhaustive* if $\bigcup_\ell F_\ell A = A$. A *filtered* $\mathbb{Q}$-algebra is a $\mathbb{Q}$-algebra endowed with a filtration.

ii) The *graded algebra* associated with a filtered algebra $(A, F_\bullet)$ is defined as

$$\operatorname{Gr}^F A = \bigoplus_{\ell \in \mathbb{Z}} F_\ell A / F_{\ell-1} A.$$

Note that the compatibility of the product and the filtration guarantees that $\operatorname{Gr}^F A$ has indeed an induced graded algebra structure.

iii) A *filtered graded* $\mathbb{Q}$-algebra is a $\mathbb{Q}$-algebra A with a filtration $F_\bullet A$ and a grading $A = \bigoplus_{k \in \mathbb{Z}} A_k$ that are compatible in the sense that the equality

$$F_\ell A = \bigoplus_{k \in \mathbb{Z}} F_\ell A_k$$

holds. Given such an algebra, we set

$$A_{k,\ell} = \operatorname{Gr}^F_\ell A_k = F_\ell A_k / F_{\ell-1} A_k$$

and form the associated *bigraded* algebra $\bigoplus_{k,\ell \in \mathbb{Z}} A_{k,\ell}$.

Remark 1.86. Let A be a $\mathbb{Q}$-algebra endowed with an exhaustive filtration F_ℓ satisfying $F_{-1}A = \{0\}$. If $\operatorname{Gr}^F A$ is a free graded algebra, then A is a free algebra. Indeed, let X_i be homogeneous free algebra generators of $\operatorname{Gr}^F A$, and pick liftings Y_i of X_i lying in the step of the filtration corresponding to the degree of X_i. The free algebra $\mathbb{Q}[Y_i]$ is then endowed with the filtration corresponding to the degree, and the natural map $\mathbb{Q}[Y_i] \to A$ is a morphism of algebras compatible with the filtrations. It induces an isomorphism on F_{-1} since both $F_{-1}\mathbb{Q}[Y_i]$ and $F_{-1}A$ vanish. Assume that $F_{\ell-1}\mathbb{Q}[Y_i] \to F_{\ell-1}A$ is an isomorphism. Then the leftmost and rightmost vertical arrows in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{\ell-1}\mathbb{Q}[Y_i] \arrow[r] \arrow[d] & F_\ell \mathbb{Q}[Y_i] \arrow[r] \arrow[d] & \operatorname{Gr}_\ell \mathbb{Q}[Y_i] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{\ell-1}A \arrow[r] & F_\ell A \arrow[r] & \operatorname{Gr}_\ell A \arrow[r] & 0
\end{tikzcd}$$

are isomorphisms, and hence so is $F_\ell\mathbb{Q}[Y_i] \to F_\ell A$ by an elementary diagram chase. Since the filtration is exhaustive, A is isomorphic to the free algebra $\mathbb{Q}[Y_i]$.

Returning to the algebra of multiple zeta values, we see that the length defines a separated and exhaustive filtration

$$F_\ell \mathcal{Z} = \langle \zeta(\boldsymbol{s}) \mid \ell(\boldsymbol{s}) \leqslant \ell \rangle_{\mathbb{Q}}.$$

Assuming Conjecture 1.68, $\mathcal{Z}$ is hence a filtered graded algebra, and

$$\mathcal{Z}_{k,\ell} = F_\ell \mathcal{Z}_k / F_{\ell-1} \mathcal{Z}_k$$

is the space of multiple zeta values of weight k and length ℓ that cannot be written as linear combinations of multiple zeta values of smaller length.

Note that the associated bigraded algebra is not free, since $\zeta(2)^2$ vanishes in $\mathcal{Z}_{4,2}$ because of the relation $\zeta(2)^2 = 5\zeta(4)/2$. To remedy this, we consider the quotient

$$\mathcal{Z}^\circ = \mathcal{Z}/\zeta(2) \cdot \mathcal{Z}$$

by the ideal generated by $\zeta(2)$. This quotient is a graded filtered algebra as well. Moreover, we equip $\mathbb{Q}[\zeta(2)]$ with the filtration

$$F_0 = \mathbb{Q} \subset F_1 = \mathbb{Q}[\zeta(2)],$$

and the grading that gives weight 2 to $\zeta(2)$.

The following is a refinement of Conjecture 1.82.

CONJECTURE 1.87.

i) $\mathrm{Gr}^F \mathcal{Z}^\circ$ *is a free bigraded algebra.*

ii) *There is an isomorphism of filtered graded algebras* $\mathcal{Z}^\circ \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta(2)] \to \mathcal{Z}$.

Definition 1.76 and Lemmas 1.77 and 1.80 extend to bigraded algebras. In particular, if $A = \bigoplus_{k,\ell} A_{k,\ell}$ is a connected free bigraded algebra, then

$$H_A(x,y) = \sum_{k,\ell \geqslant 0} (\dim_{\mathbb{Q}} A_{k,\ell}) x^k y^\ell = \prod_{k,\ell \geqslant 1} (1 - x^k y^\ell)^{-D_{k,\ell}},$$

where $D_{k,\ell}$ is the number of free algebra generators in bidegree (k,ℓ).

Extensive numerical experiments support the following refinement of Zagier's conjecture, due to Broadhurst and Kreimer [**BK97**, § 2]:

CONJECTURE 1.88 (Broadhurst–Kreimer). *Define integers* $(D_{k,\ell})_{k\geqslant 3,\ell\geqslant 1}$ *by the product expansion formula*

$$\prod_{k\geqslant 3} \prod_{\ell \geqslant 1} (1 - x^k y^\ell)^{-D_{k,\ell}} = \frac{1}{1 - O(x)y + S(x)y^2 - S(x)y^4}, \tag{1.89}$$

where $O(x)$ *and* $S(x)$ *are the formal power series*

$$O(x) = \frac{x^3}{1-x^2} = x^3 + x^5 + x^7 + x^9 + \cdots,$$

$$S(x) = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12} + x^{16} + x^{18} + x^{20} + x^{22} + 2x^{24} + \cdots.$$

Then $D_{k,\ell}$ *agrees with the number of free algebra generators of* $\mathrm{Gr}^F \mathcal{Z}^\circ$ *of weight* k *and length* ℓ.

For shorthand, write $\mathrm{BK}^0(x,y)$ for the power series expansion of

$$\frac{1}{1 - O(x)y + S(x)y^2 - S(x)y^4}.$$

Arguing as in Lemma 1.80, the numbers $D_{k,\ell}$ are given by the formula

$$D_{k,\ell} = \sum_{d \mid \gcd(k,\ell)} \frac{\mu(d)}{d} \cdot \text{coefficient of } x^{\frac{k}{d}} y^{\frac{\ell}{d}} \text{ in } \log \mathrm{BK}^0(x,y), \tag{1.90}$$

where $\gcd(k,\ell)$ denotes the greatest common divisor of k and ℓ.

Taking Conjecture 1.87 for granted, the multiplicative formula (1.89) becomes equivalent to the following additive version, which is the one that is usually found in the literature:

CONJECTURE 1.91 (Broadhurst–Kreimer). *Let* $(d_{k,\ell})_{k,\ell\geqslant 0}$ *be the sequence of non-negative integers defined by the generating series*

$$\sum_{k,\ell\geqslant 0} d_{k,\ell} x^k y^\ell = \frac{1 + E(x)y}{1 - O(x)y + S(x)y^2 - S(x)y^4},$$

where

$$E(x) = \frac{x^2}{1-x^2} = x^2 + x^4 + x^6 + x^8 + \cdots .$$

Then $d_{k,\ell}$ agrees with the dimension of the space of multiple zeta values of (precisely) weight k and length ℓ, that is

$$d_{k,\ell} = \dim_{\mathbb{Q}} \mathcal{Z}_{k,\ell}. \tag{1.92}$$

REMARK 1.93. The power series $E(x)$ "counts" even zeta values, while $O(x)$ counts odd zeta values. More interestingly, Zagier realized that $S(x)$ agrees with the generating series

$$S(x) = \sum_{k \geqslant 1} (\dim_{\mathbb{Q}} S_k) x^k,$$

where S_k stands for the $\mathbb{Q}$-vector space of cusp modular forms of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. It is a classical result that

$$\dim_{\mathbb{Q}} S_k = \begin{cases} \lfloor \frac{k}{12} \rfloor, & \text{if } k \text{ is even and } k \not\equiv 2 \bmod 12, \\ \lfloor \frac{k}{12} \rfloor - 1, & \text{if } k \equiv 2 \bmod 12, \\ 0, & \text{otherwise.} \end{cases}$$

(see [**Zag08**, §1.3 and §2.1] for an elementary proof).

Let us denote by $\mathrm{BK}(x,y)$ the power series expansion of

$$\frac{1 + E(x)y}{1 - O(x)y + S(x)y^2 - S(x)y^4}.$$

Expanding the inverse of the denominator as a geometric series and collecting the terms with lower powers of y, we obtain

$$\begin{aligned} \mathrm{BK}(x,y) = 1 &+ [E(x) + O(x)]y + [(E(x)+O(x))O(x) - S(x)]y^2 \\ &+ [(O(x)^2 - 2S(x))O(x) + (O(x)^2 - S(x))E(x)]y^3 + \cdots . \end{aligned}$$

REMARK 1.94. From this, we get $d_{k,1} = 1$ for all $k \geqslant 2$. Since $F_0\mathcal{Z} = \mathbb{Q}$, the Broadhurst–Kreimer conjecture holds in this case if and only if $\zeta(k)$ is irrational, which is only known for even k and $k = 3$.

The first values of $d_{k,2}$ and $d_{k,3}$ are given in Table 1.4 below.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$d_{k,2}$	0	0	0	0	1	1	2	2	3	3	4	3	5	5	6	5	7
$d_{k,3}$	0	0	0	0	0	0	0	1	1	3	3	6	6	9	8	14	13

TABLE 1.4. First values of $d_{k,2}$ and $d_{k,3}$

Similarly, we derive

$$\begin{aligned} \log \mathrm{BK}^0(x,y) &= -\log(1 - O(x)y + S(x)y^2 - S(x)y^4) \\ &= O(x)y + \left(\frac{1}{2}O(x)^2 - S(x)\right)y^2 + \left(\frac{1}{3}O(x)^3 - O(x)S(x)\right)y^3 + \cdots . \end{aligned}$$

REMARK 1.95. Note that $D_{k,\ell} = 0$ if k and ℓ have different parity. Indeed, in this case the integers d contributing to formula (1.90) are all odd, so k/d and ℓ/d have again different parity. However, it is clear from the above expression for the series $\log \mathrm{BK}^0(x, y)$ that only monomials in which the degree of x and the degree of y have the same parity appear.

LEMMA 1.96.

i) *If k is even, then $D_{k,2} = \lfloor \frac{k-2}{6} \rfloor$.*
ii) *If k is odd, then $D_{k,3} = \lfloor \frac{(k-3)^2-1}{48} \rfloor$.*

PROOF. Specializing (1.90) to the case $\ell = 2$, we get

$$\begin{aligned} D_{k,2} &= \text{coeff. of } x^k y^2 - \frac{1}{2}\text{coeff. of } x^{\frac{k}{2}} y \text{ in } \log BK^0(x,y) \\ &= \text{coeff. of } x^k \text{ in } \left(\frac{1}{2}O(x)^2 - S(x)\right) - \frac{1}{2} \text{ coeff. of } x^{\frac{k}{2}} \text{ in } O(x). \end{aligned}$$

Taking the equality $O(x)^2 = \sum_{\substack{k \geqslant 6 \\ \text{even}}} \frac{k-4}{2} x^k$ into account, we find the formula

$$D_{k,2} = \begin{cases} \frac{k-4}{4} - \lfloor \frac{k}{12} \rfloor, & \text{if } k \equiv 0 \bmod 4, \\ \frac{k-6}{4} - \lfloor \frac{k}{12} \rfloor, & \text{if } k \equiv 2 \bmod 4 \text{ and } k \not\equiv 2 \bmod 12, \\ \frac{k-2}{4} - \lfloor \frac{k}{12} \rfloor, & \text{if } k \equiv 2 \bmod 12. \end{cases}$$

It is then a simple matter to check that this quantity agrees with $\lfloor \frac{k-2}{6} \rfloor$. The proof of the second assertion follows the same pattern (Exercise 1.107). $\square$

REMARK 1.97.

i) The numbers $D_{k,2}$ and $D_{k,3}$ are known to be upper bounds for the number of generators of length 2 and 3 of the algebra of multiple zeta values. It is proved in [**Zag93**, § 3] for $\ell = 2$ and in [**Gon98**, Thm. 1.5] for $\ell = 3$. From this, we get the inequality

$$\dim_{\mathbb{Q}} (F_\ell \mathcal{Z}_k / F_{\ell-1} \mathcal{Z}_k) \leqslant d_{k,\ell}$$

in lengths $\ell = 1, 2, 3$.

ii) In particular, for double zeta values we get

$$\dim_{\mathbb{Q}} F_2 \mathcal{Z}_k - 1 \leqslant d_{k,2}.$$

By contrast, Proposition 1.64 yields the upper bound

$$\dim_{\mathbb{Q}} F_2 \mathcal{Z}_k - 1 \leqslant \left\lceil \frac{k-4}{2} \right\rceil.$$

The right-hand side of this last inequality agrees with the coefficient of degree k of the power series $(E(x)+O(x))O(x)$, while $d_{k,2}$ is, by definition, the coefficient of degree k in $(E(x) + O(x))O(x) - S(x)$. Therefore, the bound of Proposition 1.64 is *not* optimal for those integers k such that there exists a non-zero cusp form of weight k.

iii) Brown [**Bro21**] reformulated the Broadhurst–Kreimer conjecture in terms of the homology of a certain Lie algebra.

1.4.6. *Known results.* Not much is known about these conjectures, especially the one by Broadhurst and Kreimer. The goal of these notes is to explain in detail the following two results towards Zagier's and Hoffman's conjectures. In spite of their elementary formulation, this will carry us far away since the only known proofs rely on the theory of motives.

THEOREM 1.98 (Terasoma [**Ter02**], Deligne–Goncharov [**DG05**]). *The number d_k is an upper bound for the dimension of the $\mathbb{Q}$-vector space of multiple zeta values of weight k, that is,*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leqslant d_k.$$

THEOREM 1.99 (Brown, [**Bro12**]). *Every multiple zeta value can be written as a $\mathbb{Q}$-linear combination of those $\zeta(s_1, \ldots, s_l)$ with all entries $s_i \in \{2, 3\}$.*

REMARK 1.100. As we will see at the very end of the text (Section 5.5.1), a corollary of these two theorems is that Zagier's conjecture implies the algebraic independence of odd zeta values (Conjecture 1.11).

⋆ ⋆ ⋆

EXERCISE 1.101. Prove that, if the weight defines a grading on the algebra of multiple zeta values (Conjecture 1.68), then the inclusion

$$F_\ell \mathcal{Z}_k \subseteq F_\ell \mathcal{Z} \cap \mathcal{Z}_k$$

from Remark 1.31 is an equality.

EXERCISE 1.102. Prove that the sequence $(d_k)_{k\geqslant 0}$ defined in (1.70) satisfies

$$\lim_{k\to\infty} (d_k - \kappa r^k) = 0$$

where $\kappa = \frac{r+1}{2r+3}$ and r is the real root of $x^3 - x - 1$.

EXERCISE 1.103. For each $k \geqslant 0$, let δ_k be the number of ordered sequences of integers $(s, n_1, \ldots, n_r)$ such that $r \geqslant 0$, $s \geqslant 0$, $n_i \geqslant 1$, and

$$k = 2s + 2n_1 + 1 + \cdots + 2n_r + 1.$$

Show that $\delta_0 = 1$, $\delta_1 = 0$, $\delta_2 = 1$ and $\delta_k = \delta_{k-2} + \delta_{k-3}$ for all $k \geqslant 3$. Therefore, the equality $\delta_k = d_k$ holds.

EXERCISE 1.104. Express the conjectural basis elements $\zeta(2, 2, 2)$ and $\zeta(3, 3)$ of $\mathcal{Z}_6$ as polynomial expressions in $\zeta(2), \zeta(3), \zeta(2, 3)$.

EXERCISE 1.105. Assume that the numbers $\zeta(2), \zeta(3), \zeta(5), \ldots$ are algebraically independent, so that $\mathbb{Q}[\zeta(2), \zeta(3), \ldots]$ is a free graded algebra. Apply Lemma 1.80 to compute the dimensions of the graded pieces, and compare them to the conjectural dimensions of multiple zeta values. Then find an example of a multiple zeta value which is not expected to be in the algebra generated by Riemann zeta values.

EXERCISE 1.106. Show that either Hoffman's or the Broadhurst–Kreimer conjecture implies Zagier's conjecture.

EXERCISE 1.107. Prove the equality $D_{k,3} = \lfloor \frac{(k-3)^2-1}{48} \rfloor$.

EXERCISE 1.108. Let $A = \bigoplus_{k \geqslant 0} A_k$ be a connected graded free $\mathbb{Q}$-algebra with finite-dimensional graded pieces and $I = \bigoplus_{k>0} A_k$. Prove that the number of free algebra generators in degree k of A is the dimension of the k-th graded piece of the quotient I/I^2.

EXERCISE 1.109. Consider the $\mathbb{Q}$-algebra of formal power series $A = \mathbb{Q}[\![t]\!]$ endowed with the exhaustive filtration given by

$$F_\ell A = t^{-\ell}\mathbb{Q}[\![t]\!] \quad \text{for } \ell \leqslant 0 \quad \text{and} \quad F_\ell A = A \quad \text{for } \ell \geqslant 0.$$

Prove that the associated graded algebra $\mathrm{Gr}_F A$ is free, but that A is not. This shows that the assumption on $F_{-1}A$ from Remark 1.86 is necessary.

1.5. Integral representation of multiple zeta values. We defined multiple zeta values as sums of infinite series. Using this representation, we proved that the product of two multiple zeta values is a linear combination of multiple zeta values with coefficients given by the stuffle multiplicities. We also derived some linear relations among multiple zeta values by means of partial fraction decompositions. Kontsevich found a different representation in terms of integrals. This way of writing multiple zeta values is central to the theory. From a combinatorial point of view, it yields the *shuffle product*, a new structure from which many other linear relations are obtained in a systematic way. More importantly from a conceptual point of view, the integral representation shows that multiple zeta values are *periods* of algebraic varieties and allow us to use algebro-geometric tools to study them.

1.5.1. *Two examples.*

EXAMPLE 1.110. The identity

$$\zeta(2) = \int\limits_{1 \geqslant t_1 \geqslant t_2 \geqslant 0} \frac{\mathrm{d}t_1}{t_1} \frac{\mathrm{d}t_2}{1-t_2} = \int_0^1 \left(\frac{1}{t_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{1-t_2} \right) \mathrm{d}t_1 \tag{1.111}$$

holds. Indeed, from the geometric series expansion

$$\frac{1}{1-t_2} = \sum_{n \geqslant 1} t_2^{n-1},$$

valid for $0 \leqslant t_2 < 1$, we get the equality

$$\int_0^{t_1} \frac{\mathrm{d}t_2}{1-t_2} = \sum_{n \geqslant 1} \int_0^{t_1} t_2^{n-1} \mathrm{d}t_2 = \sum_{n \geqslant 1} \frac{t_1^n}{n}.$$

Plugging it into the integrand gives

$$\int_0^1 \left(\frac{1}{t_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{1-t_2} \right) \mathrm{d}t_1 = \int_0^1 \sum_{n \geqslant 1} \frac{t_1^n}{n} \frac{\mathrm{d}t_1}{t_1} = \sum_{n \geqslant 1} \frac{1}{n} \int_0^1 t_1^{n-1} \mathrm{d}t_1 = \sum_{n \geqslant 1} \frac{1}{n^2}.$$

EXAMPLE 1.112. The identity

$$\zeta(2,1) = \int\limits_{1 \geqslant t_1 \geqslant t_2 \geqslant t_3 \geqslant 0} \frac{\mathrm{d}t_1}{t_1} \frac{\mathrm{d}t_2}{1-t_2} \frac{\mathrm{d}t_3}{1-t_3}$$

holds. Indeed, this follows from the transformations

$$\begin{aligned}\int\limits_{1\geqslant t_1\geqslant t_2\geqslant t_3\geqslant 0}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{1-t_2}\frac{\mathrm{d}t_3}{1-t_3} &= \int\limits_{1\geqslant t_1\geqslant t_2\geqslant 0}\frac{1}{t_1}\sum_{n\geqslant 1}\frac{t_2^n}{n}\frac{\mathrm{d}t_1\,\mathrm{d}t_2}{1-t_2}\\ &= \int\limits_{1\geqslant t_1\geqslant t_2\geqslant 0}\frac{1}{t_1}\sum_{n,m\geqslant 1}\frac{t_2^{n+m-1}}{n}\mathrm{d}t_1\,\mathrm{d}t_2\\ &= \int\limits_{1\geqslant t_1\geqslant 0}\sum_{n,m\geqslant 1}\frac{t_1^{n+m}}{(n+m)n}\frac{\mathrm{d}t_1}{t_1}\\ &= \sum_{n,m\geqslant 1}\frac{1}{(n+m)^2 n}\\ &= \zeta(2,1).\end{aligned}$$

(Solve Exercise 1.141 to convince yourself that the exchange of the order of integration and summation is justified in these two examples.)

Remark 1.113. As we will see in Section 3.8, the above integrals are instances of the notion of *iterated* integral, but for the moment we will think of them just as ordinary integrals over a simplex.

1.5.2. *The integral representation.* A piece of notation is needed to describe the general integral representation of multiple zeta values.

Notation 1.114. Given a real number $0 \leqslant t \leqslant 1$, we define

$$\varDelta^p(t) = \{(t_1,\dots,t_p)\in\mathbb{R}^p \mid t\geqslant t_1\geqslant t_2\geqslant\cdots\geqslant t_p\geqslant 0\}.$$

Note that $\Delta^0(t)$ is a singleton for all t. We endow $\varDelta^p(t)$ with the standard orientation given by the order of the variables. We will simply write $\varDelta^p = \varDelta^p(1)$ for $t = 1$. Furthermore, consider the measures

$$\omega_0(t) = \frac{\mathrm{d}t}{t} \quad\text{and}\quad \omega_1(t) = \frac{\mathrm{d}t}{1-t}$$

on the open interval $(0, 1)$.

If $\boldsymbol{s} = (s_1,\dots,s_l)\in\mathbb{Z}^l$ is a positive multi-index (recall from Definition 1.20 that this means $s_i \geqslant 1$ for all i), we write $r_i = s_1+\cdots+s_i$ for each $i = 1,\dots,l$. In particular, $r_1 = s_1$ and r_l is the weight of $\boldsymbol{s}$. For convenience, we also set $r_0 = 0$. Let $\omega_{\boldsymbol{s}}$ be the measure on the interior of the simplex $\Delta^{\mathrm{wt}(\boldsymbol{s})}$ given by

$$\omega_{\boldsymbol{s}} = \prod_{i=1}^{\ell}\underbrace{\omega_0(t_{r_{i-1}+1})\cdots\omega_0(t_{r_i-1})}_{s_i-1\text{ times}}\,\omega_1(t_{r_i}). \tag{1.115}$$

For example, one has:

$$\omega_{(2)} = \frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{1-t_2},$$
$$\omega_{(2,2)} = \frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{1-t_2}\frac{\mathrm{d}t_3}{t_3}\frac{\mathrm{d}t_4}{1-t_4},$$
$$\omega_{(2,1)} = \frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{1-t_2}\frac{\mathrm{d}t_3}{1-t_3},$$
$$\omega_{(1,3)} = \frac{\mathrm{d}t_1}{1-t_1}\frac{\mathrm{d}t_2}{t_2}\frac{\mathrm{d}t_3}{t_3}\frac{\mathrm{d}t_4}{1-t_4}.$$

The following result is attributed to Kontsevich.

THEOREM 1.116. *Let $\boldsymbol{s} = (s_1, \ldots, s_\ell)$ be an admissible multi-index. The multiple zeta value $\zeta(\boldsymbol{s})$ is equal to the value of the absolutely convergent integral*

$$\zeta(\boldsymbol{s}) = \zeta(s_1, \ldots, s_\ell) = \int_{\Delta^{\mathrm{wt}(\boldsymbol{s})}} \omega_{\boldsymbol{s}}. \tag{1.117}$$

In order to easily prove this theorem, we introduce the polylogarithm functions, which will also be of use later in Chapter 3.

DEFINITION 1.118. Let $\boldsymbol{s} = (s_1, \ldots, s_\ell)$ be a positive multi-index and t a complex number with $|t| < 1$. We define

$$\mathrm{Li}_{\boldsymbol{s}}(t) = \mathrm{Li}_{s_1,\ldots,s_\ell}(t) = \sum_{n_1 > n_2 > \cdots > n_\ell \geqslant 1} \frac{t^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.$$

We call $\mathrm{Li}_{\boldsymbol{s}}$ the *(multiple) polylogarithm function (of one variable)*.

REMARK 1.119. Similarly, one can define *multiple polylogarithms of several variables* as the absolutely convergent series

$$\mathrm{Li}_{\boldsymbol{s}}(t_1, \ldots, t_\ell) = \sum_{n_1 > n_2 > \cdots > n_\ell \geqslant 1} \frac{t_1^{n_1} \cdots t_\ell^{n_\ell}}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}$$

whenever the complex numbers t_i satisfy $|t_1| < 1$ and $|t_i| \leqslant 1$ for $i = 2, \ldots, \ell$.

The following proposition is a straightforward consequence of basic results in complex analysis.

PROPOSITION 1.120. *If $\boldsymbol{s}$ is a positive multi-index, then the function $\mathrm{Li}_{\boldsymbol{s}}$ is holomorphic on the open unit disc $|t| < 1$. Moreover, if $\boldsymbol{s}$ is admissible, then $\mathrm{Li}_{\boldsymbol{s}}$ can be extended continuously to the closed disc $|t| \leqslant 1$ and satisfies*

$$\mathrm{Li}_{\boldsymbol{s}}(1) = \zeta(\boldsymbol{s}).$$

For instance, $\mathrm{Li}_1(t)$ is given by

$$\mathrm{Li}_1(t) = \sum_{n \geqslant 1} \frac{t^n}{n} = -\log(1-t) = \int_0^t \frac{\mathrm{d}t_1}{1-t_1}, \tag{1.121}$$

where $\int_0^t$ denotes the integral along the straight path from 0 to t, and $\mathrm{Li}_2(t)$ is the primitive of $\mathrm{Li}_1(t)/t$ that vanishes at $t = 0$:

$$\mathrm{Li}_2(t) = \sum_{n \geqslant 1} \frac{t^n}{n^2} = -\int_0^t \log(1-t_1)\frac{\mathrm{d}t_1}{t_1}. \tag{1.122}$$

These relations are among the simplest functional equations satisfied by polylogarithms. They generalize as follows.

PROPOSITION 1.123. *The following identities hold for all $|t| < 1$:*

$$\int_0^t \mathrm{Li}_{s_1,\dots,s_\ell}(t_1)\frac{\mathrm{d}t_1}{t_1} = \mathrm{Li}_{s_1+1,\dots,s_\ell}(t), \tag{1.124}$$

$$\int_0^t \mathrm{Li}_{s_1,\dots,s_\ell}(t_1)\frac{\mathrm{d}t_1}{1-t_1} = \mathrm{Li}_{1,s_1,\dots,s_\ell}(t). \tag{1.125}$$

PROOF. Equation (1.124) simply follows from plugging the definition of the polylogarithm in the integral and exchanging sum and integration:

$$\begin{aligned}\int_0^t \mathrm{Li}_{s_1,\dots,s_\ell}(t_1)\frac{\mathrm{d}t_1}{t_1} &= \int_0^t \sum_{n_1>n_2>\dots>n_\ell\geqslant 1} \frac{t_1^{n_1}}{n_1^{s_1}n_2^{s_2}\dots n_l^{s_\ell}}\frac{\mathrm{d}t_1}{t_1} \\ &= \sum_{n_1>n_2>\dots>n_\ell\geqslant 1} \frac{t^{n_1}}{n_1^{s_1+1}n_2^{s_2}\dots n_l^{s_\ell}} \\ &= \mathrm{Li}_{s_1+1,\dots,s_\ell}(t).\end{aligned}$$

Similarly, equation (1.125) follows from the manipulations

$$\begin{aligned}\int_0^t \mathrm{Li}_{s_1,\dots,s_\ell}(t_1)\frac{\mathrm{d}t_1}{1-t_1} &= \int_0^t \sum_{n_1>n_2>\dots>n_\ell\geqslant 1} \frac{t_1^{n_1}}{n_1^{s_1}n_2^{s_2}\dots n_\ell^{s_\ell}}\sum_{m\geqslant 0} t_1^m \mathrm{d}t_1 \\ &= \sum_{n_0>n_1>\dots>n_\ell\geqslant 1} \frac{t^{n_0}}{n_0 n_1^{s_1}n_2^{s_2}\dots n_\ell^{s_\ell}} \\ &= \mathrm{Li}_{1,s_1,\dots,s_\ell}(t),\end{aligned}$$

where we have written $n_0 = n_1 + m + 1 > n_1$. (In both cases, the exchange of the summation and the integral signs is justified by the fact that we are integrating holomorphic functions on a compact set.) □

With these preliminaries out of the way, Theorem 1.116 is a particular case of the next result.

THEOREM 1.126. *If $\boldsymbol{s}$ is a positive multi-index and $0 < t < 1$ a real number, then the following identity holds:*

$$\mathrm{Li}_{\boldsymbol{s}}(t) = \int_{\Delta^{\mathrm{wt}(\boldsymbol{s})}(t)} \omega_{\boldsymbol{s}}.$$

PROOF. We proceed by induction on the weight of $\boldsymbol{s}$. If $\mathrm{wt}(\boldsymbol{s}) = 1$, then $\boldsymbol{s} = (1)$ and the statement is just formula (1.121). The inductive step follows from the functional equations in Proposition 1.123. Indeed, let $\boldsymbol{s} = (s_1, \dots, s_\ell)$ be a positive multi-index and assume that the result is true for all multi-indices of lower weight. If $s_1 > 1$, then we write $\boldsymbol{s}' = (s_1 - 1, \dots, s_\ell)$. By the identity (1.124) and induction,

$$\mathrm{Li}_{\boldsymbol{s}}(t) = \int_0^t \mathrm{Li}_{\boldsymbol{s}'}(t_1)\frac{\mathrm{d}t_1}{t_1} = \int_0^t \int_{\Delta^{\mathrm{wt}(\boldsymbol{s}')}(t_1)} \omega_{\boldsymbol{s}'}\frac{\mathrm{d}t_1}{t_1} = \int_{\Delta^{\mathrm{wt}(\boldsymbol{s})}(t)} \omega_{\boldsymbol{s}}.$$

The case $s_1 = 1$ is similar, using equation (1.125) instead. □

1.5.3. *Shuffles.* Since multiple zeta values are integrals along simplices, certain combinatorial properties of the latter translate into relations among the former. Let us first illustrate this with an example.

EXAMPLE 1.127. The following equalities hold:

$$\begin{aligned}
\zeta(2)^2 &= \int\limits_{1\geqslant t_1\geqslant t_2\geqslant 0} \frac{\mathrm{d}t_1\,\mathrm{d}t_2}{t_1(1-t_2)} \cdot \int\limits_{1\geqslant u_1\geqslant u_2\geqslant 0} \frac{\mathrm{d}u_1\,\mathrm{d}u_2}{u_1(1-u_2)} \\
&= \int\limits_{\substack{1\geqslant t_1\geqslant t_2\geqslant 0\\ 1\geqslant u_1\geqslant u_2\geqslant 0}} \frac{\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}u_1\,\mathrm{d}u_2}{t_1(1-t_2)u_1(1-u_2)} \\
&= \sum_{i=1}^{6}\int_{U_i} \frac{\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}u_1\,\mathrm{d}u_2}{t_1(1-t_2)u_1(1-u_2)} \\
&= \zeta(3,1)+\zeta(3,1)+\zeta(2,2)+\zeta(3,1)+\zeta(3,1)+\zeta(2,2) \\
&= 4\zeta(3,1)+2\zeta(2,2),
\end{aligned}$$

where the sets U_i, for $i = 1,\dots,6$, are defined by

$$\begin{aligned}
U_1 &= \{1 \geqslant t_1 \geqslant u_1 \geqslant t_2 \geqslant u_2 \geqslant 0\},\\
U_2 &= \{1 \geqslant t_1 \geqslant u_1 \geqslant u_2 \geqslant t_2 \geqslant 0\},\\
U_3 &= \{1 \geqslant t_1 \geqslant t_2 \geqslant u_1 \geqslant u_2 \geqslant 0\},\\
U_4 &= \{1 \geqslant u_1 \geqslant t_1 \geqslant u_2 \geqslant t_2 \geqslant 0\},\\
U_5 &= \{1 \geqslant u_1 \geqslant t_1 \geqslant t_2 \geqslant u_2 \geqslant 0\},\\
U_6 &= \{1 \geqslant u_1 \geqslant u_2 \geqslant t_1 \geqslant t_2 \geqslant 0\}.
\end{aligned}$$

The third equality comes from the decomposition

$$\{(t_1,t_2,u_1,u_2) \mid 1 \geqslant t_1 \geqslant t_2 \geqslant 0,\ 1 \geqslant u_1 \geqslant u_2 \geqslant 0\} = \bigcup_{i=1}^{6} U_i,$$

and the fourth one from Theorem 1.116.

REMARK 1.128. This expression of $\zeta(2)^2$ as a linear combination of double zeta values is different from the one obtained by means of the series representation in Example 1.16. Combining both, we recover one of the relations that was proved in Corollary 1.63 using the method of partial fraction decompositions, namely:

$$\zeta(4) = 4\zeta(3,1)$$

To generalize the previous example, we introduce the notion of shuffles.

DEFINITION 1.129. Let $r, s \geqslant 0$ be integers. A *shuffle of type* (r,s) is a permutation σ of the set $\{1,2,\dots,r+s\}$ satisfying

$$\sigma(1) < \sigma(2) < \cdots < \sigma(r) \text{ and } \sigma(r+1) < \sigma(r+2) < \cdots < \sigma(r+s).$$

We denote by $ш(r,s)$ the set of all shuffles of type (r,s).

That is to say, a shuffle is a permutation that respects the ordering of two distinguished subsets. The name comes from the way gamblers shuffle a deck of cards in western saloons.

EXAMPLE 1.130. The set of shuffles of type $(2,2)$ consists of the permutations

$$⧢(2,2) = \{\mathrm{Id}, (23), (243), (123), (1243), (13)(24)\}.$$

Shuffles allow us to decompose a product of two simplices into a union of simplices, and therefore to express a product of integrals over simplices as a linear combination of integrals.

PROPOSITION 1.131. *Let $r, s \geqslant 0$ be integers and $0 < t < 1$ a real number. For each choice of $\mu_i \in \{\omega_0, \omega_1\}$ for $i = 1, \ldots, r+s$, the following holds:*

$$\int_{\Delta^r(t)} \mu_1(t_1)\cdots\mu_r(t_r) \int_{\Delta^s(t)} \mu_{r+1}(t_{r+1})\cdots\mu_{r+s}(t_{r+s}) \\ = \sum_{\sigma \in ⧢(r,s)} \int_{\Delta^{r+s}(t)} \mu_{\sigma^{-1}(1)}(t_1)\cdots\mu_{\sigma^{-1}(r+s)}(t_{r+s}).$$

PROOF. Using the decomposition

$$\Delta^r(t) \times \Delta^s(t) = \bigcup_{\sigma \in ⧢(r,s)} \left\{(t_1, \ldots, t_{r+s}) \mid t \geqslant t_{\sigma^{-1}(1)} \geqslant \cdots \geqslant t_{\sigma^{-1}(r+s)} \geqslant 0\right\},$$

together with the fact that the intersection of two simplices on the right-hand side is a set of measure zero, we obtain

$$\begin{aligned} \int_{\Delta^r(t)} \mu_1(t_1)\cdots\mu_r(t_r) &\int_{\Delta^s(t)} \mu_{r+1}(t_{r+1})\cdots\mu_{r+s}(t_{r+s}) \\ &= \int_{\Delta^r(t)\times\Delta^s(t)} \mu_1(t_1)\cdots\mu_{r+s}(t_{r+s}) \\ &= \sum_{\sigma \in ⧢(r,s)} \int_{t \geqslant t_{\sigma^{-1}(1)} \geqslant \cdots \geqslant t_{\sigma^{-1}(r+s)} \geqslant 0} \mu_1(t_1)\cdots\mu_{r+s}(t_{r+s}) \\ &= \sum_{\sigma \in ⧢(r,s)} \int_{\Delta^{r+s}(t)} \mu_{\sigma^{-1}(1)}(t_1)\cdots\mu_{\sigma^{-1}(r+s)}(t_{r+s}), \end{aligned}$$

where, in the last equality, we made the change of variables $t_i = t_{\sigma^{-1}(i)}$ to write the set $t \geqslant t_{\sigma^{-1}(1)} \geqslant \cdots \geqslant t_{\sigma^{-1}(r+s)} \geqslant 0$ as $\Delta^{r+s}(t)$. □

1.5.4. *Multi-indices and binary sequences.* To exploit Proposition 1.131 to derive relations among polylogarithms, and in particular among multiple zeta values, we need a new piece of notation.

DEFINITION 1.132. A *binary sequence* is an element $\alpha \in \{0,1\}^k$. We say that α has *weight* k, and *length* the number of $1s$ in the sequence. A sequence is called *positive* if it ends in 1, and *admissible* if it ends in 1 and starts with 0.

We will use the following notation to go from multi-indices to binary sequences and the other way around.

NOTATION 1.133. To each positive multi-index $\boldsymbol{s} = (s_1, \ldots, s_\ell)$ we attach the positive binary sequence

$$\mathrm{bs}(\boldsymbol{s}) = (0^{\{s_1-1\}}, 1, 0^{\{s_2-1\}}, \ldots, 0^{\{s_\ell-1\}}, 1),$$

where $0^{\{s\}}$ means that the entry zero is repeated s times. By convention, the empty binary sequence is admissible of weight and length both equal to zero. Clearly, bs defines a bijection between the set of positive multi-indices and the set of positive binary sequences which respects the weight and the length. Moreover, it restricts to a bijection between the subsets of admissible objects on both sides.

If $\alpha = (\varepsilon_1, \dots, \varepsilon_r)$ is a binary sequence, then we will set

$$\omega_\alpha = \omega_{\varepsilon_1} \dots \omega_{\varepsilon_r}.$$

In particular, if $\boldsymbol{s}$ is a positive multi-index, then

$$\omega_{\boldsymbol{s}} = \omega_{\mathrm{bs}(\boldsymbol{s})}.$$

Moreover, if α is positive, then we set $\mathrm{Li}_\alpha(t) = \mathrm{Li}_{\mathrm{bs}^{-1}(\alpha)}(t)$, and if it is also admissible, then we write $\zeta(\alpha) = \zeta(\mathrm{bs}^{-1}(\alpha))$.

1.5.5. *The shuffle multiplicities.*

Definition 1.134. Let $\alpha = (\varepsilon_1, \dots, \varepsilon_r)$, $\alpha' = (\varepsilon_{r+1}, \dots, \varepsilon_{r+s})$ and α'' be binary sequences of lengths r, s and t respectively. The *shuffle multiplicity* $\sqcup\!\sqcup(\alpha, \alpha'; \alpha'')$ is the number of shuffles of type (r, s) that transform $\alpha\alpha'$ into α''. That is,

$$\sqcup\!\sqcup(\alpha, \alpha'; \alpha'') = |\{\sigma \in \sqcup\!\sqcup(r, s) \mid \alpha'' = (\varepsilon_{\sigma^{-1}(1)}, \dots, \varepsilon_{\sigma^{-1}(r+s)})\}|.$$

Clearly, $\sqcup\!\sqcup(\alpha, \alpha'; \alpha'') = 0$ unless $t = r + s$.

The next result is the analogue of Lemma 1.38 for the shuffle multiplicity; it follows directly from the definition as well.

Lemma 1.135. *Let α, α', and α'' be binary sequences satisfying the condition $\sqcup\!\sqcup(\alpha, \alpha'; \alpha'') > 0$. Then the following holds:*

i) $\mathrm{wt}(\alpha'') = \mathrm{wt}(\alpha) + \mathrm{wt}(\alpha)$;
ii) $\ell(\alpha'') = \ell(\alpha) + \ell(\alpha')$;
iii) *if both α and α' are positive (resp. admissible), then so is α''.*

With this notation, Proposition 1.131 translates into the following result, which is the analogue of Lemma 1.39 for the shuffle product.

Lemma 1.136. *Let α and α' be positive binary sequences. Then*

$$\mathrm{Li}_\alpha(t)\,\mathrm{Li}_{\alpha'}(t) = \sum_{\alpha''} \sqcup\!\sqcup(\alpha, \alpha'; \alpha'')\,\mathrm{Li}_{\alpha''}(t).$$

Moreover, if α and α' are admissible, then

$$\zeta(\alpha) \cdot \zeta(\alpha') = \sum_{\alpha''} \sqcup\!\sqcup(\alpha, \alpha'; \alpha'')\zeta(\alpha'').$$

1.5.6. *An involution.* Another useful identity comes from exploiting the symmetry $t \mapsto 1 - t$ in the integral representation of multiple zeta values.

Proposition 1.137. *Let $\alpha = (\varepsilon_1, \dots, \varepsilon_r)$ be an admissible binary sequence. Write $\tilde{\omega}_0 = \omega_1$ and $\tilde{\omega}_1 = \omega_0$. Then the following holds:*

$$\int\limits_{1 \geqslant t_1 \geqslant \cdots \geqslant t_r \geqslant 0} \omega_{\varepsilon_1}(t_1) \cdots \omega_{\varepsilon_r}(t_r) = \int\limits_{1 \geqslant t_1 \geqslant \cdots \geqslant t_r \geqslant 0} \tilde{\omega}_{\varepsilon_r}(t_1) \cdots \tilde{\omega}_{\varepsilon_1}(t_r).$$

Proof. The change of variables $s_i = 1 - t_i$ transforms the measure $\omega_0(t_i)$ into $\omega_1(s_i) = \tilde{\omega}_0(s_i)$, and the measure $\omega_1(t_i)$ into $\omega_0(s_i) = \tilde{\omega}_1(s_i)$. Hence,

$$\int\limits_{1\geqslant t_1\geqslant\cdots\geqslant t_r\geqslant 0} \omega_{\varepsilon_1}(t_1)\cdots\omega_{\varepsilon_r}(t_r) = \int\limits_{0\leqslant s_1\leqslant\cdots\leqslant s_r\leqslant 1} \widetilde{\omega}_{\varepsilon_1}(s_1)\cdots\widetilde{\omega}_{\varepsilon_r}(s_r).$$

The statement follows by setting $s_i = t_{r-i}$ on the right-hand side. □

Definition 1.138. For a binary sequence $\alpha = (\varepsilon_1, \ldots, \varepsilon_r)$, we write

$$\tau(\alpha) = (1 - \varepsilon_r, \ldots, 1 - \varepsilon_1).$$

(Note the reverse order.) If α is admissible, then so is $\tau(\alpha)$.

From Proposition 1.137 and Theorem 1.116, we derive the following:

Corollary 1.139. *If α is an admissible binary sequence, then*

$$\zeta(\alpha) = \zeta(\tau(\alpha)).$$

Example 1.140. In weight 4, this corollary gives the relation

$$\zeta(4) = \zeta((0,0,0,1)) = \zeta((0,1,1,1)) = \zeta(2,1,1),$$

which was stated without proof in Section 1.4.1.

⋆ ⋆ ⋆

Exercise 1.141. Justify the exchange of the integral and the summation sign in the computations of Examples 1.110 and 1.112.

Exercise 1.142. Prove that the vector ${}^t(1\ \mathrm{Li}_1(t)\ \cdots\ \mathrm{Li}_n(t))$ of polylogarithm functions is a solution of the linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} 0 & & & & & \\ \frac{\mathrm{d}t}{1-t} & 0 & & & & \\ & \frac{\mathrm{d}t}{t} & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & \frac{\mathrm{d}t}{t} & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}.$$

Exercise 1.143. Show that there are $\binom{r+s}{r}$ shuffles of type (r, s).

Exercise 1.144. Manipulating series, prove directly the equality

$$\zeta(3) = \int\limits_{1\geqslant t_1\geqslant t_2\geqslant t_3\geqslant 0} \frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\frac{\mathrm{d}t_3}{1-t_3}$$

and, more generally,

$$\zeta(s) = \int\limits_{1\geqslant t_1\geqslant t_2\geqslant\cdots\geqslant t_s\geqslant 0} \frac{\mathrm{d}t_1}{t_1}\cdots\frac{\mathrm{d}t_{s-1}}{t_{s-1}}\frac{\mathrm{d}t_s}{1-t_s}.$$

Exercise 1.145. Use Lemma 1.136 to check the shuffle relation (1.52) for the product $\zeta(j)\zeta(k-j)$.

Exercise 1.146. Find a formula for $\zeta(s)\zeta(p,q)$ using shuffles.

EXERCISE 1.147. Check the identities

$$\zeta(5) = \zeta(2,1,1,1), \qquad \zeta(4,1) = \zeta(3,1,1),$$
$$\zeta(2,1,2) = \zeta(2,3), \qquad \zeta(2,2,1) = \zeta(3,2)$$

with the help of Proposition 1.137.

1.6. Quasi-shuffle products and the Hoffman algebra. In the previous sections, we saw two methods to express a product of multiple zeta values as a linear combination of multiple zeta values. The first one, using the series representation, gives the product in terms of the stuffle multiplicities. The second one, using the integral representation, gives the product in terms of the shuffle multiplicities. As we saw in Examples 1.16 and 1.127, these two methods may give different linear combinations for the same product of multiple zeta values, thus leading to linear relations among them. The stuffle multiplicities are easily expressed in terms of multi-indices as in Lemma 1.39, while the shuffle multiplicities are expressed more conveniently using binary sequences as in Lemma 1.136. We now want to put a little order to make the combinatorial structure of multiple zeta values clearer. To this end, we will define the *stuffle product* and the *shuffle product* as products in certain formal algebras that encode the stuffle and the shuffle multiplicities respectively.

1.6.1. *Alphabets and the quasi-shuffle product.*

NOTATION 1.148. Let $A = \{a_i\}_{i\in S}$ be a countable (possibly finite) set. The elements a_i of A will be called *letters*, and A is called an *alphabet*. Let $\mathbb{Q}A$ be the $\mathbb{Q}$-vector space with A as a basis. Let $\mathbb{Q}\langle A\rangle$ be the non-commutative polynomial algebra over A, that is,

$$\mathbb{Q}\langle A\rangle = \langle a_{i_1}a_{i_2}\cdots a_{i_n} \,|\, n \geqslant 0, i_j \in S\rangle_{\mathbb{Q}}$$

is the vector space with the set of *words* in the letters of A as a basis, along with the concatenation product

$$(a_{i_1}\cdots a_{i_n})\cdot(a_{j_1}\cdots a_{j_m}) = a_{i_1}\cdots a_{i_n}a_{j_1}\cdots a_{j_m}.$$

We say that a word $w = a_1\cdots a_n$ has *length* $\ell(w) = n$. We consider 1 as the empty word and set $\ell(1) = 0$.

DEFINITION 1.149. Let A be an alphabet, and let

$$\lozenge\colon \mathbb{Q}A \times \mathbb{Q}A \longrightarrow \mathbb{Q}A$$

be a commutative and associative product. We recursively define a new product $*_\lozenge$ on $\mathbb{Q}\langle A\rangle$ by setting $1 *_\lozenge w = w *_\lozenge 1 = w$ and

$$aw *_\lozenge bv = a(w *_\lozenge bv) + b(aw *_\lozenge v) + (a\lozenge b)(w *_\lozenge v),$$

for letters $a, b \in A$ and words $w, v \in \mathbb{Q}\langle A\rangle$. This product is extended to $\mathbb{Q}\langle A\rangle$ by $\mathbb{Q}$-linearity and is called the *quasi-shuffle product* associated with $\lozenge$.

THEOREM 1.150 (Hoffman [**Hof00**]). *The vector space $\mathbb{Q}\langle A\rangle$ equipped with the product $*_\lozenge$ is a commutative $\mathbb{Q}$-algebra.*

PROOF. Let us check the commutativity

$$u_1 *_\lozenge u_2 = u_2 *_\lozenge u_1 \tag{1.151}$$

by induction on the sum of lengths $\ell(u_1) + \ell(u_2)$. If either u_1 or u_2 is the empty word, then (1.151) holds trivially. It thus suffices to consider the case $u_1 = aw$

and $u_2 = bv$ with letters $a, b \in A$ and words $w, v \in \mathbb{Q}\langle A\rangle$. Then, by definition of the product $*_\Diamond$ and the induction hypothesis, we get

$$u_1 *_\Diamond u_2 - u_2 *_\Diamond u_1 = (a\Diamond b)(w *_\Diamond v) - (b\Diamond a)(v *_\Diamond w).$$

Since $\Diamond$ is assumed to be commutative, (1.151) follows from induction. The proof of the associativity is similar and is left as an exercise. □

We will next give two examples of quasi-shuffle products.

1.6.2. *Stuffle product.* Let Y be the alphabet with letters $y_1, y_2, y_3, \dots$, together with the product

$$\Diamond_1 \colon \mathbb{Q}Y \times \mathbb{Q}Y \longrightarrow \mathbb{Q}Y, \qquad y_i \Diamond_1 y_j = y_{i+j},$$

which is commutative and associative. The product $*_{\Diamond_1}$ on $\mathbb{Q}\langle Y\rangle$ will be denoted by $*$ and called the *stuffle product*. By definition, it is given by

$$y_i w * y_j v = y_i(w * y_j v) + y_j(y_i w * v) + y_{i+j}(w * v). \tag{1.152}$$

EXAMPLE 1.153. We have $y_i * y_j = y_i y_j + y_j y_i + y_{i+j}$ and

$$\begin{aligned} y_2 * y_3 y_4 &= y_2(y_3 y_4) + y_3(y_2 * y_4) + y_5(y_4) \\ &= y_2 y_3 y_4 + y_3(y_2 y_4 + y_4 y_2 + y_6) + y_5 y_4 \\ &= y_2 y_3 y_4 + y_3 y_2 y_4 + y_3 y_4 y_2 + y_3 y_6 + y_5 y_4. \end{aligned}$$

NOTATION 1.154. A positive multi-index $\boldsymbol{s} = (s_1, \dots, s_\ell)$ defines a word

$$y_{\boldsymbol{s}} = y_{s_1} \cdots y_{s_\ell}.$$

In fact, the set of positive multi-indices and the set of words in the alphabet Y are in bijection. We will use this bijection to identify both sets.

LEMMA 1.155. *The stuffle product is given by*

$$y_{\boldsymbol{s}} * y_{\boldsymbol{s}'} = \sum_{\boldsymbol{s}''} \operatorname{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'') y_{\boldsymbol{s}''}.$$

PROOF. The proof proceeds by induction on the length of the multi-indices $\boldsymbol{s}$ and $\boldsymbol{s}'$. If one of them, say $\boldsymbol{s}$, has length zero, then both sides of the equality to be proven are $y_{\boldsymbol{s}'}$, so it is true. Assume then that both $\boldsymbol{s}$ and $\boldsymbol{s}'$ have length $\geqslant 1$, and write $\boldsymbol{s} = (s_1, \dots)$ and $\boldsymbol{s}' = (s'_1, \dots)$, so that the associated words are of the form $y_{\boldsymbol{s}} = y_{s_1} v$ and $y_{\boldsymbol{s}'} = y_{s'_1} w$. Equation (1.152) yields

$$y_{\boldsymbol{s}} * y_{\boldsymbol{s}'} = y_{s_1}(v * y_{\boldsymbol{s}'}) + y_{s'_1}(y_{\boldsymbol{s}} * w) + y_{s_1 + s'_1}(v * w).$$

The matrices that are used to compute the stuffle indices $\operatorname{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'')$ in Definition 1.36 fall into three types, namely

$$\begin{pmatrix} s_1 & \cdots \\ 0 & \cdots \end{pmatrix}, \qquad \begin{pmatrix} 0 & \cdots \\ s'_1 & \cdots \end{pmatrix}, \qquad \begin{pmatrix} s_1 & \cdots \\ s'_1 & \cdots \end{pmatrix}.$$

Using the induction hypothesis, one sees that the matrices of the first type give rise to the term $y_{s_1}(v * y_{\boldsymbol{s}'})$, the matrices of the second type to the term $y_{s'_1}(y_{\boldsymbol{s}} * w)$, and the matrices of the third type to the term $y_{s_1+s'_1}(v * w)$. □

Since the words of the alphabet Y are related to multi-indices and the product of $\mathbb{Q}\langle Y\rangle$ is the stuffle product, one may expect a morphism of $\mathbb{Q}$-algebras

$$\begin{aligned}(\mathbb{Q}\langle Y\rangle, *) &\longrightarrow \quad (\mathcal{Z}, \cdot)\\ y_{s_1}\cdots y_{s_\ell} &\longmapsto \zeta(s_1,\dots,s_\ell).\end{aligned}$$

However, since multiple zeta values are only defined when $s_1 > 1$, we need to restrict the source of this map. Later, in Section 1.7 we will see how to extend it to the whole $(\mathbb{Q}\langle Y\rangle, *)$ by means of a regularization process.

Definition 1.156. A word $w = y_{s_1}\dots y_{s_\ell}$ is called *admissible* if $s_1 > 1$, *i.e.* if it corresponds to an admissible multi-index. We will denote by $\mathbb{Q}\langle Y\rangle^0$ the vector subspace of $\mathbb{Q}\langle Y\rangle$ spanned by admissible words.

Proposition 1.157.

i) *$(\mathbb{Q}\langle Y\rangle^0, *)$ is a subalgebra of $(\mathbb{Q}\langle Y\rangle, *)$.*

ii) *There is a morphism of $\mathbb{Q}$-algebras*

$$\mathbb{Q}\langle Y\rangle^0 \longrightarrow \mathcal{Z}$$

determined by the assignment

$$y_{s_1}\cdots y_{s_l} \longmapsto \zeta(s_1,\dots,s_l).$$

Proof. The first statement can be checked directly from the definition of the product $*$. Alternatively, it follows from Lemma 1.155 and Lemma 1.38 iii). The second statement follows from Lemmas 1.155 and 1.39. $\square$

Since we have identified positive multi-indices with words in the alphabet Y, we will often just write $\zeta(w)$ instead of $\zeta(s_1,\dots,s_l)$ for $w = y_{s_1}\dots y_{s_l}$. With this notation, the equality

$$\zeta(w * v) = \zeta(w)\zeta(v) \tag{1.158}$$

holds for all words $w, v \in \mathbb{Q}\langle Y\rangle^0$.

1.6.3. *Shuffle product.* Let $X = \{x_0, x_1\}$ be the alphabet in two letters, along with the trivial product $a\Diamond_2 b = 0$ for all $a, b \in X$. We denote by $\sqcup\!\sqcup$ the corresponding product $*_{\Diamond_2}$ and call it the *shuffle product*. By definition, it is given by

$$x_i w \sqcup\!\sqcup x_j v = x_i(w \sqcup\!\sqcup x_j v) + x_j(x_i w \sqcup\!\sqcup v).$$

Definition 1.159. We call $\mathfrak{H} = (\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup)$ the *Hoffman algebra.*

Proposition 1.160. *Given words $x_{\varepsilon_1}\dots x_{\varepsilon_r}$ and $x_{\varepsilon_{r+1}}\dots x_{\varepsilon_{r+s}}$ on the alphabet X, their shuffle product is equal to*

$$x_{\varepsilon_1}\dots x_{\varepsilon_r} \sqcup\!\sqcup x_{\varepsilon_{r+1}}\dots x_{\varepsilon_{r+s}} = \sum_{\sigma\in \sqcup\!\sqcup(r,s)} x_{\varepsilon_{\sigma^{-1}(1)}}\dots x_{\varepsilon_{\sigma^{-1}(p+q)}}.$$

Proof. Exercise 1.172. $\square$

Example 1.161. Here are two examples of shuffle products of words:

$$\begin{aligned}x_0x_1 \sqcup\!\sqcup x_0x_1 &= 2x_0x_1x_0x_1 + 4x_0^2x_1^2,\\ x_0x_1 \sqcup\!\sqcup x_0^2x_1 &= x_0x_1x_0^2x_1 + 3x_0^2x_1x_0x_1 + 6x_0^3x_1^2.\end{aligned}$$

NOTATION 1.162. There is an obvious bijection between binary sequences and words in the alphabet X that with a binary sequence $\alpha = (\varepsilon_1, \dots, \varepsilon_r)$ associates the word $x_\alpha = x_{\varepsilon_1} \dots x_{\varepsilon_r}$. Using this bijection, we can transfer the shuffle multiplicity, as introduced in Definition 1.134, to words in the alphabet X. The resulting multiplicity will be denoted by $\sqcup\!\sqcup(u, v; w)$.

With this notation, Proposition 1.160 can be rewritten as

$$u \sqcup\!\sqcup v = \sum_w \sqcup\!\sqcup(u, v; w) w. \tag{1.163}$$

Again, this equation hints at the existence of an algebra morphism from $\mathfrak{H}$ to the algebra of multiple zeta values. As was the case for the alphabet Y, one needs to restrict the source of this map to the subspace that gives rise to convergent series.

DEFINITION 1.164. A word in the alphabet X is said to be *positive* if it ends in x_1, and is said to be *admissible* if it ends in x_1 and starts with x_0.

PROPOSITION 1.165. *Let $\mathfrak{H}^1$ (resp. $\mathfrak{H}^0$) be the subspace of $\mathfrak{H}$ generated by positive (resp. admissible) words, so that there are inclusions*

$$\mathfrak{H} \supset \mathfrak{H}^1 \supset \mathfrak{H}^0.$$

Then the following properties hold:

i) $(\mathfrak{H}^0, \sqcup\!\sqcup)$ *and* $(\mathfrak{H}^1, \sqcup\!\sqcup)$ *are subalgebras of* $(\mathfrak{H}, \sqcup\!\sqcup)$.

ii) *There is a morphism of $\mathbb{Q}$-algebras*

$$\zeta\colon \mathfrak{H}^0 \longrightarrow \mathcal{Z}$$

given by the assignment

$$x_\alpha \longmapsto \zeta(\alpha).$$

(Recall that the multiple zeta value corresponding to an admissible binary sequence was defined as $\zeta(\mathrm{bs}^{-1}(\alpha))$ in Notation 1.133).

PROOF. Exercise 1.173. □

Since we are identifying binary sequences and words in the alphabet X, we will often write $\zeta(x_\alpha)$ instead of $\zeta(\alpha)$. With this notation, Proposition 1.165 becomes

$$\zeta(w \sqcup\!\sqcup v) = \zeta(w)\zeta(v) \tag{1.166}$$

for all words $w, v \in \mathfrak{H}^0$.

1.6.4. *Double shuffle relations.* In the same way that positive multi-indices can be translated into binary sequences, there is a natural map between $\mathbb{Q}\langle Y\rangle$ and $\mathfrak{H}$. This map does not transform the stuffle product on $\mathbb{Q}\langle Y\rangle$ into the shuffle product on $\mathfrak{H}$. To remedy this, we define a second product on $\mathfrak{H}$.

DEFINITION 1.167. Setting $z_p = x_0^{p-1}x_1$, the *stuffle product*

$$*\colon \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathfrak{H}$$

is recursively defined as follows:

$$\begin{aligned}
1 * w &= w * 1 = w && \text{for } w \in \mathfrak{H},\\
x_0^p * w &= w * x_0^p = w x_0^p && \text{for } p \geqslant 1 \text{ and } w \in \mathfrak{H},\\
z_p w * z_q v &= z_p(w * z_q v) + z_q(z_p w * v) + z_{p+q}(w * v) && \text{for } w, v \in \mathfrak{H}.
\end{aligned}$$

PROPOSITION 1.168.

i) $(\mathfrak{H}, *)$ *is a commutative and associative* $\mathbb{Q}$*-algebra.*

ii) *The map*

$$\begin{aligned} (\mathbb{Q}\langle Y\rangle, *) &\hookrightarrow (\mathfrak{H}, *) \\ y_i &\longmapsto z_i = x_0^{i-1}x_1 \end{aligned}$$

is an injective morphism of algebras with image the subalgebra $\mathfrak{H}^1$.

PROOF. Exercise 1.174. □

We will use this map to identify the algebras $\mathfrak{H}^1$ and $\mathbb{Q}\langle Y\rangle$.

THEOREM 1.169. *Let* $\zeta\colon \mathfrak{H}^0 \to \mathcal{Z}$ *be the map from Proposition 1.165. Then,*

$$\zeta(w \sqcup\!\sqcup v - w * v) = 0.$$

PROOF. This follows from equations (1.158) and (1.166). □

This theorem is a source of relations among multiple zeta values that are called *double shuffle relations*. Nevertheless, it is clear that double shuffle relations are not enough to describe all relations among multiple zeta values. For instance, we do not obtain Euler's relation in weight 3, and we can only produce one relation in weight 4, while there are at least 3 independent relations. To obtain the remaining relations, we will consider products with non-admissible words in the next section.

⋆⋆⋆

EXERCISE 1.170. Prove the identity

$$x_0^r \sqcup\!\sqcup x_0^s = x_0 \dots x_0 \sqcup\!\sqcup x_0 \dots x_0 = \frac{(r+s)!}{r!s!} x_0^{r+s}.$$

EXERCISE 1.171. Prove the identity

$$a \sqcup\!\sqcup wv = (a \sqcup\!\sqcup w)v + w(a \sqcup\!\sqcup v) - wav$$

for a letter a and words w and v in the alphabet $X = \{x_0, x_1\}$.

EXERCISE 1.172. Prove Proposition 1.160.

EXERCISE 1.173. Prove Proposition 1.165.

EXERCISE 1.174. Prove Proposition 1.168.

EXERCISE 1.175. Given a multi-index $\boldsymbol{s}$ and an integer $M \geqslant 0$, we set

$$\zeta_M(\boldsymbol{s}) = \sum_{M > m_1 > m_2 > \dots > m_\ell > 0} \frac{1}{m_1^{s_1} \dots m_l^{s_\ell}}.$$

i) Show that, if $\boldsymbol{s}$ is admissible, then $\lim_{M\to\infty} \zeta_M(\boldsymbol{s}) = \zeta(\boldsymbol{s})$.

ii) Recall that we identified words and multi-indices. Prove that the map

$$\zeta_M\colon (\mathfrak{H}^1, *) \longrightarrow \mathbb{Q}$$

is a morphism of algebras, *i.e.* for all $w, v \in \mathfrak{H}^1$ it satisfies

$$\zeta_M(w * v) = \zeta_M(w)\zeta_M(v).$$

EXERCISE 1.176. Using the identification between words in the alphabet X and binary sequences, we obtain a map

$$\mathrm{Li}\colon \mathfrak{H}^1 \longrightarrow \mathcal{C}^\infty((0,1))$$

that sends a word $w \in \mathfrak{H}^1$ to the polylogarithm function $\mathrm{Li}_w(t)$ from Definition 1.118. Prove that this map satisfies

$$\mathrm{Li}_{w \sqcup\!\sqcup v}(t) = \mathrm{Li}_w(t) \cdot \mathrm{Li}_v(t)$$

for all $w, v \in \mathfrak{H}^1$, and hence is a morphism of algebras

$$\mathrm{Li}\colon (\mathfrak{H}^1, \sqcup\!\sqcup) \longrightarrow (\mathcal{C}^\infty((0,1)), \cdot).$$

1.7. Regularization and the Ihara–Kaneko–Zagier theorem. In this section, we discuss two regularization processes that can be used to extend the definition of multiple zeta values to non-admissible words and derive relations among them. Conjecturally, all relations can be obtained in this way. The exposition follows the work of Ihara–Kaneko–Zagier [**IKZ06**].

1.7.1. *The stuffle algebra as a polynomial ring.* Recall from Definition 1.156 that $\mathbb{Q}\langle Y\rangle^0$ stands for the set of admissible words in the alphabet Y.

THEOREM 1.177. *The map of* $(\mathbb{Q}\langle Y\rangle^0, *)$*-algebras*

$$\begin{aligned} \varphi\colon\ \mathbb{Q}\langle Y\rangle^0[T] &\longrightarrow \mathbb{Q}\langle Y\rangle \\ T\quad &\longmapsto\ y_1 \end{aligned}$$

is an isomorphism.

PROOF. We first show that the map φ is surjective, which amounts to proving that any element $w \in \mathbb{Q}\langle Y\rangle$ can be written as a polynomial in y_1 with coefficients in $\mathbb{Q}\langle Y\rangle^0$. The bijection between the sets of multi-indices and words in the alphabet Y induces a grading by the weight wt and a filtration by the length ℓ on the space $\mathbb{Q}\langle Y\rangle$ given by

$$\begin{aligned} \mathrm{wt}(y_{s_1}\cdots y_{s_\ell}) &= s_1 + \cdots + s_\ell, \\ \ell(y_{s_1}\cdots y_{s_\ell}) &= \ell. \end{aligned}$$

The surjectivity will follow by induction on ℓ, once we show that for each word $w \in F_\ell\mathbb{Q}\langle Y\rangle$ of length ℓ, there exist $v_1 \in F_\ell\mathbb{Q}\langle Y\rangle^0$ and $v_2, v_3 \in F_{\ell-1}\mathbb{Q}\langle Y\rangle$ satisfying

$$w = v_1 + v_2 * y_1 + v_3. \tag{1.178}$$

First observe that any word of length ℓ can be written as

$$w = \underbrace{y_1\cdots y_1}_{m} y_{s_1}\cdots y_{s_{\ell-m}} = \{y_1\}^m y_{s_1}\cdots y_{s_{\ell-m}}$$

with $s_1 \neq 1$ and $m \geqslant 0$. Using this, we will prove by induction on m, that w can be written as in (1.178). For $m = 0$, we have $w \in \mathbb{Q}\langle Y\rangle^0$. Thus, we can choose $v_1 = w$ and $v_2 = v_3 = 0$. For the induction step, we compute

$$\{y_1\}^{m-1}y_{s_1}\cdots y_{s_{\ell-m}} * y_1 = m\cdot w + \sum_{i=1}^{\ell-m}\{y_1\}^{m-1}y_{s_1}\cdots y_{s_i}y_1y_{s_{i+1}}\cdots y_{s_{\ell-m}} - v_3$$

with $v_3 \in F_{\ell-1}\mathbb{Q}\langle Y\rangle$. Applying the induction hypothesis with respect to m, we deduce that w can be written as in (1.178). It follows that φ is surjective.

To prove the injectivity of φ, we write each non-zero $P \in \mathbb{Q}\langle Y\rangle^0[T]$ as

$$P = w_1 T^m + w_2$$

for non-zero $w_1 \in \mathbb{Q}\langle Y\rangle^0$ and w_2 of degree less than m in the variable T. Then

$$\varphi(P) = m! y_1^m w_1 + v_2,$$

where all the words in v_2 have less than m factors y_1 in the front. Thus, $\varphi(P) \neq 0$ and the map φ is injective as well. □

1.7.2. *The shuffle algebra as a polynomial ring.* Mutatis mutandis, one can prove the analogous result for the shuffle product. Recall the algebras $\mathfrak{H} \supset \mathfrak{H}^1 \supset \mathfrak{H}^0$ from Proposition 1.165.

THEOREM 1.179.

i) *The map of* $(\mathfrak{H}^0, ⧢)$*-algebras*

$$\begin{aligned} \psi_1\colon\ \mathfrak{H}^0[T] &\longrightarrow \mathfrak{H}^1 \\ T &\longmapsto x_1 \end{aligned}$$

is an isomorphism.

ii) *The map of* $(\mathfrak{H}^1, ⧢)$*-algebras*

$$\begin{aligned} \psi_2\colon\ \mathfrak{H}^1[U] &\longrightarrow \mathfrak{H} \\ U &\longmapsto x_0 \end{aligned}$$

is an isomorphism.

Therefore, the map of $(\mathfrak{H}^0, ⧢)$*-algebras*

$$\begin{aligned} \psi\colon\ \mathfrak{H}^0[T,U] &\longrightarrow \mathfrak{H} \\ T &\longmapsto x_1 \\ U &\longmapsto x_0 \end{aligned}$$

is an isomorphism.

PROOF. Exercise 1.214. □

1.7.3. *Regularized zeta values.* Using the previous theorems, we define the stuffle and shuffle regularization maps. Recall the identification $\mathfrak{H}^1 = \mathbb{Q}\langle Y\rangle$ from Proposition 1.168, and the isomorphism φ from Theorem 1.177.

DEFINITION 1.180. The *stuffle regularization map*

$$\mathrm{reg}_*^T\colon\ \mathfrak{H}^1 = \mathbb{Q}\langle Y\rangle \ \longrightarrow\ \mathfrak{H}^0[T] = \mathbb{Q}\langle Y\rangle^0[T]$$

is defined as $\mathrm{reg}_*^T = \varphi^{-1}$. The *shuffle regularization maps*

$$\mathrm{reg}_{⧢}^T\colon \mathfrak{H}^1 \longrightarrow \mathfrak{H}^0[T], \qquad \text{and} \qquad \mathrm{reg}_{⧢}^{T,U}\colon \mathfrak{H} \longrightarrow \mathfrak{H}^0[T,U]$$

are defined as $\mathrm{reg}_{⧢}^T = \psi_1^{-1}$ and $\mathrm{reg}_{⧢}^{T,U} = \psi^{-1}$.

Theorems 1.177 and 1.179 allow us to extend the function ζ from Proposition 1.165 in a formal way.

DEFINITION 1.181. The *stuffle regularized* zeta map is the composition

$$\zeta_*^T \colon \mathbb{Q}\langle Y\rangle \xrightarrow{\mathrm{reg}_*^T} \mathbb{Q}\langle Y\rangle^0[T] \xrightarrow{\zeta} \mathcal{Z}[T] \subset \mathbb{R}[T].$$

We denote by ζ_* the composition of ζ_*^T with the evaluation at $T=0$.

The *shuffle regularized* zeta map, denoted by $\zeta_ш^T$, is the composition

$$\mathfrak{H}^1 \xrightarrow{\mathrm{reg}_ш^T} \mathfrak{H}^0[T] \xrightarrow{\zeta} \mathcal{Z}[T] \subset \mathbb{R}[T].$$

Similarly, we write $\zeta_ш^{T,U}$ for the composition

$$\mathfrak{H} \xrightarrow{\mathrm{reg}_ш^{T,U}} \mathfrak{H}^0[T,U] \xrightarrow{\zeta} \mathcal{Z}[T,U] \subset \mathbb{R}[T,U].$$

We denote by $\zeta_ш$ the composition of $\zeta_ш^{T,U}$ with the evaluation at $T=U=0$. We will also denote by $\zeta_ш$ its restriction to $\mathfrak{H}^1$.

By identifying $(\mathbb{Q}\langle Y\rangle, *)$ with $(\mathfrak{H}^1, *)$, we will also consider ζ_*^T as the linear map from $(\mathfrak{H}^1, *)$ to $\mathbb{R}[T]$ characterized by the conditions

$$\begin{aligned} \zeta_*^T(w) &= \zeta(w) \in \mathbb{R}, \qquad \text{if } w \in \mathfrak{H}^0,\\ \zeta_*^T(x_1) &= T,\\ \zeta_*^T(v * w) &= \zeta_*^T(v)\zeta_*^T(w). \end{aligned}$$

In the same way, the map $\zeta_ш^T$ is characterized by linearity and the identities

$$\begin{aligned} \zeta_ш^T(w) &= \zeta(w) \in \mathbb{R}, \qquad \text{if } w \in \mathfrak{H}^0,\\ \zeta_ш^T(x_1) &= T,\\ \zeta_ш^T(v ш w) &= \zeta_ш^T(v)\zeta_ш^T(w). \end{aligned}$$

The maps ζ_*, $\zeta_ш$ and $\zeta_ш^{T,U}$ are determined by similar conditions. For future reference, we single out the properties characterizing $\zeta_ш$.

PROPOSITION 1.182. *The map $\zeta_ш \colon \mathfrak{H} \to \mathbb{R}$ is the only linear map satisfying*

$$\zeta_ш(w) = \zeta(w) \in \mathbb{R}, \qquad \text{if } w \in \mathfrak{H}^0, \tag{1.183}$$

$$\zeta_ш(x_0) = 0, \ \zeta_ш(x_1) = 0, \tag{1.184}$$

$$\zeta_ш(v ш w) = \zeta_ш(v)\zeta_ш(w). \tag{1.185}$$

COROLLARY 1.186. *The image of $\zeta_ш$ is equal to $\mathcal{Z}$.*

PROOF. By Theorem 1.179, every element $w \in \mathfrak{H}$ can be written as a finite sum $w = \sum a_{ij} ш x_0^i ш x_1^j$ for some $a_{ij} \in \mathfrak{H}^0$. Thanks to Proposition 1.182, the equality $\zeta_ш(w) = \zeta_ш(a_{00})$ holds, and hence $\zeta_ш(w)$ belongs to $\mathcal{Z}$. □

EXAMPLE 1.187. On the one hand, we have

$$\begin{aligned} \zeta_*^T(1,2) &= \zeta_*^T(y_1y_2)\\ &= \zeta_*^T(y_2 * y_1 - y_2y_1 - y_3)\\ &= \zeta(2)T - \zeta(2,1) - \zeta(3), \end{aligned}$$

which yields $\zeta_*(1,2) = -\zeta(2,1) - \zeta(3)$. On the other hand,

$$\begin{aligned} \zeta_ш^T(1,2) &= \zeta_ш^T(x_1x_0x_1)\\ &= \zeta_ш^T(x_0x_1 ш x_1 - 2x_0x_1x_1)\\ &= \zeta(2)T - 2\zeta(2,1). \end{aligned}$$

Therefore, $\zeta_{\sqcup\!\sqcup}(1,2) = -2\zeta(2,1)$. These are two different ways to assign a finite value to the divergent series $\sum_{n>m\geqslant 1}\frac{1}{nm^2}$.

1.7.4. *Comparing the shuffle and the stuffle regularizations.* As we just saw in the previous example, the regularizations $\zeta^T_{\sqcup\!\sqcup}(w)$ and $\zeta^T_*(w)$ are in general different from each other. In order to compare them, we introduce the formal power series

$$A(u) = e^{\gamma u}\Gamma(1+u) = \exp\left(\sum_{n\geqslant 2}\frac{(-1)^n}{n}\zeta(n)u^n\right),$$

where γ is the Euler constant, and the second identity follows from the Taylor expansion of the logarithm of the gamma function (Exercise 1.18). We write

$$A(u) = \sum_{k\geqslant 0}\gamma_k u^k. \tag{1.188}$$

Observe that γ_k is a linear combination, with rational coefficients, of multiple zeta values of weight k. The first values are given in Table 1.5 below.

k	0	1	2	3	4	5
γ_k	1	0	$\frac{\zeta(2)}{2}$	$-\frac{\zeta(3)}{3}$	$\frac{\zeta(2,2)}{4}+\frac{3\zeta(4)}{8}$	$-\frac{\zeta(2,3)}{6}-\frac{\zeta(3,2)}{6}-\frac{11\zeta(5)}{30}$

TABLE 1.5. First values of γ_k

We define an $\mathbb{R}$-linear map $\varrho\colon \mathbb{R}[T]\longrightarrow\mathbb{R}[T]$ by the formula

$$\varrho(T^n) = \frac{d^n}{du^n}\left(A(u)e^{Tu}\right)\bigg|_{u=0} = n!\sum_{k=0}^{n}\gamma_k\frac{T^{n-k}}{(n-k)!}, \tag{1.189}$$

so that the equality $\varrho(e^{Tu}) = A(u)e^{Tu}$ holds when the map ρ is extended $\mathbb{R}$-linearly to formal power series.

THEOREM 1.190 (Ihara–Kaneko–Zagier [**IKZ06**]). *The identity*

$$\zeta^T_{\sqcup\!\sqcup}(w) = \varrho(\zeta^T_*(w))$$

holds for all words $w\in\mathfrak{H}^1$.

EXAMPLE 1.191. Since $\gamma_0 = 1$ and $\gamma_1 = 0$, we have $\varrho(1) = 1$ and $\varrho(T) = T$. Combining this with Example 1.187, we find

$$\begin{aligned}\varrho(\zeta^T_*(1,2)) &= \varrho(\zeta(2)T-\zeta(2,1)-\zeta(3))\\ &= \zeta(2)T-\zeta(2,1)-\zeta(3).\end{aligned}$$

Since, on the other hand, we get

$$\zeta^T_{\sqcup\!\sqcup}(1,2) = \zeta(2)T-2\zeta(2,1),$$

Theorem 1.190 recovers Euler's relation $\zeta(2,1)=\zeta(3)$ from Corollary 1.56.

PROOF OF THEOREM 1.190. The idea is to see the equality $\zeta^T_{\sqcup\!\sqcup}(w) = \varrho(\zeta^T_*(w))$ to be proved as an identity of functions of the variable T. Let $M>0$ be an integer, and let $w = y_{s_1}\cdots y_{s_\ell}$ be a word in the alphabet Y. We write

$$\zeta_M(w) = \sum_{M>m_1>m_2>\cdots>m_\ell>0}\frac{1}{m_1^{s_1}\cdots m_\ell^{s_\ell}}.$$

By Exercise 1.175, if w is admissible, then $\lim_{M\to\infty}\zeta_M(w)=\zeta(w)$. We extend ζ_M to a map $\mathbb{Q}\langle Y\rangle \to \mathbb{R}$ by linearity. Then ζ_M satisfies the stuffle relation

$$\zeta_M(w_1)\zeta_M(w_2)=\zeta_M(w_1 * w_2),$$

using Exercise 1.175 again. From the approximation of the harmonic series

$$\zeta_M(y_1)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{M-1}=\log M+\gamma+O\left(\frac{1}{M}\right)$$

and the representation of $\zeta_*^T(w)$ as a polynomial in $\zeta_*(1)$, it follows that there exists an integer $j\geqslant 0$ such that the estimate

$$\zeta_M(w)=\zeta_*^{\log M+\gamma}(w)+O(M^{-1}\log^j M) \tag{1.192}$$

holds for large enough M. Here, the notation $\zeta_*^{\log M+\gamma}(w)$ indicates the evaluation of $\zeta_*^T(w)$ at $T=\log M+\gamma$.

Recall from Definition 1.118 the polylogarithm function $\mathrm{Li}_{\boldsymbol{s}}$ associated with a positive multi-index $\boldsymbol{s}$. Using the identification of positive multi-indices with words in the alphabet Y and linearity, we attach a function Li_w on the segment $(0,1)$ to each element $w\in\mathfrak{H}^1$. If $w\in\mathfrak{H}^0$, then

$$\lim_{t\to 1^-}\mathrm{Li}_w(t)=\zeta(w).$$

Moreover, the equality

$$\mathrm{Li}_w(t)\cdot\mathrm{Li}_{w'}(t)=\mathrm{Li}_{w ⧢ w'}(t)$$

holds for all $w,w'\in\mathfrak{H}^1$ and $t\in(0,1)$ by Exercise 1.176. From the equality

$$\mathrm{Li}_{y_1}(t)=\log\left(\frac{1}{1-t}\right),$$

we see that there exists an integer $j\geqslant 0$ depending on w such that the estimate

$$\mathrm{Li}_w(t)=\zeta_{⧢}^{\log\left(\frac{1}{1-t}\right)}(w)+O\left((1-t)\log^j\left(\frac{1}{1-t}\right)\right) \tag{1.193}$$

holds as $t\to 1^-$. As above, the notation $\zeta_{⧢}^{\log\left(\frac{1}{1-t}\right)}(w)$ stands for the evaluation of $\zeta_{⧢}^T(w)$ at $T=\log\left(\frac{1}{1-t}\right)$. By explicit calculations,

$$\begin{aligned}
\mathrm{Li}_w(t) &= \sum_{m_1>m_2>\cdots>m_\ell>0}\frac{t^{m_1}}{m_1^{s_1}\cdots m_\ell^{s_\ell}}\\
&=\sum_{m\geqslant 1}\left(\sum_{m>m_2>\cdots>m_\ell>0}\frac{1}{m^{s_1}m_2^{s_2}\cdots m_\ell^{s_\ell}}\right)t^m\\
&=\sum_{m\geqslant 1}(\zeta_{m+1}(w)-\zeta_m(w))t^m\\
&=(1-t)\sum_{m\geqslant 2}\zeta_m(w)t^{m-1},
\end{aligned}$$

where the last equality uses the vanishing $\zeta_1(w)=0$ (an empty sum).

To go further, we apply Lemma 1.194 below to the polynomials $P(T) = \zeta_*^T(w)$ and $Q(T) = \varrho(\zeta_*^T(w))$. We derive the equalities

$$\begin{aligned}
\mathrm{Li}_w(t) &= (1-t)\sum_{m\geqslant 2}\zeta_m(w)t^{m-1} \\
&\overset{(1.192)}{=} (1-t)\sum_{m\geqslant 2}\zeta_*^{\log m+\gamma}(w)t^{m-1} + (1-t)\sum_{m\geqslant 1} O\left(\frac{\log^j m}{m}\right)t^{m-1} \\
&\overset{(1.195)}{=} Q\left(\log\frac{1}{1-t}\right) + O\left((1-t)\log^j\left(\frac{1}{1-t}\right)\right).
\end{aligned}$$

Comparing this expression for with $\mathrm{Li}_w(t)$ the asymptotic expansion (1.193), we get the identity $\zeta_{ш}^T(\boldsymbol{s}) = \varrho(\zeta_*^T(\boldsymbol{s}))$ that we wanted to prove. □

The next lemma was used in the proof of Theorem 1.190.

LEMMA 1.194.

i) *Let $P(T) \in \mathbb{R}[T]$ and $Q(T) = \varrho(P(T))$. Then there exists an integer $j \geqslant 0$ such that the estimate*

$$\sum_{m\geqslant 2} P(\log(m)+\gamma)t^{m-1} = \frac{1}{1-t}Q\left(\log\frac{1}{1-t}\right) + O\left(\log^j\left(\frac{1}{1-t}\right)\right) \tag{1.195}$$

holds as $t \to 1^-$.

ii) *As $t \to 1^-$, the following estimate holds:*

$$\sum_{m\geqslant 2}\frac{\log^j m}{m}t^{m-1} = O\left(\log^{j+1}\left(\frac{1}{1-t}\right)\right). \tag{1.196}$$

PROOF. Let us prove (1.196) first. Since

$$\sum_{m\geqslant 2}\frac{t^{m-1}}{m} = -1 - \frac{1}{t}\log(1-t),$$

for $j = 0$ the left-hand side of (1.196) is of type $O\left(\log\left(\frac{1}{1-t}\right)\right)$ as $t \to 1^-$, which proves the statement in this case. We proceed by induction on j. Note the identity

$$\log^{j+1}(m) \leqslant c_j\sum_{n=1}^{m}\frac{\log^j(n)}{n}$$

for $m \geqslant 1$ and $j \geqslant 0$, which easily follows from the integral

$$\int_1^m \frac{\log^j(x)}{x}\mathrm{d}x = \frac{\log^{j+1}(m)}{j+1}.$$

Hence, for $t<1$, we obtain

$$\begin{aligned}\sum_{m\geqslant 1}\frac{\log^{j+1}(m)}{m}t^{m-1} &\leqslant c_j\sum_{m\geqslant 1}\frac{t^{m-1}}{m}\sum_{n=1}^{m}\frac{\log^j(n)}{n}\\ &= c_j\sum_{n\geqslant 1}\frac{\log^j(n)}{n}t^{n-1}\sum_{r\geqslant 1}\frac{t^{r-1}}{r+n-1}\\ &< c_j\left(\sum_{n\geqslant 1}\frac{\log^j(n)}{n}t^{n-1}\right)\left(\frac{1}{t}\log\left(\frac{1}{1-t}\right)\right).\end{aligned}$$

From this, (1.196) follows by induction on j for all $j\geqslant 0$.

We now prove identity (1.195). By construction, ϱ is a linear map on $\mathbb{R}[T]$, so it suffices to do it for $P(T)=(T-\gamma)^n$. Thus, we set $Q(T)=\varrho((T-\gamma)^n)$. Then, from equation (1.189), we get

$$Q(T)=\frac{d^n}{du^n}\left(A(u)e^{(T-\gamma)u}\right)\bigg|_{u=0}=\frac{d^n}{du^n}\left(\Gamma(1+u)e^{Tu}\right)\bigg|_{u=0},$$

and hence the equalities

$$\begin{aligned}\frac{1}{1-t}Q\left(\log\left(\frac{1}{1-t}\right)\right) &= \frac{d^n}{du^n}\left(\frac{\Gamma(1+u)}{(1-t)^{1+u}}\right)\bigg|_{u=0}\\ &= \frac{d^n}{du^n}\left(\sum_{m\geqslant 1}\frac{\Gamma(m+u)}{\Gamma(m)}t^{m-1}\right)\Bigg|_{u=0}\\ &= \sum_{m\geqslant 1}\frac{\Gamma^{(n)}(m)}{\Gamma(m)}t^{m-1},\end{aligned}$$

where $\Gamma^{(n)}(m)$ is the n-th derivative of the Γ function evaluated at m. Besides, for all n, we have the estimate

$$\frac{\Gamma^{(n)}(m)}{\Gamma(m)}=\log^n(m)+O\left(\frac{\log^{n-1}(m)}{m}\right)\tag{1.197}$$

as $m\to\infty$. Using this and (1.196), we obtain

$$\begin{aligned}\sum_{m\geqslant 1}\frac{\Gamma^{(n)}(m)}{\Gamma(m)}t^{m-1} &= \sum_{m\geqslant 1}\log^n(m)t^{m-1}+O\left(\log^n\left(\frac{1}{1-t}\right)\right)\\ &= \sum_{m\geqslant 1}P(\log(m)+\gamma)t^{m-1}+O\left(\log^n\left(\frac{1}{1-t}\right)\right),\end{aligned}$$

concluding the proof of the lemma. □

1.7.5. *The extended double shuffle relations.* We now introduce extended double shuffle relations. We first recall the two commutative diagrams

$$\begin{tikzcd}(\mathfrak{H}^1,ш) \arrow[r, "\mathrm{reg}^T_{ш}"] \arrow[dr, "\zeta^T_{ш}"'] & (\mathfrak{H}^0,ш)[T], \arrow[d, "w\mapsto\zeta(w)", "T\mapsto T"'] \\ & \mathbb{R}[T]\end{tikzcd}\qquad \begin{tikzcd}(\mathfrak{H}^1,*) \arrow[r, "\mathrm{reg}^T_{*}"] \arrow[dr, "\zeta^T_{*}"'] & (\mathfrak{H}^0,*)[T]. \arrow[d, "w\mapsto\zeta(w)", "T\mapsto T"'] \\ & \mathbb{R}[T]\end{tikzcd}\tag{1.198}$$

DEFINITION 1.199. Let $(R,\cdot)$ be a $\mathbb{Q}$-algebra and $Z_R\colon \mathfrak{H}^0 \to R$ a map. The pair (R, Z_R) satisfies *finite double shuffle relations* if Z_R is a morphism of algebras with respect to both the shuffle and the stuffle product on $\mathfrak{H}^0$, that is, the equality

$$Z_R(w \sqcup\!\sqcup\, v) = Z_R(w)\cdot Z_R(v) = Z_R(w * v) \tag{1.200}$$

holds for all $w, v \in \mathfrak{H}^0$.

A map $Z_R\colon \mathfrak{H}^0 \to R$ as above gives rise to maps

$$Z^T_{R,\sqcup\!\sqcup}\colon (\mathfrak{H}^1, \sqcup\!\sqcup) \longrightarrow R[T],$$
$$Z^T_{R,*}\colon (\mathfrak{H}^1, *) \longrightarrow R[T],$$

by composition with the regularization maps $\mathrm{reg}^T_{\sqcup\!\sqcup}$ and reg^T_*. Since R is a $\mathbb{Q}$-algebra with a map from $\mathfrak{H}^0$, we can define the formal power series

$$A_R(u) = \exp\left(\sum_{n\geqslant 2} \frac{(-1)^n}{n} Z_R(y_n)u^n\right),$$

and by analogy with ρ from (1.189), the linear map $\varrho_R\colon R[T] \to R[T]$ uniquely determined by the equality of formal power series

$$\varrho_R(e^{Tu}) = A_R(u)e^{Tu}. \tag{1.201}$$

DEFINITION 1.202. Assume that the pair (R, Z_R) satisfies the finite double shuffle relations. We say that (R, Z_R) satisfies the *extended double shuffle relations* if, in addition, the following equality holds for all words $w \in \mathfrak{H}^1$:

$$Z^T_{R,\sqcup\!\sqcup}(w) = \varrho_R(Z^T_{R,*}(w)). \tag{1.203}$$

In light of this definition, Theorems 1.169 and 1.190 can be rephrased as follows:

THEOREM 1.204. *The pair $(\mathbb{R}, \zeta)$ satisfies the extended double shuffle relations.*

This theorem is a source of new relations among multiple zeta values that cannot be obtained by comparing the usual shuffle and stuffle products. For the next result, see [**IKZ06**, Thm. 2].

COROLLARY 1.205. *Let $w_1 \in \mathfrak{H}^1$ and $w_0 \in \mathfrak{H}^0$. Then*

$$\zeta_{\sqcup\!\sqcup}(w_1 \sqcup\!\sqcup\, w_0 - w_1 * w_0) = 0.$$

For example, since $x_1 \sqcup\!\sqcup\, w - x_1 * w$ belongs to $\mathfrak{H}^0$ for every word $w \in \mathfrak{H}^0$ (Exercise 1.209), we deduce the so-called *Hoffman relation*

$$\zeta(x_1 \sqcup\!\sqcup\, w - x_1 * w) = 0, \tag{1.206}$$

which implies Euler's sum formula (1.57) by Exercise 1.210.

1.7.6. *The universal algebra satisfying the extended double shuffle relations.* By considering the quotient of $\mathfrak{H}^0$ by all extended double shuffle relations, we obtain an algebra R_{EDS} and a map $Z_{\mathrm{EDS}}\colon \mathfrak{H}^0 \to R_{\mathrm{EDS}}$ such that the pair $(R_{\mathrm{EDS}}, Z_{\mathrm{EDS}})$ satisfies the extended double shuffle relations. It is the *universal* pair in the following sense: for any (R, Z_R) satisfying the extended double shuffle relations, there exists a unique map $\varphi_R\colon R_{\mathrm{EDS}} \to R$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{H}^0 \arrow[r, "Z_{\mathrm{EDS}}"] \arrow[dr, "Z_R"'] & R_{\mathrm{EDS}}. \arrow[d, "\varphi_R"] \\ & R \end{tikzcd}$$

The following conjecture describes the combinatorial structure of the algebra of multiple zeta values.

CONJECTURE 1.207. *The map $\varphi_{\mathbb{R}}$ is injective. Equivalently, the algebra $\mathcal{Z}$ of multiple zeta values is isomorphic to R_{EDS}.*

REMARK 1.208. The finite double shuffle relations are linear and homogeneous with respect to the weight. Moreover, the extended double shuffle relations are also homogeneous (Exercise 1.212). Since the coefficients of the power series A_R are polynomials in zeta values, the extended double shuffle relations are polynomial in the mutliple zeta values. Since products of these numbers can be reduced to linear combinations of multiple zeta values using either the shuffle or the stuffle product, we can reduce the extended double shuffle relations relations to linear ones. Hence, all possible relations among multiple zeta values are conjectured to be generated by homogeneous linear relations.

⋆ ⋆ ⋆

EXERCISE 1.209. Show that $x_1 ⧢ w - x_1 * w$ belongs to $\mathfrak{H}^0$ for all words $w \in \mathfrak{H}^0$.

EXERCISE 1.210. Deduce Euler's sum formula (1.57) from the Hoffman relation. [Hint: take $w = x_0^{p-1}x_1$.]

EXERCISE 1.211. Show that the coefficient γ_k in the power series (1.188) is a polynomial in $\zeta(2), \zeta(3), \dots$, that is homogeneous of weight k.

EXERCISE 1.212. Use Exercise 1.211 to prove that the extended double shuffle relations are homogeneous.

EXERCISE 1.213. Which identities do we get from the comparison of the regularized multiple zeta values $\zeta_*(1,1,2)$ and $\zeta_{⧢}(1,1,2)$?

EXERCISE 1.214. Prove Theorem 1.179.

EXERCISE 1.215. Check the estimates

$$\begin{aligned}\zeta_M(\boldsymbol{s}) &= \sum_{k=0}^{n} a_k \left(\log M + \gamma + O\left(\frac{1}{M}\right)\right)^k \\ &= \sum_{k=0}^{n} a_k (\log M + \gamma)^k + O\left(\frac{1}{M}\log^{n-1}(M)\right).\end{aligned}$$

EXERCISE 1.216. Prove the estimate (1.197).

EXERCISE 1.217. Use Corollary 1.205 to prove that the identity

$$\zeta_{⧢}(x_1^m * w) = 0$$

holds for all $w \in \mathfrak{H}^0$ and $m \geqslant 1$.

2. Periods of mixed Hodge structures

In this chapter, we introduce the first tools from algebraic geometry that will be needed for the study of multiple zeta values. Our main goal is to show that all these numbers can be obtained by integrating an algebraic differential form over a topological cycle on an algebraic variety defined over the field of rational numbers. The extra structures carried by cohomology will then give non-trivial information about multiple zeta values. With this in mind, we begin by recalling the definition of singular homology and cohomology of a topological space M in Section 2.1. It is a classical theorem of de Rham that, whenever M is a differentiable manifold, singular cohomology can be computed using differential forms. More precisely, the map that sends a differential form to the integration functional on singular homology induces an isomorphism between de Rham cohomology and singular cohomology. If M underlies a complex algebraic variety X, this cohomology can even be computed using differential forms with polynomial coefficients. As we explain in Sections 2.2 and 2.3, it is isomorphic to algebraic de Rham cohomology by a theorem of Grothendieck. A remarkable consequence is that, when X is defined over $\mathbb{Q}$, we get two different rational structures on the same complex vector space that are not compatible. This is not bad news, quite the opposite: the comparison between them produces an interesting class of complex numbers called *periods*; we define them in various equivalent ways in Section 2.4. Then, in Section 2.5, we explain how to interpret multiple zeta values as periods of a cohomology group built out of the moduli spaces $M_{0,n}$ of n ordered distinct points in $\mathbb{P}^1$ up to projective equivalence, following Goncharov and Manin. Another important result relying on the comparison isomorphism is that the cohomology of X is equipped with two filtrations the interaction of which gives rise to a *mixed Hodge structure*, a notion developed by Deligne at the beginning of the 70s. We explain the definition in Section 2.6, and how to compute certain extension groups in the category of mixed Hodge structures in Section 2.7. In Section 2.8, we sketch some of the ideas in the proof that the cohomology of any algebraic variety carries a mixed Hodge structure and we give many examples, in particular of Hodge structures of mixed Tate type. Conjecturally, mixed Hodge structures of algebraic varieties over $\mathbb{Q}$ capture all algebraic relations between periods. As an illustration, in the final Section 2.9 we go back to the interpretation of $\zeta(2)$ as a period of a pair of algebraic varieties, we take a closer look at the mixed Hodge structure on its cohomology, and we indicate how the information we obtain may be used to prove "by pure thought" that $\zeta(2)$ is a rational multiple of π^2.

2.1. Singular homology and cohomology. We begin by briefly recalling the definition of singular homology and cohomology of a topological space; for more details, we refer the reader to Chapters 2 and 3 of Hatcher's book [**Hat02**].

For each integer $n \geqslant 0$, let

$$\Delta^n_{\mathrm{st}} = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geqslant 0 \text{ for } i = 0, \dots, n \right\}$$

be the standard n-dimensional simplex in $\mathbb{R}^{n+1}$. For each integer $i = 0, \dots, n+1$, there is a *face* map $\delta^i \colon \Delta^n_{\mathrm{st}} \to \Delta^{n+1}_{\mathrm{st}}$ given by plugging a 0 at the i-th coordinate:

$$\delta^i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n).$$

REMARK 2.1. The standard simplex Δ^n_{st} is homeomorphic to the simplex

$$\Delta^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid 1 \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_n \geqslant 0\}$$

from Notation 1.114 (see Exercise A.240). The representation we use here is more symmetric and enables one to write down the face maps in a uniform way. By contrast, working with Δ^n is more convenient for decomposing a product of simplices (see Section 1.5.3 and Example A.228).

Let M be a topological space. A continuous map $\sigma\colon \Delta^n_{\mathrm{st}} \to M$ is called a *singular n-simplex* on M. For each $n \geqslant 0$, let

$$C_n(M) = \bigoplus_{\sigma\colon \Delta^n_{\mathrm{st}} \to M} \mathbb{Z}\,\sigma$$

be the free abelian group generated by singular n-simplices on M. The elements of $C_n(M)$ are finite linear combinations with integral coefficients of continuous maps $\sigma\colon \Delta^n_{\mathrm{st}} \to M$; they are called *singular n-chains* on M, or simply singular chains when n is clear from the context. For example, a singular 0-chain is a linear combination of points of M, and a singular 1-chain is a linear combination of paths $\gamma\colon [0,1] \to M$, once we identify the interval $[0,1]$ with the simplex Δ^1_{st} through the homeomorphism $t \mapsto (t, 1-t)$. That is,

$$C_0(M) = \bigoplus_{p \in M} \mathbb{Z}\,p, \qquad C_1(M) = \bigoplus_{\gamma\colon [0,1] \to M} \mathbb{Z}\,\gamma.$$

To make the notation uniform in what follows, we also set $C_n(M) = 0$ for all $n < 0$.

For each $n \geqslant 1$, we define a *boundary* homomorphism

$$\begin{aligned} \partial_n\colon\ C_n(M) &\longrightarrow C_{n-1}(M) \\ \sigma\ &\longmapsto \sum_{i=0}^{n} (-1)^i (\sigma \circ \delta^i). \end{aligned} \tag{2.2}$$

This is well defined, since $\sigma \circ \delta^i\colon \Delta^{n-1}_{\mathrm{st}} \to \Delta^n_{\mathrm{st}} \to M$ is a singular $(n-1)$-simplex on M, namely the restriction of σ to the face of Δ^n_{st} where the coordinate t_i vanishes. We also set $\partial_n = 0$ on $C_n(M)$ for all $n \leqslant 0$. Thanks to the alternating signs in (2.2), the boundary maps satisfy

$$\partial_{n-1} \circ \partial_n = 0$$

for each integer n (Exercise 2.59), thus making

$$(C_*(M), \partial_*) = \Big[\cdots \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} C_0(M) \xrightarrow{\partial_0} 0 \longrightarrow \cdots\Big] \tag{2.3}$$

into a chain complex of abelian groups in the sense of Definition A.13 v).

DEFINITION 2.4. We call $(C_*(M), \partial_*)$ the *singular chain complex* of M.

Elements in the kernel of the boundary map ∂_n are called *closed* singular chains, or more often *cycles*, and elements in the image of ∂_{n+1} are called *boundaries.*

DEFINITION 2.5. Let n be an integer. The *singular homology* in degree n of M is the n-th homology group

$$\mathrm{H}_n(M, \mathbb{Z}) = \frac{\operatorname{Ker}(\partial_n\colon C_n(M) \longrightarrow C_{n-1}(M))}{\operatorname{Im}(\partial_{n+1}\colon C_{n+1}(M) \longrightarrow C_n(M))}$$

of the singular chain complex of M. In other words, $\mathrm{H}_n(M, \mathbb{Z})$ is the quotient of the abelian group of cycles by the subgroup consisting of boundaries.

EXAMPLE 2.6. In degree zero, $\mathrm{H}_0(M,\mathbb{Z})$ is the direct sum of copies of $\mathbb{Z}$ indexed by the set $\pi_0(M)$ of path-connected components of M. Indeed, any two points x and y in the same component define the same element of $\mathrm{H}_0(M,\mathbb{Z})$ since $y-x$ is the boundary of a path $\gamma\colon [0,1]\to M$ satisfying $\gamma(0)=x$ and $\gamma(1)=y$. Therefore, choosing a point on each path-connected component we obtain a set of generators of $\mathrm{H}_0(M,\mathbb{Z})$. On the other hand, since $[0,1]$ is path-connected, the image of a continuous map $\gamma\colon [0,1]\to M$ is contained in a path-connected component, and this implies that all elements in the previous set of generators are linearly independent.

The construction of singular homology is functorial. For each continuous map of topological spaces $f\colon M_1\to M_2$, sending a singular n-simplex $\sigma\colon \Delta^n_{\mathrm{st}}\to M_1$ on M_1 to the singular n-simplex $f\circ\sigma\colon \Delta^n_{\mathrm{st}}\to M_2$ on M_2 induces homomorphisms

$$f_*\colon C_n(M_1)\longrightarrow C_n(M_2) \tag{2.7}$$

that commute with the boundary maps ∂_n (Exercise 2.60), and hence morphisms of graded abelian groups, still denoted by

$$f_*\colon\ \mathrm{H}_*(M_1,\mathbb{Z})\longrightarrow \mathrm{H}_*(M_2,\mathbb{Z}).$$

Moreover, $(f\circ g)_*=f_*\circ g_*$ holds for all composable maps f and g, and the identity of M induces the identity of $\mathrm{H}_*(M,\mathbb{Z})$.

Dualizing the definitions of singular chains and boundary maps, we find the free abelian group of *singular n-cochains*

$$C^n(M)=\mathrm{Hom}_{\mathbb{Z}}(C_n(M),\mathbb{Z}),$$

as well as *coboundary* maps

$$\mathrm{d}^n\colon C^n(M)\to C^{n+1}(M).$$

In particular, $C^n(M)$ and d^n are zero for all $n<0$. Explicitly, d^n maps a singular n-cochain $\eta\colon C_n(M)\to\mathbb{Z}$ to the singular $(n+1)$-cochain that takes the value

$$(\mathrm{d}^n\eta)(\sigma)=\eta(\partial_{n+1}\sigma) \tag{2.8}$$

on an $(n+1)$-singular chain σ on M. These coboundary maps satisfy

$$\mathrm{d}^{n+1}\circ\mathrm{d}^n=0$$

for each integer n (Exercise 2.59), thus making

$$(C^*(M),\mathrm{d}^*)=\Big[\cdots\longrightarrow 0\longrightarrow C^0(M)\xrightarrow{\mathrm{d}^0} C^1(M)\xrightarrow{\mathrm{d}^1}\cdots\Big] \tag{2.9}$$

into a cochain complex of abelian groups in the sense of Definition A.13 i).

DEFINITION 2.10. We call $(C^*(M),\mathrm{d}^*)$ the *singular cochain complex of M*.

The elements of the kernel of d^n are called *cocycles*, and the elements of the image are called *coboundaries*.

DEFINITION 2.11. Let n be an integer. The *singular cohomology* in degree n of M is the n-th cohomology group

$$\mathrm{H}^n(M,\mathbb{Z})=\frac{\mathrm{Ker}(\mathrm{d}^n\colon C^n(M)\longrightarrow C^{n+1}(M))}{\mathrm{Im}(\mathrm{d}^{n-1}\colon C^{n-1}(M)\longrightarrow C^n(M))}$$

of the singular cochain complex of M. In other words, $\mathrm{H}^n(M,\mathbb{Z})$ is the quotient of the abelian group of cocycles by the subgroup consisting of coboundaries.

Similarly as above, singular cohomology is functorial: from a continuous map of topological spaces $f\colon M_1 \to M_2$, we get a homomorphism

$$f^*\colon C^n(M_2) \longrightarrow C^n(M_1)$$

by sending a singular n-cochain $\eta\colon C_n(M_2) \to \mathbb{Z}$ on M_2 to the singular n-cochain on M_1 obtained by composition with (2.7), namely $\eta \circ f_*\colon C_n(M_1) \to \mathbb{Z}$. The map f^* commutes with the coboundaries d^n (Exercise 2.60), and hence induces a morphism of graded abelian groups

$$f^*\colon \mathrm{H}^*(M_2, \mathbb{Z}) \longrightarrow \mathrm{H}^*(M_1, \mathbb{Z})$$

that we still denote by the same symbol. This assignment satisfies $(f \circ g)^* = g^* \circ f^*$ for all composable maps f and g, and $(\mathrm{Id}_M)^* = \mathrm{Id}_{\mathrm{H}^*(M,\mathbb{Z})}$.

Remark 2.12.

i) We defined singular homology and cohomology with integer coefficients, but the same construction extends to any abelian group G by considering

$$C_n(M,G) = \bigoplus_{\sigma\colon \Delta^n_{\mathrm{st}} \to M} G\sigma,$$
$$C^n(M,G) = \mathrm{Hom}_{\mathbb{Z}}(C_n(M,G), G).$$

If G is a ring, then the resulting objects are G-modules and the unique ring morphism $\mathbb{Z} \to G$ induces a morphism of G-modules

$$\mathrm{H}^*(M,\mathbb{Z}) \otimes_{\mathbb{Z}} G \longrightarrow \mathrm{H}^*(M,G).$$

By the *universal coefficient theorem* to be explained below, this map is an isomorphism if G is a field k of characteristic zero:

$$\mathrm{H}^*(M,k) \simeq \mathrm{H}^*(M,\mathbb{Z}) \otimes_{\mathbb{Z}} k.$$

Most of the time, it will be enough for our purposes to work with rational coefficients, that is, to take $G = k = \mathbb{Q}$.

ii) Singular homology and cohomology are defined for any topological space. When M underlies a differentiable manifold, we may use *smooth* maps instead of continuous maps, that is, maps $\sigma\colon \Delta^n_{\mathrm{st}} \to M$ admitting a $\mathcal{C}^\infty$ extension to an open neighborhood of Δ^n_{st} in $\mathbb{R}^{n+1}$. It is a standard result of differential geometry (see [**War83**, § 5.32] or [**Lee13**, Thm. 18.7]) that the singular homology and cohomology groups remain the same when one considers this restricted set of generators. In other words, the inclusion of the subcomplex of $C_*(M)$ built out of smooth chains is a quasi-isomorphism.

iii) Working with rational coefficients, we may identify singular cohomology with the linear dual of singular homology

$$\mathrm{H}^n(M,\mathbb{Q}) \simeq \mathrm{Hom}_{\mathbb{Q}}(\mathrm{H}_n(M,\mathbb{Q}),\mathbb{Q}), \tag{2.13}$$

and think of cohomology classes as linear functionals on homology; this will be useful to discuss periods. However, the isomorphism (2.13) cannot hold integrally since the group $\mathrm{Hom}(\mathrm{H}_n(M,\mathbb{Z}),\mathbb{Z})$ is always torsion free, while $\mathrm{H}^n(M,\mathbb{Z})$ may have torsion (see Exercise 2.62 for an example).

In the sequel, we will mainly consider the singular cohomology with rational coefficients of complex algebraic varieties, which deserves the special name of *Betti cohomology*. Namely, given an algebraic variety X defined over a subfield k of $\mathbb{C}$, the set of complex points $X(\mathbb{C})$ carries a natural topology coming from the euclidean topology of $\mathbb{C}$; it is usually called the *classical* or the *analytic*, or yet the *transcendental* topology to distinguish it from the Zariski topology on X.

DEFINITION 2.14. Let k be a subfield of $\mathbb{C}$ and let X be an algebraic variety over k. The *Betti cohomology* $\mathrm{H}^*_{\mathrm{B}}(X)$ of X is the singular cohomology with rational coefficients of the set of complex points $X(\mathbb{C})$ equipped with the classical topology:

$$\mathrm{H}^*_{\mathrm{B}}(X) = \mathrm{H}^*(X(\mathbb{C}), \mathbb{Q}).$$

2.1.1. *Properties of singular homology and cohomology.* Singular homology and cohomology enjoy several properties that are very useful to explicitly compute them by "decomposing" a given topological space into simpler ones. We list some of them.

• *Homotopy invariance.* A continuous map $f\colon M_1 \to M_2$ of topological spaces is called a *homotopy equivalence* if there exists a continuous map $g\colon M_2 \to M_1$ such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity maps on M_1 and M_2 respectively. This means that there exists a continuous map

$$H\colon M_1 \times [0,1] \longrightarrow M_1$$

satisfying $H(x,0) = g(f(x))$ and $H(x,1) = x$ for all $x \in M_1$, and similarly for $f \circ g$. We then say that the topological spaces M_1 and M_2 have the same *homotopy type*. When M has the same homotopy type as a point, we say that M is *contractible*.

If $f\colon M_1 \to M_2$ is a homotopy equivalence, then the maps

$$f_*\colon \mathrm{H}_*(M_1, \mathbb{Z}) \longrightarrow \mathrm{H}_*(M_2, \mathbb{Z}) \quad \text{and} \quad f^*\colon \mathrm{H}^*(M_2, \mathbb{Z}) \longrightarrow \mathrm{H}^*(M_1, \mathbb{Z})$$

are isomorphisms of abelian groups. This is a straightforward consequence of the result that two homotopic maps induce the same morphisms in homology and cohomology, which is for instance proved in [**Hat02**, Thm. 2.10 and p. 201].

• *Mayer–Vietoris long exact sequence.* For any cover $M = U \cup V$ of M by open subspaces U and V, there is a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}_n(U \cap V, \mathbb{Z}) \arrow[r, "\alpha"] & \mathrm{H}_n(U,\mathbb{Z}) \oplus \mathrm{H}_n(V,\mathbb{Z}) \arrow[r, "\beta"] & \mathrm{H}_n(M,\mathbb{Z}) \arrow[dll, out=0, in=180] \\ & \mathrm{H}_{n-1}(U \cap V, \mathbb{Z}) \arrow[r] & \cdots & \end{tikzcd} \tag{2.15}$$

Letting $\iota_{U\cap V,U}\colon U \cap V \hookrightarrow U$ denote the inclusion, and similarly for other pairs of a space and a subspace, the maps α and β are given by

$$\alpha = (-(\iota_{U\cap V,U})_*, (\iota_{U\cap V,V})_*), \qquad \beta = (\iota_{U,M})_* + (\iota_{V,M})_*.$$

The morphism connecting two lines is defined as follows: using barycentric subdivision (see, for instance, [**Hat02**, p. 150]), each n-cycle σ on M can be written as a sum $\sigma = \sigma' + \sigma''$ of n-chains σ' and σ'' on U and V respectively. The class $[\sigma]$ is then mapped to the class of $\partial\sigma' = -\partial\sigma''$, which represents a well-defined element of $\mathrm{H}_{n-1}(U \cap V, \mathbb{Z})$. Dually, there is a long exact sequence in cohomology

$$\begin{tikzcd} \cdots & \mathrm{H}^n(U \cap V, \mathbb{Z}) \arrow[l] & \mathrm{H}^n(U,\mathbb{Z}) \oplus \mathrm{H}^n(V,\mathbb{Z}) \arrow[l] & \mathrm{H}^n(M,\mathbb{Z}) \arrow[l] \\ & \mathrm{H}^{n-1}(U \cap V, \mathbb{Z}) \arrow[urr, out=180, in=0] & \cdots \arrow[l] & \end{tikzcd} \tag{2.16}$$

• *Universal coefficients theorem.* Singular homology and cohomology of a topological space with coefficients in an abelian group can be deduced from those with integer coefficients. It is in this sense that the integers are the *universal coefficients*. The precise result is as follows (see [**Hat02**, Thm. 3.2 and Thm. 3A.3]). Let M be a topological space and let G be an abelian group. For each $n \geqslant 0$, there are natural short exact sequences

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H}_{n-1}(M,\mathbb{Z}),G) \longrightarrow \mathrm{H}^n(M,G) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(M,\mathbb{Z}),G) \longrightarrow 0, \tag{2.17}$$

$$0 \longrightarrow \mathrm{H}_n(M,\mathbb{Z}) \otimes G \longrightarrow \mathrm{H}_n(M,G) \longrightarrow \mathrm{Tor}_1(\mathrm{H}_{n-1}(M,\mathbb{Z}),G) \longrightarrow 0, \tag{2.18}$$

where $\mathrm{Ext}^1(-,G)$ and $\mathrm{Tor}_1(-,G)$ denote the Ext^1 and Tor_1 functors in the category of $\mathbb{Z}$-modules, defined as the right and left derived functors in degree one of $\mathrm{Hom}(-,G)$ and $-\otimes G$ respectively (see Examples A.114 and A.124). The map

$$\mathrm{H}^n(M,G) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(M,\mathbb{Z}),G)$$

is induced from the evaluation of a singular cochain on a singular chain, and

$$\mathrm{H}_n(M,\mathbb{Z}) \otimes G \longrightarrow \mathrm{H}_n(M,G)$$

from the product of a singular chain with integer coefficients by an element of G.

The existence of the exact sequences (2.17) and (2.18) can be proved using spectral sequences as in Exercise 2.63. Moreover, they are (non-canonically) split. For us, the main important consequence of the universal coefficients theorem is that it gives isomorphisms of rational vector spaces

$$\mathrm{H}_n(M,\mathbb{Q}) = \mathrm{H}_n(M,\mathbb{Z}) \otimes \mathbb{Q}, \qquad \mathrm{H}^n(M,\mathbb{Q}) = \mathrm{Hom}(\mathrm{H}_n(M,\mathbb{Z}),\mathbb{Q}),$$

on noting that the functors $\mathrm{Hom}(-,G)$ and $-\otimes G$ are exact for $G=\mathbb{Q}$, and hence both $\mathrm{Ext}^1(\mathrm{H}_{n-1}(M,\mathbb{Z}),G)$ and $\mathrm{Tor}_1(\mathrm{H}_{n-1}(M,\mathbb{Z}),G)$ vanish.

• *Künneth formula in homology.* Singular homology is equipped with an *external product* that induces an isomorphism

$$\mathrm{H}_n(M_1 \times M_2,\mathbb{Q}) \simeq \bigoplus_{i+j=n} \mathrm{H}_i(M_1,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{H}_j(M_2,\mathbb{Q}) \tag{2.19}$$

which is usually referred to as the *Künneth formula*. For it to be true as stated, we need to consider homology with rational coefficients (see Exercise 2.68 for a counterexample), but a more sophisticated formula involving Tor groups is true with integral coefficients; see [**Hat02**, Thm. 3B.6].

The combinatorial structures involved in the construction of the external product is very much related to those that we have already seen in the study of multiple zeta values. Recall from Example A.36 that the *tensor product* of the chain complexes $C_*(M_1)$ and $C_*(M_2)$ is the complex $C_*(M_1)\otimes C_*(M_2)$ with degree n term

$$\bigoplus_{k=0}^{n} C_k(M_1) \otimes C_{n-k}(M_2)$$

and differential ∂ given on elementary tensors of $\alpha \in C_k(M_1)$ and $\beta \in C_{n-k}(M_2)$ by

$$\partial(\alpha\otimes\beta) = \partial\alpha\otimes\beta + (-1)^k \alpha\otimes\partial\beta.$$

Our goal is to show that this tensor product is quasi-isomorphic to the complex of singular chains of the product space $M_1 \times M_2$. For this, first note that a simplex in $M_1 \times M_2$ is given by a pair (σ, τ), where σ is a simplex in M_1 and τ is a simplex in M_2. We define a deconcatenation map

$$\begin{aligned} F\colon\ C_*(M_1 \times M_2) &\longrightarrow \qquad C_*(M_1) \otimes C_*(M_2) \\ (\sigma, \tau) \qquad &\longmapsto \sum_{k=0}^{n} (\sigma \circ \delta^n \circ \cdots \circ \delta^{k+1}) \otimes (\tau \circ (\delta^0)^k) \end{aligned} \tag{2.20}$$

for each $(\sigma, \tau) \in C_n(M_1 \times M_2)$. Explicitly, $\sigma \circ \delta^n \circ \cdots \circ \delta^{k+1} \in C_k(M_1)$ is the k-th front face of the simplex σ, given by

$$(\sigma \circ \delta^n \circ \cdots \circ \delta^{k+1})(t_0, \ldots, t_k) = \sigma(t_0, \ldots, t_k, \underbrace{0, \ldots, 0}_{n-k}),$$

while $\tau \circ (\delta^0)^k \in C_{n-k}(M_2)$ is the $(n-k)$-th back face of the simplex τ, given by

$$(\tau \circ (\delta^0)^k)(t_0, \ldots, t_{n-k}) = \tau(\underbrace{0, \ldots, 0}_{k}, t_0, \ldots, t_{n-k}).$$

One can check that F is a morphism of chain complexes (Definition A.13), and hence induces a morphism of homology groups

$$\mathrm{H}_n(M_1 \times M_2, \mathbb{Z}) \longrightarrow \mathrm{H}_n\left(C_*(M_1) \otimes C_*(M_2)\right). \tag{2.21}$$

Second, there is a map

$$G\colon C_*(M_1) \otimes C_*(M_2) \longrightarrow C_*(M_1 \times M_2)$$

defined using shuffles (Section 1.5.3) and the non-standard simplices Δ^k and Δ^{n-k} (see Remark 2.1). Given simplices $\alpha\colon \Delta^k \to M_1$ and $\beta\colon \Delta^{n-k} \to M_2$, we set

$$G(\alpha \otimes \beta) = \sum_{\sigma \in ш(k, n-k)} (\alpha \otimes \beta)_\sigma, \tag{2.22}$$

where $(\alpha \otimes \beta)_\sigma$ is the simplex $\Delta^n \to M_1 \times M_2$ given by

$$(\alpha \otimes \beta)_\sigma(t_1, \ldots, t_n) = \big(\alpha(t_{\sigma(1)}, \ldots, t_{\sigma(k)}), \beta(t_{\sigma(k+1)}, \ldots, t_{\sigma(n)})\big)$$

and we extend (2.22) by linearity to all singular chains. The map G is again a morphism of chain complexes, and hence induces morphisms of homology groups

$$\mathrm{H}_n\left(C_*(M_1) \otimes C_*(M_2)\right) \longrightarrow \mathrm{H}_n(M_1 \times M_2, \mathbb{Z}). \tag{2.23}$$

The compositions $F \circ G$ and $G \circ F$ are both homotopically equivalent to the identity, which implies that (2.21) and (2.23) are isomorphisms inverse to each other. This statement is a particular case of the Eilenberg–Zilber theorem. It can be proved in a completely abstract way, without writing down explicit homotopies, by means of the acyclic models theorem (see [**Spa66**, Chap. 5, § 3 Thm. 6] and Section A.1.12).

THEOREM 2.24 (Eilenberg–Zilber [**EZ53**]). *The functor $M \mapsto C_*(M)$ from the category of topological spaces to the category of chain complexes of abelian groups satisfies the following three properties:*

i) *There exists a natural transformation of functors*

$$\alpha\colon C_*(\cdot \times \cdot) \to C_*(\cdot) \otimes C_*(\cdot)$$

that is unique up to a natural homotopy.

ii) *There exists a natural transformation of functors*

$$\beta\colon C_*(\cdot)\otimes C_*(\cdot)\to C_*(\cdot\times\cdot)$$

that is unique up to a natural homotopy.

iii) *The natural transformations α and β are natural homotopy equivalences that are inverse to each other.*

In order to see that the maps F and G are candidates for the natural transformations α and β, one only needs to solve Exercise 2.65, which consists of a straightforward although cumbersome sequence of verifications. Then Theorem 2.24 implies that F and G are homotopy equivalences inverse to each other, as claimed.

The last ingredient to prove the Künneth isomorphism as stated in (2.19) is the general fact about tensor products of complexes that there is a natural map

$$\begin{aligned}\bigoplus_{k=0}^{n}\mathrm{H}_k(C_*(M_1))\otimes\mathrm{H}_{n-k}(C_*(M_2)) &\longrightarrow \mathrm{H}_n\left(C_*(M_1)\otimes C_*(M_2)\right),\\ [\sigma]\otimes[\tau] &\longmapsto [\sigma\otimes\tau]\end{aligned}\tag{2.25}$$

which is well defined since α and β are closed (Exercise 2.66). The map (2.25) is injective and becomes an isomorphism after tensoring by the field of rational numbers (this follows from [**Wei94**, Thm. 3.6.3], using that every module over a field is flat). It is only in this last step that rational coefficients are needed.

Remark 2.26. We emphasize again the similarities between the map F and the deconcatenation coproduct that will be introduced later in Example 3.64, and between the map G and the shuffle product from the same example.

• *Künneth formula in cohomology.* There is also a Künneth formula for singular cohomology, but only under the assumption that at least one of the topological spaces has finite-dimensional cohomology in all degrees:

$$\mathrm{H}^n(M_1\times M_2,\mathbb{Q})\simeq\bigoplus_{i+j=n}\mathrm{H}^i(M_1,\mathbb{Q})\otimes_{\mathbb{Q}}\mathrm{H}^j(M_2,\mathbb{Q}).\tag{2.27}$$

The need of such a restriction is already clear in degree zero: if M_1 and M_2 are infinite countable discrete spaces, then $\mathrm{H}^0(M_1\times M_2,\mathbb{Q})$ is isomorphic to $\mathbb{Q}[\![x,y]\!]$, whereas $\mathrm{H}^0(M_1,\mathbb{Q})\otimes\mathrm{H}^0(M_2,\mathbb{Q})$ is isomorphic to $\mathbb{Q}[\![x]\!]\otimes\mathbb{Q}[\![y]\!]$.

Assuming the finiteness condition, the Künneth formula (2.27) follows from that in homology (Exercise 2.67). What always exists without any assumption is a map from the right-hand side of (2.27) to the left-hand side, which is constructed as follows. Dually to F in (2.20), there is a morphism of cochain complexes

$$C^*(M_1)\otimes C^*(M_2)\longrightarrow C^*(M_1\times M_2)\tag{2.28}$$

given in each degree n by mapping a tensor product $\eta\otimes\xi$ of cochains $\eta\in C^k(M_1)$ and $\xi\in C^{n-k}(M_2)$ to the cochain in $C^n(M_1\times M_2)$ whose value at (σ,τ) is

$$\eta(\sigma\circ\delta^n\circ\cdots\circ\delta^{k+1})\cdot\xi(\tau\circ(\delta^0)^k)$$

(the middle product takes place in $\mathbb{Z}$). This map induces a morphism in cohomology

$$\mathrm{H}^n\left(C^*(M_1)\otimes C^*(M_2)\right)\longrightarrow\mathrm{H}^n(M_1\times M_2,\mathbb{Z}).\tag{2.29}$$

Besides, there is an injective homomorphism

$$\bigoplus_{k=0}^{n} \mathrm{H}^k(C^*(M_1)) \otimes \mathrm{H}^{n-k}(C^*(M_2)) \longrightarrow \mathrm{H}^n\big(C^*(M_1) \otimes C^*(M_2)\big), \tag{2.30}$$

given by the same expression as in (2.25), that becomes an isomorphism rationally. The composition of these two morphisms yields the map

$$\bigoplus_{i+j=n} \mathrm{H}^i(M_1, \mathbb{Z}) \otimes \mathrm{H}^j(M_2, \mathbb{Z}) \longrightarrow \mathrm{H}^n(M_1 \times M_2, \mathbb{Z}). \tag{2.31}$$

• *Cup-product.* One of the advantages of working with cohomology rather than homology is that combining the external product with the contravariant functoriality of cohomology one obtains a product in cohomology called the *cup-product*. Namely, consider the diagonal embedding

$$\begin{aligned} \mathrm{diag}\colon\ M &\longrightarrow M \times M. \\ x &\longmapsto (x, x) \end{aligned}$$

Composing (2.28) with the morphism induced by diag by functoriality of singular cochains, we get a cup-product

$$\smile\colon C^k(M) \otimes C^\ell(M) \longrightarrow C^{k+\ell}(M)$$

that maps $\eta \otimes \xi$ to the singular cochain whose value on $\sigma \in C_{k+\ell}(M)$ is

$$(\eta \smile \xi)(\sigma) = \eta(\sigma \circ \delta^n \circ \cdots \circ \delta^{\ell+1}) \cdot \xi(\sigma \circ (\delta^0)^k)$$

On cohomology, this map induces the composition

$$\mathrm{H}^k(M, \mathbb{Z}) \otimes \mathrm{H}^{n-k}(M, \mathbb{Z}) \longrightarrow \mathrm{H}^n(M \times M, \mathbb{Z}) \xrightarrow{\mathrm{diag}^*} \mathrm{H}^n(M, \mathbb{Z}),$$

where the first arrow is (2.31). We still call it the *cup-product*.

• *Cap-product.* For all non-negative integers $k \geqslant \ell$, there is a bilinear map

$$\frown\colon C_k(M) \times C^\ell(M) \longrightarrow C_{k-\ell}(M)$$

defined on a simplex $\sigma\colon \Delta^k_{\mathrm{st}} \to M$ and a cochain $\eta\colon C_\ell(M) \to \mathbb{Z}$ by the formula

$$\sigma \frown \eta = \eta(\sigma \circ \delta^n \circ \cdots \circ \delta^{\ell+1}) \cdot \sigma \circ (\delta^0)^\ell.$$

Using an explicit expression for the boundary of such a chain, namely

$$\partial(\sigma \frown \eta) = (-1)^\ell\big(\partial\sigma \frown \eta - \sigma \frown \partial\eta\big),$$

one checks that the map $\frown$ induces a bilinear map

$$\frown\colon\ \mathrm{H}_k(M, \mathbb{Z}) \times \mathrm{H}^\ell(M, \mathbb{Z}) \longrightarrow \mathrm{H}_{k-\ell}(M, \mathbb{Z}),$$

which is still denoted by the same symbol and called the *cap-product*. By construction, the cup-product and the cap-product are related by the formula

$$\xi(\sigma \frown \eta) = (\eta \smile \xi)(\sigma)$$

for singular cochains $\eta \in C^k(M)$ and $\xi \in C^{k-\ell}(M)$ and singular chains $\sigma \in C_k(M)$.

• *Finite-dimensionality of Betti cohomology.* Let X be a quasi-projective algebraic variety over a field $k \subset \mathbb{C}$. Then the Betti cohomology $\mathrm{H}^*_{\mathrm{B}}(X)$ is a finite-dimensional rational vector space. Indeed, let $\overline{X} \subset \mathbb{P}^n$ be a projective compactification of X, and write $Z = \overline{X} \setminus X$ for the complement. By Exercise 2.71, we can embed $\mathbb{P}^n(\mathbb{C})$ as a bounded semi-algebraic subset of $\mathbb{R}^N$ for some big enough N, *i.e.* a finite union and intersection of subsets of the form $\{f(x_1, \ldots, x_N) \geqslant 0\}$, where f is a polynomial with real coefficients. By a theorem of Łojasiewicz [**Loj64**, Thm. 3], there is a finite triangulation of $\mathbb{P}^n(\mathbb{C})$ that induces a triangulation both of $\overline{X}(\mathbb{C})$ and $Z(\mathbb{C})$. This implies that $(\overline{X}(\mathbb{C}), Z(\mathbb{C}))$ is homeomorphic to a pair (K, K'), where K and K' are finite simplicial complexes (see [**Hat02**, p. 107] for the notion of simplicial complex). After, if necessary, passing to a refinement of K, we may assume that the intersection with K' of every simplex σ of K is either empty, all of σ, or one of its faces. In that case, the set

$$K'' = \bigcup_{\substack{\sigma \in K \\ \sigma \cap K' = \emptyset}} \sigma$$

is a finite simplicial complex homotopically equivalent to $K \setminus K'$. Therefore,

$$X(\mathbb{C}) = \overline{X}(\mathbb{C}) \setminus Z(\mathbb{C})$$

has the same homotopy type as a finite CW complex, from which it follows that its singular cohomology is finite-dimensional.

Remark 2.32. In fact, using Nagata's compactification theorem [**Con07**], the fact that every variety can be embedded as a closed subvariety of a smooth one, and Grauert's embedding theorem [**Gra58**], the argument carries over to any variety (separated scheme of finite type) not necessarily quasi-projective over k.

• *Poincaré duality.* Let M be an oriented compact topological manifold of dimension n. For instance, the topological space $X(\mathbb{C})$ associated with a smooth proper variety X of dimension d over the field of complex numbers is an oriented compact topological manifold of dimension $2d$. Then the top degree homology of each connected component M_α of M is free of rank 1, generated by the so-called *fundamental class* $[M_\alpha]$ of M_α, which can be obtained by choosing a triangulation of M_α and using the orientation to turn it into a singular chain:

$$\mathrm{H}_n(M_\alpha) \simeq \mathbb{Z} \cdot [M_\alpha].$$

The fundamental class of M is defined as the sum $[M] = \sum_\alpha [M_\alpha]$. *Poincaré duality* is the statement that, in each degree $0 \leqslant j \leqslant n$, the cap product with the fundamental class $[M]$ is an isomorphism

$$[M] \frown - \colon \mathrm{H}^j(M, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}_{n-j}(M, \mathbb{Z}). \tag{2.33}$$

Using the relation between the cup product and the cap product, as well as the universal coefficients theorem, Poincaré duality can also be interpreted on the cohomology with rational coefficients as the statement that the pairing

$$\mathrm{H}^j(M, \mathbb{Q}) \otimes \mathrm{H}^{n-j}(M, \mathbb{Q}) \longrightarrow \mathbb{Q} \tag{2.34}$$

given by taking cup-product and evaluating the resulting element of $\mathrm{H}^n(M, \mathbb{Q})$ on the fundamental class $[M]$ is perfect. Indeed, on singular cochains representing cohomology classes, this operation is given by

$$(\eta \smile \xi)([M]) = ([M] \frown \eta)(\xi),$$

so if for a given η the left-hand side vanishes for all ξ, then $\eta = 0$ thanks to (2.33). As usual, the perfect pairing (2.34) translates into an isomorphism

$$\mathrm{H}^j(M, \mathbb{Q}) \simeq \mathrm{Hom}(\mathrm{H}^{n-j}(M, \mathbb{Q}), \mathbb{Q}). \tag{2.35}$$

EXAMPLE 2.36. Let $M = \mathbb{C} \setminus \{0\}$ be the punctured complex plane. Since M is path-connected, $\mathrm{H}_0(M, \mathbb{Z})$ is the free abelian group generated by the singular simplex $\sigma_0 \colon \Delta^0_{\mathrm{st}} \to M$ that maps $1 \in \Delta^0_{\mathrm{st}}$ to $1 \in M$. Using that M is homotopic to the unit circle S^1 and the Mayer–Vietoris long exact sequence, one shows that the group $\mathrm{H}_1(M, \mathbb{Z})$ is also free of rank one, generated by the singular simplex

$$\sigma_1 \colon \Delta^1_{\mathrm{st}} \longrightarrow M, \qquad (t, 1-t) \longmapsto \exp(2\pi i t),$$

and that all other homology groups vanish. Indeed, the inclusion $S^1 \hookrightarrow M$ is a homotopy equivalence (an inverse being given by $z \mapsto z/|z|$), so it suffices to compute $\mathrm{H}_1(S^1, \mathbb{Z})$. Let us cover S^1 by two open arcs of circle U and V that intersect along two small intervals. Since U and V are contractible, and $U \cap V$ has the same homotopy type as two points, the vanishing of $\mathrm{H}_n(S^1, \mathbb{Z})$ for all $n \geqslant 2$ follows from (2.15), which also yields a long exact sequence

$$0 \longrightarrow \mathrm{H}_1(S^1, \mathbb{Z}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the map α is given by the matrix $\left(\begin{smallmatrix} -1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Therefore, $\mathrm{H}_1(S^1, \mathbb{Z}) = \mathrm{Ker}(\alpha)$ is isomorphic to $\mathbb{Z}$, and is actually generated by the simplex σ_1.

2.1.2. *Relative homology and cohomology.* There is also a relative version of singular homology, in which chains are allowed to have a non-zero boundary as long as it lies in a fixed subspace. If $\iota \colon N \hookrightarrow M$ is the inclusion of a topological subspace, then the morphism of chain complexes $\iota_* \colon C_*(N) \to C_*(M)$ is injective.

DEFINITION 2.37. The *relative singular complex* $C_*(M, N)$ is the cone of i_*:

$$C_*(M, N) = \mathrm{cone}(i_*).$$

Explicitly (see Remark A.28), this is the complex given by

$$C_n(M, N) = C_{n-1}(N) \oplus C_n(M)$$

in degree n, together with the differential

$$\partial(a, b) = (-\partial a, \partial b + \iota_*(a)). \tag{2.38}$$

DEFINITION 2.39. Let n be an integer. The *relative singular homology* in degree n of a pair (M, N) consisting of a topological space M and a subspace $N \subset M$ is the n-th homology group of the relative singular complex $C_*(M, N)$. We write

$$\mathrm{H}_n(M, N; \mathbb{Z}) = \mathrm{H}_n(C_*(M, N)).$$

Exercise 2.69 presents an alternative construction of relative homology in terms of the quotient complex $C_*(M)/C_*(N)$.

REMARK 2.40. An element of $\mathrm{H}_n(M, N; \mathbb{Z})$ is represented by a pair (σ_N, σ_M) consisting of singular chains $\sigma_N \in C_{n-1}(N)$ and $\sigma_M \in C_n(M)$ which is closed for the differential (2.38), *i.e.* satisfies $\partial_{n-1}\sigma_N = 0$ and $\partial_n \sigma_M = -\iota_* \sigma_N$. Since ι_* is injective, the singular chain σ_N is determined by the latter condition, which implies the former thanks to the relation $\partial_n \circ \partial_{n-1} = 0$. In other words, relative homology classes are represented by singular chains in M which are not necessarily closed but whose boundary is constrained to lie in the subspace N.

By design, $C_*(M, N)$ fits into a short exact sequence of complexes

$$0 \longrightarrow C_*(M) \xrightarrow{\mathrm{b}} C_*(M, N) \xrightarrow{\mathrm{a}} C_*(N)[-1] \longrightarrow 0, \tag{2.41}$$

where the map b sends b to $(0,b)$, and the map a sends (a,b) to a. Above, the shifted complex $C_*(N)[-1]$ has $C_{n-1}(N)$ as degree n term, with differential $-\partial_{n-1}$, so that the relation $\mathrm{H}_n(C_*(N)[-1]) = \mathrm{H}_{n-1}(N,\mathbb{Z})$ holds. The long exact sequence in homology associated with this short exact sequence (see (A.41)) then reads

$$\cdots \longrightarrow \mathrm{H}_n(M,\mathbb{Z}) \longrightarrow \mathrm{H}_n(M,N;\mathbb{Z}) \longrightarrow \mathrm{H}_{n-1}(N,\mathbb{Z}) \longrightarrow \mathrm{H}_{n-1}(M,\mathbb{Z}) \longrightarrow \mathrm{H}_{n-1}(M,N;\mathbb{Z}) \longrightarrow \cdots, \tag{2.42}$$

and the connecting morphisms are given by

$$-\iota_* \colon \mathrm{H}_*(N,\mathbb{Z}) \longrightarrow \mathrm{H}_*(M,\mathbb{Z}),$$

the opposite of the maps induced by the inclusion $\iota\colon N \hookrightarrow M$.

Remark 2.43. The reason why the negative sign appears in the connecting morphism is explained in Remark A.86. Instead of looking at the short exact sequence (2.41), we may consider the distinguished triangle

$$C_*(N) \xrightarrow{\iota_*} C_*(M) \longrightarrow C_*(M,N) \longrightarrow C_*(N)[-1]$$

(in the language of Section A.3) to obtain the long exact sequence of abelian groups

$$\cdots \longrightarrow \mathrm{H}_n(N,\mathbb{Z}) \xrightarrow{\iota_*} \mathrm{H}_n(M,\mathbb{Z}) \longrightarrow \mathrm{H}_{n-1}(M,N;\mathbb{Z}) \longrightarrow \mathrm{H}_{n-1}(N,\mathbb{Z}) \longrightarrow \mathrm{H}_{n-1}(M,\mathbb{Z}) \longrightarrow \cdots$$

where the maps from $\mathrm{H}_n(N,\mathbb{Z})$ to $\mathrm{H}_n(M,\mathbb{Z})$ are not affected by a sign anymore.

Example 2.44. Consider $M = \mathbb{C}\setminus\{0\}$ and let $N = \{p,q\} \subseteq M$ be a subspace consisting of two distinct points p and q. Let $\sigma_2 \colon \Delta^1_{\mathrm{st}} \to M$ be any continuous map such that $\sigma_2((0,1)) = p$ and $\sigma_2((1,0)) = q$. Then

$$\partial\sigma_2 = p - q \in C_0(N),$$

so σ_2 defines a relative chain. It follows from the long exact sequence (2.42) that the only non-trivial relative homology group is $\mathrm{H}_1(M,N;\mathbb{Z})$, which has a basis given by the chain σ_1 from Example 2.36 and σ_2 (see Figure 2). Indeed, for dimension reasons the non-trivial part of the long exact sequence reads

$$0 \longrightarrow \mathrm{H}_1(M,\mathbb{Z}) \longrightarrow \mathrm{H}_1(M,N;\mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_0(N,\mathbb{Z}) \longrightarrow \mathrm{H}_0(M,\mathbb{Z}) \longrightarrow 0, \tag{2.45}$$

so the image of the generator σ_1 of $\mathrm{H}_1(M,\mathbb{Z})$ defines a non-zero class, and any relative chain whose boundary is a generator of $\mathrm{Ker}(\mathrm{H}_0(N,\mathbb{Z}) \to \mathrm{H}_0(M,\mathbb{Z}))$, for example σ_2, completes σ_1 to a $\mathbb{Z}$-basis of $\mathrm{H}_1(M,N;\mathbb{Z})$.

In a similar way, one defines *relative cohomology* groups. Let $\iota\colon N \hookrightarrow M$ be the inclusion of a topological subspace, and $\iota^*\colon C^*(M) \to C^*(N)$ the induced map on cochain complexes. We consider the total complex

$$C^*(M,N) = \mathrm{Tot}(\iota^*) = \mathrm{cone}(-\iota^*)[-1],$$

as in Example A.35. Explicitly, this is the complex given in degree n by

$$C^n(M,N) = C^n(M) \oplus C^{n-1}(N)$$

together with the differential

$$\mathrm{d}(a,b) = (\mathrm{d}a, -\mathrm{d}b + \iota^*(a)).$$

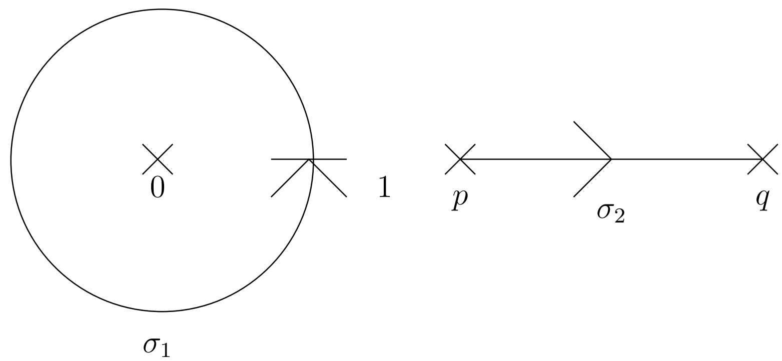


Figure 2. A basis of $\mathrm{H}_1(\mathbb{C}\setminus\{0\},\{p,q\};\mathbb{Z})$

It fits into a distinguished triangle (see Section A.3.2)

$$C^*(M,N)\longrightarrow C^*(M)\xrightarrow{\iota^*}C^n(N)\longrightarrow C^*(M,N)[1].$$

Definition 2.46. Let n be an integer. The *relative singular cohomology* in degree n of the pair (M,N) is the n-th cohomology group of the complex $C^*(M,N)$. We denote it by

$$\mathrm{H}^n(M,N;\mathbb{Z})=\mathrm{H}^n(C^*(M,N)).$$

By construction, the relative singular cohomology groups sit into a long exact sequence of abelian groups

$$\begin{aligned}\cdots\longrightarrow \mathrm{H}^n(M,N;\mathbb{Z})\longrightarrow \mathrm{H}^n(M,\mathbb{Z})\xrightarrow{\iota^*}\mathrm{H}^n(N,\mathbb{Z})&\\ \longrightarrow \mathrm{H}^{n+1}(M,N;\mathbb{Z})\longrightarrow \mathrm{H}^{n+1}(M,\mathbb{Z})\longrightarrow\cdots&\end{aligned}\tag{2.47}$$

Remark 2.48. It is instructive to compare the definitions of relative homology and cohomology. In the first case, we use the cone of the morphism ι_*, whereas in the second case we use the total complex of ι^*. Both points of view are equivalent according to Example A.35. The use of either the cone or of the total complex of a morphism of complexes depends on whether we want the degree of the resulting complex to agree with the degree of the source or the target complex.

2.1.3. *Singular cohomology as sheaf cohomology.* Under mild assumptions on the topological space M, the singular cohomology groups $\mathrm{H}^*(M,\mathbb{Z})$ can be identified with the sheaf cohomology groups of the constant sheaf. More precisely, let $\underline{\mathbb{Z}}_M$ be the sheaf associated with the presheaf that assigns to each open subset V of M the abelian group $\mathbb{Z}$, with all restriction maps equal to the identity. Its sections are

$$\underline{\mathbb{Z}}_M(V)=\{\text{locally constant functions } V\longrightarrow\mathbb{Z}\},$$

that is, those functions with the property that each point of V has an open neighborhood on which they are constant.

Theorem 2.49. *Assume that M is locally contractible and that every open subset of M is paracompact. There is a canonical isomorphism*

$$\mathrm{H}^*(M,\mathbb{Z})\simeq\mathrm{H}^*(M,\underline{\mathbb{Z}}_M),\tag{2.50}$$

where the left-hand side is the singular cohomology of M and the right-hand side is the sheaf cohomology of $\underline{\mathbb{Z}}_M$.

The proof of this result is presented in Section A.9.8 (see Theorem A.304). The main idea is to introduce the sheaves $\widetilde{C}^n$ on M associated with the presheaves of singular cochains $U \longmapsto C^n(U)$. As n varies, they form a complex along with the differentials $\mathrm{d}^n \colon \widetilde{C}^n \to \widetilde{C}^{n+1}$. The construction of the isomorphism (2.50) then relies on the following three properties of this complex:

i) The natural morphism $(C^*(M), \mathrm{d}^*) \longrightarrow (\widetilde{C}^*(M), \mathrm{d}^*)$ from the singular chain complex to the complex of global sections is a quasi-isomorphism.
ii) All the sheaves $\widetilde{C}^n$ are flasque, and hence hypercohomology can be computed using global sections: $\mathbb{H}^*(M, \widetilde{C}^*) \simeq \mathrm{H}^*(M, \widetilde{C}^*(M))$.
iii) The map of sheaves $\underline{\mathbb{Z}}_M \to \widetilde{C}^0$ associated with the map of presheaves that sends 1 to the singular 0-cochain $(\sum n_x[x] \mapsto \sum n_x)$ induces a quasi-isomorphism $\underline{\mathbb{Z}}_M \to \widetilde{C}^*$.

Using the universal coefficients theorem and the fact that tensoring by a field is an exact operation, we deduce that the same result holds for the cohomology with coefficients in a field of characteristic zero, for instance $\mathbb{Q}$.

In the same spirit, relative cohomology, as introduced in Section 2.1.2, can be written as the hypercohomology of a complex of sheaves. Indeed, let M be a topological space satisfying the assumptions of Theorem 2.49, and let $\iota \colon N \to M$ be the inclusion of a closed subspace (see Remark 2.52 below for the general case). Consider the *direct image* sheaf $\iota_* \underline{\mathbb{Z}}_N$ on M whose sections on an open subset V are

$$(\iota_* \underline{\mathbb{Z}}_N)(V) = \underline{\mathbb{Z}}_N(V \cap N).$$

Note that this definition makes sense since $V \cap N$ is an open subset of N. The stalks of the sheaf $\iota_* \underline{\mathbb{Z}}_N$ are

$$(\iota_* \underline{\mathbb{Z}}_N)_x = \begin{cases} \mathbb{Z}, & \text{if } x \in N, \\ 0, & \text{if } x \notin N. \end{cases}$$

If N is locally contractible as well, then there is an isomorphism

$$\mathrm{H}^*(N, \mathbb{Z}) \simeq \mathrm{H}^*(M, \iota_* \underline{\mathbb{Z}}_N).$$

Indeed, the functor ι_* is exact (Exercise A.323), so that we have $\iota_* \underline{\mathbb{Z}}_N = R\iota_* \underline{\mathbb{Z}}_N$, and hence $\mathrm{H}^*(M, \iota_* \underline{\mathbb{Z}}_N) = \mathrm{H}^*(N, \underline{\mathbb{Z}}_N)$ by Proposition A.278. Moreover, there is a canonical map of sheaves $\underline{\mathbb{Z}}_M \to \iota_* \underline{\mathbb{Z}}_N$ which is abstractly given by the adjunction morphism (A.286) on noting that $\underline{\mathbb{Z}}_N = \iota^{-1} \underline{\mathbb{Z}}_M$, or more concretely by sending a locally constant function on V to its restriction to $V \cap N$.

THEOREM 2.51. *Let M be a locally contractible topological space such that every open subset of M is paracompact. Let $\iota \colon N \to M$ be the inclusion of a closed locally contractible subspace. There is a canonical isomorphism*

$$\mathrm{H}^*(M, N; \mathbb{Z}) \simeq \mathbb{H}^*(M, \underline{\mathbb{Z}}_M \to \iota_* \underline{\mathbb{Z}}_N).$$

PROOF. Since the functor ι_* is exact, it maps the quasi-isomorphism $\underline{\mathbb{Z}}_N \to \widetilde{C}^*_N$ to a quasi-isomorphism $\iota_* \underline{\mathbb{Z}}_N \to \iota_* \widetilde{C}^*_N$. Besides, the sheaves $\iota_* \widetilde{C}^n_N$ are flasque, since the restriction maps $(\iota_* \widetilde{C}^n_N)(V) \to (\iota_* \widetilde{C}^n_N)(V')$ for open subsets $V' \subset V$ are by definition the restriction maps $\widetilde{C}^n_N(V \cap N) \to \widetilde{C}^n_N(V' \cap N)$, which are surjective as the sheaves $\widetilde{C}^n_N$ are flasque. To compute the hypercohomology group on the right-hand side, we can therefore replace the complex of sheaves $\underline{\mathbb{Z}}_M \to \iota_*\underline{\mathbb{Z}}_N$ with the double complex of global sections $\widetilde{C}^*_M(M) \to (\iota_*\widetilde{C}^*_N)(M) = \widetilde{C}^*_N(N)$. Since the source and the target are quasi-isomorphic to the singular cochain complexes $C^*(M)$ and $C^*(N)$ respectively, we find:

$$\mathbb{H}^*(M, \underline{\mathbb{Z}}_M \to \iota_*\underline{\mathbb{Z}}_N) \simeq \mathrm{H}^*(\mathrm{Tot}(C^*(M) \xrightarrow{\iota^*} C^*(N))),$$

which is by definition the relative singular cohomology of the pair (M, N). □

REMARK 2.52. If the subspace N is locally contractible but not closed, then the functor ι_* is not necessarily exact. A similar result still holds, upon replacing the sheaf $\iota_*\underline{\mathbb{Z}}_N$ with the complex of sheaves $R\iota_*\underline{\mathbb{Z}}_N$. That is, there is an isomorphism

$$\mathrm{H}^*(M, N; \mathbb{Z}) \simeq \mathbb{H}^*(M, \underline{\mathbb{Z}}_M \to R\iota_*\underline{\mathbb{Z}}_N)$$

Let $N \subset M$ be a closed subspace. Let $U = M \setminus N$ be the complementary open subset, and let $j\colon U \hookrightarrow M$ denote the inclusion. Consider the *extension by zero* sheaf $j_!\underline{\mathbb{Z}}_U$, which is the sheaf on M associated with the presheaf

$$V \longmapsto \begin{cases} \underline{\mathbb{Z}}_U(V), & \text{if } V \subset U, \\ 0, & \text{otherwise.} \end{cases} \tag{2.53}$$

The stalks of this sheaf are

$$(j_!\underline{\mathbb{Z}}_U)_x = \begin{cases} \mathbb{Z}, & \text{if } x \in U, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, there is a canonical map of sheaves $j_!\underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}}_M$ given by the adjunction morphism (A.299) on noting that $\underline{\mathbb{Z}}_U = j^{-1}\underline{\mathbb{Z}}_M$ (concretely, it is induced by the map of presheaves from (2.53) to $\underline{\mathbb{Z}}_M$ that, for each open subset $V \subset U$, sends a locally constant function $V \to \mathbb{Z}$ to itself regarded as a function on $V \subset M$). These maps fit into an exact sequence of sheaves of abelian groups

$$0 \longrightarrow j_!\underline{\mathbb{Z}}_U \longrightarrow \underline{\mathbb{Z}}_M \longrightarrow \iota_*\underline{\mathbb{Z}}_N \longrightarrow 0, \tag{2.54}$$

as is readily checked stalk by stalk.

REMARK 2.55. Thanks to the exact sequence (2.54), the complex $\underline{\mathbb{Z}}_M \to \iota_*\underline{\mathbb{Z}}_N$ is quasi-isomorphic to the complex concentrated in degree zero $j_!\underline{\mathbb{Z}}_U$, and hence relative cohomology can also be expressed as the cohomology of a single sheaf rather than the hypercohomology of a complex of sheaves, namely

$$\mathrm{H}^*(M, N; \mathbb{Z}) \simeq \mathrm{H}^*(M, j_!\underline{\mathbb{Z}}_U).$$

Moreover, combining (2.54) with Theorem 2.49, we recover the long exact sequence (2.47) of relative cohomology.

2.1.4. *Cohomology with compact support and Poincaré duality.* Once we have interpreted singular cohomology as sheaf cohomology, we can apply Definition A.295 to define cohomology with compact support.

DEFINITION 2.56. Let M be a locally contractible topological space such that every open subset of M is paracompact. The *singular cohomology with compact support* of M is the cohomology with compact support of the sheaf $\underline{\mathbb{Z}}$. We write

$$\mathrm{H}^*_c(M, \mathbb{Z}) = \mathrm{H}^*_c(M, \underline{\mathbb{Z}}).$$

One defines similarly cohomology with compact support for other coefficients like $\mathbb{Q}$.

REMARK 2.57. Another way to define cohomology with compact support is to consider the complex $C_*^{\mathrm{BM}}(M)$ of *locally finite chains* on a topological space M. A locally finite chain is a product $\sigma = \prod_\alpha n_\alpha \sigma_\alpha$ of singular simplices (that is, an element of the direct product over all simplices on M, as opposed to the direct sum in the definition of singular chains) such that every point of M has a neighborhood U that only intersects σ at a finite number of σ_α. This gives rise to *Borel–Moore* homology. Then the dual complex

$$C^*_{\mathrm{BM}}(M) = \mathrm{Hom}(C_*^{\mathrm{BM}}(M), \mathbb{Z})$$

gives rise to cohomology with compact support. For example, let γ_n be the straight path from e^n to e^{n+1}. Then $\prod_{n\in\mathbb{Z}} \gamma_n$ is a locally finite 1-chain on the topological space $M = \mathbb{C} \setminus \{0\}$, which defines a non-trivial class in the first Borel–Moore homology group of M. The local finiteness property is the key to integrate differential forms with compact support along locally finite chains.

The cohomology with compact support allows us to extend Poincaré duality, in the form given in (2.35), to non-compact topological manifolds.

THEOREM 2.58 (Poincaré duality). *Let M be an oriented topological manifold of dimension n with a finite number of connected components. For each $0 \leqslant j \leqslant n$, there is a canonical isomorphism*

$$\mathrm{H}^j(M, \mathbb{Q}) \simeq \mathrm{Hom}(\mathrm{H}_c^{n-j}(M, \mathbb{Q}), \mathbb{Q}).$$

⋆ ⋆ ⋆

EXERCISE 2.59. Prove that the boundary maps (2.2) in the definition of singular homology satisfy $\partial_{n-1} \circ \partial_n = 0$ for all integers $n \geqslant 1$. Deduce that the coboundary maps (2.8) in the definition of singular cohomology satisfy $\mathrm{d}^{n+1} \circ \mathrm{d}^n = 0$ for all integers $n \geqslant 0$.

EXERCISE 2.60. Prove that the map f_* from (2.7) is compatible with the boundary maps, and hence defines a homomorphism in homology. Deduce the same result for singular cochains.

EXERCISE 2.61. Use the Mayer–Vietoris long exact sequences (2.15) and (2.16) and Example 2.36 to compute the homology and the cohomology of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$.

EXERCISE 2.62. In this exercise, we compute the homology and the cohomology of the real projective plane.

i) Let M be the *Möbius band*, defined as the quotient

$$M = [0,1] \times (0,1)/\sim,$$

where $\sim$ is the equivalence relation generated by $(0,x) \sim (1, 1-x)$. Prove that the first homology group of M is generated by the singular simplex

$$\sigma\colon \Delta^1_{\mathrm{st}} \to M, \qquad \sigma(t, 1-t) = (t, 1/2),$$

and that the simplex

$$\sigma_1(t,1-t) = \begin{cases} (2t, 3/4), & \text{if } t \leqslant 1/2, \\ (2t-1, 1/4), & \text{if } t \geqslant 1/2, \end{cases}$$

is closed and represents the same class as 2σ in $\mathrm{H}_1(M, \mathbb{Z})$.

ii) Let $\mathbb{P}^2(\mathbb{R})$ be the real projective plane, viewed as the quotient

$$\mathbb{P}^2(\mathbb{R}) = S^2/\sim$$

of the two-dimensional sphere S^2 by the equivalence relation generated by $x \sim -x$. Show that $\mathbb{P}^2(\mathbb{R})$ can be covered by an open subset homeomorphic to the unit disc in $\mathbb{R}^2$ and an open subset homeomorphic to the Möbius band.

iii) Use the Mayer–Vietoris sequences in singular homology (2.15) and singular cohomology (2.16) to compute

$$\mathrm{H}_i(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } i = 1, \\ 0, & \text{if } i = 2. \end{cases}$$

$$\mathrm{H}^i(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{if } i = 1, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } i = 2. \end{cases}$$

iv) Use the universal coefficients theorem to explain the discrepancy between homology and cohomology.

Exercise 2.63. The goal of this exercise is to sketch a proof of the universal coefficients theorem. Let M be a topological space.

i) Apply the variant of Proposition A.210 for left exact contravariant functors (that uses projective resolutions) to the chain complex $C_*(M, \mathbb{Z})$ and the functor $\mathrm{Hom}(\cdot, G)$ to produce a spectral sequence

$$\mathrm{Ext}^p(\mathrm{H}_q(M, \mathbb{Z}), G) \Longrightarrow \mathrm{H}^{p+q}(M, G). \tag{2.64}$$

ii) Use the vanishing $\mathrm{Ext}^k(A, B) = 0$ for all abelian groups A, B and all $k \geqslant 2$ (Exercise A.134), to prove that the spectral sequence (2.64) degenerates at the term E_2 and gives rise to short exact sequences

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H}_{n-1}(M, \mathbb{Z}), G) \longrightarrow \mathrm{H}^n(M, G) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(M, \mathbb{Z}), G) \longrightarrow 0.$$

iii) Let $G = \mathbb{Z}$. Conclude that whenever $\mathrm{H}_{n-1}(M, \mathbb{Z})$ is torsion-free, the Ext group vanishes and we get an isomorphism between $\mathrm{H}^n(M, \mathbb{Z})$ and the linear dual of $\mathrm{H}_n(M, \mathbb{Z})$.

iv) Apply now the variant of Proposition A.210 for right exact covariant functors and bounded above complexes (that uses again projective resolutions) to the chain complex $C_*(M, \mathbb{Z})$ and the functor $\cdot \otimes G$ to show the existence of the exact sequence (2.18).

Exercise 2.65. Prove that the maps F and G defined in (2.20) and (2.22) satisfy the following properties:

i) They are morphisms of chain complexes.

ii) They are functorial in M_1 and M_2, and hence define natural transformations of functors

$$\alpha \colon C_*(\cdot \times \cdot) \to C_*(\cdot) \otimes C_*(\cdot) \quad \text{and} \quad \beta \colon C_*(\cdot) \otimes C_*(\cdot) \to C_*(\cdot \times \cdot).$$

Exercise 2.66. Prove that the map (2.25) is well defined.

Exercise 2.67. Let M_1 and M_2 be topological spaces. Assume that $\mathrm{H}^i(M_1, \mathbb{Q})$ is finite-dimensional in all degrees $i \geqslant 0$. Combine the Künneth formula in homology with the universal coefficients theorem to obtain an isomorphism

$$\mathrm{H}^n(M_1 \times M_2, \mathbb{Q}) \simeq \bigoplus_{i+j=n} \mathrm{H}^i(M_1, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{H}^j(M_2, \mathbb{Q}).$$

Exercise 2.68. Compute the homology and the cohomology of $\mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})$ and deduce that the Künneth formulas (2.19) and (2.27) do not hold with integral coefficients instead of rational coefficients.

Exercise 2.69 (An alternative definition of relative homology). We keep the notation from Section 2.1.2. Given a topological space M and a subspace N, show that the boundary maps ∂_n yield a complex

$$\cdots \longrightarrow \frac{C_n(M)}{C_n(N)} \longrightarrow \frac{C_{n-1}(M)}{C_{n-1}(N)} \longrightarrow \cdots \tag{2.70}$$

which is quasi-isomorphic to the relative singular homology complex $C_*(M, N)$ (see Exercise A.93). Therefore, one can also define the relative homology groups of the pair (M, N) as the homology groups of the complex (2.70). Note that the corresponding long exact sequence is (2.42) shifted by one.

Exercise 2.71. Let $n \geqslant 0$ be an integer. Prove that the map

$$[x_0 \colon \cdots : x_n] \longmapsto \left(\ldots, \frac{\mathrm{Re}(x_i \bar{x}_j)}{\sum_{m=0}^n x_m \bar{x}_m}, \frac{\mathrm{Im}(x_i \bar{x}_j)}{\sum_{m=0}^n x_m \bar{x}_m}, \ldots \right)$$

induces a homeomorphism from complex projective space $\mathbb{P}^n(\mathbb{C})$ onto a closed bounded semi-algebraic subset of $\mathbb{R}^{2(n+1)^2}$ (see [**HMS17**, p. 62]).

2.2. Algebraic de Rham cohomology. Inspired by ideas of Atiyah and Hodge, Grothendieck introduced the de Rham cohomology of algebraic varieties over fields of characteristic zero in his paper [**Gro66**], which was written shortly after Hironaka's proof of the resolution of singularities. In this section, we explain the definition of algebraic de Rham cohomology and compute it in a few examples.

2.2.1. *Motivation: de Rham's theorem in differential geometry.* Before going into Grothendieck's construction, we give a quick review of the more familiar objects in differential geometry. The reader is encouraged to consult the book by Bott and Tu [**BT82**] for a very nice exposition of the subject.

Let M be a differentiable manifold of dimension d. We denote by TM its tangent bundle and by T^*M its *cotangent bundle*, which is by definition the dual of TM; they are both vector bundles of rank d over M. For each integer $p \geqslant 0$, consider the p-th exterior power

$$\pi \colon \Lambda^p T^*M \longrightarrow M$$

of the cotangent bundle and the sheaf $\mathcal{E}_M^p$ of real vector spaces of smooth sections of $\Lambda^p T^*M$. For each open subset U of M, the sections of $\mathcal{E}_M^p$ are given by

$$\mathcal{E}_M^p(U) = \{\mathcal{C}^\infty\text{-maps } f \colon U \to \Lambda^p T^*M \text{ such that } \pi \circ f \text{ is the inclusion } U \hookrightarrow M\}.$$

In particular, $\mathcal{E}_M^0$ is the sheaf of smooth functions on M. Sections of $\mathcal{E}_M^p$ are called *smooth differential p-forms*, or simply differential p-forms, and can be written in a local chart with coordinates $x_1, \dots, x_d$ as

$$\omega = \sum_{1 \leqslant i_1 < i_2 < \cdots i_p \leqslant d} f_{i_1,\dots,i_p}(x_1, \dots, x_d) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \tag{2.72}$$

for some $\mathcal{C}^\infty$-functions $f_{i_1,\dots,i_p}(x_1, \dots, x_d)$ on the local chart. Write

$$E^p(M) = \Gamma(M, \mathcal{E}^p_M)$$

for the real vector space of global sections of $\mathcal{E}^p_M$, and set

$$E^*(M) = \bigoplus_{p=0}^{d} E^p(M).$$

When we want to emphasize that these are differential forms with real coefficients, we write $E^*(M, \mathbb{R})$ instead of $E^*(M)$. The space of differential forms with complex coefficients is defined as the complexification $E^*(M, \mathbb{C}) = E^*(M) \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 2.73. The *exterior derivative*

$$\mathrm{d}\colon \mathcal{E}^*_M \longrightarrow \mathcal{E}^*_M$$

is the unique $\mathbb{R}$-linear map of sheaves of degree 1 (*i.e.* that maps $\mathcal{E}^p_M$ to $\mathcal{E}^{p+1}_M$) that satisfies the following two conditions:

i) If f is a smooth function, then $\mathrm{d}f$ is given in local coordinates by

$$\mathrm{d}f = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

ii) If α is a local section of $\mathcal{E}^p_M$, then the equality

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta$$

holds for every local section β of $\mathcal{E}^*_M$.

It follows from the definition (Exercise 2.121) that the exterior derivative satisfies $\mathrm{d}^2 = 0$. Taking global sections, we thus get a complex

$$0 \longrightarrow E^0(M) \xrightarrow{\mathrm{d}} E^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} E^d(M) \longrightarrow 0$$

of real vector spaces, the *de Rham complex* of M.

Definition 2.74. The *de Rham cohomology* $\mathrm{H}^*_{\mathrm{dR}}(M)$ of M is the cohomology of this complex. When we want to emphasize that the de Rham cohomology is a real vector space, we write $\mathrm{H}^*_{\mathrm{dR}}(M, \mathbb{R})$ instead.

A classical theorem of de Rham asserts that singular cohomology with real coefficients $\mathrm{H}^*(M, \mathbb{R})$ can be computed using differential forms. As was mentioned in Remark 2.12 ii), if one replaces singular chains with smooth singular chains in the definition of singular homology the resulting groups are the same. We denote by $C^{\mathrm{sm}}_*(M, \mathbb{R})$ the complex of smooth singular chains with real coefficients and, as in Remark A.307, we define smooth singular cochains as the linear dual

$$S^n(M, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(C^{\mathrm{sm}}_n(M, \mathbb{R}), \mathbb{R}).$$

The vector spaces $S^n(M, \mathbb{R})$ form a complex that computes the singular cohomology with real coefficients of M. At this point, the advantage of working with smooth chains is that we can integrate differential forms along them: given a smooth simplex $\sigma\colon \Delta^n_{\mathrm{st}} \to M$ and a differential form $\omega \in E^*(M)$, the integral $\int_\sigma \omega$ is defined as

$$\int_\sigma \omega = \int_{\Delta^n_{\mathrm{st}}} \sigma^*\omega, \tag{2.75}$$

and extended linearly to all smooth singular chains. We thus get a map

$$E^*(M) \longrightarrow S^*(M, \mathbb{R}) \tag{2.76}$$

that associates with each differential form $\omega \in E^*(M)$ the *integration functional*

$$\begin{aligned} \textstyle\int \omega\colon\ S_*(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ \sigma &\longmapsto \textstyle\int_\sigma \omega. \end{aligned}$$

LEMMA 2.77. *The map* (2.76) *is a functorial morphism of complexes.*

PROOF. Functoriality means that, for any morphism $f\colon M \to M'$ of differentiable manifolds, the diagram

$$\begin{tikzcd} E^*(M') \arrow[r] \arrow[d, "f^*"'] & S^*(M', \mathbb{R}) \arrow[d, "f^*"] \\ E^*(M) \arrow[r] & S^*(M, \mathbb{R}) \end{tikzcd} \tag{2.78}$$

commutes. This amounts to the equality

$$\int_\sigma f^*\omega = \int_{f_*\sigma} \omega,$$

which readily follows from the definition (2.75) of the integral along a chain. Being a morphism of complexes means that the map (2.76) commutes with the differentials on the de Rham and the singular chain complexes. This amounts to the equality

$$\int_\sigma \mathrm{d}\omega = \int_{\partial\sigma} \omega,$$

which is the content of the classical Stokes's theorem. □

In view of the lemma, the morphism (2.76) induces a functorial linear map

$$\int\colon\ \mathrm{H}^*_{\mathrm{dR}}(M, \mathbb{R}) \longrightarrow \mathrm{H}^*(M, \mathbb{R}),$$

which we still call the integration functional. De Rham's theorem is the statement that this map is an isomorphism or, equivalently, that the original morphism of complexes (2.76) is a quasi-isomorphism.

THEOREM 2.79 (de Rham). *Let M be a differentiable manifold of dimension d. For each $0 \leqslant j \leqslant d$, the map*

$$\mathrm{H}^j_{\mathrm{dR}}(M, \mathbb{R}) \longrightarrow \mathrm{H}^j(M, \mathbb{R})$$

that sends the class of a differential form ω to the integration functional $\int \omega$ is an isomorphism. This isomorphism is functorial for maps of differentiable manifolds.

PROOF. Recall that $E^p(M)$ is defined, for each $p \geqslant 0$, as the real vector space of global sections of the sheaf $\mathcal{E}_M^p$ of smooth differential p-forms on M. The existence of partitions of unity implies that all the sheaves $\mathcal{E}_M^p$ are *fine*, and hence acyclic (see Example A.262 and Lemma A.263). Moreover, by the differentiable Poincaré lemma (see *e.g.* [**Lee13**, Thm. 17.14] or Theorem 2.145 below for the holomorphic Poincaré lemma, which is proven in the same way) the inclusion of sheaves $\underline{\mathbb{R}}_M \to \mathcal{E}_M^0$ that sends a locally constant function to the corresponding $\mathcal{C}^\infty$-function fits into an exact sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{R}}_M \longrightarrow \mathcal{E}_M^0 \longrightarrow \mathcal{E}_M^1 \longrightarrow \cdots . \tag{2.80}$$

Let $\tilde{S}_M^*$ be the complex of sheaves of smooth cochains on M, *i.e.* the sheaves associated with the presheaves $U \mapsto S^n(U, \mathbb{R})$, as explained in Remark A.307. The map (2.76) induces a morphism of complexes of sheaves

$$\int : \mathcal{E}_M^* \longrightarrow \tilde{S}_M^*$$

that fits into a commutative diagram

$$\begin{tikzcd} \underline{\mathbb{R}}_M \arrow[r] \arrow[dr] & \mathcal{E}_M^* \arrow[d, "\int"] \\ & \tilde{S}_M^*. \end{tikzcd}$$

For each integer $0 \leqslant j \leqslant d$, taking the cohomology in degree j of these complexes of sheaves we get a commutative diagram

$$\begin{tikzcd} \mathrm{H}^j(M, \underline{\mathbb{R}}_M) \arrow[r] \arrow[dr] & \mathrm{H}^j_{\mathrm{dR}}(M, \mathbb{R}). \arrow[d] \\ & \mathrm{H}^j(M, \mathbb{R}) \end{tikzcd}$$

The horizontal arrow is an isomorphism by the exactness of (2.80) and the fact that the sheaves $\mathcal{E}_M^p$ are acyclic; the diagonal arrow is an isomorphism by Theorem A.304 and Remark A.307. Thus, the vertical arrow is an isomorphism as well. The claim about functoriality then follows from the commutativity of diagram (2.78). $\square$

2.2.2. *Kähler differentials.* Remarkably enough, when M is the differential manifold underlying a smooth affine complex algebraic variety X, it suffices to consider differential forms with regular functions on X (as opposed to all $\mathcal{C}^\infty$ functions) as coefficients to capture all de Rham cohomology classes.

From now on, we will assume that the reader is familiar with the rudiments of the language of schemes, as can be found in the first sections of Chapter II of Hartshorne's book [**Har77**]. We first introduce the notion of Kähler differentials, the substitute for the differential forms (2.72) that will allow for a purely algebraic definition of de Rham cohomology.

Let k be a field of characteristic zero and let A be a finitely generated reduced k-algebra. Being *reduced* means that there are no non-zero nilpotent elements in A; that is, if $x^n = 0$ for some integer $n \geqslant 1$, then $x = 0$. The spectrum $X = \mathrm{Spec}(A)$ is then an affine algebraic variety over k.

DEFINITION 2.81. Let M be an A-module. A k-linear *derivation* of A into M is a k-linear map $D\colon A \to M$ satisfying the Leibniz rule

$$D(ab) = aD(b) + bD(a) \tag{2.82}$$

for all elements $a, b \in A$.

Using (2.82), the k-linearity of D is equivalent to the condition $D(r) = 0$ for all $r \in k$. In particular, elements of k are "constants" for the derivation.

DEFINITION 2.83. The A-module of *Kähler differentials* $\Omega^1_{A/k}$ is the quotient of the free A-module generated by symbols $\mathrm{d}a$, for all $a \in A$, by the submodule spanned by the elements

$$\mathrm{d}r, \qquad \mathrm{d}(a+b) - \mathrm{d}a - \mathrm{d}b, \qquad \mathrm{d}(ab) - a\mathrm{d}b - b\mathrm{d}a \tag{2.84}$$

for all $r \in k$ and all $a, b \in A$.

By construction, the map $\mathrm{d}\colon A \to \Omega^1_{A/k}$ that sends a to $\mathrm{d}a$ is a k-linear derivation. It is in fact the *universal* k-derivation: for any k-linear derivation $D\colon A \to M$, there is a unique morphism of A-modules $\varphi\colon \Omega^1_{A/k} \to M$ making the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{d}"] \arrow[dr, "D"'] & \Omega^1_{A/k} \arrow[d, "\varphi"] \\ & M \end{tikzcd} \tag{2.85}$$

commutative. The following example is at the base of many computations in algebraic de Rham cohomology.

EXAMPLE 2.86. Set $A = k[x_1, \dots, x_d]$. Then $\Omega^1_{A/k}$ is the free A-module generated by $\mathrm{d}x_1, \dots, \mathrm{d}x_d$. Indeed, let $D\colon A \to M$ be any k-linear derivation. It follows from the Leibniz rule (2.82) that the image by D of a polynomial $f \in A$ is equal to

$$D(f) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} D(x_i),$$

where $\partial f/\partial x_i \in A$ stands for the partial derivative of f with respect to x_i. From this it follows that the free A-module generated by $\mathrm{d}x_1, \dots, \mathrm{d}x_d$, along with the k-linear map $f \mapsto \sum_i \partial f/\partial x_i \, \mathrm{d}x_i$, is the universal derivation.

More generally, if A is the k-algebra

$$A = k[x_1, \dots, x_n]/(f_1, \dots, f_m) \tag{2.87}$$

for some polynomials $f_1, \dots, f_m$, then the A-module of Kähler differentials $\Omega^1_{A/k}$ has generators $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ and relations

$$\mathrm{d}f_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \mathrm{d}x_i = 0$$

for $j = 1, \dots, m$. In other words, it is given by the quotient

$$\Omega^1_{A/k} = (A\mathrm{d}x_1 \oplus \cdots \oplus A\mathrm{d}x_n)/(\mathrm{d}f_1, \dots, \mathrm{d}f_m).$$

The first part of Example 2.86, where we found that the Kähler differentials of affine space form a free module can be generalized to any smooth affine variety. In general, we will only get locally free modules. Recall that an A-module M is said to be *locally free* of some rank d if, for each maximal ideal $\mathfrak{m}$ of A, the localization $M_\mathfrak{m}$ is a free module of rank d over the ring $A_\mathfrak{m}$, *i.e.* it is isomorphic to $A_\mathfrak{m}^{\oplus d}$.

PROPOSITION 2.88. *If $X = \mathrm{Spec}(A)$ is smooth of dimension d, then the module of Kähler differentials $\Omega^1_{A/k}$ is locally free of rank d.*

PROOF. Since we are assuming that k has characteristic zero, this follows from instance from [**Har77**, Chap. II, Thm. 8.8]. Exercise 2.130 illustrates why the smoothness condition is necessary for the statement to hold. □

By the *jacobian criterion for smoothness*, an algebra A of the form (2.87) is smooth of dimension d if, for every prime ideal $\mathfrak{p}$ of A, corresponding to a prime ideal $\mathfrak{q}$ of $k[x_1, \ldots, x_n]$, there exist

- elements $g_1, \ldots, g_{n-d}$ of the ideal $I = (f_1, \ldots, f_m)$ whose images in the localization $k[x_1, \ldots, x_n]_\mathfrak{q}$ generate $I \cdot k[x_1, \ldots, x_n]_\mathfrak{q}$;
- indexes $i_1, \ldots, i_{n-d}$ such that the jacobian determinant
$$\det\big(\partial g_i / x_{i_j}\big)_{1 \leqslant i,j \leqslant n-d}$$
does not belong to $\mathfrak{q}$.

In the case where I is locally generated by a single polynomial f, this simply means that at least one partial derivative of f does not belong to $\mathfrak{q}$.

2.2.3. *Algebraic de Rham cohomology of smooth affine varieties.* In the remainder of this section, A denotes a k-algebra such that $X = \mathrm{Spec}(A)$ is a smooth variety of dimension d. For each integer $p \geqslant 0$, let
$$\Omega^p_{A/k} = \Lambda^p \Omega^1_{A/k}$$
be the p-th exterior power of $\Omega^1_{A/k}$ over A. In particular, $\Omega^0_{A/k} = A$ and $\Omega^p_{A/k} = 0$ for all $p > d$. For each $1 \leqslant p \leqslant d$, the A-module $\Omega^p_{A/k}$ is the quotient of the free A-module generated by the elements $\omega_1 \wedge \cdots \wedge \omega_p$, with $\omega_i \in \Omega^1_{A/k}$, by the submodule generated by the elements
$$\begin{aligned} &\omega_1 \wedge \cdots \wedge (a\omega_i + b\omega_i') \wedge \cdots \wedge \omega_p \\ &\qquad - a\omega_1 \wedge \cdots \wedge \omega_i \wedge \cdots \wedge \omega_p - b\omega_1 \wedge \cdots \wedge \omega_i' \wedge \cdots \wedge \omega_p, \end{aligned}$$
for all $1 \leqslant i \leqslant p$ and all $a, b \in A$, and by $\omega_1 \wedge \cdots \wedge \omega_p$ whenever two of the ω_i are equal. From this, we get the identity
$$\omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(p)} = \mathrm{sign}(\sigma)\omega_1 \wedge \cdots \wedge \omega_p$$
for each permutation $\sigma \in \mathfrak{S}_p$. We call *algebraic differential forms of degree p*, or just *p-forms*, the elements of $\Omega^p_{A/k}$. Since $\Omega^1_{A/k}$ is locally free of rank d, the p-th exterior power $\Omega^p_{A/k}$ is locally free of rank $\binom{d}{p}$.

As in the case of classical de Rham cohomology recalled in Section 2.2.1, the derivation $\mathrm{d}\colon A \to \Omega^1_{A/k}$ extends in a unique way to k-linear maps
$$\mathrm{d}^p \colon \Omega^p_{A/k} \to \Omega^{p+1}_{A/k}$$
satisfying the following properties:

i) $\mathrm{d}^p \circ \mathrm{d}^{p-1} = 0$ for all p,

ii) $\mathrm{d}^{p+q}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta$ for all p-forms α and q-forms β,

with $\mathrm{d}^0 = \mathrm{d}$. Explicitly, every 1-form is a k-linear combination of elements $a\mathrm{d}b$, for some $a, b \in A$, and one defines

$$\mathrm{d}^1(a\mathrm{d}b) = \mathrm{d}a \wedge \mathrm{d}b.$$

For p-forms, one sets

$$\mathrm{d}^p(\omega_1 \wedge \cdots \wedge \omega_p) = \sum_{j=1}^{p} (-1)^{j+1} \omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge \mathrm{d}^1\omega_j \wedge \omega_{j+1} \wedge \cdots \wedge \omega_p.$$

It is straightforward to check that these maps are well defined (*i.e.* vanish on the elements (2.84) defining relations on $\Omega^1_{A/k}$ and its exterior powers) and satisfy the above conditions i) and ii). This yields the *algebraic de Rham complex*

$$\Omega^*_{A/k}\colon \qquad A \longrightarrow \Omega^1_{A/k} \longrightarrow \Omega^2_{A/k} \longrightarrow \cdots \longrightarrow \Omega^d_{A/k}.$$

A p-form ω is said to be *closed* if $\mathrm{d}^p\omega = 0$, and *exact* if there exists a $(p-1)$-form η satisfying $\mathrm{d}^{p-1}\eta = \omega$.

Definition 2.89. The *algebraic de Rham cohomology* of $X = \mathrm{Spec}(A)$ is the cohomology of the algebraic de Rham complex

$$\mathrm{H}^*_{\mathrm{dR}}(X) = \mathrm{H}^*(\Omega^*_{A/k}).$$

In other words, $\mathrm{H}^n_{\mathrm{dR}}(X)$ is the quotient of the vector space of closed n-forms on X by the subspace of exact n-forms.

Both the space of closed forms and the space of exact forms on X have in general infinite dimension. However, we will prove below (Corollary 2.167) that the k-vector space $\mathrm{H}^*_{\mathrm{dR}}(X)$ is finite-dimensional.

Example 2.90 (Punctured affine line). Consider the affine variety

$$\mathbb{G}_m = \mathbb{A}^1_k \setminus \{0\} = \mathrm{Spec}\left(k[t,s]/(ts-1)\right) = \mathrm{Spec}\left(k[t,t^{-1}]\right),$$

which is the algebraic analogue of the punctured complex plane from Example 2.36. Set $A = k[t,t^{-1}]$. By Example 2.86, the module of Kähler differentials $\Omega^1_{A/k}$ is generated by $\mathrm{d}t$ and $\mathrm{d}s$, modulo the relation $s\mathrm{d}t + t\mathrm{d}s = 0$ obtained by differentiating the defining equation $ts = 1$. Since this relation amounts to $\mathrm{d}s = -t^{-2}\mathrm{d}t$, the module $\Omega^1_{A/k} = A\mathrm{d}t$ is free of rank one. The algebraic de Rham complex of X is therefore the two-term complex

$$\begin{aligned} \mathrm{d}\colon\ k[t,t^{-1}] &\longrightarrow k[t,t^{-1}]\mathrm{d}t. \\ t^m \quad &\longmapsto\ mt^{m-1}\mathrm{d}t \end{aligned}$$

There is non-zero cohomology at most in degrees zero and one, given by

$$\mathrm{H}^0_{\mathrm{dR}}(X) = \mathrm{Ker}(\mathrm{d}), \quad \mathrm{H}^1_{\mathrm{dR}}(X) = \mathrm{Coker}(\mathrm{d}).$$

Since only constant Laurent polynomials have zero derivative, we find that $\mathrm{H}^0_{\mathrm{dR}}(X)$ is the one-dimensional k-vector space generated by 1. To compute the cokernel of d, we note that $t^m\mathrm{d}t = \mathrm{d}(t^{m+1}/(m+1))$ lies in the image of d except for $m = -1$, in which case the function $1/t$ does not admit a primitive among Laurent polynomials.

It follows that $\mathrm{H}^1_{\mathrm{dR}}(X)$ is the one-dimensional k-vector space generated by the class of $\mathrm{d}t/t$. Here is the summary of our computation:

$$\mathrm{H}^n_{\mathrm{dR}}(\mathbb{G}_m) = \begin{cases} k, & \text{if } n = 0, \\ k[\frac{\mathrm{d}t}{t}], & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

EXAMPLE 2.91 (Affine elliptic curves). Let $a, b \in k$ be such that $4a^3 + 27b^2$ is non-zero. Then the polynomial

$$f(x) = x^3 + ax + b$$

has no double roots, and hence the equation $y^2 = f(x)$ defines a smooth affine plane curve $X \subseteq \mathbb{A}^2_k$. We will call X an *affine elliptic curve* since its projective closure $\overline{X} \subset \mathbb{P}^2_k$ is an elliptic curve, from which X is obtained by removing the point $O = [0 \colon 1 \colon 0]$ at infinity. From the point of view of schemes, the affine elliptic curve X is the spectrum of the k-algebra

$$A = k[x, y]/(y^2 - f(x)) \simeq k[x] \oplus k[x]y.$$

According to Example 2.86, the A-module of Kähler differentials is then given by

$$\Omega^1_{A/k} = (Adx \oplus Ady)/(2ydy - f'(x)dx),$$

where f' denotes the derivative of f. Since $\Omega^p_{A/k}$ vanishes for $p \geqslant 2$, the algebraic de Rham complex of X is the two-term complex $A \to \Omega^1_{A/k}$. It has non-zero cohomology at most in degrees zero and one, given by

$$\mathrm{H}^0_{\mathrm{dR}}(X) = \mathrm{Ker}(\mathrm{d} \colon A \to \Omega^1_{A/k}), \quad \mathrm{H}^1_{\mathrm{dR}}(X) = \mathrm{Coker}(\mathrm{d} \colon A \to \Omega^1_{A/k}).$$

To compute these spaces, we first give a more manageable presentation of $\Omega^1_{A/k}$. The intuition for the computation below is that, in case k is a subfield of the complex numbers, the complex points $\overline{X}(\mathbb{C})$ can be described as the quotient $\mathbb{C}/\Lambda$ of the complex plane by the action by translation of a lattice Λ. Under the explicit uniformization map that provides such an isomorphism, the differential form $\mathrm{d}z$ on $\mathbb{C}$ corresponds to $\mathrm{d}x/y$. However, it is not clear a priori that $\mathrm{d}x/y$ defines an element of $\Omega^1_{A/k}$ since the function y is not invertible in A. To show that $\mathrm{d}x$ is indeed divisible by y in $\Omega^1_{A/k}$, we resort to the following trick. Since f has no double roots, the polynomials f and f' are coprime. By Bézout's identity, there are polynomials $P, Q \in k[x]$ satisfying $Pf + Qf' = 1$. We can then consider the 1-form

$$\omega = Py\mathrm{d}x + 2Q\mathrm{d}y \in \Omega^1_{A/k}.$$

Using the equalities $y^2 = f(x)$ in A and $2y\mathrm{d}y = f'(x)\mathrm{d}x$ in $\Omega^1_{A/k}$, we find

$$y\omega = Py^2\mathrm{d}x + 2Qy\mathrm{d}y = (Pf + Qf')\mathrm{d}x = \mathrm{d}x,$$

which shows that ω *is* the form $\mathrm{d}x/y$. Similarly, the relation

$$\frac{f'(x)\omega}{2} = \frac{Pyf'(x)\mathrm{d}x}{2} + Qf'(x)\mathrm{d}y = (Py^2 + Qf'(x))\mathrm{d}y = \mathrm{d}y$$

holds in $\Omega^1_{A/k}$. From the expression of the generators $\mathrm{d}x$ and $\mathrm{d}y$ in terms of ω, we see that every element of $\Omega^1_{A/k}$ can be written uniquely as $(R + Sy)\omega$ for some polynomials $R, S \in k[x]$. In other words, there is an isomorphism $\Omega^1_{A/k} \simeq A\omega$, which shows that $\Omega^1_{A/k}$ is *free* of rank 1 in this case.

In terms of this presentation, the differential is given by the formula

$$\mathrm{d}(T+Uy) = T'\mathrm{d}x + U'y\mathrm{d}x + U\mathrm{d}y = (U'f + Uf'/2 + T'y)\omega \tag{2.92}$$

for polynomials $T, U \in k[x]$. Note that the polynomial $U'f + Uf'/2$ has degree $\geqslant 2$ for every non-zero U. Therefore, the right-hand side of (2.92) vanishes if and only if $U = 0$ and $T' = 0$. It follows that $\mathrm{H}^0_{\mathrm{dR}}(X)$ is the one-dimensional k-vector space of constant polynomials.

To compute $\mathrm{H}^1_{\mathrm{dR}}(X)$, we need to find which of the forms $(R + Sy)\omega$ are exact. This is the case for all $Sy\omega$, as we see by choosing $U = 0$ and a primitive of S for T. It remains to check when $R\omega$ is exact. For this, we try to write $x^n\omega$ as a linear combination of forms of the same shape with smaller n modulo the image of the differential. For each $m \geqslant 0$, it follows from the equality

$$\mathrm{d}\Big(\frac{2x^m y}{3+2m}\Big) = \big(x^{m+2} + \text{polynomial of degree} \leqslant m\big)\omega$$

that the class of $x^{m+2}\omega$ in the cokernel of d is a k-linear combination of the classes of $\omega, x\omega, \ldots, x^m\omega$. Therefore, $\mathrm{H}^1_{\mathrm{dR}}(X)$ is spanned by ω and $x\omega$. On the other hand, since $U'f + Uf'/2$ is either zero or has degree $\geqslant 2$, no linear combination of ω and $x\omega$ lies in the image of d. Hence, the classes of x and $x\omega$ form a basis of $\mathrm{H}^1_{\mathrm{dR}}(X)$. Here is the summary of our computation:

$$\mathrm{H}^n_{\mathrm{dR}}(X) = \begin{cases} k, & \text{if } n = 0, \\ k[\frac{\mathrm{d}x}{y}] \oplus k[\frac{x\mathrm{d}x}{y}], & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

2.2.4. *Algebraic de Rham cohomology of smooth varieties.* Let us now turn to the case where X is any variety over k, not necessarily affine. Gluing differential forms on affine open subsets, we get a sheaf for the Zariski topology on X. The following result is explained in [**Har77**, Chap. II, § 8].

Proposition 2.93. *There exists a unique coherent $\mathcal{O}_X$-module $\Omega^1_{X/k}$ whose restriction to every affine open subset U of X is the $\mathcal{O}_U$-module associated with the Kähler differentials $\Omega^1_{\mathcal{O}_X(U)/k}$, and such that the restriction maps between open affine subsets are given by the restriction of differential forms.*

Coherence is a finiteness property that allows, among other things, for vanishing results, comparison theorems between algebraic and analytic cohomology, and so on (see Section A.9.9). Many of the properties we have discussed in the affine case globalize to arbitrary smooth varieties. In particular, if X is a smooth variety of dimension d, then the sheaf Ω^1_X is locally free of rank d and is equipped with the universal k-derivation $\mathrm{d}\colon \mathcal{O}_X \to \Omega^1_X$. Let Ω^p_X denote the p-th exterior power of Ω^1_X. The differential d canonically extends to morphisms of sheaves of k-vector spaces $\mathrm{d}^p\colon \Omega^p_X \to \Omega^{p+1}_X$ satisfying $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$. The resulting complex

$$(\Omega^*_X, \mathrm{d})\colon \quad \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \Omega^2_X \xrightarrow{\mathrm{d}} \cdots \tag{2.94}$$

is called the *algebraic de Rham complex*. Observe that every term in this complex is a locally free $\mathcal{O}_X$-module but the differential d is not $\mathcal{O}_X$-linear, only k-linear. The default of $\mathcal{O}_X$-linearity is precisely measured by the Leibniz rule.

Definition 2.95. Let X be a smooth variety over a field k of characteristic zero. The *algebraic de Rham cohomology* of X is the hypercohomology of the algebraic de Rham complex. We denote it by

$$\mathrm{H}^*_{\mathrm{dR}}(X) = \mathbb{H}^*(X, \Omega^*_X).$$

Recall from Section A.9.4 that the hypercohomology of Ω^*_X is defined as the cohomology of the complex of global sections of an acyclic resolution of Ω^*_X, for example Godement's canonical resolution

$$\mathrm{Gd}(\Omega^*_X) = \mathrm{Tot}\,\mathrm{Gd}^*(\Omega^*_X)$$

from Section A.9.3. That is, the algebraic de Rham cohomology of X can be computed as the cohomology of a complex of k-vector spaces, namely

$$\mathrm{H}^*_{\mathrm{dR}}(X) = \mathrm{H}^*(\Gamma(X, \mathrm{Gd}(\Omega^*_X))).$$

Remark 2.96. When X is affine, there is no need to use hypercohomology to define algebraic de Rham cohomology. Indeed, as all coherent sheaves on an affine variety are acyclic (Theorem A.311), the complex Ω^*_X consists of acyclic sheaves, so it is an acyclic resolution of itself. It follows that its hypercohomology agrees with the cohomology of the complex of global sections $(\Omega^*_X(X), \mathrm{d})$. This last complex is called the *global de Rham complex*. In the affine case, the global de Rham complex agrees with the complex of Kähler differentials

$$\Omega^*_X(X) = \Omega^*_{X/k}$$

by Proposition 2.93, and hence Definitions 2.89 and 2.95 agree for affine varieties.

In general, when X is not affine, the cohomology of the global de Rham complex does not coincide with the algebraic de Rham cohomology. For example, the global sections $\Omega^p_X(X)$ vanish for $p > \dim X$, and hence so does the cohomology of the global de Rham complex, while a variety will in general have non-trivial de Rham cohomology up to degree $2\dim X$. Most of the varieties we will deal with in applications are affine, so we will often be able to use the global de Rham complex.

A tool to compute algebraic de Rham cohomology is to view it, as the hypercohomology of any complex of sheaves, as the abutment of a spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(X, \Omega^p_X) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X) \tag{2.97}$$

with differentials $\mathrm{d}_1 \colon E_1^{p,q} \to E_1^{p+1,q}$ induced by $\mathrm{d} \colon \Omega^p_X \to \Omega^{p+1}_X$. In this particular context, this is called the *Frölicher* or *Hodge–de Rham spectral sequence*

In practice, to compute the algebraic de Rham cohomology of X and make the Hodge–de Rham spectral sequence explicit, one chooses a cover of X by a finite collection of affine open subsets $U_1, \dots, U_n$, and forms the *Čech* double complex

$$\begin{tikzcd} \vdots & \vdots \\ \bigoplus_i \Omega^1(U_i) \arrow[r] \arrow[u] & \bigoplus_{i<j} \Omega^1(U_i \cap U_j) \arrow[r] \arrow[u] & \cdots \\ \bigoplus_i \mathcal{O}(U_i) \arrow[r] \arrow[u] & \bigoplus_{i<j} \mathcal{O}(U_i \cap U_j) \arrow[r] \arrow[u] & \cdots \end{tikzcd} \tag{2.98}$$

where the vertical differentials are the differentials in the algebraic de Rham complex, and the horizontal differentials

$$\bigoplus_{i_0<\cdots<i_q} \Omega^p(U_{i_0} \cap \cdots \cap U_{i_q}) \longrightarrow \bigoplus_{i_0<\cdots<i_{q+1}} \Omega^p(U_{i_0} \cap \cdots \cap U_{i_{q+1}})$$

send a section $\alpha \in \Omega^p(U_{i_0} \cap \cdots \cap U_{i_q})$ to the element $\partial\alpha$ with factors

$$(\partial\alpha)_{i_0,\dots,i_{q+1}} = \sum_{r=0}^{q+1} (-1)^{r+1} \alpha_{i_0,\dots,\widehat{i_r},\dots,i_{q+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{q+1}}}.$$

Thanks to Remark 2.96, the algebraic de Rham cohomology of X is the cohomology of the total complex associated with this double complex (Definition A.33). Moreover, (2.97) is the spectral sequence associated with it as in Example A.205.

Example 2.99. Let $X = \mathbb{P}^1_k$ be the projective line over k. We consider the cover by the two affine open subsets

$$U_0 = \mathbb{P}^1_k \setminus \{0\} = \operatorname{Spec}(k[t]), \quad U_1 = \mathbb{P}^1_k \setminus \{\infty\} = \operatorname{Spec}(k[s]),$$

whose coordinates are related by $s = 1/t$ on the intersection $U_0 \cap U_1$. The only non-zero terms in the double complex (2.98) are then

$$\begin{tikzcd} k[t]\mathrm{d}t \oplus k[s]\mathrm{d}s \arrow[r] & k[t,t^{-1}]\mathrm{d}t \\ k[t] \oplus k[s] \arrow[r] \arrow[u, "\mathrm{d}\oplus\mathrm{d}"] & k[t,t^{-1}], \arrow[u, "\mathrm{d}"'] \end{tikzcd}$$

where the horizontal differentials are given by

$$(f\mathrm{d}t, g\mathrm{d}s) \longmapsto (f(t) + g(1/t)t^{-2})\mathrm{d}t, \quad (f,g) \longmapsto f(t) - g(1/t).$$

We leave as Exercise 2.124 for the reader to check that the cohomology groups of the associated total complex are

$$\mathrm{H}^0_{\mathrm{dR}}(\mathbb{P}^1_k) = k, \quad \mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1_k) = 0, \quad \mathrm{H}^2_{\mathrm{dR}}(\mathbb{P}^1_k) = k\left[\frac{\mathrm{d}t}{t}\right]. \tag{2.100}$$

Alternatively, we can compute the algebraic de Rham cohomology of $\mathbb{P}^1_k$ by means of the spectral sequence (2.97). For this, we first note that the sheaf of Kähler differentials $\Omega^1_{\mathbb{P}^1}$ is the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$, since the generator $\mathrm{d}t \in \Omega^1_{\mathbb{P}^1}(\mathbb{A}^1)$ has a pole of order 2 at infinity. By the standard computation of the cohomology of line bundles on $\mathbb{P}^1$ [**Har77**, Chap. III, § 5], the sheaf $\mathcal{O}_{\mathbb{P}^1}$ has only non-vanishing cohomology in degree zero and $\mathcal{O}_{\mathbb{P}^1}(-2)$ has only non-vanishing cohomology in degree one. Therefore, the spectral sequence reads

$$\begin{array}{|ll} & 0 \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \\ & \\ \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0 & \\ \hline \end{array}$$

and all differentials vanish already at the first page. This yields isomorphisms

$$\mathrm{H}^n_{\mathrm{dR}}(\mathbb{P}^1) = \begin{cases} \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}), & \text{if } n = 0, \\ 0, & \text{if } n = 1, \\ \mathrm{H}^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}), & \text{if } n = 2. \end{cases}$$

EXAMPLE 2.101. Let X be a smooth connected projective curve over an algebraically closed field k of characteristic zero. In this example, we will use without proof a few results from the theory of algebraic curves, for which the reader can for instance consult [**Har77**, Chap. IV]. The *genus* of X is defined as the dimension

$$g = \dim \mathrm{H}^0(X, \Omega^1_X)$$

of the space of global sections of the sheaf of differential forms. Besides, since X is projective and connected, the only global sections of $\mathcal{O}_X$ are the constant functions:

$$\mathrm{H}^0(X, \mathcal{O}_X) = k.$$

Moreover, Serre's duality implies (see [**Har77**, Cor. 7.13]) the equalities

$$\dim \mathrm{H}^1(X, \mathcal{O}_X) = g, \qquad \dim \mathrm{H}^1(X, \Omega^1_X) = 1.$$

The spectral sequence (2.97) computing $\mathrm{H}^*_{\mathrm{dR}}(X)$ is given by

$$\begin{array}{|l} \mathrm{H}^1(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^1(X, \Omega^1_X) \\ \\ \mathrm{H}^0(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^0(X, \Omega^1_X) \\ \hline \end{array}$$

and the lower horizontal map is zero since the differential of a constant function vanishes. We shall show that the upper horizontal map is zero as well.

Let $f\colon X \to \mathbb{P}^1$ be a non-constant rational function. Then f is surjective, and the open subsets

$$U_0 = X \setminus f^{-1}(0) \quad \text{and} \quad U_1 = X \setminus f^{-1}(\infty)$$

are affine and cover X. We work with this affine open cover. As explained after Remark 2.96, the first page of the spectral sequence (2.97) can be computed using the commutative diagram with exact columns

$$\begin{tikzcd}
0 & 0 \\
\mathrm{H}^1(X,\mathcal{O}_X) \arrow[u] \arrow[r, "\mathrm{d}_1"] & \mathrm{H}^1(X,\Omega^1_X) \arrow[u] \\
\mathcal{O}_X(U_0\cap U_1) \arrow[u] \arrow[r, "\mathrm{d}"] & \Omega^1_X(U_0\cap U_1) \arrow[u] \\
\mathcal{O}_X(U_0)\oplus\mathcal{O}_X(U_1) \arrow[u, "\partial"] \arrow[r, "\mathrm{d}"] & \Omega^1_X(U_0)\oplus\Omega^1_X(U_1) \arrow[u, "\partial"'] \\
\mathrm{H}^0(X,\mathcal{O}_X) \arrow[u] \arrow[r, "\mathrm{d}_1"] & \mathrm{H}^0(X,\Omega^1_X) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

The zeroth cohomology groups are the kernels of the middle vertical arrows ∂, and the first cohomology groups are the cokernels of the same maps. The differentials denoted by d_1 are just the induced maps. We have already discussed that the lower d_1 is zero and our task now is to prove that the upper d_1 is zero.

To this end, we will use the *residue map*. For any Zariski open subset U of X and any k-point $P\in X\setminus U$, there is a residue map

$$\operatorname{res}_P\colon \Omega^1_X(U)\longrightarrow k$$

satisfying the following properties (see [Ser88, § II.7]):

i) for each $\omega\in\Omega^1_X(U)$, the equality $\sum_{P\in X\setminus U}\operatorname{res}_P(\omega)=0$ holds;

ii) for forms $\omega=\mathrm{d}g$, where $g\in\mathcal{O}_X(U)$ is a regular function, the vanishing $\operatorname{res}_P(\omega)=0$ holds for all k-points P of $X\setminus U$.

Building on these properties, we define the map $\varphi\colon\Omega^1_X(U_0\cap U_1)\to k$ as

$$\varphi(\omega)=\sum_{P\in(X\setminus U_0)(k)}\operatorname{res}_P(\omega)=-\sum_{P\in(X\setminus U_1)(k)}\operatorname{res}_P(\omega),$$

where the second equality follows from property i) of the residue map applied to the open set $U_0\cap U_1$. Again by property i), but this time applied to the open sets U_0 and U_1, the composition $\varphi\circ\partial$ vanishes. Therefore, the map φ factors through $\mathrm{H}^1(X,\Omega^1_X)$ and gives rise to a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(X,\mathcal{O}_X) \arrow[r, "\mathrm{d}_1"] & \mathrm{H}^1(X,\Omega^1_X) \arrow[r, "\varphi_0"] & k. \\
\mathcal{O}_X(U_0\cap U_1) \arrow[u] \arrow[r, "\mathrm{d}"] & \Omega^1_X(U_0\cap U_1) \arrow[u] \arrow[ur, "\varphi"'] & \\
& \Omega^1_X(U_0)\oplus\Omega^1_X(U_1) \arrow[u, "\partial"'] &
\end{tikzcd}$$

The differential form $\mathrm{d}f/f$ has simple poles whenever f has a zero or a pole. The residue of $\mathrm{d}f/f$ at a point is exactly the order of vanishing of f at the point. In particular, it is positive if f has a zero, and negative if f has a pole. Since f is non-constant, it has at least one zero. Since $X \setminus U_0$ is the set of zeros of f, we deduce $\varphi(\mathrm{d}f/f) \neq 0$. Hence, the map φ_0 is surjective. Since we already know that the dimension of $H^1(X, \Omega^1_X)$ is one, φ_0 is an isomorphism. By property ii) of the residue, we deduce that the composition $\varphi \circ \mathrm{d}$ vanishes. This implies the vanishing of the upper differential $\mathrm{d}_1 = 0$. All in all, the spectral sequence (2.97) degenerates at the page E_1. From this we get that

$$\mathrm{H}^0_{\mathrm{dR}}(X) = \mathrm{H}^0(X, \mathcal{O}_X) \quad \text{and} \quad \mathrm{H}^2_{\mathrm{dR}}(X) = \mathrm{H}^1_{\mathrm{dR}}(X, \Omega^1_X)$$

are both one-dimensional, and that the first de Rham cohomology group sits in the exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \Omega^1_X) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(X) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X) \longrightarrow 0.$$

In particular, $\dim \mathrm{H}^1_{\mathrm{dR}}(X) = 2g$. In the classical literature, elements of $\mathrm{H}^0(X, \Omega^1_X)$ are called *differentials of the first kind*.

In this particular example we have proved by hand that the Hodge–de Rham spectral sequence (2.97) degenerates at the term E_1. In fact, this is a general result for smooth proper varieties, which is at the heart of the Hodge decomposition.

2.2.5. *Some properties of algebraic de Rham cohomology.*

Base change: Algebraic de Rham cohomology is compatible with extensions of the field of definition.

LEMMA 2.102. *Let X be a smooth variety over a field k of characteristic zero, let K be a field extension of k, and let $X_K = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ denote the extension of scalars. There is a canonical isomorphism*

$$\mathrm{H}^*_{\mathrm{dR}}(X) \otimes_k K \xrightarrow{\sim} \mathrm{H}^*_{\mathrm{dR}}(X_K).$$

This is proved either using the compatibility with field extensions of the cohomology of coherent sheaves and the spectral sequence (2.97), or the functoriality of Godement's canonical resolution as in [**HMS17**, Lem. 3.1.11].

Functoriality: Let $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ be smooth affine varieties over k, and let $f\colon U \to V$ be a morphism between them. We still denote by f the associated morphism of k-algebras $B \to A$, which endows $\Omega^1_{A/k}$ with the structure of a B-module. Since the composition $B \to A \to \Omega^1_{A/k}$ is a k-linear derivation, by the universal property of Kähler differentials, there exists a morphism of B-modules

$$f^*\colon \Omega^1_{B/k} \longrightarrow \Omega^1_{A/k}. \tag{2.103}$$

Now let $f\colon X \to Y$ be a morphism between any smooth varieties over k. For each Zariski open subset U of Y, gluing the maps (2.103) on affine open covers of U and $f^{-1}(U)$ yields a morphism

$$\Omega^1_Y(U) \longrightarrow \Omega^1_X(f^{-1}(U)) = (f_*\Omega^1_X)(U),$$

and hence morphisms $\Omega^*_Y(U) \to (f_*\Omega^*_X)(U)$ that are compatible with the differentials. From this, we get morphisms of complexes of sheaves

$$\Omega^*_Y \xrightarrow{f^*} f_*\Omega^*_X \longrightarrow Rf_*\Omega^*_X.$$

Proposition A.278 implies that the hypercohomologies $\mathbb{H}^*(Y, Rf_*\Omega_X^*)$ and $\mathbb{H}^*(X, \Omega_X^*)$ are isomorphic, and hence the above morphisms induce a k-linear map

$$f^*\colon \mathrm{H}^*_{\mathrm{dR}}(Y) \longrightarrow \mathrm{H}^*_{\mathrm{dR}}(X).$$

Cup-product: The exterior product of differential forms gives rise to a *cup-product*

$$\mathrm{H}^*_{\mathrm{dR}}(X) \otimes \mathrm{H}^*_{\mathrm{dR}}(X) \longrightarrow \mathrm{H}^*_{\mathrm{dR}}(X)$$

in algebraic de Rham cohomology. Indeed, from the morphism of complexes

$$\mathrm{Tot}(\Omega_X^* \otimes_k \Omega_X^*) \longrightarrow \Omega_X^*$$

given by exterior product, we get a morphism of Godement resolutions

$$\mathrm{Gd}(\Omega_X^* \otimes_k \Omega_X^*) \longrightarrow \mathrm{Gd}(\Omega_X^*). \tag{2.104}$$

Since every module over a field is flat, tensoring by Ω_X^* is an exact functor, and hence the natural map

$$\Omega_X^* \otimes_k \Omega_X^* \longrightarrow \mathrm{Gd}(\Omega_X^*) \otimes_k \mathrm{Gd}(\Omega_X^*)$$

is a quasi-isomorphism. Applying the Godement resolution again to the source and the target, we obtain a quasi-isomorphism

$$\mathrm{Gd}(\Omega_X^* \otimes_k \Omega_X^*) \longrightarrow \mathrm{Gd}\left(\mathrm{Gd}(\Omega_X^*) \otimes_k \mathrm{Gd}(\Omega_X^*)\right). \tag{2.105}$$

Composing the natural map from a complex to its Godement resolution, the inverse of the quasi-isomorphism (2.105), and the product (2.104), we obtain morphisms

$$\begin{tikzcd}
\mathrm{Gd}(\Omega_X^*) \otimes_k \mathrm{Gd}(\Omega_X^*) \arrow[r] & \mathrm{Gd}(\mathrm{Gd}(\Omega_X^*) \otimes_k \mathrm{Gd}(\Omega_X^*)) & \\
& \mathrm{Gd}(\Omega_X^* \otimes_k \Omega_X^*) \arrow[u, "\sim"] \arrow[r] & \mathrm{Gd}(\Omega_X^*)
\end{tikzcd}$$

in the derived category. Taking the equality

$$\mathrm{H}^*_{\mathrm{dR}}(X) = \mathrm{H}^*(\Gamma(X, \mathrm{Gd}(\Omega_X^*)))$$

into account, this morphism and composition with the map

$$\begin{aligned}
\mathrm{H}^*\Big(\Gamma\big(X, \mathrm{Gd}(\Omega_X^*)\big)\Big) &\otimes \mathrm{H}^*\Big(\Gamma\big(X, \mathrm{Gd}(\Omega_X^*)\big)\Big) \\
&\xrightarrow{\sim} \mathrm{H}^*\Big(\Gamma\big(X, \mathrm{Gd}(\Omega_X^*)\big) \otimes \Gamma\big(X, \mathrm{Gd}(\Omega_X^*)\big)\Big) \\
&\longrightarrow \mathrm{H}^*\Big(\Gamma\big(X, \mathrm{Gd}(\Omega_X^*) \otimes_k \mathrm{Gd}(\Omega_X^*)\big)\Big)
\end{aligned}$$

induce the sought-after product in de Rham cohomology. A variant of the above construction yields an external product

$$\mathrm{H}^*_{\mathrm{dR}}(X) \otimes \mathrm{H}^*_{\mathrm{dR}}(Y) \longrightarrow \mathrm{H}^*_{\mathrm{dR}}(X \times Y)$$

for smooth varieties X and Y over k.

2.2.6. *Relative de Rham cohomology.* There is also a relative version of algebraic de Rham cohomology. For simplicity, we explain the construction only in the affine case. Let X be a smooth affine variety over k, and consider a smooth closed subvariety $\iota\colon Z \hookrightarrow X$, which is hence affine. There is a restriction morphism of complexes $\iota^*\colon \Omega^*(X) \to \Omega^*(Z)$. Let $\Omega^*(X, Z)$ denote the complex

$$\Omega^*(X, Z) = \mathrm{Tot}(\iota^*) = \mathrm{cone}(-\iota^*)[-1].$$

Explicitly, it is given by

$$\Omega^n(X, Z) = \Omega^n(X) \oplus \Omega^{n-1}(Z)$$

with differential

$$\mathrm{d}(\alpha, \beta) = (\mathrm{d}\alpha, \iota^*(\alpha) - \mathrm{d}\beta).$$

By construction, there is a short exact sequence

$$0 \longrightarrow \Omega^*(Z)[-1] \longrightarrow \Omega^*(X, Z) \longrightarrow \Omega^*(X) \longrightarrow 0$$

in the category of complexes of k-vector spaces, that induces a distinguished triangle

$$\Omega^*(X, Z) \longrightarrow \Omega^*(X) \xrightarrow{\iota^*} \Omega^*(Z) \longrightarrow \Omega^*(X, Z)[1] \tag{2.106}$$

in the corresponding derived category (see Section A.3.2).

Definition 2.107. Let X be a smooth affine variety and $Z \hookrightarrow X$ a smooth closed subvariety. The *relative de Rham cohomology* of the pair (X, Z) is the cohomology of the complex $\Omega^*(X, Z)$. We denote it by

$$\mathrm{H}^*_{\mathrm{dR}}(X, Z) = \mathrm{H}^*(\Omega^*(X, Z)).$$

A relative de Rham class is represented by a pair of differential forms (α, β) such that α is closed and the restriction of α to Z is equal to $\mathrm{d}\beta$. In general, neither α nor β is determined by the other form. Taking the long exact sequence associated with (2.106), one gets

$$\cdots \longrightarrow \mathrm{H}^{n-1}_{\mathrm{dR}}(Z) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(X, Z) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(X) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(Z) \longrightarrow \cdots$$

where $\mathrm{H}^n_{\mathrm{dR}}(X) \to \mathrm{H}^n_{\mathrm{dR}}(Z)$ is the restriction map, $\mathrm{H}^n_{\mathrm{dR}}(X, Z) \to \mathrm{H}^n_{\mathrm{dR}}(X)$ sends (α, β) to α, and $\mathrm{H}^{n-1}_{\mathrm{dR}}(Z) \to \mathrm{H}^n_{\mathrm{dR}}(X, Z)$ sends β to $(0, \beta)$.

Example 2.108. Set $X = \mathbb{G}_m = \mathrm{Spec}\left(k[x, x^{-1}]\right)$, and let $Z = \{p, q\}$ be the closed subvariety of X consisting of two distinct k-points p and q, that is, the zero locus of the polynomial $(x - p)(x - q)$. Then $\Omega^*(Z)$ is concentrated in degree zero, where it is equal to $\Omega^0(Z) = k \oplus k$, and the map

$$\iota^*\colon \Omega^0(X) = k[x, x^{-1}] \longrightarrow \Omega^0(Z) = k \oplus k$$

is given by evaluation at p and q, that is, $\iota^*(f) = (f(p), f(q))$. Therefore, the complex $\Omega^*(X, Z)$ reads

$$\begin{aligned} \mathrm{d}\colon\; k[x, x^{-1}] &\longrightarrow k[x, x^{-1}]\mathrm{d}x \oplus k \oplus k \\ f \qquad &\longmapsto (f'(x)\mathrm{d}x, f(p), f(q)). \end{aligned} \tag{2.109}$$

The differential d is injective, so that $\mathrm{H}^0_{\mathrm{dR}}(X, Z)$ vanishes. Indeed, if f lies in its kernel, then f is constant since $f'(x) = 0$, but this constant must be equal to $f(p) = f(q) = 0$. Besides, using the equalities $\mathrm{d}(1) = (0, 1, 1)$ and

$$\mathrm{d}(x^{n+1}/(n+1)) = (x^n\mathrm{d}x, p^{n+1}/(n+1), q^{n+1}/(n+1)) \qquad (n \neq -1),$$

one sees that the cokernel of d is generated by

$$\omega_1' = (0, 1, 0), \quad \omega_2 = (\mathrm{d}x/x, 0, 0).$$

As no linear combination of these two elements lies in the image of d (look at the first entry of the triple), they form a basis of Coker(d) and $\mathrm{H}^1_{\mathrm{dR}}(X, Z)$ is thus the k-vector space $\langle \omega_1', \omega_2 \rangle_k$. As you will show in Exercise 2.127, the element ω_1' is cohomologous to $\omega_1 = (\mathrm{d}x/(p-q), 0, 0)$. Therefore, we can represent $\mathrm{H}^1_{\mathrm{dR}}(X, Z)$ using the differential forms

$$\omega_1 = \mathrm{d}x/(p-q), \quad \omega_2 = \mathrm{d}x/x$$

on X that vanish when restricted to Z.

REMARK 2.110. Recall that the de Rham cohomology of affine smooth varieties vanishes above the dimension. If $n = \dim X$ and $Z \subset X$ is a closed smooth subvariety of strictly smaller dimension, then a useful part of the long exact sequence of relative de Rham cohomology is

$$\cdots \longrightarrow \mathrm{H}^{n-1}_{\mathrm{dR}}(Z) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(X, Z) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(X) \longrightarrow 0. \tag{2.111}$$

For instance, it implies that every top degree differential form on a smooth affine variety can be lifted to a relative cohomology class.

2.2.7. *The case of normal crossing divisors.* In the sequel, we will also need to consider relative de Rham cohomology in the case where Z is not a smooth subvariety, but a *simple normal crossing divisor*. Using standard tools from homological algebra, we will be able to extend Definition 2.107 to this setting. We first introduce the relevant notions from algebraic geometry following [**Kol13**, p. 9 ff.].

Let X be a smooth algebraic variety of dimension d over a field k. Given a closed point p of X, we denote by $\mathcal{O}_{X,p}$ the local ring of germs of regular functions at p, by $\mathfrak{m}_p$ its maximal ideal, and by $\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p$ its residue field. Recall that $\mathcal{O}_{X,p}$ is regular of Krull dimension d, which means that a minimal set of generators of $\mathfrak{m}_p$ contains d elements; any such set is called a *regular system of parameters*. By Nakayama's lemma, the elements $x_1, \ldots, x_d \in \mathfrak{m}_p$ form a regular system of parameters if and only if their residue classes $\bar{x}_1, \ldots, \bar{x}_d$ modulo $\mathfrak{m}_p^2$ form a $\kappa(p)$-basis of the cotangent space $T_p^* X = \mathfrak{m}_p/\mathfrak{m}_p^2$.

DEFINITION 2.112. A closed subvariety $D \subset X$ of codimension one is called a *simple normal crossing divisor* if, for each $p \in D$, there exists a Zariski open neighborhood $p \subset U_p \subset X$, a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$, and an integer $1 \leqslant r \leqslant d$ such that $D \cap U_p$ lies in the zero locus of $x_1 \cdots x_r$.

It follows from the definition that the irreducible components of a simple normal crossing divisor D are smooth and that the intersection of m distinct irreducible components is a smooth subvariety of codimension m in X.

REMARK 2.113. Simple normal crossing divisors are also called *strict normal crossing divisors* in the literature. Note that this definition is sensitive to the base field. For example, if -1 is not a square in k, the subvariety $D = \{x^2 + y^2 = 0\} \subset \mathbb{A}^2_k$ is irreducible and singular at the origin, and hence is not a simple normal crossing divisors. However, if k contains a square root of -1, then the factorization

$$x^2 + y^2 = (x + \sqrt{-1}y)(x - \sqrt{-1}y)$$

shows that D is the union of two affine lines meeting at the origin, which is the paradigmatic example of a simple normal crossing divisor.

Construction 2.114. Let X be a smooth affine variety over k and let D be a simple normal crossing divisor, with irreducible components $D_0, \dots, D_r$. Given a subset $I \subseteq \{0, \dots, r\}$, we write

$$D_I = \bigcap_{i \in I} D_i.$$

We define $D^0 = X$ and, for $p = 1, \dots, r+1$,

$$D^p = \coprod_{|I|=p} D_I.$$

Then there is a double complex of k-vector spaces

$$K^{p,q} = \Omega^q(D^p), \tag{2.115}$$

where the vertical differentials $\mathrm{d}^{\mathrm{ver}}$ are the differentials d in the de Rham complex, and the horizontal differentials $\mathrm{d}^{\mathrm{hor}}$ are alternating sums of restriction maps. More precisely, $\mathrm{d}^{\mathrm{hor}} \colon K^{p,q} \to K^{p+1,q}$ is given by

$$\bigoplus_{\substack{|I|=p \\ |J|=p+1 \\ I \subset J}} \varepsilon(I,J) \mathrm{d}_{J,I}, \tag{2.116}$$

where $\mathrm{d}_{J,I} \colon \Omega^q(D_I) \to \Omega^q(D_J)$ denotes the restriction map, and the sign $\varepsilon(I,J)$ is defined as follows: if $J = \{j_0, \dots, j_p\}$ with the indexes ordered as $j_0 < \dots < j_p$, and $I = \{j_0, \dots, \widehat{j_\ell}, \dots, j_p\}$, then $\varepsilon(I,J) = (-1)^\ell$.

Let $\Omega^*(X,D)$ denote the total complex associated with $K^{p,q}$, that is,

$$\left(\Omega^*(X,D) = \bigoplus_{p+q=*} K^{p,q},\ \partial = \mathrm{d}^{\mathrm{hor}} + (-1)^p \mathrm{d}^{\mathrm{ver}} \right).$$

Thanks to the sign $(-1)^p$ of the vertical differential, the map ∂ satisfies $\partial^2 = 0$, and hence $\Omega^*(X,D)$ is a complex. To make the link with the relative de Rham complex, as defined for smooth D, consider the double complex

$$\Omega^{p,q}(D) = \Omega^q(D^{p+1})$$

with vertical differentials given by the differentials in the de Rham complex, and horizontal differentials

$$\mathrm{d}^{\mathrm{hor}} = \bigoplus_{\substack{|I|=p \\ |J|=p+1 \\ I \subset J}} -\varepsilon(I,J) \mathrm{d}_{J,I}.$$

We define the de Rham complex of D as the total complex

$$\Omega^*(D) = \mathrm{Tot}(\Omega^{*,*}(D)).$$

Definition 2.117. The *de Rham cohomology* $\mathrm{H}^*_{\mathrm{dR}}(D)$ of D is defined as the cohomology of the complex $\Omega^*(D)$.

By construction, there is a restriction map $\iota^* \colon \Omega^*(X) \to \Omega^*(D)$ satisfying

$$\Omega^*(X,D) = \mathrm{Tot}(\iota^*).$$

In view of the similarities with the Čech double complex (2.98), one may think of $\Omega^*(D)$ as a Čech complex associated with a closed cover of D.

Definition 2.118. The *relative de Rham cohomology* $\mathrm{H}^*_{\mathrm{dR}}(X, D)$ is the cohomology of the complex $\Omega^*(X, D)$.

As for any total complex associated with a double complex, the cohomology can be computed by means of the spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(\Omega^*(D^p)) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X, D). \tag{2.119}$$

Let $d = \dim X$. By definition, a class in the top degree cohomology $\mathrm{H}^d(X, D)$ is represented by a tuple

$$(\omega_0, \dots, \omega_d) \in \bigoplus_{p=0}^{d} \Omega^{d-p}(D^p).$$

What is more, one can always choose $\omega_p = 0$ for $p = 1, \dots, d$, so that all classes in $\mathrm{H}^d(X, D)$ are indeed represented by some $\omega \in \Omega^d(X)$. The key point here is that the restriction maps $\Omega^{d-p-1}(D^p) \to \Omega^{d-p-1}(D^{p+1})$ are all surjective, as is proved in [**HMS17**, Lem. 3.3.20]. We will see in the example below how to use this result to find a representative of the sought shape; the proof of the existence of such representatives in the general case is analogous.

Example 2.120. Let $X = \mathbb{A}^2 = \mathrm{Spec}(k[x, y])$ and let $D \subset X$ be the union of three lines in general position. After an affine transformation, we may assume without loss of generality that D is the union of the lines

$$D_0 = \{y = 0\}, \qquad D_1 = \{x = 0\}, \qquad D_2 = \{x + y = 1\}.$$

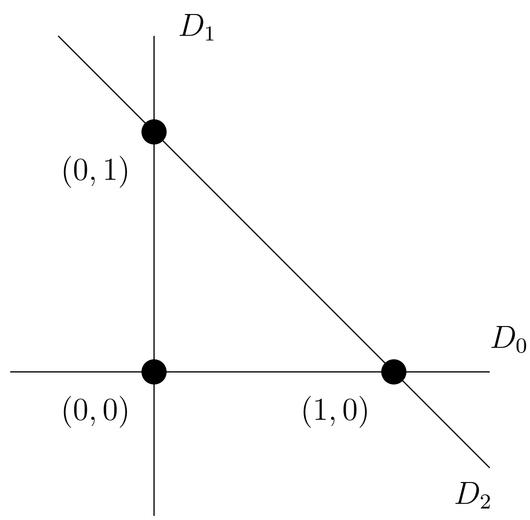


Figure 3. The triangle D

In this case, the double complex (2.115) is equal to

$$(\Omega^*(\mathbb{A}^2), d) \longrightarrow \bigoplus_{i=0}^{2} (\Omega^*(D_i), d) \longrightarrow \bigoplus_{0 \leqslant i < j \leqslant 2} (\Omega^*(D_i \cap D_j), d) \longrightarrow 0.$$

To make all the above terms and maps explicit, we write

$$D_0 = \mathrm{Spec}(k[x]), \quad D_1 = \mathrm{Spec}(k[y])$$

and we parametrize $D_2 = \mathrm{Spec}\bigl(k[x,y]/(x+y-1)\bigr)$ by the coordinate $z = x$. Then the double complex takes the form

$$\begin{tikzcd}
k[x,y]\mathrm{d}x \wedge \mathrm{d}y & & \\
k[x,y]\mathrm{d}x \oplus k[x,y]\mathrm{d}y \arrow[u, "\mathrm{d}"] \arrow[r, "c"] & k[x]\mathrm{d}x \oplus k[y]\mathrm{d}y \oplus k[z]\mathrm{d}z & \\
k[x,y] \arrow[u, "\mathrm{d}"] \arrow[r, "a"] & k[x] \oplus k[y] \oplus k[z] \arrow[u, "\mathrm{d}"] \arrow[r, "b"] & k \oplus k \oplus k,
\end{tikzcd}$$

where d is the exterior derivative, the maps a and b are given by

$$\begin{aligned}
a\colon \quad & f(x,y) \longmapsto (f(x,0), f(0,y), f(z,1-z)),\\
b\colon \quad & (f(x),g(y),h(z)) \longmapsto (g(0)-f(0), h(1)-f(1), h(0)-g(1)),
\end{aligned}$$

and c is induced from a in the obvious way. The spectral sequence (2.119) reads

$$\begin{array}{|lll}
0 & & \\
 & & \\
0 & 0 & \\
 & & \\
k \longrightarrow & k\oplus k\oplus k \longrightarrow & k\oplus k\oplus k \\
\hline
\end{array}$$

where the first map sends a to (a,a,a) and the second one is given by

$$(a,b,c) \mapsto (b-a, c-a, c-b).$$

Since the cohomology of the bottom complex is concentrated in degree two, where it is generated by the class of the element $(1,0,0)$, the second page of the spectral sequence is reduced to $E_2^{2,0} = k$. It follows that $\mathrm{H}^n_{\mathrm{dR}}(\mathbb{A}^2, D)$ vanishes for $n \neq 2$ and is one-dimensional for $n = 2$.

To produce a differential $\omega \in \Omega^2(\mathbb{A}^2)$ representing the generator $(1,0,0)$, we follow the "zig-zag" method, which consists in

- finding $\omega_1 \in k[x] \oplus k[y] \oplus k[z]$ such that $b(\omega_1) = (1,0,0)$;
- applying $\mathrm{d}^{\mathrm{ver}}$ to get $\omega_2 = \mathrm{d}\omega_1$ one row above; from the equalities

$$\partial\omega_1 = d^{\mathrm{hor}}\omega_1 - \mathrm{d}^{\mathrm{ver}}\omega_1 = b(\omega_1) - \mathrm{d}\omega_1 = (1,0,0) - \omega_2,$$

  it follows that $(1,0,0)$ and ω_2 are cohomologous;
- choosing $\omega_3 \in k[x,y]\mathrm{d}x \oplus k[x,y]\mathrm{d}y$ with $c(\omega_3) = \omega_2$; setting $\omega = -\mathrm{d}\omega_3$, we then get

$$\partial\omega_3 = c(\omega_3) + \mathrm{d}\omega_3 = \omega_2 - \omega,$$

  so that ω and ω_2, and hence ω and $(1,0,0)$, are cohomologous.

$$\begin{tikzcd}
\omega & & \\
\omega_3 \arrow[u, "-\mathrm{d}"] \arrow[r, "c"'] & \omega_2 & \\
& \omega_1 \arrow[u, "\mathrm{d}"] \arrow[r, "b"'] & (1,0,0)
\end{tikzcd}$$

FIGURE 4. The zig-zag method

It is straightforward to check that one can take

$$\omega_1 = (x-1, 0, 0), \quad \omega_2 = (\mathrm{d}x, 0, 0), \quad \omega_3 = (1-y)\mathrm{d}x + x\mathrm{d}y.$$

This yields the differential form $\omega = -2\mathrm{d}x \wedge \mathrm{d}y$ on $\mathbb{A}^2$, which defines a relative cohomology class since it has top degree. Summing up, we get

$$\mathrm{H}^n_{\mathrm{dR}}(X, D) = \begin{cases} \langle \mathrm{d}x \wedge \mathrm{d}y \rangle_k, & \text{if } n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

⋆⋆⋆

EXERCISE 2.121. Prove that the axioms i) and ii) in Definition 2.73 imply that the exterior derivative is given by

$$\mathrm{d}(f\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$$

in local coordinates $x_1, \ldots, x_d$. Deduce that $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d}$ vanishes.

EXERCISE 2.122. Let $\omega_1, \ldots, \omega_p$ be Kähler differentials, and let $\sigma \in \mathfrak{S}_p$ be a permutation. Prove the equality

$$\omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(p)} = \mathrm{sign}(\sigma)\omega_1 \wedge \cdots \wedge \omega_p.$$

EXERCISE 2.123. Let k be a field of characteristic zero. Show that $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{A}^n_k) = k$ and that all the other cohomology groups vanish.

EXERCISE 2.124. Check the identities (2.100) in Example 2.99.

EXERCISE 2.125. In Example 2.91 we saw that a basis of the de Rham cohomology of an affine elliptic curve $X \subseteq \mathbb{A}^2_k$ is given by the classes of the differential forms ω and $x\omega$. Let $\overline{X} \subseteq \mathbb{P}^2_k$ be the projective completion of X, that is, the smooth projective curve obtained by adjoining to X the point at infinity $O = [0:1:0]$. Prove that ω extends to a holomorphic differential form on $\overline{X}$, that is, to a global section of the sheaf $\Omega^1_{\overline{X}}$, whereas $x\omega$ has a double pole at O.

EXERCISE 2.126. We have defined de Rham cohomology for varieties over a field of characteristic zero. Show by means of an example that the same definition gives pathological results in positive characteristic (for instance, the cohomology of the affine line $\mathbb{A}^1$ is infinite-dimensional).

EXERCISE 2.127. We place ourselves in the situation of Example 2.108.

i) Show that the elements $-\omega_1' = (0,-1,0)$ and $\omega_1 = (\mathrm{d}x/(p-q),0,0)$ are cohomologous.

ii) Show that, for every integer $r \neq -1$, the elements
$$(x^r\mathrm{d}x, 0, 0), \quad \text{and} \quad \frac{p^{r+1}-q^{r+1}}{r+1}\omega_1$$
are cohomologous.

Exercise 2.128. Let $D \subset \mathbb{P}^n$ be the union of a finite collection of hyperplanes. Under which condition on the hyperplanes is D a simple normal crossing divisor?

Exercise 2.129. Let A be a k-algebra and let $\mu\colon A \otimes_k A \to A$ denote the multiplication map which sends an element $\sum_i a_i \otimes b_i$ to $\sum a_i b_i$. Set
$$I = \operatorname{Ker}\big(\mu\colon A \otimes_k A \to A\big).$$
The goal of the exercise is to establish an isomorphism of A-modules
$$\Omega^1_{A/k} \simeq I/I^2.$$

i) Show that the map $a \mapsto 1\otimes a - a \otimes 1$ induces a k-linear derivation $A \to I/I^2$, and hence a morphism of A-modules $\varphi\colon \Omega^1_{A/k} \to I/I^2$ by the universal property (2.85).

ii) Consider the ring $R = A \oplus \Omega^1_{A/k}$, where A acts on $\Omega^1_{A/k}$ via the A-module structure, and the product of any two elements of $\Omega^1_{A/k}$ is zero. Show that the k-bilinear map
$$A \times A \longrightarrow R \qquad (a_1, a_2) \longmapsto (a_1a_2, a_1\mathrm{d}a_2)$$
factors through $A \otimes_k A$ and sends I to $\Omega^1_{A/k}$ and I^2 to zero. Therefore, it defines a map $\psi\colon I/I^2 \to \Omega^1_{A/k}$.

iii) Prove that φ and ψ are inverse of each other.

Exercise 2.130 (The module of Kähler differentials is not locally free for singular varieties). Set $A = k[x,y]/(xy)$ and $X = \operatorname{Spec}(A)$. By Example 2.86, the module of Kähler differentials $\Omega^1_{X/k}$ has generators $\mathrm{d}x$ and $\mathrm{d}y$, which are subject to the relation $x\mathrm{d}y = -y\mathrm{d}x$. Set $\omega = x\mathrm{d}y$.

i) Show that $\omega \neq 0$ but $x\omega = y\omega = 0$.

ii) Let $z \in A$. Show that $xz = yz = 0$ implies $z = 0$. Conclude that $\Omega^1_{X/k}$ is not locally free.

iii) Show that $k \cdot \omega$ sits in an exact sequence
$$0 \longrightarrow k\cdot\omega \longrightarrow \Omega^1_{X/k} \longrightarrow k[x]\mathrm{d}x \oplus k[y]\mathrm{d}y \longrightarrow 0,$$
and that this exact sequence does not split as a sequence of A-modules.

iv) Prove the equality $\Omega^2_X = k \cdot \mathrm{d}x \wedge \mathrm{d}y$.

Exercise 2.131 (An instance of Jouanolou's trick). In this exercise, we show how to compute the algebraic de Rham cohomology of the projective line $\mathbb{P}^1$ using global differential forms on an affine variety.

i) Let $\Delta \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal and set $X = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$. Prove that X is the affine variety
$$\operatorname{Spec}\left(k[x,y,z]/(x(x-1)-yz)\right)$$

and that the projection onto the first factor $\pi\colon X \to \mathbb{P}^1$ is given in these coordinates by $\pi(x,y,z) = [x\colon y] = [x-1 : z]$. Observe that all the fibers of π are affine lines. [Hint: first identify $\mathbb{P}^1 \times \mathbb{P}^1$ with a quadric in $\mathbb{P}^3$ through the Segre embedding.]

ii) Prove that the complexes $\Omega^*_{\mathbb{P}^1}$ and $R\pi_*\Omega^*_X$ of locally free sheaves on $\mathbb{P}^1$ are quasi-isomorphic. Deduce that the algebraic de Rham cohomologies $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^1)$ and $\mathrm{H}^i_{\mathrm{dR}}(X)$ are isomorphic. [Hint: use the Leray spectral sequence and that the morphism π is affine, so that $R\pi_*\Omega^*_X = \pi_*\Omega^*_X$ holds.]

iii) Write down a global differential form $\omega \in \Omega^2(X)$ generating $\mathrm{H}^2_{\mathrm{dR}}(X)$.

2.3. The comparison isomorphism. The goal of this section is to prove an algebraic counterpart of de Rham's theorem 2.79, namely Grothendieck's theorem [**Gro66**] according to which the Betti and the de Rham cohomology of a smooth algebraic variety over a subfield of the complex numbers become canonically isomorphic after extension of scalars.

THEOREM 2.132 (Grothendieck). *Let k be a subfield of the complex numbers and let X be a smooth variety over k. There is a functorial isomorphism*

$$\mathrm{comp}_{\mathrm{B,dR}}\colon \mathrm{H}^i_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

of complex vector spaces, which for affine X is given by

$$\mathrm{comp}_{\mathrm{B,dR}}([\omega])([\sigma]) = \int_\sigma \omega.$$

We will often refer to $\mathrm{comp}_{\mathrm{B,dR}}$ as the *comparison isomorphism.* In this context, functoriality means that the diagram

$$\begin{tikzcd}
\mathrm{H}^i_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \arrow[r, "\mathrm{comp}_{\mathrm{B,dR}}"] & \mathrm{H}^i_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \\
\mathrm{H}^i_{\mathrm{dR}}(Y) \otimes_k \mathbb{C} \arrow[r, "\mathrm{comp}_{\mathrm{B,dR}}"] \arrow[u, "f^*"] & \mathrm{H}^i_{\mathrm{B}}(Y) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[u, "f^*"']
\end{tikzcd}$$

commutes for each morphism $f\colon X \to Y$ of smooth algebraic varieties over k.

2.3.1. *From algebraic to analytic: Serre's GAGA theorem.* The first step in the proof of the comparison isomorphism consists in relating the cohomology of coherent sheaves on algebraic varieties to the cohomology of their analytifications. This is the content of Serre's GAGA theorem, named after the title "Géométrie Algébrique et Géométrie Analytique" of his landmark paper [**Ser56**]. We refer the reader to Section A.9.9 of the appendix for a short reminder of the notions of coherent sheaf in algebraic and analytic geometry, as well as the vanishing theorems for coherent sheaves on affine and Stein varieties that are used below.

Let X be a smooth algebraic variety over the field of complex numbers. The set of complex points $X(\mathbb{C})$ of X, together with the topology inherited from that of $\mathbb{C}$, carries the structure of a complex manifold X^{an}. We let $\mathcal{O}_{X^{\mathrm{an}}}$ denote the sheaf of holomorphic functions on X^{an} and $\psi\colon X^{\mathrm{an}} \to X$ the map of topological spaces that sends an element of $X(\mathbb{C}) = \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}), X)$ to the corresponding closed point of X. Since the complex points of every Zariski open subset of X form an open subset of $X(\mathbb{C})$, this map is continuous. Moreover, there is a morphism

$$\psi^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_{X^{\mathrm{an}}}$$

of sheaves of rings on X^{an}. Indeed, recall from Definition A.283 that the inverse image $\psi^{-1}\mathcal{O}_X$ is the sheaf associated with the presheaf

$$U \longmapsto \varinjlim_{V} \mathcal{O}_X(V), \tag{2.133}$$

where U is an open subset of $X(\mathbb{C})$ and the limit runs over all Zariski open subsets V of X containing $\psi(U)$. Since every regular function f on V induces a holomorphic function f^{an} on $V(\mathbb{C})$, and hence on $U(\mathbb{C})$ upon restriction, there is an obvious morphism of presheaves from (2.133) to $\mathcal{O}_{X^{\mathrm{an}}}$. As the target $\mathcal{O}_{X^{\mathrm{an}}}$ is itself a sheaf, the universal property of sheafification (Proposition A.249) yields the sought morphism of sheaves $\psi^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{\mathrm{an}}}$.

Definition 2.134. Let F be a sheaf of $\mathcal{O}_X$-modules on X. The *analytification* of F is the sheaf of $\mathcal{O}_{X^{\mathrm{an}}}$-modules on X^{an} given by

$$F^{\mathrm{an}} = \psi^{-1}F \otimes_{\psi^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}.$$

In other words, F^{an} is the sheaf associated with the presheaf

$$U \longmapsto \psi^{-1}F(U) \otimes_{\psi^{-1}\mathcal{O}_X(U)} \mathcal{O}_{X^{\mathrm{an}}}(U). \tag{2.135}$$

If F is coherent in the sense of algebraic geometry, then F^{an} is coherent in the sense of analytic geometry.

Example 2.136. Let X be an algebraic variety. By its very definition, the analytification of the sheaf of regular functions on X is the sheaf of holomorphic functions on X^{an}. In symbols, $\mathcal{O}_X^{\mathrm{an}} = \mathcal{O}_{X^{\mathrm{an}}}$. Moreover, on each complex manifold M one can define the *sheaf of holomorphic differential forms*. Namely, for $p \geqslant 0$, the sheaf Ω^p_M consists of differential forms that can be written locally as

$$\omega = \sum_{i_1<i_2<\cdots<i_p} f_{i_1,\ldots,i_j} \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p},$$

where $(z_1, \ldots, z_d)$ are local complex coordinates of M and the functions $f_{i_1,\ldots,i_p}$ are holomorphic. The sheaves Ω^*_M together with the differential d form a complex

$$0 \longrightarrow \mathcal{O}_M \longrightarrow \Omega^1_M \longrightarrow \cdots \longrightarrow \Omega^d_M \longrightarrow 0,$$

that is called the *holomorphic de Rham complex*. For each $p \geqslant 1$, the analytification of the sheaf of Kähler differentials is the sheaf of holomorphic differential forms on the complex manifold X^{an}. In symbols,

$$(\Omega^p_X)^{\mathrm{an}} = \Omega^p_{X^{\mathrm{an}}}.$$

The morphism of presheaves from $\psi^{-1}F$ to (2.135) that sends a local section e to $e \otimes 1$ yields a morphism of sheaves $\psi^{-1}F \to F^{\mathrm{an}}$, and hence a morphism of sheaves $F \to \psi_* F^{\mathrm{an}}$ using the adjunction between inverse and direct image functors; see (A.285) from the appendix. Composing this with the map from a functor to its derived functor, we get morphisms of complexes of sheaves

$$F \longrightarrow \psi_* F^{\mathrm{an}} \longrightarrow R\psi_* F^{\mathrm{an}}. \tag{2.137}$$

Lemma 2.138. *If F is a coherent sheaf on X, then $R^i\psi_* F^{\mathrm{an}} = 0$ for all $i \geqslant 1$, and hence the map $\psi_* F^{\mathrm{an}} \to R\psi_* F^{\mathrm{an}}$ is an isomorphism.*

PROOF. Using Proposition A.277, the stalk at a closed point x of X of the higher direct image sheaf $R^i\psi_* F^{\mathrm{an}}$ is the inductive limit

$$\varinjlim \mathrm{H}^i(\psi^{-1}(U), F^{\mathrm{an}})$$

as U runs through all Zariski open subsets of X containing x. Since affine open subsets form a basis of the Zariski topology, we can restrict to affine U in this limit. For such an open U, the cohomology $\mathrm{H}^i(\psi^{-1}(U), F^{\mathrm{an}})$ vanishes for all $i \geqslant 0$ by Cartan's theorem A.311, since $\psi^{-1}(U)$ is Stein and F^{an} is coherent, so $R^i\psi_* F^{\mathrm{an}} = 0$ for all $i \geqslant 1$. Hence, the map $\psi_* F^{\mathrm{an}} \to R\psi_* F^{\mathrm{an}}$ is an isomorphism. □

The celebrated GAGA theorem by Serre [**Ser56**] is the result that, for X projective, the analytification functor $F \mapsto F^{\mathrm{an}}$ is an equivalence of categories from coherent sheaves on X to coherent sheaves on X^{an}, and that this functor preserves cohomology. This was later generalized to proper (not necessarily projective) varieties by Grothendieck in [**Gro03**, Exp. XII, Thm. 4.4]. In what follows, we will only need the comparison of algebraic and analytic cohomology on smooth varieties (using the language of analytic spaces instead of complex manifolds, the statement below remains valid for singular varieties).

THEOREM 2.139 (GAGA theorem). *For any smooth proper variety X over the field of complex numbers and any coherent sheaf F on X, the analytification map* (2.137) *induces isomorphisms*

$$\mathrm{H}^i(X, F) \xrightarrow{\sim} \mathrm{H}^i(X^{\mathrm{an}}, F^{\mathrm{an}}).$$

REMARK 2.140. The GAGA theorem fails dramatically when the properness assumption is dropped. For example, if $X = \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[x])$ is the affine line and $F = \mathcal{O}_X$ is the structure sheaf, in degree $i = 0$ the left-hand side is the set of polynomials $\mathbb{C}[x]$, whereas the right-hand side is the set of all entire functions.

By a limiting process, Theorem 2.139 still holds for quasi-coherent sheaves, as explained *e.g.* in [**Del70**, Lem. 6.5]. Indeed, any quasi-coherent sheaf F on X can be written as an inductive limit $F = \varinjlim F_i$ of coherent sheaves F_i; taking into account that the inverse image functor and the tensor product commute with inductive limits, its analytification is given by $F^{\mathrm{an}} = \varinjlim F_i^{\mathrm{an}}$. Besides, on a compact topological space cohomology commutes with inductive limits, and hence $\mathrm{H}^*(X, F)$ and $\mathrm{H}^*(X^{\mathrm{an}}, F^{\mathrm{an}})$ are the inductive limits of $\mathrm{H}^*(X, F_i)$ and $\mathrm{H}^*(X^{\mathrm{an}}, F_i^{\mathrm{an}})$ respectively, which are isomorphic by the GAGA theorem for coherent sheaves.

COROLLARY 2.141. *For any smooth proper variety X over the field of complex numbers and any quasi-coherent sheaf F on X, the analytification map* (2.137) *induces isomorphisms*

$$\mathrm{H}^i(X, F) \xrightarrow{\sim} \mathrm{H}^i(X^{\mathrm{an}}, F^{\mathrm{an}}).$$

2.3.2. *Algebraic and analytic de Rham cohomology.*

DEFINITION 2.142. Let M be a complex manifold. The *analytic de Rham cohomology* of M is the hypercohomology of the holomorphic de Rham complex (Example 2.136). We denote it by

$$\mathrm{H}^*_{\mathrm{dR}}(M) = \mathbb{H}^*(M, \Omega^*_M).$$

Let X be a smooth algebraic variety over $\mathbb{C}$ and let X^{an} be the associated complex manifold. Since the sheaf of holomorphic differentials $\Omega^p_{X^{\mathrm{an}}}$ is the analytification of the sheaf of Kähler differentials Ω^p_X for all $p \geqslant 0$ by Example 2.136, the morphisms (2.137) specialize to

$$\Omega^p_X \longrightarrow \psi_*\Omega^p_{X^{\mathrm{an}}} \longrightarrow R\psi_*\Omega^p_{X^{\mathrm{an}}}.$$

Moreover, since the sheaves Ω^p_X are locally free, and hence coherent, the second map is an isomorphism by Lemma 2.138. Besides, these maps are compatible with the differential, as the differential of a regular function is the same whether we consider it as an algebraic or as an analytic function, so we get a morphism $\Omega^*_X \to R\psi_*\Omega^*_X$ of complexes of sheaves on X. It induces a $\mathbb{C}$-linear map on cohomology
(2.143)

$$\mathrm{H}^*_{\mathrm{dR}}(X) = \mathbb{H}^*(X, \Omega^*_X) \longrightarrow \mathbb{H}^*(X, R\psi_*\Omega^*_X) = \mathbb{H}^*(X^{\mathrm{an}}, \Omega^*_{X^{\mathrm{an}}}) = \mathrm{H}^*_{\mathrm{dR}}(X^{\mathrm{an}}).$$

that we call the *analytification map*. We shall prove later (see the proof of Theorem 2.168 below) that this map is always an isomorphism. For the time being, we restrict ourselves to proper varieties.

Proposition 2.144. *If X is proper, then the analytification map* (2.143) *is an isomorphism:*

$$\mathrm{H}^*_{\mathrm{dR}}(X) \xrightarrow{\sim} \mathrm{H}^*_{\mathrm{dR}}(X^{\mathrm{an}}).$$

Proof. Since algebraic and analytic de Rham cohomology are defined as the hypercohomology of the complexes of sheaves Ω^*_X and $\Omega^*_{X^{\mathrm{an}}}$ respectively, they are the abutments of the spectral sequences

$$E_1^{p,q} = \mathrm{H}^q(X, \Omega^p_X) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X),$$
$$E_1^{p,q} = \mathrm{H}^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}}) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X^{\mathrm{an}}).$$

By the construction of the map between algebraic and analytic de Rham cohomology, there is a morphism of spectral sequences compatible with this map. By the GAGA theorem 2.139, the map

$$\mathrm{H}^q(X, \Omega^p_X) \longrightarrow \mathrm{H}^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}})$$

is an isomorphism for all $p, q \geqslant 0$, hence the result. □

2.3.3. *Analytic de Rham cohomology and the Poincaré lemma.* The next tool in the proof of the comparison theorem is a result relating analytic de Rham cohomology with singular cohomology.

Theorem 2.145 (Poincaré lemma). *Let M be a complex manifold. The inclusion of the sheaf of locally constant functions $\underline{\mathbb{C}}_M$ into the sheaf of holomorphic functions $\mathcal{O}_M$ induces a quasi-isomorphism $\iota\colon \underline{\mathbb{C}}_M \to \Omega^*_M$.*

Proof. Let M be a complex manifold of dimension d. Proving that the morphism of complexes of sheaves

$$\iota\colon \underline{\mathbb{C}}_M \longrightarrow \Omega^*_M$$

is a quasi-isomorphism amounts to showing that ι is a quasi-isomorphism after taking the stalk at each point of M. Since M is a complex manifold, every point has an open neighborhood biholomorphic to the polydisc

$$\mathbb{D}^d = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d \mid |z_i| < 1 \text{ for } i = 1, \ldots, d\},$$

and it suffices to show that the morphism $\iota\colon \mathbb{C} \to \Omega^*_M(\mathbb{D}^d)$ of complexes of sections on $\mathbb{D}^d$ is a quasi-isomorphism. We shall in fact prove that it is a homotopy equivalence (see Definition A.42). For this, we consider the map

$$r\colon \Omega^*_M(\mathbb{D}^d) \longrightarrow \mathbb{C}$$

that sends a function $g \in \mathcal{O}_M(\mathbb{D}^d)$ to its value at the center of the polydisc, and a differential form $\omega \in \Omega^p_M(\mathbb{D}^d)$, with $p \geqslant 1$, to zero. Clearly, $r \circ \iota = \mathrm{Id}_{\mathbb{C}}$ and the goal is to prove that $\iota \circ r$ is homotopic to the identity on $\Omega^*_M(\mathbb{D}^d)$. For this, we shall construct a map $h\colon \Omega^p_M(\mathbb{D}^d) \to \Omega^{p-1}_M(\mathbb{D}^d)$ for each $p \geqslant 0$. Let $H\colon [0,1] \times \mathbb{D}^d \to \mathbb{D}^d$ be the map given by $H(t,z) = tz$. For each differential form $\omega \in \Omega^p_M(\mathbb{D}^d)$, we define

$$h(\omega) = \int_0^1 i_{\partial_t} H^*(\omega)\mathrm{d}t, \tag{2.146}$$

where i_{∂_t} stands for the contraction of a differential form on $[0,1] \times \mathbb{D}^d$ with the vector field tangent to $[0,1]$. Recalling that a differential p-form on a manifold is a linear functional on the p-th alternating power of its tangent bundle, the contraction $\iota_{\partial_t} H^*(\omega)$ is defined as the $(p-1)$-th form which takes the value

$$(\iota_{\partial_t} H^*(\omega))(X_1 \wedge \cdots \wedge X_{p-1}) = H^*(\omega)(\partial_t \wedge X_1 \wedge \cdots \wedge X_{p-1})$$

on vector fields $X_1, \ldots, X_{p-1}$. To get a grasp on what h does to a differential form, let us compute an example. If $\omega = z_1 z_2 \mathrm{d}z_1 \wedge \mathrm{d}z_2$, then

$$\begin{aligned} H^*(\omega) &= (tz_1)(tz_2)\mathrm{d}(tz_1) \wedge \mathrm{d}(tz_2) \\ &= z_1 z_2 t^4 \mathrm{d}z_1 \wedge \mathrm{d}z_2 + z_1^2 z_2 t^3 \mathrm{d}t \wedge \mathrm{d}z_2 + z_1 z_2^2 t^3 \mathrm{d}z_1 \wedge \mathrm{d}t. \end{aligned}$$

Concretely, ι_{∂_t} deletes the summands not involving $\mathrm{d}t$ and replaces $\mathrm{d}t$ with a sign depending on its position in the remaining ones. In the case at hand,

$$i_{\partial_t} H^*(\omega) = z_1^2 z_2 t^3 \mathrm{d}z_2 - z_1 z_2^2 t^3 \mathrm{d}z_1,$$

and integration with respect to t yields

$$h(\omega) = \frac{1}{4}(z_1^2 z_2 \mathrm{d}z_2 - z_1 z_2^2 \mathrm{d}z_1).$$

If $\omega \in \Omega^p_M(\mathbb{D}^d)$, then $h(\omega)$ is a holomorphic differential $(p-1)$-form on $\mathbb{D}^d$, whence a map $h\colon \Omega^p_M(\mathbb{D}^d) \to \Omega^{p-1}_M(\mathbb{D}^d)$ for each $p \geqslant 0$. By the fundamental theorem of calculus, the equality

$$\mathrm{d}h(\omega) + h(\mathrm{d}\omega) = H^*(\omega)|_{t=1} - H^*(\omega)|_{t=0} = \omega - \iota \circ r(\omega)$$

holds, which shows that $\iota \circ r$ is indeed homotopic to the identity. □

Since the singular cohomology with complex coefficients of a complex manifold M is canonically isomorphic to the sheaf cohomology of $\underline{\mathbb{C}}_M$ by Theorem A.304, from the Poincaré lemma we immediately derive:

COROLLARY 2.147. *Singular cohomology with complex coefficients is canonically isomorphic to analytic de Rham cohomology:*

$$\mathrm{H}^*(M, \mathbb{C}) \xrightarrow{\sim} \mathrm{H}^*_{\mathrm{dR}}(M).$$

REMARK 2.148. The Poincaré lemma fails for the Zariski topology: if X is an algebraic variety over k, then the complex

$$0 \longrightarrow \mathrm{Ker}(d) \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_X \longrightarrow \cdots \tag{2.149}$$

of sheaves for the Zariski topology is not exact. For example, if $X = \mathbb{G}_m$ with coordinate t, then a non-empty Zariski open subset of X is the complement $U = X \setminus S$ of a finite set of closed points S, and the complex

$$0 \longrightarrow k \longrightarrow \mathcal{O}_X(U) \xrightarrow{d} \Omega^1_X(U) \longrightarrow 0$$

always has cohomology in degree two since the class of $\mathrm{d}t/t$ still does not vanish in $\Omega^1_X(U)/\operatorname{Im}(d)$. In fact, the smaller the Zariski open gets the bigger the cohomology group becomes (see Exercise 2.177).

2.3.4. *The case of smooth proper varieties.* We are now ready to establish the comparison isomorphism for smooth proper varieties as a mere combination of the Poincaré lemma and the isomorphism between algebraic and analytic de Rham cohomology deduced from the GAGA theorem.

Theorem 2.150. *Let k be a subfield of $\mathbb{C}$ and let X be a smooth proper variety over k. There is a canonical isomorphism*

$$\mathrm{H}^*_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} \mathrm{H}^*_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Proof. The sought isomorphism is the composition of the maps ①, ②, the inverse of ③, and ④ below:

$$\begin{tikzcd}
\mathrm{H}^*_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \arrow[r, "\sim", "①"'] & \mathrm{H}^*_{\mathrm{dR}}(X_{\mathbb{C}}) \arrow[r, "\sim", "②"'] & \mathrm{H}^*_{\mathrm{dR}}(X^{\mathrm{an}}) & \\
 & & \mathrm{H}^*(X(\mathbb{C}), \underline{\mathbb{C}}) \arrow[u, "③", "\sim"'] \arrow[r, "\sim", "④"'] & \mathrm{H}^*_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}.
\end{tikzcd}$$

The isomorphism ① comes from the compatibility of algebraic de Rham cohomology with extension of scalars (Lemma 2.102). The isomorphism ② between algebraic and analytic de Rham cohomology is the corollary of the GAGA theorem established as Proposition 2.144. The isomorphism ③ follows from the Poincaré lemma (Theorem 2.145). Finally, Theorem A.304 in the appendix and the universal coefficient theorem yield the isomorphism ④. □

In order to extend this result to arbitrary smooth varieties, we need to introduce the sheaf of differential forms with logarithmic poles.

2.3.5. *De Rham cohomology in terms of logarithmic differentials.* Recall the notion of a simple normal crossing divisor from Definition 2.112. In this paragraph, we explain how to compute algebraic de Rham cohomology of a smooth variety as the hypercohomology of the complex of sheaves of logarithmic differentials on a smooth compactification by a simple normal crossing divisor.

We begin with Hironaka's theorem on the *resolution of singularities*, which ensures that any compactification can be transformed into a smooth compactification by a simple normal crossing divisor. By a resolution of singularities of a variety Y, we mean a proper birational morphism $\pi\colon \widetilde{Y} \to Y$ from a smooth variety $\widetilde{Y}$. Recall that being birational means that there exists a dense open subset $U \subset Y$ such that the map $\pi^{-1}(U) \to U$ is an isomorphism; the closed complement $E \subset Y$ of the largest open subset with this property is called the *exceptional locus* of the morphism π.

THEOREM 2.151 (Hironaka). *Let k be a field of characteristic zero. Let Y be a variety over k and let $Z \subset Y$ be a closed subvariety. There exists a proper birational morphism $\pi\colon \widetilde{Y} \to Y$ such that*

i) *$\widetilde{Y}$ is smooth;*

ii) *the union of $\pi^{-1}(Z)$ and the exceptional locus of π is a simple normal crossing divisor;*

iii) *π is an isomorphism outside the union of Z and the singular locus of Y.*

Moreover, such a morphism can be obtained as an iterated blow-up along smooth subvarieties of Y.

The morphism $\pi\colon \widetilde{Y} \to Y$ in Theorem 2.151 is called a *resolution of singularities* of the pair (Y, Z). We will mainly use the following consequence of Hironaka's theorem. Start with a smooth variety X over k and choose a proper variety Y over k containing X as an open dense subvariety (for example, if X is quasi-projective, then one can pick as Y the closure of X into some projective space on which X embeds as a locally closed subset; the general case requires Nagata's compactification theorem). Applied to Y and $Z = Y \setminus X$, Hironaka's theorem yields a resolution of singularities $\pi\colon \widetilde{Y} \to Y$ that is an isomorphism outside Z and such that $\pi^{-1}(Z)$ is a normal crossing divisor.

COROLLARY 2.152. *Given a smooth variety X over k, there exists a smooth proper variety $\overline{X}$ over k and an open immersion $j\colon X \hookrightarrow \overline{X}$ such that $D = \overline{X} \setminus X$ is a simple normal crossing divisor.*

We shall call such an $\overline{X}$ a smooth *compactification* of X by the simple normal crossing divisor D. With these preliminaries out of the way, we now turn to the definition of the complex of logarithmic differentials.

DEFINITION 2.153 (Deligne). The *complex of sheaves of logarithmic differentials along D* is the smallest subcomplex

$$\Omega^*_{\overline{X}}(\log D) \hookrightarrow j_*\Omega^*_X$$

that is stable under wedge product of differential forms and contains $\Omega^*_{\overline{X}}$ and the logarithmic derivatives $\mathrm{d}f/f$ of all local sections f of $j_*\mathcal{O}_X^\times$.

The sheaf $j_*\mathcal{O}_X^\times$ consists of all rational functions that can be written locally as h/g with $h, g \in \mathcal{O}_{\overline{X}}$ and $h|_X, g|_X \in \mathcal{O}_X^\times$. It follows that $\Omega^1_{\overline{X}}(\log D)$ is a locally free $\mathcal{O}_{\overline{X}}$-module of rank $d = \dim X$. Indeed, if $z_1, \ldots, z_d$ is a regular system of parameters with respect to which D is cut out by the equation $z_1 \cdots z_r = 0$ (Definition 2.112), then $\Omega^1_{\overline{X}}(\log D)$ is locally generated by the differential forms

$$\mathrm{d}z_1/z_1, \ldots, \mathrm{d}z_r/z_r, \mathrm{d}z_{r+1}, \ldots, \mathrm{d}z_d.$$

Moreover, the equality $\Omega^p_{\overline{X}}(\log D) = \Lambda^p\Omega^1_{\overline{X}}(\log D)$ holds for all $p \geqslant 0$.

We will also need to consider logarithmic differentials in the analytic setting. We first define simple normal crossing divisors on complex manifolds.

DEFINITION 2.154. Let M be a complex manifold of dimension d. A closed analytic subset $Z \subset M$ of codimension 1 is called a *simple normal crossing divisor* if all its irreducible components are smooth and, for each point $p \in D$, there exist an integer $1 \leqslant r \leqslant d$ and local coordinates $x_1, \ldots, x_d$ around p on which D is defined by the equation $x_1 \cdots x_r = 0$.

REMARK 2.155. There is an analogous definition of the sheaf of holomorphic forms on a complex manifold M with logarithmic poles along a simple normal crossing divisor Z. If X is a smooth complex algebraic variety and $\overline{X}$ a smooth compactification by a simple normal crossing divisor $D = \overline{X} \setminus X$, then

$$\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}) = \Omega^*_{\overline{X}}(\log D)^{\mathrm{an}}.$$

DEFINITION 2.156. Let M be a complex manifold, $Z \subset M$ a normal crossing divisor, $U = M \setminus Z$ its complement, and $j \colon U \to M$ the inclusion. The subcomplex of $j_*\Omega^*_U$ consisting of the sheaves of holomorphic differential forms on U that are meromorphic along Z is denoted by $j^{\mathrm{m}}_*\Omega^*_U$.

In concrete terms, $j^{\mathrm{m}}_*\Omega^*_U$ is the sheaf of differential forms on M, possibly with singularities along Z, that can be written locally as ω/f for local sections ω of Ω^*_M and f of $\mathcal{O}_M$ such that the restriction $f|_U$ is invertible, *i.e.* a local section of $\mathcal{O}^\times_U$.

EXAMPLE 2.157. Let X be a smooth variety over $\mathbb{C}$ and $\overline{X}$ a smooth compactification with $D = \overline{X} \setminus X$ a simple normal crossing divisor. By an abuse of notation, we write j for both the algebraic open immersion $X \to \overline{X}$ and the analytic open immersion $X^{\mathrm{an}} \to \overline{X}^{\mathrm{an}}$. Then, we can identify

$$j^{\mathrm{m}}_*\Omega^*_{X^{\mathrm{an}}} = (j_*\Omega^*_X)^{\mathrm{an}}. \tag{2.158}$$

Indeed, both sheaves are subsheaves of $j_*\Omega^*_{X^{\mathrm{an}}}$. A local section of $j_*\Omega^*_{X^{\mathrm{an}}}$ belongs to $(j_*\Omega^*_X)^{\mathrm{an}}$ if it can be written locally as ω/f with $\omega \in \Omega^*_{\overline{X}^{\mathrm{an}}}$ and $f \in \mathcal{O}_{\overline{X}}$ such that $f|_X \in \mathcal{O}^\times_X$. Since any holomorphic function $g \in \mathcal{O}_{\overline{X}^{\mathrm{an}}}$ with $g|_{X^{\mathrm{an}}} \in \mathcal{O}^\times_{X^{\mathrm{an}}}$ can be written locally as $g = fu$ with f as before and $u \in \mathcal{O}^\times_{\overline{X}^{\mathrm{an}}}$, we deduce (2.158) from the concrete description of $j^{\mathrm{m}}_*\Omega^*_{X^{\mathrm{an}}}$ given in Definition 2.156.

The proof of the next theorem is taken from [**Del70**, Lem. 6.9].

THEOREM 2.159. *The inclusions of complexes of sheaves*

$$\iota^{\mathrm{m}} \colon \Omega^*_M(\log Z) \hookrightarrow j^{\mathrm{m}}_*\Omega^*_U \qquad \iota \colon \Omega^*_M(\log Z) \hookrightarrow j_*\Omega^*_U$$

are quasi-isomorphisms.

PROOF. To fix ideas we start by proving the result in dimension one. Since the statement is local, pick a small ball B with coordinate z such that $Z \cap B = \{z = 0\}$. Then all elements $\omega_1 \in (j_*\Omega^*_U)(B)$ and $\omega_2 \in (j^{\mathrm{m}}_*\Omega^*_U)(B)$ can be written as

$$\omega_1 = \sum_{n\in\mathbb{Z}} a_n z^n + \sum_{n\in\mathbb{Z}} b_n z^n \frac{\mathrm{d}z}{z}, \qquad \omega_2 = \sum_{n\geqslant n_0} a_n z^n + \sum_{n\geqslant n_1} b_n z^n \frac{\mathrm{d}z}{z}.$$

We consider the map $r \colon (j_*\Omega^*_U)(B) \to (\Omega^*_M(\log Z))(B)$ that forgets the polar non-logarithmic part, that is,

$$\sum_{n\in\mathbb{Z}} a_n z^n + \sum_{n\in\mathbb{Z}} b_n z^n \frac{\mathrm{d}z}{z} \longmapsto \sum_{n\geqslant 0} a_n z^n + \sum_{n\geqslant 0} b_n z^n \frac{\mathrm{d}z}{z}.$$

Let r^{m} be the restriction of r to $(j^{\mathrm{m}}_*\Omega^*_U)(B)$. Then the equalities $r \circ \iota = r^{\mathrm{m}} \circ \iota^{\mathrm{m}} = \mathrm{Id}$ hold, so we need to prove that the compositions $\iota \circ r$ and $\iota^{\mathrm{m}} \circ r^{\mathrm{m}}$ are homotopically equivalent to the identity. We define the homotopy

$$\begin{array}{rccc} h\colon & (j_*\Omega^1_U)(B) & \longrightarrow & (j_*\Omega^0_U)(B) \\ & \sum_{n\in\mathbb{Z}} b_n z^n \frac{\mathrm{d}z}{z} & \longmapsto & \sum_{n<0} \frac{b_n}{n} z^n. \end{array}$$

Let $h^{\mathrm{m}}\colon (j_*^{\mathrm{m}}\Omega_U^1)(B) \to (j_*^{\mathrm{m}}\Omega_U^0)(B)$ the restriction of h. A direct verification gives

$$\mathrm{d}\circ h + h\circ \mathrm{d} = \mathrm{Id} - \iota\circ r, \quad \mathrm{d}\circ h^{\mathrm{m}} + h^{\mathrm{m}}\circ \mathrm{d} = \mathrm{Id} - \iota^{\mathrm{m}}\circ r^{\mathrm{m}}.$$

We now assume that M is d-dimensional. Let $P\subset M$ be an open polydisc with coordinates $z_1,\dots,z_d$ such that $Z\cap P$ has equation $z_1\dots z_k=0$. For $i=1,\dots,d$, we define maps $r_i\colon (j_*\Omega_U^*)(P)\to (j_*\Omega_U^*)(P)$ by the assignment

$$\sum_{\substack{I\subset\{1,\dots,k\}\\ J\subset\{k+1,\dots,d\}\\ \alpha\in\mathbb{Z}^k\times\mathbb{N}^{d-k}}} a_{I,J,\alpha} z^{\alpha}\frac{\mathrm{d}z^I}{z^I}\mathrm{d}z^J \longmapsto \sum_{\substack{I\subset\{1,\dots,k\}\\ J\subset\{k+1,\dots,d\}\\ \alpha\in\mathbb{Z}^k\times\mathbb{N}^{d-k}\\ \alpha_i\geqslant 0}} a_{I,J,\alpha} z^{\alpha}\frac{\mathrm{d}z^I}{z^I}\mathrm{d}z^J.$$

That is, the map r_i erases the polar non-logarithmic part on the coordinate z_i. The maps r_i are commuting projectors, *i.e.* satisfy $r_i^2=r_i$ and $r_i\circ r_j = r_j\circ r_i$. Moreover, the image of $r_1\circ\cdots\circ r_k$ is $(\Omega_M^*(\log Z))(P)$.

We now define the homotopies $H_i\colon (j_*\Omega_U^*)(P)\to (j_*\Omega_U^{*-1})(P)$ by the rule

$$\sum_{\substack{I\subset\{1,\dots,k\}\\ J\subset\{k+1,\dots,d\}\\ \alpha\in\mathbb{Z}^k\times\mathbb{N}^{d-k}}} a_{I,J,\alpha} z^{\alpha}\frac{\mathrm{d}z^I}{z^I}\mathrm{d}z^J \longmapsto \sum_{\substack{I\subset\{1,\dots,k\}\\ J\subset\{k+1,\dots,d\}\\ i\in I\\ \alpha\in\mathbb{Z}^k\times\mathbb{N}^{d-k}\\ \alpha_i< 0}} \varepsilon_i\frac{a_{I,J,\alpha}}{\alpha_i} z^{\alpha}\frac{\mathrm{d}z^{I\setminus\{i\}}}{z^{I\setminus\{i\}}}\mathrm{d}z^J,$$

where $\varepsilon_i=\varepsilon(I\setminus\{i\},I)$ is the sign of the permutation $(i,I\setminus\{i\})$ of I, seing I as an ordered set, as in Construction 2.114.

The homotopies H_i satisfy

$$H_i\circ\mathrm{d}+\mathrm{d}\circ H_i = \mathrm{Id} - r_i$$

for all i and $H_i\circ r_j = r_j\circ H_i$ for all $i\neq j$. This implies that all the maps

$$(j_*\Omega_U^*)(P)\xrightarrow{r_1}\mathrm{Im}(r_1)\xrightarrow{r_2}\mathrm{Im}(r_2\circ r_1)\longrightarrow\cdots\longrightarrow(\Omega_M^*(\log Z))(P) \tag{2.160}$$

are homotopy equivalences, and hence their composition is a quasi-isomorphism from the complex $(j_*\Omega_U^*)(P)$ to the complex $(\Omega_M^*(\log Z))(P)$. Since the map ι in the statement is a right-inverse of the composition (2.160), it is a quasi-isomorphism as well. The result for $(j_*^{\mathrm{m}}\Omega_U^*)(P)$ follows from the observation that the operators r_i and H_i send this last space to itself. □

As an immediate consequence of this theorem, we see that the complexes $j_*^{\mathrm{m}}\Omega_U^*$ and $j_*\Omega_U^*$ are quasi-isomorphic.

Corollary 2.161. *The inclusion of complexes of sheaves*

$$j_*^{\mathrm{m}}\Omega_U^*\hookrightarrow j_*\Omega_U^*$$

is a quasi-isomorphism.

Another consequence of Theorem 2.159, combined with the GAGA theorem for quasi-coherent sheaves, is that the complex of logarithmic differentials computes algebraic de Rham cohomology.

Proposition 2.162. *Let k be a field of characteristic zero. Let X be a smooth variety and $\overline{X}$ a smooth compactification of X by a simple normal crossing divisor D, with everything defined over k. Let $j\colon X\hookrightarrow\overline{X}$ denote the inclusion. The morphism of complexes*

$$\Omega_{\overline{X}}^*(\log D)\hookrightarrow j_*\Omega_X^* \tag{2.163}$$

induces an isomorphism in hypercohomology, hence an isomorphism

$$\mathbb{H}^n(\overline{X}, \Omega^*_{\overline{X}}(\log D)) \xrightarrow{\sim} \mathrm{H}^n_{\mathrm{dR}}(X).$$

PROOF. We first use a standard trick in algebraic geometry to reduce to varieties over the field of complex numbers. Since the varieties X and $\overline{X}$ are of finite type, there is a finitely generated subfield $\mathbb{Q} \subset k_0 \subset k$ and varieties $(X_0, \overline{X}_0)$ defined over k_0 such that the pair $(X, \overline{X})$ is deduced from $(X_0, \overline{X}_0)$ by extension of scalars. Set $D_0 = \overline{X}_0 \setminus X_0$. Any field extension being flat, the map (2.163) induces an isomorphism in hypercohomology if and only if the same holds for the map

$$\Omega^*_{\overline{X}_0}(\log D_0) \hookrightarrow j_*\Omega^*_{X_0}. \tag{2.164}$$

The advantage is that k_0 can now be embedded into the complex numbers. We choose an embedding $k_0 \subset \mathbb{C}$ and we write $(X_{\mathbb{C}}, \overline{X}_{\mathbb{C}})$ for the pair of complex varieties obtained from $(X_0, \overline{X}_0)$ by extension of scalars. Again, the map (2.164) induces an isomorphism in hypercohomology if and only if the map

$$\Omega^*_{\overline{X}_{\mathbb{C}}}(\log D_{\mathbb{C}}) \hookrightarrow j_*\Omega^*_{X_{\mathbb{C}}}$$

does, and hence we may assume $k = \mathbb{C}$ without loss of generality.

Recall the continuous map $\psi\colon \overline{X}^{\mathrm{an}} \to \overline{X}$ from Section 2.3.1, and consider the commutative diagram

$$\begin{tikzcd}
j_*\Omega^*_X \arrow[r] & \psi_* j^{\mathrm{m}}_*\Omega^*_{X^{\mathrm{an}}} \\
\Omega^*_{\overline{X}}(\log D) \arrow[u] \arrow[r] & \psi_*\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}). \arrow[u]
\end{tikzcd}$$

Using the GAGA theorem for quasi-coherent sheaves (Corollary 2.141) and the identification $(j_*\Omega^*_X)^{\mathrm{an}} = j^{\mathrm{m}}_*\Omega^*_{X^{\mathrm{an}}}$ from Example 2.157, a spectral sequence argument similar to that in the proof of Proposition 2.144 shows that the upper horizontal arrow induces an isomorphism in hypercohomology. Again a spectral sequence argument and the GAGA theorem shows that the lower horizontal arrow also induces an isomorphism in hypercohomology. Theorem 2.159, together with the fact that the involved sheaves are ψ-acyclic, implies that the right vertical arrow is a quasi-isomorphism. We deduce that the left vertical arrow induces an isomorphism in hypercohomology. This gives the first statement and the isomorphism ③ in

$$\begin{tikzcd}
\mathrm{H}^n_{\mathrm{dR}}(X) = \mathbb{H}^n(X, \Omega^*_X) \arrow[r, "\simeq", "①"'] & \mathbb{H}^n(\overline{X}, Rj_*\Omega^*_X) \\
\mathbb{H}^n(\overline{X}, \Omega^*_{\overline{X}}(\log D)) \arrow[r, "\sim", "③"'] & \mathbb{H}^n(\overline{X}, j_*\Omega^*_X) \arrow[u, "\sim"', "②"]
\end{tikzcd} \tag{2.165}$$

The remaining isomorphisms are obtained as follows. Since the morphism j is affine (see Example A.312) and the sheaves $j_*\Omega^i_X$ are quasi-coherent, ② is an isomorphism by Theorem A.313. Finally, the isomorphism ① follows from Proposition A.278. Composing the maps ①, ②, and the inverse of ③, we get the second statement. □

REMARK 2.166. We have proved Proposition 2.162 using the GAGA principle and the corresponding result for complex manifolds. A purely algebraic proof using étale cohomology can be found in [**HMS17**, Prop. 3.1.16].

COROLLARY 2.167. *Let X be a smooth algebraic variety of dimension d over k. The algebraic de Rham cohomology $\mathrm{H}^n_{\mathrm{dR}}(X)$ is a finite-dimensional k-vector space. Moreover, $\mathrm{H}^n_{\mathrm{dR}}(X) = 0$ for all $n > 2d$.*

PROOF. By Proposition 2.162, it suffices to prove that the hypercohomology of the complex of logarithmic differentials is finite-dimensional. This cohomology group is the abutment of the spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(\overline{X}, \Omega^p_{\overline{X}}(\log D)).$$

Since the logarithmic differentials $\Omega^q_{\overline{X}}(\log D)$ are coherent sheaves on the proper variety $\overline{X}$, all terms $E_1^{p,q}$ of the spectral sequence have finite dimension and vanish unless $0 \leqslant p \leqslant d$ and $0 \leqslant q \leqslant d$. □

2.3.6. *The comparison isomorphism.* We now have all the ingredients needed to prove Grothendieck's comparison isomorphism.

THEOREM 2.168 (Grothendieck, [**Gro66**]). *Let X be a smooth variety over a subfield k of $\mathbb{C}$. There is a canonical isomorphism*

$$\mathrm{comp}_{\mathrm{B,dR}} \colon \mathrm{H}^i_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{2.169}$$

PROOF. As discussed in Section 2.3.4, one may assume $k = \mathbb{C}$ and it suffices to prove that the analytification map

$$\mathrm{H}^*_{\mathrm{dR}}(X) \longrightarrow \mathrm{H}^*_{\mathrm{dR}}(X^{\mathrm{an}})$$

is an isomorphism. Let $j \colon X \hookrightarrow \overline{X}$ be a smooth compactification by a simple normal crossing divisor. Since j is an affine morphism and Ω^*_X is a complex of coherent sheaves, the complex $Rj_*\Omega^*_X$ is reduced to $j_*\Omega^*_X$, so

$$\mathrm{H}^*_{\mathrm{dR}}(X) = \mathrm{H}^*(X, \Omega^*_X) = \mathrm{H}^*(\overline{X}, Rj_*\Omega^*_X) = \mathrm{H}^*(\overline{X}, j_*\Omega^*_X).$$

Similarly, since $j \colon X^{\mathrm{an}} \hookrightarrow \overline{X}^{\mathrm{an}}$ is a Stein morphism and the sheaves $\Omega^*_{X^{\mathrm{an}}}$ are coherent, the complex $Rj_*\Omega^*_{X^{\mathrm{an}}}$ is reduced to $j_*\Omega^*_{X^{\mathrm{an}}}$, and hence

$$\mathrm{H}^*_{\mathrm{dR}}(X^{\mathrm{an}}) = \mathrm{H}^*(X^{\mathrm{an}}, \Omega^*_{X^{\mathrm{an}}}) = \mathrm{H}^*(\overline{X}^{\mathrm{an}}, Rj_*\Omega^*_{X^{\mathrm{an}}}) = \mathrm{H}^*(\overline{X}^{\mathrm{an}}, j_*\Omega^*_{X^{\mathrm{an}}}).$$

By Corollary 2.161, the complex $j_*\Omega^*_{X^{\mathrm{an}}}$ is quasi-isomorphic to $j_*^{\mathrm{m}}\Omega^*_{X^{\mathrm{an}}}$, which is nothing but the analytification of $j_*\Omega^*_X$ (Example 2.157). Therefore, the first pages of the spectral sequences

$$\mathrm{H}^q(\overline{X}, j_*\Omega^p_X) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X)$$
$$\mathrm{H}^q(\overline{X}^{\mathrm{an}}, j_*^{\mathrm{m}}\Omega^p_{X^{\mathrm{an}}}) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X^{\mathrm{an}})$$

are isomorphic by the GAGA theorem for quasi-coherent sheaves (Corollary 2.141) , and so are the abutments. □

REMARK 2.170. When X is an affine variety, all classes in de Rham cohomology are represented by differential forms. In that case, the comparison isomorphism is induced by the pairing

$$\begin{aligned} \mathrm{H}^i_{\mathrm{dR}}(X) \otimes \mathrm{H}_i(X(\mathbb{C}), \mathbb{Q}) &\longrightarrow \mathbb{C} \\ \omega \otimes \sigma &\longmapsto \textstyle\int_\sigma \omega. \end{aligned} \tag{2.171}$$

This pairing is called the *period pairing*. The fact that it depends only on the classes of ω and σ, and is thus well defined, follows from Stokes's theorem.

REMARK 2.172. Later on, we will also need the inverse of the comparison isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$, which will be written as

$$\mathrm{comp}_{\mathrm{dR,B}}\colon \mathrm{H}^i_{\mathrm{B}}(X)\otimes_{\mathbb{Q}}\mathbb{C}\xrightarrow{\sim}\mathrm{H}^i_{\mathrm{dR}}(X)\otimes_k\mathbb{C}.$$

REMARK 2.173. The comparison isomorphism (Theorem 2.168) does not hold if the smoothness assumption is removed. For instance, if X is the affine plane curve defined by the equation $x^5+y^5+x^2y^2=0$, one can show that $\dim\mathrm{H}^1_{\mathrm{dR}}(X)$ is bigger than $\dim\mathrm{H}^1_{\mathrm{B}}(X)$; see [**AK11**, Ex. 4.4] for an example. However, the theorem remains true for singular varieties X with the "correct" definition of de Rham cohomology, as presented for instance in [**HMS17**, §3.3.3].

There is also a relative version of the comparison isomorphism:

THEOREM 2.174. *Let k be a subfield of the complex numbers, X a smooth variety, and $Z\subseteq X$ either a smooth closed subvariety or a normal crossing divisor, with everything defined over k. Then there is a canonical isomorphism*

$$\mathrm{H}^i_{\mathrm{dR}}(X,Z)\otimes_k\mathbb{C}\xrightarrow{\sim}\mathrm{H}^i_{\mathrm{B}}(X,Z)\otimes_{\mathbb{Q}}\mathbb{C}. \tag{2.175}$$

REMARK 2.176. For affine X and a smooth closed subvariety $\iota\colon Z\hookrightarrow X$, relative cohomology classes are represented by pairs (ω_X,ω_Z) and (σ_X,σ_Z) satisfying

$$\partial\sigma_X=-\iota_*\sigma_Z,\quad \iota^*\omega_X=\mathrm{d}\omega_Z,\quad \mathrm{d}\omega_X=0.$$

As in Remark 2.170, the comparison isomorphism is also given by a period pairing:

$$\begin{array}{rcl}\mathrm{H}^i_{\mathrm{dR}}(X,Z)\otimes\mathrm{H}_i^{\mathrm{B}}(X,Z) & \longrightarrow & \mathbb{C}\\ (\omega_X,\omega_Z)\otimes(\sigma_X,\sigma_Z) & \longmapsto & \int_{\sigma_X}\omega_X+\int_{\sigma_Z}\omega_Z.\end{array}$$

⋆⋆⋆

EXERCISE 2.177. A way to rephrase the fact that the Poincaré lemma fails for the Zariski topology, as explained in Remark 2.148, is by saying that, for a smooth connected variety X over k, the Zariski sheaves

$$\mathcal{H}^q_X=\frac{\mathrm{Ker}(\mathrm{d}\colon\Omega^q_X\to\Omega^{q+1}_X)}{\mathrm{Im}(\mathrm{d}\colon\Omega^{q-1}_X\to\Omega^q_X)}$$

are not zero in general. Observe that $\mathcal{H}^q_X$ is the sheaf associated with the presheaf

$$U\longmapsto\mathrm{H}^q_{\mathrm{dR}}(U).$$

As for any hypercohomology of sheaves, there is a spectral sequence

$$E_2^{p,q}=\mathrm{H}^p(X,\mathcal{H}^q_X)\Longrightarrow\mathrm{H}^{p+q}_{\mathrm{dR}}(X).$$

i) Prove that the sheaf $\mathcal{H}^0_X$ is flasque, and hence acyclic.
ii) Deduce that the presheaf $U\mapsto\mathrm{H}^1_{\mathrm{dR}}(U)$ is already a sheaf on X.

EXERCISE 2.178. Use the GAGA theorem for quasi-coherent sheaves (Corollary 2.141) to prove the following statement. Let X be a smooth complex variety and let $\overline{X}$ be a smooth compactification by a simple normal crossing divisor. Let f be a meromorphic function on $\overline{X}^{\mathrm{an}}$ that is holomorphic on X^{an}. Then f is algebraic, that is, comes from a rational function on $\overline{X}$. For example, an entire function on the complex plane that extends meromorphically to ∞ is a polynomial.

Exercise 2.179. Let k be a subfield of $\mathbb{C}$ and X a smooth variety over k. Let $\mathcal{E}^*_{X^{\mathrm{an}}_{\mathbb{C}}}$ be the sheaf of real-valued differential forms on the complex manifold $X^{\mathrm{an}}_{\mathbb{C}}$, let $E^*(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{R})$ be its complex of global sections, and $S^*(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{R})$ the complex of smooth singular cochains. Let $\psi_k \colon X^{\mathrm{an}}_{\mathbb{C}} \to X$ be the continuous map between the complex manifold and the scheme. Show that the comparison isomorphism of Theorem 2.168 is induced by the inclusion of sheaves

$$\Omega^*_{X/k} \longrightarrow (\psi_k)_* \mathcal{E}^*_{X^{\mathrm{an}}_{\mathbb{C}}} \otimes \mathbb{C}$$

and the morphism of complexes

$$\int \colon E^*(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{R}) \otimes \mathbb{C} \longrightarrow S^*(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{R}) \otimes \mathbb{C}.$$

2.4. Periods. In this section, we introduce a class of complex numbers called *periods*. They sit halfway between the algebraic and the transcendental numbers: although they tend to be transcendental, periods share with algebraic numbers the property that they contain "a finite amount of information". Moreover, this information is of geometric nature. From the modern point of view, periods appear when comparing de Rham and Betti cohomology of algebraic varieties over number fields. We refer the reader to [**HMS17**] and [**Fre24**] for more detailed expositions.

2.4.1. *An elementary definition of periods.* The following elementary definition was first written down by Kontsevich and Zagier [**KZ01**, § 1.1]:

Definition 2.180. A *period* is a complex number whose real and imaginary parts can be written as absolutely convergent integrals of the form

$$\int_S f(x_1, \dots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n, \tag{2.181}$$

where the integrand is a rational function f with rational coefficients and the integration domain is a $\mathbb{Q}$-semi-algebraic subset $S \subset \mathbb{R}^n$, *i.e.* a finite union and intersection of subsets $\{g(x_1, \dots, x_n) \geqslant 0\}$ with g a polynomial with rational coefficients.

Periods form a countable subring of the complex numbers (Exercise 2.189). Moreover, one may replace "rational function" with "algebraic function" and "rational coefficients" with "algebraic coefficients" in the above definition, and still obtain the same class of numbers. Standard examples of periods include:

- All algebraic numbers (see Exercise 2.190).
- The number $\pi = \int_{x^2+y^2 \leqslant 1} \mathrm{d}x\mathrm{d}y$.
- Logarithms of rational numbers $\log(q) = \int_1^q \frac{\mathrm{d}x}{x}$, where $q \in \mathbb{Q}_{\geqslant 1}$.
- Elliptic integrals $\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x(x-1)(x-\lambda)}}$ where $\lambda \in \mathbb{Q} \setminus \{0, 1\}$.
- Multiple zeta values, certain Feynman integrals (see Section 2.8.11), periods of modular forms, some special values of L-functions, etc.

2.4.2. *A cohomological interpretation of periods.* The comparison isomorphism of Theorem 2.168 does not respect the rational structures, as it will be clear from the next basic example. In particular, in the case where $k = \mathbb{Q}$, the vector spaces $\mathrm{H}^i_{\mathrm{dR}}(X)$ and $\mathrm{H}^i_{\mathrm{B}}(X)$ are isomorphic (they have the same dimension), but there is no canonical isomorphism between them!

EXAMPLE 2.182. Let $X = \mathbb{G}_m = \operatorname{Spec}\big(\mathbb{Q}[t, t^{-1}]\big)$, so that $X(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. We know from examples 2.36 and 2.90 that

$$\mathrm{H}^1_{\mathrm{dR}}(X) = \mathbb{Q}\tfrac{\mathrm{d}t}{t}, \qquad \mathrm{H}_1(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}\sigma,$$

where σ is the counterclockwise oriented unit circle. Then the comparison isomorphism is given by multiplication by:

$$\int_\sigma \frac{\mathrm{d}t}{t} = 2\pi i.$$

The fact that the comparison isomorphism does not respect the rational structures gives rise to the cohomological interpretation of *periods*. The following result is shown in [**HMS17**, § 11.2]

THEOREM 2.183. *The set of periods is equal to the set of complex numbers that appear as a coefficient of a matrix of the comparison isomorphism* (2.175) *between the de Rham cohomology and the Betti cohomology of some pair* (X, Z) *consisting of a smooth variety* X *and a simple normal crossing divisor* $Z \subset X$*, both defined over* $\mathbb{Q}$*, with respect to* $\mathbb{Q}$*-bases of de Rham cohomology and Betti cohomology.*

2.4.3. *Examples.*

EXAMPLE 2.184. All algebraic numbers are periods. Indeed, let k be a number field and consider the zero-dimensional variety $X = \operatorname{Spec}(k)$, which we regard as defined over $\mathbb{Q}$. Then $\mathrm{H}^0_{\mathrm{dR}}(X)$ is canonically identified with the $\mathbb{Q}$-vector space k. By its very definition,

$$X(\mathbb{C}) = \operatorname{Hom}(\operatorname{Spec}(\mathbb{C}), \operatorname{Spec}(k)) = \operatorname{Hom}(k, \mathbb{C})$$

is the set of complex embeddings of k, and hence $\mathrm{H}^0_{\mathrm{B}}(X) = \mathbb{Q}^{\operatorname{Hom}(k,\mathbb{C})}$. If we choose a basis $a_1, \dots, a_n$ of k over $\mathbb{Q}$ and $\sigma_1, \dots, \sigma_n$ denote the complex embeddings of k, then the period matrix is $(\sigma_i(a_j))_{i,j}$.

In fact this example can be generalized as follows. Let $k \subset \mathbb{C}$ be a number field. A *period over* k is defined as a complex number that appears as a coefficient of a matrix of the comparison isomorphism (2.175) between the de Rham cohomology and the Betti cohomology of some pair (X, Z) consisting of a smooth variety X defined over k and a simple normal crossing divisor $Z \subset X$ also defined over k, with respect to k-basis of de Rham cohomology and a $\mathbb{Q}$ basis of Betti cohomology. This apparently more general definition of periods does not produce new numbers.

PROPOSITION 2.185 ([**HMS17**, § 11.2]). *The set consisting of all periods over* k *all number fields agrees with the set of periods over* $\mathbb{Q}$*.*

EXAMPLE 2.186. Let $X = \mathbb{G}_{m,\mathbb{Q}}$ and $Z = \{1, q\}$ for $q \in \mathbb{Q} \setminus \{0, 1\}$. In Example 2.44 we obtained generators σ_1 and σ_2 of $\mathrm{H}^{\mathrm{B}}_1(X, Z)$, and in Example 2.108 generators ω_1 and ω_2 of $\mathrm{H}^1_{\mathrm{dR}}(X, Z)$. With respect to these bases, the period matrix is given by

$$\begin{pmatrix} \int_{\sigma_2} \omega_1 & \int_{\sigma_2} \omega_2 \\ \int_{\sigma_1} \omega_1 & \int_{\sigma_1} \omega_2 \end{pmatrix} = \begin{pmatrix} 1 & \log(q) \\ 0 & 2\pi i \end{pmatrix},$$

which shows that logarithms of rational numbers are periods in the sense of the comparison isomorphism.

2.4.4. *Compatibility with complex conjugation.* We finish this section by proving a result of compatibility of the comparison isomorphism with complex conjugation that will be used in Chapter 4. Assume that k is a subfield of the real numbers. Complex conjugation $c\colon \mathbb{C} \to \mathbb{C}$ induces a continuous map $X(\mathbb{C}) \to X(\mathbb{C})$ on the complex points of any algebraic variety X over k, and hence an involution

$$\rho\colon \mathrm{H}^i_{\mathrm{B}}(X) \longrightarrow \mathrm{H}^i_{\mathrm{B}}(X)$$

on its Betti cohomology.

Proposition 2.187. *Assume $k \subseteq \mathbb{R}$. The comparison isomorphism (2.169) is equivariant for the action of $\mathbb{Z}/2$ by $\mathrm{Id} \otimes c$ on the left-hand side (de Rham) and by $\rho \otimes c$ on the right-hand side (Betti).*

Proof. We assume, as we may without loss of generality after extension of scalars, that k is the field of real numbers. Let X be a smooth variety over $\mathbb{R}$ and let $X^{\mathrm{an}}_{\mathbb{C}}$ be the corresponding complex manifold. We denote by $\mathcal{E}^*_{X^{\mathrm{an}}_{\mathbb{C}}}$ the sheaf of real-valued differential forms on $X^{\mathrm{an}}_{\mathbb{C}}$, by $E^*(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{R})$ its complex of global sections, and by $S^*(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{R})$ the complex of smooth singular cochains. Let $\psi_{\mathbb{R}}\colon X^{\mathrm{an}}_{\mathbb{C}} \to X$ be the continuous map between the complex manifold and the real scheme. By Exercise 2.179, the comparison isomorphism is induced by the inclusion of sheaves

$$\Omega^*_{X/\mathbb{R}} \longrightarrow (\psi_{\mathbb{R}})_* \mathcal{E}^*_{X^{\mathrm{an}}_{\mathbb{C}}} \otimes \mathbb{C}$$

and the morphism of complexes

$$\int\colon E^*(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{R}) \otimes \mathbb{C} \longrightarrow S^*(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{R}) \otimes \mathbb{C}$$

given by integration. By the functoriality of the de Rham theorem for differentiable maps (Theorem 2.79), $\int$ commutes with the map $\rho \otimes c$ that is defined on both sides. Let V be a Zariski open subset of X and let $U = \psi_{\mathbb{R}}^{-1}(V)$. Then U is invariant under ρ. Then we claim that the sections of

$$\Omega^*_{X/\mathbb{R}}(V) \subset (\psi_{\mathbb{R}})_* \mathcal{E}^*(V) \otimes \mathbb{C} = E^*(U) \otimes \mathbb{C}$$

are invariant under the action of $\rho \otimes c$. To prove this claim, we can assume that V is affine and there is a closed immersion $V \subset \mathbb{A}^N_{\mathbb{R}}$. Since every differential form on V is the restriction of a differential form on $\mathbb{A}^N_{\mathbb{R}}$, we are reduced to the case $X = \mathbb{A}^N_{\mathbb{R}}$ where we can use coordinates $\underline{x} = (x_1, \ldots, x_N)$.

Let $\omega = \sum_I f_I(\underline{x}) \mathrm{d}x^I$ be a holomorphic differential form. Then

$$(\rho \otimes c)(\omega) = \sum_I \overline{f_I(\underline{\bar{x}}) \mathrm{d}\bar{x}^I}.$$

If ω is a section of $\Omega^*_{X/\mathbb{R}}$, then the f_I are polynomials with real coefficients and we deduce that $(\rho \otimes c)(\omega) = \omega$, thus proving the claim.

It follows from the claim that $\mathrm{comp}_{\mathrm{B,dR}}(\mathrm{H}^i_{\mathrm{dR}}(X))$ is invariant under the action of $\rho \otimes c$, which in turn, thanks to Exercise 2.193, implies that $\rho \otimes c$ in $\mathrm{H}^i_{\mathrm{dR}}(X) \otimes \mathbb{C}$ induces $\mathrm{Id} \otimes c$ through the comparison isomorphism. This concludes the proof. □

We illustrate the proposition in the case of $X = \mathbb{G}_m$ viewed as a variety over $\mathbb{Q}$ (see Exercise 2.192 below for another instance).

EXAMPLE 2.188. We know from Example 2.182 that the comparison isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$ sends $\mathrm{d}t/t$ to $\sigma^\vee \otimes (2\pi i)$. The differential form being rational, it is invariant under complex conjugation, so $\sigma^\vee \otimes (2\pi i)$ should also be invariant. For this, observe that the image of σ by complex conjugation is the clockwise oriented unit circle, whose cohomology class is $-\sigma$. Thus,

$$(\rho \otimes c)(\sigma \otimes (2\pi i)) = -\sigma \otimes (-2\pi i) = \sigma \otimes (2\pi i),$$

as predicted by Proposition 2.187.

⋆⋆⋆

EXERCISE 2.189. Show that the set of periods is a countable ring. [Hint: to prove that it is closed under products use Fubini's theorem. To prove that it is closed under sums, first show that a real number is a period if and only if it can be written as a difference of volumes of $\mathbb{Q}$-semi-algebraic sets, and then show that the sum of two differences of volumes can be written as a difference of volumes.]

EXERCISE 2.190. In this exercise, we show that all algebraic numbers are periods in the sense of Definition 2.180. For example, the integral representation

$$\sqrt{2} = \int_{\substack{x^2 \leqslant 2 \\ x \geqslant 0}} \mathrm{d}x$$

shows that $\sqrt{2}$ is a period.

i) Let $P \in \mathbb{Q}[x]$ be an irreducible polynomial and let $\alpha_1, \ldots, \alpha_r$ be its real roots. Generalize the above example to show that all α_i are periods.
ii) Using that the real and the imaginary part of a complex algebraic number are real algebraic numbers, deduce that all algebraic numbers are periods.

EXERCISE 2.191. Let X be a smooth affine variety of dimension d over a subfield k of the complex numbers. Prove the vanishing $\mathrm{H}^i_{\mathrm{B}}(X) = 0$ for all $i > d$.

EXERCISE 2.192. Let $C \subset \mathbb{A}^2_{\mathbb{Q}}$ be the affine conic given by $x^2 + y^2 = 1$.

i) Show that the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(C)$ is generated by the class of the differential form $x\mathrm{d}y - y\mathrm{d}x$ and that the singular homology group $\mathrm{H}_1(C(\mathbb{C}), \mathbb{Q})$ is generated by the chain
$$\sigma \colon [0,1] \longrightarrow C(\mathbb{R}), \quad t \longmapsto (\cos(2\pi t), \sin(2\pi t)).$$
ii) Prove that the associated period is equal to
$$\int_\sigma x\mathrm{d}y - y\mathrm{d}x = 2\pi$$
and check Proposition 2.187 in this case.
iii) Find generators of the singular homology of the conics C defined by the equations $x^2 + y^2 = -1$ and $x^2 - y^2 = 1$ and check Proposition 2.187 in these cases as well.

EXERCISE 2.193. Let (V, σ) be a finite-dimensional complex vector space, together with an antilinear involution. Prove that the set of fixed points $V_{\mathbb{R}} = V^\sigma$ is a real vector space, that there is a unique isomorphism $V_{\mathbb{R}} \otimes \mathbb{C} \to V$ compatible with the inclusion $V_{\mathbb{R}} \subset V$, and that the involution σ agrees with the involution $\mathrm{Id} \otimes c$ on $V_{\mathbb{R}} \otimes \mathbb{C} \to V$ under this isomorphism.

2.5. Multiple zeta values as periods of algebraic varieties. The examples from the previous section show that algebraic numbers, logarithms of rational numbers, as well as the ubiquitous $2\pi i$ are all periods. From the integral representation (1.117), it follows immediately that multiple zeta values are periods in the sense of Kontsevich and Zagier (Definition 2.180). However, it is not so easy to exhibit the corresponding algebraic varieties. The main goal of this section is to work out the example of $\zeta(2)$ in detail to give an idea of the difficulties involved.

2.5.1. *The example of* $\zeta(2)$. Recall from Example 1.110 that $\zeta(2)$ admits the integral representation

$$\zeta(2) = \int_{1 \geqslant t_1 \geqslant t_2 \geqslant 0} \frac{\mathrm{d}t_1}{t_1} \wedge \frac{\mathrm{d}t_2}{1-t_2}. \tag{2.194}$$

The integrand is the differential form

$$\omega = \frac{\mathrm{d}t_1}{t_1} \wedge \frac{\mathrm{d}t_2}{1-t_2}$$

on the affine plane, which is singular along the union of the lines

$$\ell_0 = \{t_1 = 0\} \quad \text{and} \quad \ell_1 = \{t_2 = 1\}.$$

Thus, ω is a global differential 2-form on $Y = \mathbb{A}^2 \setminus (\ell_0 \cup \ell_1)$.

The domain of integration is the simplex

$$\sigma = \{(t_1, t_2) \mid 1 \geqslant t_1 \geqslant t_2 \geqslant 0\} \subset \mathbb{A}^2(\mathbb{C}).$$

However, if we want to consider the integral (2.194) as a period of Y, relative to some divisor containing the boundary of σ, we immediately face the problem that σ is not contained in Y, as the points $p = (0,0)$ and $q = (1,1)$ belong to $\sigma \cap (\ell_0 \cup \ell_1)$ (see Figure 5).

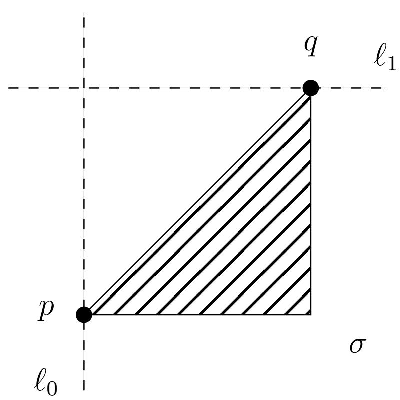


FIGURE 5. The simplex σ and the singular locus $\ell_0 \cup \ell_1$

A way to remedy this is to perform a geometric construction called *blow-up*, which replaces a point on a variety with a divisor. It is a very useful technique in the study of singularities. In our case, we have to blow up the two problematic points p and q. The blow-up of $\mathbb{A}^2$ along p and q is the closed subvariety $X \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$

defined by the equations

$$t_1\alpha_1 = t_2\beta_1,$$
$$(t_1-1)\alpha_2 = (t_2-1)\beta_2,$$

where $[\alpha_i : \beta_i]$ are homogeneous coordinates on the two copies of $\mathbb{P}^1$. The projection onto the first factor induces a proper surjective map

$$\pi \colon X \longrightarrow \mathbb{A}^2.$$

One checks that $\pi^{-1}(p)$ is the projective line

$$E_p = (0,0) \times \mathbb{P}^1 \times [1:1] \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

whereas $\pi^{-1}(q)$ is the projective line

$$E_q = (1,1) \times [1:1] \times \mathbb{P}^1 \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

They are called the *exceptional divisors* of the blow-up. Moreover, the restriction

$$\pi\mid_{X\setminus(E_p\cup E_q)} \colon X \setminus (E_p \cup E_q) \longrightarrow \mathbb{A}^2 \setminus \{p,q\}$$

is an isomorphism. For any closed subset $C \subset \mathbb{A}^2$, the *strict transform* $\widehat{C}$ of C is the closed subset of X given by

$$\widehat{C} = \overline{\pi^{-1}(C \setminus \{p,q\})}.$$

In words: we first remove the points p and q if they are in C, then we take the preimage on X by π, and finally the Zariski closure. The strict transform is contained in the *total transform* $\pi^{-1}(C)$ but it may be smaller. For instance, the strict transform of ℓ_0 is the affine line

$$L_0 = \widehat{\ell}_0 = \{((0,t_2),[1:0],[1-t_2:1]) \mid t_2 \in \mathbb{A}^1\},$$

while the total transform is $L_0 \cup E_p$. Note that L_0 and E_p intersect at the point

$$L_0 \cap E_p = \{((0,0),[1:0],[1:1])\}. \tag{2.195}$$

Similarly, the strict transform of ℓ_1 is the affine line

$$L_1 = \widehat{\ell}_1 = \{((t_1,1),[1:t_1],[0:1]) \mid t_1 \in \mathbb{A}^1\},$$

which is disjoint from the exceptional divisor E_p, intersects the line L_0 at the point $((0,1),[1:0],[0:1])$, and intersects E_q at $((1,1),[1:1],[0:1])$.

In principle, the pull-back $\pi^*(\omega)$ of ω might have singularities along the total transform of $\ell_0 \cup \ell_1$, which would only worsen the initial situation. Luckily, $\pi^*(\omega)$ is only singular on the strict transform $L_0 \cup L_1$. This can be seen using local coordinates in X. For instance, a local chart of X around the intersection of L_0 and E_p is given by the coordinates

$$t = \frac{\beta_1}{\alpha_1} = \frac{t_1}{t_2}, \quad s = t_2,$$

in which E_p and L_0 have local equations $s=0$ and $t=0$, respectively. Then

$$\pi^*(\omega) = \frac{\mathrm{d}(st)}{st} \wedge \frac{\mathrm{d}s}{1-s} = \frac{\mathrm{d}s}{s} \wedge \frac{\mathrm{d}s}{1-s} + \frac{\mathrm{d}t}{t} \wedge \frac{\mathrm{d}s}{1-s} = \frac{\mathrm{d}t}{t} \wedge \frac{\mathrm{d}s}{1-s},$$

where we have used the Leibniz rule and $\mathrm{d}s \wedge \mathrm{d}s = 0$. It follows that $\pi^*(\omega)$ is only singular along L_0 and not along $E_p \setminus L_0$. An analogous computation shows that $\pi^*(\omega)$ has singularities along L_1 but not along E_q.

The closed points of the exceptional divisor E_p can be interpreted as lines passing through the point p. This allows us to find the points of E_p that are contained in $\widehat{\sigma}$, namely

$$\widehat{\sigma} \cap E_p = \{((0,0), [m:1], [1:1]) \mid 0 \leqslant m \leqslant 1\}.$$

Combined with (2.195), this implies that $\widehat{\sigma} \cap L_0 = \emptyset$. A similar argument shows that $\widehat{\sigma} \cap L_1 = \emptyset$, so, after passing to the blow-up X, the singular locus of $\pi^*(\omega)$ and the domain of integration $\widehat{\sigma}$ are disjoint (Figure 6).

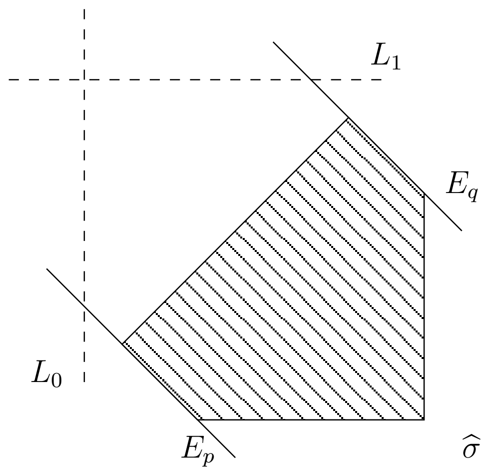


FIGURE 6. The strict transform of σ and the singular locus $L_0 \cup L_1$ of the form $\pi^*(\omega)$

Write $L = L_0 \cup L_1$. The complement $X \setminus L$ is still an affine variety; in fact, it is the closed subvariety of $\mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{A}^1$ defined by the equations

$$t_1 t = t_2,$$
$$(t_1 - 1) = (t_2 - 1)s,$$

where t, s are the coordinates of the first and the second affine lines. By the previous discussion, $\pi^*(\omega)$ is an element of $\Omega^2(X \setminus L)$.

The next issue one needs to deal with is that σ is not a closed chain. Its boundary is contained in the union of the affine lines

$$m_2 = \{t_1 = t_2\},\ m_3 = \{t_2 = 0\},\ m_4 = \{t_1 = 1\},$$

so we are naturally led to consider the normal crossing divisor

$$M = \pi^{-1}(m_2 \cup m_3 \cup m_4) = E_p \cup E_q \cup M_2 \cup M_3 \cup M_4 \subset X,$$

where M_i denotes the strict transform of m_i. One checks that the intersection $L \cap M$ is reduced to the points $L_0 \cap E_p$ and $L_1 \cap E_q$ which we have already computed.

Since $\widehat{\sigma}$ is contained in $X \setminus L$ and its boundary lies in M, using Remark 2.40 we see that $\widehat{\sigma}$ determines a relative homology class

$$\widehat{\sigma} \in \mathrm{H}_2(X \setminus L, M \setminus (L \cap M)).$$

Besides, the restriction of $\pi^*(\omega)$ to every irreducible component of M is zero for dimension reasons, so it defines a relative cohomology class

$$\pi^*(\omega) \in \mathrm{H}^2_{\mathrm{dR}}(X \setminus L, M \setminus (L \cap M)).$$

Pairing these classes through the comparison isomorphism (2.175) yields, as we wanted, the period

$$\int_{\widehat{\sigma}} \pi^*(\omega) = \int_{\pi_*(\widehat{\sigma})} \omega = \int_{\sigma} \omega = \zeta(2).$$

REMARK 2.196. This example was generalized by Terasoma in [**Ter02**]. For any multiple zeta value $\zeta(\boldsymbol{s})$, he starts with some affine space $\mathbb{A}^n$ and shows that, after blowing up some subvarieties, one obtains a smooth algebraic variety X and simple normal crossing divisors L and M on X such that $\zeta(\boldsymbol{s})$ is a period of the pair $(X \setminus L, M \setminus L \cap M)$. An important feature of Terasoma's construction is that the subvarieties one has to blow up are given by linear equations with all coefficients equal to 0 and 1. This implies that the "motive" of the pair $(X \setminus L, M \setminus L \cap M)$ is a "mixed Tate motive over $\mathbb{Z}$" and he uses this observation to prove Theorem A from the introduction. The notion of motive and mixed Tate motive over $\mathbb{Z}$ will be explained in Chapter 4. The combinatorics in Terasoma's approach is quite involved. Roughly at the same time, Goncharov and Manin [**GM04**] found a clever trick to make geometry do the hard work for us. They showed that multiple zeta values appear as periods of a suitable pair built out of a compactification of the moduli space $M_{0,n}$. We will explain this point of view in the next section with a little more detail. Thanks to the properties of these moduli spaces, they also show that multiple zeta values are periods of "mixed Tate motive over $\mathbb{Z}$". This can be used to give another proof of Theorem A. In these notes, we will rather follow a third approach due to Deligne and Goncharov [**DG05**] to show that multiple zeta values are periods associated with the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. This approach is better adapted to the question whether relations between multiple zeta values come from geometry.

2.5.2. *Multiple zeta values as periods of the moduli spaces* $\overline{M}_{0,n}$. For each integer $n \geqslant 3$, let $M_{0,n}$ be the moduli space of n ordered distinct points in $\mathbb{P}^1$ up to projective equivalence. In other words, every point of $M_{0,n}(\mathbb{C})$ is given by a tuple $(x_1, \ldots, x_n)$ of distinct complex points in $\mathbb{P}^1(\mathbb{C})$, and two tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are identified if there exists an element $g \in \mathrm{PGL}_2(\mathbb{C})$ satisfying $g(x_i) = y_i$ for all i. Since there exists a unique automorphism of $\mathbb{P}^1$ sending any given three points to $0, 1, \infty$, we can fix an identification

$$(x_1, \ldots, x_n) = (0, 1, \infty, t_1, \ldots, t_{n-3})$$

to get rid of the quotient. This induces an isomorphism

$$M_{0,n} \simeq (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \{(t_1, \ldots, t_{n-3}) \mid t_i = t_j \text{ for some } i \neq j\}, \tag{2.197}$$

which shows that $M_{0,n}$ is a smooth variety of dimension $n-3$. In particular, $M_{0,3}$ is reduced to a point and $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Even better, $M_{0,n}$ can be represented by a smooth scheme over $\mathbb{Z}$.

Deligne, Mumford, and Knudsen [**Knu83**] constructed a smooth compactification $\overline{M}_{0,n}$ of $M_{0,n}$ by a simple normal crossing divisor. The complex points $\overline{M}_{0,n}(\mathbb{C})$ are identified with *stable marked curves of genus* 0.

DEFINITION 2.198. A *stable n-marked curve of genus* 0 *over* $\mathbb{C}$ is a pair consisting of a curve X over $\mathbb{C}$ with at worst nodal singularities and an ordered tuple $(x_1, \dots, x_n)$ of distinct smooth points, such that every irreducible component of X is a $\mathbb{P}^1$, the space $X(\mathbb{C})$ is connected and simply connected, and every irreducible component has at least three special points. By special point we mean a point of intersection with another component or a marked point.

The points of $M_{0,n}(\mathbb{C})$ correspond to the irreducible curves. The space $\overline{M}_{0,n}(\mathbb{C})$ can be stratified into locally closed subsets corresponding to the different combinatorial types of the stable marked curves. In particular, the irreducible components of the boundary are in one-to-one correspondence with the partitions of the marked points into subsets of cardinality at least 2. We refer the reader to [**KV07**] for a nice introduction to these spaces and their compactifications.

EXAMPLE 2.199. The space $M_{0,4}$ can be identified with $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The compactification $\overline{M}_{0,4}$ can be identified with $\mathbb{P}^1$. Thus, there are three boundary components of dimension 0 that correspond to the three possible partitions of the set $\{1, 2, 3, 4\}$ into two subsets of size 2 (see Figure 7).

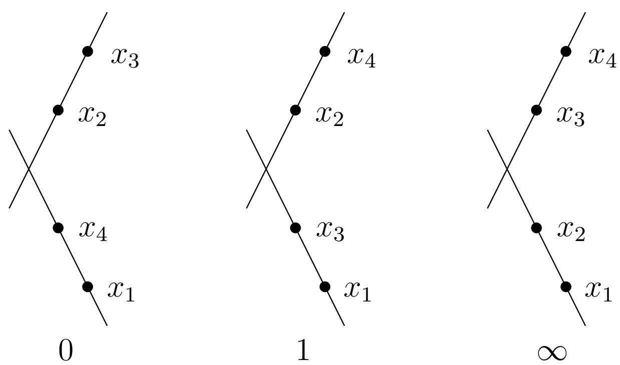


FIGURE 7. Boundary of the moduli space $M_{0,4}$

EXAMPLE 2.200. The irreducible components of the boundary of $\overline{M}_{0,5}$ are in one-to-one correspondence with the 10 ways of dividing the set $\{1, \dots, 5\}$ into two subsets, one of size 3 and the other of size 2. For instance, the component corresponding to the partition $\{1, 2, 3\} \cup \{4, 5\}$ has an open dense subset corresponding to curves with two components, one with the marked points (x_1, x_2, x_3) and the other with the points (x_4, x_5), intersecting at a non-marked point. This open set is isomorphic to $M_{0,4} \times M_{0,3}$. This component and the component corresponding to the partition $\{1, 2\} \cup \{3, 4, 5\}$ meet transversely at a single point that corresponds to a curve with three components as in Figure 8. Intuitively, we can think that, to move the point x_3 from one component to another, we have to cross the singular point. Since this is forbidden we have to add a new component.

REMARK 2.201. The Deligne–Knudsen–Mumford compactification $\overline{M}_{0,5}$ is isomorphic to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $(0, 0)$, $(1, 1)$, and (∞, ∞). The boundary $\overline{M}_{0,5} \setminus M_{0,5}$ consists of 10 smooth divisors intersecting transversally.

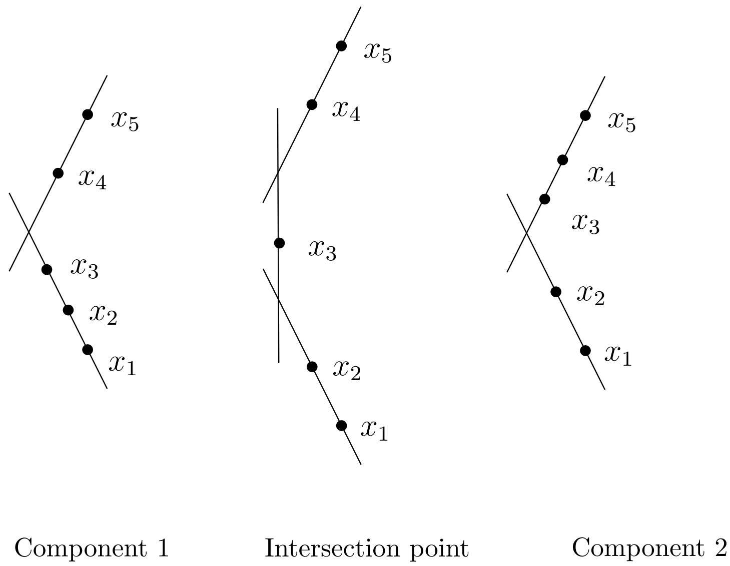


FIGURE 8. Two components of the boundary of $M_{0,5}$

They correspond to the three exceptional divisors $E_{(0,0)}$, $E_{(1,1)}$, and $E_{(\infty,\infty)}$, and the strict transforms of the divisors

$$\{t_1 = 0\}, \quad \{t_1 = 1\}, \quad \{t_1 = \infty\}, \quad \{t_2 = 0\}, \quad \{t_2 = 1\}, \quad \{t_2 = \infty\}, \quad \{t_1 = t_2\}.$$

The correspondence is given in Table 2.1. The boundary components of $\overline{M}_{0,5}$ are

Component	Partition	Divisor
$E_{(0,0)}$	$\{2,3\} \cup \{1,4,5\}$	B
$E_{(1,1)}$	$\{1,3\} \cup \{2,4,5\}$	B
$E_{(\infty,\infty)}$	$\{1,2\} \cup \{3,4,5\}$	A
$\{t_1 = 0\}$	$\{1,4\} \cup \{2,3,5\}$	A
$\{t_1 = 1\}$	$\{2,4\} \cup \{1,3,5\}$	B
$\{t_1 = \infty\}$	$\{3,4\} \cup \{1,2,5\}$	A
$\{t_2 = 0\}$	$\{1,5\} \cup \{2,3,4\}$	B
$\{t_2 = 1\}$	$\{2,5\} \cup \{1,3,4\}$	A
$\{t_2 = \infty\}$	$\{3,5\} \cup \{1,2,4\}$	A
$\{t_1 = t_2\}$	$\{4,5\} \cup \{1,2,3\}$	B

TABLE 2.1. Boundary components of $\overline{M}_{0,5}$

distributed into two simple normal crossings divisors A and B. The discussion in Section 2.5.1 shows that $\zeta(2)$ is a period of the cohomology group

$$\mathrm{H}^2(\overline{M}_{0,5} \setminus A, B \setminus (A \cap B)).$$

This example was generalized by Goncharov and Manin:

THEOREM 2.202 (Goncharov–Manin [GM04]). *Let $\boldsymbol{s}$ be an admissible multi-index $\boldsymbol{s}$ of weight n. There exist two simple normal crossing divisors $A_{\boldsymbol{s}}$ and B, supported on the boundary of $M_{0,n+3}$ and with no common irreducible components, such that the multiple zeta value $\zeta(\boldsymbol{s})$ is a period of*

$$\mathrm{H}^n(\overline{M}_{0,n+3} \setminus A_{\boldsymbol{s}}, B \setminus (A_{\boldsymbol{s}} \cap B)). \tag{2.203}$$

SKETCH OF THE PROOF. We explain the general strategy of the proof without entering in the combinatorial details that are encapsulated in two lemmas.

For any $n \geqslant 3$, we have identified in (2.197) the moduli space $M_{0,n+3}$ with an open subset of $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^n$ with coordinates $t_1, \dots, t_n$. Using this identification, the open simplex

$$(\Delta^n)^\circ = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid 1 > t_1 > t_2 > \cdots > t_n > 0\}$$

is contained in $M_{0,n+3}(\mathbb{R})$. We denote by $\widehat{\sigma}$ the closure of $(\Delta^n)^\circ$ in $\overline{M}_{0,n+3}$. On the marked points

$$x_1 = 0,\ x_2 = 1,\ x_3 = \infty,\ x_4 = t_1,\ \dots,\ x_{n+3} = t_n,$$

we consider the cyclic order $x_1 < x_3 < x_2 < x_4 < \cdots < x_{n+3} < x_1$ (note that we switched x_2 and x_3 from the standard cyclic order). Recall that the boundary components of $\overline{M}_{0,n+3}$ are indexed by the partitions of the set $\{1, \dots, n+3\}$ into two subsets of size at least 2. Let B be the union of boundary components of $\overline{M}_{0,n+3}$ indexed by the partitions such that both subsets consist of consecutive elements for the above cyclic order (see, for instance, the components labeled B in Table 2.1).

LEMMA 2.204. *For every n, the boundary of $\widehat{\sigma}$ is contained in B.*

This lemma implies that $\widehat{\sigma}$ defines a relative homology class

$$[\widehat{\sigma}] \in \mathrm{H}_n(\overline{M}_{0,n+3}, B).$$

Let now $\omega_{\boldsymbol{s}}$ be the differential form given by equation (1.115). By the explicit formula of $\omega_{\boldsymbol{s}}$, it is an algebraic differential form on $M_{0,n+3}$ of top degree n. We can see $\omega_{\boldsymbol{s}}$ as a singular form of $\overline{M}_{0,n+3}$. Let $A_{\boldsymbol{s}}$ denote the divisor where $\omega_{\boldsymbol{s}}$ is singular. That is, $\overline{M_{0,n+3}} \setminus A_{\boldsymbol{s}}$ is the maximal open set where $\omega_{\boldsymbol{s}}$ can be extended as a regular algebraic differential form. By dimension reasons, the differential form $\omega_{\boldsymbol{s}}$ vanishes when restricted to B. Therefore, $\omega_{\boldsymbol{s}}$ defines a class

$$[\omega_s] \in \mathrm{H}^n(\overline{M}_{0,n+3} \setminus A_{\boldsymbol{s}}, B \setminus (A_{\boldsymbol{s}} \cap B)).$$

The divisor $A_{\boldsymbol{s}}$ depends on $\boldsymbol{s}$. The main point in the proof of the theorem is:

LEMMA 2.205. *The divisors $A_{\boldsymbol{s}}$ and B do not have any common component. Moreover, $\widehat{\sigma}$ and $A_{\boldsymbol{s}}$ are disjoint.*

In view of this lemma, $[\widehat{\sigma}]$ can be lifted to a class

$$[\widehat{\sigma}] \in \mathrm{H}_n(\overline{M}_{0,n+3} \setminus A_{\boldsymbol{s}}, B \setminus (A_{\boldsymbol{s}} \cap B)).$$

The number $\langle [\omega_{\boldsymbol{s}}], [\widehat{\sigma}] \rangle$ is thus a period of $\mathrm{H}^n(\overline{M}_{0,n+3} \setminus A_{\boldsymbol{s}}, B \setminus (A_{\boldsymbol{s}} \cap B))$. This period is given by

$$\langle [\omega_{\boldsymbol{s}}], [\widehat{\sigma}] \rangle = \int_{\widehat{\sigma}} \omega_{\boldsymbol{s}} = \int_{\Delta^n} \omega_{\boldsymbol{s}} = \zeta(\boldsymbol{s}),$$

concluding the proof of the theorem. $\square$

REMARK 2.206. A converse to this theorem, due to Brown [**Bro09**], affirms that, for any choice of boundary divisors A and B, all periods of the cohomology groups $\mathrm{H}^n(\overline{M}_{0,n+3} \setminus A, B \setminus (A \cap B))$ are $\mathbb{Q}[2\pi i]$-linear combinations of multiple zeta values. This can now be seen as a consequence of Brown's theorem characterizing the periods of mixed Tate motives over $\mathbb{Z}$ (Corolllary 5.123).

⋆⋆⋆

EXERCISE 2.207. Show that the boundary of the Deligne–Knudsen–Mumford compactification of $M_{0,n}$ has $2^{n-1} - n - 1$ irreducible components.

EXERCISE 2.208. Draw a stable curve of genus zero with six marked points that has four components, one of them without any marked points. What is the dimension of the stratum of $\overline{M}_{0,6}$ containing the point representing this curve? Write down the components of the boundary divisor that contain that point.

EXERCISE 2.209. How many irreducible components does the boundary divisor B from Lemma 2.204 contain?

2.6. Mixed Hodge structures. Thanks to the comparison isomorphism, the Betti cohomology of an algebraic variety has richer properties than the singular cohomology of a random topological space. As we will explain in this section, it is endowed with a *mixed Hodge structure*, which can be thought of as a first approximation to the notion of motive. This theory was developed by Deligne in the 70s, taking as source of inspiration on the one hand Hodge's theorem for compact Kähler manifolds and, on the other hand, the properties of ℓ-adic cohomology of varieties over finite fields. For a more systematic treatment, we refer the reader to Deligne's original papers [**Del71, Del74**], or the monographs [**Voi02**] and [**PS08**]. The paper [**Dur83**] is a user-friendly introduction to the subject. Usually, the study of a period begins by understanding the mixed Hodge structure on the cohomology of the pair of varieties from which it arises.

2.6.1. *Pure Hodge structures.* Let M be a compact Kähler manifold of dimension d, for instance a smooth projective complex variety. For integers $p, q \geqslant 0$, let

$$\mathrm{H}^{p,q}(M) \subseteq \mathrm{H}^{p+q}(M, \mathbb{C})$$

be the subspace of cohomology classes that can be represented by a $\mathcal{C}^\infty$-closed differential $(p+q)$-form of type (p, q), *i.e.* that can be locally written as

$$\sum_{I,J} f_{I,J}(z_1, \ldots, z_d) \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q},$$

where the sum runs over subsets $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$ of $\{1, \ldots, d\}$, and $f_{I,J}$ are $\mathcal{C}^\infty$-functions in the local coordinates $z_1, \ldots, z_d$.

The starting point of Hodge theory is the following theorem:

THEOREM 2.210 (Hodge). *There is a direct sum decomposition*

$$\mathrm{H}^n(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} \mathrm{H}^{p,q}(M). \tag{2.211}$$

Complex conjugation acts on the right-hand side of (2.211) through the action on the coefficients of the left-hand side, that is,

$$\overline{\sigma \otimes w} = \sigma \otimes \overline{w} \quad \text{for all } \sigma \in \mathrm{H}^n(M, \mathbb{Q}) \text{ and } w \in \mathbb{C}.$$

This action transforms the subspace $\mathrm{H}^{p,q}(M)$ into $\mathrm{H}^{q,p}(M)$, a property commonly referred to as *Hodge symmetry*. Abstractly, the data appearing in Hodge's theorem is captured in the following definition:

DEFINITION 2.212. Let n be an integer. A *pure Hodge structure of weight n* is the data of a finite-dimensional $\mathbb{Q}$-vector space H and a bigrading

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\substack{(p,q)\in\mathbb{Z}^2\\ p+q=n}} H^{p,q}$$

of its complexification satisfying $\overline{H^{p,q}} = H^{q,p}$ for all $p, q \in \mathbb{Z}$. The set of pairs (p, q) for which $H^{p,q}$ is non-zero is called the *Hodge type* of H.

LEMMA 2.213. *The data defining a pure Hodge structure of weight n are equivalent to the datum of an exhaustive decreasing filtration F on $H_{\mathbb{C}}$ (the* Hodge filtration*) such that, for all integers p, the following equality holds:*

$$H_{\mathbb{C}} = F^p \oplus \overline{F^{n+1-p}}. \tag{2.214}$$

PROOF. Given a pure Hodge structure of weight n, one sets

$$F^p = \bigoplus_{r\geqslant p} H^{r,s},$$

which is visibly an exhaustive and decreasing filtration of $H_{\mathbb{C}}$. Since

$$\overline{F^{n+1-p}} = \bigoplus_{r\geqslant n+1-p} \overline{H^{r,s}} = \bigoplus_{s\leqslant p-1} H^{s,r}$$

by Hodge symmetry, condition (2.214) holds. Conversely, given a filtration F as in the statement, define

$$H^{p,n-p} = F^p \cap \overline{F^{n-p}}.$$

With this definition, $\overline{H^{p,q}} = H^{q,p}$ holds. It remains to prove that $H = \bigoplus_p H^{p,n-p}$. We proceed by induction on $r = \max\{p \mid F^p \neq 0\}$. If $r < n/2$, then

$$F^{\lceil \frac{n}{2}\rceil} = F^{n+1-\lceil \frac{n}{2}\rceil} = 0.$$

Therefore, $H_{\mathbb{C}} = 0$ and there is nothing to prove. Let us now assume $r \geqslant n/2$. By Exercise 2.238, there is a direct sum decomposition

$$H_{\mathbb{C}} = \overline{F^r} \oplus (F^{n+1-r} \cap \overline{F^{n+1-r}}) \oplus F^r.$$

Condition (2.214) and the definition of r readily imply the equalities

$$F^r = H^{r,n-r}, \quad \overline{F^r} = H^{n-r,r}.$$

Write $H'_{\mathbb{C}} = F^{n+1-r} \cap \overline{F^{n+1-r}}$. This complex vector space has a real structure. Using the induction hypothesis, one derives

$$H'_{\mathbb{C}} = \bigoplus_{p=r+1}^{n-r-1} H^{p,n-p},$$

thus finishing the proof of the statement. □

Definition 2.212 and Lemma 2.213 contain the definition that one usually finds in textbooks about Hodge theory. However, for the purpose of studying periods it is important to remember that, for a Hodge structure of the form $\mathrm{H}^n(M,\mathbb{Q})$, the filtration F comes from de Rham cohomology of M. If M is given by the complex points of an algebraic variety X over a subfield k of the complex numbers, then

$$\mathrm{H}^n(M,\mathbb{C}) \simeq \mathrm{H}^n_{\mathrm{dR}}(X) \otimes_k \mathbb{C}$$

and the Hodge filtration is in fact already defined on the k-vector space $\mathrm{H}^n_{\mathrm{dR}}(X)$. The following definition keeps track of all these elements:

Definition 2.215. Let k be a subfield of $\mathbb{C}$. A *pure Hodge structure over k* is the data

$$H = (H_{\mathrm{B}}, (H_{\mathrm{dR}}, F), \mathrm{comp}_{\mathrm{B,dR}})$$

of a finite-dimensional $\mathbb{Q}$-vector space H_{B}, a finite-dimensional k-vector space H_{dR}, together with an exhaustive decreasing filtration F, and an isomorphism of complex vector spaces

$$\mathrm{comp}_{\mathrm{B,dR}}\colon H_{\mathrm{dR}} \otimes_k \mathbb{C} \longrightarrow H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C},$$

such that the induced filtration on $H_{\mathbb{C}} = H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$, still denoted by F, satisfies that there exists an integer n such that, for all p, the equality

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{n-p+1} H_{\mathbb{C}}} \tag{2.216}$$

holds. We call n the *weight* of H. Abusing language, we will often say that H_{B} carries a pure Hodge structure.

Definition 2.217. A *morphism of pure Hodge structures over k*

$$f\colon H \longrightarrow H'$$

is a pair $f = (f_{\mathrm{B}}, f_{\mathrm{dR}})$ consisting of a $\mathbb{Q}$-linear map $f_{\mathrm{B}}\colon H_{\mathrm{B}} \to H'_{\mathrm{B}}$ and a k-linear map $f_{\mathrm{dR}}\colon H_{\mathrm{dR}} \to H'_{\mathrm{dR}}$ such that $f_{\mathrm{dR}}(F^p H_{\mathrm{dR}}) \subseteq F^p H'_{\mathrm{dR}}$ holds for all $p \in \mathbb{Z}$, and that the following diagram commutes:

$$\begin{tikzcd}
H_{\mathrm{dR}} \otimes_k \mathbb{C} \arrow[r, "\mathrm{comp}_{\mathrm{B,dR}}"] \arrow[d, "f_{\mathrm{dR}} \otimes_k \mathrm{Id}_{\mathbb{C}}"'] & \mathrm{H}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \arrow[d, "f_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathrm{Id}_{\mathbb{C}}"] \\
H'_{\mathrm{dR}} \otimes_k \mathbb{C} \arrow[r, "\mathrm{comp}'_{\mathrm{B,dR}}"] & H'_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}.
\end{tikzcd}$$

It follows from this definition that a morphism of Hodge structures of different weight is always zero (Exercise 2.241).

We let $\mathbf{HS}(k)$ denote the category of pure Hodge structures over k. If L is another subfield of $\mathbb{C}$ containing k, there is an "extension of scalars" functor

$$- \otimes_k L\colon \mathbf{HS}(k) \longrightarrow \mathbf{HS}(L) \tag{2.218}$$

such that $(H \otimes_k L)_{\mathrm{B}} = H_{\mathrm{B}}$ and $(H \otimes_k L)_{\mathrm{dR}} = H_{\mathrm{dR}} \otimes_k L$ together with the induced filtration and the induced comparison isomorphism via the canonical identification

$$(H_{\mathrm{dR}} \otimes_k L) \otimes_L \mathbb{C} = H_{\mathrm{dR}} \otimes_k \mathbb{C}.$$

Example 2.219 (Hodge–Tate structures). For each integer $n \in \mathbb{Z}$, set

$$\mathbb{Q}(n) = (\mathbb{Q}, (\mathbb{Q}, F), \mathrm{comp}_{\mathrm{B,dR}}),$$

where F is the filtration

$$\mathbb{Q} = F^{-n}\mathbb{Q} \supseteq F^{-n+1}\mathbb{Q} = \{0\},$$

and the isomorphism $\mathrm{comp}_{\mathrm{B,dR}}\colon \mathbb{C} \to \mathbb{C}$ is given by multiplication by $(2\pi i)^{-n}$. Then $\mathbb{Q}(n)$ is a one-dimensional pure Hodge structure of weight $-2n$ over $\mathbb{Q}$.

Upon application of the functor (2.218), we obtain a Hodge structure over any subfield k of $\mathbb{C}$ that will be still denoted by $\mathbb{Q}(n)$. Note, however, that the special role of $2\pi i$ will be more or less significant depending on the nature of k. For example, if $k = \mathbb{C}$, then the Hodge structure $\mathbb{Q}(n)$ is isomorphic to the one where $\mathrm{comp}_{\mathrm{B,dR}}$ is given by the identity, and indeed to any one-dimensional pure Hodge structure of weight $-2n$ (Exercise 2.239).

The Hodge structures $\mathbb{Q}(n)$ are called *Hodge–Tate structures*. Observe that we already encountered $\mathbb{Q}(-1)$ in our study of periods. By Example 2.182, this Hodge structure is isomorphic to the triple

$$\mathrm{H}^1(\mathbb{G}_m) = (\mathrm{H}^1_{\mathrm{B}}(\mathbb{G}_m), (\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m), F), \mathrm{comp}_{\mathrm{B,dR}}),$$

where F is the trivial filtration concentrated in degree 1, and $\mathrm{comp}_{\mathrm{B,dR}}$ stands for Grothendieck's comparison isomorphism from Theorem 2.168.

Once we have introduced these notions, we can state the following algebraic variant of Hodge's theorem:

THEOREM 2.220. *Let k be a subfield of $\mathbb{C}$ and let X be a smooth proper variety over k. The Betti cohomology $\mathrm{H}^n_{\mathrm{B}}(X)$ carries a functorial pure Hodge structure of weight n over k.*

More precisely, we consider the triple

$$\mathrm{H}^n(X) = (\mathrm{H}^n_{\mathrm{B}}(X), (\mathrm{H}^n_{\mathrm{dR}}(X), F), \mathrm{comp}_{\mathrm{B,dR}}).$$

As in the previous example, $\mathrm{comp}_{\mathrm{B,dR}}$ is the comparison isomorphism of Theorem 2.168. The Hodge filtration F is given by

$$F^p\,\mathrm{H}^n_{\mathrm{dR}}(X) = \mathrm{Im}\left(\mathbb{H}^n(X, \Omega_X^{\geqslant p}) \longrightarrow \mathbb{H}^n(X, \Omega_X^*)\right),$$

where $\Omega_X^{\geqslant p}$ stands for the *bête* truncation of the de Rham complex, namely

$$\Omega_X^{\geqslant p}\colon \quad 0 \longrightarrow \cdots 0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^{p+1} \longrightarrow \cdots .$$

The functoriality of the Hodge structure on $\mathrm{H}^n(X)$ means the following, for any morphism $f\colon X \to Y$ of smooth proper varieties,

$$f^*\colon \mathrm{H}^n(Y) \longrightarrow \mathrm{H}^n(X)$$

is a morphism of Hodge structures.

As we have already mentioned, by Exercise 2.241, there are no non-zero morphisms between pure Hodge structures of different weight. However, such maps naturally occur in geometry. For example, if $Z \hookrightarrow X$ is a smooth closed subvariety of codimension c, then there is a *Gysin morphism*

$$\mathrm{H}^n(Z) \longrightarrow \mathrm{H}^{n+2c}(X).$$

In order to turn the Gysin morphism into a morphism of Hodge structures, we introduce *Tate twists*: given a pure Hodge structure H of weight n and an integer m, we denote by $H(m)$ the pure Hodge structure of weight $n-2m$ with the same underlying vector spaces, filtration shifted by m, and comparison isomorphism multiplied by $(2\pi i)^{-m}$. In fact (see Exercise 2.242), there is a tensor product of Hodge structures and $H(m)$ is nothing but the tensor product of H with the Hodge structure $\mathbb{Q}(m)$ from Example 2.219. With this notation, the Gysin map becomes a morphism of Hodge structures $\mathrm{H}^n(Z)(-c) \to \mathrm{H}^{n+2c}(X)$. See Definition 2.301.

EXAMPLE 2.221. As a Hodge structure over $\mathbb{Q}$, the cohomology of projective space $\mathbb{P}^n$ is isomorphic to

$$\mathrm{H}^j(\mathbb{P}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q}(-j/2), & \text{if } 0 \leqslant j \leqslant 2n \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

2.6.2. *Mixed Hodge structures.* Before discussing mixed Hodge structures, we recall some terminology concerning filtrations and morphisms (see Section A.7 for a more thorough discussion in the context of an arbitrary abelian category).

DEFINITION 2.222. Let k be a field and (V, F) and (V', F) filtered k-vector spaces. A morphism $f\colon V \to V'$ is called *filtered* if $f(F^pV) \subset F^pV'$ holds for all p and *strict* (with respect to F) if, in addition,

$$f(F^pV) = F^pV' \cap \mathrm{Im}(f).$$

Hodge's theorem says that the cohomology in degree n of a smooth proper complex variety carries a pure Hodge structure of weight n. In general, this is no longer true when X fails to be smooth or proper. For instance, $\mathrm{H}^1(\mathbb{G}_m)$ is one-dimensional, so it cannot carry a pure Hodge structure of weight one because of Hodge symmetry. Nevertheless, Deligne proved that the cohomology of any complex variety is an "iterated extension" of pure Hodge structures.

THEOREM 2.223 (Deligne). *Let X be a complex algebraic variety.*

i) *There exists an increasing filtration*

$$W_{-1} = 0 \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{2n} = \mathrm{H}^n(X),$$

and a decreasing filtration

$$F^0 = \mathrm{H}^n(X, \mathbb{C}) \supseteq F^1 \supseteq \cdots \supseteq F^n \supseteq F^{n+1} = 0$$

such that F induces a pure Hodge structure of weight m on each graded piece

$$\mathrm{Gr}^W_m \mathrm{H}^n(X) = W_m/W_{m-1}.$$

ii) *Moreover, if $f\colon X \to Y$ is a morphism of complex varieties, the induced map on cohomology $f^*\colon \mathrm{H}^n(Y) \to \mathrm{H}^n(X)$ is a filtered morphism with respect to both filtrations, i.e.*

$$f^*(W_m \mathrm{H}^n(Y)) \subseteq W_m \mathrm{H}^n(X),$$
$$f^*_{\mathbb{C}}(F^p \mathrm{H}^n(Y)) \subseteq F^p \mathrm{H}^n(X).$$

iii) *If X is smooth, then $\mathrm{Gr}^W_m \mathrm{H}^n(X) = 0$ holds for all $m < n$. If X is proper, then $\mathrm{Gr}^W_m \mathrm{H}^n(X) = 0$ for all $m > n$.*

Some ingredients of the proof of this theorem will be presented in Section 2.8. This motivates the following definition:

DEFINITION 2.224. Let k be a subfield of $\mathbb{C}$. A *mixed Hodge structure over k* is a triple

$$H = ((H_{\mathrm{B}}, W^{\mathrm{B}}), (H_{\mathrm{dR}}, F, W^{\mathrm{dR}}), \mathrm{comp}_{\mathrm{B,dR}})$$

consisting of:

- a finite-dimensional $\mathbb{Q}$-vector space H_{B}, together with an increasing filtration W^{B},

- a finite-dimensional k-vector space H_{dR}, together with an increasing filtration W^{dR} and a decreasing filtration F,
- an isomorphism of complex vector spaces

$$\mathrm{comp}_{\mathrm{B,dR}}\colon H_{\mathrm{dR}} \otimes_k \mathbb{C} \xrightarrow{\sim} H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$$

that is filtered with respect to the weight filtration. That is,

$$\mathrm{comp}_{\mathrm{B,dR}}(W^{\mathrm{dR}} \otimes_k \mathbb{C}) = W^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Moreover, we require that these data verify the following: for each integer m,

$$\mathrm{Gr}_m^W H = (\mathrm{Gr}_m^W H_{\mathrm{B}}, (\mathrm{Gr}_m^W H_{\mathrm{dR}}, F), \mathrm{comp}_{\mathrm{B,dR}}) \tag{2.225}$$

is a pure Hodge structure over k of weight m.

Given a mixed Hodge structure H, we endow $H_{\mathbb{C}} = H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ with the complex conjugation coming from this rational structure and the Hodge filtration induced by the one in H_{dR} through the comparison isomorphism. Then $H_{\mathbb{C}}$ has a complex conjugate filtration $\overline{F}$.

Definition 2.226. A morphism $f\colon H \to H'$ of mixed Hodge structures over k is a pair $f = (f_{\mathrm{B}}, f_{\mathrm{dR}})$ consisting of

- a morphism of $\mathbb{Q}$-vector spaces $f_{\mathrm{B}}\colon H_{\mathrm{B}} \to H'_{\mathrm{B}}$,
- a morphism of k-vector spaces $f_{\mathrm{dR}}\colon H_{\mathrm{dR}} \to H'_{\mathrm{dR}}$

such that f_{B} is filtered with respect to the weight filtration, f_{dR} is filtered with respect to the weight and the Hodge filtrations, and both maps are compatible with the comparison isomorphisms. In symbols,

$$\begin{aligned}
f_{\mathrm{B}}(W^{\mathrm{B}} H_{\mathrm{B}}) &\subseteq W^{\mathrm{B}} H'_{\mathrm{B}},\\
f_{\mathrm{dR}}(F H_{\mathrm{dR}}) &\subseteq F H'_{\mathrm{dR}},\\
f_{\mathrm{dR}}(W^{\mathrm{dR}} H_{\mathrm{dR}}) &\subseteq W^{\mathrm{dR}} H'_{\mathrm{dR}},\\
f_{\mathrm{dR}} \circ \mathrm{comp}'_{\mathrm{B,dR}} &= \mathrm{comp}_{\mathrm{B,dR}} \circ (f_{\mathrm{B}} \otimes \mathrm{Id}_{\mathbb{C}}).
\end{aligned}$$

We shall denote by $\mathbf{MHS}(k)$ the category of mixed Hodge structures over k. When $k = \mathbb{C}$, we shall simply speak of "mixed Hodge structures" and write $\mathbf{MHS}$ instead of $\mathbf{MHS}(\mathbb{C})$.

Definition 2.227. We call *Betti fiber functor* and *de Rham fiber functor* on the category $\mathbf{MHS}(k)$ the forgetful functors

$$\begin{aligned}
\omega_{\mathrm{B}}\colon\ \mathbf{MHS}(k) &\longrightarrow \mathbf{Vec}_{\mathbb{Q}},\\
\omega_{\mathrm{dR}}\colon\ \mathbf{MHS}(k) &\longrightarrow \mathbf{Vec}_{k}
\end{aligned}$$

that map H to H_{B} and H_{dR} respectively.

The reason for the name "fiber functor" will be explained when we discuss tannakian categories in Chapter 4.

Definition 2.228. A mixed Hodge structure H over k is called *split* if there exists an isomorphism of mixed Hodge structures

$$H \xrightarrow{\sim} \bigoplus_{m \in \mathbb{Z}} \mathrm{Gr}_m^W H,$$

and hence H is a direct sum of pure Hodge structures.

As was explained in Remark 2.212, the Hodge filtration of a pure Hodge structure H induces a natural bigrading on $H_{\mathbb{C}}$. A similar, albeit more involved construction applies to mixed Hodge structures as well.

PROPOSITION 2.229 (Deligne's splitting). *Let H be a mixed Hodge structure defined over k. There exists a unique decomposition of $H_{\mathbb{C}}$ into a direct sum*

$$H_{\mathbb{C}} = \bigoplus_{p,q} H^{p,q} \tag{2.230}$$

satisfying the conditions

$$W_n H_{\mathbb{C}} = \bigoplus_{p+q\leqslant n} H^{p,q},$$
$$F^p H_{\mathbb{C}} = \bigoplus_{p'\geqslant p} H^{p',q},$$
$$\overline{H^{p,q}} \simeq H^{q,p} \quad \mathrm{mod} \bigoplus_{r<p,s<q} H^{r,s}.$$

This splitting is functorial: a morphism of mixed Hodge structures $f\colon H_1 \to H_2$ induces maps $f^{p,q}\colon H_1^{p,q} \to H_2^{p,q}$ compatible with the decomposition (2.230).

IDEA OF THE PROOF. The graded pieces are defined as

$$H^{p,q} = F^p \cap W_{p+q} \cap \left(\overline{F^q} \cap W_{p+q} + \sum_{j\geqslant 2} \overline{F^{q-j+1}} \cap W_{p+q-j} \right)$$

The proof that this decomposition satisfies the required conditions and is characterized by them can be found in [**PS08**, Lem.–Def. 3.4]. The functoriality follows from this explicit description. ☐

THEOREM 2.231 (Deligne). *The category* **MHS**(k) *is abelian.*

In [**Del71**] Deligne proves this result for $k = \mathbb{C}$, but the proof carries over to the general case. Deligne's proof of this theorem is sometimes called "a masterpiece of linear algebra". The main difficulty stems from the fact that the category of bifiltered vector spaces is not abelian. The key property that makes everything work in this setting is that any morphism of mixed Hodge structures is *strict* with respect to the weight and the Hodge filtrations. More precisely, we have the following lemma that is a consequence of Proposition 2.229.

LEMMA 2.232. *Let $f\colon H \to H'$ be a morphism of mixed Hodge structures. Then f_{B} is strict with respect to the weight filtration and f_{dR} is strict with respect to the weight and the Hodge filtrations.*

A first consequence of this lemma is the following:

COROLLARY 2.233. *The weight and the Hodge filtration are exact functors from the category of mixed Hodge structures to the category of vector spaces. That is, if*

$$0 \to H \to H' \to H'' \to 0$$

is an exact sequence of mixed Hodge structures, then

$$0 \longrightarrow W_n H \longrightarrow W_n H' \longrightarrow W_n H'' \longrightarrow 0$$

is an exact sequence of vector spaces for each n, and similarly for F^p.

Another important consequence of Lemma 2.232 is the following:

Proposition 2.234. *Let f be a morphism of mixed Hodge structures. The following are equivalent:*

i) *f is an isomorphism;*

ii) *$\omega_{\mathrm{B}}(f)$ is an isomorphism;*

iii) *$\omega_{\mathrm{dR}}(f)$ is an isomorphism.*

Proof. Thanks to the comparison isomorphism, $\omega_{\mathrm{B}}(f)$ is an isomorphism if and only if $\omega_{\mathrm{dR}}(f)$ is an isomorphism. Thus, we only need to prove that i) and iii) are equivalent. Note that this is not a general property of filtered vector spaces. A morphism $g\colon (V,F) \to (V',F)$ that induces an isomorphism on the underlying vector spaces is not necessarily an isomorphism because the filtrations, although being compatible, may not match exactly. That is,

$$g(F^pV) \subset F^pV', \qquad g(F^pV) \neq F^pV'.$$

If $f\colon H \to H'$ is a morphism of mixed Hodge structures, then the de Rham component $f_{\mathrm{dR}}\colon (H,W,F) \to (H',W,F)$ is strict with respect to both filtrations by Lemma 2.232. Hence, the are equalities

$$\begin{aligned} f_{\mathrm{dR}}(F^pH_{\mathrm{dR}}) &= F^pH'_{\mathrm{dR}} \cap \mathrm{Im}(f_{\mathrm{dR}}) = F^pH'_{\mathrm{dR}},\\ f_{\mathrm{dR}}(W_nH_{\mathrm{dR}}) &= W_nH'_{\mathrm{dR}} \cap \mathrm{Im}(f_{\mathrm{dR}}) = W_nH'_{\mathrm{dR}}, \end{aligned}$$

and this implies the result. □

2.6.3. *Mixed Hodge structures of Tate type.*

Definition 2.235. A mixed Hodge structure H over k is said to be of *Tate type* if $\mathrm{Gr}^W_{2m+1} H = 0$ holds and $\mathrm{Gr}^W_{2m} H$ is a sum of copies of the pure Hodge–Tate structure $\mathbb{Q}(-m)$ for all integers m. Mixed Hodge structures of Tate type are also called mixed Hodge–Tate structures.

We shall denote by $\mathbf{MHTS}(k)$ the full subcategory of $\mathbf{MHS}(k)$ consisting of mixed Hodge structures of Tate type over k.

Remark 2.236. One can think of mixed Hodge structures as "iterated extensions" of pure Hodge structures. Indeed, given two successive steps of the weight filtration, there is an exact sequence of vector spaces

$$0 \longrightarrow W_{m-1}H \longrightarrow W_mH \longrightarrow \mathrm{Gr}^W_m H \longrightarrow 0.$$

When m is the highest weight of H (*i.e.* $W_mH = H$), this exhibits H as an extension of the pure Hodge structure $\mathrm{Gr}^W_m H$ by $W_{m-1}H$, which in turn is an extension of the pure Hodge structure $\mathrm{Gr}^W_{m-1} H$ by $W_{m-2}H$, and so on. Then mixed Hodge–Tate structures are those obtained as iterated extensions of the simplest ones, that is, sums of copies of $\mathbb{Q}(n)$.

In the case of mixed Hodge structures H of Tate type, the bifiltered space H_{dR} admits a canonical grading. Hence, the fiber functor f_{dR} factors through the category of graded vector spaces.

Lemma 2.237. *Let H be a mixed Hodge structure of Tate type over k. Then H_{dR} is endowed with the canonical grading*

$$H_{\mathrm{dR}} = \bigoplus_p \big(F^pH_{\mathrm{dR}} \cap W_{2p}H_{\mathrm{dR}}\big),$$

and the forgetful functor $H \mapsto H_{\mathrm{dR}}$ factors through the category of graded vector spaces. Moreover, the weight and the Hodge filtration on H_{dR} can be recovered from this grading as follows:

$$F^p H_{\mathrm{dR}} = \bigoplus_{r \geqslant p} \big(F^r H_{\mathrm{dR}} \cap W_{2r} H_{\mathrm{dR}}\big),$$
$$W_{2m} H_{\mathrm{dR}} = W_{2m+1} H_{\mathrm{dR}} = \bigoplus_{r \leqslant m} \big(F^r H_{\mathrm{dR}} \cap W_{2r} H_{\mathrm{dR}}\big).$$

PROOF. Exercise 2.243. □

⋆ ⋆ ⋆

EXERCISE 2.238. Let H be a finite-dimensional real vector space along with an exhaustive and separated filtration F of $H_{\mathbb{C}} = H \otimes \mathbb{C}$ satisfying (2.214). Prove that, for every $r \in \mathbb{Z}$, there is a direct sum decomposition

$$H_{\mathbb{C}} = \overline{F^r} \oplus (F^{n+1-r} \cap \overline{F^{n+1-r}}) \oplus F^r.$$

EXERCISE 2.239. Let k be a subfield of $\mathbb{C}$. Prove that the set of isomorphism classes of one-dimensional pure Hodge structures over k is in one-to-one correspondence with the set $\mathbb{Z} \times (\mathbb{C}^\times / k^\times)$.

EXERCISE 2.240. Let H be a pure Hodge structure of weight n over k. For each integer p, we define the space of (p,p)-classes of H as

$$H^{(p,p)} = \begin{cases} \mathrm{comp}_{\mathrm{B,dR}}(F^p H_{\mathrm{dR}}) \cap (2\pi i)^p H_{\mathrm{B}}, & \text{if } n = 2p, \\ \{0\}, & \text{if } n \neq 2p. \end{cases}$$

Prove the equality

$$\mathrm{Hom}_{\mathbf{MHS}(k)}(\mathbb{Q}(-p), H) = H^{(p,p)},$$

where $\mathbb{Q}(-p)$ is the pure Hodge structure of weight $2p$ over k from Example 2.219.

EXERCISE 2.241. Let H and H' be pure Hodge structures over k of weights n and m respectively.

i) Use the definitions of Section A.7.2 to show that the vector space
$$\mathrm{Hom}_{\mathbb{Q}}(H_{\mathrm{B}}, H'_{\mathrm{B}})$$
admits a pure Hodge structure $\underline{\mathrm{Hom}}(H, H')$ over k of weight $m - n$.

ii) Show that the group of morphisms of Hodge structures from H to H' agrees with the subspace $\underline{\mathrm{Hom}}(H, H')^{(0,0)}$. [Hint: recall from Section A.7.2 that given filtered vector spaces (A, W) and (B, W) with increasing filtrations, one defines an increasing filtration on $\mathrm{Hom}(A, B)$ as
$$W_n \,\mathrm{Hom}(A, B) = \{f \in \mathrm{Hom}(A, B) \mid f(W_k A) \subset W_{k+n} B\}.$$
A similar construction is valid for decreasing filtrations.]

iii) Conclude that there are no non-zero morphisms between pure Hodge structures of different weights.

EXERCISE 2.242. Let H and H' be mixed Hodge structures over k. Define a natural mixed Hodge structure on the tensor product $H \otimes H'$. Show that for any pure Hodge structure H, there is an isomorphism

$$H(m) = H \otimes \mathbb{Q}(m).$$

EXERCISE 2.243. In this exercise, we prove Lemma 2.237. Let H be a mixed Hodge structure of Tate type.

i) Use negative induction on n to prove the equality

$$W_r H_{\mathrm{dR}} = \sum_{2p \leqslant r} \big(W_{2p} H_{\mathrm{dR}} \cap F^p H_{\mathrm{dR}}\big).$$

ii) Show that, for every $p \in \mathbb{Z}$, the following holds:

$$W_{2p-1} H_{\mathrm{dR}} \cap F^p H_{\mathrm{dR}} = 0.$$

iii) Conclude the proof of Lemma 2.237.

EXERCISE 2.244. There are two possible ways of inducing the filtration F on the graded piece $\operatorname{Gr}_m^W H$. Show that they are equivalent.

EXERCISE 2.245. Given a morphism $f\colon H \to H'$ of mixed Hodge structures, prove that the induced maps $f_m\colon \operatorname{Gr}_m^W H \to \operatorname{Gr}_m H'$ are morphism of pure Hodge structures.

EXERCISE 2.246. Let $H = (H_{\mathrm{B}}, H_{\mathrm{dR}}, \alpha)$ be a triple consisting of

- a finite-dimensional $\mathbb{Q}$-vector space H_{B}, equipped with an increasing filtration WH_{B} indexed by even integers,
- a finite-dimensional $\mathbb{Q}$-vector space H_{dR}, together with a grading indexed by even integers $H_{\mathrm{dR}} = \bigoplus_n (H_{\mathrm{dR}})_{2n}$,
- a comparison isomorphism $\alpha\colon H_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$,

subject to the condition that α maps $(H_{\mathrm{dR}})_{2n} \otimes_{\mathbb{Q}} \mathbb{C}$ to $W_{2n} H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$, and induces an isomorphism

$$\alpha_n\colon (H_{\mathrm{dR}})_{2n} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \big(W_{2n} H_{\mathrm{B}} / W_{2(n-1)} H_{\mathrm{B}}\big) \otimes_{\mathbb{Q}} \mathbb{C}$$

which sends $(H_{\mathrm{dR}})_{2n}$ to $\big(W_{2n} H_{\mathrm{B}} / W_{2(n-1)} H_{\mathrm{B}}\big) \otimes_{\mathbb{Q}} (2\pi i)^n \mathbb{Q}$.

Prove that the category $\mathbf{MHTS}(\mathbb{Q})$ is equivalent to the category whose objects are such triples and whose morphisms are the obvious ones.

2.7. Extensions of mixed Hodge structures. We now turn to the description of the extension groups in the category of mixed Hodge structures. Recall that, when no field of definition is explicitly mentioned, by a mixed Hodge structure we mean a mixed Hodge structure over $\mathbb{C}$.

2.7.1. *Definition of the group of extensions.*

DEFINITION 2.247. Let A and B be mixed Hodge structures.

i) An *extension of A by B* is a short exact sequence

$$0 \longrightarrow B \xrightarrow{\beta} H \xrightarrow{\alpha} A \longrightarrow 0, \tag{2.248}$$

where α and β are morphisms of mixed Hodge structures.

ii) Two extensions are *equivalent* if there exists a morphism of mixed Hodge structures $f\colon H \to H'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\beta"] \arrow[d, equal] & H \arrow[r, "\alpha"] \arrow[d, "f"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, "\beta'"] & H' \arrow[r, "\alpha'"] & A \arrow[r] & 0 \end{tikzcd} \tag{2.249}$$

commutes. This defines indeed an equivalence relation (see Exercise 2.261) whose set of equivalence classes will be denoted by

$$\mathrm{Ext}^1_{\mathbf{MHS}}(A, B).$$

iii) An extension $0 \to B \to H \to A$ is said to be *split* if it is equivalent to the trivial extension $0 \to B \to A \oplus B \to A \to 0$.

REMARK 2.250. An extension (2.248) is split if and only if the map $\alpha\colon H \to A$ admits a section, that is, a morphism of mixed Hodge structures $s\colon A \to H$ satisfying $\alpha \circ s = \mathrm{Id}_A$. Indeed, if the extension is split, then the projection to A of a map $f\colon H \to A \oplus B$ making the diagram (2.249) commutative provides a section. Conversely, out of a section one can form the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & A \oplus B \arrow[r] \arrow[d, "s+\beta"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, "\beta"] & H \arrow[r, "\alpha"] & A \arrow[r] & 0. \end{tikzcd}$$

Equivalently, the extension is split if and only if the map $\beta\colon B \to H$ admits a retraction, *i.e.* a morphism of mixed Hodge structures $r\colon H \to B$ with $r \circ \beta = \mathrm{Id}_B$. Concretely, if $s\colon A \to H$ is a section, one checks that the map

$$\mathrm{Id}_H - s \circ \alpha\colon H \longrightarrow H$$

takes values in B and is a retraction, and similarly a retraction gives rise to a section. Such a section or retraction is often called a *splitting*.

2.7.2. *Computation of the group of extensions.* We present Carlson's computation of the extension groups in the category of mixed Hodge structures [**Car80**]. In writing down his formula, it will be convenient to use the following filtration on the space of linear maps between filtered vector spaces already introduced in Exercise 2.241. Given vector spaces with increasing filtrations (A, W) and (B, W), we endow $\mathrm{Hom}(A, B)$ with the increasing filtration

$$W_n \mathrm{Hom}(A, B) = \{f \in \mathrm{Hom}(A, B) \mid f(W_m A) \subset W_{m+n} B \text{ for all } m\}.$$

Similarly, if (A, F) and (B, F) are vector spaces with decreasing filtrations, the space of linear maps $\mathrm{Hom}(A, B)$ is endowed with the decreasing filtration

$$F^n \mathrm{Hom}(A, B) = \{f \in \mathrm{Hom}(A, B) \mid f(F^m A) = F^{m+n} B \text{ for all } m\}.$$

THEOREM 2.251 (Carlson). *Let A and B be mixed Hodge structures. The extension group of A by B is isomorphic to*

$$\mathrm{Ext}^1_{\mathbf{MHS}}(A, B) = \frac{W_0 \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}})}{W_0 \cap F^0 \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}}) + W_0 \mathrm{Hom}_{\mathbb{Q}}(A_{\mathrm{B}}, B_{\mathrm{B}})}. \tag{2.252}$$

PROOF. Given an extension of mixed Hodge structures

$$0 \longrightarrow B \xrightarrow{\beta} H \xrightarrow{\alpha} A \longrightarrow 0,$$

we first choose a section $\varphi_1\colon A_{\mathbb{C}} \to H_{\mathbb{C}}$ of the underlying complex vector spaces that is compatible with the weight and the Hodge filtration. This is always possible, for example using Deligne's splitting (Proposition 2.229). We then choose a second section $\varphi_2\colon A_{\mathbb{C}} \to H_{\mathbb{C}}$, this time compatible with the rational structures A_{B} and H_{B} as well as the weight filtration. For every $a \in A_{\mathbb{C}}$, the element $\varphi_1(a) - \varphi_2(a)$ is mapped to zero under $\alpha_{\mathbb{C}}$, and hence there exists a unique element $b \in B_{\mathbb{C}}$ such that $\beta_{\mathbb{C}}(b) = \varphi_1(a) - \varphi_2(a)$. We set $f(a) = b$. This assignment yields a linear map $f\colon A_{\mathbb{C}} \to B_{\mathbb{C}}$ that respects the weight filtration, that is to say, an element of $W_0 \operatorname{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}})$. We thus obtain a map

$$\operatorname{Ext}^1_{\mathbf{MHS}}(A,B) \longrightarrow W_0 \operatorname{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}}). \tag{2.253}$$

This map depends on the sections φ_1 and φ_2, and we need to understand how it changes for different choices of sections.

Let φ_2' be another section compatible with the weight filtration and the rational structure, and let f' be the corresponding map. Then $f(a) - f'(a) = \varphi_2'(a) - \varphi_2(a)$, so $f - f'$ respects the rational structure and the weight filtration. In other words,

$$f - f' \in W_0 \operatorname{Hom}_{\mathbb{Q}}(A_{\mathrm{B}}, B_{\mathrm{B}}).$$

Similarly, if φ_1'' is another choice of a section compatible with the weight and the Hodge filtrations and f'' is the corresponding linear map, then

$$f - f'' \in W_0 \cap F^0 \operatorname{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}}).$$

In consequence, the class of f in the quotient of the right-hand side of (2.252) is independent of the choice of the sections φ_1 and φ_2. Let now H' be an equivalent extension. This means that there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d, "\psi"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & H' \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

If φ_1 and φ_2 are splittings of the extension H, then $\psi \circ \varphi_1$ and $\psi \circ \varphi_2$ are splittings of H' that yield the same function f. Therefore, we have constructed a map from the left-hand side of (2.252) to the right-hand side.

If the class of f in the quotient is zero, then we can modify φ_1 by an element of $W_0 \cap F^0 \operatorname{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}})$ to get a new splitting φ_1', and we can modify φ_2 by an element of $W_0 \operatorname{Hom}_{\mathbb{Q}}(A_{\mathrm{B}}, B_{\mathrm{B}})$ to get φ_2' so that $\varphi_1' = \varphi_2'$. This implies that $\varphi_1' = \varphi_2'$ defines a splitting of mixed Hodge structures, and the extension was trivial. Therefore, the map (2.253) is injective.

To see that it is surjective, we start with a function $f \in W_0 \operatorname{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}})$. We then write

$$(H_{\mathrm{B}}, W) = (A_{\mathrm{B}}, W) \oplus (B_{\mathrm{B}}, W), \quad (H_{\mathrm{dR}}, W) = (A_{\mathrm{dR}}, W) \oplus (B_{\mathrm{dR}}, W),$$

define the comparison isomorphism of H as the direct sum of the comparison isomorphisms of $A$and B, and we define the Hodge filtration on H_{dR} by

$$F^p H_{\mathrm{dR}} = F^p B_{\mathrm{dR}} \oplus \{a + f(a) \mid a \in F^p A_{\mathrm{dR}}\}.$$

The fact that the resulting tuple

$$H = ((H_{\mathrm{B}}, W), (H_{\mathrm{dR}}, W, F), \mathrm{comp}_{\mathrm{B,dR}}) \tag{2.254}$$

is a mixed Hodge structure is the content of Exercise 2.262. By construction, the function corresponding to H is the original function f. This shows that the map (2.253) is an isomorphism and concludes the proof of the theorem. □

One consequence of Carlson's formula is that the category of mixed Hodge structures has cohomological dimension one, *i.e.* all higher extension groups vanish.

THEOREM 2.255. *For any mixed Hodge structures A and B and any integer $n \geqslant 2$, the following holds:*

$$\mathrm{Ext}^n_{\mathbf{MHS}}(A, B) = 0.$$

PROOF. According to Lemma A.66 from the appendix, the vanishing of the higher extension groups would follow if we knew that $\mathrm{Ext}^1_{\mathbf{MHS}}(A, -)$ is a right exact functor for every mixed Hodge structure A. Since the functors $\mathrm{Ext}^*_{\mathbf{MHS}}(A, -)$ form a cohomological functor, it is enough to show that, if $B_1 \to B_2$ is an epimorphism of mixed Hodge structures, then

$$\mathrm{Ext}^1_{\mathbf{MHS}}(A, B_1) \longrightarrow \mathrm{Ext}^1_{\mathbf{MHS}}(A, B_2)$$

is surjective. But this is a direct consequence of Carlson's formula. □

2.7.3. *Extensions of Hodge–Tate structures.* In the case of Hodge–Tate structures, the extensions defined over $\mathbb{Q}$ can also be described.

THEOREM 2.256. *Let m and n be integers. Then*

$$\mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(m), \mathbb{Q}(n)) = \begin{cases} \mathbb{C}/(2\pi i)^{n-m}\mathbb{Q}, & \textit{if } m < n, \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. Assume first that $m > n$. Then $W_{-2m}H \subseteq H$ is a rank one sub-Hodge structure over $\mathbb{Q}$, and the composition

$$W_{-2m}H \hookrightarrow H \xrightarrow{\alpha} \mathbb{Q}(m)$$

is an isomorphism. Thus, the extension is necessarily split.

For $m = n$, the weight and the Hodge filtration of H are trivial (the corresponding subobjects are either zero or everything), and hence any section s_{B} of the map $\alpha_{\mathrm{B}} \colon H_{\mathrm{B}} \to \mathbb{Q}(n)_{\mathrm{B}}$ induces a morphism of Hodge structures $s \colon \mathbb{Q}(n) \to H$, so the extension is again split.

Now assume that $m < n$. The $\mathbb{Q}$-vector space H_{dR} has a canonical splitting

$$H_{\mathrm{dR}} = W_n H_{\mathrm{dR}} \oplus F^{-m} H_{\mathrm{dR}}.$$

Choose a basis e_0, e_1 of H_{B} satisfying $e_0 = \beta(1_n)$ and $\alpha(e_1) = 1_m$, where 1_n is the generator of $\mathbb{Q}(n)_{\mathrm{B}}$ and 1_m is the generator of $\mathbb{Q}(m)_{\mathrm{B}}$. This basis uniquely determines a basis f_0, f_1 of H_{dR} by the conditions

$$\begin{aligned} &f_0 \in W_{-2n}H_{\mathrm{dR}}, \quad \mathrm{comp}_{\mathrm{B,dR}}(f_0) = (2\pi i)^{-n} e_0, \\ &f_1 \in F^{-m}H_{\mathrm{dR}}, \quad \mathrm{comp}_{\mathrm{B,dR}}(f_1) \in (2\pi i)^{-m} e_1 + W_{-2n}H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}. \end{aligned}$$

In these bases, the morphism $\mathrm{comp}_{\mathrm{B,dR}}$ can be written as

$$\begin{pmatrix} (2\pi i)^{-n} & (2\pi i)^{-n} a \\ 0 & (2\pi i)^{-m} \end{pmatrix}$$

for a complex number a that determines the class of the extension.

We can change the basis (e_0, e_1) by an upper triangular matrix with diagonal entries 1 and a rational coefficient b in the upper right corner. The basis (f_0, f_1)

remains unchanged. In these new bases, the comparison isomorphism is given by

$$\begin{aligned}\begin{pmatrix}(2\pi i)^{-n} & (2\pi i)^{-n}a' \\ 0 & (2\pi i)^{-m}\end{pmatrix} &= \begin{pmatrix}1 & b \\ 0 & 1\end{pmatrix}\begin{pmatrix}(2\pi i)^{-n} & (2\pi i)^{-n}a \\ 0 & (2\pi i)^{-m}\end{pmatrix} \\ &= \begin{pmatrix}(2\pi i)^{-n} & (2\pi i)^{-n}(a+(2\pi i)^{n-m}b) \\ 0 & (2\pi i)^{-m}\end{pmatrix}.\end{aligned}$$

Hence, two complex numbers $a, a' \in \mathbb{C}$ determine the same extension if and only if $a - a' \in (2\pi i)^{n-m}\mathbb{Q}$, from which the result follows. □

2.7.4. *Examples.* According to Theorem 2.256, the extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ are parametrized by elements in $\mathbb{C}/(2\pi i)^n\mathbb{Q}$. It follows that, for each $n \geqslant 2$, there exists an extension of mixed Hodge structures

$$0 \longrightarrow \mathbb{Q}(n) \longrightarrow \zeta^{MHS}(n) \longrightarrow \mathbb{Q}(0) \longrightarrow 0, \tag{2.257}$$

whose class corresponds to the zeta value $\zeta(n)$. This extension is split if and only if $\zeta(n) \in (2\pi i)^n\mathbb{Q}$. By Theorem 1.3 and the fact that elements of $(2\pi i)^n\mathbb{Q}$ are purely imaginary for odd n, the extension (2.257) is split if and only if n is even. It is an open question to construct geometrically these extensions, *e.g.* as a relative cohomology group. In the case $n = 1$, however, we know that all the extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$ have geometric origin.

Example 2.258 (Kummer mixed Hodge structure). For each complex number $t \in \mathbb{C}^\times \setminus \{1\}$, consider the relative cohomology

$$\mathrm{H}_t = \mathrm{H}^1(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, t\}).$$

The long exact sequence (2.111) gives

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^1 \setminus \{0,\infty\}) \longrightarrow \mathrm{H}^0(\{1,t\}) \longrightarrow \mathrm{H}_t \longrightarrow \mathrm{H}^1(\mathbb{P}^1 \setminus \{0,\infty\}) \longrightarrow 0.$$

By Example 2.307, there is an isomorphism $\mathrm{H}^1(\mathbb{P}^1 \setminus \{0,\infty\}) = \mathbb{Q}(-1)$, and hence we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow \mathrm{H}_t \longrightarrow \mathbb{Q}(-1) \longrightarrow 0.$$

The *Kummer mixed Hodge structure* K_t^{H} is defined to be the dual of H_t, so that

$$K_t^{\mathrm{H}} \in \mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{C})}(\mathbb{Q}(0), \mathbb{Q}(1))$$

sits in an exact sequence

$$0 \longrightarrow \mathbb{Q}(1) \longrightarrow K_t^{\mathrm{H}} \longrightarrow \mathbb{Q}(0) \longrightarrow 0. \tag{2.259}$$

For $t = 1$, the Kummer extension is defined as the trivial extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. This yields a map

$$\mathbb{C}^\times \longrightarrow \mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{C})}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathbb{C}/(2\pi i)\mathbb{Q}$$

that we want to make explicit following the recipe from the proof of Theorem 2.256.

Let $t \in \mathbb{P}^1 \setminus \{0,\infty\}$ be distinct from 1. Write $X = \mathbb{P}^1 \setminus \{0,\infty\}$ and $Z = \{1, t\}$. Since K_t^{H} is defined as the dual of H_t it is best described using homology. That is,

$$K_{t,\mathrm{B}}^{\mathrm{H}} = \mathrm{H}_1(X, Z, \mathbb{Q}).$$

Therefore, the Betti part of the exact sequence (2.259) reads

$$0 \longrightarrow \mathrm{H}_1(X) \longrightarrow K_{t,\mathrm{B}}^{\mathrm{H}} \longrightarrow \widetilde{\mathrm{H}}_0(Z) \longrightarrow 0,$$

where $\widetilde{\mathrm{H}}_0(Z) = \mathrm{Ker}(\mathrm{H}_0(Z) \to \mathrm{H}_0(X))$. Let e_0 be the element of $K^{\mathrm{H}}_{t,\mathrm{B}}$ represented by the path $\gamma_0\colon [0,1] \to X$ given by $s \mapsto e^{2\pi i s}$. This element is the image of the generator of $\mathrm{H}_1(X)$. Hence $e_0 \in W_{-2}K^{\mathrm{H}}_{t,\mathrm{B}}$. Let e_1 be the element of $K^{\mathrm{H}}_{t,\mathrm{B}}$ represented by any path $\gamma_1\colon [0,1] \to X$ such that $\gamma_1(0) = 1$ and $\gamma_1(1) = t$. Note the ambiguity in the choice of e_1, to which we can add any rational multiple of e_0.

We next describe $K^{\mathrm{H}}_{t,\mathrm{dR}}$. It is simpler to describe its dual H_t. Since X and Z are both affine, we can use the method of Section 2.2.6 to represent relative de Rham cohomology. By Example 2.108, H_t is generated by the differential forms

$$\omega_1 = \frac{\mathrm{d}z}{t-1} \quad \text{and} \quad \omega_2 = \frac{\mathrm{d}z}{z}.$$

The class $[\omega_0]$ belongs to $W_0\,\mathrm{H}_{t,\mathrm{dR}}$ because it lies in the image of $\mathrm{H}^0_{\mathrm{dR}}(Z)$. The class represented by ω_2 belongs to $F^1\,\mathrm{H}_{t,\mathrm{dR}}$ because $\omega_2 \in F^1\Omega^1_{\mathbb{P}^1}(\log\{0,\infty\})$ (see equation (2.277) for the definition of the Hodge filtration in the logarithmic complex). Note that ω_1 does not belong to $\Omega^1_{\mathbb{P}^1}(\log\{0,\infty\})$ because it has a double pole at ∞.

Going back to $K^{\mathrm{H}}_{t,\mathrm{dR}}$, let $f_0 \in K^{\mathrm{H}}_{t,\mathrm{dR}}$ be the element determined by $f_0(\omega_1) = 0$ and $f_0(\omega_2) = 1$. Since $W_1\,\mathrm{H}_{t,\mathrm{dR}} = W_0\,\mathrm{H}_{t,\mathrm{dR}}$ is generated by ω_1, we deduce that f_0 lies in $W_{-2}K^{\mathrm{H}}_{t,\mathrm{dR}}$ from Example A.194. Moreover, the computations

$$\int_{e_0} \omega_1 = 0 \quad \text{and} \quad \int_{e_0} \omega_2 = 2\pi i$$

show that f_0 satisfies $\mathrm{comp}_{\mathrm{B,dR}}(f_0) = (2\pi i)^{-1}e_0$ because

Let $f_1 \in K^{\mathrm{H}}_{t,\mathrm{dR}}$ be the element determined by $f_1(\omega_1) = 1$ and $f_1(\omega_2) = 0$. Since ω_2 generates $F^1\,\mathrm{H}_{t,\mathrm{dR}}$, we deduce that $f_1 \in (F^1\,\mathrm{H}_{t,\mathrm{dR}})^{\perp} = F^0K^{\mathrm{H}}_{t,\mathrm{dR}}$ and satisfies that $\mathrm{comp}_{\mathrm{B,dR}}(f_1) = e_1 + W_{-2}$ because

$$\int_{e_1} \omega_1 = 1.$$

Finally, the equation

$$\int_{e_1} \omega_2 = \log t$$

implies that

$$\mathrm{comp}_{\mathrm{B,dR}}(f_1) = e_1 - (2\pi i)^{-1}(\log t)e_0.$$

Therefore, the class of $K^{\mathrm{H}}_{t,\mathrm{dR}}$ in $\mathbb{C}/(2\pi i)\mathbb{Q}$ is represented by the number $\log t$.

Example 2.260. As another example of how arithmetic information can be encoded in extensions of mixed Hodge structures, let us consider extensions of the first cohomology of a smooth projective curve C by $\mathbb{Q}(-1)$. In this situation, Carlson's theorem implies that extensions classes of $\mathbb{Q}(-1)$ by $\mathrm{H}^1(-1)$ are given by

$$\mathrm{Ext}^1_{\mathbf{MHS}}(\mathbb{Q}(-1), \mathrm{H}^1(C)) = \mathrm{Jac}(C)(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\mathrm{Jac}(C)(\mathbb{C}) = \mathrm{H}^0(C, \Omega^1_C)/\,\mathrm{H}_1(C(\mathbb{C}), \mathbb{Z})$ is the complex torus given by the complex points of the Jacobian of C. By Example 2.307, the cohomology of $C\backslash\{p,q\}$ for any pair of points gives such an extension. Through the above isomorphism, the class of the extension is given by the class of the divisor $[p] - [q]$ in $\mathrm{Jac}(C)(\mathbb{C})$. In particular, the extension splits if and only if this divisor is torsion.

⋆⋆⋆

EXERCISE 2.261. Show that being equivalent in the sense of Definition 2.247 defines an equivalence relation on the set of extensions of mixed Hodge structures.

EXERCISE 2.262. Let H be the structure given in (2.254). Use that A and B are mixed Hodge structures to prove the equality

$$\operatorname{Gr}_n^W H_{\mathbb{C}} = F^p \operatorname{Gr}_n^W H_{\mathbb{C}} \oplus \overline{F^{n-p+1}} \operatorname{Gr}_n^W H_{\mathbb{C}}$$

for all $n, p \in \mathbb{Z}$. Conclude that H is a mixed Hodge structure.

2.8. Construction of mixed Hodge structures. We now explain some ideas behind the construction of mixed Hodge structures on the cohomology of algebraic varieties. We start by presenting our basic tool, which is the notion of *mixed Hodge complex* introduced by Deligne in [**Del74**].

2.8.1. *Mixed Hodge complexes.* Among the different variants of the notion of mixed Hodge complex, we will deal here with the one that is the most relevant for the study of periods since it keeps track of the field of definition of the de Rham component and its filtrations. Throughout, k denotes a subfield of $\mathbb{C}$. Recall the notion of filtered quasi-isomorphism from Definition A.195.

DEFINITION 2.263. A *mixed Hodge complex over* k is a 5-tuple

$$A = ((A^*_{\mathrm{B}}, W), (A^*_{\mathrm{dR}}, W, F), (A^*_{\mathbb{C}}, W), \alpha, \beta),$$

consisting of the following data:

(a) (A^*_{B}, W) is a bounded below complex of $\mathbb{Q}$-vector spaces along with an increasing filtration W;
(b) $(A^*_{\mathrm{dR}}, W, F)$ is a bounded below complex of k-vector spaces along with an increasing filtration W and a decreasing filtration F;
(c) $(A^*_{\mathbb{C}}, W)$ is a bounded below complex of $\mathbb{C}$-vector spaces along with an increasing filtration W;
(d) $\alpha\colon (A^*_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}, W) \to (A^*_{\mathbb{C}}, W)$ is a filtered quasi-isomorphism;
(e) $\beta\colon (A^*_{\mathrm{dR}} \otimes_k \mathbb{C}, W) \to (A^*_{\mathbb{C}}, W)$ is a filtered quasi-isomorphism;

subject to the following two conditions:

i) for every integer $n \in \mathbb{Z}$, the differential induced on the complex $\operatorname{Gr}_n^W A^*_{\mathrm{dR}}$ is strict with respect to the filtration F;
ii) for all integers $n, m \in \mathbb{Z}$, the triple

$$(\mathrm{H}^m(\operatorname{Gr}_n^W A^*_{\mathrm{B}}), (\mathrm{H}^m(\operatorname{Gr}_n^W A^*_{\mathrm{dR}}), F), \mathrm{H}^m(\alpha)^{-1} \circ \mathrm{H}^m(\beta)),$$

is a pure Hodge structure over k of weight $n+m$ (Definition 2.215).

A *morphism of mixed Hodge complexes* $f\colon A \to A'$ is a triple

$$f = (f_{\mathrm{B}}, f_{\mathrm{dR}}, f_{\mathbb{C}})$$

consisting of the following data:

(a) a morphism of filtered complexes $f_{\mathrm{B}}\colon (A^*_{\mathrm{B}}, W) \to ((A')^*_{\mathrm{B}}, W')$;
(b) a morphism of bifiltered complexes $f_{\mathrm{dR}}\colon (A^*_{\mathrm{dR}}, W, F) \to ((A')^*_{\mathrm{dR}}, W', F')$;
(c) a morphism of bifiltered complexes $f_{\mathbb{C}}\colon (A^*_{\mathbb{C}}, W, F) \to ((A')^*_{\mathbb{C}}, W', F')$;

subject to the condition that f_{B} and $f_{\mathbb{C}}$ commute with the quasi-isomorphisms α and α', and f_{dR} and $f_{\mathbb{C}}$ commute with the quasi-isomorphisms β and β'.

EXAMPLE 2.264. Let H^* be a bounded below complex of mixed Hodge structures over k. Unraveling the definition, this is the data of

(a) a complex H^*_{B} of $\mathbb{Q}$-vector spaces along with an increasing filtration W, in which the differential is compatible with W and strict with respect to W;

(b) a complex H^*_{dR} of k-vector spaces along with an increasing filtration W and a decreasing filtration F, in which the differential is compatible and strict with respect to both filtrations;

(c) a comparison isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} \colon H^*_{\mathrm{dR}} \otimes_k \mathbb{C} \longrightarrow H^*_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

We explain how to associate a mixed Hodge complex with H^*. The first component is the filtered complex (A^*_{B}, W) defined by writing

$$A^m_{\mathrm{B}} = H^m_{\mathrm{B}}, \qquad W_n A^m_{\mathrm{B}} = W_{n+m} H^m_{\mathrm{B}}.$$

It is clear that the differential respects this new filtration W, but in general it is no longer strict with respect to W. Recall from Definition 2.222 that being strict means that every element of $W_n H^{m+1}_{\mathrm{B}} \cap \mathrm{Im}(d)$ lies in $d(W_n H^m_{\mathrm{B}})$. However,

$$W_n A^{m+1}_{\mathrm{B}} \cap \mathrm{Im}(d) = W_{n+m+1} H^{m+1}_{\mathrm{B}} \cap \mathrm{Im}(d) = d(W_{n+m+1} H^m_{\mathrm{B}}) = d(W_{n+1} A^m_{\mathrm{B}}),$$

and in general $W_{n+1} A^m_{\mathrm{B}}$ is bigger than $W_n A^m_{\mathrm{B}}$. The second component is the bifiltered complex $(A^*_{\mathrm{dR}}, W, F)$ given by

$$A^m_{\mathrm{dR}} = H^m_{\mathrm{dR}}, \qquad W_n A^m_{\mathrm{dR}} = W_{n+m} H^m_{\mathrm{dR}}, \qquad F^p A^m_{\mathrm{dR}} = F^p H^m_{\mathrm{dR}}.$$

Finally, we consider $(A^*_{\mathbb{C}}, W) = (A^*_{\mathrm{B}} \otimes \mathbb{C}, W)$, along with the identity isomorphism α, and $\beta = \mathrm{comp}_{\mathrm{B,dR}}$. Then

$$A = ((A^*_{\mathrm{B}}, W), (A^*_{\mathrm{dR}}, W, F), (A^*_{\mathbb{C}}, W), \alpha, \beta)$$

is a mixed Hodge complex over k. The only point to check in the definition is condition ii), as condition i) follows from the strictness of d with respect to F. By definition of the new weight filtration, $\mathrm{Gr}^W_n A^m$ is a pure Hodge structure over k of weight $n+m$, and hence the induced differential $d \colon \mathrm{Gr}^W_n A^m \to \mathrm{Gr}^W_n A^{m+1}$ is zero since the source and the target are pure of different weights. Therefore, condition ii) is satisfied. We note that this construction yields a functor from $\mathcal{C}^+(\mathbf{MHS}(k))$ to the category of mixed Hodge complexes over k.

The basic properties of mixed Hodge complexes are summarized in the following result of Deligne [**Del74**, Sch. 8.1.9]:

PROPOSITION 2.265. *Let A be a mixed Hodge complex over k.*

i) *For every n, the triple*

$$((\mathrm{H}^n(A^*_{\mathrm{B}}), W[n]), (\mathrm{H}^n(A^*_{\mathrm{dR}}), W[n], F), \mathrm{comp}_{\mathrm{B,dR}} = \mathrm{H}^n(\alpha)^{-1} \circ \mathrm{H}^n(\beta))$$

is a mixed Hodge structure over k.

ii) *A morphism of mixed Hodge complexes induces a morphism of mixed Hodge structures in cohomology.*

iii) *The spectral sequences associated with the filtered complexes (A^*_{B}, W) and (A^*_{dR}, W) degenerate at the term E_2.*

iv) *The spectral sequence associated with the filtered complex (A^*_{dR}, F) degenerates at the term E_1.*

2.8.2. *The triangulated category of mixed Hodge complexes.* We can mimic the definition of the derived category of an abelian category (Example A.81) to obtain a triangulated category of mixed Hodge complexes. Namely, if A^* is a mixed Hodge complex, then a *homotopy* on A^* is a family of morphisms of degree -1

$$\begin{aligned} s_{\mathrm{B}}\colon (A^*_{\mathrm{B}}, W) &\longrightarrow (A^*_{\mathrm{B}}, W), \\ s_{\mathrm{dR}}\colon (A^*_{\mathrm{dR}}, W, F) &\longrightarrow (A^*_{\mathrm{dR}}, W, F), \\ s_{\mathbb{C}}\colon (A^*_{\mathbb{C}}, W) &\longrightarrow (A^*_{\mathbb{C}}, W), \end{aligned}$$

that commute with the quasi-isomorphisms α and β. Two morphisms of mixed Hodge complexes $f, g\colon A^* \longrightarrow B^*$ are said to be *homotopically equivalent* if there exists a homotopy s on B^* satisfying

$$f - g = d \circ s + s \circ d.$$

The triangulated category of mixed Hodge complexes over k is defined by first identifying morphisms that are homotopically equivalent, and then inverting the quasi-isomorphisms. Thanks to a theorem by Beilinson [**Beĭ86**, Thm. 3.4], this triangulated category is the derived category of mixed Hodge structures.

Theorem 2.266. *The functor defined in Example 2.264 induces an equivalence of categories from the bounded derived category of mixed Hodge structures to the bounded derived category of mixed Hodge complexes.*

2.8.3. *dg-mixed Hodge complexes.* In practice, it will be useful to combine several mixed Hodge complexes into a single one through the notion of dg (for *differential graded*) mixed Hodge complexes and their associated total complex.

Definition 2.267. A *dg-mixed Hodge complex over k* is a 5-tuple

$$A = ((A^{*,*}_{\mathrm{B}}, W), (A^{*,*}_{\mathrm{dR}}, W, F), (A^{*,*}_{\mathbb{C}}, W), \alpha, \beta)$$

consisting of the following data:

i) $(A^{*,*}_{\mathrm{B}}, W)$ is a bounded below double complex of $\mathbb{Q}$-vector spaces along with an increasing filtration W;

ii) $(A^{*,*}_{\mathrm{dR}}, W, F)$ is a bounded below double complex of k-vector spaces along with an increasing filtration W and a decreasing filtration F;

iii) $(A^{*,*}_{\mathbb{C}}, W)$ is a bounded below double complex of $\mathbb{C}$-vector spaces along with an increasing filtration W;

iv) $\alpha\colon (A^{*,*}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}, W) \to (A^{*,*}_{\mathbb{C}}, W)$ is a filtered morphism of double complexes;

v) $\beta\colon (A^{*,*}_{\mathrm{dR}} \otimes_{k} \mathbb{C}, W) \to (A^{*,*}_{\mathbb{C}}, W)$ is a filtered morphism of double complexes;

subject to the condition that, for every integer $p \in \mathbb{Z}$, the 5-tuple

$$A = ((A^{p,*}_{\mathrm{B}}, W), (A^{p,*}_{\mathrm{dR}}, W), (A^{p,*}_{\mathbb{C}}, W, F), \alpha, \beta),$$

is a mixed Hodge complex over k in the sense of Definition 2.224.

Let A be a dg-mixed Hodge complex over k. We can construct the total complexes $\mathrm{Tot}(A_{\mathrm{B}})$, $\mathrm{Tot}(A_{\mathrm{dR}})$, and $\mathrm{Tot}(A_{\mathbb{C}})$ as in Definition A.33 from the appendix. On each of them, we will denote by L the filtration defined by the second degree, and let $\delta(W, L)$ be the diagonal filtration defined as

$$\delta(W, L)_n A^{p,*}_{?} = W_{n+p} A^{p,*}_{?}. \tag{2.268}$$

DEFINITION 2.269. Let

$$A = ((A_{\mathrm{B}}^{*,*}, W), (A_{\mathrm{dR}}^{*,*}, W), (A_{\mathbb{C}}^{*,*}, W, F), \alpha, \beta)$$

be a dg-mixed Hodge complex defined over k. Then $\mathrm{Tot}(A)$ is the 5-tuple

$$((\mathrm{Tot}(A_{\mathrm{B}})^*, \delta(W,L)), (\mathrm{Tot}(A_{\mathrm{dR}})^*, \delta(W,L), F), (\mathrm{Tot}(A_{\mathbb{C}})^*, \delta(W,L), F), \alpha, \beta).$$

PROPOSITION 2.270. *Let A be a dg-mixed Hodge complex defined over k. Then* $\mathrm{Tot}(A)$ *is a mixed Hodge complex defined over k.*

REMARK 2.271. The need to introduce the diagonal filtration instead of the induced filtration stems from the fact that the weight filtration in cohomology is not the induced weight filtration, but the shifted filtration

$$W_n H^m = \mathrm{Im}(W_{n-m}).$$

Let us see this with an example. Let A be a dg-mixed Hodge complex and let $x \in W_r A^{p,q}$ be a cycle. In the cohomology group $\mathrm{H}^q(A^{p,*})$, the class of x is an element of weight $r+q$, not r. We want all maps to preserve weights, so in $\mathrm{H}^{p+q}(\mathrm{Tot}(A))$, the element x should also have weight $r+q$. This implies that in the complex $\mathrm{Tot}(A)$, the element x should be in the piece $r+q-p-q = r-p$ of the filtration. This is exactly the role of the diagonal filtration:

$$x \in W_r A^{p,q} = \delta(W,L)_{r-p} A^{p,q}.$$

Every time we construct a simple complex from a double complex, it comes equipped with a spectral sequence that relates the cohomology of the total complex with the individual cohomologies of the columns or the rows of the double complex. The extra information in the case of dg-mixed Hodge complexes is that this spectral sequence is a spectral sequence of mixed Hodge structures.

PROPOSITION 2.272. *Let A be a dg-mixed Hodge complex over k. Let G denote the decreasing filtration induced by the first degree of* $\mathrm{Tot}(A)$*, that is,*

$$G^p \,\mathrm{Tot}(A)^n = \bigoplus_{p' \geqslant p} A^{p', n-p'}.$$

The spectral sequence associated with G converges to

$${}_G E_1^{p,q} = \mathrm{H}^q(A^{p,*}) \Longrightarrow \mathrm{H}^{p+q}(\mathrm{Tot}(A)).$$

Moreover, the following holds:

i) *all the terms ${}_G E_r^{p,q}$ carry a mixed Hodge structure and all the maps* d_r *are morphisms of mixed Hodge structures;*

ii) *the mixed Hodge structure on the graded piece $\mathrm{Gr}_G^p H^n(\mathrm{Tot}(A))$ agrees with the mixed Hodge structure on ${}_G E_\infty^{p,n-p}$.*

This proposition often allows one to prove that a spectral sequence degenerates. Indeed, since the differentials d_r are morphisms of mixed Hodge structures, they are strict with respect to the weight filtration. In particular, whenever two terms have disjoint weights, any map between them is zero.

2.8.4. *Smooth proper varieties.* Let k be a subfield of $\mathbb{C}$ and let X be a smooth proper variety over k. As a warming up, we construct a mixed Hodge complex defined over k that produces the pure Hodge structure of the cohomology of X discussed in Theorem 2.220.

The difficulties we have to overcome are twofold. First, algebraic de Rham cohomology of X is defined as the hypercohomology of the algebraic de Rham complex. Therefore, in order to compute it we need to replace this complex with a complex made out of acyclic sheaves. The second difficulty is that de Rham cohomology is computed in the algebraic scheme X with its Zariski topology, while Betti cohomology is computed as a sheaf cohomology in the analytic space $X(\mathbb{C})$ with its analytic topology. All the game of Hodge structures is to compare two cohomologies that live in completely different worlds. Luckily, the Godement resolution of Section A.9.3 behaves so well that solves both difficulties for us.

We start with the de Rham complex $\Omega^*_{X/k}$, define on it the weight filtration as the trivial filtration

$$W_{-1}\Omega^*_{X/k} = \{0\}, \quad W_0\Omega^*_{X/k} = \Omega^*_{X/k},$$

and the Hodge filtration as the bête filtration

$$F^p\Omega^*_{X/k} = \Omega^{*\geqslant p}_{X/k}.$$

For each sheaf $\Omega^p_{X/k}$, we construct the Godement resolution $\mathrm{Gd}(\Omega^p_{X/k})$. Thanks to the functorial properties of the Godement resolution, $\mathrm{Gd}^*(\Omega^*_{X/k})$ is a double complex with induced weight and Hodge filtrations. Its total complex is the Godement resolution of the de Rham complex $\mathrm{Gd}(\Omega^*_{X/k})$ (Definition A.271). Then the de Rham part of the sought mixed Hodge complex is the complex of global sections

$$(A_{\mathrm{dR}}, W, F) = \big(\Gamma(X, \mathrm{Gd}(\Omega^*_{X/k})), W, F\big).$$

We now look at the complex manifold $X(\mathbb{C})$ and let $\underline{\mathbb{Q}}$ be the constant sheaf on this manifold. Since $X(\mathbb{C})$ satisfies the hypothesis of Theorem A.304, its singular cohomology with rational coefficients agrees with the sheaf cohomology of $\underline{\mathbb{Q}}$, we define the weight filtration of $\underline{\mathbb{Q}}$ as the trivial filtration

$$W_{-1}\,\underline{\mathbb{Q}} = \{0\}, \qquad W_0\,\underline{\mathbb{Q}} = \underline{\mathbb{Q}}.$$

Then the Godement resolution $\mathrm{Gd}(\underline{\mathbb{Q}})$ has an induced weight filtration, and we define the Betti component of the mixed Hodge complex taking global sections:

$$(A_{\mathrm{B}}, W) = (\Gamma(X(\mathbb{C}), \mathrm{Gd}(\underline{\mathbb{Q}})), W).$$

Now we need to compare the de Rham and the Betti components. That is, we need a complex that receives arrows from both complexes, and these arrows are filtered quasi-isomorphisms. To this end, we use the complex of holomorphic differential forms $\Omega^*_{X(\mathbb{C})}$. Once again, we define the weight filtration as the trivial filtration, we apply the Godement resolution and take global sections:

$$(A_{\mathbb{C}}, W) = (\Gamma(X(\mathbb{C}), \mathrm{Gd}(\Omega^*_{X(\mathbb{C})})), W)$$

Next we need the comparison maps. The map α is easy because the complexes involved are both global sections of sheaves living in the same topological space. Since $\underline{\mathbb{Q}}$ agrees with the sheaf of locally constant functions on $X(\mathbb{C})$ and locally constant functions are holomorphic, we deduce a morphism

$$\underline{\mathbb{Q}} \longrightarrow \mathcal{O}_{X(\mathbb{C})} = \Omega^0_{X(\mathbb{C})} \longrightarrow \Omega^*_{X(\mathbb{C})}.$$

By the functoriality of the Godement resolution, it induces a morphism

$$\alpha\colon A_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow A_{\mathbb{C}}$$

that, thanks to the Poincaré Lemma is a quasi-isomorphism, and hence a filtered quasi-isomorphism with respect to the (trivial) weight filtration.

The map β is more complicated as we have to change not only the sheaves but also the spaces. Recall from Section 2.3.1 that there is a continuous map

$$\psi\colon X(\mathbb{C}) \longrightarrow X.$$

from the manifold $X(\mathbb{C})$ with the analytic topology to the underlying topological space of X with its Zariski topology. Applying Lemma A.279, we obtain a map

$$\psi^{-1}(\mathrm{Gd}(\Omega^p_{X/k})) \longrightarrow \mathrm{Gd}(\psi^{-1}\Omega^p_{X/k})$$

for each p. Since algebraic differential forms are always holomorphic, there is also a morphism of sheaves

$$\psi^{-1}\Omega^p_{X/k} \longrightarrow \Omega^p_{X(\mathbb{C})}.$$

Taking the Godement resolution of this last map, global sections and total complexes we deduce a map

$$\beta\colon A_{\mathrm{dR}} \otimes_k \mathbb{C} \longrightarrow A_{\mathbb{C}} \tag{2.273}$$

that, thanks to the GAGA theorem (Theorem 2.139), is a quasi-isomorphism, and hence a filtered quasi-isomorphism with respect to the (trivial) weight filtration.

For future reference, we wrap the previous complexes in a single symbol.

Definition 2.274. Let X be a smooth proper variety over k. We denote by A^{H}_X the mixed Hodge complex constructed in this section.

Proposition 2.275. *Let X be a smooth proper variety over k. The mixed Hodge complex A^{H}_X induces the Hodge structure of Theorem 2.220 on the cohomology of X. Moreover, the assignment $X \mapsto A^{\mathrm{H}}_X$ is functorial: if $f\colon X \to Y$ is a morphism of smooth proper varieties over k, then there is an induced morphism of mixed Hodge complexes $A^{\mathrm{H}}_f\colon A^{\mathrm{H}}_Y \to A^{\mathrm{H}}_X$, which induces the morphism $f^*\colon \mathrm{H}^*(Y) \to \mathrm{H}^*(X)$ of pure Hodge structures.*

2.8.5. *Smooth varieties.* Let X be a smooth variety over a subfield k of $\mathbb{C}$. By Theorem 2.168, there is a canonical comparison isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}}\colon \mathrm{H}^n_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \simeq \mathrm{H}^n_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{2.276}$$

We would like to endow $\mathrm{H}^n_{\mathrm{B}}(X)$ with a filtration W^{B} and $\mathrm{H}^n_{\mathrm{dR}}(X)$ with two filtrations W^{dR} and F making the triple

$$((\mathrm{H}^n_{\mathrm{B}}(X), W^{\mathrm{B}}), (\mathrm{H}^n_{\mathrm{dR}}(X), F, W^{\mathrm{dR}}), \mathrm{comp}_{\mathrm{B,dR}})$$

into a mixed Hodge structure over k. However, if algebraic de Rham cohomology is computed from its Definition 2.95, that is, by using the complex Ω^*_X of Kähler differentials on X, we face two problems:

- A Hodge filtration defined by means of the *bête* filtration $\Omega^{\geqslant p}_X$ will not give much information. For example, for a smooth affine variety X we saw in Remark 2.96 that $\mathrm{H}^n_{\mathrm{dR}}(X)$ is the cohomology of the global de Rham complex, so in this case the definition would yield the trivial filtration

$$F^n\,\mathrm{H}^n_{\mathrm{dR}}(X) = \mathrm{H}^n_{\mathrm{dR}}(X).$$

- There is no obvious way to get the weight filtration from Ω^*_X.

To solve these difficulties, we shall instead use the complex of logarithmic differentials, as introduced in Section 2.3.5. In view of Proposition 2.162, the strategy is to define the Hodge and the weight filtrations on the complex $\Omega^*_{\overline{X}}(\log D)$. The Hodge filtration is given by the *bête* filtration, that is

$$F^p\Omega^*_{\overline{X}}(\log D) = \Omega^{*\geqslant p}_{\overline{X}}(\log D). \tag{2.277}$$

Note that F is defined over k. The weight filtration is given by the order of poles:

$$W_m\Omega^p_{\overline{X}}(\log D) = \begin{cases} 0, & \text{if } m < 0,\\ \Omega^{p-m}_{\overline{X}} \wedge \Omega^m_{\overline{X}}(\log D), & \text{if } 0 \leqslant m \leqslant p,\\ \Omega^p_{\overline{X}}(\log D), & \text{if } m \geqslant p.\end{cases}$$

Once we have a complex of sheaves with two filtrations, in order to produce the de Rham part of a mixed Hodge complex we follow the same strategy used in the smooth proper case. Namely, we define A'_{dR} as the complex of global sections of the total complex of the Godement resolution of $\Omega^*_{\overline{X}}(\log D)$ with the induced weight and Hodge filtrations.

$$(A_{\mathrm{dR}}, W, F) = (\Gamma(\overline{X}, \mathrm{Gd}(\Omega^*_{\overline{X}}(\log D))), W, F).$$

The weight filtration defined by the order of the poles does not look a priori as a "topological" filtration, so it is not clear how to translate it to the Betti side. The key idea now is to use a different filtration that has a more topological flavour. But this new filtration is only available in the analytic topology. So let X^{an} denote the complex manifold associated with the complex variety $X \times \mathrm{Spec}(\mathbb{C})$ and, similarly, let $\overline{X}^{\mathrm{an}}$ denote the one associated with $\overline{X} \times \mathrm{Spec}(\mathbb{C})$ and $D^{\mathrm{an}} = \overline{X}^{\mathrm{an}} \setminus X^{\mathrm{an}}$. We also denote by $j\colon X^{\mathrm{an}} \to \overline{X}^{\mathrm{an}}$ the open immersion of complex manifolds.

The *canonical filtration* on $\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}})$, as defined in Example A.196 from the appendix, is the filtration

$$\tau_{\leqslant n}\Omega^p_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}_{\mathbb{C}}) = \begin{cases} \Omega^p_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}), & \text{if } p < n,\\ \operatorname{Ker} \mathrm{d}, & \text{if } p = n,\\ \{0\}, & \text{if } p > n.\end{cases}$$

Consider the complex of sheaves $j_*\Omega^*_{X^{\mathrm{an}}}$ on $\overline{X}^{\mathrm{an}}$. Note that, since j is an affine morphism and the sheaves $\Omega^p_{X^{\mathrm{an}}}$ are coherent, all higher direct images vanish. Let τ denote also the canonical filtration of this complex.

Proposition 2.278 ([**Del71**, Prop. 3.1.8]). *The arrows*

$$(\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}), W) \longleftarrow (\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}), \tau) \longrightarrow (j_*\Omega^*_{X^{\mathrm{an}}}, \tau)$$

are filtered quasi-isomorphisms.

Proposition 2.278 also suggests how to define the weight filtration on the Betti part of the mixed Hodge complex. Let $\underline{\mathbb{Q}}$ be the constant sheaf on X^{an}, and let $\mathrm{Gd}(\underline{\mathbb{Q}})$ be its Godement's resolution. Since the sheaves composing this complex are *flasque*, they are acyclic with respect to the functor j_*. Therefore, the complex $j_*\,\mathrm{Gd}(\underline{\mathbb{Q}})$ is isomorphic in the derived category of sheaves to $Rj_*\underline{\mathbb{Q}}$. Let now τ denote again the increasing canonical filtration, but this time of the complex $j_*\,\mathrm{Gd}(\underline{\mathbb{Q}})$. Since each sheaf $\mathrm{Gd}^p(\underline{\mathbb{Q}})$ is flasque, $j_*\,\mathrm{Gd}^p(\underline{\mathbb{Q}})$ is also flasque, so

$$\mathrm{H}^*(\Gamma(\overline{X}^{\mathrm{an}}, j_*\,\mathrm{Gd}(\underline{\mathbb{Q}}))) = \mathrm{H}^*(X^{\mathrm{an}}, \mathbb{Q}).$$

So, we define the Betti part of the mixed Hodge complex as

$$(A_{\mathrm{B}}, W) = \left(\Gamma(\overline{X}^{\mathrm{an}}, j_* \operatorname{Gd}(\underline{\mathbb{Q}})), \tau\right).$$

We now consider the diagram of filtered complexes of vector spaces given in Figure 9. All the arrows in that diagram exist and are filtered quasi-isomorphisms. The existence of the arrow ① follows a similar argument that the existence of the map β in (2.273). The fact that this arrow is a filtered quasi-isomorphism is a consequence of the GAGA theorem. For the arrows ② and ③, this follows from Proposition 2.278. For the arrow ④, it follows from Exercise A.329. The existence of the arrow ⑤ follows from Lemma A.279 and the fact that the sheaves $\Omega^p_{\overline{X}^{\mathrm{an}}}$ are acyclic for the functor j_*. Finally, the arrow ⑥ comes from the exactness of Gd (Lemma A.268) and Exercise A.218.

$$\begin{tikzcd}
\left(\Gamma(\overline{X}, \operatorname{Gd}(\Omega^*_{\overline{X}}(\log D))), \operatorname{Gd}(W)\right) \otimes \mathbb{C} \arrow[dr, "\textcircled{1}"] & \\
& \left(\Gamma(\overline{X}^{\mathrm{an}}, \operatorname{Gd}(\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}))), \operatorname{Gd}(W)\right) \\
\left(\Gamma(\overline{X}^{\mathrm{an}}, \operatorname{Gd}(\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}))), \operatorname{Gd}(\tau)\right) \arrow[ur, "\textcircled{2}"'] \arrow[dr, "\textcircled{3}"] & \\
& \left(\Gamma(\overline{X}^{\mathrm{an}}, \operatorname{Gd}(j_*\Omega^*_{\overline{X}^{\mathrm{an}}})), \operatorname{Gd}(\tau)\right) \\
\left(\Gamma(\overline{X}^{\mathrm{an}}, \operatorname{Gd}(j_*\Omega^*_{\overline{X}^{\mathrm{an}}})), \tau\right) \arrow[ur, "\textcircled{4}"'] \arrow[dr, "\textcircled{5}"] & \\
& \left(\Gamma(\overline{X}^{\mathrm{an}}, \operatorname{Tot}(j_* \operatorname{Gd}^*(\Omega^*_{\overline{X}^{\mathrm{an}}}))), \tau\right) \\
\left(\Gamma(\overline{X}^{\mathrm{an}}, j_* \operatorname{Gd}(\underline{\mathbb{Q}})), \tau\right) \otimes \mathbb{C} \arrow[ur, "\textcircled{6}"'] &
\end{tikzcd}$$

FIGURE 9. Diagram of filtered complexes

We are almost done with the construction of the sought Hodge complex, but we still have to solve the small technical problem that some of the arrows in the previous diagram go in the wrong direction. To invert these arrows, we apply Exercise A.217 three times, one time to the arrows ② and ③, another time to the arrows ④ and ⑤, and finally to the arrows obtained in the previous iteration. This is illustrated in Figure 10. There the complexes Ⓐ to Ⓖ are the filtered complexes appearing in Figure 9. The first two applications of Exercise A.217 produce the filtered complexes cone(②+③) and cone(④+⑤). The third application of the exercise produces the filtered complex cone(④+⑤).

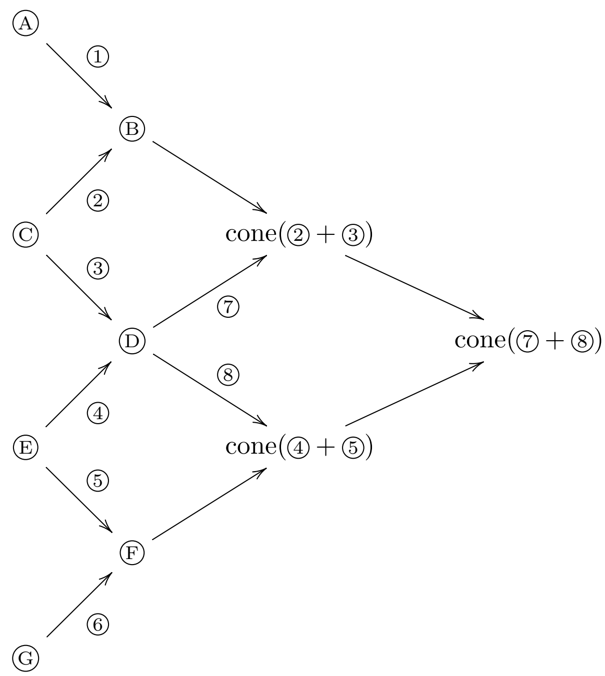


FIGURE 10. Completing the diagram of filtered complexes

In this way, we obtain a big filtered complex $(A^*_{\mathbb{C}}, W) = \operatorname{cone}(⑦+⑧)$ together with filtered quasi-isomorphisms

$$(A^*_{\mathrm{dR}}, W) \otimes \mathbb{C} \xrightarrow{\beta} (A^*_{\mathbb{C}}, W) \xleftarrow{\alpha} (A^*_{\mathrm{B}}, W) \otimes \mathbb{C}.$$

DEFINITION 2.279. Let X be a smooth variety over k and $j\colon X \to \overline{X}$ a smooth compactification with $D = \overline{X} \setminus X$ a simple normal crossing divisor. We set

$$A^{\mathrm{H}}_{\overline{X}}(\log D) = ((A_{\mathrm{dR}}, W, F), (A_{\mathrm{B}}, W), (A_{\mathbb{C}}, W), \alpha, \beta).$$

PROPOSITION 2.280. *Let X be a smooth variety defined over k and let $\overline{X}$ be a smooth compactification with $D = \overline{X} \setminus X$ a simple normal crossing divisor. Then the 5-tupe $A^{\mathrm{H}}_{\overline{X}}(\log D)$ is a mixed Hodge complex. In particular, it induces the Hodge structure of Theorem 2.223 on the cohomology of X. Moreover, the assignment*

$$X \longmapsto A^{\mathrm{H}}_{\overline{X}}(\log D)$$

is functorial with respect to pairs of compactifications: given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y, \arrow[d] \\ \overline{X} \arrow[r, "\overline{f}"] & \overline{Y} \end{tikzcd}$$

in which X and Y are smooth varieties, $\overline{X}$ and $\overline{Y}$ are smooth proper varieties, and $D_X = \overline{X} \setminus X$ and $D_Y = \overline{Y} \setminus Y$ are simple normal crossing divisors, there is a morphism of mixed Hodge complexes

$$\overline{f}^*\colon A^{\mathrm{H}}_{\overline{Y}}(\log D_Y) \longrightarrow A^{\mathrm{H}}_{\overline{X}}(\log D_X)$$

that induces the morphism of mixed Hodge structures $f^*\colon \mathrm{H}^*(Y) \to \mathrm{H}^*(X)$.

Consider the weight and the Hodge filtrations on cohomology given by

$$W^{\mathrm{dR}}_m \mathrm{H}^n_{\mathrm{dR}}(X) = \mathrm{Im}\left(\mathbb{H}^n(X, W_{m-n}\Omega^*_{\overline{X}}(\log D)) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(X)\right), \tag{2.281}$$

$$F^p_{\mathrm{dR}} \mathrm{H}^n_{\mathrm{dR}}(X) = \mathrm{Im}\left(\mathbb{H}^n(X, F^p\Omega^*_{\overline{X}}(\log D)) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(X)\right), \tag{2.282}$$

We refer the reader *e.g.* to [**Del71**] or [**PS08**, § 4] for a proof that the filtrations we have introduced define a mixed Hodge structure on $\mathrm{H}^n_{\mathrm{B}}(X)$.

Definition 2.283. We say that a mixed Hodge structure H has weights in a subset $I \subseteq \mathbb{Z}$ if $\mathrm{Gr}^W_m H = 0$ holds whenever $m \notin I$.

It follows from (2.281) that the cohomology group $\mathrm{H}^n_{\mathrm{B}}(X)$ of a smooth variety X has weights in $[n, 2n]$. Moreover, noting that $W_0\Omega^*_{\overline{X}}(\log D) = \Omega^*_{\overline{X}}$ and the shift of indices in (2.281), one finds that the first step in the weight filtration is the piece of the cohomology coming from the compactification:

$$W_n \mathrm{H}^n_{\mathrm{B}}(X) = \mathrm{Im}\left(\mathrm{H}^n_{\mathrm{B}}(\overline{X}) \longrightarrow \mathrm{H}^n_{\mathrm{B}}(X)\right).$$

In contrast, when X is proper, the mixed Hodge structure $\mathrm{H}^n(X)$ defined in [**Del74**] has weights in $[0, n]$. The combination of these two statements implies that the cohomology of a smooth proper variety carries a pure Hodge structure.

As we have seen, the definition of de Rham cohomology involves hypercohomology of sheaves; therefore, to compute it concretely, in general we cannot use directly the algebraic de Rham complex but we need a resolution of it, like the Godement resolution. As we have seen in Remark 2.96 for an affine variety X, every coherent sheaf is acyclic and we can represent de Rham cohomology with algebraic differentials directly. Nevertheless, the Hodge structure involves a hypercohomology computed on a proper compactification of X; therefore, even in the case of affine varieties, in order to compute the Hodge structure we will need an acyclic resolution of the complex of logarithmic differentials, compatible with the weight and the Hodge filtrations.

Example 2.284. Let us compute everything for $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, viewed as a variety over $\mathbb{Q}$. As for any smooth curve, there is a canonical smooth compactification, in this case $\overline{X} = \mathbb{P}^1$. Write $D = \{0, 1, \infty\}$ for the divisor at infinity. Recall that $\mathcal{O}_{\mathbb{P}^1}(D)$ stands for the sheaf of rational functions having at most simple poles at D and nowhere else. We have:

$$\Omega^0_{\mathbb{P}^1}(\log D) = \mathcal{O}_{\mathbb{P}^1}, \qquad \Omega^1_{\mathbb{P}^1}(\log D) = \mathcal{O}_{\mathbb{P}^1}(D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \Omega^1_{\mathbb{P}^1}.$$

Since $\Omega^1_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, one sees that $\Omega^1_{\mathbb{P}^1}(\log D) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. By the standard computation of the cohomology of line bundles on $\mathbb{P}^1$ [**Har77**, Chap. III, § 5], none of the terms in the complex of logarithmic differentials has higher cohomology. Besides, setting $\omega_0 = \mathrm{d}t/t$ and $\omega_1 = \mathrm{d}t/(1-t)$, one has:

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{Q}, \qquad \mathrm{H}^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D)) = \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1$$

(note that these differentials ω_0 and ω_1 have a simple pole at ∞ as well). From the spectral sequence (A.272), it follows that

$$\begin{aligned}\mathrm{H}^*_{\mathrm{dR}}(X) &= \mathbb{H}^*(\mathcal{O}_{\mathbb{P}^1} \xrightarrow{d} \mathcal{O}_{\mathbb{P}^1}(D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \Omega^1_{\mathbb{P}^1})\\ &= \mathrm{H}^*(\mathbb{Q} \longrightarrow \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1),\end{aligned}$$

where the differential in the second complex is the zero map. Thus,

$$\mathrm{H}^1_{\mathrm{dR}}(X) = \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1.$$

We now turn to the filtrations. For the Hodge filtration, (2.277) gives

$$\mathrm{H}^1_{\mathrm{dR}}(X) = F^0 = F^1 \supseteq F^2 = \{0\}.$$

Moreover, the weight filtration on the complex of logarithmic differentials is given by $\Omega^*_{\mathbb{P}^1} = W_0 \subseteq W_1 = \Omega^*_{\mathbb{P}^1}(\log D)$. Since $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1)$ vanishes, we find:

$$\{0\} = W_1 \subseteq W_2 = \mathrm{H}^1_{\mathrm{dR}}(X).$$

On the other hand, the first homology group $\mathrm{H}_1(X(\mathbb{C}), \mathbb{Q})$ has as a basis the classes of two loops σ_0 and σ_1 winding once counterclockwise around the punctures 0 and 1. By Cauchy's residue theorem, the period matrix reads:

$$\begin{pmatrix} \int_{\sigma_0} \omega_0 & \int_{\sigma_1} \omega_0 \\ \int_{\sigma_0} \omega_1 & \int_{\sigma_1} \omega_1 \end{pmatrix} = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \end{pmatrix}.$$

In other words, letting $\sigma_0^\vee$ and $\sigma_1^\vee$ denote the dual elements in cohomology, the isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$ sends ω_0 to $\sigma_0^\vee \otimes 2\pi i$ and ω_1 to $\sigma_1^\vee \otimes 2\pi i$. Comparing with Example 2.219, one concludes that there is an isomorphism

$$\mathrm{H}^1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \simeq \mathbb{Q}(-1)^{\oplus 2}$$

of mixed Hodge structures over $\mathbb{Q}$.

Observe that all the information in the de Rham part of the mixed Hodge structure over $\mathbb{Q}$ of the variety $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ can be read from the complex

$$A^* = A^0 \oplus A^1, \quad A^0 = \mathbb{Q}, \quad A^1 = \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1, \tag{2.285}$$

together with the trivial differential and the filtrations

$$\begin{aligned} &F^0 = A^* \supset F^1 = A^1 \supset F^2 = \{0\},\\ &W_{-1} = 0 \subset W_0 = A^0 \subset W_1 = A^*. \end{aligned} \tag{2.286}$$

Note that A^* has an algebra structure given by $\omega_i \wedge \omega_j = 0$, for $i, j \in \{0, 1\}$.

For later reference, we summarize the results of this example in a proposition. We say that a morphism $f\colon (A, W, F) \to (A', W', F')$ between two complexes provided with two filtrations is a *bifiltered quasi-isomorphism* if f is compatible with the filtrations and the induced maps

$$\mathrm{Gr}^p_F \mathrm{Gr}^W_n A \longrightarrow \mathrm{Gr}^q_{F'} \mathrm{Gr}^{W'}_n A'$$

are quasi-isomorphisms for all p and n.

Proposition 2.287. *Set $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ and let $A^* = A^*_{\mathbb{P}^1}$ be the filtered algebra from (2.285) and (2.286). The algebraic de Rham cohomology of X equals*

$$\mathrm{H}^*_{\mathrm{dR}}(X) = \mathrm{H}^*(A^*).$$

The Hodge and the weight filtration are induced by the filtrations (2.286)*:*

$$F^p \, \mathrm{H}^*_{\mathrm{dR}}(X) = \mathrm{H}^*(F^p A^*),$$
$$W_k \, \mathrm{H}^n_{\mathrm{dR}}(X) = H^n(W_{k-n} A^*).$$

Moreover, the inclusion of algebras

$$A^* \longrightarrow E^*_{\mathbb{P}^1(\mathbb{C})}(\log D)$$

induces a bifiltered quasi-isomorphism

$$(A^* \otimes_{\mathbb{Q}} \mathbb{C}, W, F) \longrightarrow (E^*_{\mathbb{P}^1(\mathbb{C})}(\log D), W, F).$$

2.8.6. *Simple normal crossing divisors.* Following the same method we used to define the de Rham cohomology of a simple normal crossing divisor in Section 2.2.7, we can build a mixed Hodge complex that endows it with a mixed Hodge structure.

Let X be a smooth variety over k, and let D be a simple normal crossing divisor on X. We keep notation from Section 2.2.7. In particular, D_I stands for the intersection of irreducible components indexed by a subset I, and D^p for the disjoint union of all D_I such that I has cardinal p. By resolution of singularities (Theorem 2.151), we can choose a smooth compactification $\overline{X}$ of X such that, for each subset I, the Zariski closure $\overline{D_I}$ is a smooth compactification of D_I whose complement $E_I = \overline{D_I} \setminus D_I$ is a simple normal crossing divisor. Let E^p be the disjoint union of all E_I such that I has cardinal p. The mixed Hodge complexes

$$A^{\mathrm{H}}_{\overline{D^p}}(\log E^p)$$

for $p \geqslant 1$ form a dg-mixed Hodge complex with the same differentials used in Section 2.2.7. More precisely, the double complex

$$A^{\mathrm{H}}_{\overline{D}}(\log E)^{p,q} = A^{\mathrm{H}}_{\overline{D^{p+1}}}(\log E^{p+1})^q, \qquad p, q \geqslant 0$$

is a dg-mixed Hodge complex. Its total complex

$$A^{\mathrm{H}}_{\overline{D}}(\log E) = \mathrm{Tot}(A^{\mathrm{H}}_{\overline{D}}(\log E)^{*,*}) \tag{2.288}$$

is also a mixed Hodge complex that defines a mixed Hodge structure on the cohomology of D. Since all D^p and E^p are smooth, we deduce from Proposition 2.275 a morphism of mixed Hodge complexes

$$A^{\mathrm{H}}_{\overline{X}}(\log(\overline{X} \setminus X)) \longrightarrow A^{\mathrm{H}}_{\overline{D}}(\log E).$$

In case X is proper, so that there is no need to compactify, we write A^{H}_{D} for the complex (2.288). There is then a morphism of mixed Hodge complexes $A^{\mathrm{H}}_{X} \to A^{\mathrm{H}}_{D}$.

Example 2.289. We consider Example 2.120 again. Now, instead of computing the relative cohomology $\mathrm{H}^*(X, D)$, we will compute the cohomology $\mathrm{H}^*(D)$ with its mixed Hodge structure. The added information is that, by Proposition 2.272, there is a spectral sequence of mixed Hodge structures. Taking into account that the mixed Hodge structure on the H^0 of an irreducible smooth variety is always a copy of $\mathbb{Q}(0)$, we obtain the E_1 term of the spectral sequence

$$\begin{array}{|cc}
0 & 0 \\
 & \\
\mathbb{Q}(0) \oplus \mathbb{Q}(0) \oplus \mathbb{Q}(0) \longrightarrow & \mathbb{Q}(0) \oplus \mathbb{Q}(0) \oplus \mathbb{Q}(0) \\
\hline
\end{array}$$

where the horizontal map is $(a,b,c) \mapsto (b-a, c-a, c-b)$. From this, we easily get

$$\mathrm{H}^0(D) = \mathrm{H}^1(D) = \mathbb{Q}(0).$$

In fact, using hyperresolutions, the technique we have used for a simple normal crossing divisor can be extended to any quasi-proyective variety [**Del74**].

2.8.7. *Mixed Hodge structures on cohomology with compact support and relative cohomology.* The cohomology with compact support of complex algebraic varieties or, more generally, the relative cohomology is also endowed with a mixed Hodge structure. The basic technique is the following. Let $f\colon A \to B$ be a morphism of mixed Hodge complexes. We can see f as a dg-mixed Hodge complex. Then

$$\mathrm{Tot}(f) = \mathrm{cone}(-f)[-1]$$

is a mixed Hodge complex (Proposition 2.270) and the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n(\mathrm{Tot}(f)) \longrightarrow \mathrm{H}^n(A) \longrightarrow \mathrm{H}^n(B) \longrightarrow \mathrm{H}^{n+1}(\mathrm{Tot}(f)) \longrightarrow \cdots$$

is a long exact sequence of mixed Hodge structures (Proposition 2.265 ii)). We will apply this technique to two situations: the cohomology with compact support and the cohomology with support on a subvariety.

Definition 2.290. Let k be a subfield of $\mathbb{C}$ and let X be an algebraic variety over k. The *Betti cohomology with compact support* $\mathrm{H}^*_{\mathrm{B},c}(X)$ is the cohomology with compact support (as introduced in Definition 2.56) of the set of complex points $X(\mathbb{C})$ equipped with the classical topology:

$$\mathrm{H}^*_{\mathrm{B},c}(X) = \mathrm{H}^*_c(X(\mathbb{C}), \mathbb{Q}).$$

By Exercise A.326, the Betti cohomology with compact support of X can be identified with a relative cohomology group on a compactification. Namely, if $\overline{X}$ is a proper variety containing X as a dense open subset and $D = \overline{X} \setminus X$ is the complement, there is a canonical isomorphism

$$\mathrm{H}^*_{\mathrm{B},c}(X) \simeq \mathrm{H}^*_{\mathrm{B}}(\overline{X}, D), \tag{2.291}$$

and hence a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n_{\mathrm{B},c}(X) \longrightarrow \mathrm{H}^n_{\mathrm{B}}(\overline{X}) \longrightarrow \mathrm{H}^n_{\mathrm{B}}(D) \longrightarrow \mathrm{H}^{n+1}_{\mathrm{B},c}(U) \longrightarrow \cdots$$

More generally, for any open subvariety U of X, with closed complement subset Z, there is a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n_{\mathrm{B},c}(U) \longrightarrow \mathrm{H}^n_{\mathrm{B},c}(X) \longrightarrow \mathrm{H}^n_{\mathrm{B},c}(Z) \longrightarrow \mathrm{H}^{n+1}_{\mathrm{B},c}(U) \longrightarrow \cdots \tag{2.292}$$

The Betti cohomology with compact support of an algebraic variety is equipped with a mixed Hodge structure for which (2.292) is a long exact sequence of mixed Hodge structures. We content ourselves with explaining the construction for a smooth variety X. Then X has a smooth compactification $\overline{X}$ such that $D = \overline{X} \setminus X$ is a simple normal crossing divisor. Let $A^{\mathrm{H}}_{\overline{X}}$ and A^{H}_D be the mixed Hodge complexes from Definition 2.274 and Section 2.8.6. Then there is a map $f\colon A^{\mathrm{H}}_{\overline{X}} \to A^{\mathrm{H}}_D$ and the mixed Hodge structure of $\mathrm{H}^n_c(X)$ is the one induced by $\mathrm{Tot}(f)$. Alternatively, we can use a variant of the construction we made in Section 2.8.6 to define A^{H}_D. We keep the notation from that section and consider the double complex

$$A^{\mathrm{H},p,q}_{\overline{X},D} = A^{\mathrm{H},q}_{D^p}, \quad p,q \geqslant 0.$$

This complex is a dg-mixed Hodge complex. Moreover, $\mathrm{Tot}(f) = \mathrm{Tot}(A^{\mathrm{H},*,*}_{\overline{X},D})$.

REMARK 2.293 (A spectral sequence computing the weight filtration on cohomology with compact support). One advantage of this second point of view is that it enables us to use the spectral sequence associated with the dg-mixed Hodge complex $A^{\mathrm{H}}_{X,D}$. This spectral sequence has first page

$$E_1^{p,q} = \mathrm{H}^q(D^p),$$

and the term $E_1^{p,q}$ carries a pure Hodge structure of weight q since the variety D^p is smooth and proper for each p. Since the differentials are morphisms of Hodge structures by Proposition 2.272 i), the terms of the second page

$$E_2^{p,q} = \frac{\mathrm{Ker}(\mathrm{d}_1\colon E_1^{p,q} \to E_1^{p+1,q})}{\mathrm{Im}(\mathrm{d}_1\colon E_1^{p-1,q} \to E_1^{p,q})}$$

are pure of weight q as well, and hence all differentials $\mathrm{d}_2\colon E_2^{p,q} \to E_2^{p+2,q-1}$ vanish (the source and the target are pure of different weights). Since the spectral sequence degenerates at the second page, Proposition 2.272 ii) gives

$$\mathrm{Gr}_m^W H_c^n(X) = E_2^{n-m,m}.$$

More generally, if X is a smooth variety and D a simple normal crossing divisor on X, one can find a compactification $\overline{X}$ of X as in Section 2.8.6. The relative cohomology of the pair (X, D) has a mixed Hodge structure induced by the total complex $\mathrm{Tot}(f)$ of the morphism of mixed Hodge complexes

$$f\colon A^{\mathrm{H}}_{\overline{X}}(\log \overline{X}\setminus X) \to A^{\mathrm{H}}_{\overline{D}}(\log \overline{D}\setminus D).$$

Note, however, that in this case the analogue of Remark 2.293 is no longer true since, although the varieties D^p are still smooth, they will in general not be proper, and hence the Hodge structure of $\mathrm{H}^q(D^p)$ can be mixed of different weights.

EXAMPLE 2.294. Let X and D be as in Example 2.120. Then the spectral sequence considered in that example is a spectral sequence of mixed Hodge structures whose E_1 term reads

$$\begin{array}{|ccccc}
0 & & 0 & & 0 \\
 & & & & \\
0 & & 0 & & 0 \\
 & & & & \\
\mathbb{Q}(0) & \longrightarrow & \mathbb{Q}(0)\oplus\mathbb{Q}(0)\oplus\mathbb{Q}(0) & \longrightarrow & \mathbb{Q}(0)\oplus\mathbb{Q}(0)\oplus\mathbb{Q}(0) \\
\hline
\end{array}$$

from which we derive the equalities

$$\mathrm{H}^i(X,D) = \begin{cases} \mathbb{Q}(0), & \text{if } i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, let X be a smooth variety, $Z \subset X$ a closed subvariety, and $U = X \setminus Z$ its complement. By resolution of singularities (Theorem 2.151), we can find a compactification $\overline{X}$ such that $D = \overline{X}\setminus X$ and $E = \overline{X}\setminus U$ are simple normal crossing divisors. By functoriality, there is a morphism of Hodge complexes

$$f\colon A^{\mathrm{H}}_{\overline{X}}(\log D) \longrightarrow A^{\mathrm{H}}_{\overline{X}}(\log E).$$

DEFINITION 2.295. The *mixed Hodge structure on the cohomology with support on Z* is defined as

$$\mathrm{H}^*_Z(X) = \mathrm{H}^*(X, U) = \mathrm{H}^*(\mathrm{Tot}(f)).$$

By construction, the cohomology with support $\mathrm{H}^*_Z(X)$ sits into a long exact sequence of mixed Hodge structures

$$\cdots \longrightarrow \mathrm{H}^n_Z(X) \longrightarrow \mathrm{H}^n(X) \longrightarrow \mathrm{H}^n(U) \longrightarrow \mathrm{H}^{n+1}_Z(X) \longrightarrow \cdots \tag{2.296}$$

For an example, see Exercise 2.319.

2.8.8. *Poincaré duality and the Gysin morphism.* The cup-product in cohomology is also a morphism of mixed Hodge structures.

PROPOSITION 2.297. *Let X and Y be varieties over k. The external product*

$$\mathrm{H}^n(X) \otimes \mathrm{H}^m(Y) \longrightarrow \mathrm{H}^{n+m}(X \times Y)$$

is a morphism of mixed Hodge structures over k. In particular, the cup-product

$$\mathrm{H}^n(X) \otimes \mathrm{H}^m(X) \longrightarrow \mathrm{H}^{n+m}(X)$$

is a morphism of mixed Hodge structures over k.

This is proved in [**PS08**, Section 5.4]. Since an isomorphism of mixed Hodge structure is a morphism of Hodge structures whose underlying linear map is an isomorphism of vector spaces, we immediately derive:

COROLLARY 2.298 (Künneth formula). *The external product induces an isomorphism of mixed Hodge structures*

$$\bigoplus_{i+j=n} \mathrm{H}^i(X) \otimes \mathrm{H}^j(Y) \xrightarrow{\sim} \mathrm{H}^n(X \times Y).$$

Another useful property is:

LEMMA 2.299. *Let X be a smooth irreducible proper variety of dimension d over k. There is an isomorphism of Hodge structures*

$$\mathrm{H}^{2d}(X) \simeq \mathbb{Q}(-d).$$

PROOF. Since X is irreducible, the topological space $X(\mathbb{C})$ is connected. Hence, its singular cohomology in top degree $\mathrm{H}^{2d}(X(\mathbb{C}), \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ by Poincaré duality (2.33). We first assume that X is projective, so that we can pick an embedding $X \subset \mathbb{P}^N$ into some projective space. Then the map

$$\mathrm{H}^{2d}_{\mathrm{B}}(\mathbb{P}^N) \longrightarrow \mathrm{H}^{2d}_{\mathrm{B}}(X)$$

that sends the class of a general linear subvariety of codimension d in $\mathbb{P}^N$ to its intersection with X is an isomorphism. The map $\mathrm{H}^{2d}(\mathbb{P}^N) \to \mathrm{H}^{2d}(X)$ is hence an isomorphism of mixed Hodge structures by Proposition 2.234. Since $\mathrm{H}^{2d}(\mathbb{P}^N)$ is the Hodge structure $\mathbb{Q}(-d)$, the result follows. If X is proper but non-projective, by Chow's lemma [**Gro61**, Thm. 5.6.1] there exists a birational morphism $X' \to X$ from a smooth projective irreducible variety X'. Such a morphism induces an isomorphism $\mathrm{H}^{2d}(X) \to \mathrm{H}^{2d}(X')$, and we can then argue as above. □

Putting together Proposition 2.297 and Lemma 2.299, we deduce that Poincaré duality is a morphism of mixed Hodge structures after a twist.

PROPOSITION 2.300 (Poincaré duality). *Let X be a smooth variety of dimension d over k. Poincaré duality gives an isomorphism of mixed Hodge structures*

$$\mathrm{H}^n(X) \simeq \mathrm{Hom}(\mathrm{H}_c^{2d-n}(X), \mathbb{Q}(-d)).$$

For some morphisms of algebraic varieties, such as the inclusions of smooth closed subvarieties, Poincaré duality is used to define direct images in cohomology.

DEFINITION 2.301. Let X be a smooth variety of dimension d and let $Z \subset X$ be a smooth closed subvariety of codimension p. For each $n \geqslant 0$, the *Gysin map*

$$\gamma\colon\ \mathrm{H}^n(Z)(-p) \to \mathrm{H}^{n+2p}(X)$$

is defined as the composition

$$\begin{tikzcd}
\mathrm{H}^n(Z)(-p) \arrow[d, "\simeq"] & \\
\mathrm{Hom}(\mathrm{H}_c^{2d-2p-n}(Z), \mathbb{Q}(p-d))(-p) \arrow[r, equal] & \mathrm{Hom}(\mathrm{H}_c^{2d-2p-n}(Z), \mathbb{Q}(-d)) \arrow[d] \\
\mathrm{H}^{n+2p}(X) & \mathrm{Hom}(\mathrm{H}_c^{2d-2p-n}(X), \mathbb{Q}(-d)) \arrow[l, "\simeq"']
\end{tikzcd}$$

of Poincaré duality and the dual of the map $\mathrm{H}_c^{2d-2p-n}(X) \to \mathrm{H}_c^{2d-2p-n}(Z)$ induced by the inclusion $Z \hookrightarrow X$.

The Gysin map has the following very useful property.

PROPOSITION 2.302 (Gysin long exact sequence). *Let X be a smooth variety over k and let $Z \subseteq X$ be a smooth closed subvariety of codimension p with open complement $U = X \setminus Z$. For each $n \geqslant 0$, the Gysin map can be lifted to an isomorphism of mixed Hodge structures $\mathrm{H}^n(Z)(-p) \to \mathrm{H}_Z^{n+2p}(X)$. Therefore, there is a long exact sequence of mixed Hodge structures*

$$\cdots \longrightarrow \mathrm{H}^{j-1}(X) \xrightarrow{\alpha} \mathrm{H}^{j-1}(U) \xrightarrow{\beta} \mathrm{H}^{j-2p}(Z)(-p) \xrightarrow{\gamma} \mathrm{H}^j(X) \longrightarrow \cdots \tag{2.303}$$

where α is the usual restriction map and γ is the Gysin map.

As we will see in Exercise 2.322, the smoothness assumption on Z is necessary. This proposition is proved in

2.8.9. *E-polynomials and the Grothendieck ring of varieties.* The *E-polynomial* of a complex algebraic variety X is the polynomial in two variables defined as

$$E_X(u,v) = \sum_{p,q} \Big(\sum_i (-1)^i h^{p,q}(\mathrm{Gr}_{p+q}^W \mathrm{H}_c^i(X)) \Big) u^p v^q,$$

where $h^{p,q}(\mathrm{Gr}_{p+q}^W \mathrm{H}_c^i(X))$ stands for the dimension of the piece of type (p,q) in the Hodge decomposition of the pure Hodge structure $\mathrm{Gr}_{p+q}^W \mathrm{H}_c^i(X)$ of weight $p+q$. From the fact that (2.292) is a long exact sequence of mixed Hodge structures, we get the equality of E-polynomials

$$E_X = E_Z + E_{X \setminus Z} \tag{2.304}$$

for each variety X and each closed subvariety $Z \subset X$. Besides, since the Künneth isomorphism is compatible with the mixed Hodge structures by Corollary 2.298, the E-polynomial is multiplicative:

$$E_{X\times Y} = E_X \cdot E_Y. \tag{2.305}$$

The above identities suggest to consider the following universal setting. For any field k, the *Grothendieck ring of varieties* $K_0(\mathrm{Var}_k)$ is the quotient of the free abelian group on the set of isomorphism classes of varieties over k by the subgroup generated by elements of the form

$$[X] - [Z] - [X \setminus Z],$$

where $Z \subset X$ is a closed subvariety, together with the ring operation

$$[X] \cdot [Y] = [X \times_k Y].$$

In the case where k is a subfield of $\mathbb{C}$, a compact way to summarize the equalities (2.304) and (2.305) is then to say that the E-polynomial gives rise to a ring morphism, also known as a *motivic measure*,

$$E\colon K_0(\mathrm{Var}_k) \to \mathbb{Z}[u, v]. \tag{2.306}$$

Another typical example of a motivic measure, when k is a finite field, is the function $K_0(\mathrm{Var}_k) \to \mathbb{Z}$ that sends $[X]$ to the cardinal of $X(k)$. The definition of the Grothendieck ring of varieties extends verbatim to schemes over any base (for example $\mathrm{Spec}(\mathbb{Z})$, which is particularly interesting since it provides a common framework for counting points over closed points and computing Hodge numbers over the generic point). We refer the reader to [**CLNS18**, Chap. 2] for a full-fledged treatment of the Grothendieck ring of varieties and motivic measures.

2.8.10. *More examples.* We close this section with a few more examples of mixed Hodge structures on the cohomology of algebraic varieties.

Example 2.307 (Smooth open curves). Let $\overline{C}$ be a smooth projective complex curve, and let $S \subset \overline{C}$ be a non-empty finite subset consisting of s points. In this example, we describe the mixed Hodge structure on the first cohomology group of the open curve $C = \overline{C} \setminus S$. Since S is non-empty, the curve C is affine, and hence $\mathrm{H}^2(C)$ vanishes by Exercise 2.191. Thanks to this vanishing, the Gysin long exact sequence (2.303) reads

$$0 \longrightarrow \mathrm{H}^1(\overline{C}) \longrightarrow \mathrm{H}^1(C) \longrightarrow \mathrm{H}^0(S)(-1) \xrightarrow{\gamma} \mathrm{H}^2(\overline{C}) \longrightarrow 0.$$

By Lemma 2.299, the last two non-zero terms are isomorphic to the Hodge structures $\mathbb{Q}(-1)^{\oplus s}$ and $\mathbb{Q}(-1)$. Through this identification, the Gysin map γ is given by the sum of the coordinates. From this, we get a short exact sequence

$$0 \longrightarrow \mathrm{H}^1(\overline{C}) \longrightarrow \mathrm{H}^1(C) \longrightarrow \mathbb{Q}(-1)^{\oplus(s-1)} \longrightarrow 0.$$

Since taking a step of the weight filtration is an exact functor (Corollary 2.233) and the Hodge structures $\mathrm{H}^1(\overline{C})$ and $\mathbb{Q}(-1)$ are pure of weights 1 and 2 respectively, the weight filtration on $\mathrm{H}^1(C)$ is given by

$$0 = W_0 \mathrm{H}^1(C) \subset W_1 \mathrm{H}^1(C) = \mathrm{H}^1(\overline{C}) \subset W_2 \mathrm{H}^1(C) = \mathrm{H}^1(C).$$

The graded pieces are therefore

$$\mathrm{Gr}_1^W \mathrm{H}^1(C) \simeq \mathrm{H}^1(\overline{C}), \qquad \mathrm{Gr}_2^W \mathrm{H}^1(C) \simeq \mathbb{Q}(-1)^{\oplus(s-1)},$$

which are indeed pure Hodge structures of weights 1 and 2 respectively. In particular, $\mathrm{H}^1(C)$ is of Tate type if and only if $\mathrm{H}^1(\overline{C})$ vanishes, which is equivalent to asking that the curve is a punctured projective line $C = \mathbb{P}^1 \setminus S$.

EXAMPLE 2.308 (Moduli spaces $M_{0,n}$). Recall the moduli spaces $M_{0,n}$ from Section 2.5.2. In this example, we compute the Hodge structure on their cohomology using a method we learnt from Consani and Faber [**CF06**, Lem. 3].

PROPOSITION 2.309. *For each $i \geqslant 0$ and $n \geqslant 3$, the cohomology group $\mathrm{H}^i(M_{0,n})$ carries a pure Hodge–Tate structure of weight $2i$. More precisely, there is an isomorphism of pure Hodge structures*

$$\mathrm{H}^i(M_{0,n}) \simeq \mathbb{Q}(-i)^{\oplus b_{i,n}},$$

where the Betti numbers $b_{i,n}$ are given by the generating series

$$\sum_{i \geqslant 0} b_{i,n} t^i = (1+2t)(1+3t)\cdots(1+(n-2)t).$$

PROOF. We proceed by induction on n. For $n = 3$, the moduli space is reduced to a point, and hence all cohomology groups but $\mathrm{H}^0(M_{0,3}) = \mathbb{Q}(0)$ vanish. The case $n = 4$ was already settled in Example 2.307, where we saw that the only non-trivial cohomology groups of $M_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$ are $\mathrm{H}^0(M_{0,4}) = \mathbb{Q}(0)$ and $\mathrm{H}^1(M_{0,4}) = \mathbb{Q}(-1)^{\oplus 2}$. Assume $n \geqslant 5$, and let $(0, 1, \infty, t_1, \dots, t_{n-3})$ denote the coordinates on $M_{0,n}$. The map

$$\begin{array}{ccc} M_{0,n} & \longrightarrow & M_{0,4} \times M_{0,n-1} \\ (0, 1, \infty, t_1, \dots, t_{n-3}) & \longmapsto & ((0, 1, \infty, t_1), (0, 1, \infty, t_2, \dots, t_{n-3})) \end{array}$$

induces an isomorphism from $M_{0,n}$ to the complement of the smooth closed subvariety $Z \subset M_{0,4} \times M_{0,n-1}$ defined as

$$Z = \bigsqcup_{i=2}^{n-3} \{t_1 = t_i\} \simeq \bigsqcup_{i=2}^{n-3} M_{0,n-1}.$$

We compute the cohomology of $M_{0,n} \simeq (M_{0,4} \times M_{0,n-1}) \setminus Z$ by combining the Gysin exact sequence, the Künneth formula, and the induction hypothesis. First, the Gysin sequence (2.303) gives

$$\begin{aligned} \cdots \longrightarrow \mathrm{H}^{i-2}(Z)(-1) &\xrightarrow{\alpha} \mathrm{H}^i(M_{0,4} \times M_{0,n-1}) \longrightarrow \mathrm{H}^i(M_{0,n}) \\ \longrightarrow \mathrm{H}^{i-1}(Z)(-1) &\xrightarrow{\beta} \mathrm{H}^{i+1}(M_{0,4} \times M_{0,n-1}) \longrightarrow \cdots \end{aligned} \tag{2.310}$$

By the Künneth formula (Corollary 2.298) and the induction hypothesis, there are isomorphisms of Hodge structures

$$\begin{aligned} \mathrm{H}^i(M_{0,4} \times M_{0,n-1}) &\simeq \bigoplus_{a+b=i} \mathrm{H}^a(M_{0,4}) \otimes \mathrm{H}^b(M_{0,n-1}) \\ &\simeq \mathrm{H}^i(M_{0,n-1}) \oplus \mathrm{H}^{i-1}(M_{0,n-1})(-1)^{\oplus 2} \\ &\simeq \mathbb{Q}(-i)^{\oplus(b_{i,n-1} + 2b_{i-1,n-1})}. \end{aligned} \tag{2.311}$$

It follows that the maps α and β in (2.310) are morphisms between pure Hodge structures of different weights, and hence are identically zero. From this, we derive the short exact sequence of Hodge structures

$$0 \longrightarrow \mathrm{H}^i(M_{0,4} \times M_{0,n-1}) \longrightarrow \mathrm{H}^i(M_{0,n}) \longrightarrow \mathrm{H}^{i-1}(M_{0,n-1})(-1)^{\oplus(n-4)} \longrightarrow 0.$$

Thanks to the isomorphism (2.311) and the induction hypothesis, the cohomology $\mathrm{H}^i(M_{0,n})$ is an extension of two pure Hodge–Tate structures of the same weight. Since all such extensions are split by Theorem 2.256, we deduce

$$\mathrm{H}^i(M_{0,n}) \simeq \mathbb{Q}(-i)^{b_{i,n}}, \quad b_{i,n} = b_{i,n-1} + (n-2)b_{i-1,n-1}.$$

One immediately checks that this recurrence relation amounts to the expression for the Betti numbers given in the statement. □

2.8.11. *Graph hypersurfaces.* Let $G = (V, E)$ be a finite graph with vertex set V and edge set E. Assume that G is connected. A subgraph $T \subseteq G$ is called a *spanning tree* if T is a tree (*i.e.* a connected graph with no loops) and contains all vertices of G. Consider a collection of formal variables $(x_e)_{e\in E}$ indexed by the edges of G. The first *Symanzik polynomial* of the graph G is defined as

$$\psi_G = \sum_{T\subseteq G} \prod_{e\notin T} x_e \in \mathbb{Z}\big[(x_e)_{e\in E}\big], \tag{2.312}$$

where the sum runs over all spanning trees in G. Let n_G be the number of edges of G, and h_G the number of loops. One readily checks that ψ_G is a homogeneous polynomial of degree h_G (Exercise 2.327). After choosing a numbering of the vertices, we can see ψ_G as a polynomial in the variables $x_0, \ldots, x_{n_G-1}$.

Definition 2.313. The *graph hypersurface* $X_G \subseteq \mathbb{P}^{n_G-1}$ is the vanishing locus of the polynomial ψ_G.

Graph hypersurfaces appear in perturbative quantum field theory, which associates an integral called *Feynman amplitude* with each graph describing a possible interaction between particles. In the case of *primitive log divergent* graphs, which are those satisfying the conditions $n_G = 2h_G$ and $n_\gamma > 2h_\gamma$ for all non-empty strict subgraphs $\gamma \subsetneq G$, the Feynman amplitude is given by the convergent integral

$$I_G = \int_\sigma \frac{\Omega}{\psi_G^2} \tag{2.314}$$

up to a normalization factor that will play no role in our discussion. In the above formula, Ω stands for the differential form

$$\Omega = \sum_{j=0}^{n_G-1} (-1)^j x_j \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_{n_G-1},$$

and the integration domain is the real coordinate simplex

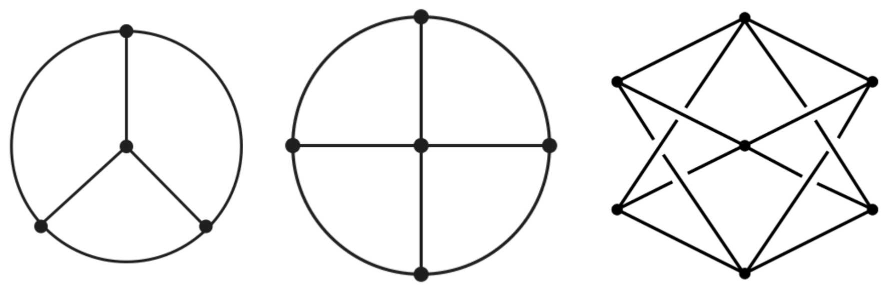
$$\sigma = \{[x_0\colon \cdots : x_{n_G-1}] \in \mathbb{P}^{n_G-1}(\mathbb{R}) \mid x_j \geqslant 0\}.$$

Note that the condition $n_G = 2h_G$ implies that the integrand of (2.314), which is written in homogeneous coordinates, is well defined. Setting $t_i = x_i/x_0$, the amplitude I_G can also be written as the affine integral

$$I_G = \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 \cdots \int_0^\infty \frac{\mathrm{d}t_{n_G-1}}{\psi_G^2(1, t_1, t_2, \ldots, t_{n_G-1})}.$$

That the integral converges is proved, for instance, in [**BEK06**, Prop. 5.2].

The extensive numerical calculations of Broadhurst and Kreimer [**BK97**] identified many Feynman amplitudes of primitive log divergent graphs, such as those shown in Figure 11, with $\mathbb{Q}$-linear combinations of multiple zeta values. This happens to be a general phenomenon for “small graphs” (for example, those with at most 6 loops), and it was believed for some time that I_G might always be a $\mathbb{Q}$-linear combination of multiple zeta values. In our current state of knowledge, giving a negative answer to this question seems completely out of reach, as it would require to prove the very strong diophantine statement that some Feynman amplitude is $\mathbb{Q}$-linearly independent from *all* multiple zeta values. Nevertheless, we can approach the question from a cohomological point of view to get compelling evidence for a negative answer. Indeed, multiple zeta values are periods of Hodge structures of mixed Tate type built on algebraic varieties over $\mathbb{Q}$. Conversely, a form of Grothendieck's period conjecture implies that, if all periods of such a mixed Hodge structure are $\mathbb{Q}$-linear combinations of multiple zeta values, then it is necessarily of mixed Tate type. One can then ask instead if the mixed Hodge structure that naturally arises from the integral representation (2.314) is of mixed Tate type. An easier question to begin with is whether the cohomology of $Y_G = \mathbb{P}^{n_G-1} \setminus X_G$ is always of mixed Tate type. This variety being smooth, the question amounts by Poincaré duality to asking whether the cohomology *with compact support* of Y_G is always of mixed Tate type or, using the long exact sequence (2.292), whether the cohomology of X_G is always of mixed Tate type.



$6\zeta(3)$ $\quad$ $20\zeta(5)$ $\quad$ $\frac{45}{4}\zeta(3,5) + \frac{333}{20}\zeta(5,3) - \frac{9}{5}\zeta(8)$

FIGURE 11. Three examples of primitive log divergent graphs and the corresponding Feynman amplitudes

This last question was answered in the negative by Belkale and Brosnan, who showed in [**BB03**] that varieties of the form Y_G are general enough to span the Grothendieck ring of varieties (see Section 2.8.9). Note that the polynomial ψ_G has integer coefficients, and hence the graph hypersurface X_G and its complement Y_G are schemes defined over $\mathrm{Spec}(\mathbb{Z})$. Their result can then be stated as follows:

THEOREM 2.315 (Belkale–Brosnan [**BB03**]). *For each scheme X of finite type over* $\mathrm{Spec}(\mathbb{Z})$*, there exist finitely many graphs G_i, polynomials $p_i \in \mathbb{Z}[T]$, and integers $n_j \geqslant 2$ such that, setting $\mathbb{L} = [\mathbb{A}^1_{\mathbb{Z}}]$, the equality*

$$[X] = \prod_j (\mathbb{L}^{n_j} - \mathbb{L}) \cdot \sum_i p_i(\mathbb{L})[Y_{G_i}]$$

holds in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{Z}})$.

By base change, the same identity holds between the classes of the corresponding varieties over the field of complex numbers. Now recall from (2.306) that there is a unique ring morphism $K_0(\mathbf{Var}_{\mathbb{C}}) \to \mathbb{Z}[u,v]$ that sends the class of a variety to its E-polynomial. If all the cohomology with compact support groups of a variety are of mixed Tate type, then its E-polynomial is a polynomial in the variable uv. This is, for example, the case for the affine line, since its only non-zero cohomology with compact support is $\mathrm{H}^2_c(\mathbb{A}^1) = \mathbb{Q}(-1)$; the E-polynomial of $\mathbb{L}$ is thus equal to uv. It then follows from Theorem 2.315 that, if the cohomology with compact support of Y_G were of mixed Tate type for all graphs G, then the same would hold for any variety. But this is clearly not true: for example, the E-polynomial of an elliptic curve is $(1-u)(1-v)$. Therefore, there exists at least one graph G such that not all cohomology with compact support groups of Y_G (or, equivalently, all cohomology groups of X_G) are of mixed Tate type.

In a different line of thought, the fact that the cohomology of X_G is of mixed Tate type is also expected to impose strong constraints on the number of points of this variety over finite fields. Indeed, since ψ_G has integral coefficients, it makes sense to consider for each finite field $\mathbb{F}_q$ of characteristic p the number of $\mathbb{F}_q$-points

$$|X_G(\mathbb{F}_q)| = \{[x_0\colon \dots : x_{n_G-1}] \in \mathbb{P}^{n_G-1}(\mathbb{F}_q) \mid \psi_G(x_0, \dots, x_{n_G-1}) = 0\}$$

of the reduction modulo p of the graph hypersurface. In 1997, Kontsevich informally conjectured that, for each graph G, the function

$$q \longmapsto |X_G(\mathbb{F}_q)| \tag{2.316}$$

is a polynomial in q. Again, this happens to be true for "small graphs" (for example, those with at most 12 edges [**Ste98**]), but the theorem of Belkale and Brosnan also disproves the expectation that it might be a general phenomenon. Indeed, Theorem 2.315 implies that the functions (2.316) span all counting functions of schemes over $\mathbb{Z}$: for each scheme of finite type X over $\mathrm{Spec}(\mathbb{Z})$, there exist finitely many graphs G_i, polynomials $p_i \in \mathbb{Z}[T]$, and integers $n_j \geqslant 2$ satisfying

$$|X(\mathbb{F}_q)| = \prod_j (q^{n_j} - q) \cdot \sum_i p_i(q) |X_{G_i}(\mathbb{F}_q)|$$

for all q. Therefore, if the function $q \mapsto |X_G(\mathbb{F}_q)|$ were a polynomial in q for each graph G, then the same would hold for all varieties. This is again false, for example, for an elliptic curve. Later on, Brown and Schnetz [**BS12**] constructed explicit graphs that are counterexamples to Kontsevich's conjecture. For example, there exists a primitive log divergent graph G with 8 loops and 16 edges such that the counting function of X_G is given modulo pq^2 by a modular form arising from a certain K3 surface with complex multiplication.

In order to interpret the Feynman amplitude I_G as a period, we observe that the integrand of (2.314) is a global top-degree differential form ω_G on $\mathbb{P}^{n_G-1} \setminus X_G$ and that the boundary of the integration domain σ is contained in the union D of the coordinate hyperplanes $\{x_i = 0\}$. In general, σ intersects the graph hypersurface X_G, so we are faced again with the problem we encountered in Section 2.5 when dealing with $\zeta(2)$ that the integration cycle does *not* define an element in the relative cohomology group

$$\mathrm{H}_{\mathrm{B}}^{n_G-1}(\mathbb{P}^{n_G-1} \setminus X_G, D \setminus D \cap X_G).$$

However, the fact that the coefficients of ψ_G are positive makes this intersection easy to describe. In fact, it is equal to

$$X_G(\mathbb{C}) \cap \sigma = \bigcup_{h_\gamma > 0} L_\gamma(\mathbb{R}_{\geqslant 0}),$$

where γ is a subgraph of G with at least one loop, L_γ is the linear subvariety of $\mathbb{P}^{n_G-1}$ defined by the equations $x_e = 0$ for all vertices e of γ, and we set

$$L_\gamma(\mathbb{R}_{\geqslant 0}) = \{[x_e]_{e\in E} \in L_\gamma(\mathbb{R}) \mid x_e \geqslant 0\}.$$

This allowed Bloch, Esnault, and Kreimer to obtain the following result:

Theorem 2.317 (Bloch–Esnault–Kreimer [**BEK06**, Prop. 7.3]). *There exists a tower of blow-ups*

$$\pi\colon P = P_r \longrightarrow \cdots \longrightarrow P_0 = \mathbb{P}^{n_G-1}$$

such that each P_i is obtained by blowing up P_{i-1} along the strict transform of a coordinate linear space L_i, and the following conditions hold:

i) *The differential $\pi^*\omega_G$ has no poles along the exceptional divisors associated with the blow-ups.*
ii) *The total transform B of D is a normal crossing divisor such that none of the non-empty intersections of its irreducible components is contained in the strict transform Y of X_G.*
iii) *The strict transform of σ does not meet Y.*

Corollary 2.318. *Keeping the notation from above, the Feynman amplitude I_G is a period of the mixed Hodge structure*

$$\mathrm{H}^{n_G-1}(P \setminus Y, B \setminus (B \cap Y)).$$

⋆⋆⋆

Exercise 2.319. Set $X = \mathbb{P}^1_{\mathbb{Q}}$ and let $Z \subset X$ be a closed subvariety consisting of a rational point. Compute the mixed Hodge structure on the cohomology with support $\mathrm{H}^*_Z(X)$ introduced in Definition 2.295.

Exercise 2.320. Let X be a smooth complex variety, let $Z \subset X$ be a smooth closed subvariety of codimension c, and write $U = X \setminus Z$. Use the Gysin long exact sequence (2.303) to prove that the restriction map $\mathrm{H}^i(X) \to \mathrm{H}^i(U)$

i) is an isomorphism for $i < 2c-1$;
ii) is injective for $i = 2c-1$.

Exercise 2.321 (Varieties which do not admit a compactification by a smooth divisor). Let U be a smooth complex variety. In this exercise, we show that the existence of a smooth compactification by a *smooth* divisor imposes strong restrictions on the mixed Hodge structure on the cohomology of U.

i) Use the Gysin exact sequence (2.303) to prove that, if $U = X \setminus D$ is the complement of a smooth divisor D on a smooth proper variety X, then the mixed Hodge structure $\mathrm{H}^n(U)$ has only weights in $[n, n+1]$.
ii) Give an example of a smooth surface which does not admit a smooth proper compactification by a smooth divisor.

EXERCISE 2.322. In this exercise, we show that Proposition 2.302 does not hold without the smoothness assumption.

i) Let $H_1 \to H_2 \to H_3$ be an exact sequence of mixed Hodge structures. Assume that H_1 has weights in $I \subset \mathbb{Z}$ and H_3 has weights in $J \subset \mathbb{Z}$. Prove that H_2 has weights in $I \cup J$.

ii) Let X be a smooth proper variety and Z a closed subvariety. Use the exact sequence (2.296) to prove that, for any $n \geqslant 0$, the mixed Hodge structure $\mathrm{H}^n_Z(X)$ has weights in $[n-1, \max(n, 2n-2)]$. Then show that weight $n-1$ does not occur.

iii) Let X be a smooth proper variety and D a simple normal crossing divisor. Show that $\mathrm{H}^{n-2}(D)(-1)$ has weights in $[2, n]$. (In fact, a similar result holds for any closed subvariety.)

iv) It follows from the above that $\mathrm{H}^n_D(X)$ and $\mathrm{H}^{n-2}(D)(-1)$ can only be isomorphic mixed Hodge structures if they are both pure of weight n. Consider $X = \mathbb{P}^2$, and let $D = D_1 \cup D_2 \cup D_3$ be the union of the coordinate hyperplanes. Prove that there are isomorphisms of mixed Hodge structures $\mathrm{H}^3_D(X) = \mathbb{Q}(-2)$ and $\mathrm{H}^1(D)(-1) = \mathbb{Q}(-1)$.

v) Even if the weights match, the Hodge structures need not be isomorphic. Consider $X = \mathbb{P}^2$, and let $D = D_1 \cup D_2$ be the union of two coordinate hyperplanes. Prove that there are isomorphisms

$$\mathrm{H}^n_D(X) = \begin{cases} \mathbb{Q}(-2), & \text{if } n = 4, \\ \mathbb{Q}(-1) \oplus \mathbb{Q}(-1), & \text{if } n = 2, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathrm{H}^n(D) = \begin{cases} \mathbb{Q}(-1) \oplus \mathbb{Q}(-1), & \text{if } n = 2, \\ \mathbb{Q}(0), & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\mathrm{H}^n_D(X)$ is not isomorphic to $\mathrm{H}^{n-2}(D)(-1)$.

EXERCISE 2.323. Let $\overline{X}$ be a smooth proper complex variety, and let Y_0, Y_1 be smooth divisors on $\overline{X}$ such that $Y_0 \cup Y_1$ has simple normal crossings. Set

$$X = \overline{X} \setminus Y_0, \quad Y = Y_1 \setminus (Y_0 \cap Y_1).$$

Show that the weight filtration on the relative cohomology group

$$M = \mathrm{H}^n(X, Y)$$

is given by the following steps:

$$\begin{aligned} W_{n-2}M &= 0, \\ W_{n-1}M &= \mathrm{Im}(\mathrm{H}^{n-1}(Y_1) \to M), \\ W_n M &= \mathrm{Ker}(M \to \mathrm{H}^{n-1}(Y_0)(-1)), \\ W_{n+1}M &= M. \end{aligned}$$

[Hint: Consider a diagram of mixed Hodge structures whose rows are Gysin long exact sequences and whose columns are long exact sequences of relative cohomology; then use the fact that W_m is an exact functor and Lemma 2.232.]

EXERCISE 2.324 (The graded pieces of the mixed Hodge structure of a smooth variety). Let X be a smooth proper variety, D a simple normal crossing divisor, and $U = X \setminus D$. As in Construction 2.114, we use the notation

$$D^0 = X, \quad D^p = \coprod_{i_1, i_2, \ldots, i_p} D_{i_1} \cap \cdots \cap D_{i_p} \quad (p \geqslant 1).$$

Prove that the weight filtration of $\mathrm{H}^n(U)$ has graded pieces

$$\mathrm{Gr}_m^W \mathrm{H}^n(U) = \mathrm{H}^{n-m}\left(\cdots \longrightarrow \mathrm{H}^{m-2}(D^1)(-1) \longrightarrow \mathrm{H}^m(X) \longrightarrow 0\right), \tag{2.325}$$

where the term $\mathrm{H}^{m-2p}(D^p)(-p)$ sits in degree $-p$ and the arrows are alternating sums of Gysin maps.

EXERCISE 2.326. Let $n \geqslant 3$ be an integer and q a power of a prime number. As the moduli space $M_{0,n}$ is defined over the integers, we can consider its base change to any finite field $\mathbb{F}_q$. Consider the polynomial of degree $n-3$ given by

$$P(T) = (T-2)(T-3)\cdots(T-n+2).$$

i) Show that the number of points of $M_{0,n}$ over $\mathbb{F}_q$ is equal to $P(q)$.
ii) Building on Proposition 2.309, prove that the E-polynomial of $M_{0,n}$ equals

$$E_{M_{0,n}}(u,v) = P(uv).$$

More generally, given any complex variety X, there exists a subring $R \subset \mathbb{C}$ that is finitely generated over $\mathbb{Z}$ and a scheme $\mathcal{X}$ over R that gives X back after extension of scalars. We say that X has *strong polynomial count* if R and $\mathcal{X}$ can be chosen in such a way that there exists a polynomial $P \in \mathbb{Z}[t]$ with the following property: for each finite field $\mathbb{F}_q$ and each ring morphism $R \to \mathbb{F}_q$, the number of points of $\mathcal{X}(\mathbb{F}_q)$ is $P(q)$. It is then a general result, proved in Katz's appendix to [**HRV08**], that the E-polynomial of a variety with strong polynomial count is equal to $P(uv)$.

EXERCISE 2.327. Prove that the first Symanzik polynomial of a graph, as defined in (2.312), is homogeneous of degree the number of loops in G.

EXERCISE 2.328 (Deletion-contraction relations). Let G be a connected graph and e an edge of G. We denote by $G \setminus e$ the graph obtained by deleting the edge e, and by G/e the graph obtained by contracting the edge e. Assuming that $G \setminus e$ is still connected and that the two endpoints of e are different, show that the identity

$$\psi_G = x_e \psi_{G\setminus e} + \psi_{G/e}$$

relating the first Symanzik polynomials of G, $G \setminus e$, and G/e holds.

EXERCISE 2.329 (The trivial Feynman amplitude). Consider the graph G with two vertices and two edges connecting them, as in Figure 12. Compute the Feynman amplitude I_G defined in (2.314), and write down a Hodge structure for which it is a period (no blow-up is needed in this case).

2.9. Back to $\zeta(2)$ and irrationality proofs. We end the chapter by showing that the relative cohomology group attached to the period $\zeta(2)$ in Section 2.5.1 is an extension of $\mathbb{Q}(-2)$ by $\mathbb{Q}(0)$. We then discuss the problem of constructing extensions of $\mathbb{Q}(-n)$ by $\mathbb{Q}(0)$ by geometric means, with a potential application to irrationality proofs. Many thanks to Clément Dupont and Peter Jossen for their help.

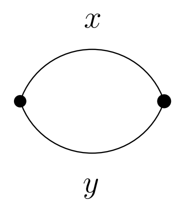


FIGURE 12. A simple graph

2.9.1. *The extension associated with* $\zeta(2)$. We prove that the relative cohomology group built in Section 2.5.1 out of the integral representation of $\zeta(2)$ is an extension of $\mathbb{Q}(-2)$ by $\mathbb{Q}(0)$. Recall that we considered the blow-up X of $\mathbb{A}^2$ at the points $p = (0,0)$ and $q = (1,1)$, together with the simple normal crossing divisors

$$L = L_0 \cup L_1, \quad M = M_0 \cup M_1 \cup M_2 \cup M_3 \cup M_4,$$

with the following irreducible components:

- L_0 and L_1 are the strict transforms of $\{t_1 = 0\}$ and $\{t_2 = 1\}$ (affine lines),
- $M_0 = E_p$ and $M_1 = E_q$ are the exceptional divisors (projective lines),
- M_2, M_3, and M_4 are the strict transforms in X of $\{t_1 = t_2\}$, $\{t_2 = 0\}$, and $\{t_1 = 1\}$ (affine lines).

PROPOSITION 2.330. *There exists a short exact sequence*

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow \mathrm{H}^2(X \setminus L, M \setminus (L \cap M)) \longrightarrow \mathbb{Q}(-2) \longrightarrow 0 \tag{2.331}$$

of mixed Hodge structures.

The proof relies on a spectral sequence computing the grading for the weight filtration on relative cohomology groups that we first discuss in a general setting. Let X be any smooth complex variety of dimension d, and L and M two simple normal crossing divisors on X with no common irreducible components and such that $L \cup M$ has simple normal crossings as well. By [**Dup17**, App. A.1], there is a spectral sequence of mixed Hodge structures

$$E_1^{p,q} = \bigoplus_{\substack{j-i=p \\ |I|=i \\ |J|=j}} \mathrm{H}^{q-2i}(L_I \cap M_J)(-i) \Longrightarrow \mathrm{Gr}^W \mathrm{H}^{p+q}(X \setminus L, M \setminus (M \cap L)), \tag{2.332}$$

where the indexes run over $-d \leqslant p \leqslant d$ and $0 \leqslant q \leqslant 2d$, and the differential

$$\mathrm{d}_1 \colon E_1^{p,q} \longrightarrow E_1^{p+1,q}$$

is the sum of the following maps:

i) the restriction maps

$$\mathrm{H}^{q-2i}(L_I \cap M_J)(-i) \longrightarrow \mathrm{H}^{q-2i}(L_I \cap M_{J\cup\{s\}})(-i)$$

induced from the inclusions

$$L_I \cap M_{J\cup\{s\}} \hookrightarrow L_I \cap M_J,$$

multiplied by the signs $\varepsilon(J, J \cup \{s\})$;

ii) the Gysin morphisms

$$\mathrm{H}^{q-2i}(L_I \cap M_J)(-i) \longrightarrow \mathrm{H}^{q-2i+2}(L_{I\setminus\{r\}} \cap M_J)(-i+1)$$

associated with the inclusions

$$L_I \cap M_J \hookrightarrow L_{I\setminus\{r\}} \cap M_J,$$

multiplied by the signs $\varepsilon(I \setminus \{r\}, I)$.

(Recall from Section 2.2.7 that, if $J = \{j_0, \dots, j_r\}$ is an index set with $j_0 < \cdots < j_r$ and I is obtained from J by removing j_ℓ, we set $\varepsilon(I, J) = (-1)^\ell$.) In particular, d_1 is a morphism of mixed Hodge structures.

Assume that all the terms $E_1^{p,q}$ in the spectral sequence (2.332) carry a pure Hodge structure of weight q. The second page is given by

$$E_2^{p,q} = \frac{\mathrm{Ker}(\mathrm{d}_1 \colon E_1^{p,q} \to E_1^{p+1,q})}{\mathrm{Im}(\mathrm{d}_1 \colon E_1^{p-1,q} \to E_1^{p,q})},$$

together with a differential $\mathrm{d}_2 \colon E_2^{p,q} \to E_2^{p+1,q-1}$. Thus, $E_2^{p,q}$ has a pure Hodge structure of weight q as well, which implies $\mathrm{d}_2 = 0$ since there are no non-trivial morphisms between Hodge structures of different weight. It follows that the spectral sequence degenerates at E_2 and

$$E_2^{p,q} = \mathrm{Gr}_q^W \mathrm{H}^{p+q}(X \setminus L, M \setminus (M \cap L)). \tag{2.333}$$

Proof. Let us now turn to our particular situation. Setting

$$R = L_0 \cap L_1, \quad S = L_0 \cap E_p, \quad T = L_1 \cap E_q, \quad M_{ij} = M_i \cap M_j,$$

the spectral sequence takes the form of Figure 13. By way of illustration, the piece $E_1^{1,2}$ is the sum of all possible $\mathrm{H}^{2-2i}(L_I \cap M_J)(-i)$ with $j = i+1$. Then necessarily $i = 0$ or $i = 1$, and the second case does not occur since there are no non-empty intersections of one component of L and two components of M. For $i = 0$, we get $\bigoplus \mathrm{H}^2(M_j) = \mathrm{H}^2(E_p) \oplus \mathrm{H}^2(E_q)$, taking into account that the remaining components are affine lines. Observe that odd values of q do not need to be considered, since all intersections $L_I \cap M_J$ have cohomology concentrated in even degrees. The assumption that $E_1^{p,q}$ has pure weight q also holds in our case.

We need to prove that

$$\mathrm{Gr}^W \mathrm{H}^2(X \setminus L, M \setminus (L \cap M)) = \mathbb{Q}(0) \oplus \mathbb{Q}(-2). \tag{2.334}$$

In this equality, the piece $\mathbb{Q}(-2)$ comes from the top-left corner of the spectral sequence, while $\mathbb{Q}(0)$ arises as the cokernel of the map $\bigoplus \mathrm{H}^0(M_i) \to \bigoplus \mathrm{H}^0(M_{ij})$, which has rank 4. Indeed, this map is given by

$$(a, b, c, d, e) \longmapsto (c-a, d-a, c-b, e-b, e-d).$$

Since the map $\mathrm{H}^0(X) \to \bigoplus \mathrm{H}^0(M_i)$ sends a to (a, a, a, a, a), the cohomology of the bottom line is concentrated in $E_2^{2,0} = \mathbb{Q}(0)$.

We are thus reduced to show that the complex $E_1^{*,2}$ is exact at the middle term. For this, we first observe that the Gysin maps induce an isomorphism of Hodge structures

$$\mathrm{H}^0(E_p)(-1) \oplus \mathrm{H}^0(E_q)(-1) \xrightarrow{\sim} \mathrm{H}^2(X). \tag{2.335}$$

$$
\begin{array}{c|ccccc}
4 & \boxed{\mathrm{H}^0(R)(-2)} & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & \begin{array}{c}\mathrm{H}^0(L_0)(-1)\\ \oplus \\ \mathrm{H}^0(L_1)(-1)\end{array} & \boxed{\begin{array}{c}\mathrm{H}^2(X)\\ \oplus\\ \mathrm{H}^0(S)(-1)\\ \oplus\\ \mathrm{H}^0(T)(-1)\end{array}} & \begin{array}{c}\mathrm{H}^2(E_q)\\ \oplus\\ \mathrm{H}^2(E_p)\end{array} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{H}^0(X) & \bigoplus \mathrm{H}^0(M_i) & \boxed{\bigoplus \mathrm{H}^0(M_{ij})} \\
\hline
 & \text{-2} & \text{-1} & 0 & 1 & 2
\end{array}
$$

FIGURE 13. The first page of the spectral sequence computing the grading $\mathrm{Gr}^W \mathrm{H}^2(X \setminus L, M \setminus (L \cap M))$

This is an instance of the general computation of the Hodge structure of a blow-up; see *e.g.* [**Voi02**, § 7.3.3]. In the case at hand, it can be seen as follows: the Gysin long exact sequence (2.303) for $U = X \setminus (E_p \cup E_q)$ reads

$$\cdots \longrightarrow \mathrm{H}^1(U) \longrightarrow \mathrm{H}^0(E_p)(-1) \oplus \mathrm{H}^0(E_q)(-1) \longrightarrow \mathrm{H}^2(X) \longrightarrow \mathrm{H}^2(U) \longrightarrow \cdots$$

Since U and $\mathbb{A}^2 \setminus \{p, q\}$ are isomorphic via the blow-up map, the cohomology groups $\mathrm{H}^1(U)$ and $\mathrm{H}^2(U)$ vanish (use Exercise 2.320). It follows that, in suitable bases compatible with the isomorphism (2.335), the differential

$$\mathrm{d}_1 \colon E_1^{0,2} \longrightarrow E_1^{1,2}$$

in the spectral sequence is given by

$$
\begin{array}{ccc}
\mathrm{H}^2(X) \oplus \mathrm{H}^0(S)(-1) \oplus \mathrm{H}^0(T)(-1) & \longrightarrow & \mathrm{H}^2(E_p) \oplus \mathrm{H}^2(E_q) \\
(a, b, c, d) & \longmapsto & (a + c, b + d).
\end{array} \tag{2.336}
$$

To compute the remaining map in the spectral sequence, we take a closer look at the cohomology classes $[L_i] \in \mathrm{H}^2(X)$. We claim that $[L_0] = -[E_p]$. Indeed, since the total transform of ℓ_0 is the union $L_0 \cup E_p$, we get

$$[L_0] + [E_p] = [\pi^{-1}(\ell_0)] = \pi^*[\ell_0] = 0,$$

where the last equality follows from the fact that $[\ell_0]$ lives in $\mathrm{H}^2(\mathbb{A}^2) = 0$. Similarly, the equality $[L_1] = -[E_q]$ holds, so that $\mathrm{d}_1 \colon E_1^{-1,2} \to E_1^{0.2}$ is given by

$$
\begin{array}{ccc}
\mathrm{H}^0(L_0)(-1) \oplus \mathrm{H}^0(L_1)(-1) & \longrightarrow & \mathrm{H}^2(X) \oplus \mathrm{H}^0(S)(-1) \oplus \mathrm{H}^0(T)(-1) \\
(a, b) & \longmapsto & (-a, -b, a, b).
\end{array} \tag{2.337}
$$

It is now obvious that the middle row of the spectral sequence is exact. Indeed, its whole second page reads

$$\begin{array}{ccccc}
\mathbb{Q}(-2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{Q}(0).
\end{array}$$

This concludes the proof of the equality (2.334) and shows, moreover, the vanishing

$$\mathrm{H}^i(X \setminus L, M \setminus (L \cap M)) = 0$$

in all degrees i different from 2. □

REMARK 2.338. A byproduct of the proof is that we have canonical identifications (see Exercise 2.344)

$$\begin{aligned}
\operatorname{Gr}_4^W \mathrm{H}^2(X \setminus L, M \setminus (L \cap M)) &= \mathrm{H}^2(X \setminus L) = \mathbb{Q}(-2), \\
\operatorname{Gr}_0^W \mathrm{H}^2(X \setminus L, M \setminus (L \cap M)) &= \mathrm{H}^2(X, M) = \mathbb{Q}(0).
\end{aligned} \tag{2.339}$$

Recall from Section 2.5.1 that the differential form $\pi^*(\omega)$ defines a class in the de Rham cohomology group $\mathrm{H}^2_{\mathrm{dR}}(X \setminus L)$ and the simplex $\widehat{\sigma}$ belongs to $\mathrm{H}^2(X, M)$. By Theorem 2.256, the class of the extension

$$[\mathrm{H}^2(X \setminus L, M \setminus (L \cap M))] \in \operatorname{Ext}^1_{\mathbf{MHS}}(\mathbb{Q}(-2), \mathbb{Q}(0)) = \mathbb{C}/(2\pi i)^2\mathbb{Q}$$

is thus given by $\int_{\widehat{\sigma}} \pi^*(\omega) = \zeta(2)$. One would like to use this information as follows: imagine that we knew by "pure thought" that all extensions of $\mathbb{Q}(-2)$ by $\mathbb{Q}(0)$ given by relative cohomology of varieties defined over $\mathbb{Q}$ are split. Then $\zeta(2)$ would have to vanish in the quotient $\mathbb{C}/(2\pi i)^2\mathbb{Q}$, which would yield a more conceptual explanation of why $\zeta(2)$ is a rational multiple of π^2. To carry out this program, one needs however to leave the category of mixed Hodge structures and work with the more abstract notion of mixed Tate motives which will be introduced in Chapter 4.

2.9.2. *Odd zeta extensions.* In general, it is a difficult problem to give a geometric construction of the extension of $\mathbb{Q}(-n)$ by $\mathbb{Q}(0)$ whose class in

$$\operatorname{Ext}^1_{\mathbf{MHS}}(\mathbb{Q}(-n), \mathbb{Q}(0)) = \mathbb{C}/(2\pi i)^n\mathbb{Q}$$

is the zeta value $\zeta(n)$. The meaning of "geometric" is vague for the moment: we may understand it as "given by a relative cohomology group of a pair of algebraic varieties over $\mathbb{Q}$", or more generally built out of such a relative cohomology by linear algebra operations such as taking the kernel and the image of maps induced by morphisms of algebraic varieties (functoriality, Gysin maps, etc.). Besides Proposition 2.330, a case where such a geometric construction is known is $n = 3$, by work of Brown [**Bro16**] and Dupont [**Dup18**]. We sketch the later, which is inspired by the integral representation

$$\zeta(n) = \int_{[0,1]^n} \frac{\mathrm{d}x_1 \dots \mathrm{d}x_n}{1 - x_1 \cdots x_n}. \tag{2.340}$$

In order to attach a relative cohomology group to the period (2.340), we start with affine space $\mathbb{A}^n$ and the hypersurfaces

$$\ell_n = \{x_1 \cdots x_n = 1\}, \quad m_n = \bigcup_{1 \leqslant i \leqslant n} \{x_i = 0\} \cup \bigcup_{1 \leqslant i \leqslant n} \{x_i = 1\}.$$

The divisor ℓ_n is smooth and m_n is a normal crossing divisor; however, their union $\ell_n \cup m_n$ fails to have simple normal crossings at the point $p_n = (1, \dots, 1)$, where $n+1$ irreducible components intersect. Let $\pi_n \colon X_n \to \mathbb{A}^n$ be the blow-up of $\mathbb{A}^n$ at p_n, and let E_n denote the exceptional divisor. We write L_n for the strict transform of ℓ_n, and M_n for the union of the strict transform of m_n and the exceptional divisor E_n. We form the relative cohomology:

$$Z_n = \mathrm{H}^n(X_n \setminus L_n, M_n \setminus (L_n \cap M_n)).$$

Dupont proves that Z_n fits into an exact sequence of mixed Hodge structures

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow Z_n \longrightarrow \mathbb{Q}(-2) \oplus \cdots \oplus \mathbb{Q}(-n)$$

and that there is a natural isomorphism

$$Z_n/\mathbb{Q}(0) \xrightarrow{\sim} \mathrm{H}^{n-1}(\ell_n, \bigcup_{1 \leqslant i \leqslant n} \{x_i = 1\})(-1). \tag{2.341}$$

Moreover, with respect to appropriate bases of de Rham and Betti cohomology, the period matrix of Z_n is given by

$$\begin{pmatrix} 1 & \zeta(2) & \zeta(3) & \dots & \dots & \zeta(n-1) & \zeta(n) \\ & (2\pi i)^2 & & & & & \\ & & (2\pi i)^3 & & & 0 & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & 0 & & & & (2\pi i)^{n-1} & \\ & & & & & & (2\pi i)^n \end{pmatrix}.$$

To separate the values of the zeta function at even and odd integers, he uses the involution

$$\tau(x_1, \dots, x_n) = (x_1^{-1}, \dots, x_n^{-1})$$

on ℓ_n. Since τ leaves each subvariety $\{x_i = 1\}$ invariant, it induces an involution on the right-hand side of (2.341). By [**Dup18**, Thm. 1.4], the invariants are given by

$$(Z_n/\mathbb{Q}(0))^{\tau=1} \simeq \bigoplus_{3 \leqslant 2k+1 \leqslant n} \mathbb{Q}(-(2k+1)).$$

Therefore, letting $p \colon Z_n \to Z_n/\mathbb{Q}(0)$ denote the quotient map, and defining

$$Z_n^{\mathrm{odd}} = p^{-1}((Z_n/\mathbb{Q}(0))^{\tau=1}),$$

we obtain an exact sequence of mixed Hodge structures

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow Z_n^{\mathrm{odd}} \longrightarrow \bigoplus_{3 \leqslant 2k+1 \leqslant n} \mathbb{Q}(-(2k+1)) \longrightarrow 0.$$

The period matrix of Z_n^{odd} is given by

$$\begin{pmatrix} 1 & \zeta(3) & \zeta(5) & \zeta(7) & \dots & \dots \\ & (2\pi i)^3 & & & & \\ & & (2\pi i)^5 & & & 0 \\ & & & (2\pi i)^7 & & \\ & 0 & & & \ddots & \\ & & & & & \ddots \end{pmatrix}.$$

This example has recently been generalized in [DF24] to construct relative cohomology groups associated with the polylogarithm.

2.9.3. *Irrationality proofs.* Here is how a typical irrationality proof works. To show that a real number α is irrational, we proceed in three steps:

i) we construct linear forms

$$I_n = a_n + b_n\alpha, \quad a_n, b_n \in \mathbb{Q}, \tag{2.342}$$

such that $0 < |I_n| < C^n$ for some $0 < C < 1$ and all sufficiently big n;

ii) if r_n is the common denominator of a_n and b_n, then we require that $r_n < D^n$ for some real number D, again when n is big enough;

iii) C and D should be related by the inequality $CD < 1$.

If one succeeds in carrying out these three steps, then α is irrational. Indeed, assume by contradiction that α is of the form p/q for some integers p and q. Multiplying I_n by $r_n q$, we get

$$0 < |r_n a_n q + r_n b_n p| < q r_n C^n < q(CD)^n,$$

so the sequence inside the absolute value converges to zero by the assumption that $CD < 1$. But then, for sufficiently big n, we would find integers strictly bigger than 0 and strictly smaller than 1, which is of course a contradiction!

Algebraic geometry could be useful in producing the linear forms (2.342). Indeed, assume that we can construct a mixed Hodge structure over $\mathbb{Q}$ which is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ with period matrix

$$\begin{pmatrix} 1 & \alpha \\ 0 & (2\pi i)^n \end{pmatrix}$$

with respect to some bases $\{\omega_0, \omega_1\}$ of H_{dR} and $\{\sigma_0, \sigma_1\}$ of H_{B}. Then, given an $\omega \in H_{\mathrm{dR}}$, there exist rational numbers a and b such that $\omega = a\omega_0 + b\omega_1$, and the integral $\int_{\sigma_0} \omega$ is equal to $a + b\alpha$. Typically, H is given by a relative cohomology group and one considers a sequence $\omega_n = f^n\omega$ where ω is a fixed differential form and f is a function vanishing on the boundary.

Example 2.343. Consider the differential form

$$\omega_{a,b,c} = \frac{(x-1)^a(t-x)^b}{x^{c+1}}\mathrm{d}x,$$

where $a, b, c \geqslant 1$ and $t \geqslant 2$ are integers. Since $\omega_{a,b,c}$ is only singular along $x = 0, \infty$ and has top degree, it defines a class in $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, t\})$. By Example 2.108, a basis of this relative cohomology group is given by the differentials $\omega_1 = \mathrm{d}x/(t-1)$ and $\omega_2 = \mathrm{d}x/x$, so there exist rational numbers A and B such that

$$[\omega_{a,b,c}] = A[\omega_1] + B[\omega_2].$$

Indeed, elementary manipulations using Exercise 2.127 ii) yield the values

$$A = \sum_{\substack{0\leqslant i\leqslant a\\ 0\leqslant j\leqslant b\\ i+j\neq c}} \frac{\binom{a}{i}\binom{b}{j}(-1)^{a-i-j}(t^{b-c+i}-t^{b-j})}{i+j-c},$$

$$B = \sum_{\substack{0\leqslant i\leqslant a\\ 0\leqslant j\leqslant b\\ i+j=c}} \binom{a}{i}\binom{b}{j}(-1)^{a-i-j}t^{b-j},$$

Note that B is an integer. In view of Example 2.186, it follows that

$$\int_1^t \omega_{a,b,c} = A + B\log(t),$$

and choosing the parameters a, b, c as functions of n gives a sequence of linear forms in 1 and $\log(t)$ as in Step i).

Let us specialize to the case $a = b = c = n$ and $t = 2$. Then

$$I_n = \int_1^2 \omega_{n,n,n} = a_n + b_n \log(2),$$

where b_n is an integer and a_n is given by the formula

$$a_n = \sum_{\substack{0\leqslant i\leqslant n\\ 0\leqslant j\leqslant n\\ i+j\neq n}} \frac{\binom{n}{i}\binom{n}{j}(-1)^{n-i-j}(2^i-2^{n-j})}{i+j-n}.$$

Since the denominators of the summands in a_n run through $[-n, n]$, one can take $r_n = \mathrm{lcm}(1, 2, \ldots, n)$. We have:

$$r_n = \prod_{\substack{p\leqslant n\\ \text{prime}}} p^{\left\lfloor \frac{\log n}{\log p}\right\rfloor} < \prod_{\substack{p\leqslant n\\ \text{prime}}} p^{\frac{\log n}{\log p}} = n^{\pi(n)},$$

where $\pi(n)$ is the number of primes smaller than or equal to n. Here is where some deep arithmetic input enters: the *prime number theorem* asserts that

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log(n)} = 1;$$

see *e.g.* [**IK04**, Chap. 2]. It follows that, for all $\varepsilon > 0$ and big enough n, the inequality $n^{\pi(n)} < e^{(1+\varepsilon)n}$ holds. Being generous, $D = 3$ thus works in Step ii). In fact, it suffices to use the estimate $r_n \leqslant 3^n$, which is easier to prove.

By the choice of the parameters, the integral I_n can be written as

$$I_n = \int_1^2 f^n \frac{\mathrm{d}x}{x}, \qquad f(x) = \frac{(x-1)(2-x)}{x}.$$

The function f is strictly positive on the open interval $(1, 2)$ and bounded above by its maximal value $3 - 2\sqrt{2}$. Therefore,

$$0 < I_n < (3-2\sqrt{2})^n \log(2) < (3-2\sqrt{2})^n,$$

so $C = 3 - 2\sqrt{2}$ satisfies the assumptions. Luckily, $CD = 0{,}5147186\ldots < 1$ and, all in all, we have proved that $\log(2)$ is irrational!

⋆⋆⋆

EXERCISE 2.344. Specialize the spectral sequence (2.332) to the cases $I = \emptyset$ and $J = \emptyset$. Deduce the identifications (2.339).

EXERCISE 2.345. Let $L = L_0 \cup L_1 \cup L_2$ and $M = M_0 \cup M_1 \cup M_2$ be two triangles in $\mathbb{P}^2$ such that no three lines intersect at a common point. Use the spectral sequence (2.332) to construct an isomorphism

$$\mathrm{Gr}^W \mathrm{H}^2(\mathbb{P}^2 \setminus L, M \setminus (L \cap M)) \cong \mathbb{Q}(0) \oplus \mathbb{Q}(-1)^{\oplus 4} \oplus \mathbb{Q}(-2).$$

The question of what happens when the lines are not in general position is studied in great detail in [**BGSV90**].

EXERCISE 2.346 (Irrationality of $\zeta(3)$). The goal of this exercise is to prove that $\zeta(3)$ is irrational following the proof given by Beukers [**Beu79**] shortly after Apéry's announcement (see Section 1.1.2). We keep the notation

$$r_n = \mathrm{lcm}(1, 2, \dots, n).$$

i) Let $n, m \geqslant 0$ be integers and $\sigma \geqslant 0$ a real number. Prove the identity

$$\int_{[0,1]^2} \frac{x^{n+\sigma} y^{m+\sigma}}{1 - xy} \mathrm{d}x\mathrm{d}y = \begin{cases} \dfrac{1}{n-m} \left(\dfrac{1}{m+1+\sigma} + \dots + \dfrac{1}{n+m} \right), & \text{if } n > m, \\ \displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+n+\sigma)^2}, & \text{if } n = m. \end{cases}$$

ii) Let $n, m \geqslant 0$ be integers and consider the integral

$$I_{n,m} = \int_{[0,1]^2} \frac{-\log xy}{1 - xy} x^n y^m \mathrm{d}x\mathrm{d}y.$$

Show that, if $n > m$, then $I_{n,m}$ is a rational number whose denominator divides r_n^3, and that

$$I_{n,n} = \begin{cases} 2\zeta(3), & \text{if } n = 0, \\ 2\left(\zeta(3) - 1 - 2^{-3} - \dots - n^{-3}\right), & \text{if } n > 0. \end{cases}$$

[Hint: differentiate the formulas of part i) with respect to σ.]

iii) For each integer $n \geqslant 1$, let $P_n \in \mathbb{Z}[x]$ be the polynomial defined by

$$n! P_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^n (1-x)^n)$$

and consider the integral

$$I_n = \int_{[0,1]^2} \frac{-\log xy}{1 - xy} P_n(x) P_n(y) \mathrm{d}x\mathrm{d}y.$$

Prove that there exist rational numbers $a_n, b_n \in \mathbb{Q}$ whose denominators divide r_n^3 such that

$$I_n = a_n + b_n \zeta(3).$$

iv) Prove that the above integral can be rewritten as

$$I_n = \int_{[0,1]^3} \frac{x^n(1-x)^n y^n (1-y)^n z^n (1-z)^n}{(1-(1-xy)z)^{n+1}} \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

[Hint: use the integral representation

$$\frac{-\log xy}{1-xy} = \int_0^1 \frac{\mathrm{d}w}{1-(1-xy)w}$$

and the change of variables $z = (1-w)(1-(1-xy)w)^{-1}$.]

v) Show that, for all $0 \leqslant x, y, z \leqslant 1$, one has

$$\frac{x(1-x)y(1-y)z(1-z)}{(1-(1-xy)z)} \leqslant (\sqrt{2}-1)^4$$

and deduce the inequalities $0 < |I_n| < 2\zeta(3)(\sqrt{2}-1)^{4n}$. [Hint: first prove that the maximum of the left-hand side occurs for $x = y$.]

vi) Conclude that $\zeta(3)$ is irrational.

3. Multiple zeta values and the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

In this chapter, we start moving towards the goal of upgrading multiple zeta values to their motivic counterparts, which are functions on an algebro-geometric construction associated with the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. To this end, we first look for homotopy functionals on the space of paths on a differentiable manifold M. By Stokes's theorem, line integrals of closed 1-forms are examples of such functionals; however, the corresponding functions on the fundamental group $\pi_1(M)$ always factor through its abelianization and cannot detect loops with trivial homology classes. Trying to go further, K-T. Chen had the fundamental insight that *iterated integrals* yield finer invariants, which are in fact sufficient to recover all finite-dimensional unipotent representations of $\pi_1(M)$ and not only the abelian ones. In Section 3.1, we present the definition and algebraic properties of iterated integrals, and tackle the question of when they define homotopy functionals. Chen's results are most conveniently phrased in terms of Hopf algebras and algebraic groups, notions that we review in Sections 3.2 and 3.3, first in the general setting and then in the case of unipotent groups. Then, in Section 3.4, we explain how to associate with any abstract group its pro-unipotent completion, an algebraic group whose finite-dimensional representations are the unipotent ones of the initial group. Chen's celebrated π_1-de Rham theorem asserts that the Hopf algebra of regular functions on the pro-unipotent completion of the fundamental group $\pi_1(M)$ is isomorphic to the cohomology in degree zero of the bar complex of any connected model of the algebra of differential forms on M. After introducing the bar complex of a dg-algebra, we state this theorem in Section 3.5. As an example, we observe that the bar complex of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the Hoffman algebra $\mathfrak{H}$ which we already encountered in the combinatorial study of multiple zeta values. The proof of Chen's theorem presented in Section 3.6 relies on a result of Beilinson identifying the algebra of functions on the pro-unipotent completion with a limit of certain relative cohomology groups. As we explain in Section 3.7, Chen's theorem has a number of important consequences, notably the fact due to Hain that the pro-unipotent completion of the fundamental group of a complex algebraic variety carries a mixed Hodge structure; thanks to Beilinson's theorem, we will even be able to show that the resulting mixed Hodge structure is motivic. In the remaining of the chapter, we focus on the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The integral representation of polylogarithms and multiple zeta values suggests that these numbers are iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and hence periods of the corresponding mixed Hodge structure on the pro-unipotent completion of the fundamental group, except for the fact that the endpoints of the integration path do not lie on the ambient space. To remedy this, we resort to the notion of tangential base points in Section 3.8. Armed with this tool, in Section 3.9 we make the comparison isomorphism in Chen's theorem explicit for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ using polylogarithms and the Drinfeld associator. In the final Section 3.10, we introduce the tangential fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and study its automorphisms. As an outcome, we obtain a new structure on the Hoffman algebra $\mathfrak{H}$ called Goncharov's coproduct that will play a pivotal role in the proof of Brown's theorem in Chapter 5.

3.1. Iterated integrals and parallel transport. Our presentation follows closely the first sections of Hain's survey [**Hai87b**]. Other nice references are Cartier's Bourbaki seminar [**Car88**] and Brown's notes [**Bro13**].

3.1.1. *The fundamental groupoid.* Let M be a connected differentiable manifold. A continuous function $\gamma\colon [0,1] \to M$ is said to be *piecewise smooth* if there exists a partition $0 = a_0 < a_1 < \ldots < a_{n+1} = 1$ of the unit interval such that the restriction of γ to each $[a_i, a_{i+1}]$ is smooth, meaning that it can be extended to a $\mathcal{C}^\infty$ function on an open neighborhood of $[a_i, a_{i+1}]$. Similarly, a continuous map $F\colon [0,1]^2 \to M$ is said to be piecewise smooth if there exists a finite decomposition into polyhedra (*i.e.* regions bounded by polygons) $[0,1]^2 = \bigcup_i C_i$ such that all the restrictions $F|_{C_i}$ are smooth, in the sense that they can be extended to a smooth function on an open neighborhood of C_i.

From now on, we will call a continuous piecewise smooth map from $[0,1]$ to M simply a *path* (see Remark 3.8 below), and denote the space of paths by

$$\mathcal{P}(M) = \{\gamma\colon [0,1] \longrightarrow M \mid \gamma \text{ continuous and piecewise smooth}\}\,.$$

Given points x and y in M, the subspace of $\mathcal{P}(M)$ consisting of paths from x to y will be denoted by

$$_y\mathcal{P}(M)_x = \{\gamma \in \mathcal{P}(M) \mid \gamma(0) = x,\ \gamma(1) = y\}.$$

When the endpoints of γ agree, we will often call it a *loop*.

Definition 3.1. Let $\gamma_1, \gamma_2 \in {_y\mathcal{P}(M)_x}$ be paths from x to y. We say that γ_1 and γ_2 are *homotopic* if there exists a continuous piecewise smooth function

$$\begin{aligned} F\colon\ [0,1]^2 &\longrightarrow M \\ (t,s) &\longmapsto F(t,s) \end{aligned}$$

satisfying the following conditions:

$$\begin{aligned} &F(t,0) = \gamma_1(t), \quad F(t,1) = \gamma_2(t), \quad \text{for all } t \in [0,1], \\ &F(0,s) = x, \qquad\ \ F(1,s) = y, \qquad\ \ \text{for all } s \in [0,1]. \end{aligned} \tag{3.2}$$

In other words, F is a continuous family of paths

$$\begin{aligned} f_s\colon\ [0,1] &\longrightarrow M \\ t &\longmapsto f_s(t) = F(t,s) \end{aligned}$$

parameterized by $s \in [0,1]$ that interpolates between γ_1 and γ_2, while keeping the endpoints fixed (see Figure 14).

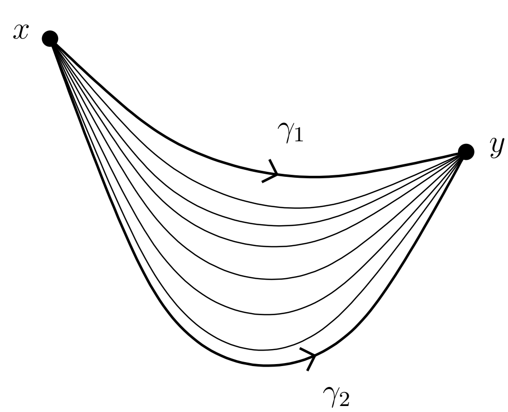


Figure 14. A homotopy between two paths

It is straightforward to check that "being homotopic" defines an equivalence relation $\sim$ on ${}_y\mathcal{P}(M)_x$. We write

$$\pi_1(M; y, x) = {}_y\mathcal{P}(M)_x/\!\sim$$

for the set of equivalence classes. When the endpoints agree, we will abbreviate this notation to $\pi_1(M, x)$. Note that there is a *reversal of paths* operation

$$\begin{array}{ccc} {}_y\mathcal{P}(M)_x & \longrightarrow & {}_x\mathcal{P}(M)_y \\ \gamma & \longmapsto & \gamma^{-1} \end{array}$$

defined by the formula $\gamma^{-1}(t) = \gamma(1-t)$. Moreover, given a point z in M, there is a *composition of paths* operation

$$\begin{array}{ccc} {}_z\mathcal{P}(M)_y \times {}_y\mathcal{P}(M)_x & \longrightarrow & {}_z\mathcal{P}(M)_x, \\ (\gamma_1, \gamma_2) & \longmapsto & \gamma_1\gamma_2 \end{array}$$

given by first going along γ_2 and then along γ_1. Explicitly, $\gamma_1\gamma_2$ is the path

$$\gamma_1\gamma_2(t) = \begin{cases} \gamma_2(2t), & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \gamma_1(2t-1), & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \tag{3.3}$$

Both the reversal and the composition of paths are compatible with the homotopy equivalence relation, and hence induce operations

$$\pi_1(M; y, x) \longrightarrow \pi_1(M; x, y) \tag{3.4}$$

$$\pi_1(M; z, y) \times \pi_1(M; y, x) \longrightarrow \pi_1(M; z, x) \tag{3.5}$$

on the sets of equivalence classes, which will be called "inverse" and "composition" respectively. It is a simple matter to check that the composition is associative and that the class of the constant path $\gamma(t) = x$ for all $t \in [0,1]$ in $\pi_1(M, x)$ is a neutral element. As such, it will be usually denoted by 1.

If the endpoints are fixed and agree, the above operations endow $\pi_1(M, x)$ with the structure of a group: the *fundamental group* of M. In general, when we allow the endpoints to vary and be distinct, we obtain a groupoid. The definition of such a structure is in fact tailored to study this example.

Definition 3.6. A *groupoid* G is the data of a set G_0 of "objects" and a set G_1 of "arrows", together with the following five operations:

- a *source* map $s\colon G_1 \to G_0$;
- a *target* map $t\colon G_1 \to G_0$;
- a *unit* map $u\colon G_0 \to G_1$ satisfying $s(u(x)) = t(u(x)) = x$ for all $x \in G_0$;
- a *composition* map $m\colon G_1 \,{}_s\!\times_t G_1 \to G_1$ defined on the set
$$G_1 \,{}_s\!\times_t G_1 = \{(f, g) \in G_1 \times G_1 \mid s(f) = t(g)\}$$
such that the equalities $s(m(f,g)) = s(g)$ and $t(m(f,g)) = t(f)$ hold for all arrows $f, g \in G_1$, and that u is a two-sided unit for m. Moreover, the composition is required to be associative;
- an *inverse* map $i\colon G_1 \to G_1$ satisfying $s(i(f)) = t(f)$ and $t(i(f)) = s(f)$ for all arrows $f \in G_1$, and which is a two-sided inverse for the composition.

Equivalently, a groupoid can be viewed as a small category in which all morphisms are isomorphisms (Exercise 3.38).

EXAMPLE 3.7 (The fundamental groupoid). The *fundamental groupoid* of M is the groupoid where G_0 is the set of points of M and G_1 is the set of homotopy classes of paths in M, that is:

$$G_1 = \coprod_{x,y \in M} \pi_1(M; y, x).$$

The source, the target, and the unit are defined in the obvious way, and the inverse and the composition maps are given by (3.4) and (3.5) respectively.

REMARK 3.8. When doing homotopy theory on a differentiable manifold, one can choose to work with continuous, piecewise smooth, or smooth paths. By Whitney's approximation theorem (see *e.g.* [**Lee13**, Thm. 6.19]), each homotopy class of continuous paths admits a smooth representative; the resulting fundamental group or groupoid is hence the same in all three cases. To make the link with differential forms, it is convenient to work with piecewise smooth or smooth paths. However, since the composition of smooth paths as defined in (3.3) is in general only piecewise smooth, we prefer to work with piecewise smooth paths from the beginning to avoid having to replace $\gamma_1\gamma_2$ with a homotopic smooth path.

3.1.2. *Homotopy functionals.* We would like to construct functions on the fundamental groupoid of a manifold.

DEFINITION 3.9. A function on $\mathcal{P}(M)$ is called a *homotopy functional* if the image of every element of $\mathcal{P}(M)$ depends only on its homotopy class, and hence induces a function on $\pi_1(M; y, x)$ for all $x, y \in M$.

The simplest method to construct homotopy functionals is by means of differential forms, as we now recall. Let k be either the field of real numbers or the field of complex numbers. We consider the k-algebra

$$E^*(M, k) = \bigoplus_{p=0}^{\dim M} E^p(M, k)$$

of smooth k-valued differential forms on M, as introduced in Section 2.2.1.

Let $\omega \in E^1(M, k)$ be a differential 1-form and $\gamma \in \mathcal{P}(M)$ a path. Since γ is assumed to be piecewise smooth, we can pullback ω to the interval $[0, 1]$; the pullback takes the form $\gamma^*\omega = f(t)\mathrm{d}t$ for some bounded function f that may be discontinuous at the points at which γ is not smooth. The *line integral* of ω along γ is then defined as

$$\int_\gamma \omega = \int_0^1 \gamma^*\omega = \int_0^1 f(t)\mathrm{d}t. \tag{3.10}$$

Since the integral converges by the assumption on γ, this yields a function

$$\begin{aligned} \textstyle\int \omega \colon\ \mathcal{P}(M) &\longrightarrow k \\ \gamma \ &\longmapsto \textstyle\int_\gamma \omega. \end{aligned}$$

LEMMA 3.11. *The function $\int \omega$ is a homotopy functional if and only if ω is a closed 1-form.*

PROOF. The result follows from Stokes's theorem. We first assume that ω is closed, and that we are given paths γ_1 and γ_2 and a homotopy F between them.

Using the conditions (3.2) in the definition of F, we find

$$\int_{\gamma_1}\omega-\int_{\gamma_2}\omega=\int_{[0,1]}\gamma_1^*\omega-\int_{[0,1]}\gamma_2^*\omega=\int_{\partial[0,1]^2}F^*\omega,$$

where $\partial[0,1]^2$ stands for the boundary of the square $[0,1]^2$. Since F is piecewise smooth, there exists a finite decomposition $[0,1]^2=\bigcup_i C_i$ into polyhedra C_i such that $F|_{C_i}$ is smooth. By Stokes's theorem and the fact that taking pullback by F commutes with the differential, we get

$$\int_{\partial[0,1]^2}F^*\omega=\sum_i\int_{\partial C_i}F^*\omega=\sum_i\int_{C_i}F^*(\mathrm{d}\omega)=0,$$

which proves that the line integral is a homotopy functional.

Conversely, assume that the 1-form ω is *not* closed, *i.e.* $\mathrm{d}\omega\neq 0$. Thanks to Exercise 3.39, we can then find a smooth map $f\colon D\to M$ from the unit disc

$$D=\{(x,y)\in\mathbb{R}^2\mid x^2+y^2\leqslant 1\}$$

to the manifold M satisfying

$$\int_D f^*(\mathrm{d}\omega)\neq 0.$$

On the one hand, the paths from $x=f(1,0)$ to $y=f(-1,0)$ given by

$$\gamma_1(t)=f(\cos(\pi t),\sin(\pi t))\quad\text{and}\quad\gamma_2(t)=f(\cos(\pi t),-\sin(\pi t))$$

are homotopic through the homotopy

$$F(x,y)=f(\cos(\pi x),(1-2y)\sin(\pi x)).$$

On the other hand, another application of Stokes's theorem gives

$$\int_{\gamma_1}\omega-\int_{\gamma_2}\omega=\int_{\partial D}f^*\omega=\int_D f^*(\mathrm{d}\omega)\neq 0,$$

which proves that ω being closed is a necessary condition as well. □

Line integrals of closed 1-forms produce, however, only a very special kind of homotopy functionals. Indeed, from (3.10) we get the relations

$$\int_{\gamma_1\gamma_2}\omega=\int_{\gamma_2}\omega+\int_{\gamma_1}\omega\quad\text{and}\quad\int_{\gamma^{-1}}\omega=-\int_\gamma\omega,$$

which together imply that the equality

$$\int_{\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2}\omega=0\tag{3.12}$$

holds for all loops $\gamma_1,\gamma_2\in\pi_1(M,x)$. From this, it follows that line integrals of closed 1-forms factor through the abelianization of the fundamental group.

Definition 3.13. The *abelianization* of a group G is the quotient

$$G^{\mathrm{ab}}=G/[G,G]$$

of G by the normal subgroup $[G,G]$ generated by the commutators

$$[g,h]=g^{-1}h^{-1}gh\qquad(g,h\in G).$$

The abelianization G^{ab} of G is an abelian group with the universal property that any homomorphism from G to an abelian group factors through G^{ab}. In particular, for every closed 1-form ω the homomorphism

$$\int \omega \colon \pi_1(M,x) \longrightarrow k$$

factors through $\pi_1(M,x)^{\mathrm{ab}}$. Now, viewing a loop $\gamma\colon [0,1] \to M$ as a closed singular 1-chain, as defined in Section 2.1, yields a canonical homomorphism

$$h\colon \pi_1(M,x) \longrightarrow \mathrm{H}_1(M,\mathbb{Z}),$$

which is often called the *Hurewicz map*. The following is a basic result from algebraic topology; see for instance [**Hat02**, Thm. 2A.1] for a proof.

THEOREM 3.14. *The kernel of the homomorphism h consists exactly of the commutator subgroup $[\pi_1(M,x), \pi_1(M,x)]$ of $\pi_1(M,x)$. Moreover, if M is connected, then h is surjective and thus induces an isomorphism*

$$\pi_1(M,x)^{\mathrm{ab}} \simeq \mathrm{H}_1(M,\mathbb{Z}).$$

Summarizing, line integrals of closed 1-forms always factors through the first homology group of the manifold. Since the fundamental group is a finer invariant, we would like to construct other homotopy functionals that are able to detect the extra information carried by $\pi_1(M,x)$.

3.1.3. *Iterated integrals.* The theory of iterated integrals started with the fundamental observation by Chen [**Che77**] that homotopy functionals obtained by successive integration of 1-forms can detect elements of $\pi_1(M,x)$ with trivial homology classes in $\mathrm{H}_1(M,\mathbb{Z})$.

DEFINITION 3.15. Let $\omega_1,\ldots,\omega_r$ be smooth k-valued 1-forms on M. The *iterated integral* of $\omega_1,\ldots,\omega_r$ is the function

$$\begin{array}{rccc} \int \omega_1 \cdots \omega_r \colon & \mathcal{P}(M) & \longrightarrow & k \\ & \gamma & \longmapsto & \int_\gamma \omega_1 \cdots \omega_r \end{array} \tag{3.16}$$

defined by the formula

$$\int_\gamma \omega_1 \cdots \omega_r = \int\limits_{1 \geqslant t_1 \geqslant \cdots \geqslant t_r \geqslant 0} f_1(t_1)\cdots f_r(t_r)\mathrm{d}t_1 \cdots \mathrm{d}t_r,$$

where $\gamma^*\omega_i = f_i(t)\mathrm{d}t$ is the pullback of ω_i to $[0,1]$ along the path γ.

More generally, we will call *iterated integral* every function on $\mathcal{P}(M)$ that can be written as a k-linear combination of (3.16) and the constant function 1, which we view as an iterated integral of *length* 0. We say that an iterated integral has *length* $\leqslant s$ if each summand is of the form $\int \omega_1 \cdots \omega_r$ with $r \leqslant s$.

REMARK 3.17. Here is an explanation of the term "iterated integral" taken from [**Del13**, p. 163]. Let S be the operator that transforms a 1-form η on the interval $[0,1]$ into the function $S[\eta](t) = \int_0^t \eta$. To obtain the iterated integral we apply S to $\gamma^*\omega_r$, then multiply the resulting function by $\gamma^*\omega_{r-1}$, apply S again, multiply by $\gamma^*\omega_{r-2}$, etc., and finally evaluate at $t=1$. That is,

$$\int_\gamma \omega_1 \cdots \omega_r = S[\gamma^*\omega_1 \cdot S[\gamma^*\omega_2 \cdots S[\gamma^*\omega_r]\cdots]](1).$$

Observe that the integral representations of multiple zeta values and polylogarithms from Theorems 1.116 and 1.126 look very similar to iterated integrals but do not quite fit into the framework of Definition 3.15 because the differential forms $\mathrm{d}x/x$ and $\mathrm{d}x/(1-x)$ are singular at the endpoints of the integration path. We will come back to this question in Section 3.8.

3.1.4. *Basic properties of iterated integrals.* The first important property is that iterated integrals are functorial and independent of the parametrization of the path. The proof is left to the reader (see Exercise 3.41).

Proposition 3.18 (Functoriality). *Let $f\colon N \to M$ be a smooth map of differentiable manifolds. For all $\gamma \in \mathcal{P}(N)$ and $\omega_1, \dots, \omega_r \in E^1(M, k)$, the equality*

$$\int_\gamma f^*\omega_1 \cdots f^*\omega_r = \int_{f\circ\gamma} \omega_1 \cdots \omega_r$$

holds. In particular, the iterated integral $\int_\gamma \omega_1 \cdots \omega_r$ does not depend on the choice of parametrization of the path γ.

The basic algebraic properties of iterated integrals are formulas for the reversal and composition of paths, as well as for the product of iterated integrals.

Theorem 3.19. *Let $\omega_1, \dots, \omega_{r+s}$ be smooth k-valued 1-forms on M and $\gamma, \gamma_1, \gamma_2$ be piecewise smooth paths on M satisfying $\gamma_2(1) = \gamma_1(0)$. Then the following holds:*

$$\int_\gamma \omega_1 \cdots \omega_r = (-1)^r \int_{\gamma^{-1}} \omega_r \cdots \omega_1, \tag{3.20}$$

$$\int_{\gamma_1\gamma_2} \omega_1 \cdots \omega_r = \sum_{i=0}^{r} \int_{\gamma_1} \omega_1 \cdots \omega_i \int_{\gamma_2} \omega_{i+1} \cdots \omega_r, \tag{3.21}$$

$$\int_\gamma \omega_1 \cdots \omega_r \int_\gamma \omega_{r+1} \cdots \omega_{r+s} = \sum_{\sigma \in ш(r,s)} \int_\gamma \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(r+s)}. \tag{3.22}$$

In the last identity, the sum runs over the subset $ш(r, s) \subset \mathfrak{S}_{r+s}$ of the symmetric group on $r + s$ elements consisting of shuffles of type (r, s), as in Definition 1.129.

Proof. The identity (3.20) follows from a simple computation, using the fact that the equality $\gamma^*\omega_i = f_i(t)\mathrm{d}t$ implies $(\gamma^{-1})^*\omega_i = -f_i(1-t)\mathrm{d}t$, and hence

$$\begin{aligned}
\int_{\gamma^{-1}} \omega_r \cdots \omega_1 &= (-1)^r \int\limits_{1\geqslant t_1 \geqslant \cdots \geqslant t_r \geqslant 0} f_r(1-t_1) \cdots f_1(1-t_r)\mathrm{d}t_1 \cdots \mathrm{d}t_r \\
&= (-1)^r \int\limits_{1\geqslant u_1 \geqslant \cdots \geqslant u_r \geqslant 0} f_r(u_r) \cdots f_1(u_1)\mathrm{d}u_1 \cdots \mathrm{d}u_r \\
&= (-1)^r \int_\gamma \omega_1 \cdots \omega_r.
\end{aligned}$$

To get the second equality above we made the change of variables $u_i = 1 - t_{r-i+1}$, whose Jacobian has absolute value equal to 1.

We next prove formula (3.21). Writing

$$(\gamma_1\gamma_2)^*\omega_i = f_i(t)\mathrm{d}t, \quad \gamma_1^*\omega_i = g_i(t)\mathrm{d}t, \quad \gamma_2^*\omega_i = h_i(t)\mathrm{d}t,$$

the functions f_i, g_i, h_i are related by

$$f_i(t) = \begin{cases} 2h_i(2t), & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ 2g_i(2t-1), & \text{if } \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \tag{3.23}$$

thanks to the composition rule (3.3). We decompose the domain of integration as a union $\Delta^r = \bigcup_{i=0}^r C_i$, where

$$C_i = \{(t_1, \dots, t_r) \in \mathbb{R}^r \mid 1 \geqslant t_1 \geqslant \cdots \geqslant t_i \geqslant \tfrac{1}{2} \geqslant t_{i+1} \cdots \geqslant t_r \geqslant 0\}.$$

Observe that projecting to the first i and the last $r-i$ coordinates yields an isomorphism $C_i \simeq \Delta^i \times \Delta^{r-i}$. Figure 15 illustrates the case $r = 2$.

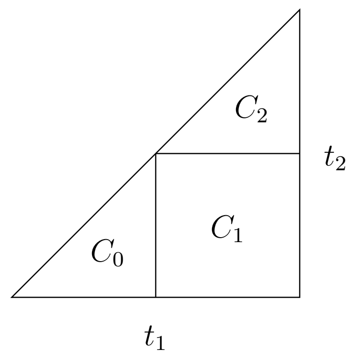


FIGURE 15. The decomposition $\Delta^2 = C_0 \cup C_1 \cup C_2$

Formula (3.21) now follows from the computation

$$\begin{aligned} \int_{C_i} f_1(t_1) \cdots f_r(t_r) \mathrm{d}t_1 \cdots \mathrm{d}t_r &= \int\limits_{\substack{1 \geqslant t_1 \geqslant \ldots \geqslant t_i \geqslant 1/2 \\ 1/2 \geqslant t_{i+1} \geqslant \cdots \geqslant t_r \geqslant 0}} f_1(t_1) \cdots f_r(t_r) \mathrm{d}t_1 \cdots \mathrm{d}t_r \\ &= \frac{2^r}{2^r} \int\limits_{\substack{1 \geqslant u_1 \geqslant \cdots \geqslant u_i \geqslant 0 \\ 1 \geqslant u_{i+1} \geqslant \cdots \geqslant u_r \geqslant 0}} g_1(u_1) \cdots g_i(u_i) h_{i+1}(u_{i+1}) \cdots h_r(u_r) \mathrm{d}u_1 \cdots \mathrm{d}u_r \\ &= \int_{\gamma_1} \omega_1 \cdots \omega_i \int_{\gamma_2} \omega_{i+1} \cdots \omega_r, \end{aligned}$$

together with the fact that the overlaps of the C_i do not contribute to the integral because they all have codimension at least 1, and hence their Lebesgue measure is zero. The second equality is obtained by the change of variables

$$u_j = \begin{cases} 2t_j - 1, & \text{if } j \leqslant i, \\ 2t_j, & \text{if } j > i. \end{cases}$$

The 2^r in the numerator comes from the identity (3.23), whereas the 2^r in the denominator is the Jacobian of the change of variables.

Finally, formula (3.22) is a consequence of the decomposition

$$\varDelta^r \times \varDelta^s = \bigcup_{\sigma \in ш(r,s)} \{(t_1, \dots, t_{r+s}) \mid 1 \geqslant t_{\sigma^{-1}(1)} \geqslant \cdots \geqslant t_{\sigma^{-1}(r+s)} \geqslant 0\},$$

which was already used in the proof of Proposition 1.131. □

3.1.5. *When are iterated integrals homotopy functionals?* We have seen that iterated integrals do not depend on the parametrization of the path (Proposition 3.18). However, even when all the forms ω_i are closed, they do not always give rise to homotopy functionals, as this example borrowed from [**Bro13**] shows:

EXAMPLE 3.24. Take $M = \mathbb{R}^2$ with the standard coordinates x and y. Let $a, b > 0$ be real numbers and consider the path $\gamma_{a,b}\colon [0,1] \to \mathbb{R}^2$ from $(0,0)$ to $(1,1)$ given by $\gamma_{a,b}(t) = (t^a, t^b)$. Let $\omega_1 = \mathrm{d}x$ and $\omega_2 = \mathrm{d}y$. Taking the equalities

$$\gamma_{a,b}^*\omega_1 = at^{a-1}\mathrm{d}t, \qquad \gamma_{a,b}^*\omega_2 = bt^{b-1}\mathrm{d}t$$

into account, the iterated integral of $\omega_1\omega_2$ along $\gamma_{a,b}$ is equal to

$$\int_{\gamma_{a,b}} \omega_1\omega_2 = \int_0^1 \left(at_1^{a-1} \int_0^{t_1} bt_2^{b-1}\mathrm{d}t_2 \right) \mathrm{d}t_1 = \frac{a}{a+b},$$

which obviously depends on the choice of a and b. However, all the paths $\gamma_{a,b}$ are homotopic to each other! This example will be revisited in Exercise 3.42.

A natural question is thus: when is an iterated integral invariant under homotopy? Theorem 3.270 gives a full answer to this question in terms of an algebraic construction called the *bar complex*. For the moment, we content ourselves with a partial answer by linking iterated integrals to connections on trivial bundles through the notion of parallel transport.

3.1.6. *Iterated integrals and connections on trivial bundles.* We continue writing k for either the real or the complex numbers. Let

$$V = k^n \times M$$

be the trivial vector bundle of rank n over M. Since V is trivial, we can identify the space of sections of V with the space of functions $x\colon M \to k^n$. We let $C^\infty(V)$ denote the space of all smooth sections of V.

DEFINITION 3.25. A *connection* on V is a k-linear map

$$\nabla\colon C^\infty(V) \longrightarrow C^\infty(V) \otimes_{C^\infty(M)} E^1(M,k)$$

that satisfies Leibniz's rule

$$\nabla(fx) = x \otimes \mathrm{d}f + f\nabla x$$

for each smooth function $f \in C^\infty(M)$ and each smooth section $x \in C^\infty(V)$.

A connection ∇ on V canonically extends to a k-linear map on the space of $C^\infty(V)$-valued p-forms on M as follows:

$$\begin{array}{ccc} C^\infty(V) \otimes_{C^\infty(M)} E^p(M,k) & \longrightarrow & C^\infty(V) \otimes_{C^\infty(M)} E^{p+1}(M,k). \\ x \otimes \eta & \longmapsto & x \otimes \mathrm{d}\eta + \nabla(x) \wedge \eta \end{array}$$

We will still denote by ∇ this extension.

DEFINITION 3.26. The operator $\nabla^2 = \nabla \circ \nabla$ is called the *curvature* of the connection, and ∇ is said to be *flat* (or integrable) if ∇^2 vanishes.

We call *global canonical frame* of V the tuple $e = (e_1, \dots, e_n)$ consisting of the constant functions $e_i\colon M \to k^n$ whose value is the i-th standard basis vector $(0, \dots, 1, \dots, 0)$ of k^n. By virtue of Leibniz's rule, the connection ∇ is uniquely determined by the image of the global canonical frame. Indeed, write

$$\nabla e_j = \sum_{i=1}^{n} e_i \otimes \eta_{ij} \qquad (j = 1, \ldots, n)$$

with $\eta_{ij} \in E^1(M, k)$. The matrix

$$\omega = (\eta_{ij}) \in E^1(M, k) \otimes_{C^\infty(M)} \operatorname{End}(V) = E^1(M, k) \otimes_k \operatorname{End}(k^n),$$

whose entries are smooth k-valued 1-forms on M, is called the matrix of the connection in the global canonical frame e. Seeing a section $x \colon M \to k^n$ as a column vector of smooth functions and invoking Leibniz's rule again, ∇ is given by

$$\nabla x = \mathrm{d}x + \omega x.$$

Sometimes, we will simply write $\nabla = d + \omega$. From this, one easily finds an expression for the curvature of the connection, namely

$$\begin{aligned}
\nabla^2 x &= \nabla(\mathrm{d}x + \omega x) \\
&= \mathrm{d}^2 x + \mathrm{d}(\omega x) + \omega \mathrm{d}x + \omega \wedge \omega x \\
&= \mathrm{d}^2 x + \mathrm{d}\omega x - \omega \mathrm{d}x + \omega \mathrm{d}x + \omega \wedge \omega x \\
&= (\mathrm{d}\omega + \omega \wedge \omega) x,
\end{aligned}$$

where $\omega \wedge \omega$ stands for the product of matrices of 1-forms induced by the usual wedge product. The sign when we apply Leibniz's rule in the third equality comes from the fact that ω is a 1-form. In explicit terms, writing $\omega = \sum M_i \eta_i$ for some forms $\eta_i \in E^1(M, k)$ and some $M_i \in \mathrm{GL}_n(k)$, this product is equal to

$$\omega \wedge \omega = \frac{1}{2} \sum_{i,j} [M_i, M_j] \eta_i \wedge \eta_j,$$

where we have used that wedge products anti-commute. The matrix

$$R = \mathrm{d}\omega + \omega \wedge \omega$$

is called the *curvature matrix* of the connection ∇.

Besides, associated with the trivial rank n vector bundle V is the bundle

$$\mathrm{GL}(V) \simeq \mathrm{GL}_n(k) \times M$$

with fibers $\mathrm{GL}_n(k)$. The action of the connection ∇ column by column gives then rise to a differential operator

$$\nabla X = \mathrm{d}X + \omega X$$

on the space of smooth functions $X \colon M \to \mathrm{GL}_n(k)$.

3.1.7. *Parallel transport.* Given a smooth path $\gamma \colon [0, 1] \to M$ and a section

$$\begin{aligned}
X \colon\ [0,1] &\longrightarrow \mathrm{GL}_n(k) \times M \\
t &\longmapsto (X(t), \gamma(t))
\end{aligned}$$

of $\mathrm{GL}(V)$ along γ, we say that X is *horizontal* if the vanishing

$$\nabla X(t) = 0 \tag{3.27}$$

holds. This amounts to the equality $\mathrm{d}X(t) = -\gamma^*(\omega) X(t)$. Writing $\gamma^*(\omega) = A(t)\mathrm{d}t$, then (3.27) becomes the linear differential equation

$$X'(t) + A(t) X(t) = 0.$$

The *parallel transport function*

$$T\colon \mathcal{P}(M) \longrightarrow \mathrm{GL}_n(k)$$

associated with the connection ∇ is defined as follows. We first restrict ourselves to smooth paths $\gamma\colon [0,1] \to M$ and set

$$T(\gamma) = X(1),$$

where $X\colon [0,1] \to \mathrm{GL}_n(k)$ is the unique section along the path $\gamma\colon [0,1] \to M$ that is horizontal with respect to ∇ and has initial value $X(0) = \mathrm{Id}_n$, the identity matrix of size n. From the theorems of existence and uniqueness of solutions to linear ordinary differential equations, one also gets (see [**KN96**, Chap. II, § 3]):

PROPOSITION 3.28. *Let γ and γ' be smooth paths in M with $\gamma'(1) = \gamma(0)$. Then the following holds:*

i) *The value $T(\gamma)$ is independent of the parametrization of γ.*
ii) *If $\gamma\gamma'$ is smooth, then the equality $T(\gamma\gamma') = T(\gamma)T(\gamma')$ holds.*

Using Proposition 3.28, we can extend the definition of parallel transport to piecewise smooth paths by reparameterizing them as a finite composition of smooth paths. The equality $T(\gamma\gamma') = T(\gamma)T(\gamma')$ remains true for piecewise smooth paths.

We now have all ingredients to state the main result relating connections and homotopy functionals. Recall that the connection $\nabla = d + \omega$ is flat if the curvature matrix $R = \mathrm{d}\omega + \omega \wedge \omega$ is zero.

THEOREM 3.29. *The connection ∇ is flat if and only if each of the entries of the parallel transport function is a homotopy functional.*

PROOF. Fix a point $x \in M$. The restricted holonomy group of ∇ is the subgroup $\mathrm{Hol}_x^0(\nabla) \subset \mathrm{GL}_n(k)$ consisting of the automorphisms $T(\gamma)$ for all contractible loops γ based at x. It is a connected Lie group, which is reduced to the identity matrix if and only if each of the entries of the paralell transport function is a homotopy functional. According to the Ambrose–Singer theorem [**KN96**, Thm. 8.1], the Lie algebra of $\mathrm{Hol}_x^0(\nabla)$ is the k-vector space generated by the matrices

$$T(\gamma)^{-1} R(\partial_1, \partial_2) T(\gamma),$$

where γ is any path from x to a point y, ∂_1 and ∂_2 are tangent vectors at y, and $R(\partial_1, \partial_2)$ is the result of applying the vectors ∂_1 and ∂_2 to the entries of the matrix of 2-forms R. Therefore, $\mathrm{Hol}_x^0(\nabla)$ is trivial if and only if ∇ is flat. $\square$

3.1.8. *Parallel transport and iterated integrals.* Using iterated integrals, one can give the following explicit formula for the parallel transport function:

PROPOSITION 3.30. *Let $\nabla = \mathrm{d} + \omega$ be a connection on the trivial bundle $k^n \times M \to M$. Then the parallel transport function of ∇ is given by*

$$T(\gamma) = \mathrm{Id}_n - \int_\gamma \omega + \int_\gamma \omega\omega - \int_\gamma \omega\omega\omega + \dots,$$

where the products in the integrands are formal products of matrices of 1-forms and the iterated integrals are computed entry by entry.

Proof. Setting $\gamma^*\omega = A(t)\mathrm{d}t$, the iterated integrals of formal products of matrices of 1-forms are given by

$$\int_\gamma \underbrace{\omega\omega\cdots\omega}_{r} = \int\limits_{1\geqslant t_1\geqslant\cdots\geqslant t_r\geqslant 0} A(t_1)A(t_2)\cdots A(t_r)\mathrm{d}t_1\cdots \mathrm{d}t_r. \tag{3.31}$$

Moreover, the parallel transport function is $T(\gamma) = X(1)$, where $X(t)$ is the unique solution of the differential equation

$$X'(t) + A(t)X(t) = 0 \tag{3.32}$$

with initial condition $X(0) = \mathrm{Id}_n$. Observe that the function $X(t)$ satisfies (3.32) and $X(0) = \mathrm{Id}_n$ if and only if the following integral equation holds

$$X(t) = \mathrm{Id}_n - \int_0^t A(s)X(s)\mathrm{d}s. \tag{3.33}$$

We will solve (3.33) by applying the method of Picard–Lindelöf. For this, we define recursively a sequence of approximations to the solution:

$$\begin{aligned} X_0(t) &= \mathrm{Id}_n, \\ X_r(t) &= \mathrm{Id}_n - \int_0^t A(s)X_{r-1}(s)\mathrm{d}s \qquad (r\geqslant 1). \end{aligned}$$

We need to show that the sequence $(X_r(t))_{r\geqslant 0}$ converges. In order to do so, we first prove by induction that the equality

$$X_r(t) - X_{r-1}(t) = (-1)^r \int\limits_{t\geqslant s_1\geqslant\cdots\geqslant s_r\geqslant 0} A(s_1)\cdots A(s_r)\mathrm{d}s_1\cdots \mathrm{d}s_r \tag{3.34}$$

holds for all $r\geqslant 1$. Indeed, by definition

$$X_1(t) - X_0(t) = -\int_0^t A(s)\mathrm{d}s,$$

which settles the case $r = 1$. Assume then that (3.34) holds for all indices smaller than r. By the induction hypothesis, the equalities

$$\begin{aligned} X_r(t) - X_{r-1}(t) &= -\int_0^t A(s)(X_{r-1}(s) - X_{r-2}(s))\mathrm{d}s \\ &= -\int_0^t A(s)(-1)^{r-1} \int\limits_{s\geqslant s_2\geqslant\cdots\geqslant s_r\geqslant 0} A(s_2)\cdots A(s_r)\mathrm{d}s_2\cdots \mathrm{d}s_r\mathrm{d}s \\ &= (-1)^r \int\limits_{t\geqslant s_1\geqslant\cdots\geqslant s_r\geqslant 0} A(s_1)\cdots A(s_r)\mathrm{d}s_1\cdots \mathrm{d}s_r \end{aligned}$$

hold. Using that the volume of the simplex Δ^r is $1/r!$, we deduce that there exists a constant $K > 0$ satisfying

$$\int\limits_{t\geqslant s_1\geqslant\cdots\geqslant s_r\geqslant 0} A(s_1)\cdots A(s_r)\mathrm{d}s_1\cdots \mathrm{d}s_r = O\left(\frac{K^r}{r!}\right)$$

as r goes to infinity. This estimate proves that $(X_r(t))_{r\geqslant 0}$ is a Cauchy sequence and that its limit is given by the convergent series

$$X_\infty(t) = \sum_{r\geqslant 0}(-1)^r \int\limits_{t\geqslant s_1\geqslant\cdots\geqslant s_r\geqslant 0} A(s_1)\cdots A(s_r)\mathrm{d}s_1\cdots \mathrm{d}s_r.$$

Clearly, $X_\infty(0) = \mathrm{Id}_n$ holds, and a telescopic argument shows that $X_\infty(t)$ satisfies the differential equation (3.32). Therefore, the parallel transport is given by

$$T(\gamma) = X_\infty(1) = \mathrm{Id}_n - \int_\gamma \omega + \int_\gamma \omega\omega - \dots,$$

which is what we wanted to prove. □

The entries of the parallel transport matrix involve an infinite series, and therefore they are not iterated integrals according to our Definition 3.15. On the contrary, if we can ensure that the products appearing in the right-hand side of equation (3.31) vanish for large enough r, then all the entries would be finite sums. One can then combine Theorem 3.29 and Proposition 3.30 to give examples of iterated integrals that are homotopy functionals.

Example 3.35. A strictly upper triangular matrix $A(t)$ is nilpotent, so there exists an integer $r_0 \geqslant 1$ satisfying $A(s_1)\dots A(s_{r_0}) = 0$. In this case, the parallel transport function reduces to the iterated integral

$$T = 1 - \int \omega + \dots + (-1)^{r_0-1}\int \underbrace{\omega\omega\cdots\omega}_{r_0-1}.$$

For instance, in the example of the connection matrix

$$\omega = \begin{pmatrix} 0 & \omega_1 & \omega_{12} \\ 0 & 0 & \omega_2 \\ 0 & 0 & 0 \end{pmatrix},$$

the parallel transport function is given by

$$T = \begin{pmatrix} 1 & -\int\omega_1 & \int\omega_1\omega_2 - \int\omega_{12} \\ 0 & 1 & -\int\omega_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and the curvature of the connection is equal to

$$\mathrm{d}\omega + \omega\wedge\omega = \begin{pmatrix} 0 & \mathrm{d}\omega_1 & \omega_1\wedge\omega_2 + \mathrm{d}\omega_{12} \\ 0 & 0 & \mathrm{d}\omega_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\nabla = \mathrm{d} + \omega$ is flat if and only if the equalities

$$\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0 \quad \text{and} \quad \mathrm{d}\omega_{12} + \omega_1\wedge\omega_2 = 0 \tag{3.36}$$

hold. It follows that the iterated integral $\int\omega_1\omega_2 - \int\omega_{12}$ is a homotopy functional if and only if the conditions (3.36) are satisfied.

More generally, the following result is proved in [**Hai87b**, Prop. 3.1]:

PROPOSITION 3.37. *Let $\omega, \omega_1, \dots, \omega_r$ be smooth k-valued 1-forms on M. Assume that all the forms ω_i are closed. An iterated integral of length two*

$$\sum_{1\leqslant i,j\leqslant r} a_{ij} \int \omega_i\omega_j - \int \omega$$

is a homotopy functional if and only if $\mathrm{d}\omega + \sum_{1\leqslant i,j\leqslant r} a_{ij}\omega_i \wedge \omega_j = 0$ holds.

⋆⋆⋆

EXERCISE 3.38 (Groupoids as categories). Let $\mathcal{C}$ be a small category in which all morphisms are isomorphisms. Show that $\mathcal{C}$ gives rise to a groupoid in the sense of Definition 3.6. Conversely, given a groupoid, construct such a category. Note that groups correspond to the case where the set of objects is a singleton.

EXERCISE 3.39. Let η be a smooth non-zero 2-form on a differential manifold M. The assumption that η is non-zero means that there exists a point $x \in M$ and two tangent vectors $u, v \in T_xM$ with $\eta(u \wedge v) \neq 0$. Use this information to construct a map $f\colon D \to M$ satisfying $\int_D f^*(\eta) \neq 0$, as in the proof of Lemma 3.11.

EXERCISE 3.40 (Integration by parts). Let $\omega_1, \dots, \omega_r$ be smooth k-valued 1-forms on a differentiable manifold M and let f be a smooth function on M. Prove that the equalities

$$\int_\gamma (\mathrm{d}f)\omega_1\cdots\omega_r = (f\circ\gamma)(1)\int_\gamma \omega_1\cdots\omega_r - \int_\gamma (f\omega_1)\omega_2\cdots\omega_r,$$

$$\int_\gamma \omega_1\cdots\omega_{i-1}(\mathrm{d}f)\omega_i\cdots\omega_r = \int_\gamma \omega_1\cdots(f\omega_{i-1})\omega_i\cdots\omega_r - \int_\gamma \omega_1\cdots\omega_{i-1}(f\omega_i)\omega_{i+1}\cdots\omega_r,$$

$$\int_\gamma \omega_1\cdots\omega_r(\mathrm{d}f) = \int_\gamma \omega_1\cdots\omega_{r-1}(f\omega_r) - (f\circ\gamma)(0)\int_\gamma \omega_1\cdots\omega_r,$$

hold for all paths $\gamma \in \mathcal{P}(M)$.

EXERCISE 3.41. Prove Proposition 3.18.

EXERCISE 3.42. Recall from Example 3.24 that the iterated integral of the 1-forms $\omega_1 = \mathrm{d}x$ and $\omega_2 = \mathrm{d}y$ on $\mathbb{R}^2$ is not a homotopy functional. According to Proposition 3.37, this is explained by the fact that $\omega_1 \wedge \omega_2$ does not vanish. Find a 1-form ω_{12} satisfying $\mathrm{d}\omega_{12} + \omega_1 \wedge \omega_2 = 0$ and check that the iterated integral

$$\int \omega_1\omega_2 - \int \omega_{12}\colon \mathcal{P}(\mathbb{R}^2) \longrightarrow \mathbb{R}$$

now takes the same value on all the paths $\gamma_{a,b}$ from Example 3.24.

EXERCISE 3.43 (Another proof of formulas (3.20) and (3.21)). Let $\omega_1, \dots, \omega_r$ be 1-forms on a differentiable manifold M. Consider the connection $\nabla = \mathrm{d} + \omega$ on the trivial vector bundle of rank $r+1$ on M given by the matrix

$$\omega = \begin{pmatrix} 0 & \omega_1 & 0 & \cdots & 0 \\ 0 & 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \omega_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

i) Show that the parallel transport associated with ∇ is the matrix $T = (T_{ij})$ of size $r+1$ with entries

$$T_{ij} = \begin{cases} \int \omega_i \cdots \omega_{j-1}, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ 0, & \text{if } i > j. \end{cases}$$

ii) Using the equality $T(\gamma_1\gamma_2) = T(\gamma_1)T(\gamma_2)$ from Proposition 3.28, deduce from the above computation another proof of the algebraic properties of iterated integrals (3.20) and (3.21).

3.2. Affine group schemes, Lie algebras, and Hopf algebras. In this section, we explain the definition of affine group schemes and of two intimately related algebraic structures: Lie algebras and Hopf algebras. The book [**Wat79**] is an excellent entry point for readers unfamiliar with these notions. A classical reference for Hopf algebras is [**Swe69**], and we also recommend [**Car07**] for a motivated introduction. As in Chapter 2, we assume that the reader is familiar with the relation between affine schemes and commutative algebras as explained, for instance, in [**Har77**, Chap. II, § 2] and very briefly recalled below.

Throughout this section, we fix a field k of characteristic zero (later, in the applications, it will always be equal to $\mathbb{Q}$). All undecorated cartesian and tensor products are assumed to be over k. When we want to emphasize that a group is simply a group and does not carry any additional structure (such as a scheme structure or a topology), we will call it an "abstract group".

3.2.1. *Affine group schemes.* The category of affine schemes over k is equivalent to the category of commutative k-algebras through the contravariant functors

$$A \longmapsto \mathrm{Spec}(A), \qquad X \longmapsto \mathcal{O}(X), \tag{3.44}$$

where $\mathcal{O}(X)$ denotes the ring of regular functions on an affine scheme X.

Definition 3.45. An *affine group scheme* G over k is the data of an affine scheme $G = \mathrm{Spec}(A)$ and of three morphisms of schemes

$$\begin{aligned} \mu &\colon G \times G \longrightarrow G \text{ (product)}, \\ e &\colon \mathrm{Spec}(k) \longrightarrow G \text{ (unit)}, \\ \iota &\colon G \longrightarrow G \text{ (inverse)}, \end{aligned}$$

satisfying the usual axioms in the definition of a group, which are expressed by the commutativity of the following three diagrams:

Associativity:

$$\begin{tikzcd} G\times G\times G \arrow[r, "\mu\times \mathrm{Id}"] \arrow[d, "\mathrm{Id}\times\mu"'] & G\times G \arrow[d, "\mu"] \\ G\times G \arrow[r, "\mu"'] & G. \end{tikzcd}$$

Unit:

$$\begin{tikzcd} G\times \mathrm{Spec}(k) \arrow[r, "\mathrm{Id}\times e"] \arrow[dr, "\mathrm{pr}_1"'] & G\times G \arrow[d, "\mu"] & \mathrm{Spec}(k)\times G \arrow[l, "e\times\mathrm{Id}"'] \arrow[dl, "\mathrm{pr}_2"] \\ & G & \end{tikzcd}$$

Inverse:

$$\begin{tikzcd} & G\times G \arrow[dr, "\mu"] & \\ G \arrow[ur, "\mathrm{Id}\times\iota"] \arrow[r, "\pi"] \arrow[dr, "\iota\times\mathrm{Id}"'] & \mathrm{Spec}(k) \arrow[r, "e"] & G, \\ & G\times G \arrow[ur, "\mu"'] & \end{tikzcd}$$

where π denotes the structural map of G as a k-scheme.

We say that G is *commutative* if the product μ is commutative, which can also be expressed as the commutativity of the diagram

$$\begin{tikzcd} G\times G \arrow[dd, "\tau"'] \arrow[dr, "\mu"] & \\ & G, \\ G\times G \arrow[ur, "\mu"'] & \end{tikzcd}$$

where τ is the map that swaps the factors. If the algebra A is finitely generated, we say that G is an *algebraic affine group scheme* or simply an *affine algebraic group*.

A *morphism of affine group schemes* $f\colon G\to H$ is a morphism of the underlying schemes such that the diagram

$$\begin{tikzcd} G\times G \arrow[r, "f\times f"] \arrow[d, "\mu"'] & H\times H \arrow[d, "\mu"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

commutes. We let $\mathbf{AGS}(k)$ denote the category of affine group schemes over k and $\mathbf{AAGS}(k)$ the full subcategory of $\mathbf{AGS}(k)$ consisting of affine algebraic groups.

REMARK 3.46. In fact, every affine group scheme is a projective limit of algebraic affine groups. What is more, $\mathbf{AGS}(k)$ is equivalent to the category of pro-algebraic affine group schemes; see Lemma 3.51 and Theorem 3.52 below.

3.2.2. *Hopf algebras.* The defining data of an affine group scheme can be transferred to its algebra of regular functions by means of the equivalence (3.44). This gives rise to the concept of Hopf algebra. We begin by explaining the definitions of algebra, coalgebra, bialgebra, and Hopf algebra.

DEFINITION 3.47. Let H be a k-vector space.

i) An *algebra* structure on H is the data of two k-linear morphisms

$$\begin{aligned} \nabla &\colon H \otimes H \longrightarrow H \text{ (product)}, \\ \eta &\colon k \longrightarrow H \text{ (unit)}, \end{aligned}$$

such that the following diagrams commute:

Associativity:

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\nabla\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\nabla"'] & H \otimes H \arrow[d, "\nabla"] \\ H \otimes H \arrow[r, "\nabla"] & H. \end{tikzcd}$$

Unit:

$$\begin{tikzcd} H \otimes k \arrow[r, "\mathrm{Id}\otimes\eta"] \arrow[dr, "\simeq"'] & H \otimes H \arrow[d, "\nabla"] & k \otimes H \arrow[l, "\eta\otimes\mathrm{Id}"'] \arrow[dl, "\simeq"] \\ & H & \end{tikzcd},$$

where the left and right diagonal maps are the canonical isomorphisms $h \otimes \lambda \mapsto \lambda h$ and $\lambda \otimes h \mapsto \lambda h$ respectively.

The algebra structure is said to be *commutative* if the diagram

$$\begin{tikzcd} H \otimes H \arrow[dd, "\tau"'] \arrow[dr, "\nabla"] & \\ & H \\ H \otimes H \arrow[ur, "\nabla"'] & \end{tikzcd}$$

commutes, where $\tau\colon H \otimes H \to H \otimes H$ is the swap of the factors.

ii) The notion of *coalgebra* is dual to that of algebra. That is, a coalgebra structure on H is the data of two k-linear morphisms

$$\begin{aligned} \Delta &\colon H \longrightarrow H \otimes H \text{ (coproduct)}, \\ \epsilon &\colon H \longrightarrow k \text{ (counit)}, \end{aligned}$$

such that the following diagrams commute:

Coassociativity:

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "\mathrm{Id}\otimes\Delta"] \\ H \otimes H \arrow[r, "\Delta\otimes\mathrm{Id}"] & H \otimes H \otimes H. \end{tikzcd}$$

Counit:

$$\begin{tikzcd} H \otimes k & H \otimes H \arrow[l, "\mathrm{Id} \otimes \epsilon"'] \arrow[r, "\epsilon \otimes \mathrm{Id}"] & k \otimes H \\ & H \arrow[ul, "\simeq"] \arrow[u, "\Delta"] \arrow[ur, "\simeq"'] & \end{tikzcd},$$

where the left and right diagonal maps are the canonical isomorphisms $h \mapsto h \otimes 1$ and $h \mapsto 1 \otimes h$ respectively.

The coalgebra is called *cocommutative* if the following diagram commutes:

$$\begin{tikzcd} & H \otimes H \arrow[dd, "\tau"] \\ H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \\ & H \otimes H. \end{tikzcd}$$

iii) A *bialgebra* structure on H is the data of an algebra and a coalgebra structure that are compatible with each other in the sense that the coproduct and the counit are morphisms of algebras and that the product and the unit are morphisms of coalgebras. This amounts to the commutativity of the following four diagrams:

Product and coproduct:

$$\begin{tikzcd} H \otimes H \arrow[r, "\nabla"] \arrow[d, "\Delta \otimes \Delta"'] & H \arrow[r, "\Delta"] & H \otimes H \\ H \otimes H \otimes H \otimes H \arrow[rr, "\mathrm{Id} \otimes \tau \otimes \mathrm{Id}"'] & & H \otimes H \otimes H \otimes H. \arrow[u, "\nabla \otimes \nabla"'] \end{tikzcd}$$

Unit and counit:

$$\begin{tikzcd} k \arrow[rr, "\mathrm{Id}"] \arrow[dr, "\eta"'] & & k. \\ & H \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

Unit and coproduct:

$$\begin{tikzcd} H \arrow[rr, "\Delta"] & & H \otimes H. \\ & k \simeq k \otimes k \arrow[ul, "\eta"] \arrow[ur, "\eta \otimes \eta"'] & \end{tikzcd}$$

Counit and product:

$$\begin{tikzcd} H \otimes H \arrow[rr, "\nabla"] \arrow[dr, "\epsilon \otimes \epsilon"] & & H. \arrow[dl, "\epsilon"'] \\ & k \otimes k \simeq k & \end{tikzcd}$$

iv) A *Hopf algebra* structure on H is the data of a bialgebra structure and a k-linear morphism

$$S\colon H \longrightarrow H \text{ (antipode)}$$

such that the following diagram commutes:
Antipode:

$$\begin{tikzcd} & H \otimes H \arrow[rr, "S\otimes \mathrm{Id}"] & & H \otimes H \arrow[dr, "\nabla"] & \\ H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\eta"] & & H. \\ & H \otimes H \arrow[rr, "\mathrm{Id}\otimes S"'] & & H \otimes H \arrow[ur, "\nabla"'] & \end{tikzcd}$$

v) A bialgebra H is called *commutative* if the product ∇ is commutative, and *cocommutative* if the coproduct Δ is commutative.

REMARK 3.48. The diagram concerning the antipode can be interpreted in terms of a *convolution product* as follows. If H is a bialgebra, then the space of k-linear endomorphisms $\mathrm{End}(H)$ of H is endowed with the product structure in which the product $f * g$ of elements $f, g \in \mathrm{End}(H)$ is given by the composition

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f\otimes g} H \otimes H \xrightarrow{\nabla} H.$$

The associativity of ∇ and the coassociativity of Δ imply that this convolution product is associative. Besides, its unit element is $\eta \circ \epsilon$. In these terms, the commutativity of the antipode diagram is equivalent to asking that S is a two-sided inverse of the identity map Id for the convolution product. In particular, the antipode S is unique provided it exists. However, a bialgebra does not always admit an antipode (see Exercise 3.132 for an example).

There is no need to impose further compatibilities between the antipode and the remaining structures in the definition of a Hopf algebra; they all follow from the axioms. A proof of the next result can be found in [**Swe69**, Prop. 4.0.1].

PROPOSITION 3.49. *Let H be a Hopf algebra.*

i) *The antipode S is an antihomomorphism of algebras. That is,*

$$S \circ \nabla = \nabla \circ \tau \circ (S \otimes S).$$

In particular, if H is commutative, then S is an algebra homomorphism.

ii) *The antipode S is an antihomomorphism of coalgebras. That is,*

$$(S \otimes S) \circ \tau \circ \Delta = \Delta \circ S.$$

If H is cocommutative, then S is a coalgebra homomorphism.

iii) *S respects the unit and the counit. That is,*

$$S \circ \eta = \eta \quad \textit{and} \quad \epsilon \circ S = \epsilon.$$

iv) *If H is commutative or cocommutative, then* $S \circ S = \mathrm{Id}$.

Given a commutative Hopf algebra A, we can use the algebra structure to define an affine scheme $\operatorname{Spec}(A)$. Then the coproduct, counit, and antipode of A give rise to the dual notions of product, unit, and inverse on $\operatorname{Spec}(A)$. From this, we immediately derive the following result:

PROPOSITION 3.50. *The assignment $A \mapsto \operatorname{Spec}(A)$ is a contravariant equivalence between the category of commutative Hopf k-algebras and the category of affine group schemes over k. A quasi-inverse equivalence is given by $G \mapsto \mathcal{O}(G)$. Moreover, the affine group scheme G is commutative if and only if the Hopf algebra $\mathcal{O}(G)$ is cocommutative.*

3.2.3. *Affine group schemes are pro-algebraic.* By way of illustration, we show how to use this correspondence to prove the promised result that affine group schemes are pro-algebraic.

LEMMA 3.51. *Every Hopf algebra is the inductive limit of its Hopf subalgebras that are finitely generated as k-algebras. Therefore, every affine group scheme is a projective limit of algebraic affine group schemes.*

PROOF. Let H be a Hopf algebra. It suffices to show that every $x \in H$ is contained in a finitely generated Hopf subalgebra of H. Choose a basis $\{h_i\}$ (maybe infinite and even uncountable) of H and write $\Delta(x) = \sum_i x_i \otimes h_i$, where only finitely many x_i are non-zero. Let $V \subseteq H$ be the vector subspace spanned by x and the x_i. We claim that there is an inclusion $\Delta(V) \subseteq V \otimes H$, which amounts of course to saying that $\Delta(x_i)$ belongs to $V \otimes H$ for all i. Indeed, writing $\Delta(h_i) = \sum_{j,\ell} a_{ij\ell} h_j \otimes h_\ell$ with $a_{ij\ell} \in k$, the equalities

$$\begin{aligned}\sum_i \Delta(x_i) \otimes h_i &= (\Delta \otimes \mathrm{Id})\Delta(x) \\ &= (\mathrm{Id} \otimes \Delta)\Delta(x) \\ &= \sum_{i,j,\ell} x_i \otimes a_{ij\ell} h_j \otimes h_\ell\end{aligned}$$

hold by the associativity of the coproduct. Comparing the coefficients of h_ℓ yields

$$\Delta(x_\ell) = \sum_{i,j} x_i \otimes a_{ij\ell} h_j \in V \otimes H,$$

as we wanted. Now let $\{v_i\}$ be a basis of V and write $\Delta(v_j) = \sum_i v_i \otimes h_{ij}$ for some elements $h_{ij} \in H$. By Exercise 3.131, it follows that $\Delta(h_{ij}) = \sum_\ell h_{i\ell} \otimes h_{\ell j}$, and hence the vector space U generated by $\{v_i\}$ and $\{h_{ij}\}$ satisfies $\Delta(U) \subseteq U \otimes U$. If W is the vector space spanned by U and $S(U)$, then $\Delta(W) \subseteq W \otimes W$ and $S(W) \subseteq W$ hold using Exercise 3.131 again. Finally, let A be the subalgebra of H generated by W. Since Δ and S are morphisms of algebras, there are inclusions $\Delta(A) \subseteq A \otimes A$ and $S(A) \subseteq A$, and A is a finitely generated Hopf subalgebra of H containing x. $\square$

Not only every affine group scheme is pro-algebraic, but the pro-algebraic structure is, in some sense, unique. This is the content of the following result. The pro-category and the notion of a compact object are reviewed in Section A.6.3.

THEOREM 3.52. *The functor $\operatorname{Pro}(\mathbf{AAGS}(k)) \to \mathbf{AGS}(k)$ given by*

$$(G_d)_{d \in D} \longmapsto \varprojlim_{d \in D} G_d$$

is an equivalence of categories.

Proof. In view of Lemma 3.51 and Theorem A.175 from the appendix, it suffices to show that $\mathbf{AAGS}(k)$ is the full subcategory of $\mathbf{AGS}(k)$ consisting of cocompact objects. By duality, this amounts to proving that the compact objects of the category of commutative Hopf algebras are the finitely generated algebras.

Let H be such a Hopf algebra. To prove that H is a compact object, we need to check that the canonical map

$$\varinjlim_{d\in D} \mathrm{Hom}(H, B_d) \longrightarrow \mathrm{Hom}(H, \varinjlim_{d\in D} B_d) \tag{3.53}$$

is a bijection for each inductive system of Hopf algebras $(B_d)_{d\in D}$. Let us first prove injectivity. Given $f \in \varinjlim_{d\in D} \mathrm{Hom}(H, B_d)$, there is an object d_0 of D and a morphism $f_{d_0}\colon H \to B_{d_0}$ whose image in the limit is f. For each $d_0 \to d$, we write $f_{d_0\to d}$ for the composition

$$H \longrightarrow B_{d_0} \longrightarrow B_d.$$

The image of f by the map (3.53) is the composition

$$H \longrightarrow B_{d_0} \longrightarrow \varinjlim_{d\in D} B_d.$$

Let $a_1, \dots, a_n$ be a set of generators of H. If the image of f under (3.53) is zero, then for each $i = 1, \dots, n$, there is an arrow $d_0 \to d_i$ such that $f_{d_0\to d_i}(a_i) = 0$. Taking a d' that receives arrows $d_i \to d'$, for $i = 1, \dots, n$, then $f_{d_0\to d'} = 0$, which implies that $f = 0$. Hence, the map (3.53) is injective.

To prove surjectivity, we use that the k-algebra H is noetherian, being finitely generated. There is hence an exact sequence of H-modules

$$\begin{array}{c} 0 \longrightarrow I \longrightarrow k[x_1, \dots, x_n] \longrightarrow H \longrightarrow 0, \\ \qquad\qquad\qquad x_i \longmapsto a_i \end{array}$$

where I is a finitely generated ideal. Let $f \in \mathrm{Hom}(H, \varinjlim_{d\in D} B_d)$. There is an object $d \in D$ such that, for $i = 1, \dots, n$, the element $f(a_i)$ belongs to $\mathrm{Im}(B_d)$. Choosing representatives in B_d, we construct a map $\widetilde{f}_d$ that makes the diagram

$$\begin{tikzcd} k[x_1, \dots, x_n] \arrow[r] \arrow[d, "\widetilde{f}_d"'] & H \arrow[d, "f"] \\ B_d \arrow[r] & \varinjlim_{d\in D} B_d \end{tikzcd}$$

commute. Since $f(I) = 0$ and I is finitely generated, there is an arrow $d \to d'$ such that I is mapped to zero by the composition

$$k[x_1, \dots, x_n] \xrightarrow{\widetilde{f}_d} B_d \longrightarrow B_{d'}.$$

The resulting map $f_{d'}\colon H \to B_{d'}$ yields an element $f' \in \varinjlim_{d\in D}(\mathrm{Hom}(H, B_d))$. By construction, this element is in the preimage of f, so the map (3.53) is surjective.

It remains to prove that, if $\mathrm{Hom}(H, -)$ commutes with direct limits, then H is a finitely generated algebra. As in the proof of Lemma 3.51, we write H as a limit

$$H = \varinjlim_{d\in D} H_d$$

of Hopf algebras H_d that are finitely generated as algebras. Consider the map

$$\mathrm{Id}_H \in \mathrm{Hom}(H,H) = \mathrm{Hom}(H, \varinjlim_{d\in D} H_d).$$

Since we are assuming that $\mathrm{Hom}(H,-)$ commutes with direct limits, the map (3.53) is an isomorphism. Therefore, there is a $d \in D$ and a map $H \to H_d$ such that the composition $H \to H_d \to H$ is Id_H. The map $H_d \to H$ is hence surjective and we deduce that H is finitely generated as an algebra, which finishes the proof. □

3.2.4. *Comodules and Hopf modules.*

Definition 3.54. Let (H,Δ,ϵ) be a coalgebra over k. A *right comodule* over H is the data of a k-vector space V and a k-linear map

$$\Delta\colon V \longrightarrow V \otimes H$$

called the *coaction* of H on V, such that the following diagrams commute:

Associativity:

$$\begin{tikzcd} V \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & V \otimes H, \arrow[d, "\mathrm{Id}\otimes\Delta"] \\ V \otimes H \arrow[r, "\Delta\otimes\mathrm{Id}"] & V \otimes H \otimes H. \end{tikzcd}$$

Compatibility with the counit:

$$\begin{tikzcd} V \arrow[r, "\Delta"] \arrow[dr, "\simeq"'] & V \otimes H \arrow[d, "\mathrm{Id}\otimes\epsilon"] \\ & V \otimes k \end{tikzcd}$$

Left comodules over H are defined similarly.

Examples 3.55. The following are examples of comodules.

i) Every coalgebra is a (right and left) comodule over itself.
ii) Let A be a finite-dimensional k-algebra and M a finite-dimensional (left or right) A-module. Then the dual space

$$A^\vee = \mathrm{Hom}(A,k)$$

is a coalgebra, and $M^\vee$ is a comodule over $A^\vee$.
iii) Consider the k-vector space

$$H = \bigoplus_{n\in\mathbb{Z}} k e_n$$

with the counit $\epsilon(e_n) = 1$ for all n and the coproduct

$$\Delta(e_n) = e_n \otimes e_n.$$

Then H is a coalgebra and every graded vector space $V^* = \bigoplus_{n\in\mathbb{Z}} V^n$ is a right comodule over H with coaction

$$\Delta v = v \otimes e_n,$$

for $v \in V^n$ (Exercise 3.130). Similarly, V^* can be viewed as a left comodule because the Hopf algebra H is cocommutative.

Definition 3.56. Let H be a commutative Hopf algebra. A *left Hopf module* is a vector space V that is a module over the algebra structure of H and a left comodule over its coalgebra structure (Definition 3.54). Moreover, both structures are required to be compatible in the sense that the equality

$$\Delta(hv) = \Delta(h)\Delta(v)$$

holds for all $h \in H$ and for all $v \in V$.

Example 3.57. In the same way that every affine group scheme gives rise to a Hopf algebra, every left action of an affine group scheme on an affine scheme gives rise to a left comodule. Namely, let G be an affine group scheme, X an affine scheme, and $\mu\colon G \times X \to X$ a left action of G on X. Then the dual μ is a map

$$\Delta\colon \mathcal{O}(X) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(X)$$

of μ is a coaction that turns $\mathcal{O}(X)$ into a left $\mathcal{O}(G)$-module. Similarly, a right action of G on X gives rise to a right $\mathcal{O}(G)$-module structure on $\mathcal{O}(X)$.

3.2.5. *Graded Hopf algebras.*

Definition 3.58.

i) A bialgebra H is said to be *graded* if the underlying k-vector space has a direct sum decomposition

$$H = \bigoplus_{n \in \mathbb{Z}} H_n$$

compatible with the operations in the sense that the inclusions

$$\nabla(H_p \otimes H_q) \subseteq H_{p+q} \quad \text{and} \quad \Delta H_n \subseteq \bigoplus_{i+j=n} H_i \otimes H_j$$

hold for all $p, q, n \geqslant 0$. If, moreover, $H_n = \{0\}$ for $n < 0$ and $H_0 = k$ we say that H is *connected*.

ii) A *graded Hopf algebra* is a Hopf algebra such that the underlying bialgebra is graded and the antipode satisfies $S(H_n) \subseteq H_n$.

One advantage of working with graded connected bialgebras is that they admit a unique antipode turning them into (graded) Hopf algebras (see Exercise 3.133).

3.2.6. *Examples.* In this paragraph, we give a few examples of affine group schemes and their corresponding Hopf algebras. Of particular interest for the sequel is the Hoffman algebra from Example 3.64.

In order to define affine group schemes, the language of representable functors will be useful. An affine group scheme over k defines a functor from the category of commutative k-algebras to the category of abstract groups: given $G = \mathrm{Spec}(A)$ as in Definition 3.45, the set-valued functor

$$R \longmapsto G(R) = \mathrm{Hom}_{k\text{-alg}}(A, R)$$

takes values in the category of groups, since the structure of a group scheme of G endows $G(R)$ with a group structure for all R.

Definition 3.59. We say that a functor F from commutative k-algebras to groups is *representable* if there exist an affine group scheme G and a natural isomorphism of functors between F and G.

EXAMPLES 3.60.

i) The *trivial* group scheme is $\operatorname{Spec}(k)$ with all operations equal to the identity. The corresponding commutative Hopf algebra is k with all operations equal to the identity, once $k \otimes k$ is identified with k.

ii) The *multiplicative group* $\mathbb{G}_m$. The functor from commutative k-algebras to groups given by $R \mapsto (R^\times, \times)$ is represented by an affine group scheme $\mathbb{G}_m$. The corresponding Hopf algebra is $k[x, x^{-1}]$, together with the coproduct uniquely determined by the formulas

$$\Delta(x) = x \otimes x, \quad \Delta(x^{-1}) = x^{-1} \otimes x^{-1},$$

the counit $\epsilon(x) = \epsilon(x^{-1}) = 1$, and the antipode determined by $S(x) = x^{-1}$ and $S(x^{-1}) = x$.

iii) The *additive group* $\mathbb{G}_a$. The functor from commutative k-algebras to groups given by $R \mapsto (R, +)$ is represented by an affine group scheme $\mathbb{G}_a$. The corresponding Hopf algebra is $k[x]$ with the coproduct, the counit, and the antipode being the only algebra morphisms satisfying

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \epsilon(x) = 0, \quad S(x) = -x.$$

iv) More generally, if V is a finite-dimensional k-vector space, then the functor $R \mapsto (V \otimes R, +)$ is representable. The corresponding Hopf algebra is the symmetric algebra $\operatorname{Sym}(V^\vee)$, which is the free associative and commutative k-algebra generated by $V^\vee$. Generalizing the previous example, the coproduct, the counit, and the antipode in $\operatorname{Sym}(V^\vee)$ are the algebra morphisms determined by $\Delta v = 1 \otimes v + v \otimes 1$, $\epsilon(v) = 0$, and $S(v) = -v$ respectively, for all $v \in V^\vee$.

v) When V is infinite-dimensional, the functor $R \mapsto V \otimes R$ is not representable (see Exercise 3.135). Nevertheless, for pro-finite-dimensional vector spaces, one can find a replacement as follows. Let V be a k-vector space that can be written as

$$V = \varprojlim_{\alpha} V_\alpha$$

with every V_α finite-dimensional. Then the functor

$$R \longmapsto \varprojlim_{\alpha} (V_\alpha \otimes R)$$

is representable by an affine group scheme whose Hopf algebra is

$$A = \varinjlim_{\alpha} \operatorname{Sym}(V_\alpha^\vee).$$

vi) The *linear group* GL_n. The functor that maps a commutative k-algebra R to the group $\mathrm{GL}_n(R)$ of invertible n by n matrices with entries in R is representable by an affine group scheme GL_n. Its Hopf algebra is

$$k[t, (x_{ij})_{i,j=1,\dots,n}]/(t \det(x_{ij}) - 1).$$

Recall that this means that the determinant $\det(x_{ij})$, which is a homogeneous polynomial of degree n in the entries x_{ij}, is invertible. Its inverse is the variable t. The coproduct is given by

$$\Delta t = t \otimes t, \qquad \Delta x_{ij} = \sum_{l=1}^{n} x_{il} \otimes x_{lj}. \tag{3.61}$$

The counit is the map

$$\epsilon(x_{ij}) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Finally, the antipode can be expressed using Cramer's rule for the inverse of a matrix in terms of cofactors, that is,

$$S(t) = t^{-1}, \qquad S(x_{ij}) = tC_{ji},$$

where C_{ij} is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the i-th row and the j-th column of $(x_{\ell m})_{\ell,m}$. Observe that C_{ij} is a homogeneous polynomial of degree $n-1$.

vii) Similarly, for every finite-dimensional k-vector space V, the functor

$$R \longmapsto \mathrm{Aut}_R(R \otimes V)$$

that with a k-algebra R associates the set of R-linear automorphisms of the R-module $R\otimes V$ is representable by an algebraic affine k-group scheme $\mathrm{GL}(V)$. If V has dimension n, and the choice of a basis of V induces an isomorphism between $\mathrm{GL}(V)$ and GL_n.

viii) Again, one needs to be cautious when working with infinite-dimensional vector spaces. In fact, given a k-vector space V, the functor

$$R \longmapsto \mathrm{Aut}_R(R \otimes V)$$

is representable by an affine group k-scheme if and only if V is finite-dimensional (see Exercise 3.136). Observe that the rule (3.61) from Example vi) above does not define a coproduct in the infinite-dimensional case since the sum appearing in the right-hand side will be infinite.

EXAMPLE 3.62. Further examples of affine group schemes arise from Zariski closed subsets of GL_n that are stable under matrix multiplication and matrix inversion, and contain the identity matrix, namely *classical algebraic groups* such as

i) the special linear group

$$\mathrm{SL}_n = \mathrm{Spec}\bigg(k[(x_{ij})_{i,j=1,\dots,n}]/(\det(x_{ij})-1)\bigg),$$

which represents the functor that sends a k-algebra R to the group $\mathrm{SL}_n(R)$ of $n\times n$ matrices with entries in R and determinant equal to 1;

ii) the group of unipotent matrices

$$\mathrm{Up}_n = \mathrm{Spec}\bigg(k[(x_{ij})_{i,j=1,\dots,n}]/((x_{ii}-1)_i,(x_{ij})_{i<j})\bigg),$$

which represents the functor that sends a k-algebra R to the group $\mathrm{Up}_n(R)$ of $n\times n$ upper triangular matrices with entries in R and all diagonal entries equal to 1;

iii) the orthogonal group

$$\mathrm{O}_n = \mathrm{Spec}\bigg(k[(x_{ij})_{i,j=1,\dots,n}]/(\sum_j x_{ji}x_{j\ell} - \delta_{i\ell})_{i,\ell=1,\dots,n}\bigg),$$

which represents the functor that sends a k-algebra R to the group $\mathrm{O}_n(R)$ of $n\times n$ matrices M with entries in R and $M^tM = 1$ (see Exercise 3.139).

As we will prove later (Corollary 3.125), every affine algebraic group is *linear*, *i.e.* can be seen as a Zariski closed subset of some GL_n.

Example 3.63. Let Γ be an abstract group. The group algebra

$$k[\Gamma] = \{\sum_{g\in\Gamma} a_g g \mid a_g \in k,\ a_g = 0 \text{ except for finitely many } g\}$$

carries the structure of a Hopf algebra. The product is determined by the group law of Γ, as follows:

$$\sum_{g\in\Gamma} a_g g \cdot \sum_{h\in\Gamma} b_h h = \sum_{g,h\in\Gamma} a_g b_h gh = \sum_{f\in\Gamma}\left(\sum_{g\in\Gamma} a_g b_{g^{-1}f}\right) f.$$

The unit is the k-linear map $k \to k[\Gamma]$ that sends 1 to the neutral element. The coproduct and the antipode on $k[\Gamma]$ are the k-linear extensions of the maps given by $\Delta g = g \otimes g$ and $S(g) = g^{-1}$ respectively, and the counit sends $\sum a_g g$ to $\sum a_g$. This Hopf algebra is cocommutative but not commutative, unless Γ is abelian.

Example 3.64. For the purpose of these notes, the main example will be the *Hoffman algebra* $\mathfrak{H}$ of Section 1.6. Recall that the underlying vector space of $\mathfrak{H}$ is the vector space $\mathbb{Q}\langle X\rangle$ generated by (non-commutative) words in two letters x_0, x_1. The Hopf algebra structure is given by

Shuffle product:

$$x_{\varepsilon_1}\cdots x_{\varepsilon_r} ⧢ x_{\varepsilon_{r+1}}\cdots x_{\varepsilon_{r+s}} = \sum_{\sigma\in ⧢(r,s)} x_{\varepsilon_{\sigma^{-1}(1)}}\cdots x_{\varepsilon_{\sigma^{-1}(p+q)}}.$$

Unit: The map $\eta\colon \mathbb{Q} \to \mathfrak{H}$ that sends 1 to the empty word.

Deconcatenation coproduct:

$$\Delta x_{\varepsilon_1}\cdots x_{\varepsilon_n} = \sum_{j=0}^{n} x_{\varepsilon_1}\cdots x_{\varepsilon_j} \otimes x_{\varepsilon_{j+1}}\cdots x_{\varepsilon_n}.$$

Counit: The map $\epsilon\colon \mathfrak{H} \to \mathbb{Q}$ that sends every non-empty word to 0 and the empty word to 1.

Antipode:

$$S(x_{\varepsilon_1}\cdots x_{\varepsilon_n}) = (-1)^n x_{\varepsilon_n}\cdots x_{\varepsilon_1}$$

For convenience, if w is a word on the letters x_0 and x_1, we will also use the notation

$$w^* = S(w). \tag{3.65}$$

Consider the grading of $\mathfrak{H}$ that gives *weight* n to $x_{\varepsilon_1}\cdots x_{\varepsilon_n}$. Since all the above operations respect the weight, $\mathfrak{H}$ is a graded Hopf algebra. Moreover, it is connected.

3.2.7. *The dual of a Hopf algebra.* Let H be a Hopf algebra over k. If H is a finite-dimensional k-vector space, then its dual

$$H^\vee = \mathrm{Hom}(H, k)$$

is again equipped with a Hopf algebra structure. The product of $H^\vee$ is the dual of the coproduct of H, the coproduct is the dual of the product, and similarly for the antipode. In other words, the axioms in Definition 3.47 are *self-dual*. This relies on the fact that the canonical morphism

$$H^\vee \otimes H^\vee \longrightarrow (H \otimes H)^\vee \tag{3.66}$$

is an isomorphism. If H is infinite-dimensional, then the morphism (3.66) fails to be an isomorphism, and hence the dual of the product does not give rise to a coproduct but only to what is called a *completed coproduct*. Let us explain why. Let V be an infinite-dimensional k-vector space and write

$$V = \varinjlim_I V_I,$$

where I runs over the directed set of finite-dimensional subspaces of V. Since the functor $\mathrm{Hom}(\cdot, k)$ exchanges inductive and projective limits, the dual of V equals

$$V^\vee = \mathrm{Hom}(V,k) = \mathrm{Hom}(\varinjlim_I V_I, k) = \varprojlim_I \mathrm{Hom}(V_I,k) = \varprojlim_I V_I^\vee. \tag{3.67}$$

Thus, $V^\vee$ has a natural structure of pro-finite-dimensional k-vector space.

Definition 3.68. Given a pro-finite-dimensional k-vector space

$$W = \varprojlim_I W_I,$$

the *completed tensor product* of W with itself is defined as

$$W \widehat{\otimes} W = \varprojlim_I (W_I \otimes W_I).$$

Note that the definition requires a structure of pro-finite-dimensional space. When dealing with the dual of an infinite-dimensional vector space V, we will tacitly assume that $V^\vee$ is endowed with the structure (3.67).

There are canonical maps

$$V^\vee \otimes V^\vee \longrightarrow (V \otimes V)^\vee \longrightarrow V^\vee \widehat{\otimes} V^\vee,$$

which are not isomorphisms in general. Hence, the dual of the product $A \otimes A \to A$ of an algebra gives only rise to a *completed coproduct*

$$A^\vee \longrightarrow (A \otimes A)^\vee \longrightarrow A^\vee \widehat{\otimes} A^\vee \tag{3.69}$$

and not necessarily to a coproduct $A^\vee \to A^\vee \otimes A^\vee$.

Definition 3.70. A *completed Hopf algebra* A is a pro-finite-dimensional vector space satisfying the analogous properties of a Hopf algebra (Definition 3.47), where all tensor products are replaced by completed tensor products and all the maps are compatible with the pro-finite-dimensional structure. In particular, a completed Hopf algebra has a completed coproduct

$$\Delta \colon A \longrightarrow A \widehat{\otimes} A.$$

Moreover, the algebra product $\nabla \colon A \otimes A \longrightarrow A$ factors through a *completed product*

$$A \otimes A \longrightarrow A \widehat{\otimes} A \xrightarrow{\widehat{\nabla}} A,$$

and the antipode S is compatible with the pro-finite-dimensional structure.

The *dual* of an infinite-dimensional Hopf algebra is a completed Hopf algebra. In practice, we will mostly consider the dual of connected graded Hopf algebra with finite-dimensional graded pieces, in which case the completed Hopf algebra can be conveniently written in terms of the topology induced by the augmentation ideal.

EXAMPLE 3.71. Let $A = k[x]$ be the Hopf algebra from Example 3.60 iii). We denote by $\langle \cdot, \cdot \rangle$ the pairing between a vector space and its dual. Let $y_m \in A^\vee$ be the element determined by $\langle y_m, x^n \rangle = \delta_{n,m}$. As a k-vector space, we have

$$A^\vee = \prod_{n \geqslant 0} k y_n.$$

Although $A^\vee$ is a product space, we will use additive notation and write $(a_n)_{n\geqslant 0}$ as a formal infinite sum $\sum_{n\geqslant 0} a_n y_n$. To determine the algebra structure, we compute

$$\begin{aligned}\langle \nabla(y_m \otimes y_n), x^j \rangle &= \langle y_m \otimes y_n, \Delta x^j \rangle \\ &= \langle y_m \otimes y_n, (1 \otimes x + x \otimes 1)^j \rangle \\ &= \begin{cases} \frac{(m+n)!}{n!m!}, & \text{if } j = n+m, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Therefore, we get an equality

$$y_n \cdot y_m = \frac{(m+n)!}{n!m!} y_{n+m}.$$

From this equation we deduce that $y_m = y_1^m/m!$ and that $A^\vee$ is the algebra of formal series on divided powers. Since we are working over a field of characteristic zero, $A^\vee$ is isomorphic to the algebra of formal power series. Thus, writing $y = y_1$, there is an isomorphism of algebras

$$A^\vee = k[\![y]\!].$$

The completed coproduct is determined by $\Delta y = 1 \otimes y + y \otimes 1$, the dual antipode by $S(y) = -y$, the unit by $\eta(1) = 1$ and the counit by $\epsilon(y^n) = \delta_{0,n}$. In particular,

$$\Delta y_m = \sum_{j=0}^{m} y_j \otimes y_{m-j}, \qquad S(y_m) = -y_m.$$

The completed coproduct cannot be factored through a true coproduct. Indeed, consider the element $\eta = \sum_{n\geqslant 0} n y_n$, so that

$$\Delta \eta = \sum_{n\geqslant 0} \sum_{j=0}^{n} n y_j \otimes y_{n-j}.$$

This element does not belong to $A^\vee \otimes A^\vee$, since an element

$$\sum_{i,j\geqslant 0} a_{i,j} y_i \otimes y_j \in A^\vee \otimes A^\vee$$

has the property that the rank of the matrix $(a_{i,j})$ is finite. By contrast, the rank of the matrix $(b_{i,j})$ with $b_{i,j} = i + j$ is not bounded.

EXAMPLE 3.72. Let $A = k[x, x^{-1}]$ be the Hopf algebra from Example 3.60 ii). As in the previous example, we denote by $\langle \cdot, \cdot \rangle$ the pairing between a vector space and its dual, and we let $y_m \in A^\vee$ be the element determined by $\langle y_m, x^n \rangle = \delta_{n,m}$. As a k-vector space, the dual $A^\vee$ is given by

$$A^\vee = \prod_{n \in \mathbb{Z}} k y_n.$$

As in the previous example, we will use additive notation and formal infinite sums. To determine the algebra structure, we compute

$$\begin{aligned}\langle \nabla(y_m \otimes y_n), x^j \rangle &= \langle y_m \otimes y_n, \Delta x^j \rangle \\ &= \langle y_m \otimes y_n, x^j \otimes x^j \rangle \\ &= \begin{cases} 1, & \text{if } j = n = m, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

This gives the formula

$$y_n \cdot y_m = \begin{cases} y_n, & \text{if } n = m, \\ 0, & \text{otherwise.} \end{cases} \tag{3.73}$$

Hence, the elements y_n are mutually orthogonal projectors. It is easy to check that the completed coproduct is given by

$$\Delta y_n = \sum_{a+b=n} y_a \otimes y_b,$$

which is not an element of $A^\vee \otimes A^\vee$ since the sum is infinite. This completed coproduct is not a true coproduct either. The dual antipode is given by $S(y_n) = y_{-n}$, the counit by $\epsilon(y_n) = \delta_{0,n}$ and the unit by

$$\eta(1) = \sum_{n \in \mathbb{Z}} y_n.$$

It is amusing to observe that $\operatorname{Spec}(A^\vee)(k)$ is isomorphic to $\mathbb{Z}$, realizing in some sense the duality between $\mathbb{G}_m$ and $\mathbb{Z}$ given by characters. The completed coproduct in $A^\vee$ is compatible with the addition in $\mathbb{Z}$, but even if $\operatorname{Spec}(A^\vee)$ is a scheme over k, it is not a group scheme, because the completed coproduct is not a true coproduct.

EXAMPLE 3.74. The dual of the Hoffman Hopf algebra $\mathfrak{H}$ from Example 3.64 is the space $\mathfrak{H}^\vee = \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$ of series on the non-commutative words in two letters e_0 and e_1. Given a binary sequence α and an element $\gamma \in \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$, the duality is given by the pairing

$$\langle x_\alpha, \gamma \rangle = \text{coefficient of } e_\alpha \text{ in } \gamma.$$

This duality and the Hopf algebra structure of $\mathfrak{H}$ endows $\mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$ with the following structures:

Concatenation product: The product $\Delta^\vee \colon \mathfrak{H}^\vee \otimes \mathfrak{H}^\vee \to \mathfrak{H}^\vee$ is given by

$$e_{\varepsilon_1} \cdots e_{\varepsilon_r} \cdot e_{\varepsilon_{r+1}} \cdots e_{\varepsilon_{r+s}} = e_{\varepsilon_1} \cdots e_{\varepsilon_{r+s}}.$$

Unit: It is the morphism

$$\epsilon^\vee \colon \mathbb{Q} \longrightarrow \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$$

that sends 1 to the empty word.

Completed coproduct: It is the unique morphism of algebras

$$\nabla^\vee \colon \mathfrak{H}^\vee \longrightarrow \mathfrak{H}^\vee \widehat{\otimes} \mathfrak{H}^\vee$$

such that the equality

$$\nabla^\vee e_\varepsilon = 1 \otimes e_\varepsilon + e_\varepsilon \otimes 1$$

holds for $\varepsilon = 0, 1$. This implies the equality

$$\nabla^\vee w = \sum_{w_1, w_2} ш(w_1, w_2; w) w_1 \otimes w_2,$$

for any word w on the alphabet $\{e_0, e_1\}$, where $ш(w_1, w_2; w)$ is the shuffle multiplicity introduced in Notation 1.162.

Counit: The map

$$\eta^\vee \colon \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle \longrightarrow \mathbb{Q}$$

sending all non-empty words to 0 and the empty word to 1.

Dual antipode: It is given by

$$S^\vee(e_{\varepsilon_1} \cdots e_{\varepsilon_n}) = (-1)^n e_{\varepsilon_n} \cdots e_{\varepsilon_1}.$$

By analogy with (3.65), for a word w in the letters e_0 and e_1, we use the notation

$$w^* = S^\vee(w).$$

3.2.8. *Hopf ideals, quotients, and completions.* Let A be a k-algebra. A left (resp. right) ideal of A is a left (resp. right) A-submodule of A, *i.e.* a vector subspace $I \subset A$ satisfying $AI \subset I$ (resp. $IA \subset I$). An ideal $I \subset A$ is a vector subspace that is both a left and a right ideal. If $I \subset A$ is an ideal, then A/I inherits a product structure. The corresponding notion for coalgebras is that of a coideal.

Definition 3.75. Let H be a coalgebra with coproduct Δ and counit ϵ. A vector subspace $I \subset H$ is

- a *left coideal* if $\Delta I \subset H \otimes I$;
- a *right coideal* if $\Delta I \subset I \otimes H$;
- a *coideal* if $\Delta I \subset H \otimes I + I \otimes H$ and $\epsilon(I) = 0$.

Lemma 3.76. *Let H be a coalgebra with coproduct Δ and counit ϵ. Let $I \subset H$ be a coideal. Then H/I inherits a coalgebra structure.*

Proof. Since $\epsilon(I) = 0$, the counit ϵ induces a map $\epsilon_{H/I} \colon H/I \to k$. For each class $[c] \in H/I$, with representative $c \in H$, we define $\Delta_{H/I}[c]$ as the class of Δc in the tensor product $H/I \otimes H/I$. The condition that Δc belongs to $H \otimes I + I \otimes H$ for all $c \in I$ implies that the coproduct $\Delta_{H/I}$ is well defined. The coassociativity of $\Delta_{H/I}$ and its compatibility with the counit $\epsilon_{H/I}$ then follow from the corresponding properties of Δ and ϵ. □

Definition 3.77. A *Hopf ideal* of a Hopf algebra H is a vector subspace $I \subset H$ that is both an ideal and a coideal and, moreover, satisfies $S(I) \subset I$.

If H is a Hopf algebra and I is a Hopf ideal, then H/I is also a Hopf algebra.

Examples 3.78. Let H be a Hopf algebra with coproduct Δ and counit ϵ.

i) The augmentation ideal $I = \operatorname{Ker} \epsilon$ is a Hopf ideal of H. Indeed, writing the coproduct of an element $a \in I$ as $\Delta a = \sum_i b_i \otimes c_i$, the condition that $(\epsilon \otimes \mathrm{Id}) \circ \Delta$ is the identity map under the canonical identification of $k \otimes H$ with H implies the equality $\sum_i \epsilon(b_i) c_i = a \in I$, and hence

$$\Delta a = \sum_i b_i \otimes c_i = \sum_i (b_i - \epsilon(b_i)) \otimes c_i + 1 \otimes a \in I \otimes H + H \otimes I.$$

Since $\epsilon \circ S = \epsilon$ by Proposition 3.49 iii), the ideal I is also stable under the antipode. This shows that I is a Hopf ideal.

ii) Let $I \subset H$ be a Hopf ideal. The powers I^n are not necessarily coideals, but they always satisfy the weaker condition

$$\Delta(I^n) \subset H \otimes I^n + I \otimes I^{n-1} + \cdots + I^{n-1} \otimes I + I^n \otimes H. \tag{3.79}$$

Definition 3.80. Let H be a Hopf algebra and $I \subset H$ a Hopf ideal. The *completion* of H with respect to I is the projective limit

$$H_I^\wedge = \varprojlim_n H/I^{n+1}.$$

When I is the augmentation ideal, we will simply write $H^\wedge$.

Proposition 3.81. *Let H be a Hopf algebra and $I \subset H$ a Hopf ideal. The completion $H_I^\wedge$ is a completed Hopf algebra.*

Proof. Since the powers of an ideal are again ideals, each H/I^{n+1} inherits an algebra structure from H, and hence so does the projective limit $H_I^\wedge$. Although the powers of a coideal are not necessarily coideals, condition (3.79) is enough to have coproducts

$$\Delta\colon H/I^{2n+1} \longrightarrow H/I^{n+1} \otimes H/I^{n+1},$$

that induce a completed coproduct

$$\varprojlim_n H/I^{n+1} \longrightarrow \varprojlim_n (H/I^{n+1} \otimes H/I^{n+1}) \longrightarrow \varprojlim_n H/I^{n+1} \otimes \varprojlim_n H/I^{n+1}.$$

The unit and the counit are also defined at each level H/I^{n+1}, and hence in the limit. Since I is stable under the antipode S, the same is true for the powers I^n. Therefore, $H_I^\wedge$ has an induced antipode. The compatibilities between the various structures of $H_I^\wedge$ follow from the compatibilities between those of H. $\square$

3.2.9. *Lie algebras.* We next introduce the notion of Lie algebra of an affine group scheme. The definition is modelled after the more classical notion of Lie algebra of a Lie group, which is the tangent space at the neutral element of the underlying differentiable manifold together with an antisymmetric product that reflects the non-commutativity of the group law. Throughout the next sections, k still denotes a field of characteristic zero.

Definition 3.82. A *Lie algebra* over k is the data $L = (L, [\cdot,\cdot])$ of a k-vector space L and a bilinear product

$$[\cdot,\cdot]\colon L \otimes L \longrightarrow L,$$

called the *Lie bracket*, that satisfies the following two conditions:

Antisymmetry: $[a,b] + [b,a] = 0$ for all $a, b \in L$.

Jacobi identity: $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$ for $a, b, c \in L$.

A *morphism of Lie algebras* is a k-linear map $\varphi\colon L \to L'$ that is compatible with the Lie brackets, in that the equality

$$\varphi([a,b]_L) = [\varphi(a), \varphi(b)]_{L'}$$

holds for all $a, b \in L$. When the underlying vector space of L has a grading

$$L = \bigoplus_{n\in\mathbb{Z}} L_n$$

such that $[L_n, L_m] \subseteq L_{n+m}$, we say that L is a *graded Lie algebra*.

An *abelian* Lie algebra is a Lie algebra with identically zero Lie bracket.

REMARK 3.83.

i) The antisymmetry of the Lie bracket implies that it factors through the exterior product $L \wedge L$.
ii) The commutator $[a, b] = ab - ba$ endows any associative k-algebra L with a Lie algebra structure. The Jacobi identity reflects the associativity of the product.
iii) There is a dual notion to that of Lie algebra called *Lie coalgebra*. We let the reader explore its properties in Exercise 3.142.

3.2.10. *The Lie algebra of an affine group scheme.* With an affine group scheme is associated a Lie algebra that is the algebraic analogue of the Lie algebra of a Lie group. This Lie algebra can be directly built out of the Hopf algebra of regular functions on the group, as we now explain. Let G be an affine group scheme over k and let $A = \mathcal{O}(G)$ be the corresponding commutative Hopf algebra. We keep the notation $(\nabla, \eta, \Delta, \epsilon, S)$ from Definition 3.47.

DEFINITION 3.84. The *augmentation ideal* of A is the kernel of the counit map $\epsilon \colon A \to k$. It will be denoted by $I = \operatorname{Ker}(\epsilon)$.

The augmentation ideal is the maximal ideal of regular functions on G that vanish at the unit $e = \eta(1)$. Thanks to the equality $\epsilon \circ \eta = \operatorname{Id}_k$, there is a canonical projection $A \to I$, and hence a canonical direct sum decomposition $A = k \oplus I$.

DEFINITION 3.85. The *tangent space* of the affine group scheme G at the unit element is the k-vector space $\mathfrak{g} = (I/I^2)^\vee$.

To make $\mathfrak{g}$ into a Lie algebra, we need a bracket

$$[\cdot,\cdot] \colon \mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}.$$

We will first define the dual map. For this, we observe that the compatibilities of the coproduct with the unit and the counit imply that, if $f \in I$, then

$$\Delta f - f \otimes 1 - 1 \otimes f \in I \otimes I. \tag{3.86}$$

We now consider the map

$$I \xrightarrow{\Delta} A \otimes A \longrightarrow (I/I^2) \otimes (I/I^2) \longrightarrow (I/I^2) \wedge (I/I^2),$$

where the second arrow is induced by the projection $A \to I \to I/I^2$ and the third arrow is the projection from the tensor product to the exterior product. It follows from property (3.86) that the composition of these maps vanishes on I^2. Therefore, we obtain a map

$$\mathrm{d} \colon I/I^2 \longrightarrow \bigwedge^2 (I/I^2). \tag{3.87}$$

By duality, we obtain a map

$$[\cdot,\cdot] \colon \mathfrak{g} \wedge \mathfrak{g} = (I/I^2)^\vee \wedge (I/I^2)^\vee \longrightarrow (I/I^2 \wedge I/I^2)^\vee \xrightarrow{\mathrm{d}^\vee} (I/I^2)^\vee = \mathfrak{g}.$$

Following Exercise 3.143, the pair $(I/I^2, \mathrm{d})$ is a Lie coalgebra over k. See Exercise 3.144 for a more down to earth formula for the bracket.

Definition 3.88. Let $G = \operatorname{Spec}(A)$ be an affine group scheme. The *Lie coalgebra* associated with the commutative Hopf algebra A is the pair $(I/I^2, \mathrm{d})$. Its dual $(\mathfrak{g}, [\cdot,\cdot])$ is called the *Lie algebra* of G and denoted by $\operatorname{Lie}(G)$.

The construction of the Lie algebra is functorial for morphisms of affine group schemes. Indeed, given $G = \operatorname{Spec}(A)$ and $H = \operatorname{Spec}(B)$, a morphism $f\colon G \to H$ of affine group schemes corresponds to a morphism of Hopf algebras $f\colon B \to A$. In particular, f is compatible with the counits of A and B, and hence maps the augmentation ideal J of B to the augmentation ideal I of A. The dual of the induced map $J/J^2 \to I/I^2$ is then a k-linear map

$$\mathrm{d}f\colon \operatorname{Lie}(G) \to \operatorname{Lie}(H)$$

which is compatible with the Lie bracket.

In practice, to compute the Lie algebra of an affine group scheme G, one looks for the elements of $G(k[\varepsilon])$ mapping to the identity in $G(k)$, which is an algebraic characterization of the tangent space at the unit. Here $k[\varepsilon] = k[x]/x^2$ denotes the *ring of dual numbers*, in which $\varepsilon^2 = 0$. We make this observation more precise.

Let $G = \operatorname{Spec}(A)$ be an affine group scheme over k and $\mathfrak{g} = \operatorname{Lie}(G)$ its Lie algebra. Recall that the set of R-points of G is given by $G(R) = \operatorname{Hom}_{k\text{-alg}}(A, R)$ for each k-algebra R. Consider the map

$$\begin{aligned} e^{\varepsilon\cdot}\colon\ \mathfrak{g}\otimes R &\longrightarrow G(R[\varepsilon]) \\ x &\longmapsto e^{\varepsilon x} \end{aligned} \tag{3.89}$$

constructed as follows. By definition of the Lie algebra, an element $x \in \mathfrak{g}\otimes R$ corresponds to a k-linear map $\mu_x\colon (I/I^2) \to R$, where $I \subset A$ is the ideal of regular functions on G that vanish at the unit element $e \in G(k)$. We then set

$$\begin{aligned} e^{\varepsilon x}\colon\ A &\longrightarrow R[\varepsilon] \\ f &\longmapsto f(e) + \mu_x(f - f(e))\varepsilon. \end{aligned} \tag{3.90}$$

One checks that the maps (3.89) are functorial with respect to morphisms of affine group schemes. The choice of the notation e^{ε} and $e^{\varepsilon x}$ is justified by the relation with the exponential map from Section 3.3.4.

Proposition 3.91. *There is a split short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g} \arrow[r, "e^{\varepsilon\cdot}"] & G(k[\varepsilon]) \arrow[r] & G(k) \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd},$$

where the arrows between $G(k[\varepsilon])$ and $G(k)$ are induced by the maps $k \to k[\varepsilon] \to k$.

Proof. We first show that the map (3.90) is a morphism of k-algebras. That it is k-linear is clear, so we have to show that it is an algebra homomorphism. This amounts to checking the equality

$$f(e)g(e) + \mu_x(fg - f(e)g(e))\varepsilon = [f(e) + \mu_x(f - f(e))\varepsilon][g(e) + \mu_x(g - g(e))\varepsilon]$$

in $k[\varepsilon]$ for all $f, g \in A$. Developing both sides and using that μ_x is linear, we see that this equality is equivalent to

$$\mu_x(fg + f(e)g(e) - g(e)f - f(e)g) = 0,$$

which is satisfied because

$$fg + f(e)g(e) - g(e)f - f(e)g = (f - f(e))(g - g(e)) \in I^2.$$

We next show that (3.89) is injective. The unit in the group $G(k[\varepsilon])$ corresponds to the algebra morphism $A \to k[\varepsilon]$ given by $f \mapsto f(e) + 0\varepsilon$, so if $x \in \mathfrak{g}$ is mapped to the unit element in $G(k[\varepsilon])$, then μ_x is identically zero, and hence $x = 0$.

To prove exactness in the middle, notice that any element in $\mathrm{Hom}_{k\text{-alg}}(A, k[\varepsilon])$ can be written as $f \mapsto \rho(f) + \lambda(f)\varepsilon$ for some k-linear maps $\rho, \lambda\colon A \to k$. Such an algebra morphism is mapped to the unit element in $G(k)$ if and only if $\rho(f) = f(e)$. One checks directly that a linear map of the form $f \mapsto f(e) + \lambda(f)\varepsilon$ is an algebra homomorphism if and only if $\lambda(1) = 0$ and $\lambda(I^2) = 0$. Therefore, there exists some element $x \in \mathfrak{g}$ satisfying $\lambda(f) = \mu_x(f - f(e))$.

Finally, since the composition $k \to k[\varepsilon] \to k$ of the inclusion and the projection is the identity, the map $G(k) \to G(k[\varepsilon])$ is a section of $G(k[\varepsilon]) \to G(k)$. In particular, the latter is surjective. □

Remark 3.92. In fact, we can define a functor from k-algebras to groups

$$\mathfrak{g}(R) = \mathrm{Ker}\big(G(R[\varepsilon]) \to G(R)\big).$$

Then part of Proposition 3.91 can be rephrased as the equality $\mathfrak{g}(k) = \mathfrak{g}$. Moreover, if G is algebraic, then $\mathfrak{g}(R) = \mathfrak{g} \otimes R$ (see [**DG70**, II §4, Prop. 4.8]), so that we can endow $\mathfrak{g}$ with the structure of an affine algebraic group as in Example 3.60 iv). By Lemma 3.51, an affine group scheme G is pro-algebraic. We write $G = \varprojlim_\alpha G_\alpha$. Hence, for every k-algebra R, we have $G(R) = \varprojlim_\alpha G_\alpha(R)$. Let $\mathfrak{g}_\alpha$ be the Lie algebra of G_α. Using the left-exactness of the projective limit, we deduce that

$$\mathfrak{g}(R) = \varprojlim_{\alpha}(\mathfrak{g}_\alpha \otimes R).$$

Therefore, we can endow $\mathfrak{g}$ with the scheme structure of Example 3.60 v).

Examples 3.93.

i) The group $G = \mathrm{GL}_n$ is the open subscheme of the affine space $\mathbb{A}^{n^2}$ defined as the complement of the determinant hypersurface $\{\det = 0\}$. Thus, the tangent space at the origin can be identified with the space $\mathrm{Mat}_n(k)$ of all $n \times n$ matrices over k, and the Lie bracket is just the usual commutator of matrices $[A, B] = AB - BA$. This Lie algebra is denoted by $\mathfrak{gl}_n$. More generally, if V is a finite-dimensional vector space over k, then the Lie algebra $\mathfrak{gl}(V)$ of $\mathrm{GL}(V)$ consists of all endomorphisms of V.

ii) The group $G = \mathrm{SL}_n$ is the closed subscheme of GL_n defined by the equation $\det = 1$. It represents the functor $R \mapsto \mathrm{SL}_n(R)$. The Lie algebra of G is a subalgebra of $\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n$. Using the characterization of Proposition 3.91, it consists of the matrices M such that $1 + \varepsilon M$ has determinant 1. From the equality

$$\det(1 + \varepsilon M) = 1 + \varepsilon\,\mathrm{Tr}(M)$$

in $k[\varepsilon]$, that can be proved for example by expanding the determinant along the first column and induction, we deduce that $\mathrm{Lie}(\mathrm{SL}_n)$ is the space of traceless $n \times n$ matrices. This Lie algebra is denoted by $\mathfrak{sl}_n$.

iii) The Lie algebra of the group $G = \mathrm{Up}_n$ of $n \times n$ unipotent matrices is the subalgebra $\mathfrak{u}_n \subset \mathfrak{gl}_n$ consisting of strictly upper triangular matrices. Indeed, G represents the functor that sends a k-algebra R to the group $\mathrm{Up}_n(R)$ of upper triangular matrices with diagonal entries equal to 1. The elements of $G(k[\varepsilon])$ can be thus written as $U+\varepsilon N$ with $U \in G(k)$ and N strictly upper triangular with entries in k, and the kernel of the map $G(k[\varepsilon]) \to G(k)$ is identified with the space of such matrices N.

3.2.11. *The universal enveloping algebra.* It is sometimes more convenient to work within the framework of associative algebras instead of Lie algebras. This is possible thanks to the construction of the universal enveloping algebra. Recall from Remark 3.83 that every associative algebra is endowed with a canonical Lie algebra structure in which the Lie bracket is given by the commutator. We keep the assumption that k is a field of characteristic zero.

Definition 3.94. Let $(L, [\cdot,\cdot])$ be a Lie algebra over k. The *universal enveloping algebra* of L is an associative k-algebra $U(L)$ with a morphism of Lie algebras

$$\iota_L \colon L \to U(L)$$

that satisfies the following universal property: for each associative k-algebra A and each morphism of Lie algebras $\iota \colon L \to A$, there exists a unique morphism of k-algebras $\varphi \colon U(L) \to A$ satisfying $\iota = \varphi \circ \iota_L$.

Concretely, that ι_L is a morphism of Lie algebras means that the equality

$$\iota_L([a,b]) = \iota_L(a)\iota_L(b) - \iota_L(b)\iota_L(a)$$

holds for all $a, b \in L$. To construct $U(L)$, we begin with the *tensor algebra*

$$T(L) = \bigoplus_{n \geqslant 0} L^{\otimes n}$$

with the associative product uniquely determined by

$$(a_1 \otimes \cdots \otimes a_r) \cdot (a_{r+1} \otimes \cdots \otimes a_{r+s}) = a_1 \otimes \cdots \otimes a_{r+s}.$$

This algebra is non-commutative if L has dimension bigger than 1. Consider the two-sided ideal $R(L) \subseteq T(L)$ generated by the elements

$$a \otimes b - b \otimes a - [a,b] \quad \text{for all} \quad a, b \in L. \tag{3.95}$$

We claim that the universal enveloping algebra of L is the quotient

$$U(L) = T(L)/R(L), \tag{3.96}$$

along with the morphism $\iota_L \colon L \to U(L)$ given by the composition of the natural inclusion $L \to T(L)$ with the quotient map $T(L) \to U(L)$. Indeed, every k-linear map $\iota \colon L \to A$ with target an associative k-algebra A factors uniquely through a morphism of k-algebras $T(L) \to A$ and, if ι is a morphism of Lie algebras, then the two-sided ideal $R(L)$ of $T(L)$ maps to zero.

Remark 3.97. The universal enveloping algebra $U(L)$ is endowed with the filtration (in the sense of Definition 1.85) by length given by

$$F_n U(L) = \text{image of} \bigoplus_{0 \leqslant m \leqslant n} L^{\otimes m} \quad \text{for } n \geqslant 0 \quad \text{and} \quad F_{-1}U(L) = \{0\}. \tag{3.98}$$

The Poincaré–Birkhoff–Witt theorem encapsulates the structure of the universal enveloping algebra. The version we give below is not the most general possible (see [**Bou60**, §1, n$^{\text{o}}$ 7, Thm. 1]), but it will be enough for our purposes.

THEOREM 3.99 (Poincaré–Birkhoff–Witt). *Assume that the Lie algebra L admits a totally ordered basis $\{y_i\}_{i\in I}$. Then a basis of the universal enveloping algebra $U(L)$ is given by the classes of the so-called standard monomials*

$$y_{i_1}\otimes\cdots\otimes y_{i_n}\in T(L)\quad \textit{with}\quad i_1\leqslant\cdots\leqslant i_n.$$

Note that the empty monomial $1\in T(L)$ is standard. In what follows, we will mainly use the Poincaré–Birkhoff–Witt theorem through some of its corollaries.

COROLLARY 3.100. *The map $\iota_L\colon L\to U(L)$ is injective.*

PROOF. The map ι_L sends a basis element y_i of L to the standard monomial y_i, which is part of a basis of $U(L)$ by Theorem 3.99. ☐

The universal enveloping algebra $U(L)$ can be naturally viewed as a Hopf algebra, with the coproduct uniquely determined by the rule

$$\Delta\iota_L(a)=\iota_L(a)\otimes 1+1\otimes\iota_L(a),$$

the counit $\epsilon\colon U(L)\to k$ induced by the zero map $L\to k$, and the antipode uniquely characterized by $S(a)=-a$ for all $a\in L$ (see Exercise 3.140 below). The counit ϵ is also called the *augmentation* of $U(L)$. This Hopf algebra structure allows one to recover the original Lie algebra L from its universal enveloping algebra $U(L)$ as the space of primitive elements. We first define them.

DEFINITION 3.101. Let H be a coalgebra with coproduct $\Delta\colon H\to H\otimes H$. An element $a\in H$ is called *primitive* if it satisfies

$$\Delta a=1\otimes a+a\otimes 1.$$

COROLLARY 3.102. *Let L be a Lie algebra that admits a totally ordered basis. Then the image $\iota_L(L)$ agrees with the set of primitive elements of $U(L)$.*

PROOF. Let $P(U(L))$ denote the subspace of primitive elements of $U(L)$. The inclusion $\iota_L(L)\subset P(U(L))$ is clear. To prove the converse inclusion, consider, for each integer $n\geqslant 0$, the vector subspace $U(L)_n$ of $U(L)$ spanned by all standard monomials of length n. As the image $\iota_L(L)$ agrees with $U(L)_1$, we need to prove the inclusion $P(U(L))\subset U(L)_1$. For this, we observe that the operator $\nabla\circ\Delta$ acts as multiplication by 2 on $P(U(L))$, whereas

$$(\nabla\circ\Delta)x=2^n x+(\text{terms of length}<n)\tag{3.103}$$

holds for all $x\in U(L)_n$ (solve Exercise 3.141). Let $a\in P(U(L))$ be a primitive element. By the Poincaré–Birkhoff–Witt theorem, we can write $a=a_0+\cdots+a_r$ for some elements $a_j\in U(L)_j$. Upon application of the operator $\nabla\circ\Delta$, we get $r=1$, then $a_0=0$ from (3.103). Thus, $P(U(L))\subset U(L)_1$ and $\iota_L(L)$ agrees with the space of primitive elements. ☐

Another consequence of the Poincaré–Birkhoff–Witt theorem is an expression for the abelianization of a Lie algebra. Let $J=\operatorname{Ker}(\epsilon)\subset U(L)$ be the augmentation ideal, that is, the quotient of $\bigoplus_{n>0}L^{\otimes n}$ by the two-sided ideal $R(L)$.

COROLLARY 3.104. *Let L be a Lie algebra that admits a totally ordered basis. The inclusion $\iota_L\colon L\to U(L)$ induces an isomorphism*

$$\iota'_L\colon L/[L,L]\xrightarrow{\sim} J/J^2.$$

PROOF. There is an inclusion $\iota_L(L) \subset J$ by definition of the augmentation ideal. Since ι_L is a morphism of Lie algebras and the Lie bracket on $U(L)$ is given by the commutator $[x, y] = xy - yx$, there is also an inclusion $\iota_L([L, L]) \subset J^2$. The map ι'_L is hence well defined. By Theorem 3.99, the universal enveloping algebra $U(L)$ is generated by standard monomials. Since every monomial of length bigger than or equal to 2 belongs to J^2, every element of J/J^2 can be represented by a monomial of length 1, that is, by an element of L. This shows that ι'_L is surjective. To prove injectivity, we need to show that $\iota_L^{-1}(J^2)$ is contained in $[L, L]$. For this, let us first observe that any linear combination of monomials of length n can be written, using the relations (3.95) defining the ideal $R(L)$, as the sum of a linear combination of standard monomials of length n and a linear combination of monomials of length at most $n - 1$ in which all terms contain at least one commutator. Now, an element $a \in J^2$ can be written as a linear combination of monomials of length at least 2. Applying repeatedly the above observation, a can be written as $a_\ell + \cdots + a_2 + b_1$, where a_n is a linear combination of standard monomials of length n and b_1 is a commutator. If a lies in the image of L, then it is a linear combination of standard monomials of length 1. Since standard monomials form a basis of $U(L)$ by Theorem 3.99, we deduce that a is equal to b_1, and hence lies in the image of $[L, L]$. The map ι'_L is thus an isomorphism. □

Applying Definition 3.80 to the Hopf algebra $U(L)$, we obtain:

DEFINITION 3.105. Let L be a Lie algebra. The *completed universal enveloping algebra* $U(L)^\wedge$ is the completion of $U(L)$ with respect to the augmentation ideal $J = \mathrm{Ker}(\epsilon)$, where ϵ is the counit of $U(L)$.

Explicitly, $\widehat{U}(L)$ is defined as the projective limit

$$U(L)^\wedge = \varprojlim_N U(L)/J^{N+1}.$$

Let us now explain the relationship between the dual of the ring of regular functions on an affine group scheme and the universal enveloping algebra of its Lie algebra. Let G be an affine group scheme over k and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Since $\mathcal{O}(G)$ is a Hopf algebra, its k-linear dual

$$\mathcal{O}(G)^\vee = \mathrm{Hom}_k(\mathcal{O}(G), k)$$

has the structure of an associative algebra, in which the product of two linear forms $\lambda, \mu \in \mathcal{O}(G)^\vee$ is given by the rule

$$(\lambda \cdot \mu)(f) = (\lambda \otimes \mu)(\Delta f)$$

on functions $f \in \mathcal{O}(G)$. The associativity of this product follows from the coassociativity of the coproduct in the definition of Hopf algebra. Using the definition of the Lie algebra as $\mathfrak{g} = (I/I^2)^\vee$, we get a canonical map

$$\varphi_G \colon \mathfrak{g} \longrightarrow \mathcal{O}(G)^\vee \tag{3.106}$$

that sends an element $X \in \mathfrak{g}$ to the composition

$$\mathcal{O}(G) \longrightarrow I/I^2 \xrightarrow{X} k,$$

where the first map sends f to the class of $f - f(e)$. From the fact that φ_G is a morphism of Lie algebras (Exercise 3.145) and the universal property of the universal enveloping algebra, we obtain a canonical map

$$U(\mathfrak{g}) \longrightarrow \mathcal{O}(G)^{\vee}. \tag{3.107}$$

In general, this map is not an isomorphism (see Example 3.108 below). In some cases (see Theorem 3.182), it can be extended to the completed universal enveloping algebra $U(\mathfrak{g})^{\wedge}$. We will see later that this is characteristic of pro-unipotent groups.

Examples 3.108.

i) Let $G = \mathbb{G}_a$ be the additive group over $\mathbb{Q}$. Its algebra of functions is the polynomial ring $\mathcal{O}(G) = \mathbb{Q}[x]$ and its Lie algebra is the abelian (*i.e.* with identically zero bracket) Lie algebra $\mathbb{Q}$. Its universal enveloping algebra is the algebra of polynomials $\mathbb{Q}[y]$, while its completed universal enveloping algebra is the algebra of formal power series $\mathbb{Q}[\![y]\!]$. The canonical map (3.107) is the map $\mathbb{Q}[y] \to \mathcal{O}(G)^{\vee}$ that sends the divided power $y^n/n!$ to the dual of x^n. This map is not an isomorphism, but it extends to an isomorphism $\mathbb{Q}[\![y]\!] \to \mathcal{O}(G)^{\vee}$.

ii) Let $G = \mathbb{G}_m$ be the multiplicative group over $\mathbb{Q}$. Its algebra of functions is the ring of Laurent polynomials $\mathcal{O}(G) = \mathbb{Q}[x, x^{-1}]$ and its Lie algebra is again is the one-dimensional abelian Lie algebra $\mathbb{Q}$. As in the previous example, the universal enveloping algebra is the algebra of polynomials $\mathbb{Q}[y]$, and its completion is the algebra of formal power series $\mathbb{Q}[\![y]\!]$. The map $\mathbb{Q}[y] \to \mathcal{O}(G)^{\vee}$ sends y to the linear form $P \mapsto P'(1)$, which in the representation of Example 3.72 corresponds to the element

$$\sum_{n\in\mathbb{Z}} n y_n$$

since y_n is dual to x^n. From the computation (3.73) that says that the y_n are orthogonal projectors, we then deduce that y^{ℓ} is sent to $\sum n^{\ell} y_n$. In this case, the map $\mathbb{Q}[y] \to \mathcal{O}(G)^{\vee}$ cannot be extended to the completed universal enveloping algebra.

3.2.12. *Universal enveloping algebras and distributions.* In this section, we explain a theorem by Cartier relating the universal enveloping algebra of the Lie algebra of an algebraic group with certain linear functionals on its regular functions that are called *distributions*.

Let $G = \operatorname{Spec}(A)$ be an affine group scheme over k with Lie algebra $\mathfrak{g}$. The unit element $e \in G(k)$ corresponds to a morphism of k-algebras $A \to k$ that we denote by $\varphi \mapsto \varphi(e)$. For each linear map $\mu\colon A \to k$ and each $\varphi \in A$, define

$$\begin{aligned} (\operatorname{ad}\varphi)\mu\colon\ A &\longrightarrow k \\ x &\longmapsto \varphi(e)\mu(x) - \mu(\varphi x). \end{aligned} \tag{3.109}$$

Definition 3.110. A *distribution* on G *centered* at e of order $\leqslant n$ is a linear map $\mu\colon A \to k$ such that the equality

$$(\operatorname{ad}\varphi_0)\dots(\operatorname{ad}\varphi_n)\mu = 0$$

holds for all $\varphi_0, \dots, \varphi_n \in A$. The space of all distributions centered at e of order $\leqslant n$ will be denoted by $\mathrm{Dist}_n(G)$. We also write

$$\begin{aligned}\mathrm{Dist}_n^+(G) &= \{\mu \in \mathrm{Dist}_n(G) \mid \mu(1) = 0\},\\ \mathrm{Dist}(G) &= \bigcup_{n\geqslant 0} \mathrm{Dist}_n(G),\\ \mathrm{Dist}^+(G) &= \bigcup_{n\geqslant 0} \mathrm{Dist}_n^+(G).\end{aligned}$$

Examples 3.111. Let $G = \mathrm{Spec}(A)$ be an affine group scheme and $\mathfrak{g}$ its Lie algebra. In this example, we compute the space of distributions in terms of the ideal $I \subset A$ of functions vanishing at the neutral element $e \in G(k)$.

i) An element $\mu \in \mathrm{Dist}_0(G)$ is determined by the value $\mu(1)$ through the rule $\mu(\varphi) = \varphi(e)\mu(1)$. In particular, μ vanishes on I, whence

$$\mathrm{Dist}_0(G) = (A/I)^\vee = k \quad \text{and} \quad \mathrm{Dist}_0^+(G) = \{0\}.$$

ii) Let $\mu \in \mathrm{Dist}_1(G)$. By definition, the linear map $(\mathrm{ad}\,\varphi)\mu$ lies in $\mathrm{Dist}_0(G)$ for all $\varphi \in A$, which by i) amounts to saying that it vanishes on I. This map sends x to $\mu((\varphi(e) - \varphi)x)$ by linearity of μ, and hence

$$\mathrm{Dist}_1(G) = (A/I^2)^\vee.$$

If μ vanishes at 1, then μ defines an element of $\mathfrak{g} = (I/I^2)^\vee$ and

$$\mathfrak{g} \simeq \mathrm{Dist}_1^+(G).$$

From the vanishing on I^2 and at 1, we see that all $\mu \in \mathrm{Dist}_1^+(G)$ satisfy

$$\mu(\varphi_0\varphi_1) = \varphi_0(e)\mu(\varphi_1) + \varphi_1(e)\mu(\varphi_0).$$

iii) More generally, μ belongs to $\mathrm{Dist}_n(G)$ if and only $(\mathrm{ad}\,\varphi)\mu$ belongs to $\mathrm{Dist}_{n-1}(G)$ for all $\varphi \in A$. By induction, this is equivalent to the condition that μ vanishes on I^{n+1}, hence an isomorphism

$$\mathrm{Dist}_n(G) = (A/I^{n+1})^\vee.$$

By duality, the coproduct and the product of the Hopf algebra $A = \mathcal{O}(G)$ induce a product and a coproduct in the space of distributions. Namely, there is a *convolution product* given by

$$(\mu_0 * \mu_1)(\varphi) = (\mu_0 \otimes \mu_1)(\Delta\varphi).$$

The coassociativity of Δ implies that this product is associative. In addition, since the ideal I is the kernel of the counit, it satisfies $\Delta I \subset A \otimes I + I \otimes A$. Therefore,

$$\Delta I^{n+1} \subset \sum_{i=0}^{n+1} I^i \otimes I^{n+1-i}. \tag{3.112}$$

By the characterization of Example 3.111 iii), the inclusion (3.112) implies

$$\mathrm{Dist}_n(G) * \mathrm{Dist}_m(G) \subset \mathrm{Dist}_{n+m}(G). \tag{3.113}$$

The coproduct

$$\Delta\colon\ \mathrm{Dist}(G) \longrightarrow \mathrm{Dist}(G) \otimes \mathrm{Dist}(G)$$

is given by the formula

$$\Delta\mu(\varphi_0 \otimes \varphi_1) = \mu(\varphi_0\varphi_1).$$

For all $\mu \in \mathrm{Dist}_1^+(G)$, the equalities

$$\Delta\mu(\varphi_0 \otimes \varphi_1) = \mu(\varphi_0\varphi_1) = \varphi_0(e)\mu(\varphi_1) + \varphi_1(e)\mu(\varphi_0) = (1 \otimes \mu + \mu \otimes 1)(\varphi_0 \otimes \varphi_1)$$

hold, and hence all elements of $\mathrm{Dist}_1^+(G)$ are primitive for this coproduct.

THEOREM 3.114 (Cartier). *Let G be an affine algebraic group and $\mathfrak{g}$ its Lie algebra. Then there is a unique isomorphism of algebras*

$$U(\mathfrak{g}) \longrightarrow \mathrm{Dist}(G)$$

that agrees with the isomorphism of Example 3.111 ii) for $\mathfrak{g}$. Moreover, this isomorphism is compatible with the coproduct and sends $F_nU(\mathfrak{g})$ to $\mathrm{Dist}_n(G)$.

SKETCH OF PROOF. The map $U(\mathfrak{g}) \longrightarrow \mathrm{Dist}(G)$ is constructed using the universal property of the universal enveloping algebra. Indeed, $\mathrm{Dist}(G)$ is an associative algebra, and Example 3.111 ii) gives map $\iota\colon \mathfrak{g} \to \mathrm{Dist}(G)$ that satisfies

$$\iota([a, b]) = a * b - b * a$$

by Exercise 3.144. Therefore, the universal property of $U(\mathfrak{g})$ yields a morphism of algebras $U(\mathfrak{g}) \to \mathrm{Dist}(G)$. Recall that the coproduct of $U(\mathfrak{g})$ is characterized by the fact that it is a morphism of algebras and that elements of $\mathfrak{g}$ are primitive. Since the coproduct of $\mathrm{Dist}(G)$ is a morphism of algebras by construction and $\mathfrak{g}$ is mapped to $\mathrm{Dist}_1^+(G)$, which consists of primitive elements, the map $U(\mathfrak{g}) \to \mathrm{Dist}(G)$ is compatible with the coproducts on the source and the target. Moreover, it sends $F_nU(\mathfrak{g})$ to $\mathrm{Dist}_n(G)$ because the elements of $F_nU(\mathfrak{g})$ are those that can be written as a linear combination of products of at most n elements of $\mathfrak{g}$. Thanks to (3.113), those elements are sent to $\mathrm{Dist}_n(G)$. Finally, the proof that the map is an isomorphism is a long computation using the Poincaré–Birkhoff–Witt theorem, whose details can be found in [**DG70**, II §6, Thm. 1.1]. □

3.2.13. *Representations.* We introduce the notion of representation of an abstract group and an affine group scheme. In the latter case, one needs to be careful because the group-valued functor $\mathrm{Aut}(V)$ is not representable by a scheme when V is an infinite-dimensional vector space (see Example 3.60 viii)).

DEFINITION 3.115. Let Γ be an abstract group. A *k-linear representation* of Γ is a k-vector space V together with a group homomorphism

$$\Gamma \longrightarrow \mathrm{Aut}_k(V).$$

DEFINITION 3.116. Let G be an affine group scheme over k. A *k-linear representation* of G is the data of a k-vector space V and a natural transformation of group-valued functors $G \to \mathrm{Aut}(V)$.

This means that we are given the data of a group homomorphism

$$G(R) \longrightarrow \mathrm{Aut}_R(R \otimes V)$$

for every k-algebra R, and a commutative diagram

$$\begin{tikzcd} G(R) \arrow[r] \arrow[d] & \mathrm{Aut}_R(R \otimes V) \arrow[d] \\ G(R') \arrow[r] & \mathrm{Aut}_{R'}(R' \otimes V). \end{tikzcd}$$

for every morphism of k-algebras $R \to R'$.

REMARK 3.117. Recall from Example 3.60 that the linear automorphisms of a finite-dimensional vector space V form an affine group scheme $\mathrm{GL}(V)$. It turns out that to give a finite-dimensional representation of G is equivalent to give a pair consisting of a k-vector space V and a morphism of group schemes $\rho\colon G \to \mathrm{GL}(V)$. Since we will be mainly interested in finite-dimensional representations, this is the point of view that we will mostly use.

Every k-linear representation of an affine group scheme determines a representation of the group of k-points $G(k)$, but of course not all representations of $G(k)$ arise this way (see Lemma 3.122 and Exercise 3.220 below for an example). Since we will be only working with k-linear representations, we will omit the adjective "k-linear" and refer to them in what follows simply as "representations".

In some cases, it is more convenient to use the point of view of comodules. For a more detailed proof of the next result, see [**Mil17**, Rem. 4.1].

LEMMA 3.118. *Let G be an affine group scheme over k, and let V be a k-vector space. There is a natural one-to-one correspondence between linear representations of G on V and right $\mathcal{O}(G)$-comodule structures on V.*

PROOF. For shorthand, we write $A = \mathcal{O}(G)$. In fact, the correspondence is in two steps. First, each linear representation of G corresponds to a left A-comodule structure on the dual vector space $V^\vee$. Then the left A-comodule structure on $V^\vee$ corresponds to a right A-comodule structure on V.

More precisely, let $\rho\colon G \to \mathrm{Aut}(V)$ be a representation of G. For each $\omega \in V^\vee$, we consider the element

$$\Delta_\rho \omega \in A \otimes V^\vee \simeq \mathrm{Hom}(V, A)$$

uniquely defined by the formula

$$\langle \Delta_\rho(\omega), v\rangle(g) = \langle \omega, \rho(g)(v)\rangle.$$

Here, $\langle \Delta_\rho(\omega), v\rangle$ is meant to be an element of $A = \mathcal{O}(G)$ that we determine by evaluating it at elements $g \in G(R)$ for any k-algebra R. By duality, this defines a right coaction $\Delta_\rho\colon V \to V \otimes A$, denoted with the same letter, by the rule

$$\langle \omega, \Delta_\rho(v)\rangle(g) = \langle \Delta_\rho(\omega), v\rangle(g) = \langle \omega, \rho(g)(v)\rangle. \tag{3.119}$$

for each $v \in V$ and $\omega \in V^\vee$. It is easy to check that the maps Δ_ρ and ρ from equation (3.119) determine each other, and that Δ_ρ is a right coaction of A on V if and only if ρ is a representation of G.

In the finite-dimensional case, this can be made more concrete by choosing basis. Indeed, let $e_1, \dots, e_r$ be a basis of V. Each linear representation ρ of G on V is an affine group homomorphism $\rho\colon G \to \mathrm{GL}(V)$. After choosing the basis, we can identify $\mathrm{GL}(V)$ with GL_n. Therefore, $\rho\colon \mathrm{Spec}(A) = G \to \mathrm{GL}_n$ defines a point $M \in \mathrm{GL}_n(A)$, *i.e.* an invertible $r \times r$ matrix $M = (M_{i,j})$ with entries in A. The fact that ρ is a group homomorphism is equivalent to

$$M_{i,j}(g \cdot g') = \sum_k M_{i,k}(g) M_{k,j}(g'), \quad M_{i,j}(e) = \delta_{i,j}, \tag{3.120}$$

for e the unit of G.

On the other hand, a right coaction $\Delta\colon V \to V \otimes A$ defines also a matrix $M_{i,j}$ with entries in A by the rule

$$\Delta e_j = \sum_i e_i \otimes M_{i,j}.$$

The fact that Δ is a coaction is again equivalent to (3.120). Both the set of linear representations of G on V and the set of right A-coactions on V are thus given by the set of $r \times r$ matrices with entries in A satisfying conditions (3.120). □

REMARK 3.121. It is interesting to see the relationship between Lemma 3.118 and Example 3.57. A right action $X \times G \to X$ gives rise to a representation of G on the vector space $\mathcal{O}(X)$, which, by Lemma 3.118 gives a right $\mathcal{O}(G)$-module structure on $\mathcal{O}(X)$. This structure is the same as the one from Example 3.57.

From the first part of the proof of Lemma 3.51, we also derive the following result (see also [**DM82**, Cor. 2.4]).

LEMMA 3.122. *Every linear representation of an affine group scheme is a directed union of finite-dimensional subrepresentations.*

DEFINITION 3.123. Let G be an affine algebraic group. A finite-dimensional representation $\rho\colon G \to \mathrm{GL}(V)$ is called *faithful* if $\mathrm{Ker}(\rho)$ is trivial.

A useful property of affine algebraic groups is the following:

PROPOSITION 3.124. *Let $f\colon G \to H$ be a morphism of affine algebraic groups.*

i) *The image $f(G)$ is closed in H.*
ii) *f is a closed immersion if and only if $\mathrm{Ker}(f)$ is zero.*

A proof is given in [**Bri17**, Prop. 2.7.1]). In view of Proposition 3.124, a representation ρ is faithful if and only if ρ is a closed immersion.

COROLLARY 3.125. *Every affine algebraic group admits a faithful finite-dimensional representation. In other words, any affine algebraic group can be realized as a closed subgroup of some general linear group $\mathrm{GL}(V)$.*

PROOF. Let G be an affine algebraic group, so that the algebra $\mathcal{O}(G)$ is finitely generated. Let $f_1, \dots, f_n$ be a set of generators. Since $\mathcal{O}(G)$ is a linear representation of G, Lemma 3.122 implies that there exist finite-dimensional subrepresentations $W_1, \dots, W_n$ of $\mathcal{O}(G)$ such that W_i contains f_i. Consider their sum

$$W = W_1 + \cdots + W_n$$

and the dual representation $\rho\colon G \to \mathrm{GL}(W^\vee)$. Since W contains a set of generators of $\mathcal{O}(G)$, the map $\mathrm{Sym}(W) \to \mathcal{O}(G)$ given by the universal property of the symmetric algebra is surjective. Therefore, there is a closed G-equivariant immersion

$$G \longrightarrow \mathrm{Spec}(\mathrm{Sym}(W)) = W^\vee,$$

which implies that $\mathrm{Ker}(\rho)$ is trivial. □

We next introduce Lie algebra representations.

DEFINITION 3.126. Let L be a Lie algebra over k. A *representation* of L is a k-vector space V along with a morphism of Lie algebras

$$\rho\colon L \longrightarrow \mathrm{End}(V),$$

i.e. a k-linear map such that, for all $x, y \in L$, the following equality holds:

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x). \tag{3.127}$$

PROPOSITION 3.128. *Let L be a Lie algebra. The category of representations of L is equivalent to the category of $U(L)$-modules.*

PROOF. Let $\rho\colon L \to \mathrm{End}(V)$ be a representation of L. Thanks to the universal property of the enveloping algebra $U(L)$, the representation ρ factors through a unique k-algebra morphism $U(L) \to \mathrm{End}(V)$, which turns V into a $U(L)$-module. Conversely, a $U(L)$-module is the data of a k-vector space V and a k-algebra morphism $U(L) \to \mathrm{End}(V)$. Precomposing with the natural map from L to $U(L)$, we obtain a Lie algebra morphism $L \to \mathrm{End}(V)$. It is straightforward to check that these constructions are functorial and quasi-inverse to each other. $\square$

We next relate Lie group representations with Lie algebra representations.

DEFINITION 3.129. Let G be an affine group scheme and $\mathfrak{g}$ its Lie algebra. With a representation ρ of G on a vector space V is associated a representation

$$\mathrm{d}\rho\colon \mathfrak{g} \to \mathrm{End}_k(V),$$

namely the unique map that fits in the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{d}\rho"] & G(k[\varepsilon]) \arrow[r] \arrow[d, "\rho"] & G(k) \arrow[d, "\rho"] \\
0 \arrow[r] & \mathrm{End}_k(V) \arrow[r] & \mathrm{Aut}_{k[\varepsilon]}(V \otimes k[\varepsilon]) \arrow[r] & \mathrm{Aut}_k(V).
\end{tikzcd}$$

See Exercise 3.138 for the exactness of the bottom row.

⋆ ⋆ ⋆

EXERCISE 3.130. Prove that the space H of Example 3.55 is a coalgebra and that V^* is an H-comodule.

EXERCISE 3.131. Let H be a Hopf algebra.

i) Consider a finite-dimensional subvector space V of H satisfying
$$\Delta(V) \subseteq V \otimes H.$$
Pick a basis $\{v_i\}$ of V and write $\Delta(v_j) = \sum_i v_i \otimes h_{ij}$. Prove the equality
$$\Delta(h_{ij}) = \sum_{\ell} h_{i\ell} \otimes h_{\ell j}.$$

ii) Show that $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$ holds, where τ is the swap of the factors of $H \otimes H$. Concretely, if $\Delta(h) = \sum_i a_i \otimes b_i$, then
$$\Delta(S(h)) = \sum_i S(b_i) \otimes S(a_i).$$

EXERCISE 3.132 (A bialgebra without antipode). Let $H = k[x]$ be the polynomial algebra in one variable. The coproduct $\Delta(x) = x \otimes x$ and the counit $\epsilon(x) = 1$ endow H with the structure of a cocommutative bialgebra. Show that H does not admit an antipode.

EXERCISE 3.133 (A connected graded bialgebra has an antipode). Let H be a connected graded bialgebra.

i) Use the commutativity of diagram (2) in Definition 3.47 to prove that the counit $\epsilon\colon H \to k$ vanishes on H_n for all $n \geqslant 1$, and hence induces an isomorphism $H_0 \simeq k$.

ii) Show that the antipode $S\colon H \to H$ is the unique algebra morphism such that $S_{|H_0} = \mathrm{Id}$ and, if $x \in H_n$ for $n \geqslant 1$, then

$$S(x) = -x - \sum \nabla(S(x') \otimes x''),$$

where the sum runs over all elements x'' appearing in the coproduct

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''.$$

Exercise 3.134. Let $\mathfrak{H}$ be the Hoffman algebra.

i) Verify that the operations described in Example 3.64 endow $\mathfrak{H}$ with a Hopf algebra structure.
ii) Recall that $\mathfrak{H}$ is graded by assigning weight n to $x_{\varepsilon_1} \cdots x_{\varepsilon_n}$. Prove by induction on n that the recipe to compute the antipode presented in Exercise 3.133 yields $S(x_{\varepsilon_1} \cdots x_{\varepsilon_n}) = (-1)^n x_{\varepsilon_n} \cdots x_{\varepsilon_1}$.

Exercise 3.135. Let V be an infinite-dimensional vector space over a field k. In this exercise, we show that the functor $F(R) = V \otimes R$ from k-algebras to sets is not representable by an affine scheme.

i) Assume that F is representable by an affine scheme $\mathrm{Spec}(A)$, so that the equality $F(R) = \mathrm{Hom}(A, R)$ holds. Prove that the natural map

$$F(k[\![t]\!]) \longrightarrow \varprojlim_n F(k[t]/t^{n+1})$$

is then bijective.

ii) Show that the natural map

$$V \otimes k[\![t]\!] \longrightarrow \varprojlim_n \big(V \otimes k[t]/t^{n+1}\big)$$

is not surjective.

iii) By contrast, prove that in case V is a projective limit $V = \varprojlim_\alpha V_\alpha$ of finite-dimensional vector spaces V_α, the natural map

$$\varprojlim_\alpha \big(V_\alpha \otimes k[\![t]\!]\big) \longrightarrow \varprojlim_n \varprojlim_\alpha \big(V_\alpha \otimes k[t]/t^{n+1}\big)$$

is a bijection.

Exercise 3.136. Let V be an infinite-dimensional vector space over a field k. In this exercise, we show that the functor

$$R \longmapsto \mathrm{Aut}_R(R \otimes V) \tag{3.137}$$

is not representable by an affine group scheme over k.

i) Use [**Har77**, Lem. II.2.1] to prove that, if A is any ring, then the topological space underlying $S = \mathrm{Spec}(A)$ is quasi-compact. That is, every open covering of S admits a finite subcovering.
ii) Consider the functor $\mathbb{A}^1_k$ that sends a k-algebra R to the set R. Similarly to Example 3.60 iii), it is represented by the scheme $\mathrm{Spec}(k[x])$. Let A be a k-algebra and $S = \mathrm{Spec}(A)$. Denote by $\underline{S}$ the corresponding functor. Use Yoneda's lemma to prove that we can recover A from $\underline{S}$ as

$$A = \mathrm{Hom}_{\mathrm{functors}}(\underline{S}, \mathbb{A}^1).$$

Therefore, if we want to prove that a functor is representable by an affine scheme, there is a unique candidate for the corresponding algebra.

iii) Let V be an infinite-dimensional k-vector space and let $\{v_\alpha\}_{\alpha\in I}$ be a basis. For every $\alpha \in I$, define the dual element $\omega_\alpha \in V^\vee = \mathrm{Hom}_k(V,k)$ given by

$$\omega_\alpha(v_\beta) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

Assume that the functor $R \mapsto \mathrm{Aut}_R(R \otimes V)$ is representable by an affine scheme $S = \mathrm{Spec}(A)$. By ii), for all $\alpha, \beta \in I$ the morphism of functors $x_{\alpha,\beta} \colon \underline{S} \to \mathbb{A}^1$ given by

$$x_{\alpha,\beta}(\psi) = \omega_\alpha(\psi(v_\beta))$$

determines an element of the algebra A. Let $D(x_{\alpha,\beta})$ be the open subset of S where $x_{\alpha,\beta}$ does not vanish (see [**Har77**, § II.2]). Then prove that

$$S = \bigcup_{\alpha,\beta\in I} D(x_{\alpha,\beta})$$

but that this covering does not admit a finite subcovering. This contradiction with i) shows that the functor (3.137) is not representable by an affine k-scheme.

Exercise 3.138. Let V be a k-vector space of arbitrary dimension. Prove that there is an exact sequence of groups

$$0 \longrightarrow \mathrm{End}_k(V) \longrightarrow \mathrm{Aut}_{k[\varepsilon]}(V \otimes k[\varepsilon]) \longrightarrow \mathrm{Aut}_k(V) \longrightarrow 1,$$

where $k[\varepsilon]$ is the ring of dual numbers and the first non-zero map is given by

$$\varphi \mapsto e^{\varepsilon\varphi} = \mathrm{Id}_V + \varepsilon\varphi.$$

Here, the endomorphisms $\mathrm{End}_k(V)$ are given an additive group structure, and the automorphisms $\mathrm{Aut}_{k[\varepsilon]}(V \otimes k[\varepsilon])$ and $\mathrm{Aut}_k(V)$ a multiplicative group structure.

Exercise 3.139. Let k be a field and let A be an invertible $n \times n$ matrix with coefficients in k. Prove that the functor

$$R \longmapsto G(R) = \{B \in \mathrm{GL}_n(R) \mid B^t AB = A\}$$

from commutative k-algebras to groups is representable by an affine algebraic group scheme over k, and write down the associated Hopf algebra. In the case where A is the identity matrix, one obtains the orthogonal group from Example 3.62.

Exercise 3.140. Let $(L, [\cdot,\cdot])$ be a Lie algebra. Recall the tensor algebra $T(L)$, the two-sided ideal $R(L)$, and the universal enveloping algebra $U(L) = T(L)/R(L)$ from Section 3.2.11. Consider the coproduct

$$\Delta \colon T(L) \longrightarrow T(L) \otimes T(L)$$

defined as the unique morphism of k-algebras satisfying $\Delta a = 1 \otimes a + a \otimes 1$ for all $a \in L$. Prove that there is an inclusion

$$\Delta R(L) \subset R(L) \otimes T(L) + T(L) \otimes R(L),$$

hence an induced coproduct $\Delta \colon U(L) \to U(L) \otimes U(L)$.

Exercise 3.141. Prove identity (3.103) in the proof of Corollary 3.102.

Exercise 3.142 (Lie coalgebras). We introduce Lie coalgebras.

i) Let L be a finite-dimensional Lie algebra over k. The dual of the Lie bracket $[\cdot,\cdot]\colon L\otimes L\to L$ is a map $\mathrm{d}\colon L^\vee\to L^\vee\otimes L^\vee$. Write down the properties dual to the anti-symmetry and the Jacobi identity of $[\cdot,\cdot]$.

ii) A *Lie coalgebra* over k is a pair (C,d) consisting of a k-vector space C and a k-linear map $\mathrm{d}\colon C\to C\wedge C$ such that $\mathrm{d}\circ\mathrm{d}=0$ holds (when d is appropriately extended to $C\wedge C$). Prove that the dual of a Lie coalgebra, not necessarily of finite dimension, is a Lie algebra.

Exercise 3.143. In this exercise, we show that I/I^2 is a Lie coalgebra, and hence L is a Lie algebra.

i) Check property (3.86).

ii) Extend d to an operator

$$\mathrm{d}\colon \bigwedge^n(I/I^2)\longrightarrow\bigwedge^{n+1}(I/I^2)$$

by using the Leibniz rule with appropriate signs. Then show that $\mathrm{d}^2=0$ holds. This implies that I/I^2 is a Lie coalgebra. Deduce from Exercise 3.142 that L is a Lie algebra.

Exercise 3.144. Let G be an affine group scheme and $\mathfrak{g}$ its Lie algebra. Let Δ be the coproduct in $\mathcal{O}(G)$. Let $a,b\in\mathfrak{g}$ and $f\in I\subset\mathcal{O}(G)$. Show that the bracket in $\mathfrak{g}$ is given explicitly by

$$[a,b](f)=(a\otimes b-b\otimes a)(\Delta f).$$

Exercise 3.145. Show that the map φ_G from (3.106) satisfies

$$\varphi_G([X,Y])=\varphi_G(X)\cdot\varphi_G(Y)-\varphi_G(Y)\cdot\varphi_G(X)$$

for all elements $X,Y\in\mathfrak{g}$ of the Lie algebra of G.

Exercise 3.146. Prove the equality

$$\Delta(x^n)=\sum_{r=0}^n\binom{n}{r}x^r\otimes x^{n-r}$$

in the Hopf algebra $k[x]$ associated with the additive group $\mathbb{G}_a$.

Exercise 3.147 (The Hopf algebra of rooted trees). In this exercise, we describe the *Hopf algebra of rooted trees* introduced by Connes and Kreimer, in connection with the renormalization of quantum field theories [**CK98**].

A *rooted tree* is an oriented finite graph which is connected and simply connected (in other words, a tree), and has a distinguished vertex with no incoming edges called the root. Continuing the metaphor, the vertices with no outcoming edges are called the leaves. A *rooted forest* is a disjoint union of rooted trees.

Let $\mathcal{H}_R$ be the $\mathbb{Q}$-algebra of polynomials in rooted trees, *i.e.* $\mathcal{H}_R$ is the free commutative $\mathbb{Q}$-algebra with unit generated by (isomorphism classes of) rooted trees. The product of two rooted trees can be identified with their disjoint union, and the unit is the empty tree 1. Therefore, as vector space,

$$\mathcal{H}_R=\mathbb{Q}\{\text{rooted forests up to isomorphism}\}.$$

Let t be a rooted tree. An *admissible cut* c of t is the choice of a non-empty subset of the edges such that any path from the root to the leaves meets at most one of them. Deleting the edges in c, one gets a rooted forest $W^c(t)$. Among the connected components of $W^c(t)$, there is a unique tree $R^c(t)$ containing the original root. The rooted forest consisting of the remaining components will be denoted by $P^c(t)$. Writing $\mathrm{Adm}_*(t)$ for the set admissible cuts of t, we define

$$\Delta t = 1 \otimes t + t \otimes 1 + \sum_{c \in \mathrm{Adm}_*(t)} P^c(t) \otimes R^c(t). \tag{3.148}$$

Since $\mathcal{H}_R$ is the free algebra in rooted trees, (3.148) extends uniquely to a coproduct

$$\Delta \colon \mathcal{H}_R \to \mathcal{H}_R \otimes \mathcal{H}_R.$$

Figure 16 below gives an example of an admissible cut of a rooted tree and its contribution to the coproduct.

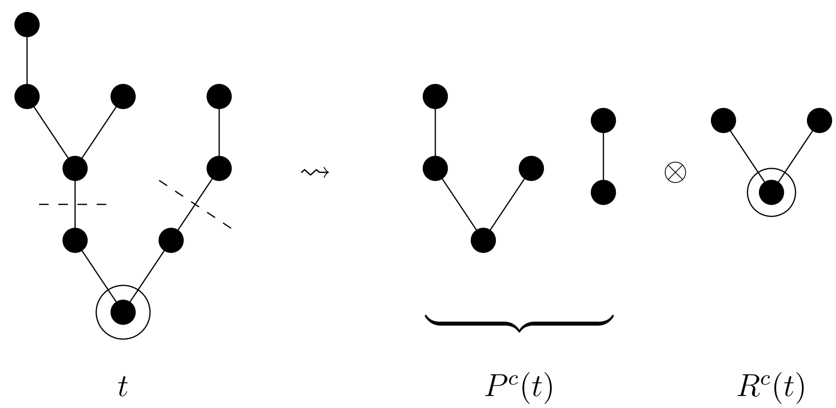


FIGURE 16. Coproduct of rooted trees

The counit is the map $\epsilon \colon \mathcal{H}_R \to \mathbb{Q}$ which sends the empty tree to 1 and everything else to zero.

i) Prove that Δ and ϵ satisfy the associativity and counit axioms from Definition 3.47. In other words, $\mathcal{H}_R$ is a bialgebra.
ii) For each integer $n \geqslant 0$, let $\mathcal{H}_R(n) \subseteq \mathcal{H}_R$ be the vector subspace generated by rooted forests with n vertices, so that
$$\mathcal{H}_R = \bigoplus_{n \geqslant 0} \mathcal{H}_R(n).$$
Observe the inclusion $\Delta \mathcal{H}_R(n) \subseteq \bigoplus_{i+j=n} \mathcal{H}_R(i) \otimes \mathcal{H}_R(j)$. Since $\mathcal{H}_R$ is obviously a graded connected algebra, by Exercise 3.133 there is a unique antipode S turning $\mathcal{H}_R$ into a Hopf algebra.
iii) Given a rooted tree t and a cut c, write n_c for the number of edges in c. Prove that the antipode is given by
$$S(t) = -t - \sum_{c \in \mathrm{Adm}_*(t)} (-1)^{n_c} W_c(t).$$

EXERCISE 3.149. Let G be an affine group scheme. By Lemma 3.51, we can write G as a projective limit $G = \varprojlim_\alpha G_\alpha$ of affine algebraic groups G_α. Use Proposition 3.124 to prove that we can assume that all maps $G \to G_\alpha$ are surjective.

3.3. Unipotent and pro-unipotent groups. In this section, we gather various properties of unipotent affine algebraic groups and pro-unipotent affine group schemes and their Lie algebras that will enter the study of the pro-unipotent completion of an abstract group in the next section.

3.3.1. *Definition of unipotent and pro-unipotent groups.* Recall that we proved in Lemma 3.51 that every affine group scheme is pro-algebraic, that is, can be written as a projective limit of affine algebraic groups.

DEFINITION 3.150. An affine algebraic group G over k (resp. an affine group scheme over k) is called *unipotent* (resp. *pro-unipotent*) if every non-zero representation of G has a non-zero fixed vector.

REMARK 3.151. In view of Lemma 3.122, to check that an affine algebraic group is unipotent or an affine group scheme is pro-unipotent, it is enough to check that non-zero *finite-dimensional* representations have a non-zero fixed vector.

EXAMPLE 3.152. Let Up_n be the affine algebraic group from Example 3.62 ii), that is, the functor that associates with each k-algebra R the subgroup of $\mathrm{GL}_n(R)$ consisting of upper triangular matrices with all diagonal entries equal to 1. For every k-vector space V of dimension n, a choice of a basis of V induces a closed immersion $\mathrm{Up}_n \to \mathrm{GL}(V)$. We will prove in Corollary 3.163 below that an affine algebraic group is unipotent if and only if it is isomorphic to a closed subgroup of some Up_n. One implication follows from the existence of faithful finite-dimensional representations; the other relies on the characterization of unipotent groups in terms of the conilpotency filtration proved in Section 3.3.2.

PROPOSITION 3.153. *Every unipotent affine algebraic group is isomorphic to a closed subgroup of some* Up_n.

PROOF. Let G be a unipotent algebraic group and let $\rho\colon G \to \mathrm{GL}(V)$ be a faithful representation of G, which exists by Corollary 3.125. Let n be the dimension of V. By definition of unipotency, V contains a non-zero fixed vector v_1. Let us choose inductively vectors $v_2, \dots, v_n$ in such a way that the image of v_{i+1} in the quotient representation $V/\langle v_1, \dots, v_i\rangle$ is a non-zero fixed vector. Then $v_1, \dots, v_n$ form a basis of V and, via the corresponding identification of $\mathrm{GL}(V)$ with GL_n, the image of ρ is contained in Up_n. Since the representation ρ is faithful, we get a closed immersion $G \to \mathrm{Up}_n$ by Proposition 3.124. □

DEFINITION 3.154. Let G be either an abstract group or an affine group scheme. A finite-dimensional representation $\rho\colon G \to \mathrm{GL}(V)$ is called *unipotent* if there exists a basis of V with respect to which the image of ρ lies inside Up_n.

It follows readily from Definitions 3.150 and 3.154 that an affine algebraic group is unipotent if and only if all its non-zero finite-dimensional representations are unipotent. The terminology pro-unipotent is justified by the following result.

PROPOSITION 3.155. *An affine group scheme G is pro-unipotent if and only if it can be written as a projective limit of unipotent affine algebraic groups.*

PROOF. Let G be a pro-unipotent affine group scheme. By Lemma 3.51, it can be written as a projective limit

$$G = \varprojlim_{\alpha} G_\alpha \tag{3.156}$$

of affine algebraic groups G_α. We assume that all the morphisms $\pi_\alpha \colon G \to G_\alpha$ are surjective, as we may by Exercise 3.149. Let V be a finite-dimensional representation of G_α. Then V defines a finite-dimensional representation of G through the map π_α. Since G is pro-unipotent, V has a fixed vector v for the action of G. Using the surjectivity of π_α, this vector is also fixed for the action of G_α, and hence the affine algebraic group G_α is unipotent.

Conversely, assume that each G_α in the limit (3.156) is unipotent, and let V be a non-zero finite-dimensional representation of G. Let $H \subset \mathrm{GL}(V)$ be the closure of the image of the representation. Then the map $\mathcal{O}(H) \to \mathcal{O}(G)$ is injective and $\mathcal{O}(H)$ is finitely generated. Taking the equality $\mathcal{O}(G) = \bigcup_\alpha \mathcal{O}(G_\alpha)$ into account, there is an inclusion $\mathcal{O}(H) \subset \mathcal{O}(G_\alpha)$ for some α. This means that the representation V factors through G_α, and hence has a non-zero fixed vector since G_α is unipotent. □

3.3.2. *The conilpotency filtration.* We give an alternative characterization of pro-unipotent affine group schemes. Let $G = \mathrm{Spec}(A)$ be an affine group scheme over k. Recall from Section 3.2.7 that $A^\vee = \mathrm{Hom}(A, k)$ has the structure of a completed Hopf algebra. In particular, it is endowed with a counit $\epsilon \colon A^\vee \to k$, that is also called an *augmentation*. Its kernel

$$J = \mathrm{Ker}(\epsilon) \subset A^\vee$$

is called the *augmentation ideal.* Denote by J^n the n-th power of the ideal J defined using the algebra structure of $A^\vee$, and consider the *annihilator*

$$C_i = \mathrm{Ann}_A\, J^{i+1},$$

that is, the set of elements $a \in A$ satisfying $\langle a, x\rangle = 0$ for all $x \in J^{i+1} \subset A^\vee$. Here we are again using the notation $\langle \cdot, \cdot \rangle$ for the pairing between a vector space and its dual. The *conilpotency filtration* is the filtration of A defined by

$$0 \subset C_0 \subset C_1 \cdots \subset C_i \subset \cdots$$

It is easy to see that $C_0 = k \cdot 1$, where 1 is the unit of A, and that the conilpotency filtration is compatible with the coproduct, in that there is an inclusion

$$\Delta C_i \subset \sum_{a+b=i} C_a \otimes C_b. \tag{3.157}$$

Proposition 3.158. *The affine group scheme $G = \mathrm{Spec}(A)$ is pro-unipotent if and only if the conilpotency filtration of A is exhaustive, that is:*

$$A = \bigcup_{i=0}^{\infty} C_i.$$

Proof. We first assume that the conilpotency filtration is exhaustive. Let V be a non-zero representation of $G = \mathrm{Spec}(A)$ and denote by $\Delta \colon V \to V \otimes A$ the corresponding comodule structure given by Lemma 3.118. The filtration $\{C_i\}_{i \geqslant 0}$ of A being exhaustive, we deduce that the filtration $\{V_i\}_{i \geqslant 0}$ of V given by

$$V_i = \{v \in V \mid \Delta v \in V \otimes C_i\}$$

is also exhaustive. By the compatibility with the counit in the axioms defining a comodule (see Definition 3.54), if $v \in V_0$, then $\Delta v = v \otimes 1$. Any vector $v \in V_0$ is hence a fixed vector for the representation, and to prove that G is pro-unipotent is enough to show that V_0 is non-zero. To this end, we show that the vanishing of V_i implies that of V_{i+1}. Indeed, assume that $V_i = 0$ and let $v \in V_{i+1}$. Using the inclusion (3.157), we get

$$(1 \otimes \Delta)\Delta v \in \sum_{a+b=i+1} V \otimes C_a \otimes C_b.$$

Since a and b cannot be both bigger than i, the vector v is sent to zero by the map

$$V \xrightarrow{\Delta} V \otimes A \xrightarrow{1\otimes\Delta} V \otimes A \otimes A \longrightarrow V \otimes A/C_i \otimes A/C_i.$$

But, by the associativity property of comodules, this map agrees with the map

$$V \xrightarrow{\Delta} V \otimes A \xrightarrow{\Delta\otimes 1} V \otimes A \otimes A \longrightarrow V \otimes A/C_i \otimes A/C_i,$$

which is an injection since $V_i = 0$. Thus, $v = 0$, and hence $V_{i+1} = 0$.

Conversely, assume that every non-zero representation of G has a non-zero fixed vector. Then every representation V has an increasing filtration $\{V_i\}_{i\geqslant 0}$ recursively defined by taking the set of fixed vectors as V_0, and the set of elements whose image in the representation V/V_i are fixed vectors as V_{i+1}. This filtration is exhaustive by Lemma 3.122. Let now V be the representation given by A itself with the coaction determined by the coproduct of A. Then

$$V_0 = \{a \in A \mid \Delta a = a \otimes 1\} = k \quad \text{and} \quad V_i = \{a \in A \mid \Delta a \in a \otimes 1 + V_{i-1} \otimes A\}$$

for all $i \geqslant 1$. We show by induction that $V_i \subset C_i$, that is, the conilpotency filtration contains this filtration. The case $n = 0$ follows from the equalities $V_0 = k = C_0$. Assume that $V_i \subset C_i$ and let $a \in V_{i+1}$. By the characterization of V_{i+1}, we see that Δa can be written as

$$\Delta a = a \otimes 1 + \sum_j b_j \otimes c_j$$

with $b_j \in V_i$. Let $x \in J^{i+1}$ and $y \in J$. Then

$$\langle a, xy\rangle = \langle \Delta a, x \otimes y\rangle = \langle a, x\rangle\langle 1, y\rangle + \sum_j \langle b_j, x\rangle\langle c_j, y\rangle = 0$$

because $\langle 1, y\rangle = 0$ and $\langle b_i, x\rangle = 0$. Therefore, $a \in \mathrm{Ann}(J^{i+2}) = C_{i+1}$. Since the filtration $\{V_i\}_{i\geqslant 0}$ is exhaustive, the same is true for the filtration $\{C_i\}_{i\geqslant 0}$. $\square$

EXAMPLE 3.159. The multiplicative group $\mathbb{G}_m$ is isomorphic to GL_1 and the standard representation of GL_1 on the one-dimensional vector space does not have any non-zero fixed vector. Thus, $\mathbb{G}_m$ is not unipotent. By contrast, the additive group $\mathbb{G}_a$ is isomorphic to Up_2 through the morphism $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, and hence it should be unipotent by Proposition. In this example, we illustrate Proposition 3.158 using both groups.

As seen in Example 3.72, the dual $\mathcal{O}(\mathbb{G}_m)^\vee$ is the algebra

$$\prod_{n\in\mathbb{Z}} k y_n$$

with product $(y_n)^2 = y_n$ and $y_n y_m = 0$ for $n \neq m$. The augmentation is given by $\epsilon(y_0) = 1$ and $\epsilon(y_n) = 0$ for $n \neq 0$. Hence, the augmentation ideal is equal to

$$J = \prod_{n\neq 0} k y_n$$

and all its powers J^i are equal to J, whence

$$C_i = \operatorname{Ann} J^{i+1} = \operatorname{Ann} J = k.$$

It follows that the conilpotency filtration is not exhaustive, in agreement with the fact that $\mathbb{G}_m$ is not a unipotent group.

On the other hand, as shown in Example 3.71, the dual $\mathcal{O}(\mathbb{G}_a)^\vee$ is the algebra of power series $k[\![y]\!]$ with augmentation $\epsilon(1) = 1$ and $\epsilon(y^n) = 0$ for $n \neq 0$. Therefore, the augmentation ideal is equal to $J = yk[\![y]\!]$ and the equality

$$J^i = y^i k[\![y]\!]$$

holds for all $i \geqslant 1$. Since $y^n/n!$ is the dual of x^n, the annihilator

$$C_i = \operatorname{Ann} J^{i+i} = k[x]_{\leqslant i}$$

is the space of polynomials of degree less than or equal to i. The conilpotency filtration is hence exhaustive, in agreement with the fact that $\mathbb{G}_a$ is unipotent.

Although the conilpotency filtration has the advantage of being canonical, a closer inspection of the proof of Proposition 3.158 shows that any filtration with similar properties would be enough to characterize pro-unipotent groups. This is made explicit in the following lemma, which is useful in situations where it might be tricky to determine the exact shape of the conilpotency filtration. For example, we will use it in Proposition 3.161 to prove that the subgroups of Up_n are unipotent.

Lemma 3.160. *Let $G = \operatorname{Spec}(A)$ be an affine group scheme. Assume that there exists an increasing exhaustive filtration $\{F_i\}_{i\geqslant 0}$ of the k-vector space A satisfying the conditions $F_0 = k$ and $\Delta F_i \subseteq \sum_{a+b=i} F_a \otimes F_b$. Then the conilpotency filtration on A is exhaustive, and hence G is pro-unipotent.*

Proof. Consider the decreasing filtration $\{W^{i+1}\}_{i\geqslant 0}$ on $A^\vee$ defined by

$$W^{i+1} = \operatorname{Ann}_{A^\vee} F_i.$$

That is, $W^{i+1} \subset A^\vee$ consists of those linear forms that vanish on all elements of F_i. Since J is the kernel of the counit, which is nothing but the evaluation map at constants inside A, the condition $F_0 = k$ is equivalent to $W^1 = J$. Moreover, the compatibility between the filtration and the coproduct implies $W^a \cdot W^b \subset W^{a+b}$, for the algebra structure on $A^\vee$ dual to the coalgebra structure on A. Indeed, the coproduct of an element $z \in F_{a+b-1}$ takes the form $\Delta z = \sum_{r+s=a+b-1} z_r \otimes z_s$ for some $z_r \in F_r$ and $z_s \in F_s$; hence, $\langle z, x \cdot y\rangle = \langle \Delta z, x \otimes y\rangle = 0$ holds for all $x \in W^a$ and $y \in W^b$, since the terms appearing in the sum either satisfy $r < a$, in which case $\langle z_r, x\rangle = 0$, or $s < b$, in which case $\langle z_s, y\rangle = 0$. From $W^1 = J$ and the multiplicativity of the filtration W^*, we deduce an inclusion $J^{i+1} \subseteq W^{i+1}$, and hence the annihilator $C_i = \operatorname{Ann}_A J^{i+1}$ contains F_i. The latter filtration being exhaustive by assumption, so is the conilpotency filtration. $\square$

Proposition 3.161. *Every closed subgroup of Up_n is unipotent.*

Proof. Recall that the algebraic group Up_n is the spectrum of the Hopf algebra $A = k[(x_{ij})_{1\leqslant i<j\leqslant n}]$, on which the coproduct is given by the formula

$$\Delta x_{ij} = 1 \otimes x_{ij} + \sum_{i<k<j} x_{ik} \otimes x_{kj} + x_{ij} \otimes 1 \tag{3.162}$$

for all $i < j$ (see Example 3.60 vi) for the case of GL_n). Let F_* be the increasing filtration of A associated with the algebra grading $A = \bigoplus_{r \geqslant 0} A_r$ determined by the properties $A_0 = k$ and $x_{ij} \in A_{j-i}$, that is, $F_i = \bigoplus_{r \leqslant i} A_r$. Clearly, F_* is exhaustive and satisfies $F_0 = k$. Moreover, the condition $\Delta F_i \subset \sum_{a+b=i} F_a \otimes F_b$ follows from formula (3.162). It then results from Lemma 3.160 that Up_n is unipotent.

Let us now consider a closed subgroup $H \subset \mathrm{Up}_n$, given by $H = \mathrm{Spec}(B)$ for a quotient $B = A/I$ of A. The filtration F_* of A induces a filtration $\overline{F}_*$ of B, which satisfies the conditions $\overline{F}_0 = k$ and $\Delta \overline{F}_i \subseteq \sum_{a+b=i} \overline{F}_a \otimes \overline{F}_b$ because $A \to B$ is a morphism of Hopf algebras. Since F_* is exhaustive and A surjects onto B, the filtration $\overline{F}_*$ is also exhaustive, and hence H is unipotent. □

Combined with Proposition 3.153, this yields the following characterization of unipotent affine algebraic groups.

Corollary 3.163. *An affine algebraic group is unipotent if and only if it is isomorphic to a closed subgroup of some* Up_n.

3.3.3. *Nilpotent and quasi-nilpotent Lie algebras.* In this section, we isolate a class of Lie algebras that are the counterpart of pro-unipotent affine group schemes.

Definition 3.164. Let L be a Lie algebra over k. The *lower central series* of L is the descending filtration

$$L \supset L^{(1)} \supset L^{(2)} \supset \cdots$$

given by $L^{(0)} = L$ and $L^{(i+1)} = [L, L^{(i)}]$ for all integers $i \geqslant 0$.

Definition 3.165. A Lie algebra L over k is said to be *nilpotent* if its lower central series $L^{(i)}$ is eventually zero, *i.e.* there exists an integer n such that

$$[x_1, [x_2, [\cdots [x_n, y] \cdots] = 0 \text{ for all } x_1, \ldots, x_n, y \in L. \tag{3.166}$$

It is said to be *quasi-nilpotent* if the lower central series is a separated filtration:

$$\bigcap_{i \geqslant 0} L^{(i)} = \{0\}.$$

Every nilpotent Lie algebra is obviously quasi-nilpotent, and the two notions agree in finite dimension. Free Lie algebras (see Definition 3.195 below) provide examples of infinite-dimensional quasi-nilpotent Lie algebras that are not nilpotent.

Example 3.167. For each $n \geqslant 1$, the vector subspace $\mathfrak{u}_n \subset \mathfrak{gl}_n$ consisting of all strictly upper triangular matrices is a nilpotent Lie algebra. Indeed, $\mathfrak{u}_n$ is a Lie subalgebra since the property of being strictly upper triangular is preserved under products and sums of matrices. It is nilpotent because in the product of r elements of $\mathfrak{u}_n$, and hence in all elements of the $(r-1)$th step of the lower central series, all the entries x_{ij} with $j - i \leqslant r$ are zero.

For finite-dimensional Lie algebras L, condition (3.166) holds as soon as, for each $x \in L$, there exists an integer $n \geqslant 1$ such that

$$\underbrace{[x, [\cdots [x}_{n}, y] \cdots] = 0$$

holds for all $y \in L$. This is the statement of *Engel's theorem*. One of its consequences, sometimes also referred to by the same name, is the following.

PROPOSITION 3.168. *Let V be a vector space of dimension n and let $L \subset \mathfrak{gl}(V)$ be a Lie subalgebra consisting of nilpotent endomorphisms. Then there exists a basis of V such that, under the identification $\mathfrak{gl}(V) = \mathfrak{gl}_n$, the subalgebra L is contained in the Lie algebra $\mathfrak{u}_n$ from Example 3.167. In other words, by a single choice of basis, all the endomorphisms of L can be turned into strictly upper triangular matrices.*

A proof is given in [**Jac62**, Chap. 2, §3].

LEMMA 3.169. *The Lie algebra of a unipotent affine algebraic group is nilpotent.*

PROOF. Let U be a unipotent affine algebraic group and let $\mathfrak{u} = \mathrm{Lie}(U)$ be its Lie algebra. By Proposition 3.153, the group U can be embedded as a closed subgroup of Up_n for some integer n, whose Lie algebra is $\mathfrak{u}_n$, the Lie algebra of strictly upper triangular matrices, by Example 3.93 iii). Hence, $\mathfrak{u}$ is a Lie subalgebra of $\mathfrak{u}_n$. The result then follows from the fact that $\mathfrak{u}_n$ is nilpotent by Example 3.167 and that any Lie subalgebra of a nilpotent algebra is again nilpotent. □

COROLLARY 3.170. *The Lie algebra of a pro-unipotent affine group scheme is quasi-nilpotent.*

PROOF. Let U be a pro-unipotent affine group scheme, written as

$$U = \varprojlim_{\alpha} U_\alpha$$

for a projective system of unipotent affine algebraic groups U_α. By functoriality, the Lie algebra $\mathfrak{u}$ of U can then be written as the projective limit $\mathfrak{u} = \varprojlim_\alpha \mathfrak{u}_\alpha$ of the finite-dimensional Lie algebras $\mathfrak{u}_\alpha = \mathrm{Lie}(U_\alpha)$. In particular, there is an injection

$$\mathfrak{u} \hookrightarrow \prod_{\alpha} \mathfrak{u}_\alpha.$$

By Lemma 3.169, each Lie algebra $\mathfrak{u}_\alpha$ is nilpotent, which means that the space $\mathfrak{u}^{(n_\alpha)}$ of the lower central series vanishes for some n_α. Therefore, an element $x \in \bigcap_{i \geqslant 0} \mathfrak{u}^{(i)}$ has zero image in all $\mathfrak{u}^{(\alpha)}$, and is hence zero itself. This shows that the Lie algebra $\mathfrak{u}$ is quasi-nilpotent. □

DEFINITION 3.171. A Lie algebra L is called *pro-nilpotent* if it can be written as a projective limit of finite-dimensional nilpotent Lie algebras:

$$L = \varprojlim_{\alpha} L_\alpha.$$

This definition is analogous to that of pro-unipotent affine group scheme in that we enforce the finite-dimensionality condition on each Lie algebra L_α. The same argument as in the proof of Corollary 3.170 implies that every pro-nilpotent Lie algebra is quasi-nilpotent.

DEFINITION 3.172. Let L be a Lie algebra with finite-dimensional abelianization $L/[L, L]$. The *pro-nilpotent completion* of L is the pro-nilpotent Lie algebra

$$\widehat{L} = \varprojlim_{i} L/L^{(i+1)}.$$

REMARK 3.173. Every finitely generated Lie algebra (for example, a free Lie algebra on a finite set of generators) satisfies the assumption of this definition. By Exercise 3.221 the finite-dimensionality of the abelianization $L/[L, L]$ implies that of $L/L^{(i+1)}$. Moreover, the lower central series of $L/L^{(i+1)}$ is given by $L^{(i)}/L^{(i+1)}$, so $L/L^{(i+1)}$ is nilpotent. Hence, the pro-nilpotent completion $\widehat{L}$ is indeed pro-nilpotent. If L is quasi-nilpotent, then the canonical map $L \to \widehat{L}$ is injective.

The completion could be defined without the assumption that $L/[L, L]$ is finite-dimensional, but in general it will not be a pro-nilpotent Lie algebra. A variant of this construction for graded Lie algebras will be introduced in Section 3.3.8.

3.3.4. *The exponential map.* One of the main tools in the classical theory of Lie groups is the exponential map from $\mathrm{Lie}(G)$ to G. Even if G is algebraic, this map is not algebraic in general (for example, it is the usual exponential function for $G = \mathbb{G}_m$), so it is not straightforward how to construct it in the setting of affine group schemes and their Lie algebras. In the case of unipotent group schemes, however, the exponential map turns to be algebraic. In what follows, we write

$$R[\varepsilon] = R[x]/x^2 = R[\![x]\!]/x^2$$

for the *ring of dual numbers* associated with a k-algebra R.

Proposition 3.174. *Let G be an affine group scheme over k and $\mathfrak{g}$ its Lie algebra. For each k-algebra R and each $x \in \mathfrak{g} \otimes R$, there exists a unique element*

$$\exp(Tx) \in G(R[\![T]\!])$$

satisfying the following conditions:

i) *the image $\exp(\varepsilon x)$ of $\exp(Tx)$ in $G(R[\varepsilon])$ is equal to $e^{\varepsilon x}$, where $e^{\varepsilon x}$ is the element defined in (3.90);*

ii) *the equality $\exp((T + T')x) = \exp(Tx)\exp(T'x)$ holds in $G(R[\![T, T']\!])$. Here $\exp((T+T')x)$ represents the image of $\exp(Tx)$ in $G(R[\![T, T']\!])$ under the map induced by the algebra morphism $R[\![T]\!] \to R[\![T, T']\!]$ that sends T to $T + T'$.*

This result is proved in [**DG70**, II §6, Prop. 3.1]. We now look for conditions ensuring that the element $\exp(Tx)$ lies in $G(R[T])$.

Definition 3.175. Let G be an affine algebraic group with Lie algebra $\mathfrak{g}$. An element $x \in \mathfrak{g} \otimes R$ is said to be *nilpotent* if there exists a faithful representation $\rho\colon G \to \mathrm{GL}(V)$ such that $\mathrm{d}\rho(x)$ is a nilpotent endomorphism of $V \otimes R$.

The next result is [**DG70**, II §6 Cor. 3.5].

Proposition 3.176. *If x is nilpotent, then $\exp(Tx)$ belongs to $G(R[T])$.*

Thanks to Proposition 3.176, we can make the following definition.

Definition 3.177. Let $x \in \mathfrak{g}\otimes R$ be a nilpotent element. Then we define $\exp(x)$ as the image of $\exp(Tx)$ under the map $G(R[T]) \to G(R)$ induced by $T \mapsto 1$.

Let us now assume that U is a unipotent affine algebraic group, and let $\mathfrak{u}$ denote its Lie algebra. Thanks to Proposition 3.153, there exists a faithful representation $\rho\colon U \to \mathrm{GL}(V)$ such that $\rho(U)$ lies in Up_n for a suitable choice of basis of V. Every element of $\mathfrak{u} \otimes R$ is hence nilpotent, and we obtain maps

$$\exp\colon \mathfrak{u} \otimes R \longrightarrow U(R) \tag{3.178}$$

thanks to Proposition 3.176. These maps are functorial with respect to morphisms of k-algebras. Since $\mathfrak{u}$ is finite-dimensional, the functor $R \mapsto \mathfrak{u}\otimes R$ is represented by a k-scheme that we will also denote by $\mathfrak{u}$, and (3.178) is the evaluation on R-points of a morphism of schemes $\exp\colon \mathfrak{u} \to U$.

THEOREM 3.179. *Let U be a unipotent affine algebraic group and $\mathfrak{u}$ its Lie algebra, viewed as a scheme. Then* $\exp\colon \mathfrak{u} \to U$ *is an isomorphism of schemes. In particular, $\mathcal{O}(U)$ is a polynomial algebra (i.e. a free commutative algebra).*

This result is proved in [**DG70**, IV §2 Prop. 4.1].

COROLLARY 3.180. *Let U be a pro-unipotent affine group scheme and $\mathfrak{u}$ its Lie algebra. If $\mathfrak{u}$ is given the pro-algebraic scheme structure from Remark 3.92, then the exponential* $\exp\colon \mathfrak{u} \to U$ *is an isomorphism of pro-algebraic schemes.*

PROOF. Write U as a projective limit $U = \varprojlim U_\alpha$ of unipotent affine algebraic groups U_α, and let $\mathfrak{u}_\alpha = \mathrm{Lie}(U_\alpha)$. Since the isomorphisms of schemes $\exp\colon \mathfrak{u}_\alpha \to U_\alpha$ from Theorem 3.179 are functorial, they induce an isomorphism of pro-algebraic schemes $\exp\colon \mathfrak{u} \to U$. □

DEFINITION 3.181. Let U be a pro-unipotent affine group scheme and $\mathfrak{u}$ its Lie algebra, with the structure of a pro-algebraic scheme from Remark 3.92. The *logarithm* is the isomorphism of schemes $\log\colon U \to \mathfrak{u}$ inverse to exp.

The phenomenon we saw in Example 3.108 i), namely the existence of an isomorphism from the completed universal enveloping algebra of the additive group to the dual of its ring of functions, is characteristic of pro-unipotent groups.

THEOREM 3.182. *Let U be a pro-unipotent affine group scheme and let $\mathfrak{u}$ be its Lie algebra. Assume that $\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$ is finite-dimensional. Then the canonical morphism $U(\mathfrak{u}) \to \mathcal{O}(U)^\vee$ from (3.107) extends to an isomorphism $U(\mathfrak{u})^\wedge \to \mathcal{O}(U)^\vee$ on the completed universal enveloping algebra.*

PROOF. We first show that the canonical map $U(\mathfrak{u}) \to \mathcal{O}(U)^\vee$ can be extended to a map $U(\mathfrak{u})^\wedge \to \mathcal{O}(U)^\vee$. Indeed, let $f \in \mathcal{O}(U)$ and $\mu \in U(\mathfrak{u})^\wedge$. Since the conilpotency filtration on $\mathcal{O}(U)$ is exhaustive by Proposition 3.158, there exists an integer i such that f lies in the annihilator $C_i = \mathrm{Ann}_{\mathcal{O}(U)} J^{i+1}$. We can then define $\mu(f)$ as the image of f by the image of μ in $U(\mathfrak{u})/J^{i+1}$. Since the original map $U(\mathfrak{u}) \to \mathcal{O}(U)^\vee$ is compatible with the product and the (completed) coproduct on the Hopf algebras $U(\mathfrak{u})$ and $\mathcal{O}(U)^\vee$, the same holds for this extension.

Let us now show that the map $U(\mathfrak{u})^\wedge \to \mathcal{O}(U)^\vee$ is an isomorphism, for which it suffices to check that it is an isomorphism of vector spaces. We first consider the case of a unipotent affine algebraic group U. Let $A = \mathcal{O}(U)$ be its ring of regular functions and $I \subset A$ the ideal of those vanishing at the neutral element $e \in U(k)$. By Theorem 3.179, A is a polynomial algebra, say $A = k[T_1, \ldots, T_r]$. Up to translating the variables by elements of k, we may assume that I is generated by $T_1, \ldots, T_r$. Let $A_{\leqslant n}$ denote the vector space of polynomials of degree less than or equal to n. As vector spaces, there is a decomposition

$$A = A_{\leqslant n} \oplus I^{n+1}, \tag{3.183}$$

hence isomorphisms of vector spaces $A_{\leqslant n} \simeq A/I^{n+1}$. Since $A = \varinjlim A_{\leqslant n}$, we have

$$A^\vee \simeq \varprojlim (A_{\leqslant n})^\vee.$$

By Theorem 3.114, the universal enveloping algebra of $\mathfrak{u}$ and the space of distributions on U are isomorphic: $U(\mathfrak{u}) \simeq \mathrm{Dist}(U)$. This isomorphism sends $F_n U(\mathfrak{u})$ to $\mathrm{Dist}_n(U)$. Besides, we saw in Example 3.111 iii) that the space $\mathrm{Dist}_n(U)$ is isomorphic to the dual of A/I^{n+1}. Hence, we obtain an identification

$$F_n U(\mathfrak{u}) \simeq (A/I^{n+1})^\vee. \tag{3.184}$$

There is a decomposition

$$U(\mathfrak{u}) = F_n U(\mathfrak{u}) \oplus J^{n+1}.$$

Together with the decomposition (3.183) and the identification (3.184), it gives an identification

$$U(\mathfrak{u})/J^{n+1} \simeq (A_{\leqslant n})^{\vee}.$$

By construction, the map $U(\mathfrak{u})^{\wedge} \to A^{\vee}$ fits into a commutative diagram

$$\begin{tikzcd}
U(\mathfrak{u})^{\wedge} \arrow[r] \arrow[d] & U(\mathfrak{u})/J^{n+1} \arrow[d, "\simeq"] \\
A^{\vee} \arrow[r] & (A_{\leqslant n})^{\vee}
\end{tikzcd}$$

for each n, which implies that it is an isomorphism.

Let us now move to the general case where U is a pro-unipotent affine group scheme such that $\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$ is finite-dimensional. Let $J \subset U(\mathfrak{u})$ be the augmentation ideal. By Corollary 3.104, the space J/J^2 is finite-dimensional. It follows that all quotients J^N/J^{N+1} are finite-dimensional, since the map $(J/J^2)^{\otimes N} \to J^N/J^{N+1}$ is surjective. Then, induction and the short exact sequence

$$0 \longrightarrow J^N/J^{N+1} \longrightarrow U(\mathfrak{u})/J^{N+1} \longrightarrow U(\mathfrak{u})/J^N \longrightarrow 0$$

imply that $U(\mathfrak{u})/J^{N+1}$ is finite-dimensional for all N.

Let us now write the pro-unipotent group U as

$$U = \varprojlim_{\alpha} U_\alpha$$

with the groups U_α unipotent and the maps $U \to U_\alpha$ surjective (Exercise 3.149). Let $\mathfrak{u}_\alpha$ be the Lie algebra of U_α and J_α the augmentation ideal of $U(\mathfrak{u}_\alpha)$. The fact that $U(\mathfrak{u})/J^{N+1}$ is finite-dimensional implies that, for each N there is a big enough index α_0 such that, for $\alpha \geqslant \alpha_0$ we have $U(\mathfrak{u}_\alpha)/J_\alpha^{N+1} = U(\mathfrak{u})/J^{N+1}$. Therefore, using that projective limits commute with each other, we get

$$\begin{aligned}
U(\mathfrak{u})^{\wedge} &= \varprojlim_{N} U(\mathfrak{u})/J^{N+1} \\
&= \varprojlim_{N} \varprojlim_{\alpha} U(\mathfrak{u}_\alpha)/J_\alpha^{N+1} \\
&= \varprojlim_{\alpha} \varprojlim_{N} U(\mathfrak{u}_\alpha)/J_\alpha^{N+1} \\
&= \varprojlim_{\alpha} U(\mathfrak{u}_\alpha)^{\wedge}.
\end{aligned} \tag{3.185}$$

Since the maps $U \to U_\alpha$ are surjective, the maps $A_\alpha = \mathcal{O}(U_\alpha) \to \mathcal{O}(U) = A$ are injective and we have isomorphisms

$$U(\mathfrak{u})^{\wedge} = \varprojlim_{\alpha} U(\mathfrak{u}_\alpha)^{\wedge} \simeq \varprojlim_{\alpha} A_\alpha^{\vee} = A^{\vee}.$$

This finishes the proof of the theorem. □

3.3.5. *Primitive elements and group-like elements.* In any completed Hopf algebra H, such as the completed universal enveloping algebra $U(\mathfrak{u})^{\wedge}$, the notion of primitive element is the same as in Definition 3.101. Namely, an element $v \in H$ is called primitive if it satisfies

$$\Delta v = 1 \otimes v + v \otimes 1.$$

We let $P(H)$ denote the set of primitive elements of H.

LEMMA 3.186. *Let H be a completed Hopf algebra. If v is a primitive element of H, then $\epsilon(v) = 0$. In other words, $v \in \operatorname{Ker}(\epsilon)$.*

PROOF. We use the relation $(\mathrm{Id} \otimes \epsilon) \circ \Delta = \mathrm{Id}$ to obtain

$$v = (\mathrm{Id} \otimes \epsilon)(1 \otimes v + v \otimes 1) = \epsilon(v) + v,$$

and hence $\epsilon(v) = 0$. □

COROLLARY 3.187. *Let U be a pro-unipotent affine group scheme and let $\mathfrak{u}$ be its Lie algebra. Assume that $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ is finite-dimensional. Then the composition*

$$\mathfrak{u} \longrightarrow U(\mathfrak{u}) \longrightarrow U(\mathfrak{u})^{\wedge}$$

is injective and identifies $\mathfrak{u}$ with the set of primitive elements of $U(\mathfrak{u})^{\wedge}$.

PROOF. We identify $U(\mathfrak{u})^{\wedge}$ with $\mathcal{O}(G)^{\vee}$ through the isomorphism from Theorem 3.182. Under this identification, the map $\mathfrak{u} \to U(\mathfrak{u})^{\wedge}$ corresponds to the inclusion $(I/I^2)^{\vee} \to \mathcal{O}(G)^{\vee}$, which is injective. Clearly, the image of $\mathfrak{u}$ is contained in the space of primitive elements of $U(\mathfrak{u})^{\wedge}$. Conversely, let us show that each primitive element $a \in \mathcal{O}(G)^{\vee}$ vanishes on I^2 and on $\eta(k)$. Indeed, if $x, y \in I$, then

$$\langle a, xy \rangle = \langle \Delta a, x \otimes y \rangle = \langle 1 \otimes a + a \otimes 1, x \otimes y \rangle = \epsilon(x)\langle a, y \rangle + \epsilon(y)\langle a, x \rangle = 0.$$

Besides, by Lemma 3.186, we have

$$\langle a, \eta(1) \rangle = \langle \epsilon(a), 1 \rangle = 0.$$

We deduce that a primitive element lies in $\mathfrak{u}$, as wanted. □

Let now H be a Hopf algebra, with augmentation ideal $J = \operatorname{Ker} \epsilon$, and let $H^{\wedge}$ be its completion with respect to J.

DEFINITION 3.188. An element $x \in H^{\wedge}$ is called *group-like* if the equalities

$$\epsilon(x) = 1 \quad \text{and} \quad \Delta x = x \otimes x$$

hold. We write $\mathcal{G}(H^{\wedge})$ for the set of group-like elements of $H^{\wedge}$.

LEMMA 3.189. *Group-like elements of $H^{\wedge}$ form a group under product.*

PROOF. Since the counit and the coproduct are morphisms of algebras, the product of two group-like elements is also group-like. The compatibility

$$\nabla \circ (S \otimes \mathrm{Id}) \circ \Delta = \eta \circ \epsilon$$

for a group-like element reads $S(x)x = 1$. Therefore, $S(x)$ is the inverse of x. □

For each $n \geqslant 1$, we define the ideal

$$J_n = \operatorname{Ker}(H^{\wedge} \longrightarrow H/J^n)$$

of $H^{\wedge}$. In particular, $J_1 = \operatorname{Ker}(\epsilon\colon H^{\wedge} \to k)$. There is an inclusion $(J_1)^n \subset J_n$ and

$$H^{\wedge} = \varprojlim H^{\wedge}/J_{n+1}. \tag{3.190}$$

With a primitive element $v \in H^\wedge$ we associate its exponential

$$\exp(v) = \sum_{n=0}^{\infty} \frac{v^n}{n!} \in H^\wedge, \tag{3.191}$$

which is to be understood as the element of the projective limit (3.190) consisting of the class of the partial sum $\sum_{m=0}^{n} \frac{v^m}{m!}$ in $H^\wedge/J_{n+1}$ for each n. This makes sense because v^n belongs to J_1^n by Lemma 3.186.

With a group-like element $x \in H^\wedge$ we associate its logarithm

$$\log(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \in H^\wedge, \tag{3.192}$$

understood as above since it is part of the definition of a group-like element (Definition 3.188) that $x - 1$ belongs to J_1.

PROPOSITION 3.193. *The series* exp *and* log *determine bijections*

$$\begin{tikzcd} \mathcal{G}(H^\wedge) \arrow[r, shift left, "\log"] & P(H^\wedge) \arrow[l, shift left, "\exp"] \end{tikzcd}$$

inverse of each other.

PROOF. The fact that the exponential and the logarithm are inverse of each other is standard, so that we only need to check that the exponential sends primitive elements to group-like elements and that the logarithm sends group-like elements to primitive elements. For this, let $v \in H^\wedge$ be a primitive element. Since the exponential series starts with 1, the condition $\epsilon(\exp(v)) = 1$ holds Moreover,

$$\begin{aligned} \Delta \sum_{n=0}^{\infty} \frac{v^n}{n!} &= \sum_{n=0}^{\infty} \frac{(1 \otimes v + v \otimes 1)^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{p=0}^{n} \binom{n}{p} \frac{v^p \otimes v^{n-p}}{n!} \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{v^p}{p!} \otimes \frac{v^q}{q!} \\ &= \exp(v) \otimes \exp(v). \end{aligned}$$

Thus, $\exp(v)$ is group-like. Let now $x \in H^\wedge$ be a group-like element. The identity

$$\log(1 + a + b + ab) = \log((1 + a)(1 + b)) = \log(1 + a) + \log(1 + b)$$

translates into the equality of power series in commuting variables

$$\sum_{n=1}^{\infty} (-1)^n \frac{(a + b + ab)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{a^n}{n} + \sum_{n=1}^{\infty} (-1)^n \frac{b^n}{n}. \tag{3.194}$$

Using this identity, we compute

$$\begin{aligned}
\Delta \sum_{n=1}^{\infty}(-1)^n \frac{(x-1)^n}{n} &= \sum_{n=1}^{\infty}(-1)^n \frac{(x\otimes x - 1\otimes 1)^n}{n} \\
&= \sum_{n=1}^{\infty}(-1)^n \frac{(1\otimes(x-1)+(x-1)\otimes 1+(x-1)\otimes(x-1))^n}{n} \\
&= \sum_{n=1}^{\infty}(-1)^n \frac{(1\otimes(x-1))^n}{n} + \sum_{n=1}^{\infty}(-1)^n \frac{((x-1)\otimes 1)^n}{n},
\end{aligned}$$

which implies that $\Delta \log(x) = 1\otimes\log(x)+\log(x)\otimes 1$. Hence, $\log(x)$ is primitive. □

3.3.6. *Free Lie algebras and the Baker–Campbell–Hausdorff formula.* We now study some properties of free Lie algebras and free associative algebras, and use them to derive the Baker–Campbell–Hausdorff formula.

Definition 3.195. Let S be a set. The *free Lie algebra* $\mathrm{Lie}\langle S\rangle$ generated by S is the unique Lie algebra with a map $S \to \mathrm{Lie}\langle S\rangle$ that satisfies the following universal property: for each Lie algebra L and each map $S \to L$, there exists a unique morphism of Lie algebras $\mathrm{Lie}\langle S\rangle \to L$ that makes the diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[dr] & \mathrm{Lie}\langle S\rangle \arrow[d] \\
 & L
\end{tikzcd}$$

commutative. Similarly, the *free associative algebra* $k\langle S\rangle$ generated by S is the unique associative k-algebra along with a map $S \to k\langle S\rangle$ such that, for each associative k-algebra A and each map $S \to A$, there exists a unique morphism of k-algebras $k\langle S\rangle \to A$ that makes the following diagram commutative:

$$\begin{tikzcd}
S \arrow[r] \arrow[dr] & k\langle S\rangle. \arrow[d] \\
 & A
\end{tikzcd}$$

Observe that the free associative algebra generated by S is nothing but the algebra of non-commuting polynomials introduced in Notation 1.148. Therefore, if $\langle S\rangle_k$ is the k-vector space with basis S, then

$$k\langle S\rangle = T(\langle S\rangle_k)$$

is the tensor algebra generated by $\langle S\rangle_k$. As the following result shows, the free Lie algebra and the associative algebra generated by the same set of elements are related by means of the universal enveloping algebra.

Proposition 3.196. *Let S be a set. Then*

$$U(\mathrm{Lie}\langle S\rangle) = k\langle S\rangle.$$

Proof. Let X be an associative k-algebra, that we will also regard as Lie algebra (Remark 3.83) or even as a bare set. We have the following chain of equalities that follow from the universal properties of the involved objects.

$$\begin{aligned}\mathrm{Hom}_{k\text{-alg}}(U(\mathrm{Lie}\langle S\rangle), X) &= \mathrm{Hom}_{\mathrm{Lie}}(\mathrm{Lie}\langle S\rangle, X)\\ &= \mathrm{Hom}_{\mathrm{sets}}(S, X)\\ &= \mathrm{Hom}_{k\text{-alg}}(k\langle S\rangle, X).\end{aligned}$$

By Yoneda's Lemma, $U(\mathrm{Lie}\langle S\rangle) = k\langle S\rangle$. □

We assume from now on that S is at most countable. From Proposition 3.196 and Corollary 3.102, we recover a theorem by Friedrichs (see [**Jac62**, V Thm. 9]).

Corollary 3.197. *Let S be a countable set and let $F = k\langle S\rangle$ be the free associative algebra generated by S. Let Δ be the coproduct on F determined by being an algebra morphism and satisfying $\Delta x = 1\otimes x + x\otimes 1$ for all $x\in S$. Then the space of primitive elements $P(F)$ is the free Lie algebra generated by S.*

Proof. By Proposition 3.196, there is an isomorphism $k\langle S\rangle = U(\mathrm{Lie}\langle S\rangle)$ that respects the coproduct, and in particular sends primitive elements to primitive elements. By Corollary 3.102, the primitive elements of $U(\mathrm{Lie}\langle S\rangle)$ are given by

$$P(U(\mathrm{Lie}\langle S\rangle)) = \mathrm{Lie}\langle S\rangle,$$

which is what we needed to prove. □

We are now ready to prove the existence of the Baker–Campbell–Hausdorff formula in the unipotent setting.

Theorem 3.198. *There exists a power series*

$$H(X,Y) = X + Y + \frac{1}{2}[X,Y] + \cdots \in \mathbb{Q}\langle\langle X, Y\rangle\rangle,$$

whose entries are iterated commutators between X and Y with rational coefficients, such that, for every pro-unipotent group U with Lie algebra $\mathfrak{u}$, every k-algebra R, and every pair of elements $x, y\in \mathfrak{u}\otimes R$, the following identity holds:

$$\exp(x)\exp(y) = \exp(H(x,y)). \tag{3.199}$$

Proof. We consider the case $k=\mathbb{Q}$ and let $S=\{X,Y\}$ be the set with two elements. We define the power series $H(X,Y)\in U(\mathrm{Lie}\langle S\rangle)^\wedge$ as

$$H(X,Y) = \log(\exp(X)\exp(Y)).$$

By Proposition 3.193 and Lemma 3.189, the element $\exp(X)\exp(Y)$ is group-like, so $H(X,Y)$ is well-defined. By Proposition 3.193 again, $H(X,Y)$ is a primitive element. If we induce a grading where X and Y are both of degree 1, the coproduct is a graded morphism. Therefore, each homogeneous term in the power series $H(X,Y)$ is primitive as well. By Corollary 3.197, each homogeneous term in $H(X,Y)$ belongs to $\mathrm{Lie}\langle S\rangle$. Therefore, $H(X,Y)$ is a formal power series whose entries are linear combinations of iterated commutators of X and Y with rational coefficients.

Since the equality (3.199) is true in the pro-nilpotent completion of the free Lie algebra in two elements, it is also true for x and y two nilpotent matrices with entries in any k-algebra R. Since any unipotent group is a closed subgroup of Up_n for some n, we deduce from Exercise 3.222 that formula (3.199) is true for any unipotent group U with Lie algebra $\mathfrak{u}$, with exp now denoting the exponential map from $\mathfrak{u}\otimes R$ to $U(R)$. Note that, since $\mathfrak{u}$ is nilpotent, $H(x,y)$ is a polynomial in commutators between x and y, and hence a well defined element of $\mathfrak{u} \otimes R$. Let now U be a pro-unipotent group and $\mathfrak{u}$ its Lie algebra. Write

$$U = \varprojlim_{\alpha} U_\alpha, \qquad \mathfrak{u} = \varprojlim_{\alpha} \mathfrak{u}_\alpha.$$

Then there is a commutative diagram

$$\begin{tikzcd} \mathfrak{u} \arrow[r] \arrow[d, "\exp"'] & \prod_\alpha \mathfrak{u}_\alpha \arrow[d, "\prod \exp"] \\ U \arrow[r] & \prod_\alpha U_\alpha. \end{tikzcd} \tag{3.200}$$

For an element x in $\mathfrak{u} \otimes R$ or in $U(R)$, we write (x_α) for its image on the right-hand side of the diagram (3.200). For $x, y \in \mathfrak{u} \otimes R$, the chain of equalities

$$(\exp(x)\exp(y))_\alpha = \exp(x_\alpha)\exp(y_\alpha) = \exp(H(x_\alpha, y_\alpha)) = \exp(H(x,y))_\alpha$$

and the injectivity of the horizontal maps in diagram (3.200) show that (3.199) is also true for U. This finishes the proof. $\square$

Corollary 3.201. *Let U_1 and U_2 be pro-unipotent groups and $\mathfrak{u}_1$ and $\mathfrak{u}_2$ their Lie algebras. Any morphism of Lie algebras $\mathfrak{u}_1 \to \mathfrak{u}_2$ induces a morphism of affine group schemes $U_1 \to U_2$.*

Proof. Recall the map log from Definition 3.181. We define a morphism of schemes as the composition

$$U_1 \xrightarrow{\log} \mathfrak{u}_1 \longrightarrow \mathfrak{u}_2 \xrightarrow{\exp} U_2. \tag{3.202}$$

The fact that this map is a group morphism follows from the universality of the Baker–Campbell–Hausdorff formula holds universally. More concretely, let φ be the original morphism of Lie algebras and ψ the composition (3.202). Let R be a k-algebra and $x, y \in U_1(R)$. Since formula (3.199) holds for any pro-unipotent group, using that φ is a morphism of Lie algebras, we deduce

$$\begin{aligned} \psi(x \cdot y) &= \psi(\exp(H(\log(x), \log(y)))) \\ &= \exp(\varphi(H(\log(x), \log(y)))) \\ &= \ \exp(H(\varphi(\log(x)), \varphi(\log(y)))) \\ &= \exp(H(\log(\psi(x)), \log(\psi(y)))) \\ &= \psi(x) \cdot \psi(y), \end{aligned}$$

proving the compatibility of ψ and the product. $\square$

We can use the previous ideas to recover the group of k-valued points of a pro-unipotent group from its Lie algebra.

Proposition 3.203. *Let U be a pro-unipotent group with Lie algebra $\mathfrak{u}$. Assume that $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ is finite-dimensional. Then the group $U(k)$ is canonically isomorphic to the group $\mathcal{G}(U(\mathfrak{u})^\wedge)$.*

Proof. The exponential map yields an isomorphism of schemes $\exp\colon \mathfrak{u} \to U$, hence a bijection $\mathfrak{u}(k) \to U(k)$. On the other hand, the power series exp gives a bijection between the space of primitive elements of $U(\mathfrak{u})^\wedge$ and the set of group-like elements of $U(\mathfrak{u})^\wedge$. By Corollary 3.187, we can identify the space of primitive elements of $U(\mathfrak{u})^\wedge$ with $\mathfrak{u}(k)$. Through this identification, the abstract exponential map gets identified with the power series exp, as in the proof of Theorem 3.198. Hence, we obtain a bijection between $U(k)$ and $\mathcal{G}(U(\mathfrak{u})^\wedge)$. Since the Baker–Campbell–Hausdorff formula holds both for $U(k)$ and $\mathcal{G}(U(\mathfrak{u})^\wedge)$, and the identification of $\mathfrak{u}$ with $P(U(\mathfrak{u})^\wedge)$ respects the Lie bracket, we deduce that the bijection $U(k) \to \mathcal{G}(U(\mathfrak{u})^\wedge)$ is a group isomorphism. $\square$

3.3.7. *Nilpotent representations.* Let U be a unipotent affine algebraic group and $\mathfrak{u}$ its nilpotent Lie algebra. As the following example shows, a finite-dimensional representation of $\mathfrak{u}$ does not need to come from a representation of U.

Example 3.204. Consider the groups $G = \mathbb{G}_m$ and $\mathbb{G}_a$. In both cases, the Lie algebra of G is the one-dimensional algebra k with trivial bracket. Consider the one-dimensional representation given by the identification $\mathbb{G}_m = \mathrm{GL}_1$. This defines a representation of k that sends an element $a \in k$ to the matrix (a). This representation of k does not come from an algebraic representation of $\mathbb{G}_a$, as can be seen, for instance, arguing that the exponential map $\exp\colon k \to \mathbb{G}_a(k) = k$ is the identity, while as matrices, $\exp(a)$ does not need to be an element of k.

This example shows that an obstruction to lifting a representation ρ from $\mathfrak{u}$ to a representation of U comes from the fact that $\rho(x)$ may be non-nilpotent and the exponential of non-nilpotent elements does not need to be algebraic.

Definition 3.205. Let $\mathfrak{u}$ be a finite-dimensional nilpotent Lie algebra. We say that a finite-dimensional representation $\rho\colon \mathfrak{u} \to \mathfrak{gl}(V)$ is *nilpotent* if $\rho(x)$ is a nilpotent endomorphism of V for all $x \in \mathfrak{u}$. We let $\mathbf{Rep}^{\mathrm{nil}}(\mathfrak{u})$ denote the category of finite-dimensional nilpotent representations.

Nilpotent representations solve the lifting problem.

Proposition 3.206. *Let U be a unipotent Lie group and $\mathfrak{u}$ its nilpotent Lie algebra. There is a canonical equivalence of categories*

$$\mathbf{Rep}(U) \longrightarrow \mathbf{Rep}^{\mathrm{nil}}(\mathfrak{u})$$

given by $\rho \mapsto \mathrm{d}\rho$.

Proof. Let $\rho\colon U \to \mathrm{GL}(V)$ be a representation of U. Since all representations of U are unipotent, we can choose a basis of V such that the image of ρ is contained in Up_n. Therefore, $\mathrm{d}\rho$ is contained in $\mathrm{Lie}(\mathrm{Up}_n) = \mathfrak{u}_n$, the space of strictly upper triangular matrices, which implies thatdρ is an object of $\mathbf{Rep}^{\mathrm{nil}}(\mathfrak{u})$. Conversely, let $\rho\colon \mathfrak{u} \to \mathfrak{gl}(V)$ be a nilpotent representation of $\mathfrak{u}$. By Engel's theorem (Proposition 3.168), there is a basis of V such that $\rho(\mathfrak{u}) \subset \mathfrak{u}_n$. By Corollary 3.201, there is an algebraic group morphism $\mu\colon U \to \mathrm{Up}_n$ satisfying $\mathrm{d}\mu = \rho$. $\square$

3.3.8. *Graded Lie algebras.* In the sequel, we will need to deal with Lie algebras L with infinite-dimensional abelianization $L/[L, L]$, for which the pro-nilpotent completion from Definition 3.171 does not have the properties we want. However, in most cases of interest, L comes with a grading with finite-dimensional graded pieces, and one can use the grading to define a better behaved notion.

DEFINITION 3.207. Let $L = \bigoplus_{n<0} L_n$ be a negatively graded Lie algebra such that L_n is finite-dimensional for all $n < 0$, and let

$$F^n L = \bigoplus_{m \leqslant -n} L_m$$

be the associated decreasing filtration. The *pro-nilpotent completion* of L (with respect to the grading) is the projective limit

$$\widehat{L} = \varprojlim_n L/F^n L.$$

The next lemma shows that this is a reasonable definition.

LEMMA 3.208. *Let $L = \bigoplus_{n<0} L_n$ be a negatively graded Lie algebra.*

i) *If $\dim L_n < \infty$ for all $n < 0$, then $\widehat{L}$ is pro-nilpotent.*
ii) *If $\dim L/[L,L] < \infty$, then $\dim L_n < \infty$ for all $n < 0$ and the pro-nilpotent completions from Definitions 3.171 and 3.207 are canonically isomorphic to each other.*

PROOF. The equality $L/F^m L = \bigoplus_{i=1}^{m-1} L_{-i}$ implies that the vector space L_n is finite-dimensional for all n if and only if $L/F^n L$ is finite-dimensional for all n. Moreover, $L^{(i)}$ is contained in F^{i+1} for all $i \geqslant 0$ since L is negatively graded, and hence $L/F^n L$ is nilpotent. This proves i). For ii), we first observe that $L/L^{(n)}$ surjects onto $L/F^{n+1}L$ for all $n \geqslant 0$. Since the finite-dimensionality of $L/[L,L]$ implies that of $L/L^{(n)}$, we deduce that all graded pieces L_i are finite-dimensional. Besides, since $L/[L,L]$ is finite-dimensional, there exists a finite set of homogeneous Lie algebra generators of L. Letting r denote the largest degree among them, we get $F^{r(i+1)} \subset L^{(i)}$ for all $i \geqslant 0$. By a standard argument using projective limits, the inclusions $F^{r(i+1)} \subset L^{(i)} \subset F^{i+1}$ imply that the two pro-nilpotent completions are canonically isomorphic to each other. $\square$

Let L be a graded Lie algebra satisfying the conditions of Definition 3.207. The Poincaré–Birkhoff–Witt theorem (Theorem 3.99) implies that the universal enveloping algebra $U(L)$ is a graded algebra with $U(L)_0 = k$, $U(L)_n = 0$ for all $n > 0$ and $\dim U(L)_n < \infty$ for all $n < 0$. The decreasing filtration $F^n U(L)$ is defined as before as

$$F^n U(L) = \bigoplus_{n' \leqslant -n} U(L)_n.$$

DEFINITION 3.209. The *graded completion* of $U(L)$ is the projective limit

$$U(L)^\vee = \varprojlim_n U(L)/F^n U(L).$$

REMARK 3.210. One can give an analogous definition in the case when L is concentrated in positive degrees.

The theory of graded Lie algebras applies notably to the following situation.

DEFINITION 3.211. A *graded pro-unipotent group* is a a pro-unipotent group U on which the group $\mathbb{G}_m$ acts compatibly with the pro-unipotent structure and the group structure.

The fact that the action is compatible with the pro-unipotent structure means that we can write

$$U = \varprojlim_{\alpha} U_\alpha,$$

with each U_α unipotent and that the group $\mathbb{G}_m$ acts on each U_α is such a way that the maps $\pi_{\alpha,\beta}\colon U_\beta \to U_\alpha$ are $\mathbb{G}_m$-equivariant.

On k-points, the compatibility with the group structures means that

$$\begin{aligned} t(g_1 g_2) &= t(g_1)t(g_2),\\ t(g_1^{-1}) &= t(g_1)^{-1}, \\ t(e) &= e \end{aligned} \tag{3.212}$$

hold for all $t \in k^\times$ and $g_1, g_2 \in U_\alpha(k)$. As usual, these conditions can be translated into diagrams of group schemes and diagrams of Hopf algebras.

Let U be a graded pro-unipotent group with Hopf algebra $A = \mathcal{O}(U)$. Then A is a graded algebra. Indeed, as seen in Example 3.60, the Hopf algebra of $\mathbb{G}_m$ is $k[x, x^{-1}]$ with coaction $\Delta x = x \otimes x$ and counit $\epsilon(x) = 1$. The action of $\mathbb{G}_m$ on U gives a coaction $\Delta\colon A \to A \otimes k[x, x^{-1}]$, and we set $A_n = \Delta^{-1}(A \otimes x^n)$. For $a \in A$, there is a decomposition into a finite sum

$$\Delta a = \sum_{n \in \mathbb{Z}} a_n x^n.$$

The properties of a coaction readily imply that

$$a = \sum_n a_n, \qquad a_n \in A_n.$$

Therefore, $A = \bigoplus_n A_n$. For $a \in A_n$ and $b \in A_m$, the computation

$$\Delta(ab) = \Delta(a)\Delta(b) = a \otimes x^n \cdot b \otimes x^m = ab \otimes x^{n+m}$$

shows that ab belongs to A_{n+m}, so A is a graded algebra. Since e is a fixed point for the action of $\mathbb{G}_m$, we deduce that the ideal I is homogeneous. Hence, I and I/I^2 inherit a structure of graded vector space.

Definition 3.213. Let U be a graded pro-unipotent affine group scheme. The *graded Lie algebra* of U is defined as

$$\mathfrak{u}^{\mathrm{gr}} = \bigoplus_n \mathfrak{u}_n = \bigoplus_n ((I/I^2)_{-n})^\vee.$$

In general, $\mathfrak{u}^{\mathrm{gr}}$ is not the Lie algebra of U but the following holds.

Lemma 3.214. *Let U be a graded pro-unipotent group such that $A = \mathcal{O}(U)$ is connected (that is, $A_n = 0$ for $n < 0$ and $A_0 = k$) and satisfies $\dim A_n < \infty$ for all n. If $\mathfrak{u} = \mathrm{Lie}(U)$, then*

$$\mathfrak{u} = \widehat{\mathfrak{u}^{\mathrm{gr}}},$$

where $\widehat{\mathfrak{u}^{\mathrm{gr}}}$ denotes the pro-nilpotent completion from Definition 3.207.

Proof. Write $\mathfrak{u} = \varprojlim \mathfrak{u}_\alpha$ as a limit of finite-dimensional graded Lie algebras, and set $K_\alpha = \mathrm{Ker}(\mathfrak{u} \to \mathfrak{u}_\alpha)$. Since the Lie algebra $\mathfrak{u}_\alpha$ is finite-dimensional, there is an $n \geqslant 0$ such that $F^n \mathfrak{u}_\alpha = 0$. This implies that $F^n \mathfrak{u} \subset K_\alpha$. On the other direction, since the projective limit is compatible with the grading,

$$\mathfrak{u}/F^n \mathfrak{u} = \varprojlim_{\alpha} \mathfrak{u}_\alpha / F^n \mathfrak{u}_\alpha$$

and, by the hypothesis, $\mathfrak{u}/F^n\mathfrak{u}$ is finite-dimensional, for each $n > 0$ there is an α such that $\mathfrak{u}/F^n\mathfrak{u} = \mathfrak{u}_\alpha/F^n\mathfrak{u}_\alpha$, hence $K_\alpha \subset F^n\mathfrak{u}$. The two inclusions we have proved imply that the limits

$$\widehat{\mathfrak{u}^{\mathrm{gr}}} = \varprojlim_n \mathfrak{u}/F^n\mathfrak{u} \quad \text{and} \quad \mathfrak{u} = \varprojlim_\alpha \mathfrak{u}/K_\alpha$$

are canonically isomorphic. □

Several results that we stated in the case when $L/[L,L]$ is finite-dimensional can be generalized to the graded case.

PROPOSITION 3.215. *Let U be a graded pro-unipotent group such that $A = \mathcal{O}(U)$ is connected and satisfies* $\dim A_n < \infty$ *for all n. If* $\mathfrak{u} = \mathrm{Lie}(U)$*, then*

i) *there is a canonical isomorphism $U(\mathfrak{u}^{\mathrm{gr}})^\wedge = A^\vee$;*
ii) *the group $U(k)$ of k-points of U is canonically isomorphic to the group of group-like elements $\mathcal{G}(U(\mathfrak{u}^{\mathrm{gr}})^\wedge)$.*

We end this section discussing finite-dimensional graded representations.

DEFINITION 3.216. Let U be a graded pro-unipotent affine group scheme. A graded representation of U is a graded finite-dimensional k-vector space V along with a $\mathbb{G}_m$-equivariant morphism $\rho\colon U \to \mathrm{GL}(V)$, *i.e.* satisfying

$$\rho(tg) = t\rho(g)t^{-1}$$

for all $t \in k^\times$ and all $g \in U$. We denote by $\mathbf{Rep}_{\mathbb{G}_m}(U)$ the category of graded finite-dimensional representations of U. Similarly, if L is a graded Lie algebra, then $\mathbf{Rep}_{\mathbb{G}_m}(L)$ denotes the category of graded finite-dimensional vector spaces V along with a representation $\rho\colon L \to \mathrm{End}(V)$ satisfying

$$\rho(L_n)V_m \subset V_{n+m}.$$

THEOREM 3.217. *Let U be a graded pro-unipotent group such that $A = \mathcal{O}(U)$ is connected and satisfies* $\dim A_n < \infty$ *for all $n > 0$. Consider the pro-algebraic group $G = U \rtimes \mathbb{G}_m$. Then there are equivalences of categories*

$$\mathbf{Rep}(G) \simeq \mathbf{Rep}_{\mathbb{G}_m}(U) \simeq \mathbf{Rep}_{\mathbb{G}_m}(\mathfrak{u}^{\mathrm{gr}}).$$

PROOF. For simplicity of the exposition, we will argue using k-points of G and U. Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G. Since G is a semidirect product $U \rtimes \mathbb{G}_m$, there is a group morphism $\mathbb{G}_m \to G$ that we use to identify $\mathbb{G}_m$ with a subtorus of G. Therefore, there is an induced representation of $\mathbb{G}_m$ on V. By Exercise 3.219, this representation induces a grading on V. Moreover, the action of $\mathbb{G}_m$ on U is given by $t(g) = tgt^{-1}$ for $t \in \mathbb{G}_m(k)$ and $g \in U(k)$. Consequently, $\rho(t(g)) = t\rho(g)t^{-1}$ and ρ defines an element of $\mathbf{Rep}_{\mathbb{G}_m}(U)$. Conversely, if $(\rho, V) \in \mathbf{Rep}_{\mathbb{G}_m}(U)$, then V is a graded vector space, and hence carries an action of $\mathbb{G}_m$. The condition $\rho(t(g)) = t\rho(g)t^{-1}$ implies that the action of U and the action of $\mathbb{G}_m$ on V are compatible with the semi-direct product $G = U \rtimes \mathbb{G}_m$ and hence they define an element of $\mathbf{Rep}(G)$.

The existence of a functor $\mathbf{Rep}_{\mathbb{G}_m}(U) \to \mathbf{Rep}_{\mathbb{G}_m}(\mathfrak{u}^{\mathrm{gr}})$ is clear. The interesting part is to show that every graded finite-dimensional representation of $\mathfrak{u}^{\mathrm{gr}}$ comes from a representation of U. Let V be such a representation. Let m be the difference between the maximal degree and the minimal degree of V. Then the action of $F^{m+1}\mathfrak{u}^{\mathrm{gr}}$ on V is trivial. By Lemma 3.214, there is a surjection

$$\mathfrak{u} \longrightarrow \mathfrak{u}^{\mathrm{gr}}/F^{m+1}\mathfrak{u}^{\mathrm{gr}} \tag{3.218}$$

compatible with the action of $\mathbb{G}_m$, and V factors through a representation of the finite-dimensional nilpotent Lie algebra $\mathfrak{u}^{\mathrm{gr}}/F^{m+1}\mathfrak{u}^{\mathrm{gr}}$. Write

$$U = \varprojlim_{\alpha} U_\alpha$$

with $\mathbb{G}_m$ acting on each U_α and the maps $U \to U_\alpha$ surjective. Let $\mathfrak{u}_\alpha$ be the Lie algebra of U_α. Since $\mathfrak{u}_n^{\mathrm{gr}}$ is finite-dimensional for all n, there is an α such that the surjection (3.218) factors as

$$\begin{tikzcd} \mathfrak{u} \arrow[r] \arrow[dr] & \mathfrak{u}_\alpha \arrow[d] \\ & \mathfrak{u}^{\mathrm{gr}}/F^{m+1}\mathfrak{u}^{\mathrm{gr}}. \end{tikzcd}$$

Therefore, the representation V defines a graded representation of $\mathfrak{u}_\alpha$. Since $\mathfrak{u}_\alpha$ is a graded Lie algebra with only negative grades, we deduce that any graded finite-dimensional representation of $\mathfrak{u}_\alpha$ is nilpotent. By Proposition 3.206, this representation of $\mathfrak{u}_\alpha$ comes from a graded representation of U_α, and hence defines a graded representation of U. □

⋆ ⋆ ⋆

Exercise 3.219. Let $V = \bigoplus_{n\in\mathbb{Z}} V^n$ be a graded k-vector space. Then there is an induced left action of $\mathbb{G}_m$ on V given by $\lambda \cdot v = \lambda^n v$ on each $v \in V^n$. In fact, giving a $\mathbb{Z}$-grading on V is equivalent to giving an action of $\mathbb{G}_m$.

i) Prove that the coalgebra $\mathcal{O}(\mathbb{G}_m)$ is isomorphic to the coalgebra H from Example 3.55.
ii) Prove that the coaction of $\mathcal{O}(\mathbb{G}_m)$ on V determined by Lemma 3.118 agrees with the coaction of H from Example 3.55.

Exercise 3.220. Let G be an affine group scheme. In this exercise, we show that not every *linear* representation of the abstract group $G(k)$ thas has "geometric origin" is an *algebraic* representation of G. For instance, consider the complex vector space $V = K(\mathbb{P}^1_{\mathbb{C}})$ of rational functions on the complex projective line. The group $G(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d},$$

and hence linearly on the vector space V.

i) Let $W \subset V$ be a finite-dimensional vector subspace. Show that the set of poles of the functions belonging to W is finite.
ii) Show that the set of poles of the functions in the G-orbit of t is infinite.
iii) Use Lemma 3.122 to conclude that the linear representation V of $G(\mathbb{C})$ does not come from a representation of the affine algebraic group $G = \mathrm{SL}_2$.

Exercise 3.221. Let L be a Lie algebra. Construct a surjective linear map

$$(L/[L,L])^{\otimes n+1} \longrightarrow L^{(n)}/L^{(n+1)},$$

and prove that, if $L/[L,L]$ is finite-dimensional, then so is $L/L^{(n+1)}$ for all $n \geqslant 0$.

EXERCISE 3.222. Consider the unipotent group Up_n with Lie algebra $\mathfrak{u}_n$. Prove that the exponential map $\exp\colon \mathfrak{u}_n \to \mathrm{Up}_n$ from Section 3.3.4 can be written explicitly as a truncated exponential series. Namely, for $N \in \mathfrak{u}_n$, it is given by

$$\exp(N) = \sum_{k=0}^{n-1} \frac{N^k}{k!}.$$

EXERCISE 3.223. Translate conditions (3.212) into the existence of commutative diagrams of affine schemes and the corresponding dual diagrams of algebras.

3.4. The pro-unipotent completion of a group. In this section, we develop some abstract machinery that will be used in the sequel to rephrase the constructions from Section 3.1 in a more conceptual way. There we saw that iterated integrals carry information about the fundamental group of a differentiable manifold. The question we would like to address now is how much of it can be recovered using differential forms. Stated in a vaguer form: what information about the fundamental group is "cohomological", or even "motivic" if we are dealing with algebraic varieties? Throughout, k still denotes a field of characteristic zero.

3.4.1. *The abelianization of the fundamental group.* The obvious piece of information that can be recovered via differential forms is the abelianization of the fundamental group. Indeed, recall the isomorphism

$$\pi_1(M,x)^{\mathrm{ab}} \simeq \mathrm{H}_1(M,\mathbb{Z})$$

from Theorem 3.14. Passing to the dual, Theorem 2.79 yields an isomorphism

$$\mathrm{H}^1_{\mathrm{dR}}(M,\mathbb{R}) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(M,x)^{\mathrm{ab}},\mathbb{R}).$$

Moreover, in the case where k is a subfield of $\mathbb{C}$ and $M = X(\mathbb{C})$ is the set of complex points of a smooth variety X over k, we get

$$\mathrm{H}^1_{\mathrm{dR}}(X) \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{Hom}(\pi_1(M,x)^{\mathrm{ab}},\mathbb{C}),$$

where the left-hand side stands for algebraic de Rham cohomology (as in Definition 2.95) and has thus a purely algebraic definition, in particular a k-structure.

However, the abelianization of the fundamental group is a very crude invariant that only knows about abelian representations. We should be able to understand much more than just the abelianization of the fundamental group using differential forms. A glimpse of this appeared in Section 3.1, when we saw that iterated integrals are related to nilpotent flat connections, in turn related to unipotent representations of the fundamental group. In the next paragraphs, we elaborate on this idea.

3.4.2. *The pro-unipotent completion.* The central concept of the whole section is the following construction attached to an abstract group.

DEFINITION 3.224. Let Γ be an abstract group. The *pro-unipotent completion* Γ^{un} of Γ over k is the universal pro-unipotent affine group scheme G over k endowed with a morphism of abstract groups $\Gamma \to G(k)$. More precisely,

- Γ^{un} is a pro-unipotent affine group scheme over k with a morphism

$$\Gamma \longrightarrow \Gamma^{\mathrm{un}}(k);$$

- for each pro-unipotent affine group scheme G over k along with a morphism of groups $\Gamma \to G(k)$, there exists a unique morphism of affine group schemes $\Gamma^{\mathrm{un}} \to G$ such that the following diagram commutes:

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[dr] & \Gamma^{\mathrm{un}}(k). \arrow[d] \\ & G(k) \end{tikzcd}$$

The pro-unipotent completion of Γ over $\mathbb{Q}$ will be simply called the pro-unipotent completion of Γ.

The pro-unipotent completion was introduced by Quillen in [**Qui69**] based on work by Malcev [**Mal49**]; it is also called *the Malcev completion* in the literature. As it is always the case with universal objects, when the pro-unipotent completion exists it is unique up to unique isomorphism. Following the same path, we can also define the pro-algebraic completion of a group.

Definition 3.225. The *pro-algebraic completion* Γ^{alg} over k of an abstract group Γ is an affine group scheme Γ^{alg} over k endowed with a morphism of abstract groups $\Gamma \to \Gamma^{\mathrm{alg}}(k)$ such that, for each affine group scheme G over k along with a morphism of groups $\Gamma \to G(k)$, there exists a unique morphism of affine group schemes $\Gamma^{\mathrm{alg}} \to G$ making the following diagram commutative:

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[dr] & \Gamma^{\mathrm{alg}}(k) \arrow[d] \\ & G(k). \end{tikzcd}$$

When $k = \mathbb{Q}$, we will simply call it the pro-algebraic completion of Γ.

Remark 3.226. Whenever the pro-unipotent completion exists, the groups Γ and Γ^{un} have the same finite-dimensional unipotent representations. Therefore, one cannot recover Γ by just looking at this kind of representations.

We now present Quillen's construction of the pro-unipotent completion of a group satisfying a finiteness condition. Basically, the idea is to build an object that looks like the completed universal enveloping algebra of a quasi-nilpotent Lie algebra L such that $L/[L, L]$ is finite-dimensional. For the moment, let Γ be any abstract group and consider the Hopf algebra

$$k[\Gamma] = \{\sum_{g\in\Gamma} a_g g \mid a_g \in k,\ a_g = 0 \text{ except for finitely many } g\}$$

from Example 3.63, which is cocommutative but in general non-commutative. Its counit is also called augmentation.

Definition 3.227. The *augmentation* of $k[\Gamma]$ is the algebra morphism

$$\begin{array}{rccc} \epsilon\colon & k[\Gamma] & \longrightarrow & k \\ & \sum_{g\in\Gamma} a_g g & \longmapsto & \sum_{g\in\Gamma} a_g \end{array}$$

and its kernel $J = \mathrm{Ker}(\epsilon)$ is called the *augmentation ideal*:

$$J = \{\sum_{g\in\Gamma} a_g g \mid \sum_{g\in\Gamma} a_g = 0\}.$$

The completion of $k[\Gamma]$ with respect to J is the projective limit

$$k[\Gamma]^\wedge = \varprojlim_N k[\Gamma]/J^{N+1},$$

where the transition maps are the projections $k[\Gamma]/J^{M+1} \to k[\Gamma]/J^{N+1}$ induced by the inclusions $J^{M+1} \subseteq J^{N+1}$ for $M \geqslant N$. The Hopf algebra structure on $k[\Gamma]$ induces a completed Hopf algebra structure on $k[\Gamma]^\wedge$ in the sense of Definition 3.70.

The space $k[\Gamma]^\wedge$ being infinite-dimensional, unless Γ is finite, its linear dual will be intractable for most of our purposes. We will instead work with the inductive limit of the linear duals of the quotients $k[\Gamma]/J^{N+1}$, which is best behaved when they are all finite-dimensional; for example, we will see that under this assumption it carries the structure of a Hopf algebra.

Definition 3.228. Let $V = \varprojlim_N V_N$ be a projective limit of finite-dimensional k-vector spaces. The *topological dual* of V is the inductive limit

$$V^{\vee_{\mathrm{top}}} = \varinjlim_N V_N^\vee$$

of the linear duals $V_N^\vee = \mathrm{Hom}(V_N, k)$ of the k-vector spaces V_N.

Let us from now on assume that Γ satisfies the finiteness condition that $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ is a finite-dimensional k-vector space. By Theorem 3.14, this is for instance satisfied when Γ is the fundamental group of a topological space with the homotopy type of a finite CW-complex, *e.g.* when Γ is the fundamental group of the space of complex points $X(\mathbb{C})$ of an algebraic variety X over $\mathbb{C}$.

Lemma 3.229. *If the vector space $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ is finite-dimensional, then all the quotients $k[\Gamma]/J^{N+1}$ are finite-dimensional as well.*

Proof. Taking the equality $k[\Gamma] = k \oplus J$ into account, it suffices to prove that the quotient J/J^{N+1} is finite-dimensional for all $N \geqslant 0$. Looking at the filtration

$$J^{N+1} \subseteq J^N \subseteq \cdots \subseteq J^2 \subseteq J,$$

this amounts to proving that the successive quotients J^i/J^{i+1} are finite-dimensional for all $i \geqslant 1$. To treat the case $i = 1$, we note that the map

$$\begin{aligned} \Gamma &\longrightarrow J/J^2 \\ g &\longmapsto (g-1) + J^2 \end{aligned}$$

factors through the abelianization of Γ, as can be seen by writing

$$gh - 1 = (g-1) + (h-1) + (g-1)(h-1).$$

In fact, it induces an isomorphism

$$\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k \xrightarrow{\sim} J/J^2$$

(its inverse sends the class of a generator $g-1$ to the class of g in Γ^{ab}). This shows that J/J^2 is finite-dimensional. Taking into account that the multiplication map

$$(J/J^2)^{\otimes i} \longrightarrow J^i/J^{i+1}$$

is surjective for all $i \geqslant 1$, the general result follows. $\square$

PROPOSITION 3.230. *Let Γ be a group such that $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ is finite-dimensional. Then the topological dual of $k[\Gamma]^\wedge$, given by the inductive limit*

$$A = (k[\Gamma]^\wedge)^{\vee_{\mathrm{top}}} = \varinjlim_N (k[\Gamma]/J^{N+1})^\vee,$$

carries the structure of a Hopf algebra.

PROOF. The proof relies on two key points. First, J is a Hopf ideal of $k[\Gamma]$ (see Example 3.78 i)), and hence $k[\Gamma]^\wedge$ is a completed Hopf algebra by Proposition 3.81. Second, each term $k[\Gamma]/J^{N+1}$ appearing in the inductive limit is finite-dimensional by Lemma 3.229, which implies that the completed Hopf algebra structure of $k[\Gamma]^\wedge$ defines a Hopf algebra structure in the limit.

We first explain the construction of the coproduct of A. The product of $k[\Gamma]$ induces products

$$k[\Gamma]/J^{N+1} \otimes k[\Gamma]/J^{N+1} \longrightarrow k[\Gamma]/J^{N+1}$$

for each $N \geqslant 0$ that, using finite-dimensionality, give rise to coproducts

$$\Delta_N \colon (k[\Gamma]/J^{N+1})^\vee \longrightarrow (k[\Gamma]/J^{N+1})^\vee \otimes (k[\Gamma]/J^{N+1})^\vee$$

for each $N \geqslant 0$. For each $a \in A$, there exists a sufficiently large N such that a is the image of an element $a_N \in (k[\Gamma]/J^{N+1})^\vee$. Then Δa is defined as the image of $\Delta_N a_N$ in $A \otimes A$; the resulting element is independent of the choices of N and a_N since the transition maps $k[\Gamma]/J^{M+1} \to k[\Gamma]/J^{N+1}$ are morphisms of algebras.

We next explain the construction of the product. The ideal J is also a *coideal*, which implies that $k[\Gamma]/J$ is a coalgebra. By contrast, J^{N+1} is not a coideal, since it only satisfies the inclusion

$$\Delta(J^{N+1}) \subset k[\Gamma] \otimes J^{N+1} + J^N \otimes J + \cdots + J \otimes J^N + J^{N+1} \otimes k[\Gamma].$$

As a result, $k[\Gamma]/J^{N+1}$ is not a coalgebra in general. Nevertheless, for $M \geqslant 2N+1$, the coproduct on $k[\Gamma]$ induces maps

$$\Delta \colon k[\Gamma]/J^M \longrightarrow k[\Gamma]/J^{N+1} \otimes k[\Gamma]/J^{N+1}$$

that commute with the transition maps. Their duals give maps

$$(k[\Gamma]/J^{N+1})^\vee \otimes (k[\Gamma]/J^{N+1})^\vee \longrightarrow (k[\Gamma]/J^{N+1} \otimes k[\Gamma]/J^{N+1})^\vee \longrightarrow (k[\Gamma]/J^M)^\vee. \tag{3.231}$$

This is enough to define a product

$$\varinjlim_N (k[\Gamma]/J^{N+1})^\vee \otimes \varinjlim_N (k[\Gamma]/J^{N+1})^\vee \longrightarrow \varinjlim_N (k[\Gamma]/J^{N+1})^\vee.$$

Indeed, given $a, b \in A$ choose representatives $a_N, b_N \in (k[\Gamma]/J^{N+1})^\vee$ for some big enough N, and define the product $a \cdot b$ as the image of $a_N \cdot b_N \in (k[\Gamma]/J^{2N+1})^\vee$ in A obtained by applying (3.231). The compatibility of (3.231) and the transition maps implies that the result is independent of the choices.

The counit of A is induced from the maps $(k[\Gamma]/J^{N+1})^\vee \to k$, which are the duals of the compositions of the unit $k \to k[\Gamma]$ and the projection $k[\Gamma] \to k[\Gamma]/J^{N+1}$, and are compatible with the transition maps.

The unit of A is induced from the maps $k[\Gamma]/J^{N+1} \to k$ obtained from the augmentation ε on noting that $\varepsilon(J^{N+1}) = 0$, by dualizing and composing with the natural map $(k[\Gamma]/J^{N+1})^\vee \to A$.

Finally, the antipode $S(g) = g^{-1}$ on $k[\Gamma]$ satisfies $S(J^{N+1}) \subseteq J^{N+1}$ for all N, and hence induces a map

$$S\colon k[\Gamma]/J^{N+1} \longrightarrow k[\Gamma]/J^{N+1}$$

compatible with the inductive system.

The compatibilities between the various operations in the definition of Hopf algebra are easily deduced from the compatibilities between those on $k[\Gamma]$. □

We now turn to Quillen's construction of the pro-unipotent completion of a group. The following result can be deduced from [**Qui69**, App. A], although the language there is different. A translation into the language of algebraic groups is given in [**Hai93**, Thm. 3.3]. We sketch the proof.

THEOREM 3.232 (Quillen [**Qui69**]). *Let Γ be an abstract group such that the vector space $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ is finite-dimensional. Then the pro-unipotent completion of Γ over k is the affine group scheme $G = \mathrm{Spec}((k[\Gamma]^{\wedge})^{\vee_{\mathrm{top}}})$.*

PROOF. As before, write $A = (k[\Gamma]^{\wedge})^{\vee_{\mathrm{top}}}$ and $G = \mathrm{Spec}(A)$. The conilpotency filtration of A is given by $\mathrm{Ann}_A\, J^{N+1} k[\Gamma]^{\wedge}$. By the definition of $k[\Gamma]^{\wedge}$ as a projective limit, is clear that the equality

$$\bigcap_{N\geqslant 0} J^{N+1} k[\Gamma]^{\wedge} = \{0\}$$

holds. Therefore, the conilpotency filtration of A is exhaustive. By Proposition 3.158, we deduce that G is pro-unipotent.

Let now $H = \mathrm{Spec}(B)$ be a pro-unipotent group and $f\colon \Gamma \to H(k)$ a group morphism. Let $B^{\vee}$ be the non-commutative algebra dual to the coalgebra B. There is an inclusion $H(k) \to B^{\vee}$ given by evaluating functions at points. The map f extends to a map $k[\Gamma] \to B^{\vee}$ also denoted by f. The augmentations of $k[\Gamma]$ and $B^{\vee}$ are compatible with f. Thus, we obtain maps

$$k[\Gamma]/J^{N+1} \longrightarrow B^{\vee}/J^{N+1},$$

where J denotes the augmentation ideal in both algebras. Dualizing, we get

$$\mathrm{Ann}_B\, J^{N+1} \longrightarrow \mathrm{Ann}_A\, J^{N+1} \hookrightarrow A.$$

Since H is pro-unipotent, by Proposition 3.158 the conilpotency filtration of B is exhaustive and we obtain a map $B \to A$, and hence a morphism of pro-unipotent groups $G \to H$. By construction, this is the only morphism of pro-unipotent groups that preserves the image of Γ, so G satisfies the universal property defining Γ^{un}. □

The Lie algebra $\mathfrak{g}$ of G satisfies the finiteness condition $\dim \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] < \infty$, so Theorem 3.182 provides an isomorphism $k[\Gamma]^{\wedge} = U(\mathfrak{g})^{\wedge}$. Therefore, Proposition 3.203 and Corollary 3.102 yield the following:

PROPOSITION 3.233. *Let Γ be a group with finite-dimensional $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$. Then*

$$\mathcal{G}(k[\Gamma]^{\wedge}) = \Gamma^{\mathrm{un}}(k)$$

and the natural map $\Gamma \to k[\Gamma]^{\wedge}$ agrees with the structural map $\Gamma \to \Gamma^{\mathrm{un}}(k)$. Moreover, the Lie algebra of Γ^{un} agrees with $P(k[\Gamma]^{\wedge})$.

EXAMPLE 3.234. Let us illustrate the above proposition for $\Gamma = \mathbb{Z}$. As we will see in Exercise 3.242, the pro-unipotent completion of Γ is the additive group $\mathbb{G}_a$ over k, so we need to show that group-like elements in $k[\![y]\!]$ are in one-to-one correspondence with k. Let $\sum_{n\geqslant 0} a_n y^n$ be a group-like element. Then $a_0 = 1$ and

$$\nabla^\vee\left(\sum_{n\geqslant 0} a_n y^n\right) = \left(\sum_{n\geqslant 0} a_n y^n\right) \otimes \left(\sum_{n\geqslant 0} a_n y^n\right). \tag{3.235}$$

Since $\nabla^\vee y = 1 \otimes y + y \otimes 1$, we have

$$\nabla^\vee y^n = \sum_{k=0}^{n} \binom{n}{k} y^k \otimes y^{n-k}.$$

Equation (3.235) is thus equivalent to the relation

$$a_k a_m = \binom{k+m}{k} a_{k+m}$$

for all $k, m \geqslant 0$. In particular, all the coefficients a_n are determined by the first one through the equality $a_n = a_1^n/n!$. Hence, the group-like element $\sum_{n\geqslant 0} a_n y^n$ is of the form $\exp(a_1 y)$ and this gives the correspondence $\mathcal{G}(k[\Gamma]^\wedge) = \Gamma^{\mathrm{un}}(k)$.

From the compatibility between the antipode, the product and the completed coproduct we easily deduce the following (see Lemma 3.189 for a similar statement and solve Exercise 3.244):

LEMMA 3.236. *If x is a primitive element, then $S(x) = -x$ holds. If g is a group-like element, then g is invertible in the algebra $k[\Gamma]^\wedge$ and satisfies $S(g) = g^{-1}$.*

EXAMPLE 3.237. Let Γ be the free group on two generators γ_0 and γ_1. In this example, we compute the pro-unipotent completion of Γ over $\mathbb{Q}$. Since the elements $\gamma_0 - 1$ and $\gamma_1 - 1$ belong to the augmentation ideal, we can define

$$\log(\gamma_0) = \log(1 + (\gamma_0 - 1)) = \gamma_0 - 1 - \frac{(\gamma_0 - 1)^2}{2} + \cdots$$
$$\log(\gamma_1) = \log(1 + (\gamma_1 - 1)) = \gamma_1 - 1 - \frac{(\gamma_1 - 1)^2}{2} + \cdots$$

as elements of $\mathbb{Q}[\Gamma]^\wedge$. Recall the algebra $\mathbb{Q}\langle\!\langle e_0, e_1\rangle\!\rangle$ from Example 3.74. We define a morphism of algebras $\mathbb{Q}\langle\!\langle e_0, e_1\rangle\!\rangle \to \mathbb{Q}[\Gamma]^\wedge$ by sending e_0 to $\log(\gamma_0)$ and e_1 to $\log(\gamma_1)$. It is easy to verify that this map is an isomorphism compatible with all the extra structures (unit, counit, completed coproduct, and antipode) carried by these completed Hopf algebras. The pro-unipotent completion of Γ is hence

$$\Gamma^{\mathrm{un}} = \mathrm{Spec}(\mathfrak{H}),$$

where $\mathfrak{H}$ is the Hoffman algebra from Example 3.64. In particular, we can identify the group of rational points $\Gamma^{\mathrm{un}}(\mathbb{Q})$ with the set of group-like elements of $\mathbb{Q}\langle\!\langle e_0, e_1\rangle\!\rangle$, the Lie algebra $\mathrm{Lie}(\Gamma^{\mathrm{un}})$ with the set of primitive elements of $\mathbb{Q}\langle\!\langle e_0, e_1\rangle\!\rangle$, and the completed universal enveloping algebra of $\mathrm{Lie}(\Gamma^{\mathrm{un}})$ with $\mathbb{Q}\langle\!\langle e_0, e_1\rangle\!\rangle$.

3.4.3. *The pro-unipotent completion of a torsor.* Quillen's construction can be extended to define the pro-unipotent completion of a torsor.

DEFINITION 3.238. Let Γ be a group. A *left* Γ*-torsor* is a non-empty set P together with a free and transitive action $\Gamma \times P \to P$. Similarly, a *right* Γ*-torsor* is a non-empty set P together with a free and transitive action $P \times \Gamma \to P$.

VARIANT 3.239. Assume that $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ is finite-dimensional and let A be as in Proposition 3.230. Let P be a left Γ-torsor. We write $k[P]$ for the k-vector space with basis P, which has the structure of a left $k[\Gamma]$-module. Moreover, there is a commutative coproduct on $k[P]$ determined by $\Delta(p) = p \otimes p$.

The completion of $k[P]$ is defined as

$$k[P]^{\wedge} = \varprojlim_{N} k[P]/J^{N+1}k[P]. \tag{3.240}$$

Consider its topological dual, defined as

$$R = (k[P]^{\wedge})^{\vee_{\mathrm{top}}} = \varinjlim_{N} (k[P]\,/J^{N+1}k[P]\,)^{\vee}.$$

Arguing as in the proof of Proposition 3.230, we deduce that R is a commutative algebra provided with a comodule structure

$$\Delta\colon R \longrightarrow A \otimes R, \tag{3.241}$$

where we set again $A = (k[P]^{\wedge})^{\vee}$. In other words, R is a Hopf module over A. The unipotent completion of P is defined as the spectrum

$$P^{\mathrm{un}} = \mathrm{Spec}(R).$$

The coaction (3.241) induces an action

$$\Gamma^{\mathrm{un}} \times P^{\mathrm{un}} \longrightarrow P^{\mathrm{un}}$$

that turns P^{un} into a left Γ^{un}-torsor. Mutatis mutandis, the same construction can be made for a right Γ-torsor P'. Our basic example will be the case when Γ is the fundamental group $\pi_1(M, x)$ and P and P' are the torsors of paths $\pi_1(M; x, y)$ and $\pi_1(M; y, x)$ respectively. In this case, there is also an antipode-like map

$$S\colon (\mathbb{Q}[P]^{\wedge})^{\vee_{\mathrm{top}}} \longrightarrow (\mathbb{Q}[P']^{\wedge})^{\vee_{\mathrm{top}}}$$

induced by the rule $S(\gamma) = \gamma^{-1}$ for a path $\gamma \in P$.

⋆ ⋆ ⋆

EXERCISE 3.242. Consider the group $\Gamma = \pi_1(S^1, 1) \simeq \mathbb{Z}$. Let γ_0 be a generator of Γ and consider $X_0 = \log(\gamma_0)$ as a power series in $(\gamma_0 - 1) \in J$. Use γ_0 and X_0 to describe explicitly all the following objects:

$$\mathbb{Q}[\pi_1(S^1, 1)]/J^{N+1}, \quad \mathbb{Q}[\pi_1(S^1, 1)]^{\wedge}, \quad \mathcal{O}(\pi_1(S^1, 1)^{\mathrm{un}}),$$
$$\pi_1(S^1, 1)^{\mathrm{un}}, \quad \mathrm{Lie}(\pi_1(S^1, 1)^{\mathrm{un}}).$$

In particular, deduce that the pro-unipotent completion of $\mathbb{Z}$ is given by the additive group $\mathbb{G}_a$. Compare this with Exercise 4.74 in the next chapter.

EXERCISE 3.243. Prove that the pro-unipotent completion of the group $\mathbb{Z}/2\mathbb{Z}$ is the trivial group $\mathrm{Spec}(\mathbb{Q})$.

EXERCISE 3.244. Prove Lemma 3.236 using the diagram for the antipode in Definition 3.47.

3.5. The bar complex and Chen's π_1-de Rham theorem. In this section, we make the relation between differential forms and the pro-unipotent completion of the fundamental group of a smooth manifold precise. If one views the latter as the *singular* side of a picture (or the Betti side if we are dealing with algebraic varieties), then the de Rham side is given by the cohomology of the bar complex. Both points of view will be related through Chen's π_1-de Rham Theorem 3.273.

3.5.1. *The bar complex of a dg-algebra.* We start with the definition of a differential graded algebra.

DEFINITION 3.245. Let k be a field of characteristic zero. A *differential graded algebra* (*dg-algebra* for short) over k is the data $A = (A^*, \wedge, d)$ of

- a graded k-vector space
$$A^* = \bigoplus_{n\in\mathbb{Z}} A^n;$$
- a multiplication map
$$\wedge\colon A^n \otimes A^m \longrightarrow A^{n+m}$$
for all integers $n, m \in \mathbb{Z}$ that makes A into an associative k-algebra with unit $1 \in A^0$;
- a differential $\mathrm{d}\colon A^* \to A^{*+1}$ such that $\mathrm{d}^2 = 0$, $\mathrm{d}(A^n) \subseteq A^{n+1}$, and
$$\mathrm{d}(a \wedge b) = \mathrm{d}a \wedge b + (-1)^n a \wedge \mathrm{d}b, \qquad a \in A^n.$$

We say that A is *(graded) commutative* if the relation
$$a \wedge b = (-1)^{nm} b \wedge a$$
holds for $a \in A^n$ and $b \in A^m$, and *connected* if $A^n = 0$ for $n < 0$ and $A^0 = k$. The sign operator $\mathrm{J}\colon A \to A$ is defined on homogeneous elements by
$$\mathrm{J}\, a = (-1)^{\deg(a)} a$$
and extended to the whole A by linearity.

A *morphism of dg-algebras* is a linear map compatible with the grading, the multiplication, and the differential. The field k has the structure of a dg-algebra concentrated in degree 0 with zero differential.

An *augmentation* of a dg-algebra A is a morphism of dg-algebras $\epsilon\colon A \to k$. It follows immediately from the definitions that a connected dg-algebra has a unique augmentation.

An example to keep in mind throughout this section, when $k = \mathbb{R}$ or $\mathbb{C}$, is the algebra $E^*(M, k)$ of smooth k-valued differential forms on a smooth manifold M, together with the wedge product $\wedge$ and the exterior differential d (see Section 2.2.1). A typical augmentation of this dg-algebra is the evaluation map on an point of M. Note that $E^*(M, k)$ is not connected. Similarly, for an arbitrary field k, if X is a smooth affine variety over k, then $\Omega^*(X)$ is also a dg-algebra and every k-rational point induces an augmentation by evaluation. As we will mainly apply the general constructions to this setting, we chose the notation $\wedge$ for the product in A. We refer the reader to Section A.8 of the appendix for a crash course on simplicial objects.

DEFINITION 3.246. Let A be a dg-algebra with augmentations ϵ_1 and ϵ_2. The *simplicial bar complex* $B_\bullet(A, \epsilon_2, \epsilon_1)$ is the simplicial complex of k-vector spaces
$$B_n(A, \epsilon_2, \epsilon_1) = A^{\otimes n} = A \otimes \overset{n}{\dots} \otimes A,$$

with faces

$$\begin{aligned}
\delta_0[x_1|\dots|x_n] &= \epsilon_2(x_1)[x_2|\dots|x_n],\\
\delta_i[x_1|\dots|x_n] &= [\mathrm{J}\,x_1|\dots|\,\mathrm{J}\,x_i\wedge x_{i+1}|\dots|x_n] \quad \text{for} \quad i=1,\dots,n-1,\\
\delta_n[x_1|\dots|x_n] &= \epsilon_1(\mathrm{J}\,x_n)[\mathrm{J}\,x_1|\dots|\,\mathrm{J}\,x_{n-1}],
\end{aligned}$$

and degeneracies

$$s_i[x_1|\dots|x_n] = [\mathrm{J}\,x_1|\dots|\,\mathrm{J}\,x_{i-1}|1|x_i|\dots|x_n] \quad \text{for} \quad i=1,\dots,n.$$

In these formulas, we use the bar notation

$$[x_1|\dots|x_n] = x_1\otimes\cdots\otimes x_n, \qquad [\,]=1.$$

The differential in the tensor complex $A^{\otimes n}$ is defined as

$$\mathrm{d}^{\mathrm{ver}}[x_1|\dots|x_n] = \sum_{i=1}^{n}(-1)^{n-i}[\mathrm{J}\,x_1|\dots|\,\mathrm{J}\,x_{i-1}|\mathrm{d}x_i|x_{i+1}|\dots|x_n].$$

Note that $\mathrm{d}^{\mathrm{ver}}$ is not the usual differential in the tensor algebra, but has an extra sign $(-1)^{n-i}$ in each term. This makes it commute with the faces and degeneracies, so that $B_\bullet(A,\epsilon_2,\epsilon_1)$ is a simplicial complex in the category of complexes of k-vector spaces, and is also natural from the point of view of Leibniz's rule (Remark 3.254).

Since $B_\bullet(A,\epsilon_2,\epsilon_1)$ is a simplicial cochain complex, there are two complexes that one can associate with it. On the one hand, we can take the associated chain complex $CB(A,\epsilon_2,\epsilon_1)_*$ as in Definition A.230, which is a chain complex of cochain complexes. We transform it into a double cochain complex by changing the sign of the chain degree as in Definition A.13 vi), and then form the total complex (Definition A.33). On the other hand, we can take the normalized chain complex $\mathcal{N}B(A,\epsilon_2,\epsilon_1)$ as in Definition A.231, change it to a double cochain complex, and then form the total complex. In both cases, we consider the simplicial degree as the horizontal or first degree when forming the total complex.

Definition 3.247. The *bar complex* of $(A,\epsilon_1,\epsilon_2)$ is defined as

$$B^*(A,\epsilon_2,\epsilon_1) = \operatorname{Tot} CB(A,\epsilon_2,\epsilon_1).$$

The *reduced bar complex* is defined as

$$\widetilde{B}^*(A,\epsilon_2,\epsilon_1) = \operatorname{Tot}\mathcal{N}B(A,\epsilon_2,\epsilon_1).$$

A direct consequence of Theorem A.232 is:

Theorem 3.248. *There are natural quasi-isomorphisms*

$$\widetilde{B}^*(A,\epsilon_2,\epsilon_1)\longrightarrow B^*(A,\epsilon_2,\epsilon_1)\longrightarrow \widetilde{B}^*(A,\epsilon_2,\epsilon_1).$$

The bar complex is enriched with the following extra structures. The grading and the differential come from the construction as a total complex, while the others are inherited from the tensor algebra.

Grading: An element of $B^*(A,\epsilon_2,\epsilon_i)$ of the form $[x_1|\dots|x_n]$ is homogeneous if all the components $x_i\in A$ are homogeneous. If this is the case, its degree is defined by

$$\deg([x_1|\dots|x_n]) = \sum_{i=1}^{n}\deg(x_i) - n, \tag{3.249}$$

where $\deg(x_i)$ is the degree of x_i in A^*. In particular, $\deg([x_i]) = n-1$ for an element $x_i \in A^n$, so the map $x_i \mapsto [x_i]$ does not preserve the degree.

Length filtration: It is the increasing filtration where

$$L_m B^*(A, \epsilon_2, \epsilon_1) \subseteq B^*(A, \epsilon_2, \epsilon_1)$$

is the subspace generated by elements $[x_1|\cdots|x_n]$ with $n \leqslant m$.

Differential: The differential is that of a total complex and is built out of the differential, the product, and the augmentations of A as follows:

$$\begin{aligned} \mathrm{d}[x_1|\cdots|x_n] &= \sum_{i=0}^{n}(-1)^i \delta_i[x_1|\cdots|x_n] + (-1)^n \mathrm{d}^{\mathrm{ver}}[x_1|\cdots|x_n] \\ &= \epsilon_2(x_1)[x_2|\cdots|x_n] + (-1)^n \epsilon_1(\mathrm{J}\,x_n)[\mathrm{J}\,x_1|\cdots|\,\mathrm{J}\,x_{n-1}] \\ &\quad + \sum_{i=1}^{n-1}(-1)^i[\mathrm{J}\,x_1|\cdots|\,\mathrm{J}\,x_i \wedge x_{i+1}|\cdots|x_n] \\ &\quad + \sum_{i=1}^{n}(-1)^i[\mathrm{J}\,x_1|\cdots|\,\mathrm{J}\,x_{i-1}|\mathrm{d}x_i|x_{i+1}|\cdots|x_n]. \end{aligned} \tag{3.250}$$

Product: It is the shuffle product

$$\nabla([x_1|\cdots|x_r] \otimes [x_{r+1}|\cdots|x_{r+s}]) = \sum_{\sigma \in ш(r,s)} \eta(\sigma)[x_{\sigma^{-1}(1)}|\cdots|x_{\sigma^{-1}(r+s)}].$$

Here, $\eta(\sigma)$ is the sign determined by the equation

$$a_1 \wedge \cdots \wedge a_{r+s} = \eta(\sigma) a_{\sigma^{-1}(1)} \wedge \cdots \wedge a_{\sigma^{-1}(r+s)}, \tag{3.251}$$

where $\deg(a_i) = \deg(x_i) - 1 = \deg([x_i])$. Although $\eta(\sigma)$ is not determined by σ alone, but also depends on the degrees of the involved elements, this abusive notation is the standard one.

Coproduct: If ϵ_3 is a third augmentation (that may agree with the previous ones), there is a *coproduct*

$$\Delta\colon B^*(A, \epsilon_3, \epsilon_1) \longrightarrow B^*(A, \epsilon_3, \epsilon_2) \otimes B^*(A, \epsilon_2, \epsilon_1)$$

given by deconcatenation

$$\Delta[x_1|\cdots|x_n] = \sum_{i=0}^{n}[x_1|\cdots|x_i] \otimes [x_{i+1}|\cdots|x_n].$$

Antipode: It is given by

$$S([x_1|\cdots|x_n]) = (-1)^n \eta(\tau_n)[x_n|\cdots|x_1], \tag{3.252}$$

where the sign $\eta(\tau_n)$ is determined by equation (3.251) as before, for the permutation given by $\tau_n(i) = n - i$.

The above structures induce the corresponding structures on the reduced bar complex $\widetilde{B}(A, \epsilon_2, \epsilon_1)$.

REMARK 3.253. The differential (3.250) can be rewritten, for homogeneous elements, using the total degrees $\deg([x_i]) = \deg(x_i) - 1$ as follows:

$$\begin{aligned}\mathrm{d}[x_1|\cdots|x_n] = \epsilon_2(x_1)[x_2|\cdots|x_n] + (-1)^{\sum_{j=1}^{n}\deg([x_j])}\epsilon_1(x_n)[x_1|\cdots|x_{n-1}]\\ + \sum_{i=1}^{n-1}(-1)^{\sum_{j=1}^{i}\deg([x_j])}[x_1|\cdots|x_i \wedge x_{i+1}|\cdots|x_n]\\ - \sum_{i=1}^{n}(-1)^{\sum_{j=1}^{i-1}\deg([x_j])}[x_1|\cdots|\mathrm{d}x_i|\cdots|x_n].\end{aligned}$$

In checking the compatibility of the differential with other structures, it might be useful to remember that $\epsilon_1(x_n)$ is zero unless x_n has degree 0.

REMARK 3.254. There are many possible choices of signs in the definition of the bar complex. For instance, in [**BK94**] the faces and degeneracies are not affected by any sign, and the differential $\mathrm{d}^{\mathrm{ver}}$ is the usual differential in the tensor complex. By contrast, we follow the sign convention of [**EM53**] and [**Hai87a**] because in this other convention, the coproduct defined below does not have any sign and the total differential satisfies Leibniz's rule with respect to the product in the tensor algebra and the degree in the bar complex. See [**EM53**, § 10] for a discussion.

REMARK 3.255. The bar complex $B^*(A, \epsilon_1, \epsilon_2)$ only depends on the semisimplicial structure (*i.e.* the faces and not the degeneracies) of $B_\bullet(A, \epsilon_1, \epsilon_2)$ that does not use the unit of A. Hence, it can be extended to non-unital algebras. By contrast, the reduced bar complex also depends on the degeneracies and needs a unit.

3.5.2. *The reduced bar complex of a connected dg-algebra.* Let A be a dg-algebra and ϵ an augmentation. We consider the non-unital algebra

$$IA = \mathrm{Ker}(\epsilon).$$

By Remark 3.255, we can define the bar complex

$$B^*(IA, \epsilon_2|_{IA}, \epsilon_1|_{IA}).$$

The augmentation, together with the unit of A defines a splitting $A = k \oplus IA$, from which we deduce the following result:

LEMMA 3.256. *Let (A, ϵ) be an augmented k-algebra, and let ϵ_1 and ϵ_2 be augmentations. Then the splitting defined by the augmentation induces an isomorphism*

$$\widetilde{B}^*(A, \epsilon_2, \epsilon_1) \xrightarrow{\sim} B^*(IA, \epsilon_2|_{IA}, \epsilon_1|_{IA})$$

that, together with Theorem 3.248, gives us a quasi-isomorphism

$$B^*(IA, \epsilon_2|_{IA}, \epsilon_1|_{IA}) \longrightarrow B^*(A, \epsilon_2, \epsilon_1).$$

PROOF. By the definition of the degeneracies, the subcomplex $DB_*(A, \epsilon_2, \epsilon_1)$ consisting of degenerate elements (Definition A.231) is given by

$$DB_n(A, \epsilon_2, \epsilon_1) = \sum_{i=0}^{n-1} A^{\otimes i} \otimes k \otimes A^{\otimes n-1-i} \subset A^{\otimes n}.$$

Therefore, the splitting induces an isomorphism

$$B_n(A, \epsilon_2, \epsilon_1)/DB_n(A, \epsilon_2, \epsilon_1) \xrightarrow{\sim} B_n(IA, \epsilon_2|_{IA}, \epsilon_1|_{IA}).$$

From Theorem A.232, we derive an isomorphism of graded k-vector spaces. Since all structures on both sides are given by the same formulas, we see that the isomorphism respects all the structures. □

We now make the definition of the reduced bar complex explicit in the case of a connected dg-algebra. An advantage of working in this setting is that connected dg-algebras have a unique augmentation ϵ, whose kernel is concentrated in positive degrees. All augmentations (those entering the definition of the faces and the coproduct and the one used to construct the splitting) will thus be equal. We omit them from the notation, and write simply

$$B^*(A) = \widetilde{B}^*(A, \epsilon, \epsilon).$$

DEFINITION 3.257. Let $(A^*, \wedge, \mathrm{d})$ be a connected dg-algebra over k. Set

$$A^+ = \bigoplus_{n>0} A^n.$$

The *reduced bar complex* $B^*(A)$ associated with A is the total tensor algebra of A^+:

$$B^*(A) = k \oplus A^+ \oplus (A^+ \otimes A^+) \oplus (A^+ \otimes A^+ \otimes A^+) \oplus \dots$$

The reduced bar complex is provided with the following structures:

Grading: An element of $B^*(A)$ of the form $[x_1|\dots|x_n]$ is homogeneous if all the components $x_i \in A^+$ are homogeneous. If this is the case, its degree is defined by

$$\deg([x_1|\dots|x_n]) = \sum_{i=1}^{n} \deg(x_i) - n,$$

where $\deg(x_i)$ is the degree of x_i in A. In particular, if $x_i \in A^n$, then $\deg([x_i]) = n-1$, so the map $x_i \mapsto [x_i]$ does not preserve the degree.

Length filtration: It is the increasing filtration where

$$L_m B^*(A) \subseteq B^*(A)$$

is the subspace generated by elements $[x_1|\cdots|x_n]$ with $n \leqslant m$.

Differential: The differential takes both the differential and the product structures of A into account:

$$\begin{aligned}\mathrm{d}[x_1|\cdots|x_n] = &-\sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i-1} \deg([x_j])} [x_1|\cdots|\mathrm{d}x_i|\cdots|x_n] \\ &+ \sum_{i=1}^{n-1} (-1)^{\sum_{j=1}^{i} \deg([x_j])} [x_1|\cdots|x_i \wedge x_{i+1}|\cdots|x_n].\end{aligned} \tag{3.258}$$

It is easy to check that d is homogeneous of degree 1 and satisfies $\mathrm{d} \circ \mathrm{d} = 0$. We will write $\mathrm{d} = \mathrm{d}_I - \mathrm{d}_C$, where

$$\mathrm{d}_I[x_1|\cdots|x_n] = -\sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i-1} \deg([x_j])} [x_1|\cdots|\mathrm{d}x_i|\cdots|x_n], \tag{3.259}$$

$$\mathrm{d}_C[x_1|\cdots|x_n] = -\sum_{i=1}^{n-1} (-1)^{\sum_{j=1}^{i} \deg([x_j])} [x_1|\cdots|x_i \wedge x_{i+1}|\cdots|x_n]. \tag{3.260}$$

Here, I stands for "internal" and C for "combinatorial".

Product: It is the shuffle product

$$\nabla([x_1|\cdots|x_r]\otimes[x_{r+1}|\cdots|x_{r+s}]) = \sum_{\sigma\in ш(r,s)} \eta(\sigma)[x_{\sigma^{-1}(1)}|\cdots|x_{\sigma^{-1}(r+s)}].$$

Here, $\eta(\sigma)$ is the sign determined by the equation (3.251).

Coproduct: The coproduct is the *deconcatenation* coproduct

$$\Delta[x_1|\cdots|x_n] = \sum_{i=0}^{n}[x_1|\cdots|x_i]\otimes[x_{i+1}|\cdots|x_n].$$

Antipode: It is given again by equation (3.252).

Since A^+ is concentrated in positive degrees, we deduce that $B^*(A)$ is concentrated in non-negative degrees. This has the following consequence:

LEMMA 3.261. *Let $A = (A^*, \mathrm{d}, \wedge)$ be a connected commutative dg-algebra. Then the above operations endow $\mathrm{H}^0(B^*(A))$, the zeroth cohomology group of the reduced bar complex, with the structure of a commutative Hopf algebra.*

PROOF. The bar complex $B^*(A)$ is a commutative differential graded Hopf algebra. This means that the product, coproduct, and antipode are compatible with the grading and the differential. The latter compatibility is written as

$$\begin{aligned} \mathrm{d}\circ\nabla &= \nabla\circ\mathrm{d}_{\otimes},\\ \Delta\circ\mathrm{d} &= \mathrm{d}_{\otimes}\circ\Delta,\\ S\circ\mathrm{d} &= \mathrm{d}\circ S, \end{aligned}$$

where $\mathrm{d}_{\otimes}$ is the induced differential in $B^*(A)\otimes B^*(A)$, which carries the usual sign. All these statements can be checked directly. Once we know that all these operations are compatible with the differential, they are transferred to cohomology. Since they are compatible with the grading, they induce operations on H^0, except, maybe, the coproduct. In principle, the coproduct would give a map

$$\Delta\colon\ \mathrm{H}^0(B^*(A)) \longrightarrow \bigoplus_{i+j=0} \mathrm{H}^i(B^*(A))\otimes\mathrm{H}^j(B^*(A)).$$

However, since $B^*(A)$ is non-negatively graded, the only non-zero term on the right-hand side is $\mathrm{H}^0(B^*(A))\otimes\mathrm{H}^0(B^*(A))$, and hence the coproduct is also well defined at the level of zeroth cohomology. $\square$

REMARK 3.262.

i) The commutativity of the product in A^* is essential in the previous proof. In fact, if the product on A^* is not commutative, it is not true that the shuffle product in $B^*(A)$ is compatible with the differential.

ii) Since the complex $B^*(A)$ is concentrated in non-negative degrees, the cohomology we are interested in is simply

$$\mathrm{H}^0(B^*(A)) = \mathrm{Ker}(\mathrm{d}\colon B^0(A)\longrightarrow B^1(A)).$$

Note that $B^0(A)$ consists of k-linear combinations of $[\ ]$ and $[x_1|\cdots|x_n]$ with $n\geqslant 1$ and $\deg(x_i) = 1$ for all $i = 1,\ldots,n$. Also, observe that the restrictions of the differentials to $B^0(A)$ are given by the formulas

$$\mathrm{d}_I[x_1|\cdots|x_n] = -\sum_{i=1}^{n}[x_1|\cdots|\mathrm{d}x_i|\cdots|x_n],$$
$$\mathrm{d}_C[x_1|\cdots|x_n] = -\sum_{i=1}^{n-1}[x_1|\cdots|x_i\wedge x_{i+1}|\cdots|x_n].$$

In practice, one would like to use the de Rham dg-algebra $E^*(M,k)$ that is not connected, even if the manifold M is connected. In order to use Lemma 3.261, it is convenient to replace it with a quasi-isomorphic connected dg-algebra.

LEMMA 3.263. *Take $k=\mathbb{R}$ or $\mathbb{C}$ and let M be a connected differentiable manifold. Let $x,y\in M$ and let $A^*\subset E^*(M,k)$ be a connected dg-subalgebra such that the inclusion $A^*\to E^*(M,k)$ is a quasi-isomorphism. Let ϵ_x and ϵ_y be the augmentations given by evaluation at the points x and y respectively. Then there is a quasi-isomorphism $B^*(A)\to B(E^*(M,k),\varepsilon_y,\varepsilon_x)$. In particular,*

$$\mathrm{H}^0(B^*(A^*)) = \mathrm{H}^0(B(E^*(M,k),\varepsilon_y,\varepsilon_x)).$$

PROOF. Let $f\colon A\longrightarrow B$ be a morphism of dg-algebras, and let ϵ_1 and ϵ_2 be two augmentations on B. By composition, they induce augmentations ϵ_1' and ϵ_2' on A. Then there is a map

$$B(f)\colon B(A,\epsilon_1',\epsilon_2')\longrightarrow B(B,\epsilon_1,\epsilon_2).$$

A consequence of a result by Chen (see [**Hai87a**, Cor. 1.2.3]) is that if f is a quasi-isomorphism, then $B(f)$ is also a quasi-isomorphism. This, together with Theorem 3.248 implies the result. $\square$

3.5.3. *The reduced bar complex and iterated integrals.* Let M be a connected differentiable manifold with the homotopy type of a finite CW complex. Recall the dg-algebra $E^*(M,\mathbb{C})$ of complex smooth differential forms on M. For simplicity of the exposition, we will assume that we have chosen a complex dg algebra A^* provided with an injective morphism of dg-algebras $\varphi\colon A^*\longrightarrow E^*(M)$ such that

i) A^* is connected, that is $A^0=\mathbb{C}$ and $A^n=0$ for $n<0$;

ii) the induced map in cohomology

$$\varphi\colon\ \mathrm{H}^*(A^*)\longrightarrow \mathrm{H}^*(E^*(M,\mathbb{C}))$$

is an isomorphism.

By Lemma 3.263, the reduced bar complex of A^* is quasi-isomorphic to the bar complex of $E^*(M,\mathbb{C})$ with respect to any pair of augmentations, so it will be enough to consider the reduced bar complex of A^*. A similar discussion can be made with the bar complex of the whole dg-algebra $E^*(M,\mathbb{C})$; see Exercise 3.280.

The condition of A^* being connected implies that the elements of degree zero of $B^0(A^*)$ are linear combinations of the shape

$$\sum[\eta_1|\cdots|\eta_r]$$

for 1-forms $\eta_i\in A^1\subset E^1(M)$. Thus, with any element $x\in B^0(A^*)$ we can associate an iterated integral

$$[\eta_1|\cdots|\eta_r]\longmapsto\left(\gamma\longmapsto\int_\gamma\eta_1\cdots\eta_r\right).$$

For each pair of points $x, y \in M$, we define a pairing

$$\begin{aligned}\langle\ ,\ \rangle\colon\ B^0(A^*)\otimes \mathbb{Q}[{}_y\mathcal{P}(M)_x] &\longrightarrow \qquad \mathbb{C}\\ [\eta_1|\cdots|\eta_r]\otimes\gamma \qquad &\longmapsto \int_\gamma \eta_1\cdots\eta_r,\end{aligned}\tag{3.264}$$

where ${}_y\mathcal{P}(M)_x$ is the set of piecewise smooth paths as in Section 3.1, and $\mathbb{Q}[{}_y\mathcal{P}(M)_x]$ denotes the $\mathbb{Q}$-vector space with basis ${}_y\mathcal{P}(M)_x$.

We can now translate Theorem 3.19 into the language of the bar complex and the pairing (3.264).

THEOREM 3.265. *Let $\eta, \eta_1, \eta_2 \in B^0(A^*)$ be degree zero elements of the reduced bar complex of A^*, and let $\gamma, \gamma_1, \gamma_2$ be piecewise smooth paths in M. Then the following three equalities are satisfied:*

$$\langle S(\eta),\gamma\rangle = \langle \eta, S(\gamma)\rangle,\tag{3.266}$$

$$\langle \eta,\gamma_1\gamma_2\rangle = \langle \Delta\eta,\gamma_1\otimes\gamma_2\rangle,\tag{3.267}$$

$$\langle \eta_1\otimes\eta_2,\nabla^\vee\gamma\rangle = \langle \eta_1 \sqcup \eta_2,\gamma\rangle.\tag{3.268}$$

It follows from this theorem that the length filtration on $B^0(A^*)$ is dual to the filtration by the augmentation ideal on the group algebra of paths.

PROPOSITION 3.269. *Let x and y be points of M. Let $J \subset \mathbb{Q}[{}_x\mathcal{P}(M)_x]$ be the augmentation ideal, $N \geqslant 0$ an integer, and γ an element of either $J^{N+1}\mathbb{Q}[{}_x\mathcal{P}(M)_y]$ or $\mathbb{Q}[{}_y\mathcal{P}(M)_x]J^{N+1}$. If $\eta \in L_N B^0(A^*)$ has length less than or equal to N, then*

$$\langle \eta,\gamma\rangle = 0.$$

PROOF. We only treat the case $\gamma \in J^{N+1}\mathbb{Q}[{}_x\mathcal{P}(M)_y]$ (the other one is completely analogous). The proof proceeds by induction on N.

For $N = 0$, every element $\gamma \in J\mathbb{Q}[{}_x\mathcal{P}(M)_y]$ can be written as

$$\gamma = \sum_{i=1}^r q_i\gamma_i,\quad q_i\in\mathbb{Q},\quad \sum_{i=1}^r q_i = 0,\quad \gamma_i\in {}_x\mathcal{P}(M)_y.$$

If $\eta \in L_0B^0(A^*)$, then $\eta = \alpha[\ ]$ for some $\alpha \in \mathbb{C}$. Since

$$\langle [\ ],\gamma_i\rangle = 1,\quad \text{for } \gamma_i\in {}_x\mathcal{P}(M)_y,$$

the result in the case $N = 0$ follows from the condition $\sum_{i=1}^r q_i = 0$.

Now fix $N > 0$ and assume that the result holds for all integers $N' < N$. To prove it for N, we may assume that $\gamma = \gamma_1\gamma_2$ with $\gamma_1 \in J$ and $\gamma_2 \in J^N\mathbb{Q}[{}_x\mathcal{P}(M)_y]$, and $\eta = [\omega_1|\cdots|\omega_N]$. Then the relation (3.267) yields

$$\begin{aligned}\langle\eta,\gamma\rangle &= \langle\Delta\eta,\gamma_1\otimes\gamma_2\rangle\\ &= \sum_{i=0}^N \langle[\omega_1|\cdots|\omega_i],\gamma_1\rangle\langle[\omega_{i+1}|\cdots|\omega_N],\gamma_2\rangle\\ &= \langle[\ ],\gamma_1\rangle\langle[\omega_1|\cdots|\omega_N],\gamma_2\rangle + \sum_{i=1}^N \langle[\omega_1|\cdots|\omega_i],\gamma_1\rangle\langle[\omega_{i+1}|\cdots|\omega_N],\gamma_2\rangle.\end{aligned}$$

The first summand in the last equality vanishes since $\langle[\ ],\gamma_1\rangle = 0$, and all the factors $\langle[\omega_{i+1}|\cdots|\omega_N],\gamma_2\rangle$ in the second sum vanish by the induction hypothesis. Thus, $\langle\eta,\gamma\rangle = 0$, as we wanted to show. $\square$

3.5.4. *The reduced bar complex and the pro-unipotent completion of the fundamental group.* One of the reasons why the reduced bar complex is interesting is that it provides us with a criterion to decide whether an iterated integral is a homotopy functional, thus solving the question raised in Section 3.1.

THEOREM 3.270. *Let $\omega \in B^0(A^*)$. If $\mathrm{d}\omega = 0$, then the iterated integral associated with ω is a homotopy functional.*

PROOF. Let $x, y \in M$ be points. Consider two homotopic paths γ_1 and γ_2 from x to y, and let F be a homotopy between them. Recall from Definition 3.1 that $F\colon [0,1]^2 \to M$ satisfies the conditions

$$F(t,0) = \gamma_1(t), \quad F(t,1) = \gamma_2(t), \quad F(0,s) = x, \quad F(1,s) = y. \tag{3.271}$$

For simplicity, we will assume that F is smooth; the general case follows by taking a polyhedral decomposition of $[0,1]^2$, as in the proof of Lemma 3.11. Set

$$\begin{aligned} F_i\colon \quad [0,1]^n \times [0,1] &\longrightarrow \quad M, \\ ((t_1,\dots,t_n),s) &\longmapsto F(t_i,s). \end{aligned}$$

The elements of $B^1(A^*)$ are linear combinations of $\nu = [\nu_1|\cdots|\nu_n]$, where one of the ν_i is a 2-form and all the others are 1-forms. Given such a ν, with the 2-form in the i-th position, we define the *integral along F* as

$$\int_F \nu = (-1)^i \int_{[0,1]\times\Delta^n} F_1^*\nu_1 \wedge \cdots \wedge F_n^*\nu_n,$$

where the second integral is oriented by $\mathrm{d}s \wedge \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$. As in Notation 1.114, in this integral Δ^n denotes the simplex

$$\Delta^n = \{(t_1,\dots,t_n) \mid 1 \geqslant t_1 \geqslant \cdots \geqslant t_n \geqslant 0\}.$$

The definition of the integral along F extends to $B^1(A^*)$ by $\mathbb{C}$-linearity. We claim that the equality

$$\int_{\gamma_2} \omega - \int_{\gamma_1} \omega = \int_F \mathrm{d}\omega, \tag{3.272}$$

holds, from which the theorem follows at once.

Formula (3.272) is proved by a careful application of Stokes's theorem. To prove it, we can assume that $\omega = [\omega_1 \mid \cdots \mid \omega_n] \in B^0(A^*)$. Note the equality

$$\mathrm{d}(F_1^*\omega_1 \wedge \cdots \wedge F_n^*\omega_n) = \sum_{i=1}^n (-1)^{i+1} F_1^*\omega_1 \wedge \cdots \wedge F_i^*(\mathrm{d}\omega_i) \wedge \cdots \wedge F_n^*\omega_n$$

derived from the properties defining the exterior derivative (see Section 2.2.1) and the commutativity of d and F_i^*. Combining this with the definitions of d_I and the integral along F, one gets:

$$\int_F \mathrm{d}_I\omega = \int_{[0,1]\times\Delta^n} \mathrm{d}(F_1^*\omega_1 \wedge \cdots \wedge F_n^*\omega_n).$$

We now apply Stokes's theorem. Set $\Omega = F_1^*\omega_1 \wedge \cdots \wedge F_n^*\omega_n$. Then,

$$\begin{aligned} \int_F \mathrm{d}_I\omega &= \int_{\partial([0,1]\times\Delta^n)} F_1^*\omega_1 \wedge \cdots \wedge F_n^*\omega_n \\ &= \int_{s=1} \Omega - \int_{s=0} \Omega - \int_{t_1=1} \Omega + \sum_{i=1}^{n-1} (-1)^{i+1} \int_{t_i=t_{i+1}} \Omega - (-1)^n \int_{t_n=0} \Omega. \end{aligned}$$

By the relations satisfied by F,

$$\begin{aligned}
\Omega|_{s=1} &= \gamma_2^*\omega_1 \wedge \cdots \wedge \gamma_2^*\omega_n, \\
\Omega|_{s=0} &= \gamma_1^*\omega_1 \wedge \cdots \wedge \gamma_1^*\omega_n, \\
\Omega|_{t_i=t_{i+1}} &= F_1^*\omega_1 \wedge \cdots \wedge F_i^*(\omega_i \wedge \omega_{i+1}) \wedge F_n^*\omega_n,
\end{aligned}$$

and $\Omega|_{t_1=1}$ (resp. $\Omega|_{t_n=0}$) vanishes since in that case F_1 (resp. F_n) is a constant function. Besides,

$$\int_F \mathrm{d}_C\omega = \sum_{i=1}^{n}(-1)^{i+1}\int_{[0,1]\times\Delta^{n-1}} F_1^*\omega_1 \wedge \cdots \wedge F_i^*(\omega_i \wedge \omega_{i+1}) \wedge \cdots \wedge F_n^*\omega_n.$$

Putting everything together yields

$$\int_F \mathrm{d}_I\omega = \int_{\gamma_2}\omega - \int_{\gamma_1}\omega + \int_F \mathrm{d}_C\omega,$$

which is exactly the content of the claim (3.272) noting that $\mathrm{d} = \mathrm{d}_I - \mathrm{d}_C$. □

Let $x \in M$ and write $\Gamma = \pi_1(M, x)$. The condition that M has the homotopy type of a finite CW complex implies that $\mathrm{H}_1(M)$ is finite-dimensional. Thus, Γ satisfies the hypothesis of Theorem 3.232 and its pro-unipotent completion is given by $\mathrm{Spec}((\mathbb{Q}[\Gamma]^\wedge)^\vee)$.

Recall that the zero cohomology group of the reduced bar complex of A^* is just the kernel of the differential map,

$$\mathrm{H}^0(B^*(A^*)) = \mathrm{Ker}\left(\mathrm{d}\colon B^0(A^*) \longrightarrow B^1(A^*)\right),$$

which, by Theorem 3.270, consists of homotopy functionals.

Putting together Theorem 3.270 and Proposition 3.269, we obtain a map

$$\mathrm{H}^0(L_N B^*(A^*)) \longrightarrow ((\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)]J^{N+1}) \otimes \mathbb{C})^\vee.$$

Theorem 3.273 (Chen's π_1-de Rham theorem). *For each integer $N \geqslant 0$ and points $x, y \in M$, the integration map gives an isomorphism*

$$\mathrm{H}^0(L_N B^*(A^*)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_1(M; y, x)]/\mathbb{Q}[\pi_1(M; y, x)]J^{N+1}, \mathbb{C}),$$

and consequently it induces an isomorphism of ind-vector spaces between

$$\mathrm{H}^0(B^*(A^*)) = \varinjlim_N \mathrm{H}^0(L_N B^*(A^*))$$

and

$$(\mathbb{C}[\pi_1(M; y, x)]^\wedge)^\vee = \varinjlim_N (\mathbb{C}[\pi_1(M; y, x)]/\mathbb{C}[\pi_1(M; y, x)]J^{N+1})^\vee.$$

We will give a proof of this result in the next section. Note moreover, that Theorem 3.265 implies that the last isomorphism of Theorem 3.273 is compatible with the Hopf algebra structures on both sides.

Corollary 3.274. *For every point $x \in M$, the iterated integral induces an isomorphism of Hopf algebras*

$$\mathrm{H}^0(B^*(A^*)) \xrightarrow{\sim} \mathcal{O}(\pi_1(M, x)^{\mathrm{un}}) \otimes \mathbb{C}.$$

Remark 3.275. The isomorphism of Corollary 3.274 depends on the choice of a base point x.

3.5.5. *The case of* $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The main example to which we would like to apply Corollary 3.274 is the manifold $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$. This example will be central for the remainder of the book. The fundamental group of M is the free group in two generators. Thus, its pro-unipotent completion is isomorphic to the spectrum of the Hoffman algebra $\mathfrak{H}$ (Example 3.237). We want to recover this fact as a particular case of Chen's theorem. For this, consider the differential forms

$$\omega_0 = \frac{\mathrm{d}t}{t}, \quad \omega_1 = \frac{\mathrm{d}t}{1-t}. \tag{3.276}$$

Let $A^*_{\mathbb{C}}$ be the dg-algebra over $\mathbb{C}$ given by

$$A^0_{\mathbb{C}} = \mathbb{C}, \qquad A^1_{\mathbb{C}} = \mathbb{C}\,\omega_0 \oplus \mathbb{C}\,\omega_1, \qquad A^{\geqslant 2}_{\mathbb{C}} = 0,$$

together with the trivial differential and the obvious multiplication. This algebra has the rational structure $A^*_{\mathbb{C}} = A^* \otimes \mathbb{C}$, where A is the $\mathbb{Q}$-algebra introduced in Example 2.284. In particular, the inclusion $A^*_{\mathbb{C}} \subset E^*(M,\mathbb{C})$ is a quasi-isomorphism.

Since $\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0$ and $\omega_0 \wedge \omega_1 = 0$, formula (3.258) shows that the differential in the reduced bar complex $B^*(A^*)$ is identically zero, and hence

$$\mathrm{H}^0(B^*(A^*)) = B^0(A^*). \tag{3.277}$$

Moreover, there is an isomorphism of Hopf algebras

$$\begin{aligned} \mathrm{H}^0(B^*(A^*)) &\longrightarrow \mathfrak{H} \\ \omega_0 \quad &\longmapsto x_0 \\ \omega_1 \quad &\longmapsto x_1 \end{aligned}$$

that induces an isomorphism of Hopf algebras

$$\mathrm{H}^0(B^*(A^*_{\mathbb{C}})) \longrightarrow \mathfrak{H} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Following Notation 1.162, for a binary sequence α, we will denote by ω_α the element of $\mathrm{H}^0(B^*(A^*_{\mathbb{C}}))$ corresponding to x_α.

⋆⋆⋆

Exercise 3.278. Show that the differentials d_I and d_C from equations (3.259) and (3.260) in the definition of the bar complex satisfy

$$\mathrm{d}_I^2 = \mathrm{d}_C^2 = 0 \quad \text{and} \quad \mathrm{d}_I\mathrm{d}_C + \mathrm{d}_C\mathrm{d}_I = 0.$$

Deduce that $\mathrm{d} = \mathrm{d}_I - \mathrm{d}_C$ satisfies $\mathrm{d}^2 = 0$ as well.

Exercise 3.279. Let η_1, η_2, and η_{12} be 1-forms on a differentiable manifold. What conditions should they satisfy for $[\eta_1|\eta_2] - [\eta_{12}]$ to be closed?

Exercise 3.280. Let M be a connected differentiable manifold with the homotopy type of a finite CW complex and let $E^*(M,\mathbb{C})$ be the differential graded algebra of smooth complex-valued differential forms on M. Consider the projection

$$E^1(M,\mathbb{C}) \longrightarrow E^1(M,\mathbb{C})/dE^0(M,\mathbb{C}).$$

Let r be any retraction of this projection as complex vector spaces. Show that

$$\mathbb{C} \oplus \mathrm{Im}(r) \oplus \bigoplus_{n \geqslant 2} E^n(M,\mathbb{C}) \;\subset\; E^*(M,\mathbb{C})$$

inherits the structure of a complex dg-algebra. It is connected and the inclusion is a quasi-isomorphism.

3.6. A geometric description of the pro-unipotent completion of the fundamental group. We now explain a proof of Chen's π_1-de Rham theorem (Theorem 3.273). This is not the classical proof that one can find for example in Hain's paper [**Hai87b**, § 4], but the strategy we follow will later enable us to exhibit the *motivic* nature of the pro-unipotent completion of the fundamental group of an algebraic variety. The first step in the proof is to show that the reduced bar complex of the de Rham complex of a differentiable manifold can be seen as the de Rham complex of a cosimplicial manifold.

3.6.1. *The normalized cochain complex and the reduced bar complex.* Let M be a connected differentiable manifold with the homotopy type of a finite CW complex, and let $x, y \in M$ be points.

Construction 3.281. We denote by ${}_yM_x^\bullet$ the cosimplicial manifold with:

- components
$$ {}_yM_x^n = M \times \overset{n}{\cdots} \times M; $$
- coface maps
$$ \delta^i \colon {}_yM_x^n \longrightarrow {}_yM_x^{n+1}, \quad i = 0, \dots, n+1, $$
given by the formulas
$$ \delta^i(x_1, \dots, x_n) = \begin{cases} (y, x_1, \dots, x_n), & \text{if } i = 0, \\ (x_1, \dots, x_i, x_i, \dots, x_n), & \text{if } 0 < i < n+1, \\ (x_1, \dots, x_n, x), & \text{if } i = n+1; \end{cases} $$
- codegeneracy maps
$$ \sigma^i \colon {}_yM_x^{n+1} \longrightarrow {}_yM_x^n, \quad i = 0, \dots, n, $$
given by the formula

$$ \sigma^i(x_1, \dots, x_{n+1}) = (x_1, \dots, x_i, x_{i+2}, \dots, x_{n+1}). \tag{3.282} $$

As in Section 3.5.3, let $E^*(M, \mathbb{C})$ be the dg-algebra of smooth complex-valued differential forms on M. For simplicity, we will assume that we have chosen a connected differential graded $\mathbb{C}$-algebra $A^*(M) \subseteq E^*(M, \mathbb{C})$ such that the inclusion is a quasi-isomorphism (see Exercise 3.280). We set
$$ A^*({}_yM_x^n) = A^*(M) \otimes \overset{n}{\cdots} \otimes A^*(M). $$
In particular, $A^*({}_yM_x^0) = \mathbb{C}$. The assignment $\Delta_n \rightsquigarrow A^*({}_yM_x^n)$ being functorial (Exercise 3.338), these complexes define a simplicial dg-algebra $A^*({}_yM_x^\bullet)$.

Remark 3.283. Since $A^*(M)$ is connected, the dg-algebra $A^*({}_yM_x^\bullet)$ does not depend on the base points $x, y \in M$. The reason to keep them in the notation is that, later on, we will build out of $A^*({}_yM_x^\bullet)$ the de Rham component of a mixed Hodge structure whose Betti component does depend on x and y.

The normalization $\mathcal{N}A^*({}_yM_x^\bullet)$ (Definition A.231) is a chain complex of cochain complexes. We transform it into a double cochain complex by changing the sign of the chain degree as in Definition A.13 vi). We denote by $\operatorname{Tot}\mathcal{N}A^*({}_yM_x^\bullet)$ the associated total complex (Definition A.33), given by
$$ \operatorname{Tot}^n \mathcal{N}A^*({}_yM_x^\bullet) = \bigoplus_{q-p=n} \mathcal{N}A^q({}_yM_x^p). $$

The subcomplexes $\sigma_{\leqslant N}\mathcal{N}A^*({}_yM_x^\bullet)$ define a filtration of $\operatorname{Tot}\mathcal{N}A^*({}_yM_x^\bullet)$. Be aware that the index N in the bête filtration refers only to the chain degree and not to the total degree. Thus,

$$\operatorname{Tot}^n \sigma_{\leqslant N}\mathcal{N}A^*({}_yM_x^\bullet) = \bigoplus_{\substack{q-p=n\\ p\leqslant N}} \mathcal{N}A^q({}_yM_x^p).$$

LEMMA 3.284. *The map*

$$\begin{aligned} \psi\colon\ B^*(A^*(M)) &\longrightarrow \operatorname{Tot}\mathcal{N}A^*({}_yM_x^\bullet)\\ [\omega_1|\cdots|\omega_n] &\longmapsto (-1)^{\sum_{i=1}^n (n-i)\deg(\omega_i)}\omega_1\otimes\cdots\otimes\omega_n \end{aligned}$$

is an isomorphism of complexes that sends $L_NB^*(A^*(M))$ to $\operatorname{Tot}\sigma_{\leqslant N}\mathcal{N}A^*({}_yM_x^\bullet)$. Similarly, if ε is the unique augmentation of $A^*(M)$, then the same formula gives us an isomorphism

$$B^*(A^*(M),\varepsilon,\varepsilon) \longrightarrow \operatorname{Tot} CA^*({}_yM_x^\bullet),$$

where C denotes the associated chain complex (Definition A.230)

PROOF. Recall the identification

$$B^*(A^*(M)) = \operatorname{Tot}\mathcal{N}B_\bullet(A^*(M))$$

and that the constituents of $B_\bullet(A^*(M))$ and of $A^*({}_yM_x^\bullet)$ are the same. The only difference are some signs in the faces, degeneracies, and de Rham differential on $B_\bullet(A^*(M))$. It follows that the map ψ is an isomorphism of graded vector spaces that respects the filtrations. To compute the differential in the total complex $\operatorname{Tot}\mathcal{N}A^*({}_yM_x^\bullet)$, let

$$\omega = \omega_1\otimes\cdots\otimes\omega_n \in \mathcal{N}A^m({}_yM_x^n) \subset \operatorname{Tot}^{m-n}\mathcal{N}A^*({}_yM_x^\bullet).$$

Then $\mathrm{d}\omega = \mathrm{d}_1\omega + (-1)^n\mathrm{d}_2\omega$, where d_1 is the differential in the normalized complex and d_2 is the differential in the de Rham complex. Therefore,

$$\begin{aligned} \mathrm{d}\omega = &\sum_{i=1}^{n-1}(-1)^i\omega_1\otimes\cdots\otimes\omega_i\wedge\omega_{i+1}\otimes\cdots\otimes\omega_n\\ &+(-1)^n\sum_{i=1}^{n}(-1)^{\sum_{j=1}^{i-1}\deg(\omega_j)}\omega_1\otimes\cdots\otimes\mathrm{d}\omega_i\otimes\cdots\otimes\omega_n. \end{aligned} \tag{3.285}$$

Comparing this formula with the differential from Definition 3.257 one checks the equality $\psi\circ\mathrm{d} = \mathrm{d}\circ\psi$. This finishes the proof. □

3.6.2. *A Mayer–Vietoris complex of sheaves.* The next step is to construct a complex of sheaves that computes certain relative cohomology groups. In fact, we are giving a variant of Construction 2.114 that was used to compute the relative de Rham cohomology in the case of a simple normal crossing divisor.

As in the previous sections, let M be a connected differentiable manifold which has the homotopy type of a finite CW complex, and let $Y_0,\dots,Y_N$ be a finite collection of closed subsets of M. Write

$$Y = Y_0\cup\cdots\cup Y_N.$$

NOTATION 3.286. The following notation will be used:

- Recall that Δ_N stands for the index set $\{0,\dots,N\}$.

- For each subset $I \subseteq \Delta_N$, we write Y_I for the intersection $\bigcap_{j\in I} Y_j$. We also write $|I|$ for the cardinal of I.
- Given a topological space T, we denote by $\underline{\mathbb{Q}}_T$ the constant sheaf on T with stalk $\mathbb{Q}$. If there is a clear closed immersion $\iota\colon T \to M$, by abuse of notation, we will denote also by $\underline{\mathbb{Q}}_T$ the extension by zero $\iota_*\underline{\mathbb{Q}}_T$. For example, if $x \in M$ is a point, we will write $\underline{\mathbb{Q}}_x$ for the skyscraper sheaf with stalk $\mathbb{Q}$ at x. For shorthand, in the situation at hand, we write $\underline{\mathbb{Q}}_I$ for the constant sheaf on Y_I extended by zero to M, that is,
$$\underline{\mathbb{Q}}_I = (\iota_I)_*\underline{\mathbb{Q}}_{Y_I},$$
where $\iota_I\colon Y_I \hookrightarrow M$ is the inclusion. In particular, $\underline{\mathbb{Q}}_\emptyset = \underline{\mathbb{Q}}_M$.
- Given subsets $I \subseteq K \subseteq \Delta_N$, there is an inclusion $Y_K \subseteq Y_I$. We denote by $\mathrm{d}_{K,I}\colon \underline{\mathbb{Q}}_I \to \underline{\mathbb{Q}}_K$ the corresponding restriction map.
- Given $K = \{i_0, \ldots, i_p\}$ with the indices i_ℓ ordered as $i_0 < \cdots < i_p$, and if $I = \{i_0, \ldots, \hat{i}_i, \ldots, i_p\}$, we set
$$\varepsilon(I, K) = (-1)^i$$
as in equation (2.116). We also write
$$\varepsilon(K) = \prod_{i\in K}(-1)^i. \tag{3.287}$$

For $0 \leqslant p \leqslant N$, we define a morphism of sheaves
$$\mathrm{d}\colon \bigoplus_{|I|=p} \underline{\mathbb{Q}}_I \longrightarrow \bigoplus_{|K|=p+1} \underline{\mathbb{Q}}_K \quad \text{as} \quad \mathrm{d} = \bigoplus_{\substack{I\subset K\\ p=|I|=|K|-1}} \varepsilon(I,K)\mathrm{d}_{K,I}.$$

We define the complex of sheaves $\widetilde{K}(M; Y_0, \ldots, Y_N)$ as
$$0 \longrightarrow \bigoplus_{|I|=0} \underline{\mathbb{Q}}_I \longrightarrow \bigoplus_{|I|=1} \underline{\mathbb{Q}}_I \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=N} \underline{\mathbb{Q}}_I \longrightarrow \bigoplus_{|I|=N+1} \underline{\mathbb{Q}}_I \longrightarrow 0, \tag{3.288}$$
where the sheaf $\bigoplus_{|I|=p}\underline{\mathbb{Q}}_I$ sits in degree p.

We also define the complex $K(M; Y_0, \ldots, Y_N)$ as
$$0 \longrightarrow \bigoplus_{|I|=0} \underline{\mathbb{Q}}_I \longrightarrow \bigoplus_{|I|=1} \underline{\mathbb{Q}}_I \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=N} \underline{\mathbb{Q}}_I \longrightarrow 0. \tag{3.289}$$
Note that the second complex agrees with the first one except for the last term $\underline{\mathbb{Q}}_{\Delta_N}$ that has been deleted.

Lemma 3.290. *If Y is locally contractible, then*
$$\mathbb{H}^n(M, \widetilde{K}(M; Y_0, \ldots, Y_N)) = \mathrm{H}^n(M, Y; \mathbb{Q}),$$
where the right-hand side is relative singular cohomology.

Proof. By Exercise 3.339, the sequence of sheaves
$$0 \longrightarrow \underline{\mathbb{Q}}_Y \longrightarrow \bigoplus_{|I|=1} \underline{\mathbb{Q}}_I \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=p} \underline{\mathbb{Q}}_I \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=N+1} \underline{\mathbb{Q}}_I \longrightarrow 0$$
is exact, so the complex $\widetilde{K}(M; Y_0, \ldots, Y_N)$ is quasi-isomorphic to
$$0 \longrightarrow \underline{\mathbb{Q}}_M \longrightarrow \underline{\mathbb{Q}}_Y \longrightarrow 0.$$

Being a differential manifold, M is a paracompact, as well as all of its closed and open subsets. Moreover, M is locally contractible. Since Y is also locally contractible, the result then follows then from Theorem 2.51. □

We now specialize the previous construction to a particular case. Let $x, y \in M$ be points and $N \geqslant 0$ an integer. Let M^N be the N-fold cartesian product of M. Given a point of M^N, we denote by $x_1, \dots, x_N$ its components. Consider the union $Y = Y_0 \cup \dots \cup Y_N$ of the closed subspaces $Y_i \subset M^N$ given by:

$$\begin{aligned} Y_0 &= \{y = x_1\}, \\ Y_i &= \{x_i = x_{i+1}\}, \quad i = 1, \dots, N-1, \\ Y_N &= \{x_N = x\}. \end{aligned}$$

Sometimes, it will useful to introduce the notation $x_0 = y$ and $x_{N+1} = x$, and write

$$Y_i = \{x_i = x_{i+1}\}$$

for all $i = 0, \dots, N$. Applying the previous construction, we define the complexes

$$\begin{aligned} {}_y\mathcal{K}_x\langle N\rangle &= K(M^N; Y_0, \dots, Y_N), \\ {}_y\widetilde{\mathcal{K}}_x\langle N\rangle &= \widetilde{K}(M^N; Y_0, \dots, Y_N). \end{aligned}$$

If the base points x and y are different from each other, then $Y_0 \cap \dots \cap Y_N = \emptyset$ and hence the two complexes agree: ${}_y\mathcal{K}_x\langle N\rangle = {}_y\widetilde{\mathcal{K}}_x\langle N\rangle$. By Lemma 3.290, the hypercohomology of ${}_y\mathcal{K}_x\langle N\rangle$ also computes the relative cohomology group:

$$\mathbb{H}^*(M^N, {}_y\mathcal{K}_x\langle N\rangle) = \mathrm{H}^*(M^N, Y; \mathbb{Q}), \quad \text{when } x \neq y.$$

In the case where $x = y$, the intersection $Y_0 \cap \dots \cap Y_N = \{(x, \dots, x)\}$ consists of a single point and there is a short exact sequence of complexes

$$0 \longrightarrow \underline{\mathbb{Q}}_{(x,\dots,x)}[-N-1] \longrightarrow {}_x\widetilde{\mathcal{K}}_x\langle N\rangle \longrightarrow {}_x\mathcal{K}_x\langle N\rangle \longrightarrow 0. \tag{3.291}$$

Note that the hypercohomology of the leftmost complex is concentrated in degree $N+1$, where it is isomorphic to $\mathrm{H}^0(M^N, \underline{\mathbb{Q}}_{(x,\dots,x)}) = \mathbb{Q}$. Thus, taking hypercohomology from (3.291) yields a long exact sequence

$$0 \longrightarrow \mathrm{H}^N(M^N, Y; \mathbb{Q}) \longrightarrow \mathbb{H}^N(M^N, {}_x\mathcal{K}_x\langle N\rangle) \xrightarrow{f} \mathbb{Q} \longrightarrow \cdots \tag{3.292}$$

The map f is surjective because it fits into a commutative diagram

$$\begin{tikzcd} \bigoplus_{|I|=N} \mathrm{H}^0(Y_I, \mathbb{Q}) \arrow[r] \arrow[dr] & \mathbb{H}^N(M^N, {}_x\mathcal{K}_x\langle N\rangle) \arrow[d, "f"] \\ & \mathbb{H}^0(M^N, \underline{\mathbb{Q}}_{(x,\dots,x)}) \end{tikzcd}$$

where the diagonal arrow is surjective. The kernel of f is thus $\mathrm{H}^N(M^N, Y; \mathbb{Q})$ and we get a short exact sequence

$$0 \longrightarrow \mathrm{H}^N(M^N, Y; \mathbb{Q}) \longrightarrow \mathbb{H}^N(M^N, {}_x\mathcal{K}_x\langle N\rangle) \xrightarrow{f} \mathbb{Q} \longrightarrow 0. \tag{3.293}$$

3.6.3. *An isomorphism of cohomology groups.* The next step is to relate the cohomology of the cosimplicial manifold ${}_yM_x^\bullet$ with that of the sheaf ${}_y\mathcal{K}_x\langle N\rangle$. Since we want an isomorphism on the level of singular cohomology over $\mathbb{Q}$, we will use smooth cochains instead of differential forms.

Given a differentiable manifold M, we denote by $S^*(M,\mathbb{Q})$ the complex of smooth singular cochains on M with rational coefficients as in Remark A.307. Recall that the complex $S^*(M,\mathbb{Q})$ computes the singular cohomology of M and that there is a quasi-isomorphism

$$E^*(M,\mathbb{C}) \longrightarrow S^*(M,\mathbb{Q})\otimes\mathbb{C}$$

given by integration of differential forms over smooth chains, that represents the comparison isomorphism between de Rham and singular cohomology in the differentiable case. As a consequence, there are quasi-isomorphisms

$$A^*(M) \xrightarrow{\sim} S^*(M,\mathbb{Q})\otimes\mathbb{C}, \quad A^*({}_yM_x^n) \xrightarrow{\sim} S^*(M^n,\mathbb{Q})\otimes\mathbb{C}.$$

Consider the simplicial object

$$S^*_\bullet = S^*({}_yM_x^\bullet,\mathbb{Q})$$

in the category of complex of $\mathbb{Q}$-vector spaces and there is a quasi-isomorphism

$$\operatorname{Tot}\mathcal{N}_*A^*({}_yM_x^\bullet) \xrightarrow{\sim} \operatorname{Tot}\mathcal{N}_*S^*_\bullet\otimes\mathbb{C}. \tag{3.294}$$

Moreover, we can apply to $S^*_\bullet$ the functor $C_*(\Delta_N,\cdot)$ defined at the end of Section A.8.2. To describe the resulting complex, for each $\emptyset\neq I\subset\Delta_N$ we set

$${}_yM_x^I = Y_{I^c}\subset M^N,$$

where $I^c=\Delta_N\setminus I$. Then

$${}_yM_x^I \simeq M^{|I|-1} \tag{3.295}$$

To realize this isomorphism, we just delete the redundant coordinates. For $i\in I^c$, the i-th coordinate of $(x_0,\dots,x_n)\in{}_yM_x^I$ coincides with the coordinate x_{i+1}, so is not needed. The first coordinate is always equal to y, so it is also redundant. Hence, if $I=(i_0,\dots,i_k)$ we only need to keep the coordinates $x_{i_1},\dots,x_{i_k}$. More precisely, we denote by $\iota_I\colon M^{|I|-1}\to M^N$ the composition of the inverse of the isomorphism (3.295) with the inclusion $Y_{I^c}\to M^N$. Then, if $I=(i_0,\dots,i_n)$ and

$$p=(x_{i_1},\dots,x_{i_n})\in{}_yM_x^I,$$

writing $x_{i_0}=y$ and $x_{i_{n+1}}=x$, we have

$$\iota_I(p)=(y_1,\dots,y_N),$$

where the coordinates y_i are defined as

$$y_i = x_{\min\{j\in I\cup\{n+1\}\mid j\geqslant i\}}.$$

For instance, if $N=6$ and $I=\{2,3,5\}$, the map $\iota_I\colon M^2\to M^6$ is given by

$$\iota_I(x_3,x_5)=(y,y,x_3,x_5,x_5,x).$$

For $K=\{j_0,\dots,j_n\}$ and $I=\{j_0,\dots,\widehat{j_i},\dots,j_n\}$, there is a face map

$$\delta_{I,K}\colon{}_yM_x^I\longrightarrow{}_yM_x^K$$

defined by the commutative diagram

$$\begin{tikzcd} {}_yM_x^I \arrow[r, "\delta_{I,K}"] \arrow[d, equal] & {}_yM_x^K \arrow[d, equal] \\ Y_{I^c} \arrow[r, hook] & Y_{K^c}. \end{tikzcd} \tag{3.296}$$

Explicitly,

$$\delta_{I,K}(x_{j_1},\dots,\widehat{x_{j_i}},\dots,x_{j_n}) = \begin{cases} (y, x_{j_2},\dots,x_{j_n}), & \text{if } i=0,\\ (\dots,x_{j_{i-1}},x_{j_{i+1}},x_{j_{i+1}},\dots), & \text{if } 0<i<n,\\ (x_{j_1},\dots,x_{j_{n-1}},x), & \text{if } i=n. \end{cases}$$

We now write

$$S_I^* = S^*({}_yM_x^I) \simeq S^*(M^{|I|-1}) = S_{|I|-1}^*.$$

Then, for each $p \geqslant 0$, we have

$$C_p(\Delta_N, S_\bullet^*) = \bigoplus_{\substack{I\subset \Delta_N\\ |I|=p+1}} S_I^*$$

with differential $\mathrm{d}\colon C_p(\Delta_N, S_\bullet^*) \to C_{p-1}(\Delta_N, S_\bullet^*)$ given, as in equation (A.237), by

$$\mathrm{d} = \bigoplus_{I\subset K} \varepsilon(I,K)\delta_{I,K}^*.$$

By Proposition A.238, there is a functorial homotopy equivalence

$$\phi\colon \operatorname{Tot}\sigma_{\leqslant N}\mathcal{N}_*S_\bullet^* \xrightarrow{\sim} \operatorname{Tot} C_*(\Delta_N, S_\bullet^*). \tag{3.297}$$

Explicitly, this morphism is given as follows. Let

$$\omega = (\omega_i)_{0\leqslant i\leqslant N} \in \operatorname{Tot}^*\sigma_{\leqslant N}\mathcal{N}_*S_\bullet^*.$$

Then $\phi(\omega) = (\omega_I)_{I\subset\Delta_N}$, where

$$\omega_I = \begin{cases} \omega_i, & \text{if } I=\Delta_i,\\ 0, & \text{otherwise.}\end{cases}$$

We next compare the cohomology of the complex $\operatorname{Tot} C_*(\Delta_N, S_\bullet^*)$ with that of the complex of sheaves ${}_y\mathcal{K}_x\langle N\rangle$. To this end, we represent the latter using also smooth singular cochains. We define the double complex $S_Y^{*,*}$ by

$$S_Y^{p,q} = \bigoplus_{|I|=p} S^q(Y_I,\mathbb{Q}), \quad p\geqslant 0, \quad 0\leqslant p<N,$$

with horizontal differential

$$\mathrm{d}'\colon \bigoplus_{|I|=p-1} S^q(Y_I,\mathbb{Q}) \longrightarrow \bigoplus_{|K|=p} S^q(Y_K,\mathbb{Q}), \quad \mathrm{d}' = \bigoplus_{I\subset K}\varepsilon(I,K)\mathrm{d}_{K,I}.$$

Let $\operatorname{Tot}^* S_Y$ be the associated total complex. By construction, there is an isomorphism of rational vector spaces

$$\mathbb{H}^*(M^N, {}_y\mathcal{K}_x\langle N\rangle[N]) = \mathrm{H}^*(\operatorname{Tot}^* S_Y[N]).$$

LEMMA 3.298. *There is a functorial isomorphism*

$$\operatorname{Tot} C_*(\Delta_N, S^*_\bullet) \xrightarrow{\sim} \operatorname{Tot}^* S_Y[N]$$

that induces an isomorphism

$$\mathrm{H}^*(\operatorname{Tot} C_*(\Delta_N, S^*_\bullet)) \xrightarrow{\sim} \mathbb{H}^*(M^N, {}_y\mathcal{K}_x\langle N\rangle[N]).$$

PROOF. For each $\emptyset \neq I \subset \Delta_N$, we have ${}_yM^I_x = Y_{I^c}$. Denote by

$$f_I \colon S^*({}_yM^I_x, \mathbb{Q}) \longrightarrow S^*(Y_{I^c}, \mathbb{Q})$$

the identity at the level of smooth singular cochains. As I runs through non-empty subsets of Δ_N, the morphisms f_I define an isomorphism between the graded $\mathbb{Q}$-vector spaces $\operatorname{Tot} C_*(\Delta_N, S^*_\bullet)$ and $\operatorname{Tot}^* S_Y[N]$.

To obtain an isomorphism of complexes, we also need to check the compatibility with the differentials. Let $\emptyset \neq I \subset K \subset \Delta_N$ be subsets with $|K| = |I| + 1$. The component of the horizontal differential of $C_*(\Delta_N, S^*_\bullet)$ from S^*_K to S^*_I is $\varepsilon(I, K)\delta^*_{I,K}$, while the component of the horizontal differential of $S^{*,*}_Y[N]$ from $S^*(Y_{K^c})$ to $S^*(Y_{I^c})$ is $(-1)^N\varepsilon(K^c, I^c)\mathrm{d}_{I^c,K^c}$. By the commutativity of diagram (3.296), the maps $\delta^*_{I,K}$ and d_{I^c,K^c} agree. Hence, we only need to adjust the signs.

Let $\varepsilon(I) = \prod_{i\in I}(-1)^i$ be the sign introduced in (3.287). The equality

$$\varepsilon(I, K)\varepsilon(K^c, I^c) = \varepsilon(I)\varepsilon(K)$$

is immediate to check. It follows that the map

$$\operatorname{Tot} C^*(\Delta_N, S^*_\bullet) \longrightarrow \operatorname{Tot} S^{*,*}_Y[N]$$

that sends $S^p({}_yM^I_x, \mathbb{Q})$ to $S^p(Y_{I^c}, \mathbb{Q})$ through the map

$$(-1)^{N|I|}\varepsilon(I)f_I \tag{3.299}$$

is an isomorphism of complexes. The sign $(-1)^{N|I|}$ accounts for the extra sign $(-1)^N$ coming from the shift in the differential of the complex $\operatorname{Tot} S^{*,*}_Y[N]$. $\square$

Combining Lemma 3.298 with Lemma 3.284, the homotopy equivalence (3.297), and the fact that $E^*(M, \mathbb{C}) \to S^*(M, \mathbb{Q}) \otimes \mathbb{C}$ is a quasi-isomorphism for any differentiable manifold M, we deduce the following result.

COROLLARY 3.300. *There is a functorial morphism*

$$L_N B^*(A^*(M)) \longrightarrow \operatorname{Tot}^* S_Y \otimes \mathbb{C}$$

that induces an isomorphism

$$\mathrm{H}^0(L_N B^*(A^*(M))) \simeq \mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle) \otimes \mathbb{C}.$$

In particular, if $y \neq x$ we deduce an isomorphism

$$\mathrm{H}^0(L_N B^*(A^*(M))) \simeq \mathrm{H}^N(M^N, Y; \mathbb{Q}) \otimes \mathbb{C}.$$

3.6.4. *Beilinson's theorem, global version.* The goal is to prove that the map

$$\mathrm{H}^0(L_N B^*(A^*(M))) \longrightarrow \big((\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1})\otimes\mathbb{C}\big)^\vee$$

discussed before Theorem 3.273 is an isomorphism. In view of Corollary 3.300, we deduce a morphism

$$\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)\otimes\mathbb{C} \longrightarrow \big((\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1})\otimes\mathbb{C}\big)^\vee.$$

We claim that it is defined over $\mathbb{Q}$. To prove this, we directly construct another morphism

$$\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle) \longrightarrow \big((\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1})\big)^\vee$$

compatible with the previous one.

Let $\gamma\colon [0,1]\to M$ be a smooth path satisfying $\gamma(0)=x$ and $\gamma(1)=y$. For each subset $\emptyset\neq I\subset\Delta_N$, consider the map

$$\begin{aligned}\sigma^{N,I}_{\gamma,y}\colon \quad \Delta^{|I|-1} &\longrightarrow {}_yM^I_x = Y_{I^c}\simeq M^{|I|-1}\\ (t_1,\dots,t_{|I|-1}) &\longmapsto \big(\gamma(t_1),\dots,\gamma(t_{|I|-1})\big).\end{aligned}$$

If $|I|=1$, then Y_{I^c} is reduced to a single point and the map $\sigma^{N,I}_{\gamma,y}$ is constant. Using the maps $\sigma^{N,I}_{\gamma,y}$, we define a map

$$\sigma^N_{\gamma,y}\colon S^{*,*}_Y[N]\longrightarrow\mathbb{Q}$$

that sends $\omega=(\omega_I)_{I\subsetneq\Delta_N}$ to

$$\sigma^N_{\gamma,y}(\omega)=\sum_{I\subsetneq\Delta_N}(-1)^{\frac{(|I^c|-1)(|I^c|-2)}{2}}(-1)^{N(|I^c|)}\varepsilon(I^c)\omega_I(\sigma^{N,I^c}_{\gamma,y}). \tag{3.301}$$

Observe the sign relation

$$(-1)^{\frac{(|I^c|-1)(|I^c|-2)}{2}}(-1)^{N(|I^c|)}=(-1)^{\frac{|I|(|I|+1)}{2}}(-1)^{\frac{N(N-1)}{2}}. \tag{3.302}$$

The reason for the complicated sign in equation (3.301) will be apparent in the proof of the next proposition.

Proposition 3.303. *The maps $\sigma^N_{\gamma,y}$ for varying γ define a morphism*

$$\sigma_y\colon \mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle) \longrightarrow \big((\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1})\big)^\vee$$

such that the diagram

$$\begin{tikzcd}\mathrm{H}^0(L_N B^*(A^*(M))) \arrow[r]\arrow[d] & \big((\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1})\otimes\mathbb{C}\big)^\vee \arrow[d]\\ \mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)\otimes\mathbb{C} \arrow[r] & \big((\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1})\otimes\mathbb{C}\big)^\vee\end{tikzcd}$$

is commutative.

Proof. We start by proving that, if ω is exact then $\sigma^N_{\gamma,y}(\omega)=0$. For this, we compute the boundary of the singular chain $\sigma^{N,K}_{\gamma,y}$ and we obtain

$$\partial\sigma^{N,K}_{\gamma,y}=\sum_{\substack{I\subset K\\ |K|=|I|+1}}\varepsilon(I,K)\sigma^{N,I}_{\gamma,y}. \tag{3.304}$$

If $\omega = \mathrm{d}\eta$ is exact in the complex $S_Y^{*,*}[N]$, then

$$(-1)^N \omega_I = (-1)^{|I|}\mathrm{d}\eta_I + \sum_{\substack{K\subset I\\ |K|=|I|-1}} \varepsilon(K,I)\eta_K|_{Y_I}. \tag{3.305}$$

Using the sign relation (3.302),

$$\begin{aligned}
(-1)^{\frac{N(N+1)}{2}}\sigma^N_{\gamma,y}(\omega) &= (-1)^N\sum_I (-1)^{\frac{|I|(|I|+1)}{2}}\varepsilon(I^c)\omega_I(\sigma^{N,I^c}_{\gamma,y})\\
&= \sum_I (-1)^{\frac{|I|(|I|+1)}{2}}\varepsilon(I^c)(-1)^{|I|}\mathrm{d}\eta_I(\sigma^{N,I^c}_{\gamma,y})\\
&\quad + \sum_I \sum_{\substack{K\subset I\\ |K|=|I|-1}} (-1)^{\frac{|I|(|I|+1)}{2}}\varepsilon(I^c)\varepsilon(K,I)\eta_K(\sigma^{N,I^c}_{\gamma,y})\\
&= \sum_I \sum_{\substack{K\supset I\\ |K|=|I|+1}} (-1)^{\frac{|I|(|I|+1)}{2}}\varepsilon(I^c)(-1)^{|I|}\varepsilon(K^c,I^c)\eta_I(\sigma^{N,K^c}_{\gamma,y})\\
&\quad + \sum_I \sum_{\substack{K\supset I\\ |K|=|I|+1}} (-1)^{\frac{|K|(|K|+1)}{2}}\varepsilon(K^c)\varepsilon(I,K)\eta_I(\sigma^{N,K^c}_{\gamma,y}).
\end{aligned}$$

In the above computation, the first equality is the definition of $\sigma^N_{\gamma,y}(\omega)$, the second equality is equation (3.305), and in the third equality we appy equation (3.304) to the first term and we interchange the roles of I and K in the second term. The vanishing $\sigma^N_{\gamma,y}(\omega)=0$ follows from the sign identities

$$\varepsilon(I,K)\varepsilon(K^c,I^c) = \varepsilon(I^c)\varepsilon(K^c), \text{ and } (-1)^{\frac{|I|(|I|+1)}{2}}(-1)^{|I|} = -(-1)^{\frac{|K|(|K|+1)}{2}},$$

for $I\subset K$ with $|K|=|I|+1$. Therefore, for ω closed, the value $\sigma^N_{\gamma,y}(\omega)$ only depends on the class of ω in $\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)$. In consequence, we obtain a pairing

$$\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)\otimes\mathbb{Q}[{}_y\mathcal{P}(M)_x] \longrightarrow \mathbb{Q}.$$

We next show that this pairing is compatible with (3.264). To write down some morphisms explicitly, we recall the notation of external product. Let $M_1,\dots,M_n$ be differential manifolds and ω_i a differential form on M_i for each i. Denote by

$$\mathrm{pr}_i\colon M_1\times\cdots\times M_n \longrightarrow M_i$$

the projection to the i-th component. Then the external product of the differential forms $\omega_1,\dots,\omega_n$ is the differential form on $M_1\times\cdots\times M_n$ given by

$$\omega_1\boxtimes\cdots\boxtimes\omega_n = \mathrm{pr}_1^*\omega_1\wedge\cdots\wedge\mathrm{pr}_n^*\omega_n.$$

Consider an element

$$[\omega_1|\dots|\omega_n]\in \mathrm{H}^0\left(L_NB^*(A^*(M))\right).$$

Since the total degree of this element is zero, every component form ω_i has degree one, *i.e.* $\omega_i\in A^1(M)$. Then the isomorphism

$$\mathrm{H}^0\left(L_NB^*(A^*(M))\right)\longrightarrow \mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)\otimes\mathbb{C}$$

sends $[\omega_1|\dots|\omega_n]$ to the form $\omega = (\omega_I)_{I\subsetneq\Delta_N}$, where $\omega_I\in S^n(Y_I,\mathbb{C})$ is given by

$$\omega_I = \begin{cases}(-1)^{\frac{n(n-1)}{2}}(-1)^{N(n+1)}\varepsilon(\Delta_n)\omega_1\boxtimes\cdots\boxtimes\omega_n, & \text{if } I=\Delta_n^c,\\ 0, & \text{otherwise.}\end{cases}$$

From this we get equalities

$$\begin{aligned}\sigma^N_{\gamma,y}(\omega) &= (-1)^{\frac{n(n-1)}{2}}(-1)^{N(n+1)}\varepsilon(\Delta_n)\omega_{\Delta_n^c}(\sigma^{N,\Delta_n}_{\gamma,y}) \\ &= \omega_1 \boxtimes \cdots \boxtimes \omega_n(\sigma^{N,\Delta_n}_{\gamma,y}) \\ &= \langle [\omega_1|\dots|\omega_n],\gamma\rangle,\end{aligned} \tag{3.306}$$

where the last term is the pairing (3.264). The signs in equation (3.301) are chosen to obtain the sign cancellation in (3.306) and the vanishing $\sigma^N_{\gamma,y}(\omega) = 0$ for exact ω.

Once we have stablished the compatibility with the pairing (3.264), Theorem 3.270, Proposition 3.269, and Corollary 3.300 imply the proposition. □

The following result gives a cohomological interpretation of the finite-dimensional pieces in the pro-unipotent completion of the fundamental group.

THEOREM 3.307 (Beilinson). *The map σ_y from Proposition 3.303 is an isomorphism. In particular, when $x \neq y$ there is an isomorphism*

$$\sigma_y\colon \mathbb{H}^N(M^N, Y;\mathbb{Q}) \longrightarrow \big((\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1})\big)^\vee.$$

The proof proceeds by induction on N and relies on a relative version of the morphism σ_y that we discuss next.

3.6.5. *Beilinson's theorem, relative version.* We next introduce a relative version of the complex ${}_y\mathcal{K}_x\langle N\rangle$, where we fix the point x but let y vary. For this, we consider the $(N+1)$-fold product

$$M^{1,N} = M \times M^N = M^{N+1}$$

regarded as a fibration over M with fiber M^N. That is to say, we consider coordinates $x_0, \dots, x_N$ on $M^{1,N}$, and denote by

$$\pi\colon M^{1,N} \longrightarrow M$$

the projection to the first factor. We introduce the closed subsets

$$Z_i = \{x_i = x_{i+1}\} \subseteq M^{1,N}, \qquad i = 0,\dots N,$$

where we are still using the convention $x_{N+1} = x$. For $y \in M$ and $i \in \Delta_N$, under the identification $\pi^{-1}(y) = M^N$, we have

$$Y_i = Z_i \cap \pi^{-1}(y),$$

so we can see Z_i as the family of sets Y_i for moving y (in fact, only Y_0 depends on y). Moreover, for any subset $I \subseteq \Delta_N$, we have

$$Z_I \cap \pi^{-1}(y) = Y_I.$$

We now define the complexes of sheaves

$$\begin{aligned}{}_\bullet\mathcal{K}_x\langle N\rangle &= K(M^{1,N}; Z_0, \dots, Z_N), \\ {}_\bullet\widetilde{\mathcal{K}}_x\langle N\rangle &= \widetilde{K}(M^{1,N}; Z_0, \dots, Z_N)\end{aligned}$$

on $M^{1,N}$, so that the following holds:

$$ {}_\bullet\mathcal{K}_x\langle N\rangle|_{\pi^{-1}(y)} = {}_y\mathcal{K}_x\langle N\rangle. \tag{3.308}$$

It is in this sense that ${}_\bullet\mathcal{K}_x\langle N\rangle$ is a relative version of ${}_y\mathcal{K}_x\langle N\rangle$.

The complexes ${}_\bullet\mathcal{K}_x\langle N\rangle$ and ${}_y\mathcal{K}_x\langle N\rangle$ satisfy a recurrence relation that will be useful later. The identity morphism between $M^{1,(N-1)}$ and M^N changes the numbering of the components, because in the convention we are using the coordinates of $M^{1,(N-1)}$ start with x_0 while those of M^N start with x_1. This identification sends the subset $Z_i \subset M^{1,(N-1)}$ to the subset $Y_{i+1} \subset M^N$ for $i = 0, \dots, N-1$.

Let $\iota_y \colon M^{N-1} \to M^N$ be the map

$$\iota_y(x_1, \dots, x_{N-1}) = (y, x_1, \dots, x_{N-1}).$$

For each $N \geqslant 1$, there is an exact sequence of sheaves of complexes

$$0 \longrightarrow (\iota_y)_* {}_y\mathcal{K}_x\langle N-1\rangle[-1] \longrightarrow {}_y\mathcal{K}_x\langle N\rangle \longrightarrow {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle \longrightarrow 0. \tag{3.309}$$

To describe this sequence, we use the following notation: given $I = (i_1, \dots, i_k)$, we denote by $I+1$ the multi-index

$$I + 1 = (i_1 + 1, \dots, i_k + 1).$$

Then, in degree $0 \leqslant j \leqslant N$, the sequence (3.309) reads

$$0 \longrightarrow \bigoplus_{\substack{I \subset \{0,\dots,N-1\} \\ |I| = j-1}} \underline{\mathbb{Q}}_{Y_{\{0\}\cup(I+1)}} \longrightarrow \bigoplus_{\substack{I \subset \{0,\dots,N\} \\ |I| = j}} \underline{\mathbb{Q}}_{Y_I} \longrightarrow \bigoplus_{\substack{I \subset \{1,\dots,N\} \\ |I| = j}} \underline{\mathbb{Q}}_{Y_I} \longrightarrow 0.$$

To identify the rightmost term of this sequence with a piece of ${}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle$ we are using the identification between the sheaf $\underline{\mathbb{Q}}_{Z_I}$ on $M^{1,N-1}$ and the sheaf $\underline{\mathbb{Q}}_{Y_{I+1}}$ on M^N. Finally, we have to be sure that the map

$$(\iota_y)_* {}_y\mathcal{K}_x\langle N-1\rangle[-1] \longrightarrow {}_y\mathcal{K}_x\langle N\rangle$$

is compatible with the differential. This amount to the sign relation

$$-\varepsilon(I, K) = \varepsilon\big(\{0\} \cup (I+1), \{0\} \cup (K+1)\big)$$

for subsets $I \subset K$ satisfying $|I| + 1 = |K|$.

The exact sequence (3.291) induces an exact sequence

$$0 \longrightarrow \underline{\mathbb{Q}}_{(x,\dots,x)}[-N] \to {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle \longrightarrow {}_\bullet\mathcal{K}_x\langle N-1\rangle \longrightarrow 0. \tag{3.310}$$

When considering a relative situation, such as the family $\pi \colon M^{1,N} \to M$, the analogue of the hypercohomology groups of the complex ${}_y\mathcal{K}_x\langle N\rangle$ are the higher direct image sheaves $R^i\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)$. As explained in Section A.9.5, they are defined as the sheaves of vector spaces associated with the presheaves that, to an open subset $U \subseteq M$, assign the vector space

$$\mathbb{H}^i(\pi^{-1}(U), {}_\bullet\mathcal{K}_x\langle N\rangle).$$

To understand them, we shall use the following concrete description of the cohomology. As in the previous sections, let $S^*(T, \mathbb{Q})$ denote the complex of smooth singular cochains on a differentiable manifold T. Using the construction (3.289) applied to $M^{1,N}$ and the subsets $Z_0, \dots, Z_N$, we obtain a double complex

$$\begin{aligned} 0 \longrightarrow \bigoplus_{|I|=0} S^*(Z_I \cap \pi^{-1}(U), \mathbb{Q}) \to \bigoplus_{|I|=1} S^*(Z_I \cap \pi^{-1}(U), \mathbb{Q}) \longrightarrow \dots \\ \dots \to \bigoplus_{|I|=N} S^*(Z_I \cap \pi^{-1}(U), \mathbb{Q}) \longrightarrow 0, \end{aligned} \tag{3.311}$$

which will be denoted by $S^*(Z_\bullet \cap \pi^{-1}(U), \mathbb{Q})$. The associated total complex computes the hypercohomology

$$\mathbb{H}^i(\pi^{-1}(U), {}_\bullet\mathcal{K}_x\langle N\rangle) = \mathrm{H}^i(\mathrm{Tot}(S^*(Z_\bullet \cap \pi^{-1}(U), \mathbb{Q}))).$$

Lemma 3.312. *For every contractible open subset U of M and every $y \in U$, the inclusion $\pi^{-1}(y) \subset \pi^{-1}(U)$ and the identification $\pi^{-1}(y) \simeq M^N$ induce an isomorphism*

$$\mathbb{H}^i(\pi^{-1}(U), {}_{\bullet}\mathcal{K}_x\langle N\rangle) \xrightarrow{\sim} \mathbb{H}^i(M^N, {}_y\mathcal{K}_x\langle N\rangle).$$

Proof. For every $I \subseteq \Delta_N$ with $|I| \leqslant N$, the morphism $\pi|_{Z_I} \colon Z_I \to M$ is a fibration. Therefore, given a contractible open subset $U \subseteq M$ and a point $y \in U$, the inclusion $Z_I \cap \pi^{-1}(y) \to Z_I \cap \pi^{-1}(U)$ is a homotopy equivalence. The induced morphism of complexes

$$S^*(Z_I \cap \pi^{-1}(U), \mathbb{Q}) \longrightarrow S^*(Z_I \cap \pi^{-1}(y), \mathbb{Q}) = S^*(Y_I, \mathbb{Q})$$

is a homotopy equivalence as well, from which the statement follows. □

Thanks Lemma 3.312, the sheaf $R^i\pi_*({}_{\bullet}\mathcal{K}_x\langle N\rangle)$ is a local system on M whose fiber at a point y is given by the hypercohomology group

$$R^i\pi_*({}_{\bullet}\mathcal{K}_x\langle N\rangle)_y = \mathbb{H}^i(M^N, {}_y\mathcal{K}_x\langle N\rangle).$$

Said differently, the sheaf $R^i\pi_*({}_{\bullet}\mathcal{K}_x\langle N\rangle)$ "glues together" the hypercohomology groups $\mathbb{H}^i(M^N, {}_y\mathcal{K}_x\langle N\rangle)$ for all possible base points y. We refer the reader to Section A.9.10 for a quick reminder on the different ways to think about local systems. In particular, The map f in the exact sequence (3.292) yields a morphism from this local system to a skyscraper sheaf

$$R^N\pi_*({}_{\bullet}\mathcal{K}_x\langle N\rangle) \longrightarrow \underline{\mathbb{Q}}_x. \tag{3.313}$$

We now describe some local systems defined using the fundamental groupoid instead of relative cohomology. The point $x \in M$ continues to be fixed. There are left actions of $\pi_1(M, x)$ on the $\mathbb{Q}$-vector spaces

$$\mathbb{Q}[\pi_1(M, x)] \quad \text{and} \quad \mathbb{Q}[\pi_1(M, x)]/J^{N+1}$$

given by path composition. These actions define local systems (see Section A.9.10)

$$\mathbb{Q}[\pi_1(M; \bullet, x)] \quad \text{and} \quad \mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1}.$$

The first local system may be of infinite rank, but the second one is always of finite rank. The fiber at a point y of the first local system is given by

$$\mathbb{Q}[\pi_1(M; \bullet, x)]_y = \mathbb{Q}[\pi_1(M; y, x)].$$

For every contractible open subset U, the sections of $\mathbb{Q}[\pi_1(M; \bullet, x)](U)$ are functions

$$s \colon U \longrightarrow \coprod_{y \in U} \mathbb{Q}[\pi_1(M; y, x)]$$

such that, for all points $y, y' \in U$ and classes of paths $\gamma \in \pi_1(U; y, y')$, the relation

$$s(y) = \gamma \cdot s(y')$$

holds. Note that, since U is assumed to be contractible, $\pi_1(U; y, y')$ contains a single element. Therefore, for every point $y \in U$, there is a canonical identification

$$\mathbb{Q}[\pi_1(M; \bullet, x)](U) = \mathbb{Q}[\pi_1(M; y, x)].$$

The description of $\mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1}$ is similar. Moreover, the unit of $\pi_1(M; x)$ induces maps

$$\begin{aligned} &\mathbb{Q} \longrightarrow \mathbb{Q}[\pi_1(M; \bullet, x)]_x, \\ &\mathbb{Q} \longrightarrow \left(\mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1}\right)_x. \end{aligned} \tag{3.314}$$

Our next goal is to construct a morphism of local systems from $R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)$ to $\big(\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1}\big)^\vee$ that should be thought of as a relative version of the map σ_y from Proposition 3.303.

LEMMA 3.315. *The maps σ_y of Proposition 3.303 for varying y glue together to a morphism of local systems*

$$\sigma\colon R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle) \longrightarrow \big(\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1}\big)^\vee.$$

PROOF. We have two local systems and a collection of morphisms between their fibers. To see that they glue together to a morphism of local systems, we need to prove that they are compatible with *parallel transport.* Given local systems F and G on M and linear maps $f_y\colon F_y \to G_y$ for each $y \in M$, to glue all these morphism we need to see that they are compatible with parallel transport. Namely, given a contractible open subset $U \subset M$ and points $y, y' \in U$, the diagram

$$\begin{tikzcd} F_y \arrow[d, "f_y"'] & F(U) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & F_{y'} \arrow[d, "f_{y'}"] \\ G_y & G(U) \arrow[l, "\simeq"] \arrow[r, "\simeq"'] & G_{y'} \end{tikzcd}$$

should commute. If this is the case, one can show that the family f_y defines a morphism of representations of the fundamental group and apply Theorem A.315.

Let $U \subset M$ be a contractible subset and $y, y' \in U$ two points. An element of $\big(\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1}\big)_y$ is represented by a linear combination of paths from x to y and the parallel transport is given by composition of paths. The fiber of the first local system is

$$R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)_y = \mathbb{H}^N(\pi^{-1}(y), {}_y\mathcal{K}_x\langle N\rangle)$$

and the parallel transport is the composition

$$\mathbb{H}^N(\pi^{-1}(y), {}_y\mathcal{K}_x\langle N\rangle) \xleftarrow{\sim} \mathbb{H}^N(\pi^{-1}(U), {}_\bullet\mathcal{K}_x\langle N\rangle) \xrightarrow{\sim} \mathbb{H}^N(\pi^{-1}(y'), {}_{y'}\mathcal{K}_x\langle N\rangle).$$

Let $\omega \in S^N(Z_\bullet \cap \pi^{-1}(U), \mathbb{Q})$ be a closed singular cochain, and write ω_y and $\omega_{y'}$ for the restrictions of ω to $S^*(Z_\bullet \cap \pi^{-1}(y), \mathbb{Q})$ and $S^*(Z_\bullet \cap \pi^{-1}(y'), \mathbb{Q})$ respectively. Let $\gamma \in {}_y\mathcal{P}(M)_x$ and $\gamma' \in {}_{y'}\mathcal{P}(U)_y$ be paths. By the previous discussion, to establish the lemma, it suffices to prove the equality

$$\sigma^N_{\gamma,y}(\omega_y) = \sigma^N_{\gamma'\gamma,y'}(\omega_{y'}).$$

The idea is to construct a singular chain with boundary $\sigma^N_{\gamma,y} - \sigma^N_{\gamma'\gamma,y'}$.

Recall that any oriented polyhedron P defines a singular chain after choosing a triangulation, and that the chains obtained from different triangulations are cohomologous. Any facet of P (*i.e.* a codimension one face) inherits an orientation from the orientation of P, and a triangulation of P defines a triangulation of the facets. Fixing a triangulation of P and identifying P and its facets with the corresponding singular chains, the equality

$$\partial P = \sum_{F \text{ facet of } P} F$$

holds. In this expression, the signs of the boundary of a chain are concealed in the orientation of the facets.

We next observe that the map $H\colon [0,1]\times[0,1]\to M$ defined as

$$H(s,t)=\gamma'\gamma\big((1+s)t/2\big)$$

satisfies the following (use the definition (3.3) of composition of paths):

$$\begin{aligned} H(0,t)&=\gamma(t), & H(1,t)&=\gamma'\gamma(t),\\ H(s,0)&=x, & H(s,1)&=\gamma'(s). \end{aligned}$$

In the previous section, we identified Y_{I^c} with $M^{|I|-1}$ by deleting the redundant coordinates. In the same way, we can identify Z_{I^c} with $M^{|I|}$. With this identification, the projection $Z_{I^c}\to M$ is the projection over the first coordinate.

For each $\emptyset\neq I\subset\Delta^N$, we denote by

$$H^{N,I}_{\gamma,\gamma'}\colon [0,1]\times\Delta^{|I|-1}\to Z_{I^c}\cap\pi^{-1}(U)$$

the map given by

$$H^{N,I}_{\gamma,\gamma'}(s,t_1,\dots,t_{|I|-1})=\Big(\gamma'(s),\gamma'\gamma\big((1+s)t_1/2\big),\dots,\gamma\big((1+s)t_{|I|-1}/2\big)\Big).$$

After triangulating $[0,1]\times\Delta^{|I|-1}$, this defines a singular chain in $Z_{I^c}\cap\pi^{-1}(U)$. Viewed as a chain in $\pi^{-1}(U)$, it satisfies the boundary identity

$$\partial H^{N,K}_{\gamma,\gamma'}=\sigma^{N,K}_{\gamma,y}-\sigma^{N,K}_{\gamma'\gamma,y'}-\sum_{\substack{I\subset K\\|K|=|I|+1}}\varepsilon(I,K)H^{N,I}_{\gamma,\gamma'}.$$

For a form $\eta\in S^{N+1}(Z_\bullet\cap\pi^{-1}(U),\mathbb{Q})$, we write

$$H^N_{\gamma,\gamma'}(\eta)=\sum_{I\subsetneq\Delta_N}(-1)^{\frac{(|I^c|-1)(|I^c|-2)}{2}}(-1)^{N(|I^c|)}\varepsilon(I^c)\eta_I(H^{N,I^c}_{\gamma,\gamma'}).$$

Then computing as in the first part of the proof of Proposition 3.303, we get

$$0=H^N_{\gamma,\gamma'}(\mathrm{d}\omega)=\sigma^N_{\gamma,y}(\omega_y)-\sigma^N_{\gamma'\gamma,y'}(\omega_{y'}),$$

which concludes the proof of the lemma. □

The following result is the relative version of Theorem 3.307 that implies it. It gives a cohomological interpretation of the finite-dimensional pieces in the pro-unipotent completion of the fundamental group. There are two proofs of this theorem, due to Goncharov [**Gon01**, § 4] and Deligne–Goncharov [**DG05**, §3.3].

THEOREM 3.316 (Beilinson).

i) *The sheaf $R^i\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)$ vanishes for all $i\leqslant N-1$. In particular,*

$$\mathbb{H}^i(M^N,{}_y\mathcal{K}_x\langle N\rangle)=0,\quad i\leqslant N-1.$$

ii) *The map σ defined in Lemma 3.315 is an isomorphism of local systems*

$$\sigma\colon R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)\longrightarrow\big(\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1}\big)^\vee.$$

In particular, there are natural isomorphisms

$$\mathbb{H}^N(M^N,{}_y\mathcal{K}_x\langle N\rangle)\longrightarrow\big(\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1}\big)^\vee.$$

iii) *The diagram of sheaves on* M

$$\begin{tikzcd} R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle) \arrow[r, "\sigma"] \arrow[dr] & \left(\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1}\right)^\vee, \arrow[d] \\ & \underline{\mathbb{Q}}_x \end{tikzcd}$$

where the diagonal arrow is (3.313) *and the vertical arrow is induced by the dual of the unit* (3.314), *is commutative.*

Proof. We start with statement iii) in the theorem. Since $(\underline{\mathbb{Q}}_x)_y = 0$ holds for $y \neq x$, we only need to check what happens for $y = x$, in which case the statement reduces to the commutativity of the diagram

(3.317)
$$\begin{tikzcd} \mathbb{H}^N(M^N, {}_x\mathcal{K}_x\langle N\rangle) \arrow[r] \arrow[d] & (\mathbb{Q}[\pi_1(M,x)]/J^{N+1})^\vee. \arrow[d] \\ \mathbb{H}^N(M^N, \underline{\mathbb{Q}}_{(x,\dots,x)}[-N]) \arrow[r] & \mathbb{Q} \end{tikzcd}$$

For simplicity, we compute $\mathbb{H}^N(M^N, {}_x\mathcal{K}_x\langle N\rangle)$ as the cohomology of the complex

$$C^* = \bigoplus_{\substack{I\subset\Delta_N\\ |I|\leqslant N}} \mathrm{Tot}^*(\mathcal{N}S^*(Y_I,\mathbb{Q})),$$

where $\mathcal{N}S^*$ denotes the normalized complex of smooth cochains (see Section A.8.2 for the definition of the normalized complex associated with a simplicial abelian group). The advantage of this point of view is that the elements of C^* vanish on degenerate chains, simplifying slightly the argument below.

Let γ_x be the constant path x in M. Since γ_x is constant, for a subset $I \subset \Delta_N$ with $|I| \leqslant N$, the singular chain $\sigma^{N,I}_{\gamma_x}$ is supported on $Z_I \cap \pi^{-1}(x) \subset M^{1,N}$. Identifying $\pi^{-1}(x)$ with M^N, we can see it as a singular chain on Y_I. This chain is degenerate unless $|I| = N$. When $|I| = N$, the chain $\sigma^{N,I}_{\gamma_x}$ is the zero-dimensional simplex supported at the point $(x,\dots,x)$.

Let $\omega = \sum_I \omega_I \in C^N$ be a closed element. The left vertical map on diagram (3.317) followed by the bottom arrow sends ω to

$$\sum_{|I|=N} \varepsilon(I,\Delta_N)\omega_I(x,\dots,x). \tag{3.318}$$

We apply the top arrow followed by the right vertical arrow to ω, using equation (3.301) and taking into account that we are working in the complex of normalized cochains, and we obtain the element

$$\sum_{|I|=N} \varepsilon(I^c)\omega_I(\sigma^{N,I}_{\gamma_x}). \tag{3.319}$$

The equality between (3.318) and (3.319) follows from the identity

$$\varepsilon(I^c) = \varepsilon(I,\Delta_N)$$

for $|I| = N$, which proves iii).

We now turn to the proof of statements i) and ii) in the theorem. We proceed by induction on N. The case $N = 0$ is obvious. Since we already now that σ is a morphism of sheaves, it is enough to prove the statements fiberwise. Let $y \in M$. From the exact sequence (3.309), we deduce a long exact sequence

$$\begin{tikzcd} & & & \mathbb{H}^{N-1}({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \arrow[dlll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ \mathbb{H}^{N-1}({}_y\mathcal{K}_x\langle N-1\rangle) \arrow[r, "\iota"] & \mathbb{H}^N({}_y\mathcal{K}_x\langle N\rangle) \arrow[r] & \mathbb{H}^N({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \arrow[dll, "g"] & \\ \mathbb{H}^N({}_y\mathcal{K}_x\langle N-1\rangle) & & & \end{tikzcd} \tag{3.320}$$

In this exact sequence, we have omitted the spaces as they can be deduced from the sheaves. We now use the sequence (3.320) to write down the diagram

$$\begin{tikzcd} & \mathbb{H}^{N-1}({}_y\mathcal{K}_x\langle N-1\rangle) \arrow[r, "\iota"] \arrow[d, "\sigma"] & \mathbb{H}^N({}_y\mathcal{K}_x\langle N\rangle) \arrow[r] \arrow[d, "\sigma"] & \operatorname{Ker}(g) \arrow[r] & 0 \\ 0 \arrow[r] & (\mathbb{Q}[\pi_1(M;y,x)]/J^N)^\vee \arrow[r] & (\mathbb{Q}[\pi_1(M;y,x)]/J^{N+1})^\vee \arrow[r] & (J^N/J^{N+1})^\vee \arrow[r] & 0 \end{tikzcd} \tag{3.321}$$

in which the first and the second rows are exact.

Claim: The left square in the above diagram is commutative.

Indeed, the first horizontal map sends a closed smooth cochain

$$\omega = \sum_{I \subseteq \Delta_{N-1}} \omega_I$$

representing a class in $\mathbb{H}^{N-1}(M^N, {}_y\mathcal{K}_x\langle N-1\rangle)$ to the cochain

$$\iota(\omega) = \sum_{I \subseteq \Delta_{N-1}} \omega_I,$$

where ω_I is now seen as a cochain in $Y_{\{0\}\cup(I+1)}$. For every class $\gamma \in \mathbb{Q}[\pi_1(M;y,x)]$, we then get the equalities

$$\begin{aligned} \sigma(\omega)(\gamma) &= \sum_{I \subseteq \Delta_{N-1}} (-1)^{\frac{|I|(|I|+1)}{2}\frac{(N-1)(N-2)}{2}} \varepsilon(I^c)\omega_I(\sigma_\gamma^{N-1,I}), \\ \sigma(\iota(\omega))(\gamma) &= \sum_{I \subseteq \Delta_{N-1}} (-1)^{\frac{(|I|+1)(|I|+2)}{2}\frac{N(N-1)}{2}} \varepsilon((I')^c)\omega_I(\sigma_\gamma^{N,I'}), \end{aligned}$$

where $I' = \{0\} \cup (I+1)$. Since the chains $\sigma_\gamma^{N-1,I}$ and $\sigma_\gamma^{N,\{0\}\cup I}$ agree and the equality of signs

$$(-1)^{\frac{|I|(|I|+1)}{2}\frac{(N-1)(N-2)}{2}}(-1)^{\frac{(|I|+1)(|I|+2)}{2}\frac{N(N-1)}{2}} = (-1)^{N+|I|} = \varepsilon(I^c)\varepsilon((I')^c)$$

is satisfied, we deduce that the square commutes. $\square$

Once we know this, an easy diagram chase yields a map

$$\sigma\colon \operatorname{Ker}(g) \longrightarrow (J^N/J^{N+1})^\vee,$$

which makes the rightmost square of the diagram (3.321) commutative.

Lemma 3.322.

i) *The equality* $\mathbb{H}^i(M^{1,N-1}, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) = 0$ *holds for all* $i \leqslant N-1$.

ii) *The map* $\sigma\colon \operatorname{Ker}(g) \to (J^N/J^{N+1})^\vee$ *is an isomorphism.*

PROOF. Let $\pi\colon M^{1,N-1} \to M$ be the projection to the first factor. We compute the hypercohomology $\mathbb{H}^i(M^{1,N-1}, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)$ using the Leray spectral sequence associated with π, that is

$$E_2^{p,q} = \mathrm{H}^p(M, R^q\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \Longrightarrow \mathbb{H}^{p+q}(M^{1,N-1}, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle). \tag{3.323}$$

Taking higher direct images with respect to π from the exact sequence of complexes (3.310) yields isomorphisms

$$R^i\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \simeq R^i\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle), \quad i \leqslant N-2,$$

and an exact sequence of sheaves

$$0 \longrightarrow R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \longrightarrow R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle) \longrightarrow \underline{\mathbb{Q}}_x \longrightarrow 0. \tag{3.324}$$

The exactness on the right follows, after passing to the fiber at x, from the surjectivity of the map f in the sequence (3.292).

Now recall that the induction hypothesis in the proof of the theorem is that the sheaf $R^i\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)$ vanishes for all $i \leqslant N-2$, and hence

$$R^i\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) = 0 \text{ for all } i \leqslant N-2.$$

Therefore, the Leray spectral sequence (3.323) looks as depicted in Figure 17. There

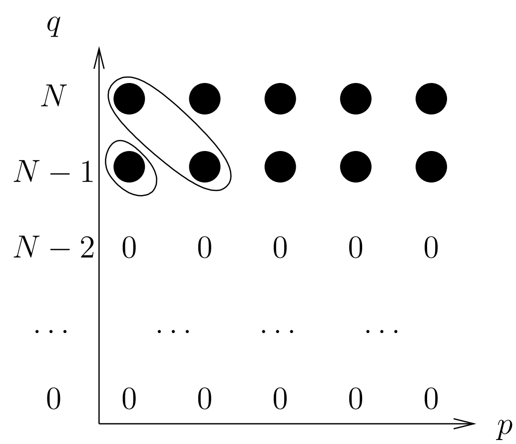


FIGURE 17. The Leray spectral sequence for ${}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle$.

can also be non-zero terms above the Nth row, but the important feature is that all rows strictly below the $(N-1)$th row are zero.

From the shape of the spectral sequence, we deduce the equality

$$\mathbb{H}^i(M^N, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) = \begin{cases} 0, & \text{if } i \leqslant N-2, \\ \mathrm{H}^0\left(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right), & \text{if } i = N-1, \end{cases} \tag{3.325}$$

and a short exact sequence of vector spaces

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1\left(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \to \mathbb{H}^N(M^N, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \longrightarrow \\ \longrightarrow \mathrm{H}^0\left(M, R^N\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \longrightarrow 0. \end{aligned} \tag{3.326}$$

To prove statement i) in the lemma, it remains to show that

$$\mathrm{H}^0\left(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) = 0. \tag{3.327}$$

The long exact sequence of cohomology associated with the short exact sequence of sheaves (3.324) yields

$$\begin{gathered}0 \to \mathrm{H}^0\left(R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \to \mathrm{H}^0\left(R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)\right) \xrightarrow{a} \mathbb{Q} \to \\ \mathrm{H}^1\left(R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \xrightarrow{b} \mathrm{H}^1\left(R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)\right) \to 0.\end{gathered} \tag{3.328}$$

We shall prove that the map a is an isomorphism. From this, we will derive the vanishing (3.327) and the fact that the map b is an isomorphism as well. For this we need to compute the cohomology of the sheaf $R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)$. By the induction hypothesis in the theorem, the map

$$\sigma\colon R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle) \longrightarrow (\mathbb{Q}[\pi_1(M;\bullet,x)]/J^N)^\vee \tag{3.329}$$

is an isomorphism and, in particular, the sheaf $R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)$ is a local system on M with fiber

$$R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)_x \simeq (\mathbb{Q}[\pi_1(M,x)]/J^N)^\vee. \tag{3.330}$$

Set $\Gamma = \pi_1(M,x)$. By Theorem A.317, the cohomology of $R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)$ in degrees $i=0$ and 1 can be computed as the group cohomology of Γ acting on (3.330). In symbols,

$$\mathrm{H}^i(M, R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)) = \mathrm{H}^i(\Gamma, (\mathbb{Q}[\Gamma]/J^N)^\vee) \quad (i=0,1).$$

Consider the short exact sequence of Γ-modules

$$0 \longrightarrow (\mathbb{Q}[\Gamma]/J^N)^\vee \longrightarrow \mathbb{Q}[\Gamma]^\vee \longrightarrow (J^N)^\vee \longrightarrow 0. \tag{3.331}$$

The Γ-module $\mathbb{Q}[\Gamma]^\vee$ being injective, its cohomology is concentrated in degree zero and there is an exact sequence

$$\begin{gathered}0 \to \mathrm{H}^0(\Gamma, (\mathbb{Q}[\Gamma]/J^N)^\vee) \to \mathrm{H}^0(\Gamma, \mathbb{Q}[\Gamma]^\vee) \to \mathrm{H}^0(\Gamma, (J^N)^\vee) \to \\ \mathrm{H}^1(\Gamma, (\mathbb{Q}[\Gamma]/J^N)^\vee) \longrightarrow 0.\end{gathered} \tag{3.332}$$

The cohomology in degree zero $\mathrm{H}^0(\Gamma, A)$ of a Γ-module A is the group of invariants A^Γ. From this one easily checks:

- The cohomology $\mathrm{H}^0(\Gamma, \mathbb{Q}[\Gamma]^\vee)$ is the one-dimensional $\mathbb{Q}$-vector space generated by the function
$$\sum_{\gamma\in\Gamma} a_\gamma[\gamma] \longmapsto \sum_{\gamma\in\Gamma} a_\gamma,$$
and the dual of the unit (3.314) induces an isomorphism
$$\mathrm{H}^0(\Gamma, \mathbb{Q}[\Gamma]^\vee) \xrightarrow{\sim} \mathbb{Q}.$$
- The cohomology $\mathrm{H}^0(\Gamma, (J^N)^\vee)$ is equal to $(J^N/J^{N+1})^\vee$, and the map
$$\mathrm{H}^0(\Gamma, \mathbb{Q}[\Gamma]^\vee) \longrightarrow (J^N/J^{N+1})^\vee$$
in the long exact sequence (3.332) is the zero map.

Putting together the above facts, the isomorphism (3.330), and the long exact sequence (3.332), we deduce

$$\mathrm{H}^0(M, R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)) \simeq \mathbb{Q}, \tag{3.333}$$

$$\mathrm{H}^1(M, R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)) \simeq (J^N/J^{N+1})^\vee. \tag{3.334}$$

Besides, the map a in (3.328) agrees with the isomorphism (3.333) by statement iii) in the theorem. From this and (3.325), we derive

$$\mathbb{H}^{N-1}(M^{1,N-1}, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) = \mathrm{H}^0(M, R^{N-1}\pi_*{}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) = 0,$$

thus concluding the proof of statement i) in the lemma.

We now turn to the proof of ii). Combining the fact that the map b in (3.328) is an isomorphism with (3.334), we get an isomorphism

$$\mathrm{H}^1\left(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \simeq (J^N/J^{N+1})^\vee.$$

Besides, by the exact sequence (3.326), there is an inclusion

$$\mathrm{H}^1\left(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \subseteq \mathbb{H}^N(M^N, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle).$$

Claim: This subspace is equal to $\mathrm{Ker}(g)$.

To prove the claim, we consider the long exact sequence of sheaves

$$\begin{aligned} \cdots \longrightarrow & R^{N-1}\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle) \xrightarrow{\varphi} R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle) \\ & \longrightarrow R^N\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \longrightarrow R^N\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle) \longrightarrow \cdots \end{aligned} \tag{3.335}$$

deduced from (3.309) by taking higher direct images. In this exact sequence, the sheaves $R^q\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)$ are all skyscraper sheaves supported at $y \in M$, and hence their cohomology is concentrated in degree zero. Thus, in the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1\left(M, R^{N-1}\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)\right) \longrightarrow \mathbb{H}^N(M^{N-1}, {}_y\mathcal{K}_x\langle N-1\rangle) \\ \longrightarrow \mathrm{H}^0\left(M, R^N\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)\right) \longrightarrow 0 \end{aligned} \tag{3.336}$$

obtained by applying the Leray spectral sequence to $(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)$, the leftmost term vanishes and the last but one map is an isomorphism.

Let us introduce the sheaf $\mathcal{F} = \mathrm{Coker}(\varphi)$ and consider the commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
 & \mathrm{H}^1\left(R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & \mathbb{H}^N(M^{1,N-1}, {}_\bullet\mathcal{K}_x\langle N-1\rangle) \arrow[r, "g"] \arrow[d] & \mathbb{H}^N(M^{N-1}, {}_y\mathcal{K}_x\langle N-1\rangle) \arrow[d] \\
\mathrm{H}^0(\mathcal{F}) \arrow[r] & \mathrm{H}^0\left(R^N\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \arrow[r, "e"] \arrow[d] & \mathrm{H}^0\left(R^N\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)\right), \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

where the first column is (3.326), the second column is (3.336), and the last row is part of the exact sequence obtained by taking cohomology from (3.335). The above diagram immediately gives an inclusion

$$\mathrm{H}^1\left(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \subseteq \mathrm{Ker}(g)$$

and to prove that this is an equality it is enough to show that $\mathrm{H}^0(M, \mathcal{F}) = 0$. To get this vanishing, we will show that the sheaf $\mathcal{F}$ is the extension by zero of a local system on $M \setminus \{y\}$. We need to distinguish the cases when the base points x and y are distinct or equal.

Case $x \neq y$. Write $U = M^{1,N-1} \setminus \pi^{-1}(y)$. Since the complex $(\iota_y)_{*\,y}\mathcal{K}_x\langle N-1\rangle$ is supported at $\pi^{-1}(y)$, one first gets from (3.309) an isomorphism

$${}_y\mathcal{K}_x\langle N\rangle|_U \simeq {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle|_U$$

and combining this with (3.310) one obtains a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Q}}_{(x,\dots,x)}[-N] \longrightarrow {}_y\mathcal{K}_x\langle N\rangle|_U \longrightarrow {}_\bullet\mathcal{K}_x\langle N-1\rangle|_U \longrightarrow 0.$$

In the associated long exact sequence

$$R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle|_U) \xrightarrow{h} R^N\pi_*\underline{\mathbb{Q}}_{(x,\dots,x)}[-N] \longrightarrow$$
$$R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle|_U) \longrightarrow R^N\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle|_U) \longrightarrow 0,$$

the map h is surjective, by repeating the argument that yields the surjectivity of the map f in (3.292). We thus get an isomorphism

$$R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)|_{M\setminus\{y\}} \longrightarrow R^N\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)|_{M\setminus\{y\}}.$$

Since the right-hand side is a local system by Lemma 3.312, the same is true for the left-hand side. Let now $V \subseteq M$ be a contractible open subset containing y but not x. Then the restrictions of ${}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle$ and ${}_\bullet\mathcal{K}_x\langle N-1\rangle$ to $\pi^{-1}(V)$ are isomorphic, so that (3.309) induces a long exact sequence

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \mathbb{H}^i(\pi^{-1}(V), {}_y\mathcal{K}_x\langle N\rangle) \arrow[r] & \mathbb{H}^i(\pi^{-1}(V), {}_\bullet\mathcal{K}_x\langle N-1\rangle) \arrow[dll, "j"] \\
\mathbb{H}^i(\pi^{-1}(y), {}_y\mathcal{K}_x\langle N-1\rangle) \arrow[r] & \cdots &
\end{tikzcd}$$

By Lemma 3.312, the map j is an isomorphism in all degrees $i \geqslant 0$. This implies, in particular, the vanishing $\mathbb{H}^N(\pi^{-1}(V), {}_y\mathcal{K}_x\langle N\rangle) = 0$ for all contractible open sets V containing the point y, and hence

$$R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)_y = 0.$$

Finally, since the source of the map

$$\varphi\colon R^{N-1}\pi_*(\iota_y)_*({}_y\widetilde{\mathcal{K}}_x\langle N-1\rangle) \longrightarrow R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)$$

is a skyscraper sheaf supported at the point y, it follows that φ is identically zero. We have thus shown that $\mathcal{F} = R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)$ is the extension by zero of a local system on $M \setminus \{y\}$.

Case $x = y$. On $U = M^{1,N-1} \setminus \pi^{-1}(x)$, the exact sequence (3.310) yields an isomorphism

$${}_x\mathcal{K}_x\langle N\rangle|_U \simeq {}_\bullet\mathcal{K}_x\langle N-1\rangle|_U,$$

which implies that $\mathcal{F}|_{M\setminus\{x\}} = R^N\pi_*({}_x\mathcal{K}_x\langle N\rangle)|_{M\setminus\{x\}}$ is a local system. Let V be a contractible open subset of M containing x. In this case, it is no longer true that ${}_y\mathcal{K}_x\langle N\rangle|_{\pi^{-1}(V)}$ has vanishing hypercohomology. Identifying $Y_{\{1,\dots,N\}}$ with the point $(x,\dots,x)$, there is a map

$$\underline{\mathbb{Q}}_{(x,\dots,x)}[-N] \longrightarrow {}_y\mathcal{K}_x\langle N\rangle|_{\pi^{-1}(V)}.$$

Using that V is contractible, this map induces an isomorphism in hypercohomology

$$\mathbb{Q} = \mathbb{H}^N(\pi^{-1}(V), \underline{\mathbb{Q}}_{(x,\dots,x)}[-N]) \simeq \mathbb{H}^N(\pi^{-1}(V), {}_y\mathcal{K}_x\langle N\rangle).$$

Therefore, the higher direct image sheaf satisfies

$$R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)_x = \mathbb{Q} \neq 0.$$

In this case, the map

$$R^{N-1}\pi_*(\iota_x)_*({}_x\widetilde{\mathcal{K}}_x\langle N-1\rangle)_x \xrightarrow{\varphi} R^N\pi_*({}_x\mathcal{K}_x\langle N\rangle)_x$$

is surjective and we again deduce that $\mathcal{F}_x = 0$. Therefore, $\mathrm{H}^0(M,\mathcal{F}) = 0$ and

$$\mathrm{Ker}(g) = \mathrm{H}^1\left(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)\right) \simeq (J^N/J^{N+1})^\vee.$$

This ends the proof of the claim. To prove the lemma, it remains to check that the above isomorphism is compatible with the map σ. We leave this to the reader. $\square$

End of the proof of Beilinson's Theorem 3.316. Recall that statement i) is the vanishing $\mathbb{H}^i(M^N, {}_y\mathcal{K}_x\langle N\rangle) = 0$ in all degrees $i \leqslant N-1$. By (3.320), this group fits into a long exact sequence

$$\cdots \longrightarrow \mathbb{H}^{i-1}(M^{N-1}, {}_y\mathcal{K}_x\langle N-1\rangle) \longrightarrow \mathbb{H}^i(M^N, {}_y\mathcal{K}_x\langle N\rangle) \longrightarrow \mathbb{H}^i(M^{1,N-1}, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \longrightarrow \cdots$$

For $i \leqslant N-1$, the leftmost term vanishes by the induction hypothesis, and the rightmost term vanishes by Lemma 3.322, and hence so does the middle term.

Finally, to prove statement ii) we combine the long exact sequence (3.320), Lemma 3.322, and the induction hypothesis to obtain that in the diagram

(3.337)
$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}^{N-1}(M^{N-1}, {}_y\mathcal{K}_x\langle N-1\rangle) \arrow[r, "\iota"] \arrow[d, "\sigma"'] & \mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle) \arrow[r] \arrow[d, "\sigma"'] & \mathrm{Ker}(g) \arrow[r] \arrow[d, "\sigma"'] & 0 \\
0 \arrow[r] & (\mathbb{Q}[\pi_1(M;y,x)]/J^N)^\vee \arrow[r] & (\mathbb{Q}[\pi_1(M;y,x)]/J^{N+1})^\vee \arrow[r] & (J^N/J^{N+1})^\vee \arrow[r] & 0
\end{tikzcd}$$

the rows are exact, and the first and third vertical maps are isomorphisms. By the five lemma, the second vertical arrow is also an isomorphism, as wanted. $\square$

3.6.6. *Proof of Chen's π_1-de Rham theorem.* We are now in position to prove Chen's π_1-de Rham theorem using Beilinson's theorem 3.316.

PROOF OF THEOREM 3.273. If $N = 0$, then $L_0B^*(A^*(M)) = \mathbb{C}$ is given by the constant functions, while

$$\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J = \mathbb{Q}.$$

Moreover, the morphism in Theorem 3.273 sends the constant function $a \in \mathbb{C}$ to the map that sends $1 \in \mathbb{Q}$ to a, that is clearly an isomorphism.

Let us now fix $N > 0$. Applying Lemma 3.284 and Proposition A.238, we obtain a quasi-isomorphism

$$L_N B^*(A^*(M)) \xrightarrow{\sim} \operatorname{Tot} C_*(\Delta_N, A^*({}_yM_x^\bullet)).$$

For each n, the composition

$$A^*(M)^{\otimes n} \otimes \mathbb{C} \longrightarrow E^*({}_yM_x^n, \mathbb{C}) \longrightarrow S^*({}_yM_x^n, \mathbb{Q}) \otimes \mathbb{C}$$

is a quasi-isomorphism, functorial in n, from which we deduce a quasi-isomorphism

$$L_N B^*(A^*(M)) \otimes \mathbb{C} \xrightarrow{\sim} \operatorname{Tot} C_*(\Delta_N, S_\bullet^*) \otimes \mathbb{C}.$$

Combining this quasi-isomorphism with Lemma 3.298 and Theorem 3.316, we get an isomorphism

$$\mathrm{H}^0(L_N B^*(A^*(M)) \otimes \mathbb{C}) \longrightarrow \big(\mathbb{C}[\pi(M;y,x)]/\mathbb{C}[\pi(M;y,x)]J^{N+1}\big)^\vee.$$

Therefore, we get an isomorphism

$$\mathrm{H}^0\left(B^*(A^*(M)) \otimes \mathbb{C}\right) = \varinjlim_N \mathrm{H}^0\left(L_N B^*(A^*(M)) \otimes \mathbb{C}\right) \longrightarrow \big(\varprojlim_N \mathbb{C}[\pi(M;y,x)]/\mathbb{C}[\pi(M;y,x)]J^{N+1}\big)^\vee = \big(\mathbb{C}[\pi(M;y,x)]^\wedge\big)^\vee,$$

as we wanted to prove. □

⋆⋆⋆

Exercise 3.338. Let $n, m \geqslant 0$ be integers and $f\colon \Delta_n \to \Delta_m$ a non-decreasing map. Using the fact that $A^*(M) \subset E^*(M, \mathbb{C})$ is a subalgebra, prove that

$$f^*\colon E^*(M^m, \mathbb{C}) \longrightarrow E^*(M^n, \mathbb{C})$$

restricts to a morphism of dg-algebras

$$f^*\colon A^*({}_yM_x^m) \longrightarrow A^*({}_yM_x^n),$$

thus making the assignment $\Delta_n \rightsquigarrow A^*({}_yM_x^n)$ functorial.

Exercise 3.339. Recall the finite ordered set $\Delta_n = \{0, \dots, n\}$. For each $i \geqslant 0$, let $\mathcal{P}_i(\Delta_n)$ denote the set of subsets of Δ_n of cardinal i. Consider the complex

$$0 \longrightarrow \mathbb{Q}^{\mathcal{P}_0(\Delta_n)} \xrightarrow{\mathrm{d}} \mathbb{Q}^{\mathcal{P}_1(\Delta_n)} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathbb{Q}^{\mathcal{P}_n(\Delta_n)} \longrightarrow 0, \tag{3.340}$$

where the differential $\mathrm{d}\colon \mathbb{Q}^{\mathcal{P}_i(\Delta_n)} \to \mathbb{Q}^{\mathcal{P}_{i+1}(\Delta_n)}$ sends a function $\varphi\colon \mathcal{P}_i(\Delta_n) \to \mathbb{Q}$ to

$$\mathrm{d}\varphi(K) = \sum_{\substack{I \subset K \\ |I| = |K|-1}} \varepsilon(I, K)\varphi(I).$$

(a) Show that (3.340) is exact [Hint: consider the homotopy

$$s\colon \mathbb{Q}^{\mathcal{P}_i(\Delta_n)} \to \mathbb{Q}^{\mathcal{P}_{i-1}(\Delta_n)}$$

defined by $s\varphi(I) = \varphi(\{0\} \cup I)$ if $0 \notin I$ and $s\varphi(I) = 0$ otherwise].

(b) Let M be a topological space, let $Y_0, \dots, Y_n$ be closed subspaces of M, and set $Y = Y_0 \cup \cdots \cup Y_n$. Consider the complex

$$0 \longrightarrow \underline{\mathbb{Q}}_Y \longrightarrow \bigoplus_{|I|=1} \underline{\mathbb{Q}}_I \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=p} \underline{\mathbb{Q}}_I \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=k+1} \underline{\mathbb{Q}}_I \longrightarrow 0$$

from Lemma 3.290. Show that this complex is exact.

3.7. A mixed Hodge structure on the pro-unipotent completion of the fundamental group.

3.7.1. *Construction of the mixed Hodge structure.* Let $M = X(\mathbb{C})$ be the set of complex points of a smooth algebraic variety X. Hain [**Hai87a**] and Morgan [**Mor78**] show that each of the quotients of the pro-unipotent completion of the fundamental group of M carries a natural mixed Hodge structure. Using the geometric interpretation of such quotients provided by Beilinson's theorem (Theorem 3.316), one can improve this result a little bit, showing that if X is defined over a subfield $k \subset \mathbb{C}$, then we obtain a mixed Hodge structure over k. We will later see that Beilinson's theorem allows us to upgrade these mixed Hodge structures to motives. For the time being, the precise statement is the following.

THEOREM 3.341. *Let k be a subfield of $\mathbb{C}$, let X be a smooth algebraic variety over k, let $M = X(\mathbb{C})$ be the set of complex points of X viewed as a differentiable manifold, and let $x, y \in X(k) \subseteq M$ be k-rational points. For each $N \geqslant 0$, the finite-dimensional $\mathbb{Q}$-vector space*

$$\mathbb{Q}[\pi_1(M; y, x)]/J^{N+1}\mathbb{Q}[\pi_1(M; y, x)] \tag{3.342}$$

carries a mixed Hodge structure over k which is functorial with respect to morphisms of pointed varieties. Moreover, given integers $N_1 \geqslant N_2 \geqslant 0$, the quotient map

$$\mathbb{Q}[\pi_1(X(\mathbb{C}); y, x)]/J^{N_1+1} \longrightarrow \mathbb{Q}[\pi_1(X(\mathbb{C}); y, x)]/J^{N_2+1}$$

is a morphism of mixed Hodge structures over k.

PROOF. The result is a direct consequence of Beilinson's theorem. In fact, Theorem 3.316 gives an isomorphism

$$\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle) \longrightarrow (\mathbb{Q}[\pi(M; y, x)]/J^{N+1}\mathbb{Q}[\pi(M; y, x)])^\vee$$

for each $N \geqslant 0$. When $x \neq y$, the groups $\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)$ can be interpreted as relative singular cohomology groups of algebraic varieties over k by Lemma 3.290, and hence they can be endowed with a mixed Hodge structure over k. For $x = y$, the short exact sequence (3.292) can be upgraded to an extension of mixed Hodge structures. Alternatively, we can use Lemma 3.298 and Proposition A.238 to identify the groups $\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)$ with certain singular cohomology groups of a simplicial manifold ${}_yM_x^\bullet$. All the maps involved in ${}_yM_x^\bullet$ are algebraic and defined over k, and hence ${}_yM_x^\bullet$ is the simplicial manifold obtained by taking complex points of a simplicial smooth variety over k. Using a variant over k of the main construction of [**Del74**], we endow $\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)$ with a mixed Hodge structure over k. By duality, the groups (3.342) are endowed with mixed Hodge structures defined over k. The functoriality properties follow from the functorial properties of the mixed Hodge structures on the cohomology of simplicial varieties. □

Taking the projective limit over N in Theorem 3.341, we obtain a pro-mixed Hodge structure on the pro-unipotent completion of the fundamental group by abstract means. Following Hain [**Hai87a**], Chen's theorem provides us with a very clear and transparent way to understand such mixed Hodge structure. We now explain how to define the Hodge and weight filtrations when X is a smooth complex variety. Consider the dg-algebra $E^*_{\overline{X}^{\mathrm{an}}}(\log D)$ from Section 2.8.5. It has two augmentations ε_1 and ε_2 given by evaluating at x and y respectively. The Hodge and weight filtrations of $E^*_{\overline{X}^{\mathrm{an}}}(\log D)$ determine the Hodge and weight filtration on $B^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))$ as follows: if $\omega_i \in F^{p_i}$ for $i = 1, \dots, r$, then

$$[\omega_1|\cdots|\omega_r] \in F^{p_1+\cdots+p_r},$$

while, if $\omega_i \in W_{n_i}$, then

$$[\omega_1|\cdots|\omega_r] \in W_{n_1+\cdots+n_r+r}.$$

In words, the Hodge type is the sum of Hodge types, while the weight is the sum of weights plus the length of the element. Then

$$\begin{aligned} F^p\,\mathrm{H}^0\left(B^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))\right) &= \mathrm{Im}\left(\mathrm{H}^0(F^pB^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))\right), \\ W_m\,\mathrm{H}^0\left(B^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))\right) &= \mathrm{Im}\left(\mathrm{H}^0(W_mB^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))\right). \end{aligned}$$

3.7.2. *The case of* $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. We now specialize the general discussion to the varieties $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ and $M = X(\mathbb{C})$, as in Section 3.5.5, and two rational points $x, y \in X(\mathbb{Q})$. As we have seen in Example 2.284, we do not need to work with the whole infinite-dimensional dg-algebra $E^*_{\overline{X}^{\mathrm{an}}}(\log D)$, but we can work with the smaller $\mathbb{Q}$-algebra

$$A = \mathbb{Q} \oplus \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1.$$

The Hodge and the weight filtrations are given by

$$\begin{gathered} F^0A = W_1A = A, \quad F^2A = W_{-1}A = 0, \\ F^1A = \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1 \quad W_0A = \mathbb{Q}. \end{gathered} \tag{3.343}$$

In this case, both augmentations ε_1 and ε_2 given by evaluating at x and y respectively agree with the trivial augmentation

$$\begin{aligned} \varepsilon\colon\ A &\longrightarrow \mathbb{Q} \\ 1 &\longmapsto 1 \\ \omega_0 &\longmapsto 0 \\ \omega_1 &\longmapsto 0. \end{aligned} \tag{3.344}$$

One advantage of working with A is that we obtain a mixed Hodge structure over $\mathbb{Q}$. Since A is connected, we can use the reduced bar complex. Arguing as in Section 3.5.5, the Hopf algebra $\mathrm{H}^0(B^*(A))$ is isomorphic to the Hoffman algebra.

By (3.343), the Hodge filtration in each finite-dimensional subspace

$$\mathrm{H}^0(L_NB^*(A_{\mathbb{C}}))$$

is the decreasing filtration determined by

$$[\omega_{i_1}|\cdots|\omega_{i_p}] \in F^p,$$

and the weight filtration is the increasing filtration determined by

$$[\omega_{i_1}|\cdots|\omega_{i_n}] \in W_{2n}.$$

We now describe an ind-mixed Hodge structure $\{{}_yA_x^{\mathrm{H},N}\}_{N\geqslant 0}$ that corresponds to the algebra of functions over the pro-unipotent completion of the fundamental group, as well as a dual pro-mixed Hodge structure $\{{}_yU_x^{\mathrm{H},N}\}_{N\geqslant 0}$ that corresponds to the universal enveloping algebra of the Lie algebra of the pro-unipotent completion of the fundamental group.

For the Betti part of ${}_yA_x^{\mathrm{H},N}$, we write

$$ {}_yA_x^{\mathrm{B},N} = \big(\mathbb{Q}[\pi(M;y,x)]/J^{N+1}\mathbb{Q}[\pi(M;y,x)]\big)^\vee $$

with the weight filtration given, for $-1 \leqslant p \leqslant N$, by

$$ \begin{aligned} W_{2p}({}_yA_x^{\mathrm{B},N}) &= W_{2p+1}({}_yA_x^{\mathrm{B},N}) \\ &= \big(J^{p+1}\mathbb{Q}[\pi(M;y,x)]/J^{N+1}\mathbb{Q}[\pi(M;y,x)]\big)^\perp. \end{aligned} $$

For the de Rham side, we have

$$ {}_yA_x^{\mathrm{dR},N} = L_N\,\mathrm{H}^0(B^*(A^*)) $$

with the weight filtration given, for $-1 \leqslant p \leqslant N$, by

$$ W_{2p}({}_yA_x^{\mathrm{dR},N}) = W_{2p+1}({}_yA_x^{\mathrm{dR},N}) = L_p\,\mathrm{H}^0(B^*(A^*)). $$

The Hodge filtration is given by defining

$$ F^p({}_yA_x^{\mathrm{dR},N}) $$

as the subspace generated by words of length ℓ with $p \leqslant \ell \leqslant N$. Note that only the Betti part depends on the points x, y.

By duality, we write

$$ \begin{aligned} {}_yU_x^{\mathrm{B},N} &= \mathbb{Q}[\pi(M;y,x)]/J^{N+1}\mathbb{Q}[\pi(M;y,x)] \\ {}_yU_x^{\mathrm{dR},N} &= L_N\,\mathrm{H}^0(B^*(A^*))^\vee, \end{aligned} $$

and endow these spaces with the dual filtrations.

We denote by $\mathrm{comp}_{\mathrm{dR,B}}$ the isomorphism of Theorem 3.273 and by $\mathrm{comp}_{\mathrm{B,dR}}$ its dual. Then the mixed Hodge structures

$$ {}_yA_x^{\mathrm{H},N} = (({}_yA_x^{\mathrm{B},N}, W), ({}_yA_x^{\mathrm{dR},N}, W, F), \mathrm{comp}_{\mathrm{dR,B}}^{-1}) $$

form an inductive system of mixed Hodge structures over $\mathbb{Q}$, and

$$ {}_yU_x^{\mathrm{H},N} = (({}_yU_x^{\mathrm{B},N}, W), ({}_yU_x^{\mathrm{dR},N}, W, F), \mathrm{comp}_{\mathrm{B,dR}}) $$

form a projective system of mixed Hodge structures over $\mathbb{Q}$.

The mixed Hodge structure we have constructed is of Hodge–Tate type.

Proposition 3.345. *The filtrations F and W of ${}_yA_x^{\mathrm{dR},N}$ satisfy the conditions*

$$ W_{2p} = W_{2p+1}, \quad W_{2p-1} \cap F^p = \{0\}, \quad W_{2p} \subset W_{2p-1} + F_p. $$

Therefore, the Hodge structure on Gr_{2p}^W is of type (p,p) and the mixed Hodge structure ${}_yA_x^{\mathrm{H},N}$ is of Hodge–Tate type for all x and y.

Proof. This is clear because the subspaces W_{2p} and W_{2p+1} agree with the subspaces of length less than or equal to p, while F^p is generated by monomials of length greater than or equal to p. □

In fact, we could have guessed the previous result by pure thought from Theorem 3.307. Since the varieties M^N, as well as the components of Y and their intersections, that appear when applying that theorem to our case are constructed from products of $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, whose cohomology is of Hodge–Tate type, it is clear that the mixed Hodge structure we have constructed is of the same kind.

3.7.3. *Iterated integrals as periods of the fundamental group.* We now show that iterated integrals along paths between x and y are periods of the mixed Hodge structure ${}_yA_x^{\mathrm{H},N}$. We keep the notation $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $M = X(\mathbb{C})$.

EXAMPLE 3.346. Let $\boldsymbol{s} = (s_1, \dots, s_n)$ be a positive multi-index of weight N and write $\mathrm{bs}(\boldsymbol{s}) = (\varepsilon_1, \dots, \varepsilon_N)$ for the associated binary sequence. On the one hand, we consider the element

$$[\omega_{\varepsilon_1}|\cdots|\omega_{\varepsilon_N}] \in {}_yA_x^{\mathrm{dR},N},$$

where $\omega_0 = \mathrm{d}t/t$ and $\omega_1 = \mathrm{d}t/(1-t)$, as usual. Besides, every path $\gamma\colon [0,1] \to M$ with endpoints $\gamma(0) = x$ and $\gamma(1) = y$ determines an element

$$[\gamma] \in \mathbb{Q}[\pi(M; y, x)]/J^{N+1}\mathbb{Q}[\pi(M; y, x)] = ({}_yA_x^{\mathrm{B},N})^\vee.$$

By the shape of the comparison isomorphism in Theorem 3.273, we deduce that the period associated with these two classes is the iterated integral

$$\langle [\omega_{\varepsilon_1}|\cdots|\omega_{\varepsilon_N}], [\gamma]\rangle = \int_\gamma \omega_{\varepsilon_1}\cdots\omega_{\varepsilon_N}. \tag{3.347}$$

Here, we have used points $x, y \in X(\mathbb{Q})$. In order to obtain multiple zeta values as values of the integral 3.347, we need to consider the case $x = 0$ and $y = 1$, but these points do not belong to $X(\mathbb{Q})$. For this reason we will need to consider tangential base points in the next section.

EXAMPLE 3.348. There is a more “classical” interpretation of the period of Example 3.346 in terms of relative cohomology. For simplicity, we assume $x \neq y$ and let $\boldsymbol{s}$ and $\mathrm{bs}(\boldsymbol{s})$ and γ be as in that example. We consider the differential form

$$\omega = \mathrm{pr}_1^*\omega_{\varepsilon_1} \wedge \cdots \wedge \mathrm{pr}_N^*\omega_{\varepsilon_N}$$

on X^N, where $\mathrm{pr}_i\colon X^N \to X$ denote the various projections. Since ω has maximal degree, it defines a class $[(\omega, 0)]$ in the relative de Rham cohomology $\mathrm{H}^N_{\mathrm{dR}}(X^N, Y)$, where Y is as in Section 3.6.5. From Lemmas 3.290, 3.298, and 3.284, as well as Proposition A.238, we derive an isomorphism

$$\mathrm{H}^N_{\mathrm{dR}}(X^N, Y) \xrightarrow{\simeq} {}_yA_x^{\mathrm{dR},N}$$

that sends ω to $[\omega_{\varepsilon_1}|\cdots|\omega_{\varepsilon_N}]$.

The path γ determines a singular simplex

$$\begin{aligned} \sigma\colon \quad \Delta^N &\longrightarrow M^N \\ (t_1, \dots, t_N) &\longmapsto (\gamma(t_1), \dots, \gamma(t_N)), \end{aligned}$$

where Δ^N is the simplex of Notation 1.114. Clearly, the chain $\partial\sigma$ is supported on Y, so σ determines a class $[\sigma]$ in the relative singular homology group $\mathrm{H}_N(M^N, Y, \mathbb{Q})$. By Lemma 3.290 and Theorem 3.316, there is an isomorphism

$$\mathrm{H}_N(M^N, Y) = ({}_yA_x{}^{\mathrm{B},N})^\vee$$

that sends the class of σ to the class of γ.

The period associated with these two classes is the iterated integral

$$\langle [(\omega, 0)], [\sigma]\rangle = \int_\sigma \omega = \int_\gamma \omega_{\varepsilon_1}\cdots\omega_{\varepsilon_N},$$

which we have in this way realized the iterated integral as a period of the relative cohomology $\mathrm{H}^N(X^N, Y)$.

⋆⋆⋆

EXERCISE 3.349. Give an explicit description of the mixed Hodge structure on the pro-unipotent completion of the fundamental group of $\mathbb{G}_m$.

3.8. Tangential base points. In this section, we keep working with the manifold $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, the differential forms ω_0 and ω_1, and the dg-algebra $A^*_{\mathbb{C}}$ from Section 3.5.5. Theorems 1.116 and 1.126 show that multiple zeta values and polylogarithms can be seen as iterated integrals. Nevertheless, we face the following technical problem: the differential forms ω_0 and ω_1 that appear in these theorems have singularities at the points 0, 1 and ∞. They are differential forms on M, but to obtain multiple zeta values we need to integrate along the straight path

$$\begin{aligned} \mathbf{dch}\colon\ [0,1] &\longrightarrow \mathbb{P}^1(\mathbb{C}), \\ t &\longmapsto t \end{aligned} \tag{3.350}$$

which is not contained in M because the endpoints are 0 and 1. Since **dch** is not a path in M, the formulas in Theorems 1.116 and 1.126 are not strictly speaking iterated integrals. Thus, to see multiple zeta values and polylogarithms as iterated integrals we have to consider *tangential base points*. As we will see, these are related to the regularization discussed in Section 1.7. Tangential base points will also play an important role later when we consider the algebraic structure of $\mathbb{P}^1$: since the variety $\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}$ does not contain any integral point (Exercise 3.382), we will need tangential base points to have a motivic version of the fundamental group of $\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}$ defined over $\mathbb{Z}$.

3.8.1. *Paths with tangential base points.* For simplicity, we will introduce tangential base points only in the case of $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, the only one we need, but the reader should be aware that the constructions extend easily to any smooth projective curve minus a finite number of points.

DEFINITION 3.351. Let $x \in \{0, 1\}$ be either the point zero or the point one in $\mathbb{P}^1(\mathbb{C})$. A *tangential base point* is a pair (x, v), where v is a non-zero tangent vector to $\mathbb{P}^1(\mathbb{C})$ at x. We will make special use of the tangential base points

$$\mathbf{0} = (0, 1) \quad \text{and} \quad \mathbf{1} = (1, -1), \tag{3.352}$$

i.e. the tangent vector 1 at the point 0 and the tangent vector -1 at the point 1.

Intuitively, a path has an endpoint at a tangential base point (x, v) if the endpoint is x and the tangent vector at the endpoint is v. However, the presence of tangential base points causes a nuisance. On the one hand, in order to be able to compose paths we need to allow tangential points to be reached by the paths at intermediate points. On the other hand, to define a homotopy between paths in an easy way it is better to avoid tangential points at intermediate points along the path. To remedy this problem, we define two kind of paths: the ones that allow tangential points at intermediate steps (and hence can be composed) and the ones that avoid tangential points. The former will be called *cuspidal* paths because of the shape we will impose at the tangential points, while the latter will be called *clean* paths. Then we define a homotopy equivalence of clean paths and a map from the space of cuspidal paths to the space of homotopy classes of clean paths.

The definition of piecewise smooth map given at the beginning of Section 3.1.1 implies the following: if $\gamma\colon [0, 1] \to M$ is a piecewise smooth map, then the right and left derivatives of γ exist at every point $t \in (0,1)$, although they may not agree. They are denoted by

$$\frac{\mathrm{d}^+\gamma}{\mathrm{d}t}(t) \quad \text{and} \quad \frac{\mathrm{d}^-\gamma}{\mathrm{d}t}(t)$$

respectively. The right derivative at $t = 0$ and the left derivative at $t = 1$ also exist.

Definition 3.353. Let $\boldsymbol{x} = (x, v)$ and $\boldsymbol{y} = (y, w)$ be two tangential base points. A *cuspidal path from $\boldsymbol{x}$ to $\boldsymbol{y}$* is a piecewise smooth map

$$\gamma\colon [0,1] \longrightarrow M \cup \{0,1\}$$

satisfying the following conditions:

i) the endpoints of the path are

$$\gamma(0) = x, \quad \frac{\mathrm{d}^+\gamma}{\mathrm{d}t}(0) = v,$$
$$\gamma(1) = y, \quad \frac{\mathrm{d}^-\gamma}{\mathrm{d}t}(1) = -w;$$

ii) the set $\{t \in (0,1) \mid \gamma(t) \in \{0,1\}\}$ is finite. Moreover, if t_0 belongs to this set, then the left and right tangent vectors to γ at t_0 are non-zero and opposed to each other, that is

$$0 \neq \frac{\mathrm{d}^+\gamma}{\mathrm{d}t}(t_0) = -\frac{\mathrm{d}^-\gamma}{\mathrm{d}t}(t_0).$$

This set is called the *set of cusps* of γ.

When the set of cusps is empty, γ is called a *clean path from $\boldsymbol{x}$ to $\boldsymbol{y}$*.

The space of cuspidal paths from $\boldsymbol{x}$ to $\boldsymbol{y}$ is denoted by ${}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$, while the subspace of clean paths is denoted ${}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$. For instance, the path $\mathbf{dch}(t) = t$ from (3.350) belongs to ${}_{\mathbf{1}}\mathcal{P}(M)_{\mathbf{0}}$.

3.8.2. *Composition of paths with tangential base points.* The composition of paths (3.3) cannot be applied directly to define

$${}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{y}} \otimes {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}} \longrightarrow {}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{x}}$$

for tangential base points $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ because condition i) imposes that the derivative of the path at zero and one is a fixed vector, while the parametrization used in (3.3) would multiply this vector by 2. Thus, to define the composition of paths we consider the functions

$$\phi_1(t) = t + 2t^2, \quad \phi_2(t) = 5t - 2 - 2t^2.$$

These functions are smooth and satisfy the properties

$$\begin{gathered}
\phi_1(0) = 0, \qquad \phi_1(1/2) = 1, \qquad \phi_1'(0) = 1,\\
\phi_2(1/2) = 0, \qquad \phi_2(1) = 1, \qquad \phi_2'(1) = 1,\\
\phi_1'(t) > 0,\ t \in [0,1/2], \quad \phi_2'(t) > 0,\ t \in [1/2,1],\\
\phi_1'(1/2) = \phi_2'(1/2).
\end{gathered}$$

Their graphs are depicted in Figure 18. In fact, any pair of smooth functions satisfying all the above properties would serve for our purposes.

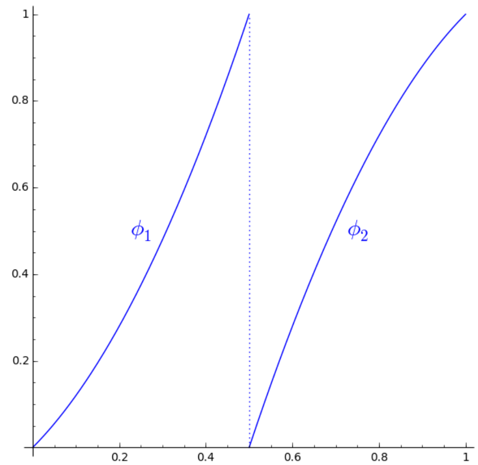


FIGURE 18. The functions ϕ_1 and ϕ_2

We define the composition of paths as

$$\begin{array}{ccc} {}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{y}} \otimes {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}} & \longrightarrow & {}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{x}}, \\ (\gamma_1, \gamma_2) & \longmapsto & \gamma_1\gamma_2 \end{array}$$

where $\gamma_1\gamma_2$ is the cuspidal path

$$\gamma_1\gamma_2(t) = \begin{cases} \gamma_2(\phi_1(t)), & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \gamma_1(\phi_2(t)), & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \tag{3.354}$$

3.8.3. *Homotopy of paths.* Let $\gamma_1, \gamma_2 \in {}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$ be two clean paths. A homotopy between γ_1 and γ_2 is a map

$$F\colon [0,1] \times [0,1] \longrightarrow M \cup \{0,1\}$$

satisfying the following conditions:

$$\begin{array}{lll} F(t,0) = \gamma_1(t), & F(t,1) = \gamma_2(t), & t \in [0,1] \\ F(0,s) = x, & F(1,s) = y, & s \in [0,1] \\ \dfrac{\partial F}{\partial t}(0,s) = v, & \dfrac{\partial F}{\partial t}(1,s) = -w, & s \in [0,1] \\ F(t,s) \in M, & 0 < t < 1, & 0 \leqslant s \leqslant 1. \end{array}$$

The space $\pi(M; \boldsymbol{y}, \boldsymbol{x})$ is the set of homotopy classes of clean paths from $\boldsymbol{x}$ to $\boldsymbol{y}$. Similar notation will be used when only one of the base points is tangential.

We next construct a map ψ from ${}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$ to $\pi(M; \boldsymbol{y}, \boldsymbol{x})$. Let $d(x,y)$ be the standard Euclidean distance in $\mathbb{C} = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$. Let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$. For each t_i in the set of cusps of γ, we can find real numbers $\varepsilon_i, \eta_i, \eta'_i \in (0, \frac{1}{2})$ such that:

i) t_i is the only cusp in the interval $[t_i - \eta'_i, t_i + \eta_i]$ and γ is smooth in the intervals $[t_i - \eta'_i, t_i)$ and $(t_i, t_i + \eta_i]$;
ii) the intervals $[t_i - \eta'_i, t_i + \eta_i]$ are disjoint and do not contain 0 or 1;

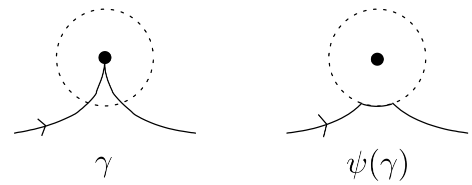


FIGURE 19. Retraction at a cusp

iii) the image of $[t_i - \eta_i', t_i + \eta_i]$ satisfies

$$d(\gamma(t_i + \eta_i), \gamma(t_i)) = \varepsilon_i, \quad d(\gamma(t), \gamma(t_i)) < \varepsilon_i, \text{ for } t_i < t < t_i + \eta_i$$
$$d(\gamma(t_i - \eta_i'), \gamma(t_i)) = \varepsilon_i, \quad d(\gamma(t), \gamma(t_i)) < \varepsilon_i, \text{ for } t_i - \eta_i' < t < t_i;$$

iv) the tangent vector to γ satisfies

$$\left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) - \frac{\mathrm{d}^-\gamma}{\mathrm{d}t}(t_i) \right\| \leqslant \frac{1}{2} \left\| \frac{\mathrm{d}^-\gamma}{\mathrm{d}t}(t_i) \right\|, \text{ for } t \in [t_i - \eta_i', t_i)$$
$$\left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) - \frac{\mathrm{d}^+\gamma}{\mathrm{d}t}(t_i) \right\| \leqslant \frac{1}{2} \left\| \frac{\mathrm{d}^+\gamma}{\mathrm{d}t}(t_i) \right\|, \text{ for } t \in (t_i, t_i + \eta_i'].$$

Note that condition iv) implies that the path γ cannot turn around the point $\gamma(t_i)$ between $t_i - \eta_i'$ and $t_i + \eta_i$.

For each cusp t_i, let $B(\gamma(t_i), \varepsilon_i)$ be the open ball of centre $\gamma(t_i)$ and radius ε_i, and let $r_i \colon \mathbb{C} \setminus \{\gamma(t_i)\} \to \mathbb{C} \setminus B(\gamma(t_i), \varepsilon_i)$ be the radial retraction. Then we define a new path γ° outside the cusps by

$$\gamma^\circ(s) = \begin{cases} \gamma(s), & \text{if } s \notin [t_i - \eta_i', t_i + \eta_i] \text{ for all } i, \\ r_i(\gamma(s)), & \text{if } s \in [t_i - \eta_i', t_i + \eta_i] \text{ and } s \neq t_i. \end{cases} \tag{3.355}$$

Condition ii) in the Definition 3.353 implies that γ° can be extended continuously to the cusps t_i, thus defining a clean path which is also denoted by γ°. The retraction at a cusp is represented in Figure 19.

PROPOSITION 3.356. *The homotopy class of clean paths of γ° does not depend on the choice of the numbers $\varepsilon_i, \eta_i, \eta_i'$.*

SKETCH OF PROOF. First observe that the numbers ε_i determine η_i and η_i' by means of condition iii). To prove that the homotopy class of clean paths of γ° is independent of the choices, let $\varepsilon_i' \geqslant \varepsilon_i$ be another choice satisfying the same requirements. Then the obvious homotopy that shows that the inclusion

$$\mathbb{C} \setminus \bigcup_i B(\gamma(t_i), \varepsilon_i') \subset \mathbb{C} \setminus \bigcup_i B(\gamma(t_i), \varepsilon_i)$$

is a deformation retract proves the claim. □

The homotopy class of γ° in $\pi(M; \boldsymbol{y}, \boldsymbol{x})$ is denoted by $\psi(\gamma)$. Using the map ψ, we can define a composition of clean paths

$$\pi(M; \boldsymbol{z}, \boldsymbol{y}) \times \pi(M; \boldsymbol{y}, \boldsymbol{x}) \longrightarrow \pi(M; \boldsymbol{z}, \boldsymbol{x}).$$

DEFINITION 3.357. Let $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ be base points (tangential or not). Given classes $\gamma_1 \in \pi(M; \boldsymbol{z}, \boldsymbol{y})$ and $\gamma_2 \in \pi(M; \boldsymbol{y}, \boldsymbol{x})$, choose representatives $\widetilde{\gamma}_1 \in {}_{\boldsymbol{z}}\mathcal{P}(M)^0_{\boldsymbol{y}}$ and $\widetilde{\gamma}_2 \in {}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$. Then $\widetilde{\gamma}_1\widetilde{\gamma}_2$ belongs to ${}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{x}}$, and we define

$$\gamma_1\gamma_2 = \psi(\widetilde{\gamma}_1\widetilde{\gamma}_2).$$

A slight variant of the proof of the same result for ordinary base points gives:

Proposition 3.358. *The composition of clean paths given in Definition 3.357 does not depend on the choice of representatives, and turns* $\pi(M;\boldsymbol{x},\boldsymbol{x})$ *into a group and* $\pi(M;\boldsymbol{y},\boldsymbol{x})$ *(resp.* $\pi(M;\boldsymbol{x},\boldsymbol{y})$*) into a right (resp. left)* $\pi(M;\boldsymbol{x},\boldsymbol{x})$*-torsor.*

The fact that the fundamental groups with different base points are isomorphic can be easily extended to tangential base points. The next proposition is proved like the corresponding one for ordinary base points.

Proposition 3.359. *Let* $\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4$ *be base points of* M *(tangential or not). Let* $\gamma_1 \in {}_{\boldsymbol{x}_4}\mathcal{P}(M)_{\boldsymbol{x}_3}$ *and* $\gamma_2 \in {}_{\boldsymbol{x}_2}\mathcal{P}(M)_{\boldsymbol{x}_1}$*. Then the map*

$$\begin{array}{ccc} \pi(M;\boldsymbol{x}_3,\boldsymbol{x}_2) & \longrightarrow & \pi(M;\boldsymbol{x}_4,\boldsymbol{x}_1) \\ \gamma & \longmapsto & \gamma_1\gamma\gamma_2 \end{array}$$

is a bijection. It is a group isomorphism if $\boldsymbol{x}_1=\boldsymbol{x}_4$*,* $\boldsymbol{x}_2=\boldsymbol{x}_3$*, and* $\gamma_2=\gamma_1^{-1}$*.*

3.8.4. *Logarithmic asymptotic developments.* We would like to extend the notion of iterated integral to tangential base points. The main problem is that the integral may diverge, so one needs to regularize it. We start by discussing some preliminaries about asymptotic developments.

Definition 3.360. Let $0<\tau\leqslant 1$ be a real number and $f\colon (0,\tau)\to\mathbb{C}$ a continuous function. We say that f admits a *logarithmic asymptotic development (of degree less than or equal to* r*)* if it can be written as

$$f(t)=f_0(t)+\sum_{k=0}^{r} a_k\log(t)^k$$

with $|f_0(t)|=O(t^\delta)$ as $t\to 0$ for some $\delta>0$ and $a_k\in\mathbb{C}$.

Lemma 3.361. *Let* $0<\tau\leqslant 1$ *be a real number and* $f\colon(0,\tau)\to\mathbb{C}$ *a continuous function. If* f *admits a logarithmic asymptotic development, then it is unique.*

Proof. Let $f\colon(0,\tau)\to\mathbb{C}$ be a continuous function that admits an asymptotic development

$$f(t)=f_0(t)+\sum_{k=0}^{r} a_k\log(t)^k.$$

We can recover the coefficient a_r as

$$a_r=\lim_{t\to 0}\frac{f(t)}{\log(t)^r}.$$

Once we know $a_{s+1},\dots,a_r$, we can recover a_s as

$$a_s=\lim_{t\to 0}\frac{f(t)-\sum_{k=s+1}^{r} a_k\log(t)^k}{\log(t)^s}.$$

Finally, $f_0=f(t)-\sum_{k=0}^{r} a_k\log(t)^k$, so the logarithmic development is unique. $\square$

3.8.5. *Asymptotic developments of iterated integrals.* Recall the two tangential base points $\mathbf{0} = (0,1)$ and $\mathbf{1} = (1,-1)$. Let $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\} \cup M$ be base points (tangential or not), $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$ a piecewise smooth clean path, and $(\varepsilon_1, \dots, \varepsilon_r)$ a binary sequence with $\varepsilon_i \in \{0,1\}$. We consider the iterated integral

$$\int_\gamma \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}.$$

Since the form ω_0 has a pole at 0 and the form ω_1 has a pole at 1, this integral may diverge. This is, for instance, the case for the integral

$$\int_{\mathbf{dch}} \omega_0 = \infty.$$

However, if the form ω_{ε_1} has no pole at the point $\boldsymbol{y}$ and the form ω_{ε_r} has no pole at the point $\boldsymbol{x}$, then the above integral is convergent. For instance, if $\gamma = \mathbf{dch}$, the integral will be convergent when $\varepsilon_1 = 0$ and $\varepsilon_r = 1$, that is, when the binary sequence is admissible.

We now describe the regularization process. Let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$ be a clean path. For $0 < \eta < \frac{1}{2}$, we write

$$\gamma_\eta(t) = \gamma(t(1-\eta) + (1-t)\eta). \tag{3.362}$$

This is a path from $\gamma(\eta)$ to $\gamma(1-\eta)$, and hence completely contained in M.

LEMMA 3.363. *Let $(\varepsilon_1, \dots, \varepsilon_r)$ be a binary sequence. The function*

$$\begin{aligned} (0,1/2) &\longrightarrow \mathbb{C} \\ \eta &\longmapsto \int_{\gamma_\eta} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} \end{aligned}$$

admits a logarithmic asymptotic development of degree $\leqslant r$.

PROOF. We write

$$\begin{aligned} \gamma_{\eta,1}(t) &= \gamma(t(1-\eta) + (1-t)/2), \\ \gamma_{\eta,2}(t) &= \gamma(t/2 + (1-t)\eta). \end{aligned}$$

The path $\gamma_{\eta,2}$ goes from $\gamma(\eta)$ to $\gamma(1/2)$, and $\gamma_{\eta,1}$ is a path from $\gamma(1/2)$ to $\gamma(1-\eta)$. Moreover, $\gamma_\eta = \gamma_{\eta,1}\gamma_{\eta,2}$ (recall that, according to our convention for the composition of paths (3.3), this means that we first walk along $\gamma_{\eta,2}$, then along $\gamma_{\eta,1}$). Using equations (3.20) and (3.21) in Theorem 3.19, it suffices to show that the functions

$$\eta \longmapsto \int_{\gamma_{\eta,i}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}, \quad i = 1,2$$

admit a logarithmic asymptotic development of degree less than or equal to r. Since both cases are analogous, we will only consider $i = 2$. We prove the existence of a logarithmic asymptotic development by induction on r. The result is clear for $r = 0$. Let us assume that it holds for a binary sequence of length less than r. If $\gamma^*_{\eta,2}\omega_{\varepsilon_i} = g_{\varepsilon_i}(t)\mathrm{d}t$ and $\gamma^*\omega_{\varepsilon_i} = h_{\varepsilon_i}(t)\mathrm{d}t$, then:

$$\begin{aligned} \int_{\gamma_{\eta,2}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} &= \int\limits_{1 \geqslant t_1 \geqslant \cdots \geqslant t_r \geqslant 0} g_{\varepsilon_1}(t_1) \cdots g_{\varepsilon_r}(t_r) \mathrm{d}t_1 \cdots \mathrm{d}t_r \\ &= \int\limits_{1/2 \geqslant t_1 \geqslant \cdots \geqslant t_r \geqslant \eta} h_{\varepsilon_1}(t_1) \cdots h_{\varepsilon_r}(t_r) \mathrm{d}t_1 \cdots \mathrm{d}t_r. \end{aligned}$$

Now we compute

$$\begin{aligned} I(\eta) &= \int\limits_{1/2\geqslant t_1\geqslant\cdots\geqslant t_r\geqslant\eta} h_{\varepsilon_1}(t_1)\cdots h_{\varepsilon_r}(t_r)\mathrm{d}t_1\cdots\mathrm{d}t_r \\ &= \int\limits_{1/2\geqslant t_r\geqslant\eta} h_{\varepsilon_r}(t_r)\Big(\int\limits_{1/2\geqslant t_1\geqslant\cdots\geqslant t_{r-1}\geqslant t_r} h_{\varepsilon_1}(t_1)\cdots h_{\varepsilon_{r-1}}(t_{r-1})\mathrm{d}t_1\cdots\mathrm{d}t_{r-1}\Big)\mathrm{d}t_r. \end{aligned}$$

By the shape of ω_{ε_r}, we deduce the estimate

$$h_{\varepsilon_r}(t_r) = \alpha/t_r + O(1) \quad \text{as } t_r \to 0,$$

where α is non-zero if ω_{ε_r} has a pole at the point $\boldsymbol{x}$ and is zero otherwise. We also apply the induction hypothesis to the inner integral to get

$$I(\eta) = \int\limits_{1/2\geqslant t_r\geqslant\eta} \Big(\frac{\alpha}{t_r} + O(1)\Big)\Big(O(t_r^\delta) + \sum_{k=0}^{r-1} b_k \log(t_r)^k\Big)\mathrm{d}t_r \quad \text{as } t_r \to 0.$$

Estimating this integral, we deduce that $I(\eta)$ admits a logarithmic asymptotic development as $\eta \to 0$ of the sought shape, thus proving the result. $\square$

3.8.6. *Regularized iterated integrals.*

Definition 3.364. Let $(\varepsilon_1, \dots, \varepsilon_r)$ be a binary sequence and let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$ be a clean path. Let

$$\int_{\gamma_\eta} \omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r} = f_0(\eta) + \sum_{k=0}^{r} a_k \log(\eta)^k$$

be the logarithmic asymptotic development provided by Lemma 3.363. Then the *regularized iterated integral* along γ is defined as

$$\int_\gamma^{\mathrm{reg}} \omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r} = a_0.$$

Proposition 3.365. *Let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$ be a cuspidal path and γ° a representative of the class $\psi(\gamma)$ obtained as in (3.355). The regularized integral*

$$\int_{\gamma^\circ}^{\mathrm{reg}} \omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r}$$

does not depend on the choice of γ°.

Proof. Let γ_1° and γ_2° be two choices. Since γ_1° and γ_2° only differ from γ in a small neighborhood of the cusps, for small enough η, the equalities

$$\gamma_1^\circ(\eta) = \gamma_2^\circ(\eta), \qquad \gamma_1^\circ(1-\eta) = \gamma_2^\circ(1-\eta)$$

hold. Moreover, $\gamma_{1,\eta}^\circ$ and $\gamma_{2,\eta}^\circ$ are homotopic. As we saw in Section 3.5.5, there is an equality $\mathrm{H}^0(B^*(A^*)) = B^0(A)$. Thus, all the iterated integrals that can be constructed from ω_0 and ω_1 are homotopy functionals. Therefore,

$$\int_{\gamma_{1,\eta}^\circ} \omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r} = \int_{\gamma_{2,\eta}^\circ} \omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r}$$

from which the result follows. $\square$

Definition 3.366. Let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$ be a cuspidal path. Let γ° be a representative of the class $\psi(\gamma)$ obtained as in (3.355). We define

$$\int_{\gamma}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \int_{\gamma^\circ}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}.$$

Clearly, when the iterated integral is convergent, the value of the regularized integral agrees with the value of the integral. Regularized iterated integrals share many of the properties of iterated integrals. In particular, Theorem 3.19 can be extended to the new setting.

Theorem 3.367. *Let $\gamma, \gamma_1, \gamma_2$ be cuspidal paths in M whose endpoints are either $\mathbf{0}, \mathbf{1}$, or belong to M and such that $\gamma_2(1) = \gamma_1(0)$ holds. Let $(\varepsilon_1, \dots, \varepsilon_{r+s})$ be a binary sequence. Then:*

i)
$$\int_{\gamma}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = (-1)^r \int_{\gamma^{-1}}^{\mathrm{reg}} \omega_{\varepsilon_r} \cdots \omega_{\varepsilon_1};$$

ii)
$$\int_{\gamma_1\gamma_2}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \sum_{i=0}^{r} \int_{\gamma_1}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_i} \int_{\gamma_2}^{\mathrm{reg}} \omega_{\varepsilon_{i+1}} \cdots \omega_{\varepsilon_r};$$

iii)
$$\int_{\gamma}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} \int_{\gamma}^{\mathrm{reg}} \omega_{\varepsilon_{r+1}} \cdots \omega_{\varepsilon_{r+s}} = \sum_{\sigma \in \sqcup\!\sqcup(r,s)} \int_{\gamma}^{\mathrm{reg}} \omega_{\varepsilon_{\sigma^{-1}(1)}} \cdots \omega_{\varepsilon_{\sigma^{-1}(r+s)}}.$$

Proof. We first prove i). If γ is cuspidal and γ° is a clean path in the homotopy class $\psi(\gamma)$ obtained as in (3.355), then $(\gamma^\circ)^{-1}$ is a clean path in the homotopy class $\psi(\gamma^{-1})$ obtained as in (3.355). Therefore, we can assume that γ is a clean path. By construction (see formula (3.362)), the equality $(\gamma^{-1})_\varepsilon = (\gamma_\varepsilon)^{-1}$ holds. By Theorem 3.19, the asymptotic expansions of

$$\int_{\gamma_\varepsilon} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} \quad \text{and} \quad (-1)^r \int_{\gamma_\varepsilon^{-1}}^{\mathrm{reg}} \omega_{\varepsilon_r} \cdots \omega_{\varepsilon_1}$$

agree. Thus, we have the equality of regularized integrals.

Statement iii) also follows from the corresponding statement in Theorem 3.19.

Statement ii) is slightly more tricky due to the possibility that the joining point is a tangential base point. The proof goes as follows. Assume that γ_1 and γ_2 are clean paths. Let $\gamma = \gamma_1\gamma_2$ be their composition and γ^0 a clean path representing γ as in (3.355). For sufficiently small η, the path $(\gamma^0)_\eta$ is homotopic to $\gamma_{1,\eta}\gamma_{0,\eta}\gamma_{2,\eta}$, where $\gamma_{0,\eta}$ denotes the straight path from $\gamma_2(1-\eta)$ to $\gamma_1(\eta)$ (see Figure 20 below).

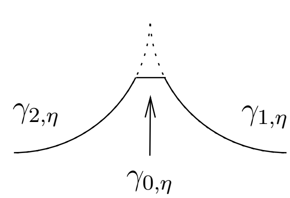


Figure 20. $(\gamma^0)_\eta \sim \gamma_{2,\eta}\gamma_{0,\eta}\gamma_{1,\eta}$

By the usual formula for the composition of paths, the following holds:

$$\int_{(\gamma^0)_\eta} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \sum_{j=0}^{r} \sum_{k=j}^{r} \int_{\gamma_{1,\eta}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_j} \int_{\gamma_{0,\eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_k} \int_{\gamma_{2,\eta}} \omega_{\varepsilon_{k+1}} \cdots \omega_{\varepsilon_r}. \tag{3.368}$$

LEMMA 3.369. *One has* $\int_{\gamma_{0,\eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_k} = O(\eta^{k-j})$ *as* η *tends to* 0.

PROOF. The key point is that there are power series expansions

$$\gamma_2(1-\eta) = \gamma_2(1) - \gamma_2'(1)\eta + O(\eta^2),$$
$$\gamma_1(\eta) = \gamma_1(0) + \gamma_1'(0)\eta + O(\eta^2).$$

From the equalities $\gamma_2(1) = \gamma_1(0)$ and $\gamma_2'(1) = -\gamma_1'(0)$, we derive

$$|\gamma_2(1-\eta) - \gamma_1(\eta)| = O(\eta^2).$$

Using the equation $\gamma_{0,\eta} = t\gamma_1(\eta) + (1-t)\gamma_2(1-\eta)$, one sees that

$$\gamma_{0,\eta}^* \frac{dz}{z} = \frac{(\gamma_1(\eta) - \gamma_2(1-\eta))\mathrm{d}t}{t\gamma_2(\eta) + (1-t)\gamma_2(1-\eta)}.$$

Since the numerator of this expression is $O(\eta^2)$ and the denominator is $O(\eta)$, we get the estimate $\gamma_{0,\eta}^*\omega = O(\eta)\mathrm{d}t$, and hence

$$\int_{\gamma_{0,\eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_k = O(\eta^{k-j}),$$

which proves the lemma. □

To conclude the proof of the theorem, we observe that $\int_{\gamma_{0,\eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_k}$ does not contribute to the constant term in the logarithmic asymptotic development of (3.368) when $k > j$. Therefore,

$$\underline{\mathrm{const}} \int_{(\gamma^0)_\eta} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \sum_{j=0}^{r} \left(\underline{\mathrm{const}} \int_{\gamma_{1,\eta}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_j} \right) \left(\underline{\mathrm{const}} \int_{\gamma_{2,\eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_r} \right),$$

from which the result follows. Here, const means the constant term a_0 in the logarithmic asymptotic expansion. □

As we did for "honest" base points, the properties of iterated integrals can be concisely rephrased in terms of a pairing between the bar complex and the vector space generated by paths. If γ is a piecewise smooth path and $\eta \in B^0(A^*)$, set

$$\langle \eta, \gamma \rangle^{\mathrm{reg}} = \int_\gamma^{\mathrm{reg}} \eta.$$

THEOREM 3.370. *Let* $\gamma, \gamma_1, \gamma_2$ *be piecewise smooth paths with any base points and let* η, η_1, η_2 *be elements of* $B^0(A^*)$. *Then the following holds:*

i) $\langle \eta, \gamma \rangle^{\mathrm{reg}} = \langle S(\eta), \gamma^{-1} \rangle^{\mathrm{reg}}$;
ii) $\langle \eta, \gamma_1\gamma_2 \rangle^{\mathrm{reg}} = \langle \Delta\eta, \gamma_1 \otimes \gamma_2 \rangle^{\mathrm{reg}}$;
iii) $\langle \eta_1, \gamma \rangle^{\mathrm{reg}} \cdot \langle \eta_2, \gamma \rangle^{\mathrm{reg}} = \langle \eta_1 \sqcup\!\sqcup \eta_2, \gamma \rangle^{\mathrm{reg}}$.

3.8.7. *Regularized iterated integrals and regularized zeta values.*

EXAMPLE 3.371. Let us compute an example of a regularized iterated integral in length 3, namely

$$\zeta(1,2)^{\mathrm{reg}} = \int_{\mathbf{dch}}^{\mathrm{reg}} \omega_1\omega_0\omega_1.$$

By definition, this is the constant term in the asymptotic logarithmic development of the function

$$\eta \longmapsto \int\limits_{1-\eta\geqslant t_1\geqslant t_2\geqslant t_3\geqslant 0} \frac{\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3}{(1-t_1)t_2(1-t_3)}.$$

To be completely precise, according to Definition 3.364 we should have also imposed the condition $t_3 \geqslant \eta$. Note, however, that the last form ω_1 has no pole at 0, so the constant term of the two asymptotic logarithmic developments agree.

We first compute the integral following the method of Examples 1.110 and 1.112 to obtain

$$\int\limits_{1-\eta\geqslant t_1\geqslant t_2\geqslant t_3\geqslant 0} \frac{\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3}{(1-t_1)t_2(1-t_3)} = \sum_{m>n>0} \frac{(1-\eta)^m}{n^2 m}. \tag{3.372}$$

This power series converges for $0<\eta<1$ but diverges for $\eta=0$ and we have to find an asymptotic expansion in $\log\eta$. To this end, we use the equality

$$\begin{aligned}\int\limits_{1-\eta\geqslant t_1\geqslant t_2\geqslant t_3\geqslant 0} \frac{\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3}{(1-t_1)t_2(1-t_3)} = \\ \int\limits_{\substack{1-\eta\geqslant t_2\geqslant t_3\geqslant 0\\ 1-\eta\geqslant t_1\geqslant 0}} \frac{\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3}{(1-t_1)t_2(1-t_3)} - 2\int\limits_{1-\eta\geqslant t_2\geqslant t_1\geqslant t_3\geqslant 0} \frac{\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3}{(1-t_1)t_2(1-t_3)},\end{aligned} \tag{3.373}$$

which is a simple consequence of the decomposition of the integration domain, together with the fact that the integrand is symmetric in t_1 and t_3 (this explains why the last term appears twice). Observe the equality

$$\int_{1-\eta\geqslant t_1\geqslant 0} \frac{\mathrm{d}t_1}{1-t_1} = \sum_{k\geqslant 1}\frac{(1-\eta)^k}{k} = -\log(\eta).$$

Combining this with the power series expansions as in Example 1.112, one sees that the right-hand side of (3.373) is equal to

$$-\log(\eta)\sum_{n\geqslant 1}\frac{(1-\eta)^n}{n^2} - 2\sum_{m>n\geqslant 1}\frac{(1-\eta)^m}{m^2 n}. \tag{3.374}$$

One can directly check (see Exercise 3.383) that this expansion agrees with the right-hand side of (3.372).

To see that the power expansion (3.374) is useful we need to prove that the series appearing in that expansion define continuous functions of η.

LEMMA 3.375. *The estimates*

$$\sum_{n\geqslant 1}\frac{(1-\eta)^n}{n^2}=\zeta(2)+O(\eta\log\eta), \tag{3.376}$$

$$\sum_{m>n\geqslant 1}\frac{(1-\eta)^m}{m^2 n}=\zeta(2,1)+O(\eta\log^2\eta) \tag{3.377}$$

hold when η goes to 0^+.

PROOF. To prove estimate (3.376), we need to study

$$\zeta(2)-\sum_{n\geqslant 0}\frac{(1-\eta)^n}{n^2}=\sum_{n\geqslant 0}\frac{1-(1-\eta)^n}{n^2}.$$

For $0<\eta<1$, using the inequalities

$$0<1-(1-\eta)^n<1, \qquad 0<1-(1-\eta)^n<n\eta,$$

we get the following:

$$0<\sum_{n\geqslant 1}\frac{1-(1-\eta)^n}{n^2}<\sum_{n=1}^{\lfloor\frac{1}{\eta}\rfloor}\frac{\eta}{n}+\sum_{n>\lfloor\frac{1}{\eta}\rfloor}\frac{1}{n^2}.$$

Since the first sum is $O(\eta\log\eta)$ and the second is $O(\eta)$, the first estimate follows. The second one is obtained in a similar way. $\square$

From Lemma 3.375, we obtain

$$\int\limits_{1-\eta\geqslant t_1\geqslant t_2\geqslant t_3\geqslant 0}\frac{\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3}{(1-t_1)t_2(1-t_3)}=-2\zeta(2,1)-\zeta(2)\log\eta+O(\eta\log^2\eta),$$

from which it follows that

$$\zeta(1,2)^{\mathrm{reg}}=-2\zeta(2,1).$$

Note that the value of $\zeta(1,2)^{\mathrm{reg}}$ is equal to the one obtained by *shuffle regularization* in Example 1.187. This is of course no coincidence, as we now prove.

THEOREM 3.378. *Let $(\varepsilon_1,\dots,\varepsilon_r)$ be a binary sequence and consider the corresponding word $w=x_{\varepsilon_1}\cdots x_{\varepsilon_r}$. Then:*

$$\zeta_{ш}(w)=\int_\gamma^{\mathrm{reg}}\omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r}.$$

PROOF. Thanks to Proposition 1.182, it suffices to show that the integral on the right-hand side satisfies the conditions determining $\zeta_{ш}(w)$. Condition (1.183) follows from Theorem 1.116 combined with the observation that for an admissible binary sequence the regularized integral agrees with the usual integral. Condition (1.184) is checked by a direct computation. Finally, condition (1.185) is the content of Theorem 3.367 iii). $\square$

3.8.8. *Chen's theorem for tangential base points.* We finish this section by stating a version of Chen's theorem with tangential base points. Recall that we are writing $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, and $A^*_{\mathbb{C}}$ is the dg-algebra from Section 3.5.5.

THEOREM 3.379 (Chen's π_1 theorem for tangential base points). *For each integer $N \geqslant 0$ and each pair of points $\boldsymbol{x}, \boldsymbol{y}$ (tangential or not), regularized iterated integrals induce an isomorphism*

$$L_N \, \mathrm{H}^0(B^*(A^*_{\mathbb{C}})) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}}\big(\mathbb{Q}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]/J^{N+1}\mathbb{Q}[\pi(M; \boldsymbol{x})], \mathbb{C}\big).$$

Passing to the limit, we deduce an isomorphism between $\mathrm{H}^0(B^0(A^*_{\mathbb{C}}))$ and the topological dual $(\mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge)^\vee$.

PROOF. We need to show that the pairing between the spaces $L_N \, \mathrm{H}^0(B^*(A^*_{\mathbb{C}}))$ and $\pi_1(M; \boldsymbol{y}, \boldsymbol{x})/J^{N+1}$ is non-degenerate. Since both are finite-dimensional, it suffices to prove that there is no non-zero $\gamma \in \pi_1(M; \boldsymbol{y}, \boldsymbol{x})/J^{N+1}$ such that $\langle \omega, \gamma \rangle = 0$ holds for all ω. Indeed, assume that such a γ exists. Choose usual base points x' and y' and paths γ_1 and γ_2 going from x' to $\boldsymbol{x}$ and from $\boldsymbol{y}$ to y'. Then, by Theorem 3.370 ii), for $\omega \in L_N \, \mathrm{H}^0(B^*(A^*))$

$$\langle \omega, \gamma_2 \gamma \gamma_1 \rangle = \sum \langle \omega_1, \gamma_2 \rangle \langle \omega_2, \gamma \rangle \langle \omega_3, \gamma_1 \rangle,$$

where all the elements $\omega_1, \omega_2, \omega_3$ are of length $\leqslant N$. Thus, $\langle \omega, \gamma_2 \gamma \gamma_1 \rangle = 0$ holds for all $\omega \in L_N \, \mathrm{H}^0(B^*(A^*))$. From the usual Chen Theorem 3.273, we get $\gamma_2 \gamma \gamma_1 = 0$, and hence the same is true for γ. $\square$

3.8.9. *A mixed Hodge structure in the case of tangential base points.* We now extend the definition of the ind-mixed Hodge structure $\{{}_yA_x^{\mathrm{H},N}\}_{N \geqslant 0}$ to the case of tangential base points $\boldsymbol{x}$ and $\boldsymbol{y}$. For the Betti part of ${}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{H},N}$, we write

$${}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{B},N} = \big(\mathbb{Q}[\pi(M; \boldsymbol{y}, \boldsymbol{x})]/J^{N+1}\mathbb{Q}[\pi(M; \boldsymbol{y}, \boldsymbol{x})]\big)^\vee$$

with the weight filtration given by

$$\begin{aligned} W_{2p}({}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{B},N}) &= W_{2p+1}({}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{B},N}) \\ &= \big(J^{p+1}\mathbb{Q}[\pi(M; \boldsymbol{y}, \boldsymbol{x})]/J^{N+1}\mathbb{Q}[\pi(M; \boldsymbol{y}, \boldsymbol{x})]\big)^\perp \end{aligned}$$

for $-1 \leqslant p \leqslant N$. For the de Rham side ${}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{dR},N}$, we just copy the definition of ${}_yA_x^{\mathrm{dR},N}$ for honest base points x and y as the de Rham side is independent of this choice. Now the comparison map $\mathrm{comp}_{\mathrm{dR,B}}$ is given by Theorem 3.379 and the inverse is denoted by $\mathrm{comp}_{\mathrm{B,dR}}$.

For usual base points, Beilinson's theorem implies that the above structures define a mixed Hodge structure. But Beilinson's theorem is not available for tangential base points, so we need to prove this fact in the case at hand.

PROPOSITION 3.380. *For every $N \geqslant 0$, the triple*

$$H = \big(({}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{B},N}, W), ({}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{dR},N}, W, F), \mathrm{comp}_{\mathrm{B,dR}}\big)$$

is a mixed Hodge structure over $\mathbb{Q}$.

PROOF. The fact that $\mathrm{comp}_{\mathrm{dR,B}}$ is an isomorphism of filtered vector spaces is the content of Theorem 3.379, so we only need to prove that the triple

$$\mathrm{Gr}_m^W H = (\mathrm{Gr}_m^W H_{\mathrm{B}}, (\mathrm{Gr}_m^W H_{\mathrm{dR}}, F^\bullet), \mathrm{comp}_{\mathrm{B,dR}})$$

is a pure Hodge structure of weight m. Thanks to Proposition 3.345, we know that $\mathrm{Gr}^W_{2p+1} H_{\mathrm{dR}} = \{0\}$ and that the filtration $F^\bullet$ induced on $\mathrm{Gr}^W_{2p} H_{\mathrm{dR}}$ satisfies

$$F^p \operatorname{Gr}^W_{2p} H_{\mathrm{dR}} = \operatorname{Gr}^W_{2p} H_{\mathrm{dR}} \quad \text{and} \quad F^{p+1} \operatorname{Gr}^W_{2p} H_{\mathrm{dR}} = \{0\}.$$

Therefore, regardless of the precise action of complex conjugation, the equality

$$\operatorname{Gr}^W_m H_{\mathrm{dR}} = F^p \oplus \overline{F^{m+1-p}}$$

holds for all m and p, which completes the proof. □

REMARK 3.381. As a consequence of Proposition 3.380 and Theorem 3.378, we have exhibit all regularized multiple zeta values as periods of mixed Hodge structures. Nevertheless, since we do not have Beilinson's theorem for tangential base points we do not know yet that these mixed Hodge structures *come from geometry* as in Theorem 2.183. This will be discussed in Section 4.3.6.

⋆ ⋆ ⋆

EXERCISE 3.382. An *integral point* of the affine scheme $X = \mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}$ is a morphism of schemes $\operatorname{Spec}(\mathbb{Z}) \to X$, or equivalently a ring morphism

$$\mathbb{Z}[t, t^{-1}, (t-1)^{-1}] \longrightarrow \mathbb{Z}.$$

Prove that X does not have integral points.

EXERCISE 3.383. By expanding the logarithm $\log(\eta)$ as a power series in $(1-\eta)$, prove the equality

$$\sum_{m>n>0} \frac{(1-\eta)^m}{n^2 m} = -\log(\eta) \sum_{n \geqslant 0} \frac{(1-\eta)^n}{n^2} - 2 \sum_{m>n \geqslant 1} \frac{(1-\eta)^m}{m^2 n}$$

of functions of the variable $0 < \eta < 1$.

EXERCISE 3.384. Let $n \geqslant 2$ be an integer. Adapt Example 3.371 to compute the regularized iterated integral

$$\int_{\mathbf{dch}}^{\mathrm{reg}} \omega_1 \omega_0^{n-1} \omega_1$$

and show that the result coincides with $\zeta_{Ш}(1, n)$.

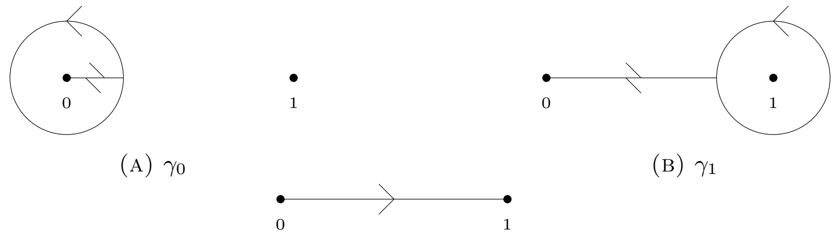
3.9. Polylogarithms and their monodromy. In this section, we explain how to make the isomorphism of Chen's Theorem 3.379 more explicit in the case of $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ by using polylogarithms.

3.9.1. *Generators of the fundamental group of M.* Recall the tangential base points $\mathbf{0}$ and $\mathbf{1}$ from (3.352). The fundamental group $\pi_1(M, \mathbf{0})$ is generated by the paths γ_0 and γ_1 of Figure 21. The space of paths $\pi(M; \mathbf{1}, \mathbf{0})$ is generated as a right $\pi_1(M, \mathbf{0})$-module by the straight path **dch** also represented in Figure 21.

The fundamental group $\pi_1(M, \mathbf{1})$ is generated by the paths

$$\gamma_0' = \mathbf{dch} \cdot \gamma_0 \cdot \mathbf{dch}^{-1} \quad \text{and} \quad \gamma_1' = \mathbf{dch} \cdot \gamma_1 \cdot \mathbf{dch}^{-1},$$

and the space $\pi(M; \mathbf{0}, \mathbf{1})$ is generated either as a right $\pi_1(M, \mathbf{1})$-module or as a left $\pi_1(M, \mathbf{0})$-module by the path $\mathbf{dch}^{-1}$.



(A) γ_0 (B) γ_1

(C) **dch**

FIGURE 21. Generators

3.9.2. *The dual of Chen's map.* We saw in Section 3.5.5 that the cohomology in degree zero of the reduced bar complex associated with $A^*_{\mathbb{C}}$ is isomorphic, as a Hopf algebra, to the complex Hoffman algebra $\mathfrak{H} \otimes \mathbb{C}$. In Example 3.74, we identified the dual $\mathfrak{H}^\vee$ with the algebra $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. We extend Notation 1.162 as follows.

NOTATION 3.385. Given a binary sequence α, we denote by x_α the corresponding word in the Hoffman algebra $\mathfrak{H}$, by ω_α the differential form ω_α in $B^0(A^*) \simeq \mathfrak{H}$, and by e_α the dual element to x_α in $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be base points (tangential or not) of $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. Given a path γ from $\boldsymbol{x}$ to $\boldsymbol{y}$ and $\omega \in B^0(A^*_{\mathbb{C}})$, we define

$$L_\omega(\gamma) = \int_\gamma^{\mathrm{reg}} \omega \in \mathbb{C}.$$

In the notation of Theorem 3.370, this amounts to

$$L_\omega(\gamma) = \langle \omega, \gamma\rangle^{\mathrm{reg}}.$$

For a binary sequence α, we set

$$L_\alpha(\gamma) = L_{\omega_\alpha}(\gamma).$$

Consider the generating series

$$L(\gamma) = \sum_\alpha L_\alpha(\gamma) e_\alpha \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle,$$

which satisfies $L(\gamma)(\omega) = L_\omega(\gamma)$ for all $\omega \in B^0(A^*_{\mathbb{C}}) \simeq \mathfrak{H} \otimes \mathbb{C} \simeq \mathbb{C}\langle\langle e_0, e_1\rangle\rangle^\vee$.

3.9.3. *The map L and polylogarithms.* Let $\boldsymbol{s}$ be a positive multi-index. Recall from Definition 1.118 the polylogarithm $\mathrm{Li}_{\boldsymbol{s}}$, a complex-valued function defined on the open unit disc $|z| < 1$. The relation between the polylogarithm and the generating series L is explained by the following lemma, whose proof is parallel to that of Theorem 1.126. We leave the details to the reader.

LEMMA 3.386. *Let z be a complex number such that $0 < |z| < 1$, γ a path from $\mathbf{0}$ to z contained in the disk $|z| < 1$, and $\mathbf{s}$ a positive multi-index. Let $\mathrm{bs}(\boldsymbol{s})$ denote the associated binary sequence. Then the following equality holds:*

$$\mathrm{Li}_{\mathbf{s}}(z) = L_{\mathrm{bs}(\mathbf{s})}(\gamma).$$

3.9.4. *Computation of* $L(\gamma_0)$. For any $z \in \mathbb{C} \setminus \{0,1\}$, any path γ from $\mathbf{0}$ to z, and any binary sequence α, the complex number $L_\alpha(\gamma)$ is defined. By abuse of notation, we will write $L_\alpha(z)$ and think of it as a *multivalued function*.

EXAMPLE 3.387. Let $z \in \mathbb{C} \setminus \{0,1\}$. Let us show that, for each $n \geqslant 1$, the following equality of multivalued functions holds:

$$L_{0^n}(z) = \frac{1}{n!}(\log z)^n. \tag{3.388}$$

Let γ be any path from $\mathbf{0}$ to z. We argue by induction on n. First, for $n = 1$, to compute the value

$$L_0(\gamma) = \int_\gamma^{\mathrm{reg}} \frac{\mathrm{d}t}{t},$$

one needs to find a logarithmic asymptotic development for

$$\begin{aligned}\eta \longmapsto \int_\eta^{1-\eta} \gamma^*(\tfrac{\mathrm{d}t}{t}) &= \int_\eta^{1-\eta} \frac{\gamma'}{\gamma}\mathrm{d}t \\ &= \log\gamma(1-\eta) - \log\gamma(\eta).\end{aligned}$$

Since $\gamma(0) = 0$ and $\gamma'(0) = 1$, one has $\gamma(\eta) = \eta(1 - O(\eta))$ as η goes to zero. On the other hand, $\gamma(1-\eta) = z + O(\eta)$. Thus,

$$\log\gamma(1-\eta) - \log\gamma(\eta) = \log z + O(\eta) - \log\eta$$

and the regularization assigns the value

$$L_0(z) = \log z.$$

Let us now assume that the identity (3.388) holds for $n - 1$. Since the number of shuffles of type $(1, n-1)$ is n by Exercise 1.143, Theorem 3.367 iii) gives the result we want:

$$nL_{0^n}(z) = \int_\gamma^{\mathrm{reg}} \omega_0 \int_\gamma \underbrace{\omega_0 \cdots \omega_0}_{n-1} = \frac{1}{(n-1)!}(\log z)^n.$$

EXAMPLE 3.389. We are now ready to compute $L(\gamma_0)$. Arguing as in Example 3.387, one gets

$$L_{0^n}(\gamma_0) = \frac{1}{n!}(2\pi i)^n.$$

If α is a non-empty positive binary sequence, then Lemma 3.386 implies

$$L_\alpha(\gamma_0) = 0.$$

In fact, it follows from Theorem 3.370 iii) and the compatibility with the shuffle product that $L_{\alpha 0^k}(\gamma_0) = 0$ for all $\alpha \neq \emptyset$ and all $k \geqslant 0$. Summing up, we deduce

$$L(\gamma_0) = \sum_\alpha L_\alpha(\gamma_0) e_\alpha = \sum_{n \geqslant 0} \frac{(2\pi i)^n}{n!} e_0^n = \exp(2\pi i e_0). \tag{3.390}$$

Thanks to the symmetry $z \mapsto 1 - z$, we then get the equality

$$L(\gamma_1') = \exp(2\pi i e_1). \tag{3.391}$$

3.9.5. *L evaluated at* **dch** *and the Drinfeld associator.*

EXAMPLE 3.392. Let α be a binary sequence. Theorem 3.378 implies the equality $L_\alpha(\mathbf{dch}) = \zeta_{\sqcup\!\sqcup}(x_\alpha)$, whence

$$L(\mathbf{dch}) = \sum_\alpha \zeta_{\sqcup\!\sqcup}(x_\alpha) e_\alpha. \tag{3.393}$$

We write $\Phi(e_0, e_1)$ for this power series with real coefficients. We also write

$$\Phi_{\mathrm{KZ}}(e_0, e_1) = \Phi(e_0, -e_1) = \sum_\alpha (-1)^{l(\alpha)} \zeta_{\sqcup\!\sqcup}(x_\alpha) e_\alpha, \tag{3.394}$$

where $l(\alpha)$ is the number of entries equal to 1 in α as in Definition 1.132.

DEFINITION 3.395. The power series $\Phi_{\mathrm{KZ}}(e_0, e_1) \in \mathbb{R}\langle\langle e_0, e_1 \rangle\rangle$ is called the *Drinfeld associator*.

REMARK 3.396. The name *associator* comes from the theory of quantum groups, where one seeks for "quantum" deformations of the Hopf algebra of a group. In this context, asking for an associative coproduct is too rigid. Instead, one defines the notion of a *quasi-bialgebra*, in which the coproduct is only associative up to conjugation by an element called the *associator*. The Drinfeld associator is some kind of universal associator that allows us to construct associators in many particular instances. We refer the reader to the book by Etingof and Schiffmann [**ES02**], especially to Chapters 14 to 16, for more details.

3.9.6. *Chen's theorem revisited.*

THEOREM 3.397. *For any two base points $\boldsymbol{x}$ and $\boldsymbol{y}$, the map L can be extended to a continuous $\mathbb{C}$-linear isomorphism*

$$L\colon \mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge \longrightarrow \mathbb{C}\langle\langle e_0, e_1 \rangle\rangle = \mathrm{Hom}(\mathfrak{H}, \mathbb{C}).$$

The following properties hold:

i) *If $u \in \mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge$, then*
$$S^\vee(L(u)) = L(S(u)).$$
In particular, if $\gamma \in \pi_1(M; \boldsymbol{y}, \boldsymbol{x})$ is a path, then $S^\vee(L(\gamma)) = L(\gamma^{-1})$.

ii) *Given three points $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$, and elements $v \in \mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge$ and $u \in \mathbb{C}[\pi_1(M; \boldsymbol{z}, \boldsymbol{y})]^\wedge$, one has*
$$L(uv) = L(u)L(v).$$

iii) *If $u \in \mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge$, then*
$$\nabla^\vee(L(u)) = (L \otimes L)(\Delta(u)).$$
In particular, if $\gamma \in \pi_1(M; \boldsymbol{y}, \boldsymbol{x})$ is a path, then $L(\gamma)$ is group-like.

PROOF. We first extend L by linearity to $\mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]$. By construction, for any path γ, the series $L(\gamma)$ starts by one. Therefore, any element in the augmentation ideal of $\mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]$ is sent to an element of the ideal generated by e_0 and e_1. Thus, it can be extended uniquely to a morphism

$$L\colon \mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge \longrightarrow \mathbb{C}\langle\langle e_0, e_1 \rangle\rangle = \mathrm{Hom}(\mathfrak{H}, \mathbb{C}).$$

That this yields an isomorphism is a reformulation of Theorem 3.379. It is enough to check properties i) to iii) on paths, all of which follow from Theorem 3.367.

We start proving i) by using Theorem 3.370 i):

$$\begin{aligned} L(\gamma^{-1}) &= \sum_\alpha \langle \omega_\alpha, \gamma^{-1}\rangle^{\mathrm{reg}} e_\alpha \\ &= \sum_\alpha \langle S(\omega_\alpha), \gamma\rangle^{\mathrm{reg}} e_\alpha \\ &= \sum_\alpha \langle \omega_\alpha, \gamma\rangle^{\mathrm{reg}} S^\vee(e_\alpha) \\ &= S^\vee(L(\gamma)). \end{aligned}$$

We next prove ii) using 3.370 ii):

$$\begin{aligned} L(\gamma_1\gamma_2) &= \sum_\alpha \langle \omega_\alpha, \gamma_1\gamma_2\rangle^{\mathrm{reg}} e_\alpha \\ &= \sum_\alpha \langle \Delta\omega_\alpha, \gamma_1\otimes\gamma_2\rangle^{\mathrm{reg}} e_\alpha \\ &= \sum_{\alpha',\alpha''} \langle \omega_{\alpha'}\otimes\omega_{\alpha''}, \gamma_1\otimes\gamma_2\rangle^{\mathrm{reg}} e_{\alpha'}e_{\alpha''} \\ &= L(\gamma_1)L(\gamma_2). \end{aligned}$$

Finally, we prove iii) using 3.370: iii).

$$\begin{aligned} \nabla^\vee(L(\gamma)) &= \sum_\alpha \langle \omega_\alpha, \gamma\rangle^{\mathrm{reg}} \nabla^\vee e_\alpha \\ &= \sum_\alpha \langle \omega_\alpha, \gamma\rangle^{\mathrm{reg}} \sum_{\alpha',\alpha''} ⧢(\alpha',\alpha'';\alpha) e_{\alpha'}\otimes e_{\alpha''} \\ &= \sum_{\alpha',\alpha''} \langle \omega_{\alpha'} ⧢ \omega_{\alpha''}, \gamma\rangle^{\mathrm{reg}} e_{\alpha'}\otimes e_{\alpha''} \\ &= \sum_{\alpha',\alpha''} \langle \omega_{\alpha'}, \gamma\rangle^{\mathrm{reg}} \langle \omega_{\alpha''}, \gamma\rangle^{\mathrm{reg}} e_{\alpha'}\otimes e_{\alpha''} \\ &= L(\gamma)\otimes L(\gamma). \end{aligned}$$

This concludes the proof. □

Example 3.398. From Theorem 3.397 iii), we deduce that

$$\Phi(e_0, e_1) = L(\mathbf{dch})$$

is a group-like element. In particular, it is the exponential of a primitive element, and its inverse as power series is given by its antipode

$$L(\mathbf{dch}^{-1}) = \Phi(e_0,e_1)^{-1} = S^\vee(\Phi(e_0,e_1)). \tag{3.399}$$

From Examples 3.389 and 3.392, as well as the compatibility of L with the composition of paths in Theorem 3.397 ii), we can compute L on the remaining generators of $\pi_1(M,\mathbf{0})$ and $\pi_1(M,\mathbf{1})$. We get:

$$\begin{aligned} L(\gamma_1) &= \Phi(e_0,e_1)^{-1}\exp(2\pi i e_1)\Phi(e_0,e_1), \\ L(\gamma_0') &= \Phi(e_0,e_1)\exp(2\pi i e_o)\Phi(e_0,e_1)^{-1}. \end{aligned}$$

3.9.7. *The Knizhnik–Zamolodchikov equation.* Theorem 3.397 encodes all the properties of the series L, and hence of polylogarithms. The first property we can extract from it is that L satisfies the so-called Knizhnik–Zamolodchikov equation:

PROPOSITION 3.400. *$L(z)$ satisfies the differential equation*

$$\frac{\mathrm{d}}{dz}L(z) = \left(\frac{e_0}{z} + \frac{e_1}{1-z}\right)L(z). \tag{3.401}$$

PROOF. Fix $z \in M$, let γ be a path with endpoint z, and let $\gamma_\varepsilon(t) = z + t\varepsilon$. To compute the derivative of $L(z)$, we need to evaluate the limit

$$\lim_{\varepsilon\to 0}\frac{L(\gamma_\varepsilon\gamma) - L(\gamma)}{\varepsilon}.$$

From Theorem 3.397 ii) we get

$$L(\gamma_\varepsilon\gamma) - L(\gamma) = (L(\gamma_\varepsilon) - 1)L(\gamma).$$

Moreover, the equality

$$L(\gamma_\varepsilon) - 1 = \int_{\gamma_\varepsilon}^{\mathrm{reg}} \omega_0 e_0 + \int_{\gamma_\varepsilon}^{\mathrm{reg}} \omega_1 e_1 + O(\varepsilon^2)$$

holds. Taking the values of the limits

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\gamma_\varepsilon}\omega_0 = \frac{1}{z} \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\gamma_\varepsilon}\omega_1 = \frac{1}{1-z}$$

into account, we conclude

$$\frac{\mathrm{d}}{dz}L(z) = \left(\frac{e_0}{z} + \frac{e_1}{1-z}\right)L(z),$$

as we wanted to show. □

3.9.8. *The monodromy of L.* The second property we want to derive is an explicit description of the monodromy of L as a multivalued function.

THEOREM 3.402. *Let $z \in M$ and γ a path from $\mathbf{0}$ to z. Then,*

$$L(\gamma\cdot\gamma_0) = L(\gamma)\exp(2\pi i e_0),$$
$$L(\gamma\cdot\gamma_1) = L(\gamma)\Phi(e_0,e_1)^{-1}\exp(2\pi i e_1)\Phi(e_0,e_1).$$

PROOF. The statement follows immediately from Theorem 3.397 ii) and Examples 3.389 and 3.398. □

3.9.9. *Further properties of the Drinfeld associator.* We next derive the basic properties of the Drinfeld associator Φ_{KZ}. Let $U\mathfrak{a}_4$ be the universal enveloping algebra of the Lie algebra of the pro-unipotent completion of the pure braid group on 4 strings. Concretely, it is the algebra of power series in letters

$$t_{i,j} \quad \text{for} \quad 1 \leqslant i,j \leqslant 4,$$

subject to the relations

$$t_{i,i} = 0, \qquad t_{i,j} = t_{j,i},$$
$$[t_{i,j}, t_{i,k} + t_{j,k}] = 0, \ \text{for } i,j,k \text{ different},$$
$$[t_{i,j}, t_{k,l}] = 0, \ \text{for } i,j,k,l \text{ different}.$$

THEOREM 3.403 (Drinfeld [**Dri90**]). *The Drinfeld associator Φ_{KZ} satisfies the following relations:*

i) *Symmetry relation:* $\Phi_{\mathrm{KZ}}(e_0,e_1)\Phi_{\mathrm{KZ}}(e_1,e_0)=1$.

ii) *Hexagon relation: set* $e_\infty=-e_0-e_1$. *Then*
$$e^{i\pi e_0}\Phi_{\mathrm{KZ}}(e_\infty,e_0)e^{i\pi e_\infty}\Phi_{\mathrm{KZ}}(e_1,e_\infty)e^{i\pi e_1}\Phi_{\mathrm{KZ}}(e_0,e_1)=1.$$

iii) *Pentagon relation: For* $t_{i,j}\in U\mathfrak{a}_4$, *we have*
$$\Phi_{\mathrm{KZ}}(t_{1,2},t_{2,3}+t_{2,4})\Phi_{\mathrm{KZ}}(t_{1,3}+t_{2,3},t_{3,4})$$
$$=\Phi_{\mathrm{KZ}}(t_{2,3},t_{3,4})\Phi_{\mathrm{KZ}}(t_{1,2}+t_{1,3},t_{2,4}+t_{3,4})\Phi_{\mathrm{KZ}}(t_{1,2},t_{2,3}).$$

PROOF. We start proving i). Consider the automorphism $z\mapsto 1-z$ of M. This automorphism sends the form ω_i to $-\omega_{1-i}$ for $i=0,1$, and hence it sends e_0 to $-e_1$, and e_1 to $-e_0$. Moreover, it sends $\mathbf{dch}$ to $\mathbf{dch}^{-1}$. Therefore, we get the equalities $L(\mathbf{dch}^{-1})=\Phi(-e_1,-e_0)$ and
$$1=L(\mathbf{dch})L(\mathbf{dch}^{-1})=\Phi(e_0,e_1)\Phi(-e_1,-e_0),$$
which is equivalent to i).

To prove ii), we need to introduce more tangential points and paths:

- Let $\mathbf{0}^-=(0,-1)$ be the tangent vector -1 at 0, and $\mathbf{1}^-=(1,1)$ be the tangent vector 1 at 1.
- Consider the point ∞ with local coordinate $u=1/z$ and write
$$\boldsymbol{\infty}=(\infty,1)\quad\text{and}\quad\boldsymbol{\infty}^-=(\infty,-1)$$
for the tangent point 1 and -1 at ∞ with respect to this coordinate
- We denote by $\delta_0\in\pi(M;\mathbf{0},\mathbf{0}^-)$ the path that starts in $\mathbf{0}^-$, gives half a turn around zero in the counterclockwise direction, and ends in $\mathbf{0}$.
- Similarly, $\delta_1\in\pi(M;\mathbf{1}^-,\mathbf{1})$ is the path that starts in $\mathbf{1}$, gives half a turn in the counterclockwise direction and ends in $\mathbf{1}^-$ and $\delta_\infty\in\pi(M;\boldsymbol{\infty}^-,\boldsymbol{\infty})$ is the path that starts in $\boldsymbol{\infty}$, gives half a turn in the counterclockwise direction, and ends in $\boldsymbol{\infty}^-$.
- Finally, we denote by $\mathbf{dch}_{\infty,1}\in\pi(M;\boldsymbol{\infty},\mathbf{1}^-)$ the straight path that starts in $\mathbf{1}^-$ and ends in $\boldsymbol{\infty}$ through the real numbers greater than one and, similarly, by $\mathbf{dch}_{0,\infty}\in\pi(M;\mathbf{0}^-,\boldsymbol{\infty}^-)$ the straight path that starts in $\boldsymbol{\infty}^-$ and ends in $\mathbf{0}^-$ through the negative real numbers.

All these paths are represented in Figure 22.

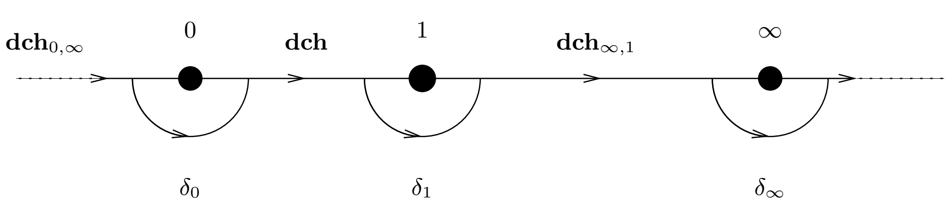


FIGURE 22. More tangential base points and paths

Clearly, the composition
$$\delta_0\cdot\mathbf{dch}_{0,\infty}\cdot\delta_\infty\cdot\mathbf{dch}_{\infty,1}\cdot\delta_1\cdot\mathbf{dch}$$

is homotopically equivalent to the trivial path. Therefore, from Theorem 3.397 ii) we get the equality

$$L(\delta_0)L(\mathbf{dch}_{0,\infty})L(\delta_\infty)L(\mathbf{dch}_{\infty,1})L(\delta_1)L(\mathbf{dch}) = 1. \tag{3.404}$$

Arguing as in Example 3.389, we can see that

$$L(\delta_0) = \exp(\pi i e_0).$$

We now consider the automorphism $z \mapsto 1/(1-z)$ of M. This map sends δ_0 to δ_1, and δ_1 to δ_∞. It also sends $\mathbf{dch}$ to $\mathbf{dch}_{\infty,1}$, and $\mathbf{dch}_{\infty,1}$ to $\mathbf{dch}_{0,\infty}$.

Moreover, the pull back by this automorphism sends the form ω_0 to the form ω_1, and the form ω_1 to the form $-\omega_0-\omega_1$. Dualizing, we deduce that this automorphism sends e_0 to $-e_1$, and e_1 to $e_0 - e_1$. We deduce the equalities

$$\begin{aligned} L(\delta_1) &= \exp(-\pi i e_1), & L(\delta_\infty) &= \exp(\pi i(e_1 - e_0)), \\ L(\mathbf{dch}_{\infty,1}) &= \Phi(-e_1, e_0 - e_1), & L(\mathbf{dch}_{0,\infty}) &= \Phi(e_1 - e_0, -e_0). \end{aligned}$$

Thus, equation (3.404) reads

$$e^{i\pi e_0}\Phi(e_1 - e_0, -e_0)e^{i\pi(e_1-e_0)}\Phi(-e_1, e_0 - e_1)e^{-i\pi e_1}\Phi(e_0, e_1) = 1,$$

which is equivalent to

$$e^{i\pi e_0}\Phi_{\mathrm{KZ}}(e_1 - e_0, e_0)e^{i\pi(e_1-e_0)}\Phi_{\mathrm{KZ}}(-e_1, e_1 - e_0)e^{-i\pi e_1}\Phi_{\mathrm{KZ}}(e_0, -e_1) = 1.$$

The hexagon relation is obtained by replacing e_1 with $-e_1$.

The proof of iii) involves considering a path in the moduli space $M_{0,5}$ which is a complex surface. To write it properly, we would need to discuss tangential base points and local monodromy in higher dimensions, so we will omit it. □

3.9.10. *The associator relations and the extended double shuffle relations.* We close this section by quoting the following result:

Theorem 3.405 (Furusho [**Fur10**], [**Fur11**]).

i) *Let $(\zeta^s(\alpha))_\alpha$ be a collection of real numbers, one for each binary sequence. Denote by $\zeta^s \colon \mathfrak{H}^0 \to \mathbb{R}$ the map obtained from these numbers by linearity. If the power series*

$$\sum_\alpha (-1)^\alpha \zeta^s(\alpha) e_\alpha$$

is group-like and satisfies the associator relations from Theorem 3.403, then the pair $(\mathbb{R}, \zeta^s)$ satisfies the extended double shuffle relations from Definition 1.202.

ii) *Let $\varphi \in \mathbb{R}\langle\langle e_0, e_1\rangle\rangle$ be a group-like element such that the coefficient of e_0e_1 is equal to $-\zeta(2) = -\pi^2/6$. If φ satisfies the pentagon relation from Theorem 3.403 iii), then it satisfies the symmetry relation i) and the hexagon relation ii) from the same theorem.*

⋆ ⋆ ⋆

Exercise 3.406. Compute explicitly the terms up to degree 5 of the Drinfeld associator $\Phi_{\mathrm{KZ}}(e_0, e_1)$. Show that, with the exception of the unit in degree 0, they can be all written as commutators.

EXERCISE 3.407. In this exercise, we show how Theorem 3.402 encodes the monodromy of multiple polylogarithms in one variable. We start with Li_3, which is the coefficient of $e_0e_0e_1$ in L. Let $z \in \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$ and γ a path from $\mathbf{0}$ to z.

i) Find the coefficient of $e_0e_0e_1$ in $L(\gamma\gamma_0)$ and $L(\gamma\gamma_1)$. The resulting expressions give the monodromy of Li_3.
ii) Compute the monodromy along γ_0 and γ_1 of the functions

$$L_\emptyset = 1, \quad L_0, \quad L_1, \quad L_{0001}, \quad L_{01001}.$$

3.10. The fundamental groupoid of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. We continue studying the manifold $M = \mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$, but we now view it as the set of complex points of the algebraic variety $X = \mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}$ defined over $\mathbb{Q}$. Recall the dg-algebra A^* from Example 2.284 that computes the algebraic de Rham cohomology of X.

3.10.1. *Summary of structures.* For convenience and to fix notation, we start by summarizing some results from the previous sections.

SUMMARY 3.408. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \{\mathbf{0},\mathbf{1}\} \cup X(\mathbb{Q})$ be base points (tangential or not). We have at our disposal the following structures.

Betti side: An affine pro-algebraic scheme over $\mathbb{Q}$

$${}_{\boldsymbol{y}}\Pi^{\mathrm{B}}_{\boldsymbol{x}} = \pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}; \boldsymbol{y}, \boldsymbol{x})^{\mathrm{un}},$$

a pro-$\mathbb{Q}$-vector space

$${}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}} = \mathbb{Q}\big[\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}; \boldsymbol{y}, \boldsymbol{x})\big]^\wedge,$$

the subspace of primitive elements

$${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{B}}_{\boldsymbol{x}} = \{x \in {}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}} \mid \nabla^\vee x = 1\otimes x + x \otimes 1\},$$

and an ind-$\mathbb{Q}$-algebra

$${}_{\boldsymbol{y}}A^{\mathrm{B}}_{\boldsymbol{x}} = \mathcal{O}({}_{\boldsymbol{y}}\Pi^{\mathrm{B}}_{\boldsymbol{x}}) = ({}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}})^\vee.$$

De Rham side: An affine pro-algebraic scheme over $\mathbb{Q}$

$${}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}} = \mathrm{Spec}(\mathfrak{H}),$$

a pro-$\mathbb{Q}$-vector space

$${}_{\boldsymbol{y}}U^{\mathrm{dR}}_{\boldsymbol{x}} = \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle,$$

the subspace of primitive elements

$${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{dR}}_{\boldsymbol{x}} = \{x \in {}_{\boldsymbol{y}}U^{\mathrm{dR}}_{\boldsymbol{x}} \mid \nabla^\vee x = 1\otimes x + x \otimes 1\},$$

and an ind-$\mathbb{Q}$-algebra

$${}_{\boldsymbol{y}}A^{\mathrm{dR}}_{\boldsymbol{x}} = \mathfrak{H}.$$

Comparison isomorphisms:

$$\begin{aligned}
\mathrm{comp}^{\Pi}_{\mathrm{dR,B}} &\colon {}_{\boldsymbol{y}}\Pi^{\mathrm{B}}_{\boldsymbol{x}} \times_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}} \times_{\mathbb{Q}} \mathbb{C},\\
\mathrm{comp}^{U}_{\mathrm{dR,B}} &\colon {}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}} \widehat{\otimes}_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}_{\boldsymbol{y}}U^{\mathrm{dR}}_{\boldsymbol{x}} \widehat{\otimes}_{\mathbb{Q}} \mathbb{C},\\
\mathrm{comp}^{\mathcal{L}}_{\mathrm{dR,B}} &\colon {}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{B}}_{\boldsymbol{x}} \widehat{\otimes}_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{dR}}_{\boldsymbol{x}} \widehat{\otimes}_{\mathbb{Q}} \mathbb{C},\\
\mathrm{comp}^{A}_{\mathrm{B,dR}} &\colon {}_{\boldsymbol{y}}A^{\mathrm{dR}}_{\boldsymbol{x}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}_{\boldsymbol{y}}A^{\mathrm{B}}_{\boldsymbol{x}} \otimes_{\mathbb{Q}} \mathbb{C}.
\end{aligned}$$

Recall that ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{B}}$ and ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{dR}}$ are affine schemes over $\mathbb{Q}$. The notation $\times_{\mathbb{Q}}\mathbb{C}$ is a shorthand for $\times_{\mathrm{Spec}(\mathbb{Q})}\mathrm{Spec}(\mathbb{C})$. All the comparison isomorphisms $\mathrm{comp}_{\mathrm{B,dR}}^{?}$ are given by the regularized iterated integrals. For instance,

$$\mathrm{comp}_{\mathrm{dR,B}}^{U} \colon {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{B}} \longrightarrow {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{dR}}$$

agrees with the map L from Theorem 3.397.

Additional structures: For $? = \mathrm{B}, \mathrm{dR}$, there are morphisms

$$ {}_{\boldsymbol{z}}\Pi_{\boldsymbol{y}}^{?} \times {}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{?} \longrightarrow {}_{\boldsymbol{z}}\Pi_{\boldsymbol{x}}^{?} \tag{3.409}$$

of pro-algebraic schemes induced from the composition of paths on the Betti side and the coproduct of $\mathfrak{H}$ on the de Rham side. These morphisms turn ${}_{\boldsymbol{x}}\Pi_{\boldsymbol{x}}^{?}$ into a pro-unipotent group scheme, and ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{?}$ into a right ${}_{\boldsymbol{x}}\Pi_{\boldsymbol{x}}^{?}$-torsor and a left ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{y}}^{?}$-torsor. Therefore, the pro-$\mathbb{Q}$-vector spaces ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?}$ are endowed with the following structures:

i) a composition of paths

$$\Delta^{\vee} \colon {}_{\boldsymbol{z}}U_{\boldsymbol{y}}^{?} \otimes {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?} \longrightarrow {}_{\boldsymbol{z}}U_{\boldsymbol{x}}^{?};$$

ii) unit

$$\eta_{\boldsymbol{x}}^{\vee} \colon \mathbb{Q} \longrightarrow {}_{\boldsymbol{x}}U_{\boldsymbol{x}}^{?};$$

iii) a completed coproduct

$$\nabla^{\vee} \colon {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?} \longrightarrow {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?} \widehat{\otimes} {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?};$$

iv) counit

$$\epsilon^{\vee} \colon {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?} \longrightarrow \mathbb{Q};$$

v) a dual antipode

$$S^{\vee} \colon {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?} \longrightarrow {}_{\boldsymbol{x}}U_{\boldsymbol{y}}^{?}.$$

The ind-algebras ${}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{?}$ are endowed with the dual structures. These additional structures are compatible with all the comparison isomorphisms.

Mixed Hodge structures: Extending the construction of Section 3.7.2 to tangential base points as in Section 3.8.9, we see that ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{B}}$ and ${}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{B}}$ are equipped with a weight filtration W, and ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{dR}}$ and ${}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{dR}}$ with a weight filtration W and a Hodge filtration F in such a way that

$${}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{H}} = (({}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{B}}, W), ({}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{dR}}, W, F), \mathrm{comp}_{\mathrm{B,dR}})$$

is an object of ind-$\mathbf{MHS}(\mathbb{Q})$, and

$${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}} = (({}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{B}}, W), ({}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{dR}}, W, F), \mathrm{comp}_{\mathrm{dR,B}}^{-1})$$

is an object of pro-$\mathbf{MHS}(\mathbb{Q})$.

The filtrations on ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?}$ induce a weight filtration on ${}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{B}}$, and a weight and a Hodge filtration on ${}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{dR}}$, so that

$${}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}} = (({}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{B}}, W), ({}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{dR}}, W, F), \mathrm{comp}_{\mathrm{dR,B}}^{-1})$$

is also an object of pro-$\mathbf{MHS}(\mathbb{Q})$.

Moreover, all the previous additional structures of ${}_{\bullet}A_{\bullet}^{\mathrm{H}}$ and ${}_{\bullet}U_{\bullet}^{\mathrm{H}}$ underlie morphisms of ind-$\mathbf{MHS}(\mathbb{Q})$ and pro-$\mathbf{MHS}(\mathbb{Q})$ respectively.

REMARK 3.410. Observe that the de Rham side on Summary 3.408 is independent of the base points. In fact, there is a canonical de Rham path ${}_{\boldsymbol{y}}1^{\mathrm{dR}}_{\boldsymbol{x}}$ in ${}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}}$ (it is the unit element in the affine group scheme $\mathrm{Spec}(\mathfrak{H})$ and corresponds to the kernel of the counit $\varepsilon\colon \mathfrak{H} \to \mathbb{Q}$). Therefore, for base points $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$, there are canonical isomorphisms

$$\begin{aligned} {}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}} &\longrightarrow {}_{\boldsymbol{z}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}}, \qquad & {}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}} &\longrightarrow {}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{z}}. \\ \gamma &\longmapsto {}_{\boldsymbol{z}}1^{\mathrm{dR}}_{\boldsymbol{y}} \cdot \gamma & \gamma &\longmapsto \gamma \cdot {}_{\boldsymbol{x}}1^{\mathrm{dR}}_{\boldsymbol{z}} \end{aligned}$$

Since the pro-algebraic scheme ${}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}}$ is independent of the base points, we will suppress them from the notation, and simply write $\Pi^{\mathrm{dR}} = \mathrm{Spec}(\mathfrak{H})$.

For future reference, we make the ind-mixed Hodge structure on ${}_{\boldsymbol{y}}A^{\mathrm{H}}_{\boldsymbol{x}}$ more explicit. This result follows directly from Proposition 3.345 and its proof.

PROPOSITION 3.411. *The ind-mixed Hodge structure* ${}_{\boldsymbol{y}}A^{\mathrm{H}}_{\boldsymbol{x}}$ *is of Hodge–Tate type. Moreover, the associated grading (Lemma 2.237) on*

$${}_{\boldsymbol{y}}A^{\mathrm{dR}}_{\boldsymbol{x}} = \mathfrak{H} = \mathbb{Q}\langle x_0, x_1\rangle$$

is the multiplicative grading that assigns degree 1 *to the elements* x_0 *and* x_1*, and degree* 0 *to the constants.*

VARIANT 3.412. The same structures are available for other varieties. For instance, everything can be easily generalized to any variety of the form $X' = \mathbb{P}^1_{\mathbb{Q}} \setminus S$ for a finite subset $S \subset \mathbb{P}^1(\mathbb{Q})$. In this case, we will use ${}_{\boldsymbol{y}}\Pi(X')^{\mathrm{B}}_{\boldsymbol{x}}$ to denote the pro-algebraic scheme in the Betti side and similar notation for the other structures. In the sequel, we will only need the case $X' = \mathbb{G}_m$, in which case we have

$$\begin{aligned} {}_{\boldsymbol{y}}\Pi(\mathbb{G}_m)^{\mathrm{dR}}_{\boldsymbol{x}} &= \mathbb{A}^1_{\mathbb{Q}} = \mathbb{G}_a, \\ {}_{\boldsymbol{y}}A(\mathbb{G}_m)^{\mathrm{dR}}_{\boldsymbol{x}} &= \mathbb{Q}[x_0], \\ {}_{\boldsymbol{y}}U(\mathbb{G}_m)^{\mathrm{dR}}_{\boldsymbol{x}} &= \mathbb{Q}[\![e_0]\!], \\ {}_{\boldsymbol{y}}\mathcal{L}(\mathbb{G}_m)^{\mathrm{dR}}_{\boldsymbol{x}} &= \mathbb{Q}e_0. \end{aligned}$$

We will also denote by L the morphism

$$\mathrm{comp}^{U}_{\mathrm{dR,B}}\colon {}_{\boldsymbol{y}}U(\mathbb{G}_m)^{\mathrm{B}}_{\boldsymbol{x}} \longrightarrow {}_{\boldsymbol{y}}U(\mathbb{G}_m)^{\mathrm{dR}}_{\boldsymbol{x}}.$$

3.10.2. *The fundamental groupoid and the local monodromy.* From now on, we focus our attention on the pro-unipotent group picture ${}_{\boldsymbol{y}}\Pi^{?}_{\boldsymbol{x}}$. The reader will have no difficulty in writing the analogous statements for $U^?$, $\mathcal{L}^?$, and $A^?$.

DEFINITION 3.413. The diagram consisting of the four schemes

$${}_{\boldsymbol{y}}\Pi^{?}_{\boldsymbol{x}}, \qquad \boldsymbol{x}, \boldsymbol{y} \in \{\boldsymbol{0}, \boldsymbol{1}\}$$

along with the composition of paths will be called the *tangential fundamental groupoid of* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. It is represented schematically in Figure 23.

We want to enrich this construction of the tangential fundamental groupoid by adding the *local monodromies* around 0 and 1. We start with the local monodromy around 0 in the de Rham side. The morphism of Hopf algebras $\mathfrak{H} \to \mathbb{Q}[x]$ that sends every word containing x_1 to zero, and $\underbrace{x_0 \dots x_0}_{n}$ to $x^n/n!$ can be seen as a map

$${}_{\boldsymbol{0}}A^{\mathrm{dR}}_{\boldsymbol{0}} \longrightarrow {}_{\boldsymbol{0}}A(\mathbb{G}_m)^{\mathrm{dR}}_{\boldsymbol{0}}$$

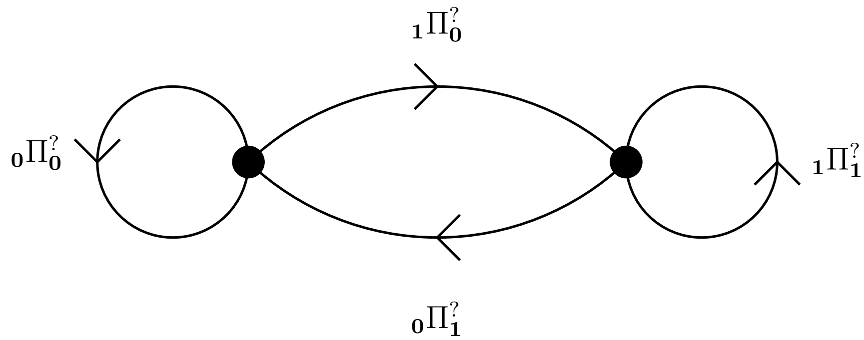


FIGURE 23. The fundamental groupoid

that induces maps

$$\mathbb{G}_a = {}_{\mathbf{0}}\Pi(\mathbb{G}_m)^{\mathrm{dR}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}} \quad \text{and} \quad {}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{dR}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}U^{\mathrm{dR}}_{\mathbf{0}}.$$

The local monodromy around 0 in the Betti side is obtained topologically as follows. Let Δ^* be a small punctured disc around zero in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. The local monodromy is the composition of the inverse of the isomorphism

$$\pi_1(\Delta^*, \mathbf{0})^{\mathrm{un}} \longrightarrow \pi_1(\mathbb{G}_m, \mathbf{0})^{\mathrm{un}} = \mathbb{G}_a$$

with the natural map

$$\pi_1(\Delta^*, \mathbf{0})^{\mathrm{un}} \longrightarrow \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \mathbf{0})^{\mathrm{un}}.$$

Similarly, the de Rham side of the local monodromy around 1 is induced by the morphism of Hopf algebras $\mathfrak{H} \to \mathbb{Q}[x]$ that sends every word containing x_0 to zero, and $x_1 \cdots x_1$ to $x^n/n!$, and the Betti side is obtained by considering a small punctured disc around 1.

The local monodromy maps are morphisms of ind-$\mathbf{MHS}(\mathbb{Q})$ in the case of A and morphism of pro-$\mathbf{MHS}(\mathbb{Q})$ in the case of U. This means that the pair of maps

$${}_{\mathbf{0}}A^{\mathrm{dR}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}A(\mathbb{G}_m)^{\mathrm{dR}}_{\mathbf{0}} \qquad {}_{\mathbf{0}}A^{\mathrm{B}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}A(\mathbb{G}_m)^{\mathrm{B}}_{\mathbf{0}}$$

is a morphism of ind-$\mathbf{MHS}(\mathbb{Q})$

$${}_{\mathbf{0}}A^{\mathrm{H}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}A(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}},$$

while the pair of maps

$${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{dR}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}U^{\mathrm{dR}}_{\mathbf{0}} \qquad {}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{B}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}U^{\mathrm{B}}_{\mathbf{0}}$$

is a morphism of pro-$\mathbf{MHS}(\mathbb{Q})$

$${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}U^{\mathrm{H}}_{\mathbf{0}},$$

and the same is true for the local monodromy maps around 1.

DEFINITION 3.414. We will denote by D^{dR} the diagram consisting of the four schemes ${}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}}$, for $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$, the morphisms given by the composition of paths, the scheme $\mathbb{G}_a$, and the two local monodromies

$$\mathbb{G}_a \longrightarrow {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}} \quad \text{and} \quad \mathbb{G}_a \longrightarrow {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{1}}.$$

Similarly, we write D^{dR}_U and D^{dR}_A for the corresponding diagram for the vector spaces U and the algebras A together with all the additional structures discussed in Summary 3.408. That is, the unit, counit, product, completed coproduct and dual antipode for U, and the dual structures for A. Similarly, we will denote by $D^{\mathrm{B}}_{?}$ the corresponding diagrams on the Betti side. Finally, we will denote by D^{H}_{U} the pair of diagrams D^{B}_{U} and D^{dR}_{U} viewed together as a diagram of pro-$\mathbf{MHS}(\mathbb{Q})$.

We will see in Section 4.5 that the diagram D^{H}_{U} is "motivic".

3.10.3. *The automorphisms of* D^{dR}. We denote by $\mathrm{Aut}(D^{\mathrm{dR}})$ the group of automorphisms of D^{dR} in the following sense: to give an element of $\mathrm{Aut}(D^{\mathrm{dR}})$ amounts to giving an automorphism of pro-algebraic schemes of each ${}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}}$ and an automorphism of $\mathbb{G}_a$ that are compatible with the composition of paths (3.409) and the local monodromy maps. The group $\mathrm{Aut}(D^{\mathrm{dR}})$ is a pro-algebraic group.

Similarly, we denote by $\mathrm{Aut}(D^{\mathrm{dR}}_{U})$ the automorphisms of the diagram D^{dR}_{U} compatible with all the structures. Since the diagrams D^{dR} and D^{dR}_{U} determine each other, there is a canonical identification between $\mathrm{Aut}(D^{\mathrm{dR}}_{U})$ and $\mathrm{Aut}(D^{\mathrm{dR}})$. Hence, we will only work with the latter. We denote by $\mathrm{Aut}^0(D^{\mathrm{dR}})$ the subgroup of $\mathrm{Aut}(D^{\mathrm{dR}})$ that acts as the identity on $\mathbb{G}_a$. There is an exact sequence

$$0 \longrightarrow \mathrm{Aut}^0(D^{\mathrm{dR}}) \longrightarrow \mathrm{Aut}(D^{\mathrm{dR}}) \longrightarrow \mathbb{G}_m \longrightarrow 0.$$

Using the tools from the next sections, one can show that the group $\mathrm{Aut}^0(D^{\mathrm{dR}})$ is pro-unipotent (Exercise 3.435).

Lemma 3.415. *There is an isomorphism of schemes*

$$\begin{aligned} \mathrm{Aut}^0(D^{\mathrm{dR}}) &\longrightarrow {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}, \\ f &\longmapsto \gamma_f \end{aligned}$$

where γ_f *is determined by the equation*

$$f({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \gamma_f.$$

Proof. Recall that the dual of the Hoffman algebra $\mathfrak{H}$ is $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. It also agrees with the completed universal enveloping algebra of $\mathrm{Lie}({}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}})$. Let R be a $\mathbb{Q}$-algebra. The elements of ${}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R)$ are the group-like elements of $R\langle\langle e_0, e_1\rangle\rangle$. Moreover, there are identities

$$\begin{aligned} {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) &= {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R), \\ {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{1}}(R) &= {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) \cdot {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}, \\ {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{1}}(R) &= {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) \cdot {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}. \end{aligned} \tag{3.416}$$

Let $f \in \mathrm{Aut}^0(D^{\mathrm{dR}})(R)$. Since f is the identity on $\mathbb{G}_a$, we deduce the equalities

$$\begin{gathered} f(\exp(e_0)) = \exp(e_0), \\ f({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \exp(e_1) \cdot {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \exp(e_1) \cdot {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}. \end{gathered}$$

We also have $f({}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{0}}) = {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{0}}$ and ${}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}} \cdot {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} = {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{0}}$. Therefore, the fact that f is compatible with the composition of paths implies that it is determined by the image of ${}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}$. We write

$$f({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \gamma_f$$

for an element $\gamma_f \in {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) \subset R\langle\langle e_0, e_1\rangle\rangle$.

Conversely, every $\gamma \in {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) = \mathrm{Spec}(\mathfrak{H})(R)$ is a group-like element of the algebra $R\langle\langle e_0, e_1\rangle\rangle$. To give an element of $\mathrm{Aut}({}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}})(R)$ is equivalent to give a continuous automorphism of $R\langle\langle e_0, e_1\rangle\rangle$ that is compatible with the completed coproduct and the antipode. We define

$$f_\gamma(e_0) = e_0, \qquad f_\gamma(e_1) = \gamma^{-1} \cdot e_1 \cdot \gamma. \tag{3.417}$$

This determines a continuous automorphism of $R\langle\langle e_0, e_1\rangle\rangle$. To show that it is compatible with the completed coproduct, it is enough to check that this property holds for the generator e_1. On the one hand,

$$\begin{aligned} f_\gamma(\nabla^\vee(e_1)) &= f_\gamma(1 \otimes e_1 + e_1 \otimes 1) \\ &= 1 \otimes (\gamma^{-1} \cdot e_1 \cdot \gamma) + (\gamma^{-1} \cdot e_1 \cdot \gamma) \otimes 1. \end{aligned}$$

On the other hand, using that γ is group-like, we get

$$\begin{aligned} \nabla^\vee(f_\gamma(e_1)) &= \nabla^\vee(\gamma^{-1} \cdot e_1 \cdot \gamma) \\ &= \gamma^{-1} \otimes \gamma^{-1} \cdot (1 \otimes e_1 + e_1 \otimes 1) \cdot \gamma \otimes \gamma \\ &= 1 \otimes (\gamma^{-1} \cdot e_1 \cdot \gamma) + (\gamma^{-1} \cdot e_1 \cdot \gamma) \otimes 1. \end{aligned}$$

The fact that f_γ is compatible with the dual antipode follows from $S^\vee(\gamma) = \gamma^{-1}$, which holds by Lemma 3.236 since γ is group-like.

To sum up, f_γ determines an element of $\mathrm{Aut}({}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}})(R)$. Writing

$$f_\gamma({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \gamma, \qquad f_\gamma({}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}) = \gamma^{-1} \cdot {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}$$

and using the identities (3.416), we obtain R-automorphisms of the schemes ${}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}}$ for $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$. By construction, these automorphisms are compatible with the composition of paths. Moreover, they are compatible with the identity automorphism of $\mathbb{G}_a$ through any of the two local monodromies. They thus define an element $f_\gamma \in \mathrm{Aut}^0(D^{\mathrm{dR}})(R)$.

Clearly, the assignments $f \mapsto \gamma_f$ and $\gamma \mapsto f_\gamma$ are inverse to each other, and this concludes the proof of the lemma. □

3.10.4. *A new product structure.* The isomorphism of schemes of Lemma 3.415 is not a morphism of groups, and hence induces a new group structure on $\mathrm{Spec}(\mathfrak{H})$.

Definition 3.418. We denote by $(\Pi, \circ)$ the scheme $\Pi = \mathrm{Spec}(\mathfrak{H})$ with the product structure induced by the isomorphism from Lemma 3.415.

This new product structure on the scheme $\Pi = \Pi^{\mathrm{dR}} = {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}$ gives rise to

- a new Lie bracket on the Lie algebra of Π, *i.e.* the set of Lie-like elements of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$, that is called the *Ihara bracket*;
- a new coproduct on $\mathfrak{H} = \mathbb{Q}\langle x_0, x_1\rangle$ that is called the *Goncharov coproduct*.

We now make all these structures explicit. We start by computing the new product $\circ$ of Π. This product is determined by the equation

$$f_\gamma(f_\mu({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}})) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot (\gamma \circ \mu).$$

Given a group-like element γ, we write $\langle\gamma\rangle_0$ for the restriction of f_γ to ${}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}$, as well as for the corresponding continuous automorphism of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. According to equation (3.417), it is given by

$$\langle\gamma\rangle_0(e_0) = e_0, \qquad \langle\gamma\rangle_0(e_1) = \gamma^{-1} \cdot e_1 \cdot \gamma. \tag{3.419}$$

Since f_γ is compatible with the composition of paths, we get

$$f_\gamma(f_\mu({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}})) = f_\gamma({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \mu) = f_\gamma({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}) \cdot f_\gamma(\mu) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \gamma \cdot \langle\gamma\rangle_0(\mu),$$

and hence the equality

$$\gamma \circ \mu = \gamma \cdot \langle \gamma \rangle_0(\mu). \tag{3.420}$$

3.10.5. *The Ihara bracket.* We now compute the new bracket induced on the set of primitive elements of $\mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$. Recall the notion of derivation from Definition 2.81. Given a primitive element $x \in \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$, consider the derivation

$$\begin{aligned} \partial_x \colon \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle &\longrightarrow \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle \\ y &\longmapsto \partial_x(y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle \exp(tx) \rangle_0(y) \Big)\Big|_{t=0}. \end{aligned} \tag{3.421}$$

Explicitly, this derivation is determined by

$$\partial_x e_0 = 0, \qquad \partial_x e_1 = -x \cdot e_1 + e_1 \cdot x,$$

and the continuity of the map ∂_x, which allows for its computation term by term.

Let x and y be primitive elements of $\mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$. We denote by

$$[x, y] = x \cdot y - y \cdot x$$

the Lie bracket corresponding to the composition of paths. The Lie bracket induced by $\circ$ will be denoted by $\{x, y\}$. It is determined by the equality

$$\{x, y\} = \frac{\mathrm{d}}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}v}\big(\exp(ux) \circ \exp(vy) \circ \exp(-ux) \circ \exp(-vy) \big)\Big|_{\substack{u=0 \\ v=0}}.$$

Explicitly, it is given by the formula

$$\{x, y\} = [x, y] + \partial_x y - \partial_y x. \tag{3.422}$$

3.10.6. *The Goncharov coproduct.* We now turn to the computation of the coproduct on the algebra $\mathfrak{H} = \mathbb{Q}\langle x_0, x_1 \rangle$. Following Notation 3.385, if α is a binary sequence, we will write $x_\alpha \in \mathfrak{H}$ for the corresponding word in the alphabet $\{x_0, x_1\}$, and $e_\alpha \in \mathfrak{H}^\vee = \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$ for the corresponding word in the alphabet $\{e_0, e_1\}$. Regarded as a function $x_\alpha \in \mathfrak{H} = \mathcal{O}(\Pi)$, the word x_α sends a group-like element of $\mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$ to the coefficient of the word e_α.

By Lemma 3.236 and Example 3.74, the dual antipode of a group-like element γ is given by $S^\vee(\gamma) = \gamma^{-1}$, and that of a word $w = e_{\varepsilon_1} \dots e_{\varepsilon_n}$ by

$$S^\vee(w) = w^* = (-1)^n e_{\varepsilon_n} \dots e_{\varepsilon_1}.$$

We deduce that, if $\gamma = \sum_w \gamma_w w$ is a group-like element, then

$$\gamma^{-1} = \sum_w \gamma_w w^*. \tag{3.423}$$

The *Goncharov coproduct* Δ^Γ is the coproduct induced in $\mathfrak{H}$ by the product $\circ$ and is determined by the equation

$$\Delta^\Gamma(x)(\gamma \otimes \mu) = x(\gamma \circ \mu) = x(\gamma \cdot \langle \gamma \rangle_0(\mu)). \tag{3.424}$$

Note that the product $\circ$ can be defined, for a group-like element γ and an arbitrary element $e \in \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$ by

$$\gamma \circ e = \gamma \cdot \langle \gamma \rangle_0(e). \tag{3.425}$$

This product is linear in the variable e. Using the explicit description (3.419) we see that, for a word w in the alphabet $\{e_0, e_1\}$, the product $\gamma \circ w$ is described as follows:

i) if the word w starts with e_0, then add γ at the beginning, while if the word starts with e_1, then do not add anything at the beginning;
ii) if the word ends with e_1, then add γ at the end, while if the word ends with e_0, then do not add anything at the end;
iii) between e_0 and e_1, insert γ^{-1}, and between e_1 and e_0, insert γ;
iv) between two consecutive occurrences of e_0 or two consecutive occurrences of e_1, do not insert anything.

For instance, this procedure yields

$$\gamma \circ (e_0e_0e_1e_0e_1e_1) = \gamma e_0e_0\gamma^{-1}e_1\gamma e_0\gamma^{-1}e_1e_1\gamma.$$

To give a more compact description of this product, we introduce the notation

$${}_1\gamma_0 = \gamma, \qquad {}_0\gamma_1 = \gamma^{-1}, \qquad {}_0\gamma_0 = 1, \qquad {}_1\gamma_1 = 1.$$

For a binary sequence $\alpha = (\varepsilon_1, \dots, \varepsilon_n)$, we then have

$$\gamma \circ e_\alpha = {}_1\gamma_{\varepsilon_1} \cdot e_{\varepsilon_1} \cdot {}_{\varepsilon_1}\gamma_{\varepsilon_2} \cdot e_{\varepsilon_2} \cdots e_{\varepsilon_n} \cdot {}_{\varepsilon_n}\gamma_0. \tag{3.426}$$

Given the shape (3.426) of the product $\circ$ and the inversion formula (3.423), for any binary sequence α, we introduce the following symbols:

$$\begin{aligned} &I(1;\alpha;0) = x_\alpha, \\ &I(0;\alpha;1) = x_\alpha^*, \\ &I(0;\alpha;0) = I(1;\alpha;1) = \begin{cases} 1, & \text{if } \alpha = \emptyset, \\ 0, & \text{if } \alpha \neq \emptyset. \end{cases} \end{aligned} \tag{3.427}$$

All of them are elements of $\mathfrak{H}$, and hence functions on Π. For a binary sequence α, a group-like element $\gamma \in \Pi(\mathbb{Q})$, and elements $\varepsilon, \varepsilon' \in \{0,1\}$, we have the duality

$$x_\alpha({}_{\varepsilon'}\gamma_\varepsilon) = I(\varepsilon';\alpha;\varepsilon)(\gamma). \tag{3.428}$$

Armed with this notation, we can give an explicit formula for Goncharov's coproduct. Let α be a binary sequence, let γ and μ be group-like elements of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. Write $\mu = \sum_w \mu_w w$. Then, from equation (3.424), we derive

$$\begin{aligned} (\Delta^\Gamma x_\alpha)(\gamma \otimes \mu) &= x_\alpha(\gamma \circ \mu) \\ &= x_\alpha(\sum_w \mu_w \gamma \circ w) \\ &= x_\alpha\big[\sum_w \mu_w\big({}_1\gamma_{\varepsilon_1(w)} \cdot e_{\varepsilon_1(w)} \cdots e_{\varepsilon_{\mathrm{wt}(w)}} \cdot {}_{\varepsilon_{\mathrm{wt}(w)}(w)}\gamma_0\big)\big], \end{aligned}$$

where $\mathrm{wt}(w)$ denotes the weight of w as in Definition 1.132, and $\varepsilon_i(w)$ is defined to be 0 or 1 depending wether the i-th letter appearing in w is e_0 or e_1. Let us write $\alpha = \varepsilon_1 \cdots \varepsilon_n$, and set $\varepsilon_0 = 1$ and $\varepsilon_{n+1} = 0$.

We need to compute the coefficient of the word e_α in the above bracketed expression. We will get a contribution for each subword of e_α corresponding to a binary subsequence $\varepsilon_{i_1} \cdots \varepsilon_{i_k}$ of α. It is easy to see that the coefficient we are looking for is given by:

$$\sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=n+1}} I(\varepsilon_0;\varepsilon_{i_1}\cdots\varepsilon_{i_k};\varepsilon_{n+1})(\mu)\prod_{p=0}^{k} I(\varepsilon_{i_p};\varepsilon_{i_p+1}\cdots\varepsilon_{i_{p+1}-1};\varepsilon_{i_{p+1}})(\gamma).$$

The upshot of these computations is the following result, which was first obtained by Goncharov [Gon05, Thm. 1.2].

PROPOSITION 3.429. *Let $\varepsilon_0 \cdots \varepsilon_{n+1}$ be a binary sequence. By transport of structure, the isomorphism from Lemma 3.415 induces the coproduct*

$$\Delta^\Gamma I(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_n; \varepsilon_{n+1}) = \sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=n+1}} \prod_{p=0}^{k} I(\varepsilon_{i_p}; \varepsilon_{i_p+1} \cdots \varepsilon_{i_{p+1}-1}; \varepsilon_{i_{p+1}}) \otimes I(\varepsilon_0; \varepsilon_{i_1} \cdots \varepsilon_{i_k}; \varepsilon_{n+1})$$

on the Hoffman algebra $\mathfrak{H}$.

PROOF. The case $\varepsilon_0 = 1$ and $\varepsilon_{n+1} = 0$ was settled above. The other cases follow immediately from (3.427). □

EXAMPLE 3.430. For $n = 1$, the formula specializes to

$$\begin{aligned}\Delta^\Gamma I(\varepsilon_0;\varepsilon_1;\varepsilon_2) &= I(\varepsilon_0;\varepsilon_1;\varepsilon_2) \otimes I(\varepsilon_0;\varepsilon_2) + I(\varepsilon_0;\varepsilon_1)I(\varepsilon_1;\varepsilon_2) \otimes I(\varepsilon_0;\varepsilon_1;\varepsilon_2) \\ &= I(\varepsilon_0;\varepsilon_1;\varepsilon_2) \otimes 1 + 1 \otimes I(\varepsilon_0;\varepsilon_1;\varepsilon_2).\end{aligned}$$

Indeed, $I(\varepsilon';\varepsilon)$ is always equal to 1 regardless of the values of ε and ε'.

EXAMPLE 3.431. For $n = 2$, we get contributions from $k = 0, 1, 2$. As before, the term indexed by $k = 0$ corresponds to the choice of the empty subsequence and gives the value $I(\varepsilon_0;\varepsilon_1\varepsilon_2;\varepsilon_3) \otimes 1$, whereas $k = 2$ represents the choice of the whole sequence and contributes with $1 \otimes I(\varepsilon_0;\varepsilon_1\varepsilon_2;\varepsilon_3)$. For $k = 1$, we obtain two terms, corresponding to $i_1 = 1$ and $i_1 = 2$. In both cases, the product contains only one non-trivial factor ($p = 1$ if $i_1 = 1$ and $p = 0$ if $i_1 = 2$). Putting everything together, the expression for the coproduct becomes

$$\begin{aligned}\Delta^\Gamma I(\varepsilon_0;\varepsilon_1\varepsilon_2;\varepsilon_3) = I(\varepsilon_0;\varepsilon_1\varepsilon_2;\varepsilon_3) &\otimes 1 \\ + I(\varepsilon_1;\varepsilon_2;\varepsilon_3) &\otimes I(\varepsilon_0;\varepsilon_1;\varepsilon_3) \\ + I(\varepsilon_0;\varepsilon_1;\varepsilon_2) &\otimes I(\varepsilon_0;\varepsilon_2;\varepsilon_3) \\ + 1 &\otimes I(\varepsilon_0;\varepsilon_1\varepsilon_2;\varepsilon_3).\end{aligned} \tag{3.432}$$

Specializing formula (3.432) to the cases $(1;1,0;0)$ and $(1;0,1;0)$, we get

$$\begin{aligned}\Delta^\Gamma(x_0x_1) &= x_0x_1 \otimes 1 + x_0 \otimes x_1 + x_1 \otimes x_0 + 1 \otimes x_0x_1, \\ \Delta^\Gamma(x_1x_0) &= x_1x_0 \otimes 1 + 1 \otimes x_1x_0.\end{aligned}$$

Just for fun, let us verify the compatibility with shuffle product. On the one hand,

$$\begin{aligned}\Delta^\Gamma&(x_0 \sqcup\!\sqcup\, x_1) \\ &= \Delta^\Gamma(x_0x_1 + x_1x_0) \\ &= (x_0x_1 + x_1x_0) \otimes 1 + 1 \otimes (x_0x_1 + x_1x_0) + x_0 \otimes x_1 + x_1 \otimes x_0.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\Delta^\Gamma x_1) &\sqcup\!\sqcup\, (\Delta^\Gamma x_2) \\ &= (1 \otimes x_0 + x_0 \otimes 1) \sqcup\!\sqcup\, (1 \otimes x_1 + x_1 \otimes 1) \\ &= 1 \otimes (x_0 \sqcup\!\sqcup\, x_1) + x_0 \otimes x_1 + x_0 \otimes x_1 + (x_0 \sqcup\!\sqcup\, x_1) \otimes 1,\end{aligned}$$

and we see that the expressions are equal.

As the previous examples show, the formula for Goncharov's coproduct in Proposition 3.429 contains many trivial factors. Later in Chapter 5 we will give a linearization which is more suitable for computation.

⋆⋆⋆

Exercise 3.433. Prove formula (3.422).

Exercise 3.434. Calculate the number of terms appearing in Goncharov's coproduct.

Exercise 3.435. The goal of this exercise is to prove that the group $\mathrm{Aut}^0(D^{\mathrm{dR}})$ is pro-unipotent.

i) Let $\circ$ denote the product on $\mathfrak{H}^\vee = \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ induced by the Goncharov coproduct on $\mathfrak{H}$. Prove the inequality
$$\mathrm{length}(a \circ b) \geqslant \mathrm{length}(a) + \mathrm{length}(b).$$
ii) Let C^i be the conilpotency filtration on the algebra $\mathfrak{H}$. Prove that C^i contains all the monomials of length less than i.
iii) Use Proposition 3.158 to conclude that $\mathrm{Aut}^0(D^{\mathrm{dR}})$ is pro-unipotent.

Exercise 3.436. Prove that the derivation ∂_x from (3.421) is continuous with respect to the natural topology on $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$.

4. Mixed Tate motives

The ultimate goal of this chapter is to give a precise meaning to the statement that the diagram D_U^H from Definition 3.414 has motivic origin. This will keep us busy for a while. We start with the definition of tannakian category in Section 4.1. This notion is an abstraction of the properties of the category of representations of an affine group scheme endowed with the forgetful functor to vector spaces. The main theorem of the theory is then that each tannakian category is indeed equivalent to the category of representations of an affine group scheme. This is very useful in cases where this is not obvious a priori, for it allows one to see objects as representations and translate categorical properties into properties of the group. One of Grothendieck's motivations to develop the theory of tannakian categories was to study motives, which were envisioned by him as a universal cohomology theory for algebraic varieties. Ideally, a category of motives should be a tannakian category equipped with a Betti realization functor in which Hom groups are described in terms of algebraic cycles. In Section 4.2, we sketch the construction of Voevodsky's triangulated category of motives, which is a candidate for the derived category of motives. We explain how to compute Hom groups in this category in terms of K-theory, and how the Hodge realization to the derived category of mixed Hodge structures works. It is expected that there exists a t-structure on $\mathbf{DM}(k)$ whose heart yields the sought after category of motives. As we explain in Section 4.3, at the time of writing, it is only known how to do that when k is a number field and $\mathbf{DM}(k)$ is replaced by the subcategory consisting of iterated extensions of the simplest objects $\mathbb{Q}(n)$. This yields the abelian category of mixed Tate motives over k. The construction relies on Borel's computation of the K-theory of number fields. We introduce the t-structure constructed by Levine and prove that the Hodge realization functor is fully faithful. The rest of the chapter deals with $k = \mathbb{Q}$. For certain purposes, the category $\mathbf{MT}(\mathbb{Q})$ is still too big. In Section 4.4, we introduce a subcategory $\mathbf{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$, which has the advantage that all extension groups are finite-dimensional. It is a tannakian category endowed with two fiber functors: Betti and de Rham. We determine the structure of the associated tannakian group: it is a semidirect product of $\mathbb{G}_m$ and a pro-unipotent group whose graded Lie algebra is free with one generator in each negative odd degree $\leqslant -3$. We also explain the torsor of motivic periods. In Section 4.5, we construct after Deligne and Goncharov a pro-mixed Tate motive over $\mathbb{Z}$ that contains all multiple zeta values among its periods and whose Hodge realization is the fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ from Chapter 3. As a pro-object of the category $\mathbf{MT}(\mathbb{Z})$, this motivic fundamental groupoid is acted upon by the tannakian fundamental group. The structure of this representation will yield the additional structures that are used in the proofs of Theorems A and B about multiple zeta values. A remarkable side effect is that this object generates the whole category.

4.1. Tannakian formalism.

The link between structural properties of mixed Tate motives over $\mathbb{Z}$ and numerical properties of multiple zeta values is made through the "group of symmetries" of the former. To make this idea precise, we will need the formalism of tannakian categories that we summarize in this section.

4.1.1. *Motivation.* One of the major inspirations for the theory of tannakian categories is the Tannaka–Krein *reconstruction theorem*, which roughly says that a compact topological group can be recovered from the category of its continuous finite-dimensional complex representations. We begin this motivational section by a brief discussion of this theorem, following the presentation in [JS91].

Let G be a locally compact abelian topological group. A *unitary character* on G is a continuous homomorphism $\chi\colon G \to S^1$. The set $G^\vee$ of unitary characters forms an abelian group; endowed with the topology of uniform convergence, it is a locally compact topological group. For example, the unitary characters on S^1 are all of the form $z \mapsto z^n$ for some integer n, and $(S^1)^\vee$ is isomorphic to $\mathbb{Z}$ with the discrete topology. It is a general feature that $G^\vee$ is discrete if and only if G is compact. *Pontryagin's duality* is the theorem that the map

$$\begin{aligned} G &\longrightarrow \quad (G^\vee)^\vee \\ g &\longmapsto \big(\chi \mapsto \chi(g)\big) \end{aligned}$$

is an isomorphism of topological groups. In particular, G can be reconstructed from the unitary characters on $G^\vee$.

In case G is not abelian, characters are not enough to recover the group, and one needs to bring all representations into play. A completely satisfactory theory only seems to exist for compact groups. Let G be a compact topological group, and let $\mathbf{Rep}_{\mathbb{C}}(G)$ be the category of complex linear continuous representations of G. Its objects are pairs (V, π_V) consisting of a finite-dimensional $\mathbb{C}$-vector space V and a continuous homomorphism $\pi_V\colon G \to \mathrm{GL}(V)$. Consider the forgetful functor

$$\begin{aligned} \omega\colon\ \mathbf{Rep}_{\mathbb{C}}(G) &\longrightarrow \mathbf{Vec}_{\mathbb{C}}\,. \\ (V, \pi_V) &\longmapsto \quad V \end{aligned}$$

We define the endomorphisms of ω as the set $\mathrm{End}(\omega)$ of families

$$\lambda = (\lambda_V)_{V \in \mathrm{Ob}(\mathbf{Rep}_{\mathbb{C}}(G))}$$

of $\mathbb{C}$-linear maps $\lambda_V\colon V \to V$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\lambda_V"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ W \arrow[r, "\lambda_W"] & W \end{tikzcd}$$

commutes for each morphism $f\colon V \to W$ in the category $\mathbf{Rep}_{\mathbb{C}}(G)$, that is, for each linear map satisfying $f(\pi_V(g)) = \pi_W(f(g))$ for all $g \in G$. For example, the very definition of a morphism of representations amounts to saying that the family

$$\pi(g) = (\pi_V(g))_{V \in \mathrm{Ob}(\mathbf{Rep}_{\mathbb{C}}(G))}$$

belongs to $\mathrm{End}(\omega)$ for all $g \in G$, hence a map

$$\pi\colon G \longrightarrow \mathrm{End}(\omega).$$

We endow $\mathrm{End}(\omega)$ with the coarsest topology making all projections $\lambda \mapsto \lambda_V$ from $\mathrm{End}(\omega)$ to $\mathrm{End}(V)$ continuous.

One of the crucial insights of the theory is that the image of G inside $\mathrm{End}(\omega)$ can be characterized using the tensor product of representations. Namely, one introduces the subset $\mathcal{T}(G) \subset \mathrm{End}(\omega)$ of those endomorphisms satisfying the following:

i) $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$;

ii) $\lambda_{\mathbf{1}} = \mathrm{Id}_{\mathbf{1}}$, where $\mathbf{1}$ is the one-dimensional trivial representation;

iii) $\lambda = \overline{\lambda}$, where $\overline{\lambda} = (\overline{\lambda}_V)$ is defined in terms of the conjugate vector space $\overline{V}$ and the antilinear "identity" map $x \mapsto \overline{x}$ as $\overline{\lambda}_V(x) = \overline{\lambda_{\overline{V}}(\overline{x})}$.

In contrast with the whole $\mathrm{End}(\omega)$, this subset $\mathcal{T}(G)$ is a topological *group* (each endomorphism λ_V is invertible thanks to the existence of the dual representation $V^\vee$ and the compatibilities i) and ii); see [**JS91**, Prop. 3]). It is even compact (one proves that each λ_V preserves a positive definite hermitian form by using iii) as well; see [**JS91**, Prop. 6]). The map π takes values in $\mathcal{T}(G)$.

Theorem 4.1 (Tannaka–Krein). *The map*

$$\begin{aligned} \pi\colon\ G &\longrightarrow \mathcal{T}(G) \\ g &\longmapsto (\pi_V(g))_{V\in \mathrm{Ob}(\mathbf{Rep}_{\mathbb{C}}(G))} \end{aligned}$$

is an isomorphism of topological groups.

The proof relies on two incarnations of the Peter–Weyl theorem for compact groups [**Kow14**, §5.4]. The first one is the fact that finite-dimensional complex representations "separate points": for each $g \in G\backslash\{e\}$, there exists an object (V, π_V) of $\mathbf{Rep}_{\mathbb{C}}(G)$ such that $\pi_V(g)$ is not the identity Id_V; this immediately implies that the map π is injective. The second one is the fact that every continuous function on a compact group G can be uniformly approached by *matrix coefficients*, that is, by complex linear combinations of functions of the form

$$\begin{aligned} G &\longrightarrow \mathbb{C} \\ g &\longmapsto f(\pi_V(g)(v)) \end{aligned}$$

where (V, π_V) is a finite-dimensional complex representation, $v \in V$ is a vector, and $f \in V^\vee = \mathrm{Hom}(V, k)$ is a linear form. This is the main tool in proving the equality

$$\int_{\mathcal{T}(G)} \varphi(u)du = \int_G \varphi(\pi(g))dg$$

for all continuous functions $\varphi\colon \mathcal{T}(G) \to \mathbb{C}$, and hence the surjectivity of π since otherwise one could pick a function with support on $\mathcal{T}(G)\backslash G$ and non-zero integral that will contradict the above; see [**JS91**, Thm. 20]. The theory of tannakian categories plays the same role for affine group schemes over an arbitrary field instead of compact topological groups. As we will see, one key idea is to reconstruct the Hopf algebra of regular functions on the group from matrix coefficients.

Another major inspiration of the theory of tannakian categories is Grothendieck's approach to Galois theory in terms of étale algebras. Let k be a field and let k^{sep} be a separable closure of k. A k-algebra F of finite degree over k is said to be *étale* if F is isomorphic to a product of finite separable field extensions of k; one then says that $Y = \mathrm{Spec}(F)$ is a finite étale k-scheme. Then the k^{sep}-points of Y form a finite set $Y(k^{\mathrm{sep}})$ of cardinality $[F\colon k]$, on which the absolute Galois group

$$\mathrm{Gal}(k^{\mathrm{sep}}/k) = \{\sigma \in \mathrm{Aut}(k^{\mathrm{sep}}) \mid \sigma|_k = \mathrm{Id}\} = \varprojlim_{\substack{k\subset L\subset k^{\mathrm{sep}} \\ \text{finite Galois}}} \mathrm{Gal}(L/k)$$

acts continuously, *i.e.* the action factors though a finite quotient of this profinite group. Grothendieck noticed that the Galois group can be recovered as the group of automorphisms of the functor

$$\begin{aligned} \{\text{finite étale } k\text{-schemes}\} &\longrightarrow \{\text{finite sets}\}. \\ Y &\longmapsto Y(k^{\mathrm{sep}}) \end{aligned} \tag{4.2}$$

That is, $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ is canonically isomorphic to the group of families $\sigma = (\sigma_Y)_Y$, where Y runs through finite étale k-schemes and σ_Y is a permutation of $Y(k^{\mathrm{sep}})$, which are compatible with all morphisms of finite étale k-schemes, and the functor (4.2) induces an equivalence of categories

$$\{\text{finite étale } k\text{-schemes}\} \longrightarrow \left\{ \begin{array}{c} \text{finite sets with a continuous} \\ \text{action of } \mathrm{Gal}(k^{\mathrm{sep}}/k) \end{array} \right\}.$$

Several natural generalizations arise from this point of view. First, by replacing finite étale k-schemes with finite étale covers of a general base scheme X and (4.2) with the fibre functor

$$\begin{aligned} \mathrm{Fib}_{\bar{x}} \colon \{\text{finite étale covers of } X\} &\longrightarrow \{\text{finite sets}\} \\ (Y \to X) &\longmapsto Y_{\bar{x}} = Y \times_X \mathrm{Spec}(\Omega) \end{aligned}$$

associated with a geometric point $\bar{x} \colon \mathrm{Spec}(\Omega) \to X$, Grothendieck defines the *étale fundamental group* $\pi_1^{\text{ét}}(X, \bar{x})$ as the automorphism group of $\mathrm{Fib}_{\bar{x}}$. It is again a profinite group, which in case we start with an algebraic variety X over the field of complex numbers, turns out to be isomorphic to the profinite completion of the topological fundamental group of its complex points:

$$\pi_1^{\text{ét}}(X, x) \cong \varprojlim_{\substack{N \triangleleft G \\ \text{finite index}}} \pi_1^{\text{top}}(X(\mathbb{C}), x)/N.$$

Another natural generalization is to work with finite-dimensional $\mathbb{Q}$-vector spaces instead of finite sets, that is, to consider the linearization functor

$$\left\{ \begin{array}{c} \text{finite sets with a} \\ \text{continuous action of} \\ \mathrm{Gal}(k^{\mathrm{sep}}/k) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{finite-dimensional} \\ \mathbb{Q}\text{-vector spaces with a} \\ \text{continuous action of} \\ \mathrm{Gal}(k^{\mathrm{sep}}/k) \end{array} \right\}.$$

that maps a $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-set S to the vector space $\mathbb{Q}^S$ of functions $f \colon S \to \mathbb{Q}$ together with the action $(g \cdot f)(s) = f(g^{-1}s)$. The right-hand side is now equipped with a tensor product, and the Galois group can be recovered as the group of automorphisms of the forgetful functor

$$\left\{ \begin{array}{c} \text{finite-dimensional } \mathbb{Q}\text{-vector} \\ \text{spaces with a continuous action} \\ \text{of } \mathrm{Gal}(k^{\mathrm{sep}}/k) \end{array} \right\} \longrightarrow \mathbf{Vec}_{\mathbb{Q}}$$

that are compatible with this tensor product. In Grothendieck's vision, this is the category of *Artin motives*, a universal cohomology theory for algebraic varieties of dimension *zero* (note that, for $k \subset \mathbb{C}$, the vector space $\mathbb{Q}^{Y(\bar{k})}$ is nothing but the Betti cohomology $\mathrm{H}^0_{\mathrm{B}}(Y)$). His goal was then to construct a category of motives for all smooth projective varieties and prove that it is equivalent to the category of finite-dimensional representations of an affine group scheme over $\mathbb{Q}$: the *motivic Galois group*. In order to realize this program, his student Saavedra-Rivano [**SR72**] and later Deligne [**Del90**] gave an abstract characterization of the categories of representations of affine group schemes that allows one to recognize them among all tensor categories; this is the notion of tannakian category.

We will mainly follow the exposition in [**DM82**], to which the reader is referred for further details. Throughout this section, we fix a field k (of any characteristic unless explicitly mentioned), that will play the role of field of coefficients.

4.1.2. *Tensor categories.* The definition of a tannakian category gathers the characteristic properties of the category $\mathbf{Rep}_k(G)$ of finite-dimensional k-linear representations of an affine group scheme G, as introduced in Section 3.2.13. First of all, since morphisms between k-linear representations form a k-vector space, we need the concept of a k-linear category. Recall the notion of additive category and additive functor from Definitions A.1 and A.2 of the appendix.

DEFINITION 4.3. A *k-linear category* $\mathcal{C}$ is an additive category in which the abelian groups $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are endowed with the structure of a k-vector space for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, in such a way that the composition maps

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

are bilinear for all objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

Let $\mathcal{C}$ and $\mathcal{C}'$ be k-linear categories. A *k-linear functor* $F\colon \mathcal{C} \to \mathcal{C}'$ is an additive functor such that the induced map on morphisms

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(F(X),F(Y))$$

is k-linear for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$.

The tensor product of two k-linear representations carries a “diagonal” action of G, making it into another k-linear representation. Therefore, a tannakian category should be endowed with some tensor product, which is a special kind of bilinear functor. To explain bilinear functors, we begin with the product category $\mathcal{C} \times \mathcal{C}$, which has objects $\mathrm{Ob}(\mathcal{C} \times \mathcal{C}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$, and morphisms

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{C}}((X,Y),(X',Y')) = \mathrm{Hom}_{\mathcal{C}}(X,X') \times \mathrm{Hom}_{\mathcal{C}}(Y,Y'). \tag{4.4}$$

DEFINITION 4.5. Let $\mathcal{C}$ be a k-linear category. A functor $F\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is said to be *bilinear* if the induced map on morphisms

$$\mathrm{Hom}_{\mathcal{C}}(X,X') \times \mathrm{Hom}_{\mathcal{C}}(Y,Y') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(F(X,Y),F(X',Y'))$$

is a bilinear map of k-vector spaces.

We now introduce an abstract version of the usual associativity and commutativity of the tensor product of vector spaces.

DEFINITION 4.6. Let $\mathcal{C}$ be a k-linear category, together with a bilinear functor

$$\otimes\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}.$$

i) An *associativity constraint* for $(\mathcal{C}, \otimes)$ is a natural transformation

$$\phi = \phi_{\cdot,\cdot,\cdot}\colon \ \cdot \otimes (\cdot \otimes \cdot) \longrightarrow (\cdot \otimes \cdot) \otimes \cdot$$

of functors from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ such that:

a) For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the map $\phi_{X,Y,Z}$ is an isomorphism.

b) (*Pentagon axiom*) For all $X, Y, Z, T \in \mathrm{Ob}(\mathcal{C})$, the following diagram commutes:

$$\begin{tikzcd}
& & X \otimes (Y \otimes (Z \otimes T)) \arrow[dll, "\mathrm{Id} \otimes \phi_{Y,Z,T}"'] \arrow[drr, "\phi_{X,Y,Z\otimes T}"] & & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[dr, "\phi_{X,Y\otimes Z,T}"'] & & & & (X \otimes Y) \otimes (Z \otimes T). \arrow[dl, "\phi_{X\otimes Y,Z,T}"] \\
& (X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\phi_{X,Y,Z}\otimes \mathrm{Id}"'] & & ((X \otimes Y) \otimes Z) \otimes T &
\end{tikzcd}$$

ii) A *commutativity constraint* is a natural transformation

$$\psi = \psi_{\cdot,*} \colon \ \cdot \otimes * \longrightarrow * \otimes \cdot$$

of functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ such that:

a) For all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the map $\psi_{X,Y}$ is an isomorphism.

b) The composition

$$\psi_{Y,X} \circ \psi_{X,Y} \colon X \otimes Y \longrightarrow X \otimes Y$$

is equal to the identity.

iii) (*Hexagon axiom*) An associativity and a commutativity constraint are said to be *compatible* with each other if the diagram

$$\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "\phi_{X,Y,Z}"] \arrow[dl, "\mathrm{Id} \otimes \psi_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[dr, "\psi_{X\otimes Y,Z}"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\phi_{X,Z,Y}"'] & & & Z \otimes (X \otimes Y) \arrow[dl, "\phi_{Z,X,Y}"] \\
& (X \otimes Z) \otimes Y \arrow[r, "\psi_{X,Z}\otimes \mathrm{Id}"'] & (Z \otimes X) \otimes Y &
\end{tikzcd}$$

commutes for all objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

iv) Finally, we say that a pair (U, u) consisting of an object $U \in \mathrm{Ob}(\mathcal{C})$ and an isomorphism $u \colon U \to U \otimes U$ is an *identity object* if the functors from $\mathcal{C}$ to $\mathcal{C}$ given on objects by

$$X \longmapsto U \otimes X \quad \text{and} \quad X \longmapsto X \otimes U$$

and on morphisms by tensoring with Id_U are equivalences of categories.

All the ingredients needed to define tensor categories, which are one of the underlying structures of tannakian categories, have now been introduced.

Definition 4.7. A *k-linear tensor category* is a quadruple

$$(\mathcal{C}, \otimes, \phi, \psi)$$

consisting of a k-linear category $\mathcal{C}$, a bilinear functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and compatible associativity and commutativity constraints ϕ and ψ, such that $\mathcal{C}$ contains an identity object. The constraints ϕ and ψ are usually omitted from the notation, and one simply writes $(\mathcal{C}, \otimes)$ for a k-linear tensor category.

Remark 4.8 (Extending the tensor product). The significance of the pentagon axiom in the definition of an associativity constraint is that it allows for an essentially unique extension of the tensor product $\otimes$ to a functor

$$\otimes\colon \underbrace{\mathcal{C}\times\cdots\times\mathcal{C}}_{n}\longrightarrow\mathcal{C}$$

for each $n\geqslant 1$; see [**DM82**, Prop. 1.5] for a precise statement. In what follows, we will sometimes consider tensor products of more than three objects, and it will then be tacitly understood that such an extension has been chosen.

Remark 4.9 (Uniqueness of identity objects). Let (U,u) be an identity object. For each object $X\in\mathrm{Ob}(\mathcal{C})$, there exist canonical isomorphisms

$$\alpha_X^U\colon X\xrightarrow{\sim}X\otimes U\quad\text{and}\quad\beta_X^U\colon U\otimes X\xrightarrow{\sim}X$$

that are functorial in X. Indeed, since the functor $X\mapsto X\otimes U$ is an equivalence of categories by definition of an identity object, the map

$$\mathrm{Hom}_{\mathcal{C}}(X,X\otimes U)\longrightarrow\mathrm{Hom}_{\mathcal{C}}(X\otimes U,(X\otimes U)\otimes U)$$

is a bijection. Using the associativity constraint, one gets a bijection

$$\mathrm{Hom}_{\mathcal{C}}(X,X\otimes U)\longrightarrow\mathrm{Hom}_{\mathcal{C}}(X\otimes U,X\otimes(U\otimes U)).$$

The isomorphism α_X^U is defined as the preimage of $\mathrm{Id}_X\otimes u$ under this map. In particular, α_U^U agrees with the morphism u which defines the identity object.

Similarly, β_X^U is the preimage of $u^{-1}\otimes\mathrm{Id}_X$ under the bijection

$$\mathrm{Hom}_{\mathcal{C}}(U\otimes X,X)\to\mathrm{Hom}_{\mathcal{C}}((U\otimes U)\otimes X,U\otimes X)$$

given by the equivalence of categories $X\mapsto U\otimes X$ and the associativity constraint. In particular, β_U^U agrees with u^{-1}.

From the existence of these isomorphisms, it follows that two identity objects of $\mathcal{C}$ are canonically isomorphic: given (U,u) and (U',u'), the morphism

$$\beta_{U'}^U\circ\alpha_U^{U'}\colon U\longrightarrow U'$$

is an isomorphism. In fact, it is the unique morphism $f\colon U\to U'$ making

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "u"'] & U' \arrow[d, "u'"] \\ U\otimes U \arrow[r, "f\otimes f"] & U'\otimes U' \end{tikzcd}$$

a commutative diagram.

From now on, we will fix an identity object and denote it by $(\mathbf{1},e)$.

Definition 4.10. Let $\mathcal{C}$ be a k-linear tensor category. An object $L\in\mathrm{Ob}(\mathcal{C})$ is called *invertible* if the functor from $\mathcal{C}$ to $\mathcal{C}$ given on objects by

$$X\longmapsto L\otimes X \tag{4.11}$$

and on morphisms by tensoring with Id_L is an equivalence of categories.

An object $L\in\mathrm{Ob}(\mathcal{C})$ is invertible if and only if there exists an object $L'\in\mathrm{Ob}(\mathcal{C})$ and an isomorphism $L\otimes L'\simeq\mathbf{1}$. Then, L' is also invertible. Indeed, if (4.11) is an equivalence of categories, then the identity object $\mathbf{1}$ belongs to its essential image, hence the existence of L' and $L\otimes L'\simeq\mathbf{1}$. Conversely, given such an object and an isomorphism, the functor $X\mapsto L'\otimes X$ is a quasi-inverse to (4.11).

DEFINITION 4.12. Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be k-linear tensor categories. We call k-linear *tensor functor* from $(\mathcal{C}, \otimes)$ to $(\mathcal{C}', \otimes')$ a pair (F, c) consisting of a k-linear functor $F \colon \mathcal{C} \to \mathcal{C}'$ and a natural transformation

$$c = c_{\cdot,\cdot} \colon F(\cdot) \otimes F(\cdot) \longrightarrow F(\cdot \otimes \cdot)$$

of functors from $\mathcal{C}$ to $\mathcal{C}'$ such that:

i) For all $X, Y \in \mathrm{Ob}(\mathcal{C})$, the map $c_{X,Y}$ is an isomorphism.

ii) (*Compatibility with associativity constraints*) If ϕ and ϕ' are the associativity constrains of $\mathcal{C}$ and $\mathcal{C}'$ respectively, then the diagram

$$\begin{tikzcd}[column sep=small]
& F(X) \otimes \big(F(Y) \otimes F(Z)\big) \arrow[rr, "\phi'"] \arrow[dl, "\mathrm{Id} \otimes c_{Y,Z}"'] & & \big(F(X) \otimes F(Y)\big) \otimes F(Z) \arrow[dr, "c_{X,Y} \otimes \mathrm{Id}"] & \\
F(X) \otimes F(Y \otimes Z) \arrow[dr, "c_{X, Y \otimes Z}"'] & & & & F(X \otimes Y) \otimes F(Z) \arrow[dl, "c_{X \otimes Y, Z}"] \\
& F\big(X \otimes (Y \otimes Z)\big) \arrow[rr, "F(\phi_{X,Y,Z})"'] & & F\big((X \otimes Y) \otimes Z\big) &
\end{tikzcd}$$

commutes for all objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

iii) (*Compatibility with commutativity constraints*) If ψ and ψ' are the commutativity constrains of $\mathcal{C}$ and $\mathcal{C}'$ respectively, then the diagram

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "c_{X,Y}"] \arrow[d, "\psi'_{F(X),F(Y)}"'] & F(X \otimes Y) \arrow[d, "F(\psi_{X,Y})"] \\
F(Y) \otimes F(X) \arrow[r, "c_{Y,X}"] & F(Y \otimes X)
\end{tikzcd}$$

commutes for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$.

iv) (*Compatibility with identity objects*) If (U, u) is an identity object of $\mathcal{C}$, then $(F(U), F(u))$ is an identity object of $\mathcal{C}'$.

4.1.3. *Rigid categories.* The vector space of k-linear maps between two representations carries a natural action of the group; in particular, a representation on a vector space induces a representation on the dual vector space. Thus, a tannakian category should contain *internal Hom* objects and *duals*, as we now define them.

DEFINITION 4.13. Let $(\mathcal{C}, \otimes)$ be a k-linear tensor category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. The functor $T \mapsto \mathrm{Hom}_{\mathcal{C}}(T \otimes X, Y)$ from $\mathcal{C}$ to the category of k-vector spaces is said to be *representable* if there exist an object $Z \in \mathrm{Ob}(\mathcal{C})$ and functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(T, Z) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(T \otimes X, Y)$$

for all $T \in \mathrm{Ob}(\mathcal{C})$. If this is the case, then we denote Z by $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ and we call it the *internal Hom* between the objects X and Y. Thus, there are isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(T, \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) \cong \mathrm{Hom}_{\mathcal{C}}(T \otimes X, Y). \tag{4.14}$$

Whenever it exists, the internal Hom object $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ is unique up to unique isomorphism, which makes it functorial in Y. In the language of adjoint functors (see Section A.1.4), the isomorphism (4.14) means that $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, -)$ is a right adjoint of $-\otimes X$. Plugging $T = \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ in (4.14), the image of the identity morphism $\mathrm{Id}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)}$ is a morphism which will be denoted by

$$\mathrm{ev}_{X,Y}\colon \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \otimes X \longrightarrow Y.$$

This is the value on Y of the general natural adjoint transformation $FG \to \mathrm{Id}$ associated with a pair of adjoint functors (F, G).

Example 4.15. The notation $\mathrm{ev}_{X,Y}$ is justified by the example of the category of k-vector spaces, in which the internal Hom is the k-vector space $\mathrm{Hom}_k(X, Y)$ of k-linear maps from X to Y, and $\mathrm{ev}_{X,Y}$ is the *evaluation map* given by

$$\mathrm{ev}_{X,Y}(f \otimes x) = f(x)$$

on a linear map $f\colon X \to Y$ and a vector $x \in X$.

Note that Yoneda's lemma implies that $\mathrm{ev}_{\mathbf{1},Y}$ is an isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathbf{1}, Y) \xrightarrow{\sim} Y \tag{4.16}$$

for all $Y \in \mathrm{Ob}(\mathcal{C})$, as both objects represent the functor $T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, Y)$ after taking the canonical isomorphism $T \otimes \mathbf{1} \cong T$ into account.

Definition 4.17. Let $(\mathcal{C}, \otimes)$ be a k-linear tensor category with identity object $\mathbf{1}$. When it exists, the *dual* $X^\vee$ of $X \in \mathrm{Ob}(\mathcal{C})$ is defined as the internal Hom

$$X^\vee = \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \mathbf{1}).$$

Therefore, there is an evaluation morphism

$$\mathrm{ev}_X\colon X^\vee \otimes X \longrightarrow \mathbf{1}. \tag{4.18}$$

If both $X^\vee$ and $(X^\vee)^\vee$ exist, then there are isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(X, (X^\vee)^\vee) &= \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}_{\mathcal{C}}(\underline{\mathrm{Hom}}_{\mathcal{C}}(X, \mathbf{1}), \mathbf{1})) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(X \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \mathbf{1}), \mathbf{1}) \\
&= \mathrm{Hom}_{\mathcal{C}}(X \otimes X^\vee, \mathbf{1})
\end{aligned}$$

thanks to the bijection (4.14). We thus obtain a natural morphism

$$X \longrightarrow (X^\vee)^\vee$$

from ev_X and the commutativity constraint. If $\mathcal{C}$ is the category of k-vector spaces, this is nothing but the usual map that sends $x \in X$ to the linear form $X^\vee \to k$ given by evaluating a linear form $\ell\colon X \to k$ at the vector x.

Definition 4.19. An object X is called *reflexive* if the morphism $X \to (X^\vee)^\vee$ is an isomorphism.

Example 4.20. In Section 4.1.4, we will see that a vector space is reflexive if and only if it is finite-dimensional.

We next introduce the concept of a rigid tensor category. As above, let $(\mathcal{C}, \otimes)$ be a k-linear tensor category. Assume that all internal Hom objects (Definition 4.13) exist, and consider the morphism

$$(\underline{\mathrm{Hom}}_{\mathcal{C}}(X_1, Y_1) \otimes X_1) \otimes (\underline{\mathrm{Hom}}_{\mathcal{C}}(X_2, Y_2) \otimes X_2) \xrightarrow{\mathrm{ev}_{X_1,Y_1} \otimes \mathrm{ev}_{X_2,Y_2}} Y_1 \otimes Y_2.$$

Upon identifying its source with

$$(\underline{\mathrm{Hom}}_{\mathcal{C}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X_2, Y_2)) \otimes (X_1 \otimes X_2)$$

through the associativity and commutativity constraints, this morphism corresponds by means of the bijection (4.14) to a morphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X_2, Y_2) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(X_1 \otimes X_2, Y_1 \otimes Y_2). \tag{4.21}$$

Definition 4.22. A k-linear tensor category $(\mathcal{C}, \otimes)$ is said to be *rigid* if the following three conditions hold:

i) the internal Hom object $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ exists for all $X, Y \in \mathrm{Ob}(\mathcal{C})$. In particular, the dual object $X^\vee$ exists for all objects $X \in \mathrm{Ob}(\mathcal{C})$;

ii) the map (4.21) is an isomorphism for all $X_1, X_2, Y_1, Y_2 \in \mathrm{Ob}(\mathcal{C})$;

iii) all objects of $\mathcal{C}$ are reflexive.

Remark 4.23. Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be objects of a rigid k-linear tensor category. Plugging $X_1 = X$, $Y_1 = X_2 = \mathbf{1}$, and $Y_2 = Y$ into (4.21) and using (4.16), we obtain an isomorphism

$$X^\vee \otimes Y \cong \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y).$$

Remark 4.24. Let $(\mathcal{C}, \otimes)$ be a rigid k-linear tensor category and let $X \in \mathrm{Ob}(\mathcal{C})$ be an object. The functor $Y \mapsto Y \otimes X^\vee$ is right adjoint to $Y \mapsto Y \otimes X$. Indeed,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(T, Y \otimes X^\vee) &= \mathrm{Hom}_{\mathcal{C}}(T, Y \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \mathbf{1})) \\ &= \mathrm{Hom}_{\mathcal{C}}(T, \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) \\ &= \mathrm{Hom}_{\mathcal{C}}(T \otimes X, Y), \end{aligned}$$

where the second equality follows from (4.21) applied to the objects $X_1 = Y_2 = \mathbf{1}$, $Y_1 = Y$, and $X_2 = X$. Then ev_X can be deduced from the adjunction morphism

$$Y \otimes X^\vee \otimes X \longrightarrow Y$$

applied to the object $\mathbf{1}$. Similarly, the adjuntion $Y \to Y \otimes X \otimes X^\vee$ applied to $\mathbf{1}$ gives rise to a coevaluation map

$$\mathrm{coev}_X \colon \mathbf{1} \longrightarrow X \otimes X^\vee. \tag{4.25}$$

Remark 4.26 (Another point of view on duality). Let X be an object of a rigid k-linear tensor category $(\mathcal{C}, \otimes)$. To find a dual of X, it is enough to find an object $Y \in \mathrm{Ob}(\mathcal{C})$ along with morphisms $\mathrm{ev} \colon Y \otimes X \to \mathbf{1}$ and $\mathrm{coev} \colon \mathbf{1} \to X \otimes Y$ such that the compositions

$$X \cong \mathbf{1} \otimes X \xrightarrow{\mathrm{coev} \otimes \mathrm{Id}_X} (X \otimes Y) \otimes X \cong X \otimes (Y \otimes X) \xrightarrow{\mathrm{Id}_X \otimes \mathrm{ev}} X \otimes \mathbf{1} \cong X,$$

$$Y \cong Y \otimes \mathbf{1} \xrightarrow{\mathrm{Id}_X \otimes \mathrm{coev}} Y \otimes (X \otimes Y) \cong (Y \otimes X) \otimes Y \xrightarrow{\mathrm{ev} \otimes \mathrm{Id}_Y} \mathbf{1} \otimes Y \cong Y$$

are the identity maps on X and Y respectively. In fact, this property characterizes rigid categories. Namely, a k-linear tensor category $\mathcal{C}$ is rigid if and only if every object X admits a dual in the sense of this remark; see [**Del90**, §2.1 to §2.5].

In a rigid category $\mathcal{C}$, the assignment $X \mapsto X^\vee$ underlies a contravariant *duality* functor, given on morphisms by sending $f \colon X \to Y$ to the unique ${}^t f \colon Y^\vee \to X^\vee$ that makes the following diagram commutative:

$$\begin{tikzcd} Y^\vee \otimes X \arrow[r, "{}^t f \otimes \mathrm{Id}_X"] \arrow[d, "\mathrm{Id}_{Y^\vee} \otimes f"'] & X^\vee \otimes X. \arrow[d, "\mathrm{ev}_X"] \\ Y^\vee \otimes Y \arrow[r, "\mathrm{ev}_Y"] & \mathbf{1} \end{tikzcd}$$

For k-vector spaces, ${}^t f$ maps a linear form $\ell\colon Y \to k$ to the linear form $\ell \circ f\colon X \to k$, and the commutativity of the diagram amounts to the equality ${}^t f(\ell)(x) = \ell(f(x))$.

4.1.4. *Vector spaces.* The first example of a rigid k-linear tensor category is that of finite-dimensional vector spaces.

EXAMPLE 4.27. The category $\mathbf{Vec}_k$ of finite-dimensional vector spaces over k, along with the usual tensor product of vector spaces and the obvious associativity and commutativity constraints

$$\phi_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z, \quad \psi_{X,Y}(x \otimes y) = y \otimes x \tag{4.28}$$

forms a k-linear rigid tensor category. An identity object is the one-dimensional vector space k with the k-linear map $u\colon k \to k \otimes k$ that sends 1 to $1 \otimes 1$.

The condition of being finite-dimensional is necessary for the category to be rigid. In fact, in the category of vector spaces, finite-dimensional vector spaces are reflexive, whereas infinite-dimensional ones are not. The former is a standard fact that appears in any textbook on linear algebra. The latter is less standard; we sketch a proof from [**Jac75**, Chap. IX §5]. Let V be a vector space over a field k, and let B be a basis of V. A basis always exist if one assumes the axiom of choice, in the form of Zorn's lemma. Every element of V can be written in a unique way as a finite linear combination of elements of B, so that we can identify the set underlying V with the set $k^{(B)}$ of functions from B to k with finite support (here, the support of a function is the set of elements on which it takes non-zero values). The set underlying the dual vector space $V^\vee$ can then be identified with the set k^B of *all* functions from B to k. Indeed, a function $f \in k^B$ corresponds to the element $\omega_f \in V^\vee$ defined by the formula

$$\omega_f(\sum_{j=1}^r a_j e_j) = \sum_{j=1}^r a_j f(e_j)$$

for all finite k-linear combinations of basis elements e_j.

To prove that an infinite-dimensional vector space and its dual are not isomorphic, we will use a few facts about cardinals. In what follows, we denote by $|A|$ the cardinality of a set A. We first observe that it is not enough to compare the cardinality of V to that of $V^\vee$, since it may happen that $|k^{(B)}|$ and $|k^B|$ are equal when k is big compared to B. We will instead compare the cardinality of bases of V and $V^\vee$. We define the dimension of a vector space as the cardinality of a basis. This is well defined, since it can be proved that any two basis have the same cardinality. The dimension of V is greater than or equal to the cardinality of any set of linearly independent vectors.

Let V be an infinite-dimensional vector space with basis B. We first claim that

$$|V| = |k^{(B)}| = \max(\dim V, |k|). \tag{4.29}$$

This is seen by writing $k^{(B)}$ as the union of the sets $k_n^{(B)}$ of functions whose support consists of n elements, and using the equalities of cardinals

$$\begin{aligned} |k_n^{(B)}| &= |B^n \times k^n| = |B \times k| = \max(|B|, |k|), \\ |k^{(B)}| &= \Big|\bigcup_{n \geqslant 0} k_n^{(B)}\Big| = |B \times k| = \max(|B|, |k|). \end{aligned}$$

Besides, since a field has at least two elements, there is an inequality

$$|k^B| \geqslant |2^B| = |P(B)| > |B|,$$

where $P(B)$ denotes the set of subsets of B, whose cardinality is bigger than that of B. Combined with (4.29), this gives

$$\dim(V) = |B| < |k^B| = |V^\vee| = \max(\dim V^\vee, |k|).$$

We are thus reduced to proving the inequality $\dim V^\vee \geqslant |k|$. Since B is infinite, it contains a countable subset $B_0 = \{e_n\}_{n \geqslant 0}$. For each $c \in k$, consider the linear form $\omega_c \in V^\vee$ uniquely determined by

$$\omega_c(e_n) = c^n, \qquad \omega_c(v) = 0, \text{ for } v \in B \setminus B_0.$$

The non-vanishing of Vandermonde determinants implies that the set $\{\omega_c\}_{c \in k}$ consists of linearly independent elements of $V^\vee$. Therefore, $\dim V^\vee \geqslant |k|$ holds.

Putting everything together, we get $\dim V^\vee > \dim V$, and hence

$$\dim(V^\vee)^\vee > \dim V^\vee > \dim V,$$

which shows that the vector space V is not reflexive. This negative result is one of the reasons why, when dealing with infinite-dimensional vector spaces, it is often convenient to endow them with a topology.

4.1.5. *Neutral tannakian categories.* The category $\mathbf{Rep}_k(G)$ of finite-dimensional k-linear representations of an affine group scheme G over k has other relevant properties besides those that we have already discussed. First, it is not only a k-linear category but an abelian category (Definition A.5). Second, the one-dimensional representation given by the vector space k with trivial G-action is an identity object $\mathbf{1}$ that satisfies $\operatorname{End}(\mathbf{1}) = k$. Finally, the functor from $\mathbf{Rep}_k(G)$ to the category $\mathbf{Vec}_k$ of finite-dimensional vector spaces that forgets the action of G is exact, faithful, and compatible with the tensor structure on both categories (*i.e.* a tensor functor in the sense of Definition 4.12). As we will see in the next section, these are all the necessary ingredients to identify the categories of finite-dimensional representations of affine groups schemes among all tensor categories.

Definition 4.30. A *neutral tannakian category* over k is a rigid k-linear abelian tensor category $(\mathcal{C}, \otimes)$ with identity object $\mathbf{1}$ satisfying $\operatorname{End}_{\mathcal{C}}(\mathbf{1}) = k$ and such that there exists an exact k-linear tensor functor

$$\omega \colon \mathcal{C} \longrightarrow \mathbf{Vec}_k\,.$$

Any such functor is called a *fiber functor*.

Remark 4.31. There is a more general notion of *tannakian category* in which one only requires the existence of an exact k-linear tensor functor $\omega \colon \mathcal{C} \to \mathbf{Vec}_F$ with values in the category of finite-dimensional vector spaces over some unspecified field extension F of k. Some natural categories such as pure motives over a finite field are expected to be tannakian in this more general sense without being neutral tannakian. Since we will never consider non-neutral tannakian categories in the sequel, we will drop the adjective “neutral” and call them “tannakian categories”.

The following compatibilities are part of the definition of tensor functor:

$$\omega(X \oplus Y) \cong \omega(X) \oplus \omega(Y), \quad \omega(X \otimes Y) \cong \omega(X) \otimes \omega(Y), \quad \omega(\mathbf{1}) \cong k. \tag{4.32}$$

Combining them with rigidity, it follows that ω is also compatible with duals, and hence with internal Homs in view of Remark 4.23. Indeed, using the point of view from Remark 4.26 that the dual $X^\vee$ is an object Y such that there exist morphisms $\mathrm{ev}\colon Y \otimes X \to \mathbf{1}$ and $\mathrm{coev}\colon \mathbf{1} \to X \otimes Y$ satisfying the compatibilities of loc. cit., one finds that $\omega(Y)$ along with the morphisms $\omega(\mathrm{ev})\colon \omega(Y) \otimes \omega(X) \to k$ and $\omega(\mathrm{coev})\colon k \to \omega(X) \otimes \omega(Y)$ is a dual of $\omega(X)$, hence an isomorphism

$$\omega(X^\vee) \cong \omega(X)^\vee. \tag{4.33}$$

The assumption $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$ in Definition 4.30 ensures that the identity object $\mathbf{1}$ is simple, *i.e.* has no non-trivial subobjects. Indeed, [**DM82**, Prop. 1.17] proves that there is a one-to-one correspondence between subobjects of $\mathbf{1}$ and idempotents in the ring $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$.

Proposition 4.34. *A fiber functor is faithful.*

Proof. We first notice that it suffices to show that ω maps a non-zero object of $\mathcal{C}$ to a non-zero vector space. Indeed, let $f\colon X \to Y$ be a morphism in the abelian category $\mathcal{C}$. Applying the exact functor ω to the epimorphism $f\colon X \to \mathrm{Im}(f)$, we find that $\omega(f)\colon \omega(X) \to \omega(\mathrm{Im}(f))$ is also epimorphism, and hence

$$\mathrm{Im}(\omega(f)) = \omega(\mathrm{Im}(f)).$$

Therefore, if $\omega(f)$ is zero, then $\mathrm{Im}(f)$ and hence f is zero as well, so that the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_k(\omega(X), \omega(Y))$$

is injective. Let now $X \in \mathrm{Ob}(\mathcal{C})$ be a non-zero object. Then the evaluation morphism $\mathrm{ev}_X\colon X^\vee \otimes X \to \mathbf{1}$ from (4.18) is non-zero, and since $\mathbf{1}$ has no non-trivial subobject, it is an epimorphism. The functor ω maps ev_X to the epimorphism

$$\mathrm{ev}_{\omega(X)}\colon \omega(X)^\vee \otimes \omega(X) \to k,$$

and this prevents $\omega(X)$ from being zero. $\square$

Remark 4.35. The implication of Proposition 4.34 has the following converse, proved in [**CEOP21**, Thm. 2.4.1]. Let $\mathcal{C}$ be a rigid k-linear tensor category satisfying $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$ and $\omega\colon \mathcal{C} \to \mathbf{Vec}_k$ a tensor functor. If ω is faithful, then ω is exact, so it is a fiber functor in the sense of Definition 4.30.

Examples 4.36.

i) The rigid k-linear abelian tensor category of finite-dimensional k-vector spaces $\mathbf{Vec}_k$ (Example 4.27), along with the identity as a fiber functor, is a tannakian category.

ii) Let $\mathbf{GrVec}_k$ be the category of finite-dimensional graded k-vector spaces. Its objects are pairs $(V, (V_n)_{n\in\mathbb{Z}})$ consisting of a finite-dimensional k-vector space V and a direct sum decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$, and its morphisms are k-linear maps $f\colon V \to W$ satisfying $f(V_n) \subseteq W_n$ for all $n \in \mathbb{Z}$. The tensor structure comes from the tensor product of vector spaces, graded as

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j.$$

Together with the obvious associativity and commutative constraints (those from Example 4.27) and the one-dimensional vector space k sitting in degree 0 as an identity object, $\mathbf{GrVec}_k$ forms a k-linear abelian tensor category. The dual of a graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is the usual dual vector space $V^\vee$ with the grading

$$V^\vee = \bigoplus_{n\in\mathbb{Z}} (V_n^\vee)_{-n}$$

(that is, $V_n^\vee$ sits in degree $-n$). With this structure, $\mathbf{GrVec}_k$ forms a rigid k-linear abelian tensor category. The forgetful functor

$$\begin{aligned} \omega\colon \quad \mathbf{GrVec}_k &\longrightarrow \mathbf{Vec}_k \\ (V,(V_n)_{n\in\mathbb{Z}}) &\longmapsto \quad V \end{aligned} \tag{4.37}$$

is a fiber functor making $\mathbf{GrVec}_k$ into a tannakian category.

iii) Let us consider the category $\mathbf{GrVec}_k$ again, but replace the usual commutativity constraint with Koszul's sign rule, given by

$$\psi_{X\otimes Y}(x\otimes y) = (-1)^{mn}(y\otimes x)$$

on homogenous elements x and y of degrees m and n respectively (that is, when swapping x and y we also change sign if their degrees have different parity). We denote it by $\mathbf{SGrVec}_k$, as in *super* graded vector spaces, to distinguish it from $\mathbf{GrVec}_k$. Then $\mathbf{SGrVec}_k$ is a rigid k-linear abelian tensor category satisfying $\mathrm{End}(\mathbf{1}) = k$, but the forgetful functor (4.37) is not a fiber functor, as the compatibility with commutativity constraints (Definition 4.12 iii)) fails. In fact, $\mathbf{SGrVec}_k$ does not admit any fiber functor, and hence is not a tannakian category (see Exercise 4.67).

iv) Let Γ be an abstract group and let $\mathbf{Rep}_k(\Gamma)$ denote the category of finite-dimensional k-linear representations of Γ. Let

$$\omega\colon\ \mathbf{Rep}_k(\Gamma) \longrightarrow \mathbf{Vec}_k$$

be the functor that forgets the action of Γ. Then $\mathbf{Rep}_k(\Gamma)$ is a tannakian category over k and ω is a fiber functor.

v) Let F and L be subfields of the complex numbers. Let us consider the category $\mathbf{Vec}_{F,L}$ whose objects are triples

$$H = (H_{\mathrm{dR}}, H_{\mathrm{B}}, c_H)$$

consisting of a finite-dimensional F-vector space H_{dR}, a finite-dimensional L-vector space H_{B}, and an isomorphism

$$c_H\colon H_{\mathrm{dR}}\otimes_F \mathbb{C} \xrightarrow{\sim} H_{\mathrm{B}}\otimes_L \mathbb{C}$$

of complex vector spaces. A morphism $f\colon H\to H'$ between such objects is a pair $f = (f_{\mathrm{dR}}, f_{\mathrm{B}})$ consisting of an F-linear map $f_{\mathrm{dR}}\colon H_{\mathrm{dR}}\to H'_{\mathrm{dR}}$ and an L-linear map $f_{\mathrm{B}}\colon H_{\mathrm{B}}\to H'_{\mathrm{B}}$ such that the diagram

$$\begin{tikzcd} H_{\mathrm{dR}}\otimes_F \mathbb{C} \arrow[r, "f_{\mathrm{dR}}\otimes_F \mathrm{Id}_{\mathbb{C}}"] \arrow[d, "c_H"] & H'_{\mathrm{dR}}\otimes_F \mathbb{C} \arrow[d, "c_{H'}"] \\ H_{\mathrm{B}}\otimes_L \mathbb{C} \arrow[r, "f_{\mathrm{B}}\otimes_L \mathrm{Id}_{\mathbb{C}}"] & H'_{\mathrm{B}}\otimes_L \mathbb{C} \end{tikzcd}$$

commutes. The category $\mathbf{Vec}_{F,L}$ is $(F\cap L)$-linear. It is endowed with the tensor product

$$H\otimes H' = (H_{\mathrm{dR}}\otimes H'_{\mathrm{dR}}, H_{\mathrm{B}}\otimes H'_{\mathrm{B}}, c_H\otimes c'_H),$$

for which $\mathbf{1} = (F, L, \mathrm{Id}_{\mathbb{C}})$ is an identity object. It satisfies $\mathrm{End}(\mathbf{1}) = F \cap L$. The forgetful functors

$$\omega_{\mathrm{dR}} \colon \mathbf{Vec}_{F,L} \longrightarrow \mathbf{Vec}_F, \quad H \longmapsto H_{\mathrm{dR}},$$
$$\omega_{\mathrm{B}} \colon \mathbf{Vec}_{F,L} \longrightarrow \mathbf{Vec}_L, \quad H \longmapsto H_{\mathrm{B}}$$

are exact faithful tensor functors, and hence $\mathbf{Vec}_{F,L}$ is an eventually non-neutral tannakian category. If the field $F \cap L$ is equal to either F or L, then $\mathbf{Vec}_{F,L}$ is a neutral tannakian category. For example, if $L = \mathbb{Q}$, then $\mathbf{Vec}_{F,L}$ is a neutral tannakian category over $\mathbb{Q}$ with fiber functor ω_{B}.

vi) Let $\mathbf{MHS}(k)$ be the category of mixed Hodge structures over k, and let

$$\omega_{\mathrm{dR}} \colon \mathbf{MHS}(k) \longrightarrow \mathbf{Vec}_k$$
$$\omega_{\mathrm{B}} \colon \mathbf{MHS}(k) \longrightarrow \mathbf{Vec}_{\mathbb{Q}}$$

be the forgetful functors from Definition 2.227. Then $\mathbf{MHS}(k)$ is a tannakian category over $\mathbb{Q}$ and ω_{B} is a fiber functor. In case $k = \mathbb{Q}$, another fiber functor is given by ω_{dR}.

vii) Let M be a path-connected topological space. The category $\mathbf{Loc}_k(M)$ of locally constant sheaves (also known as *local systems*) of finite-dimensional k-vector spaces is a tannakian category. The tensor product $V \otimes V'$ of local systems V and V' is the sheaf associated with the presheaf

$$U \longmapsto V(U) \otimes V'(U),$$

which is again locally constant. The identity object is the trivial local system $\underline{k}_M$ given by locally constant k-valued functions on M. The condition that M is connected guarantees that $\mathrm{End}(\underline{k}_M)$ is reduced to k (otherwise, one can multiply locally constant functions by different scalars on different connected components). For each point $x \in M$, the functor

$$\begin{aligned} \omega_x \colon \mathbf{Loc}_k(M) &\longrightarrow \mathbf{Vec}_k \\ V \quad &\longmapsto \quad V_x \end{aligned}$$

that sends a local system V to its fiber at x is a fiber functor. Indeed, ω_x is an exact k-linear tensor functor, since taking stalks is an exact operation on the category of sheaves and the stalk of a sheafification is equal to that of the presheaf, namely $(V \otimes V')_x = V_x \otimes V'_x$. Note that the functor ω_x is not faithful if M is not connected. The fiber functors ω_x and ω_y associated with distinct points $x, y \in M$ are non-canonically isomorphic: every path γ from x to y induces a functorial isomorphism $V_x \cong V_y$ by means of parallel transport.

4.1.6. *The fundamental group of a tannakian category.* Now that we have singled out the notion of tannakian category, our next goal consists in proving that all tannakian categories are equivalent to $\mathbf{Rep}_k(G)$ for some affine group scheme G over k. As in the case of the Tannaka–Krein theorem discussed in the motivational section, the group will arise as the automorphisms of a fiber functor.

Definition 4.38. Let $(\mathcal{C}, \otimes)$ be a tannakian category over k, along with a fiber functor $\omega \colon \mathcal{C} \to \mathbf{Vec}_k$. For every k-algebra R, let $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ denote the set of families $\lambda = (\lambda_X)_{X \in \mathrm{Ob}(\mathcal{C})}$ of R-linear automorphisms

$$\lambda_X \colon \omega(X) \otimes R \longrightarrow \omega(X) \otimes R$$

such that, for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the following three diagrams are commutative:

$$\begin{tikzcd}
\omega(X \otimes Y) \otimes R \arrow[rr, "\lambda_{X\otimes Y}"] \arrow[d, "\wr"'] & & \omega(X \otimes Y) \otimes R \arrow[d, "\wr"] \\
\omega(X) \otimes \omega(Y) \otimes R \arrow[d, "\wr"'] & & \omega(X) \otimes \omega(Y) \otimes R \arrow[d, "\wr"] \\
(\omega(X) \otimes R) \otimes_R (\omega(Y) \otimes R) \arrow[rr, "\lambda_X \otimes_R \lambda_Y"] & & (\omega(X) \otimes R) \otimes_R (\omega(Y) \otimes R),
\end{tikzcd} \tag{4.39}$$

$$\begin{tikzcd}
\omega(\mathbf{1}) \otimes R \arrow[r, "\lambda_{\mathbf{1}}"] \arrow[d, "\wr"'] & \omega(\mathbf{1}) \otimes R \arrow[d, "\wr"] \\
R \arrow[r, "\mathrm{Id}"] & R,
\end{tikzcd} \tag{4.40}$$

$$\begin{tikzcd}
\omega(X) \otimes R \arrow[r, "\lambda_X"] \arrow[d, "\omega(\alpha)\otimes \mathrm{Id}"'] & \omega(X) \otimes R \arrow[d, "\omega(\alpha)\otimes \mathrm{Id}"] \\
\omega(Y) \otimes R \arrow[r, "\lambda_Y"] & \omega(Y) \otimes R.
\end{tikzcd} \tag{4.41}$$

In the above diagrams, all unlabeled tensor products of vector spaces are over k, and the unnamed arrows are isomorphisms obtained from the compatibilities (4.32) of the fiber functor and the obvious properties of tensor products.

We think of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ as the set of automorphisms of ω with coefficients in R. In particular, we define

$$\mathrm{Aut}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega)(k).$$

This is the group of k-linear automorphisms of the functor ω.

The main theorem of the theory of tannakian categories is

Theorem 4.42 ([**DM82**, Thm. 2.11]). *Let $(\mathcal{C}, \otimes)$ be a tannakian category over k, together with a fiber functor ω. Then:*

i) *the functor $R \mapsto \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ is representable by an affine group scheme over k that we denote by $\underline{\mathrm{Aut}}^{\otimes}(\omega)$;*
ii) *for every object $X \in \mathrm{Ob}(\mathcal{C})$, the group $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ acts naturally on the k-vector space $\omega(X)$, and the functor*

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathbf{Rep}_k(\underline{\mathrm{Aut}}^{\otimes}(\omega)) \\ X &\longmapsto \omega(X) \end{aligned}$$

is an equivalence of categories.

Definition 4.43. The affine group scheme $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is called the *Tannaka group* of $(\mathcal{C}, \omega)$. Whenever we want to stress the category we are considering, we will write $\underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega)$ instead of $\underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Given a second fiber functor ω', we denote by $\underline{\mathrm{Iso}}^{\otimes}(\omega,\omega')(R)$ the set of families $\mu = (\mu_X)_{X\in\mathrm{Ob}(\mathcal{C})}$ of R-linear isomorphisms

$$\mu_X\colon \omega(X)\otimes R \longrightarrow \omega'(X)\otimes R$$

such that the diagrams of Definition 4.38, with ω replaced by ω' on the right-hand side, commute. There are right and left group actions

$$\begin{aligned}\underline{\mathrm{Iso}}^{\otimes}(\omega,\omega')(R)\times\underline{\mathrm{Aut}}^{\otimes}(\omega)(R) &\longrightarrow \underline{\mathrm{Iso}}^{\otimes}(\omega,\omega')(R)\\ \underline{\mathrm{Aut}}^{\otimes}(\omega')(R)\times\underline{\mathrm{Iso}}^{\otimes}(\omega,\omega')(R) &\longrightarrow \underline{\mathrm{Iso}}^{\otimes}(\omega,\omega')(R)\end{aligned}\tag{4.44}$$

given by precomposition and postcomposition respectively.

Theorem 4.45 ([**DM82**, Thm. 3.2]). *The functor from k-algebras to sets*

$$R\longmapsto \underline{\mathrm{Iso}}^{\otimes}(\omega,\omega')(R)$$

is representable by an affine scheme $\underline{\mathrm{Iso}}^{\otimes}(\omega,\omega')$ over k, which is a right torsor under $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ and a left torsor under $\underline{\mathrm{Aut}}^{\otimes}(\omega')$.

4.1.7. *Matrix coefficients.* Instead of proving Theorem 4.42, we will content ourselves with a description of the Hopf algebra of the Tannaka group using the notion of matrix coefficients from [**Del90**, § 4.7] (see also [**Bro17**], and compare this with the notion of framed objects from [**BGSV90**]).

Definition 4.46. Let $\mathcal{C}$ be a tannakian category over k, together with fiber functors ω_1 and ω_2. A *matrix coefficient* in $(\mathcal{C},\omega_1,\omega_2)$ is a triple

$$(X,f,v)$$

consisting of an object X of $\mathcal{C}$ and elements

$$v\in\omega_1(X)\quad\text{and}\quad f\in\omega_2(X)^{\vee}=\mathrm{Hom}(\omega_2(X),k).$$

Let H^{ω_1,ω_2} be the k-vector space generated by all matrix coefficients, and let

$$V^{\omega_1,\omega_2}\subseteq H^{\omega_1,\omega_2}$$

be the sub-vector space spanned by:

Bilinearity relations: for matrix coefficients (X,f,v_1) and (X,f,v_2), and scalars $\lambda,\mu\in k$, the relation

$$(X,f,\lambda v_1+\mu v_2)-\lambda(X,f,v_1)-\mu(X,f,v_2)\in V^{\omega_1,\omega_2}.$$

For matrix coefficients (X,f_1,v) and (X,f_2,v), and $\lambda,\mu\in k$, the relation

$$(X,\lambda f_1+\mu f_2,v)-\lambda(X,f_1,v)-\mu(X,f_2,v)\in V^{\omega_1,\omega_2}.$$

Compatibility relations: for all $X,X'\in\mathrm{Ob}(\mathcal{C})$, all $\phi\in\mathrm{Hom}_{\mathcal{C}}(X,X')$, and elements $v\in\omega_1(X)$ and $f'\in\omega_2(X')^{\vee}$, the relation

$$(X,\omega_2(\phi)^{\vee}f',v)-(X',f',\omega_1(\phi)v)\in V^{\omega_1,\omega_2}.\tag{4.47}$$

We consider the quotient

$$A^{\omega_1,\omega_2}=H^{\omega_1,\omega_2}/V^{\omega_1,\omega_2},$$

and write $[X,f,v]$ for the class of a matrix coefficient (X,f,v) in A^{ω_1,ω_2}. Whenever all fiber functors are equal $\omega_1=\omega_2=\omega$, we will write A^{ω} instead of A^{ω_1,ω_2}.

The vector space A^{ω_1,ω_2} is equipped with the following structures:

Product: The tensor structure of $\mathcal{C}$ induces the product

$$[X, f, v] \cdot [X', f', v'] = [X \otimes X', f \otimes f', v \otimes v']. \tag{4.48}$$

The associativity and commutativity constraints together with the compatibility relation imply that this product is associative and commutative.

Unit: Let $\mathbf{1}$ be an identity object of $\mathcal{C}$. For every fiber functor ω on $\mathcal{C}$, there is a canonical isomorphism $\omega(\mathbf{1}) \simeq k$. Let $v \in \omega_1(\mathbf{1})$ and $f \in \omega_2(\mathbf{1})^\vee$ be the elements corresponding to $1 \in k$ and to its dual respectively. Then $[\mathbf{1}, f, v]$ is a unit for the product.

Coaction: For any fiber functor ω_3 on $\mathcal{C}$, there is a map

$$\Delta\colon A^{\omega_1,\omega_3} \longrightarrow A^{\omega_2,\omega_3} \otimes A^{\omega_1,\omega_2}$$

given as follows: for each object X of $\mathcal{C}$, choose a basis $(e_1, \ldots, e_n)$ of $\omega_2(X)$, let $(e_1^*, \ldots, e_n^*)$ denote the dual basis, and define

$$\Delta[X, f, v] = \sum_{j=1}^{n} [X, f, e_j] \otimes [X, e_j^*, v]. \tag{4.49}$$

One checks that (4.49) does not depend on the choice of the basis.

In the case where the fiber functors ω_1, ω_2 and ω_3 are all equal, say to ω, the coaction gives rise to a coproduct

Coproduct:

$$\Delta[X, f, v] = \sum_{j=1}^{n} [X, f, e_j] \otimes [X, e_j^*, v]. \tag{4.50}$$

Moreover, for $\omega = \omega_1 = \omega_2$, there are two extra structures:

Counit: The counit is the map $\epsilon\colon A^\omega \to k$ given by

$$\epsilon([X, f, v]) = f(v). \tag{4.51}$$

Antipode: The antipode is the map $S\colon A^\omega \to A^\omega$ given by

$$S([X, f, v]) = [X^\vee, v, f] \tag{4.52}$$

under the identifications $\omega(X^\vee) \simeq \omega(X)^\vee$ and $\omega(X^\vee)^\vee \simeq \omega(X)$ that allow one to swap v and f.

It is straightforward to prove the following result.

Proposition 4.53. *Together with the above structures,*

i) *A^{ω_1} and A^{ω_2} are commutative Hopf algebras;*
ii) *A^{ω_1,ω_2} is a right A^{ω_1}-Hopf module and a left A^{ω_2}-Hopf module.*

Taking Theorem 4.42 for granted, we can show that $A = A^\omega$ is the Hopf algebra of the Tannaka group $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$. More precisely,

Proposition 4.54. *The map $\varphi\colon A \to \mathcal{O}(G)$ given by*

$$\varphi([X, f, v])(\lambda) = f(\lambda_X(v))$$

is an isomorphism of Hopf algebras.

PROOF. We only prove that φ is bijective, leaving to the reader the task of carefully checking that φ is a morphism of Hopf algebras.

By Theorem 4.42, the category $\mathcal{C}$ is equivalent to the category $\mathbf{Rep}_k(G)$ of finite-dimensional k-linear representations of G, and we can identify ω with the forgetful functor $\mathbf{Rep}_k(G) \to \mathbf{Vec}_k$.

We first prove that φ is surjective. The right action of G on itself given by multiplication induces a left group action of G on $\mathcal{O}(G)$ given by

$$(\lambda h)(\mu) = h(\mu\lambda).$$

By Lemma 3.122, $\mathcal{O}(G)$ is the union of its finite-dimensional subrepresentations: given $h \in \mathcal{O}(G)$, there exists a finite-dimensional subrepresentation (V, ρ) of $\mathcal{O}(G)$ containing h. It determines an object X of $\mathcal{C}$ such that h belongs to $\omega(X) = V$. Let $f \in V^\vee$ be the element determined by $f(u) = u(e)$, where e is the unit of G and $u \in V \subseteq \mathcal{O}(G)$. Then, for each element $\lambda \in G$, the equality

$$[X, f, h](\lambda) = f(\lambda h) = (\lambda h)(e) = h(e\lambda) = h(\lambda)$$

holds. Therefore, $\varphi([X, f, h]) = h$ and φ is surjective.

Let us now prove that φ is injective. Assume that $\varphi([X, f, v]) = 0$. We identify X with a finite-dimensional representation (V, ρ) of G such that $v \in V$. Let V' be the smallest subrepresentation of V containing v. As vector space, V' is generated by elements of the form λv for $\lambda \in G$. From $\varphi([X, f, v]) = 0$, we get $f|_{V'} = 0$. Let X' be the object of $\mathcal{C}$ corresponding to (V', ρ). By the compatibility relations for matrix coefficients (4.47), the equality

$$[X, f, v] = [X', f|_{V'}, v] = [X', 0, v] = 0$$

holds, thus completing the proof. $\square$

The same techniques used in Proposition 4.54 also give the next result.

PROPOSITION 4.55. *The map* $A^{\omega_1,\omega_2} \to \mathcal{O}(\underline{\mathrm{Iso}}^{\otimes}(\omega_1, \omega_2))$ *given by*

$$\varphi([X, f, v])(\lambda) = f(\lambda_X(v))$$

is an isomorphism of algebras. Moreover, the Hopf module structures from Proposition 4.53 are compatible with the actions (4.44).

EXAMPLE 4.56 (Graded vector spaces). Consider the tannakian category

$$(\mathbf{GrVec}_k, \omega)$$

of finite-dimensional graded k-vector spaces along with the forgetful functor from Example 4.36 ii). In this category, every object is a direct sum of one-dimensional objects k_n, one for each $n \in \mathbb{Z}$, which are concentrated in degree n and satisfy

$$\mathrm{Hom}(k_n, k_m) = \begin{cases} k, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases} \qquad k_n \otimes k_m = k_{n+m}, \qquad \omega(k_n) \simeq k.$$

This implies that we can identify k_0 with the identity object, and k_{-n} with the dual of k_n. Hence, from the compatibilities (4.32) and (4.33) of a tensor functor, we get canonical identifications $\omega(k_0) \simeq k$ and $\omega(k_n) \simeq \omega(k_{-n})^\vee$. Choose a non-zero element $u_1 \in \omega(k_1)$, and write

$$u_n = \begin{cases} u_1^{\otimes n}, & \text{if } n > 0, \\ 1 \in k, & \text{if } n = 0, \\ u_{-n}^\vee, & \text{if } n < 0. \end{cases}$$

For $n > 0$, the vector $u_n^\vee \in \omega(k_n)^\vee = \omega(k_{-n})$ is defined by $u_n^\vee(u_n) = 1$; we extend the notation $u_n^\vee$ to $n \leqslant 0$ as well. Then every matrix coefficient can be written as a linear combination of the elements

$$[k_n, u_n^\vee, u_n], \quad n \in \mathbb{Z}.$$

Moreover, the product (4.48) reads

$$[k_n, u_n^\vee, u_n] \cdot [k_m, u_m^\vee, u_m] = [k_{n+m}, u_{n+m}^\vee, u_{n+m}],$$

so that, setting $t = [k_1, u_1^\vee, u_1]$, there is an isomorphism of k-algebras

$$\mathcal{O}(\underline{\mathrm{Aut}}^\otimes(\omega)) = k[t, t^{-1}].$$

Since the coproduct (4.50), the counit (4.51), and the antipode (4.52) are given by

$$\Delta t = t \otimes t, \quad \epsilon(t) = \epsilon(t^{-1}) = 1, \quad S(t) = t^{-1},$$

we deduce from Example 3.60 ii) that $\underline{\mathrm{Aut}}^\otimes(\omega)$ is the multiplicative group $\mathbb{G}_m$. We have thus seen that the main theorem of tannakian categories yields in this case the equivalence of categories

$$\mathbf{GrVec}_k \cong \mathbf{Rep}_k(\mathbb{G}_m).$$

Example 4.57 (Split real mixed Hodge structures). A *split real mixed Hodge structure* is a finite-dimensional real vector space H equipped with a bigrading

$$H_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}$$

that is symmetric with respect to complex conjugation:

$$\overline{H^{p,q}} = H^{q,p}. \tag{4.58}$$

Together with morphisms of real vector spaces preserving the given bigradings, these objects form a category $\mathcal{C}$. Let

$$\omega \colon \mathcal{C} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$$

be the forgetful functor that sends a pair $(H, (H^{p,q}))$ to H. It is a simple matter of unraveling the definitions to check that $(\mathcal{C}, \omega)$ is a neutral tannakian category over $\mathbb{R}$. In what follows, we determine its Tannaka group. For a different determination of the group using matrix coefficients, see Exercise 4.71.

The subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices of the form $\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$ can be identified with $\mathbb{C}^\times = \mathbb{G}_m(\mathbb{C})$ through the group isomorphism

$$\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right) \longmapsto x + iy.$$

Being a closed algebraic subgroup of $\mathrm{GL}_2(\mathbb{R})$, it can also be identified with the real points of a real affine algebraic group $\mathbb{S}$ over $\mathbb{R}$ called the *Deligne torus*. That is, the Deligne torus is a real affine algebraic group over $\mathbb{R}$ whose real points are the complex points of $\mathbb{G}_m$. A way to define it is as the *Weil restriction*

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m),$$

which means that $\mathbb{S}$ is the functor mapping an $\mathbb{R}$-algebra A to the group

$$\mathbb{S}(A) = \mathbb{G}_m(A \otimes_{\mathbb{R}} \mathbb{C}) = (A \otimes_{\mathbb{R}} \mathbb{C})^\times.$$

In particular, $\mathbb{S}(R) = \mathbb{C}^\times$. The complex points of $\mathbb{S}$ consist of those matrices of $\mathrm{GL}_2(\mathbb{C})$ of the form $\left(\begin{smallmatrix} u & v \\ -v & u \end{smallmatrix}\right)$ and there is an isomorphism

$$\begin{aligned} \mathbb{S}(\mathbb{C}) &\longrightarrow \mathbb{C}^* \times \mathbb{C}^* \\ \left(\begin{smallmatrix} u & v \\ -v & u \end{smallmatrix}\right) &\longmapsto (u+iv, u-iv) \end{aligned}$$

through which the action of complex conjugation on $\mathbb{S}(\mathbb{C})$ corresponds to

$$(\alpha, \beta) \longmapsto (\bar{\beta}, \bar{\alpha}).$$

In particular, the group of real points $\mathbb{S}(\mathbb{R})$ is the subgroup of elements $(\alpha, \overline{\alpha})$.

We claim that the category of split real mixed Hodge structures is equivalent to the category of finite-dimensional real representations of $\mathbb{S}$. That is,

$$\underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega) = \mathbb{S}.$$

Indeed, if V is such a representation, then $\mathbb{S}(\mathbb{C})$ acts on $V_{\mathbb{C}}$. Since $\mathbb{S}(\mathbb{C})$ is the torus $\mathbb{C}^* \times \mathbb{C}^*$, the complex representation $V_{\mathbb{C}}$ decomposes as a direct sum

$$V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q},$$

where $V^{p,q}$ is the eigenspace on which $\mathbb{S}(\mathbb{C})$ acts through

$$(\alpha, \beta) \cdot v = \alpha^p \beta^q v.$$

Note that the subgroup $\mathbb{S}(\mathbb{R})$ acts on $V^{p,q}$ through $\alpha \cdot v = \alpha^p \overline{\alpha}^q v$. Since the representation we started with is defined over $\mathbb{R}$, we have

$$(\alpha, \beta) \cdot \bar{v} = \overline{\overline{(\alpha, \beta)} \cdot v} = \overline{(\bar{\beta}, \bar{\alpha}) \cdot v} = \overline{\bar{\beta}^p \bar{\alpha}^q v} = \alpha^q \beta^p \bar{v}$$

for each $v \in V^{p,q}$, and hence $\overline{V^{p,q}} = V^{q,p}$ holds. This yields a functor

$$\mathbf{Rep}_{\mathbb{R}}(\mathbb{S}) \longrightarrow \mathcal{C}, \tag{4.59}$$

which is compatible with the forgetful functors on both sides.

Conversely, given a split real mixed Hodge structure H, we define an action of $\mathbb{S}(\mathbb{C})$ on $H_{\mathbb{C}}$ by letting $(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^*$ act on $H^{p,q}$ as multiplication by $\alpha^p \beta^q$. The condition (4.58) readily implies that the resulting representation comes from a real representation $\mathbb{S} \to \mathrm{GL}(H)$ by extension of scalars, and we obtain in this way a quasi-inverse to the functor (4.59).

4.1.8. *The tannakian dictionary.* One of the reasons why the tannakian formalism is so powerful is that properties of a tannakian category $\mathcal{C} \cong \mathbf{Rep}_k(G)$ can be rephrased in terms of the affine group scheme G. Here are some examples:

- The affine group scheme G is *finite* if and only if there exists $X \in \mathrm{Ob}(\mathcal{C})$ such that every object of $\mathcal{C}$ is isomorphic to a subquotient of $X^{\oplus n}$ for some integer $n \geqslant 0$ (see [**DM82**, Prop. 2.20 (a)]).
- The affine group scheme G is *algebraic* (*i.e.* of finite type over k) if and only if there exists an object $X \in \mathrm{Ob}(\mathcal{C})$ such that every object of $\mathcal{C}$ is isomorphic to a subquotient of a finite direct sum

  $$\bigoplus_i X^{\otimes m_i} \otimes (X^\vee)^{\otimes n_i}$$

  for some integers $m_i, n_i \geqslant 0$ (see [**DM82**, Prop. 2.20 (b)]).

- If k is of characteristic zero, then the affine group scheme G is *connected* if and only if the category $\mathcal{C}$ does not contain any non-trivial (*i.e.* which is not a sum of copies of $\mathbf{1}$) object X such that the full subcategory of $\mathcal{C}$ consisting of subquotients of $X^{\oplus n}$ for all $n \geqslant 0$ is stable under tensor product (see [**DM82**, Cor. 2.22]).
- If k is of characteristic zero, then the affine group scheme G is *pro-reductive* if and only if the category $\mathcal{C}$ is *semisimple*, *i.e.* every object is isomorphic to a finite direct sum of simple objects (see [**DM82**, Prop. 2.23]).

The properties of functors between tannakian categories can also be interpreted in terms of the corresponding affine group schemes. Let $F\colon \mathcal{C}' \to \mathcal{C}$ be an exact k-linear tensor functor (Definition 4.12) of tannakian categories, and let $\omega\colon \mathcal{C} \to \mathbf{Vec}_k$ be a fiber functor on $\mathcal{C}$. Then

$$\omega' = \omega \circ F\colon \mathcal{C}' \longrightarrow \mathbf{Vec}_k$$

is a fiber functor on $\mathcal{C}'$, and there is an induced morphism of affine group schemes

$$f = F^*\colon G = \underline{\mathrm{Aut}}^{\otimes}(\omega) \longrightarrow G' = \underline{\mathrm{Aut}}^{\otimes}(\omega')$$

given on R-points by sending $\lambda = (\lambda_X)_{X \in \mathrm{Ob}(\mathcal{C})}$ to $f(\lambda) = (\lambda_{F(Y)})_{Y \in \mathrm{Ob}(\mathcal{C}')}$, which taking the equality $\omega' = \omega \circ F$ into account is a family of R-linear automorphisms of $\omega'(Y) \otimes R$. Conversely, every morphism $f\colon G \to G'$ of affine group schemes gives rise to an exact k-linear tensor functor of tannakian categories

$$F = f^*\colon \mathbf{Rep}_k(G') \longrightarrow \mathbf{Rep}_k(G)$$

which is compatible with the forgetful functors.

Proposition 4.60 ([**DM82**, Prop. 2.21]). *Let $F\colon \mathcal{C}' \to \mathcal{C}$ be an exact k-linear tensor functor of tannakian categories, and let $f\colon G \to G'$ be the associated morphism between their Tannaka groups.*

i) *f is a monomorphism (i.e. a closed immersion) if and only if every object $X \in \mathrm{Ob}(\mathcal{C})$ is isomorphic to a subquotient of an object $F(Y)$ for some object $Y \in \mathrm{Ob}(\mathcal{C}')$;*

ii) *f is an epimorphism (i.e. faithfully flat) if and only if F is fully faithful and its essential image is stable under taking subobjects (i.e. for each object $Y \in \mathrm{Ob}(\mathcal{C}')$, each subobject of $F(Y)$ is isomorphic to the image by F of a subobject of Y).*

Recall from Example 4.56 that the tannakian category $\mathbf{GrVec}_k$ of finite-dimensional graded k-vector spaces is equivalent to $\mathbf{Rep}_k(\mathbb{G}_m)$. A straightforward, yet very useful, application of the above proposition is the general fact that the present of a grading in a tannakian category results in a *cocharacter* of the Tannaka group, *i.e.* a morphism from $\mathbb{G}_m$.

Example 4.61. Let $\mathcal{C}$ be a tannakian category, and let $\omega\colon \mathcal{C} \to \mathbf{Vec}_k$ be a fiber functor. Write $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$. Assume that a functorial grading is given on all vector spaces $\omega(X)$, so that ω factors as

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "\omega"'] & \mathbf{GrVec}_k. \arrow[d, "\mathrm{forg}"] \\ & \mathbf{Vec}_k \end{tikzcd}$$

We then obtain a morphism of affine group schemes $F^*\colon \mathbb{G}_m \to G$. Recall from Example 4.56 that the category $\mathbf{GrVec}_k$ is semisimple with simple objects k_n. According to Proposition 4.60 i), the morphism F^* is a closed immersion if and only if for every integer n, there exists an object $X \in \mathrm{Ob}(\mathcal{C})$ such that $\omega(X)$ contains a non-trivial graded piece of degree n. Using the tensor product and the existence of duals, this is equivalent to asking that there exists a single object X such that $\omega(X)$ contains a non-trivial graded piece of degree 1. For example, applied to the tannakian category of split real mixed Hodge structures from Example 4.57, this produces the subtorus $\mathbb{G}_m \subset \mathbb{S}$ of diagonal matrices.

4.1.9. *Tannakian subcategories.* Given an object Y of a tannakian category $\mathcal{C}$, we denote by $\langle Y\rangle$ the full subcategory of $\mathcal{C}$ that contains Y and is stable by taking sums, tensor products, dual, and subquotients. Its objects are all subquotients of all finite direct sums $\bigoplus Y^{\otimes m_i} \otimes (Y^\vee)^{\otimes n_i}$ for all integers $m_i, n_i \geqslant 0$. Together with the restriction of any fiber functor ω on $\mathcal{C}$, the category $\langle Y\rangle$ is again tannakian. The action of $G = \underline{\mathrm{Aut}}^\otimes_{\mathcal{C}}(\omega)$ on the vector space $\omega(Y)$ induces a map $G \to \mathrm{GL}(\omega(Y))$. The following is shown in the proof of [**DM82**, Prop. 2.8].

LEMMA 4.62. *The image $G_Y \subset \mathrm{GL}(\omega(Y))$ of G by the above map is a closed subgroup which agrees with the Tannaka group $\underline{\mathrm{Aut}}^\otimes_{\langle Y\rangle}(\omega)$ of the subcategory $\langle Y\rangle$.*

We can order the subcategories of the form $\langle Y\rangle$ for Y an object of $\mathcal{C}$ by inclusion, so that they form a directed system. Indeed, if Y, Z are objects of $\mathcal{C}$, then

$$\langle Y\rangle \subset \langle Y \oplus Z\rangle \supset \langle Z\rangle.$$

Assume $\langle Y\rangle \subset \langle Z\rangle$. Then, by restricting a family $(\lambda_X)_{X\in\mathrm{Ob}(\langle Z\rangle)}$ to $(\lambda_X)_{X\in\mathrm{Ob}(\langle Y\rangle)}$, we obtain a morphism

$$\underline{\mathrm{Aut}}^\otimes_{\langle Z\rangle}(\omega) \longrightarrow \underline{\mathrm{Aut}}^\otimes_{\langle Y\rangle}(\omega). \tag{4.63}$$

The following lemma exhibits the pro-algebraic nature of G.

LEMMA 4.64. *Let $\mathcal{C}$ be a tannakian category with fiber functor ω. Then:*

$$\underline{\mathrm{Aut}}^\otimes_{\mathcal{C}}(\omega) = \varprojlim_{\langle Y\rangle} \underline{\mathrm{Aut}}^\otimes_{\langle Y\rangle}(\omega) = \varprojlim_{\langle Y\rangle} G_Y.$$

PROOF. By Lemma 4.62, there is a surjection $G \to \underline{\mathrm{Aut}}^\otimes_{\langle Y\rangle}(\omega)$ for every object Y of $\mathcal{C}$. These surjections are compatible with the maps $\underline{\mathrm{Aut}}^\otimes_{\langle Z\rangle}(\omega) \to \underline{\mathrm{Aut}}^\otimes_{\langle Y\rangle}(\omega)$ induced by an inclusion $\langle Y\rangle \subset \langle Z\rangle$. Therefore, there is a surjection

$$G \longrightarrow \varprojlim_{\langle Y\rangle} \underline{\mathrm{Aut}}^\otimes_{\langle Y\rangle}(\omega).$$

This map is also injective, because if an element of G is sent to the unit, then it acts trivially on $\omega(Y)$ for every object Y, and is thus the unit of G. $\square$

EXAMPLE 4.65. Let Γ be an abstract group, and let $\mathbf{Rep}_k(\Gamma)$ be the tannakian category of finite-dimensional k-linear representations of Γ, along with the forgetful fiber functor $\omega\colon \mathbf{Rep}_k(\Gamma) \to \mathbf{Vec}_k$ from Example 4.36 iv). The main theorem of tannakian categories (Theorem 4.42) yields an equivalence

$$\mathbf{Rep}_k(\Gamma) \cong \mathbf{Rep}_k(\underline{\mathrm{Aut}}^\otimes(\omega)).$$

In general, the groups Γ and $\underline{\mathrm{Aut}}^\otimes(\omega)$ are not isomorphic, since $\underline{\mathrm{Aut}}^\otimes(\omega)$ is an affine group scheme over k and Γ is only an abstract group. Thanks to Lemma 4.64, the Tannaka group $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ admits the following description. Let $Y = (V, \rho)$ be a finite-dimensional k-linear representation of Γ. The algebraic group G_Y from Lemma 4.64 is the Zariski closure of the image of $\rho\colon \Gamma \to \mathrm{GL}(V)$, that is, the smallest closed algebraic subgroup H of $\mathrm{GL}(V)$ defined over k such that $\rho(\Gamma)$ is contained in the k-points $H(k)$. Let $Y' = (V', \rho')$ be another representation such that there is an inclusion $\langle Y'\rangle \subset \langle Y\rangle$. By (4.63), there is a restriction map

$$G_Y = \overline{\rho(\Gamma)}^{\mathrm{Zar}} \longrightarrow G_{Y'} = \overline{\rho'(\Gamma)}^{\mathrm{Zar}},$$

and $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is isomorphic to the projective limit

$$\underline{\mathrm{Aut}}^{\otimes}(\omega) \cong \varprojlim_{\langle (V,\rho)\rangle} \overline{\rho(\Gamma)}^{\mathrm{Zar}},$$

taken with respect to the subcategories $\langle (V, \rho)\rangle$ ordered by inclusion. According to Exercise 4.72, this is an equivalent description of the *pro-algebraic completion* Γ^{alg} over k introduced in Definition 3.225.

4.1.10. *Tannakian categories and the fundamental group.* We finish this section by explaining what information about the fundamental group of a topological space can be recovered by means of the tannakian formalism. This namely includes the pro-unipotent completion from Section 3.4.

Let M be a path-connected topological space that is sufficiently nice to have a well-behaved notion of fundamental group, so that giving a locally constant sheaf on M is equivalent to giving a representation of the fundamental group. In the examples of interest for us, M will be a Hausdorff, second countable, locally compact and locally contractible topological space (see Theorem A.315).

Let x be a point of M, and let $\pi_1(M, x)$ be the fundamental group of M with base point x. By Example 4.36 vii), the category $\mathbf{Loc}_k(M)$ of local systems of finite-dimensional k-vector spaces over M is a tannakian category with fiber functor $\omega_x\colon \mathbf{Loc}_k(M) \to \mathbf{Vec}_k$. Recall that $\pi_1(M, x)$ acts on the fiber at x of each local system V and that associating with V the *monodromy representation*

$$\rho_V\colon \pi_1(M, x) \longrightarrow \mathrm{GL}(\omega_x(V))$$

yields an equivalence of categories from $\mathbf{Loc}_k(M)$ to $\mathbf{Rep}_k(\pi_1(M, x))$. Thus, we are in the situation of Example 4.65, and we find that the Tannaka group of the category $\mathbf{Loc}_k(M)$ is the *pro-algebraic completion* over k of the fundamental group:

$$\underline{\mathrm{Aut}}^{\otimes}(\omega_x) = \pi_1(M, x)^{\mathrm{alg}}.$$

Similarly, the *pro-unipotent completion* of the fundamental group can be recovered from the tannakian formalism. A local system V is said to be *unipotent* if the monodromy representation ρ_V is unipotent (Definition 3.154). Since being unipotent is stable under direct sums, tensor products, duals, and subquotients, the full subcategory $\mathbf{ULoc}_k(M)$ of $\mathbf{Loc}_k(M)$ consisting of unipotent local systems is a tannakian subcategory. It is equivalent to the category of finite-dimensional unipotent k-linear representations of $\pi_1(M, x)$, and its Tannaka group is the quotient

$$\pi_1(M, x)^{\mathrm{un}} = \varprojlim_{\langle (V,\rho_V)\rangle \text{ unip.}} \overline{\rho_V(\pi_1(M, x))}^{\mathrm{Zar}}$$

of the pro-algebraic completion $\pi_1(M, x)^{\mathrm{alg}}$ in which the limit only runs through the subcategories generated by unipotent representations. This is an alternative description of the pro-unipotent completion of $\pi_1(M, x)$ over k.

⋆⋆⋆

EXERCISE 4.66. Prove that the condition $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$ in the definition of a tannakian category (Definition 4.30) is necessary to deduce that an exact k-linear tensor functor $\omega\colon \mathcal{C} \to \mathbf{Vec}_k$ is faithful.

EXERCISE 4.67. Let $(\mathcal{C}, \otimes)$ be a rigid k-linear tensor category with identity object $\mathbf{1}$ satisfying $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. Recall the evaluation ev_X and coevaluation coev_X morphisms from (4.18) and (4.25). The *dimension* of X is defined as the composition

$$\mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee \xrightarrow{\psi_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbf{1},$$

viewed as an element $\dim(X) \in \mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$.

i) Prove that in the tensor category of finite-dimensional k-vector spaces, this agrees with the usual notion of dimension.

ii) Prove that every fiber functor $\omega\colon \mathcal{C} \to \mathbf{Vec}_k$ satisfies

$$\dim(X) = \dim(\omega(X))$$

for all objects $X \in \mathrm{Ob}(\mathcal{C})$, and deduce that a necessary condition for a rigid k-linear tensor category with $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$ to be tannakian is that the dimension of every object is a non-negative integer.

iii) Prove that the dimension of a super graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$, *i.e.* an object of the category $\mathbf{SGrVec}_k$ from Example 4.36 iii), equals

$$\dim(V) = \sum_{n \text{ even}} \dim V_n - \sum_{n \text{ odd}} \dim V_n.$$

Deduce that $\mathbf{SGrVec}_k$ is not a tannakian category.

EXERCISE 4.68. Consider the tannakian category $\mathbf{Vec}_k$ with the identity as the fiber functor ω. Prove that $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is the trivial group $\mathrm{Spec}(k)$.

EXERCISE 4.69. Prove the equality of matrix coefficients

$$[X \oplus Y, f \oplus g, u \oplus v] = [X, f, u] + [Y, g, v].$$

EXERCISE 4.70. Give a direct construction of an equivalence between the categories $\mathbf{GrVec}_k$ and $\mathbf{Rep}_k(\mathbb{G}_m)$.

EXERCISE 4.71. In this exercise, we give a new presentation of Example 4.57 using matrix coefficients. Let $\mathcal{C}$ be the tannakian category of split real mixed Hodge structures with fiber functor ω. Let V be a real vector space of dimension 2 along with a bigrading $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ satisfying $\overline{V^{1,0}} = V^{0,1}$.

i) Prove that all split real mixed Hodge structures of the same type as V are isomorphic to each other in $\mathcal{C}$.

ii) Show that any object $H \in \mathrm{Ob}(\mathcal{C})$ admits a decomposition

$$H = \bigoplus_{k\in\mathbb{Z}} \bigoplus_{j\geqslant 0} H_{k,j},$$

where each $H_{k,j}$ is a direct summand of $V^{\otimes k}$.

iii) Conclude from ii) that $G = \underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega)$ is a closed subgroup of $\mathrm{GL}(V)$.

iv) Let $w \in V^{1,0}$ be a non-zero vector, and write

$$v_1 = (w + \bar{w})/2, \quad v_2 = (w - \bar{w})/2i,$$

so that $\{v_1, v_2\}$ is a basis of V. Let $\{f_1, f_2\}$ be the dual basis. Prove that the $\mathbb{R}$-algebra $\mathcal{O}(G)$ is generated by the matrix coefficients

$$\alpha_{i,j} = [V, f_i, v_j] \qquad (1 \leqslant i, j \leqslant 2).$$

v) To find relations among the matrix coefficients $\alpha_{i,j}$, we study the automorphisms of V in $\mathcal{C}$. Let $\varphi\colon V \to V$ be the linear map represented by the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the basis $\{v_1, v_2\}$. Prove that φ is an automorphism of the object $V \in \mathrm{Ob}(\mathcal{C})$ if and only if $a = d$, $b = -c$, and $a^2 + b^2 \neq 0$.

vi) From v), the linear map represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is an automorphism of V in $\mathcal{C}$. Use the compatibility relations (4.47) of matrix coefficients to obtain the equalities

$$\alpha_{1,1} = \alpha_{2,2}, \quad \alpha_{1,2} = -\alpha_{2,1}.$$

Deduce that $\underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega)$ is a closed subgroup of the Deligne torus $\mathbb{S}$.

vii) Use v) and ii) to prove that there is an inclusion

$$S(\mathbb{R}) \subset \underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega)(\mathbb{R}),$$

when both groups are viewed as subsets of $\mathrm{GL}_2(\mathbb{R})$, and then deduce the equality $\underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega) = \mathbb{S}$ from vi).

Exercise 4.72 (The pro-algebraic completion of a group). Let k be a field and let Γ be an abstract group. In this exercise, we present three equivalent constructions of the *pro-algebraic completion* of Γ, which is an affine group scheme $G = \Gamma^{\mathrm{alg}}$ over k together with a group morphism $\Gamma \to G(k)$.

i) Let $\mathcal{C}$ be the category of finite-dimensional k-linear representations of Γ. Equipped with the forgetful functor, it is a tannakian category. Let G be its Tannaka group. A k-point of G is thus a collection $(\lambda_V)_{V \in \mathrm{Ob}(\mathcal{C})}$ of automorphisms $\lambda_V\colon V \to V$ satisfying the constraints of Definition 4.38. To each element $\gamma \in \Gamma$ one associates the collection of automorphisms $\lambda^\gamma = (\lambda^\gamma_V)_V$ defined as $\lambda^\gamma_V(v) = \lambda \cdot \gamma$. This yields the map $\Gamma \to G(k)$.

ii) Consider the collection of pairs (H, φ_H) consisting of an affine group scheme H over k and a group morphism $\varphi_H\colon \Gamma \to H(k)$ with Zariski dense image. We define a partial order by setting $(H, \varphi_H) \leqslant (H', \varphi_{H'})$ whenever there exists a morphism $f\colon H \to H'$ such that the induced map on k-points commutes with φ_H and $\varphi_{H'}$ and we define the pro-algebraic completion G as the projective limit

$$G = \varprojlim_{(H, \varphi_H)} H.$$

iii) The pro-algebraic completion G is an affine group scheme over k with a group morphism $\varphi\colon \Gamma \to G(k)$ such that, for any affine group scheme H over k and any group morphism $\varphi_H\colon \Gamma \to H(k)$, there exists a unique morphism of affine group schemes $f\colon G \to H$ satisfying $f \circ \varphi = \varphi_H$.

Prove that the three constructions give the same pro-algebraic group.

Exercise 4.73. Prove that a local system is unipotent if and only if can be written as an iterated extension of trivial local systems.

EXERCISE 4.74. Consider the unit circle S^1 as a topological space. Its fundamental group $\pi_1(S^1, 1)$ is isomorphic to $\mathbb{Z}$. Prove that the pro-algebraic completion $\mathbb{Z}^{\mathrm{alg}}$ is an affine group scheme that is not algebraic, while the pro-unipotent completion is the additive group:

$$\mathbb{Z}^{\mathrm{un}} \simeq \mathbb{G}_a.$$

[Hint: for the second part, use that giving a unipotent representation of $\mathbb{Z}$ is equivalent to giving a finite-dimensional vector space V together with a unipotent endomorphism of V, as well as the explicit description of the Hopf algebra of the Tannaka group from Proposition 4.54].

EXERCISE 4.75. Let $\mathcal{C}$ be a tannakian category with fiber functor ω. Show that the action of $\underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega)$ on the objects of $\mathcal{C}$ extends in a unique way to the objects of the pro and the ind-categories $\mathrm{Pro}(\mathcal{C})$ and $\mathrm{Ind}(\mathcal{C})$.

4.2. Voevodsky's category of motives.

4.2.1. *A universal cohomology.* Different cohomology theories have been proved useful in the study of algebraic varieties. For instance, as we saw in Chapter 2, to any variety X over a subfield k of $\mathbb{C}$, it is attached the Betti cohomology

$$\mathrm{H}^*_{\mathrm{B}}(X) = \mathrm{H}^*(X(\mathbb{C}), \mathbb{Q}),$$

which is a finite-dimensional graded $\mathbb{Q}$-vector space. If, in addition, X is smooth, there is also the de Rham cohomology

$$\mathrm{H}^*_{\mathrm{dR}}(X) = \mathbb{H}^*(X, \Omega^*_X),$$

which is a finite-dimensional graded k-vector space. Recall from Theorem 2.168 that de Rham and Betti cohomology are related by the comparison isomorphism

$$\mathrm{H}^*_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} \mathrm{H}^*_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.76}$$

Another important example is ℓ-adic cohomology defined, for a variety X over a field k of arbitrary characteristic p, a choice of a separable closure k^{s} of k, and a prime number ℓ different from p, as the limit

$$\mathrm{H}^*_{\ell}(X) = \varprojlim \mathrm{H}^*_{\text{ét}}(X_{k^{\mathrm{s}}}, \mathbb{Z}/\ell^n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

When k^{s} is embeddable into $\mathbb{C}$, Artin proved that there is a canonical isomorphism

$$\mathrm{H}^*_{\ell}(X) \simeq \mathrm{H}^*_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \tag{4.77}$$

A fundamental feature of these cohomology theories is that the corresponding vector spaces are usually enriched with extra structures. We have already seen that Betti cohomology can be provided with a mixed Hodge structure. Similarly, ℓ-adic cohomology carries a continuous $\mathbb{Q}_\ell$-linear action of the Galois group $\mathrm{Gal}(k^{\mathrm{s}}/k)$. This suggests to think of cohomology theories as functors from the category of varieties over k into a tannakian category.

All the cohomology theories we have mentioned satisfy similar properties, such as homotopy invariance, Poincaré duality, Künneth formulas, Mayer–Vietoris exact sequences, etc. The similarities between them, as well as the existence of comparison isomorphisms such as (4.76) or (4.77), led Grothendieck to postulate the existence of a universal cohomology theory which factors all the others: this should be the *motive* of the variety. Since its introduction by Grothendieck, the theory of motives has inspired a wealth of research. Although tremendous progress has been made, many fundamental questions remain still unanswered.

Restricting to smooth proper varieties, Grothendieck constructed a category of *pure* motives over a field k with some of the desired properties. However, in order to prove that it has all of them, he stated a set of conjectures, the *standard conjectures*, that have proved to be extremely challenging and seem to be still out of reach. Nevertheless, some of the sought properties of the category of pure motives, like Jannsen's theorem [**Jan92**] that the category of motives up to *numerical equivalence* is semisimple, have been proved without the standard conjectures.

The terminology "pure" comes from the fact that smooth proper varieties always have certain properties that are encoded in the statement "the n-th cohomology $\mathrm{H}^n(X)$ is pure of weight n". For instance, if X is a smooth proper complex variety, the group $\mathrm{H}^n_{\mathrm{B}}(X, \mathbb{C})$ has a Hodge decomposition

$$\mathrm{H}^n_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{p+q=n} \mathrm{H}^{p,q}(X).$$

The fact that only factors with $p+q=n$ appear means that its Hodge structure is pure of weight n. For varieties over a finite field, the corresponding purity is reflected by the fact that the eigenvalues of the action of Frobenius on the étale cohomology $\mathrm{H}^*_\ell(X)$ have absolute value $q^{n/2}$.

When resolution of singularities is at disposal, the cohomology of a singular non-necessarily proper variety can be expressed in terms of the cohomology of smooth proper varieties, but in this expression cohomologies of different degrees get mixed. As we have seen in Section 2.6.2 this gives rise to a mixed Hodge structure in the cohomology of X. Thus, the motive of a smooth proper variety should be *pure* while the motive of a singular or non-proper variety should be *mixed*. Since Grothendieck, there has been a great effort to develop a theory of mixed motives.

Abstractly we can think of a cohomology theory in the following way. Fix a field k, denote by $\mathbf{Var}_k$ the category of varieties over k, and let $\mathcal{A}$ be an abelian category (or more precisely a tannakian category). The derived category $\mathcal{D}^b(\mathcal{A})$ of $\mathcal{A}$ is a triangulated category provided with a t-structure (see Section A.5 for a definition) that allows us to recover $\mathcal{A}$ from $\mathcal{D}^b(\mathcal{A})$. A cohomology theory (with values in $\mathcal{A}$) is a contravariant functor

$$\mathrm{H}\colon \mathbf{Var}_k \longrightarrow \mathcal{D}^b(\mathcal{A})$$

satisfying certain properties. Out of this functor, we can recover the "cohomology groups" of X using the t-structure:

$$\mathrm{H}^n(X) = t_{\leqslant n} t_{\geqslant n} \mathrm{H}(X) \in \mathrm{Ob}(\mathcal{A}).$$

Voevodsky was able to define a triangulated category $\mathbf{DM}_{gm}(k)$ as a candidate for the derived category of mixed motives over k. The main missing piece is a suitable "motivic" t-structure. Beilinson [**Beĭ12**] showed that, when k has characteristic zero, the existence of such motivic t-structure implies the standard conjectures. Conversely, Hanamura proved in [**Han99**] that, over any field k, the conjunction of the standard conjectures and conjectures by Murre and Beilinson–Soulé implies the existence of the motivic t-structure. Thus, we seem to be back to Grothendieck's insight that a full theory of motives relies on the standard conjectures.

4.2.2. *The triangulated category of mixed motives.* Let k be a field. In what follows, we give a sketch of Voevodsky's construction of a triangulated category of mixed motives over k with rational coefficients, which will be denoted by

$$\mathbf{DM}(k) = \mathbf{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}.$$

Among the various possible approaches to this category, we present the one based on complexes of smooth varieties and finite correspondences which is the most elementary and will be tailored for the study of the motivic fundamental group. However, let us emphasize that the important theorems of the theory are proved using a different point of view, namely that of sheaves for a suitable topology on the category of smooth varieties. For more details, we refer the reader to Voevodsky's original paper [**Voe00**], the lecture notes [**MVW06**], or André's book [**And04**].

4.2.3. *First step: the category of finite correspondences.* We start with the category $\mathbf{Sm}(k)$ of smooth varieties over k. This category is not additive, for it does not make sense to "sum" two morphisms of varieties. The first step of the construction will be to enlarge the set of morphisms through the notion of finite correspondence. Grothendieck's main insight on the theory of motives was that morphisms should be related to algebraic cycles, generalizing morphisms of algebraic varieties $f\colon X \to Y$ seen as graphs $\Gamma_f \subset X \times Y$. Composition of morphisms then corresponds to intersection of cycles, which is in general only defined after "moving" them through an adequate equivalence relation. This is how the standard conjectures enter the picture. One of Voevodsky's main insight was that if we only consider a special type of algebraic cycles, then composition is defined without any equivalence relation.

DEFINITION 4.78. Let X and Y be smooth varieties over k. A *finite correspondence* from X to Y is a $\mathbb{Z}$-linear combination of integral closed subschemes

$$W \subseteq X \times Y$$

such that the projection $W \to X$ is finite and surjective over an irreducible component of the variety X.

Finite correspondences form an abelian subgroup of the group of algebraic cycles $\mathcal{Z}^{\dim Y}(X \times Y)$, which will be denoted by $c(X, Y)$.

EXAMPLE 4.79. The graph $\Gamma_f \subseteq X \times Y$ of a morphism of schemes $f\colon X \to Y$ is a finite correspondence. In general, we can think of finite correspondences as multivalued maps on an irreducible component of X.

Given smooth varieties X, Y, Z over k, we will denote by p_{XY}, p_{XZ}, and p_{YZ} the projections from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$ respectively:

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "p_{XY}"'] \arrow[d, "p_{YZ}"] \arrow[dr, "p_{XZ}"] & \\ X \times Y & Y \times Z & X \times Z. \end{tikzcd}$$

LEMMA 4.80. *Let X, Y, Z be smooth varieties over k. Given finite correspondences $W \in c(X, Y)$ and $W' \in c(Y, Z)$, the cycles $p_{XY}^*(W)$ and $p_{YZ}^*(W')$ intersect properly on $X \times Y \times Z$. Moreover, the projection of the cycle*

$$(p_{XZ})_*(p_{XY}^*\alpha \cdot p_{YZ}^*\beta)$$

is finite over X and surjective over an irreducible component.

This is proved, for instance, in [**MVW06**, Lem. 1.7]. Thanks to the this lemma, we can define a composition of finite correspondences as follows

$$\begin{aligned} \circ\colon\ c(X, Y) \times c(Y, Z) &\longrightarrow c(X, Z) \\ (\alpha, \beta) \qquad &\longmapsto \ \ \alpha \circ \beta \ = (p_{XZ})_*(p_{XY}^*\alpha \cdot p_{YZ}^*\beta). \end{aligned} \tag{4.81}$$

The category $\mathbf{SmCor}(k)$ has the same objects as $\mathbf{Sm}(k)$, but the morphisms are given by finite correspondences with $\mathbb{Q}$-coefficients:

$$\mathrm{Hom}_{\mathbf{SmCor}(k)}(X, Y) = c(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

There is a functor $\mathbf{Sm}(k) \to \mathbf{SmCor}(k)$ that is the identity on objects and sends a map $f\colon X \to Y$ to its graph Γ_f. By Exercise 4.109, the composition of maps is compatible with the composition (4.81) of finite correspondences. We denote by $[X]$ the image in $\mathbf{SmCor}(k)$ of a smooth variety X.

The direct sum in $\mathbf{SmCor}(k)$ is given by the disjoint union of varieties. This category is also equipped with the tensor product

$$[X] \otimes [Y] = [X \times_k Y].$$

Along with these structures, $\mathbf{SmCor}(k)$ is a $\mathbb{Q}$-linear tensor category.

4.2.4. *Second step: a triangulated category satisfying homotopy invariance and Mayer–Vietoris.* The second step is similar to the construction of the derived category of an abelian category. We start with the category

$$C^b(\mathbf{SmCor}(k))$$

of bounded chain complexes in $\mathbf{SmCor}(k)$. The objects are diagrams

$$\cdots \longrightarrow [X_n] \xrightarrow{\partial_n} [X_{n-1}] \longrightarrow \cdots,$$

where X_i is an object of $\mathbf{Sm}(k)$ and $\partial_n \in c(X_n, X_{n-1}) \otimes \mathbb{Q}$ are finite correspondences such that $\partial_{n-1} \circ \partial_n = 0$. Then we define the homotopy category

$$K^b(\mathbf{SmCor}(k))$$

as the one having the same objects as $C^b(\mathbf{SmCor}(k))$, and morphisms given by homotopy classes of morphisms of complexes. It is a triangulated category as explained in Section A.3.2.

Two examples of objects of $K^b(\mathbf{SmCor}(k))$ are:

$\mathbb{A}^1$-homotopy complex: for any X in $\mathbf{Sm}(k)$, the complex

$$\underset{1}{[X \times \mathbb{A}^1]} \xrightarrow{\mathrm{pr}} \underset{0}{[X]}$$

placed in degrees 1 and 0, as the indices show. In the future, we will use such kind of indices to indicate the degree.

Mayer–Vietoris complex: for any smooth scheme X in $\mathbf{Sm}(k)$ and any open cover $X = U \cup V$, the complex

$$\underset{2}{[U \cap V]} \xrightarrow{i_{U\cap V,U} + i_{U\cap V,V}} \underset{1}{[U] \oplus [V]} \xrightarrow{i_{U,X} - i_{V,X}} \underset{0}{[X]},$$

where the arrows $i_{U,X}, i_{V,X}, i_{U\cap V,U}$, and $i_{U\cap V,V}$ are the inclusions.

We want to force the homotopy invariance and the Mayer–Vietoris property, which mean that the above two complexes should become acyclic. To this end, we take the Verdier localization (see Section A.3.3) of $K^b(\mathbf{SmCor}(k))$ with respect to the *thick* triangulated subcategory generated by all homotopy and Mayer–Vietoris complexes. As in Proposition A.90, this localization is again endowed with the structure of a triangulated category.

4.2.5. *Third step: the pseudo-abelian envelope.* We next turn the quotient from the previous step into a category in which some morphisms have kernels. The relevant notion is that of a *pseudo-abelian category*, an additive category in which every idempotent (*i.e.* every morphism $p \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ satisfying $p^2 = p$) admits a kernel. There is a canonical way to make an additive category pseudo-abelian, which roughly speaking consists in formally adding those kernels.

DEFINITION 4.82. Let $\mathcal{C}$ be an additive category. The *pseudo-abelian envelope* of $\mathcal{C}$ is the category $\mathcal{C}_{pa}$ with

- *objects*: (X, p) with $X \in \mathrm{Ob}(\mathcal{C})$ and $p \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ an idempotent;
- *morphisms*: $\mathrm{Hom}_{\mathcal{C}_{pa}}((X, p), (Y, q)) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the subgroup of those morphisms $f \colon X \to Y$ satisfying $f = q \circ f \circ p$.

The functor $\mathcal{C} \to \mathcal{C}_{pa}$ sending X to (X, Id) is fully faithful. Moreover, each idempotent p defines a morphism $p \colon (X, p) \to (X, \mathrm{Id})$ in $\mathcal{C}_{pa}$, which is a monomorphism. Indeed, if some morphisms $g, h \colon (Z, q) \to (X, p)$ satisfy $p \circ g = p \circ h$, then the equalities $g = p \circ g \circ q$, $h = p \circ h \circ q$, and $p^2 = p$ imply $g = h$. If p is an idempotent, then so is $\mathrm{Id} - p$, and the object $(X, \mathrm{Id} - p)$ is a kernel of $p \colon X \to X$ viewed in the category $\mathcal{C}_{pa}$. In fact, there is a direct sum decomposition

$$(X, \mathrm{Id}) = (X, p) \oplus (X, \mathrm{Id} - p).$$

This will be crucial when we want to talk about "pieces of the cohomology".

DEFINITION 4.83. The category $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is defined as the pseudo-abelian envelope of the category obtained in the previous step.

REMARK 4.84. By a result of Balmer and Schlichting [BS01], the pseudo-abelian envelope $\mathcal{T}_{pa}$ of a triangulated category $\mathcal{T}$ has a unique structure of triangulated category such that the functor $\mathcal{T} \to \mathcal{T}_{pa}$ is triangulated. Thus, $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is still a triangulated category.

There is a functor $M \colon \mathbf{Sm}(k) \to \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ sending X to $[X]$, regarded as a complex concentrated in degree zero. The category $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is also equipped with a tensor product that is characterized by the property

$$M(X) \otimes M(Y) = M(X \times Y).$$

The identity object is the motive of the base field, which will be denoted by

$$\mathbb{Q}(0) = M(\mathrm{Spec}(k)).$$

Note also that there is a functor

$$C^b(\mathbf{SmCor}(k)_{\mathrm{pa}}) \longrightarrow \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \tag{4.85}$$

from the category of bounded complexes in the pseudo-abelian envelope of the category $\mathbf{SmCor}(k)$ to the category of effective motives $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

4.2.6. *Fourth step: inversion of the Tate motive.* Given a smooth variety X over k, we can think of the structure morphism $X \to \mathrm{Spec}(k)$ as a complex

$$\underset{0}{[X]} \longrightarrow \underset{-1}{[\mathrm{Spec}(k)]}. \tag{4.86}$$

DEFINITION 4.87. The *reduced motive* of X is the object $\widetilde{M}(X)$ of $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ determined by the complex (4.86).

When X has a k-rational point, the motive of X decomposes as a direct sum

$$M(X) = \mathbb{Q}(0) \oplus \widetilde{M}(X),$$

and this decomposition is independent of the choice of the point (see Exercise 4.110).

Definition 4.88. The *Tate motive* $\mathbb{Q}(1)$ is $\widetilde{M}(\mathbb{P}^1_k)[-2]$. For each integer $n \geqslant 0$, one defines $\mathbb{Q}(n)$ as $\mathbb{Q}(1)^{\otimes n}$.

The last step of the construction of $\mathbf{DM}(k)$, necessary to obtain a rigid tensor category, is to formally invert the motive $\mathbb{Q}(1)$. By this we mean the following: an object of the new category $\mathbf{DM}(k)$ is a pair (M, m), where M is an object of $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ and $m \in \mathbb{Z}$. Morphisms are given by

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{DM}(k)}((M,m),(N,n))\\ &\qquad = \varinjlim_{r \geqslant -m,-n} \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M \otimes \mathbb{Q}(m+r), N \otimes \mathbb{Q}(n+r)).\end{aligned}$$

The resulting category has the following property:

Theorem 4.89 (Voevodsky). *The category $\mathbf{DM}(k)$ is a rigid tensor $\mathbb{Q}$-linear triangulated category.*

Proof. See [**MVW06**, Thm. 20.17]. □

4.2.7. *Properties of* $\mathbf{DM}(k)$. All the usual machinery to compute the homology of algebraic varieties is still available in Voevodsky's category.

Künneth: $M(X \times Y) = M(X) \otimes M(Y)$.

$\mathbb{A}^1$**-homotopy invariance:** The projection map $X \times \mathbb{A}^1 \to X$ induces an isomorphism

$$M(X \times \mathbb{A}^1) = M(X).$$

Mayer–Vietoris: For $X = U \cup V$ as before, there is a distinguished triangle

$$M(U \cap V) \longrightarrow M(U) \oplus M(V) \longrightarrow M(X) \longrightarrow M(U \cap V)[1].$$

Gysin: If $Z \subset X$ is a smooth closed subscheme of codimension c of a smooth scheme X, then there is a distinguished triangle

$$M(X \setminus Z) \longrightarrow M(X) \longrightarrow M(Z)(c)[2c] \longrightarrow M(X \setminus Z)[1], \tag{4.90}$$

where the first morphism is the one induced by the immersion $Z \hookrightarrow X$.

Blow-up: Let $Z \subseteq X$ be a smooth closed subscheme of a smooth scheme. Let $\mathrm{Bl}_Z X$ the the blow-up of X along Z and E the exceptional divisor. Then there is a distinguished triangle

$$M(E) \longrightarrow M(\mathrm{Bl}_Z X) \oplus M(Z) \longrightarrow M(X) \longrightarrow M(E)[1].$$

Moreover, if Z has codimension c, this yields a canonical isomorphism

$$M(\mathrm{Bl}_Z X) = M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)(i)[2i].$$

Duality: There is a duality functor $A \mapsto A^\vee$ that, for X smooth and proper of dimension d, satisfies

$$M(X)^\vee = M(X)(-d)[-2d].$$

Adjunction: The duality and tensor product are related by

$$\operatorname{Hom}(A \otimes B^\vee, C) = \operatorname{Hom}(A, C \otimes B),$$
$$\operatorname{Hom}(A \otimes B, C) = \operatorname{Hom}(B, A^\vee \otimes C).$$

In other words, these functors are adjoint to each other.

REMARK 4.91. The functor from $\mathbf{Sm}(k)$ to $\mathbf{DM}(k)$ is covariant, and hence is a "homological" functor, in contrast to the contravariant functor chosen by Grothendieck for pure motives that was cohomological.

EXAMPLE 4.92 (Motive of projective space). Let us combine some of the above properties to prove that the motive of projective space $\mathbb{P}^n$ is equal to

$$M(\mathbb{P}^n) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2] \oplus \cdots \oplus \mathbb{Q}(n)[2n]$$

for each $n \geqslant 0$. This equality is to be compared with the computation of the Hodge structure on the cohomology of $\mathbb{P}^n$ (Example 2.221), but keeping in mind that $M(\mathbb{P}^n)$ is the universal *homology* of $\mathbb{P}^n$ rather than its cohomology.

We proceed by induction on n, the cases $n = 0$ and $n = 1$ being reduced to the definitions of $\mathbb{Q}(0)$ and $\mathbb{Q}(1)$. For $n \geqslant 2$, choosing a closed immersion $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ with complement $\mathbb{P}^n \setminus \mathbb{P}^{n-1} = \mathbb{A}^n$, we get the Gysin distinguished triangle

$$M(\mathbb{A}^n) \longrightarrow M(\mathbb{P}^n) \longrightarrow M(\mathbb{P}^{n-1})(1)[2] \longrightarrow M(\mathbb{A}^n)[1]. \tag{4.93}$$

Note that the structure morphism $\mathbb{A}^n \to \operatorname{Spec}(k)$ induces an isomorphism

$$M(\mathbb{A}^n) \simeq M(\operatorname{Spec}(k)) = \mathbb{Q}(0),$$

as one can prove by repeatedly applying the $\mathbb{A}^1$-homotopy invariance. Factorizing that structure morphism as $\mathbb{A}^n \to \mathbb{P}^n \to \operatorname{Spec}(k)$, we also see that the composition

$$M(\mathbb{A}^n) \longrightarrow M(\mathbb{P}^n) \longrightarrow M(\operatorname{Spec}(k)) = M(\mathbb{A}^n)$$

is the identity map $\mathbb{Q}(0) \to \mathbb{Q}(0)$. Thus, the triangle (4.93) is split (Definition A.75 and Proposition A.76) and

$$M(\mathbb{P}^n) = \mathbb{Q}(0) \oplus M(\mathbb{P}^{n-1})(1)[2].$$

The result then follows by induction.

REMARK 4.94. To understand the different roles of the twist and the shift, it is instructive to compare the reduced motives of $\mathbb{P}^1$ and $\mathbb{G}_m$. In the first case,

$$\widetilde{M}(\mathbb{P}^1) = \mathbb{Q}(1)[2]$$

by the definition of the right-hand side. In the second case, the Mayer–Vietoris triangle for the open cover $\mathbb{P}^1 = U \cup V$, with $U = \mathbb{P}^1 \setminus \{0\}$ and $V = \mathbb{P}^1 \setminus \{\infty\}$ reads

$$M(\mathbb{G}_m) \longrightarrow \mathbb{Q}(0) \oplus \mathbb{Q}(0) \longrightarrow \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2] \longrightarrow M(\mathbb{G}_m)[1].$$

From the octahedron axiom (see the version in Remark A.72), we deduce a distinguished triangle

$$M(\mathbb{G}_m) \longrightarrow \mathbb{Q}(0) \longrightarrow \mathbb{Q}(1)[2] \longrightarrow M(\mathbb{G}_m)[1].$$

By the discussion in Example 4.92, the middle arrow in the previous triangle is zero, from which it follows that $M(\mathbb{G}_m) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1]$, and hence

$$\widetilde{M}(\mathbb{G}_m) = \mathbb{Q}(1)[1]. \tag{4.95}$$

This can be compared with the fact that, for any of the classical cohomology theories, the groups $\mathrm{H}^1(\mathbb{G}_m)$ and $\mathrm{H}^2(\mathbb{P}^1)$ are isomorphic, but they lie in different degree. In particular, the Hodge structure $\mathrm{H}^2(\mathbb{P}^1)$ is pure of weight 2 and Hodge type $(1,1)$. The same is true for $\mathrm{H}^1(\mathbb{G}_m)$, but, since this last group lies in degree one, we consider it as a mixed Hodge structure.

EXAMPLE 4.96 (Relative motive). Let $\iota\colon Z \to X$ be a closed immersion of smooth schemes. We define the relative motive of the pair (X,Z) as

$$M(X,Z) = M([Z \to X]),$$

where Z sits in degree 1 and X in degree 0. In Exercise 4.111, you will see that there is a distinguished triangle

$$M(Z) \longrightarrow M(X) \longrightarrow M(X,Z) \longrightarrow M(Z)[1].$$

EXAMPLE 4.97 (Motive of a union of smooth closed subschemes in good position). Let X be a smooth scheme and let $Y_0, Y_1, \dots, Y_n$ be smooth closed subschemes of X such that, for each $I \subset \{0,\dots,n\}$, the schematic intersection

$$Y_I = \bigcap_{i \in I} Y_i$$

is smooth. A variant of the construction from Section 3.6.2 allows us to define the motive of $Y = \bigcup Y_i$. Namely, consider the complex

$$C_Y = \left[0 \longrightarrow \bigoplus_{|I|=n+1} Y_I \longrightarrow \bigoplus_{|I|=n} Y_I \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=1} Y_I \longrightarrow 0 \right],$$

where the piece $\bigoplus_{|I|=k} Y_I$ sits in degree k and the differentials are the same as in Section 3.6.2. Then we define

$$M(Y) = M(C_Y)[-1].$$

The relative motive $M(X,Y)$ is defined as the motive of the complex

$$C_{X,Y} = \left[0 \longrightarrow \bigoplus_{|I|=n+1} Y_I \longrightarrow \bigoplus_{|I|=n} Y_I \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=1} Y_I \longrightarrow X \longrightarrow 0 \right].$$

By Exercise 4.111, there is a distinguished triangle

$$M(Y) \longrightarrow M(X) \longrightarrow M(X,Y) \longrightarrow M(Y)[1]. \tag{4.98}$$

4.2.8. *Motivic cohomology.* Voevodsky also computed some morphism groups in the category $\mathbf{DM}(k)$. In particular, he defined the following:

DEFINITION 4.99. The *motivic cohomology* of X is

$$\mathrm{H}^n_{\mathcal{M}}(X, \mathbb{Q}(p)) = \mathrm{Hom}_{\mathbf{DM}(k)}(M(X), \mathbb{Q}(p)[n]).$$

Before stating the next result, we need to make a short digression about algebraic K-theory. For more details on algebraic K-theory, see for instance [**Wei13**] and the references therein.

REMARK 4.100. With every smooth variety X over k, Quillen [**Qui73**] has associated a graded ring

$$K_*(X) = \bigoplus_{n \geqslant 0} K_n(X).$$

There is a family $(\psi^\ell)_{\ell\geqslant 0}$, of functorial ring endomorphisms of $K_*(X)$ called the *Adams operations*, which satisfy the relations $\psi^\ell\circ\psi^m=\psi^{\ell m}$. In particular, the Adams operations commute among themselves. These operations allow us to decompose the K-groups into eigenspaces. More concretely, we write

$$(K_n(X)\otimes_{\mathbb{Z}}\mathbb{Q})^{(p)} \tag{4.101}$$

for the maximal subspace of $K_n(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ where the endomorphism $\psi^\ell-\ell^p\,\mathrm{Id}$ is nilpotent. The subspaces (4.101) are called the eigenspaces for the Adams operations. Then there is a decomposition

$$K_n(X)\otimes_{\mathbb{Z}}\mathbb{Q}=\bigoplus_{p\geqslant 0}(K_n(X)\otimes_{\mathbb{Z}}\mathbb{Q})^{(p)}.$$

In this book, we will only need the case $X=\mathrm{Spec}(k)$.

The eigenspaces for the Adams operations acting on algebraic K-theory, can be seen as a universal cohomology theory with coefficients in $\mathbb{Q}$ that has characteristic classes for vector bundles. In fact, using Bloch's formula relating *higher Chow groups* and K-theory ([**Voe02**], [**Blo86**], [**Lev94**]), Voevodsky proves the following:

THEOREM 4.102 (Voevodsky). *Given a smooth variety X over k, the motivic cohomology groups of X are isomorphic to*

$$\mathrm{H}^n_{\mathcal{M}}(X,\mathbb{Q}(p))=(K_{2p-n}(X)\otimes_{\mathbb{Z}}\mathbb{Q})^{(p)}.$$

4.2.9. *The normalization of a cosimplicial scheme.* To every variety X over a field k, not necessarily smooth, one attaches a motive $M(X)$ in Voevodsky's category $\mathbf{DM}(k)$. Using tools from homological algebra, one can construct more general motives, for instance the motive of a cosimplicial variety.

Recall that in Section A.8.2 we defined the normalized complex associated with a cosimplicial object in an abelian category. It turns out that it is enough to work in a pseudo-abelian category.

LEMMA 4.103. *Let $X^\bullet$ be a cosimplicial object of the category $\mathbf{Sm}(k)$. Given integers $m>n\geqslant 0$, the endomorphism*

$$p_n=(1-\delta^0\sigma^0)(1-\delta^1\sigma^1)\cdots(1-\delta^n\sigma^n):[X^m]\longrightarrow[X^m]$$

in an idempotent in the category $\mathbf{SmCor}(k)$.

PROOF. We argue by induction on n. For $n=0$, the relation $\sigma^0\delta^0=\mathrm{Id}$ implies that $\delta^0\sigma^0$ is an idempotent, and hence the same holds for $1-\delta^0\sigma^0$. Let us now assume that p_{n-1} is idempotent. We next observe that, for $i=0,\dots,n-1$, the face σ^n commutes with $\delta^i\sigma^i$. Indeed, by relations (c) and (b) in (A.224),

$$\sigma^n(\delta^i\sigma^i)=\delta^i\sigma^{n-1}\sigma^i=(\delta^i\sigma^i)\sigma^n.$$

Moreover, relation (d) in (A.224) implies the equality $\sigma^n(1-\delta^n\sigma^n)=0$. These two equations together imply

$$\sigma^n(1-\delta^0\sigma^0)\cdots(1-\delta^n\sigma^n)=0. \tag{4.104}$$

We now compute, using equation (4.104), and the induction hypothesis,

$$\begin{aligned} p_n^2 = &\underbrace{(1-\delta^0\sigma^0)\cdots(1-\delta^{n-1}\sigma^{n-1})}_{p_{n-1}}(1-\delta^n\sigma^n) \\ &\qquad\underbrace{(1-\delta^0\sigma^0)\cdots(1-\delta^{n-1}\sigma^{n-1})}_{p_{n-1}}(1-\delta^n\sigma^n) \\ &\qquad\qquad = p_{n-1}^2(1-\delta^n\sigma^n) = p_{n-1}(1-\delta^n\sigma^n) = p_n, \end{aligned}$$

as we wanted to show. □

Since p_n is idempotent, $\mathrm{Im}(p_n)$ is an object of the pseudo-abelian envelope of $\mathbf{SmCor}(k)$. By convention, we write $p_{-1} = \mathrm{Id}$.

DEFINITION 4.105. Let $X^\bullet$ be a cosimplicial object in $\mathbf{Sm}(k)$. The *normalization* of $X^\bullet$ is the complex in $\mathbf{SmCor}(k)_{\mathrm{pa}}$ given by

$$\mathcal{N}(X^\bullet)^n = \mathrm{Im}(p_{n-1}\colon [X^n] \longrightarrow [X^n]),$$

together with the differential

$$\mathrm{d} = \sum_{i=0}^{n+1}(-1)^i\delta^i\colon \mathcal{N}(X^\bullet)^n \longrightarrow \mathcal{N}(X^\bullet)^{n+1}.$$

If the cosimplicial object $X^\bullet$ is not bounded, then the complex $\mathcal{N}(X^\bullet)$ is not bounded in general. To obtain a bounded complex, we consider the *bête* truncation $\sigma_{\leqslant N}\mathcal{N}(X^\bullet)$ defined as

$$\sigma_{\leqslant N}\mathcal{N}(X^\bullet)^n = \begin{cases} \mathcal{N}(X^\bullet)^n, & \text{if } n \leqslant N, \\ 0, & \text{if } n > N. \end{cases}$$

This is now an element of $C^b(\mathbf{SmCor}(k)_{\mathrm{pa}})$. For each $N \geqslant 0$, applying the functor (4.85), we obtain a motive

$$[\sigma_{\leqslant N}\mathcal{N}(X^\bullet)].$$

Clearly, given integers $M \geqslant N \geqslant 0$, there is a morphism of complexes

$$\sigma_{\leqslant M}\mathcal{N}(X^\bullet) \longrightarrow \sigma_{\leqslant N}\mathcal{N}(X^\bullet).$$

The system $([\sigma_{\leqslant N}\mathcal{N}(X^\bullet)])_{N\geqslant 0}$ is a pro-object in $\mathbf{DM}(k)$.

REMARK 4.106. The advantage of using Lemma 4.103 is that it provides us with an explicit idempotent cutting out the normalized complex from the cochain complex. However, we could have also constructed it directly by abstract means, as J. Ayoub pointed to us. Recall from Definition A.1 that a category is said to be *preadditive* if the morphism sets are abelian groups and the composition of maps is bilinear. Given a preadditive category $\mathcal{A}$, let $\mathbf{Ab}(\mathcal{A})$ denote the category of presheaves of abelian groups on $\mathcal{A}$, by which we simply mean additive contravariant functors from $\mathcal{A}$ to the category $\mathbf{Ab}$ of abelian groups. Then $\mathbf{Ab}(\mathcal{A})$ is an abelian category, and Yoneda's lemma ensures that the natural functor

$$\begin{aligned} h\colon\ \mathcal{A} &\longrightarrow \mathbf{Ab}(\mathcal{A}) \\ X &\longmapsto \mathrm{Hom}(-,X) \end{aligned}$$

is fully faithful. Assume now that $\mathcal{A}$ is pseudo-abelian. If Y' is a direct factor of an object of the form $h(X)$, then projecting to the complement one gets an idempotent p of $h(X)$ such that $Y' = \mathrm{Ker}(p)$. By fully-faithfulness, we can see p as an idempotent of X, and the object $Y = \mathrm{Ker}(p)$ in $\mathcal{A}$, determined up to unique isomorphism, satisfies $h(Y) = Y'$. If $X^\bullet$ is a cosimplicial object in $\mathcal{A}$, then the associated cochain complex CX^* is a complex in $\mathcal{A}$ whose formation commutes with the functor h, in the sense that $h(CX^*) = C^*(h(X^\bullet))$ holds. Since $\mathbf{Ab}(\mathcal{A})$ is abelian, the normalized complex $\mathcal{N}^*(h(X^\bullet))$, as introduced in Section A.8.2, is a direct factor of $C^*(h(X^\bullet))$. Proceeding as above, one gets a complex (up to unique isomorphism) $\mathcal{N}X^*$ such that $h(\mathcal{N}X^*) = \mathcal{N}^*(h(X^\bullet))$.

4.2.10. *Hodge realization.* We now assume that k is a subfield of the complex numbers, and discuss the construction of a Hodge realization functor with values in the derived category of mixed Hodge structures over k.

THEOREM 4.107. *There is a $\mathbb{Q}$-linear tensor triangulated functor*

$$R^{\mathrm{H}}\colon \mathbf{DM}(k) \longrightarrow \mathcal{D}^b(\mathbf{MHS}(k)).$$

We sketch a proof of this theorem following [**DG05**, § 1.5]. The main difficulty is the covariance for finite correspondences. As in loc. cit., we will give a cohomological version of the construction which fits better within the framework we have developed to construct mixed Hodge structures.

Let X_* be a bounded homological complex in $\mathbf{SmCor}(k)$. We can assume that each term X_m is a quasi-projective smooth scheme over k. The differential

$$d\colon X_m \longrightarrow X_{m-1}$$

is given by a correspondence Γ_m. The first step is the following result that follows from [**DG05**, Lem. 1.5.1].

LEMMA 4.108. *For each m, there exists a smooth projective scheme $\overline{X}_m$ and an open immersion $X_m \to \overline{X}_m$ such that $D_m = \overline{X}_m \setminus X_m$ is a simple normal crossing divisor, and a correspondence $\overline{\Gamma}_m\colon \overline{X}_m \to \overline{X}_{m-1}$ extending the correspondence Γ_m and such that $\overline{X}_*$ is still a complex.*

After applying Lemma 4.108 to the complex X_*, each pair $(\overline{X}_m, D_m)$ gives rise to a mixed Hodge complex

$$A^{\mathrm{H}}_{\overline{X}_m}(\log D_m) = ((A_{\mathrm{dR}}, W, F), (A_{\mathrm{B}}, W), (A_{\mathbb{C}}, W), \alpha, \beta)$$

by the construction of Definition 2.279. Here,

$$(A_{\mathrm{dR}}, W, F) = (\Gamma(\overline{X}_m, \mathrm{Gd}(\Omega^*_{\overline{X}_m}(\log D_m))), W, F),$$

$$(A_{\mathrm{B}}, W) = \left(\Gamma(\overline{X}^{\mathrm{an}}_m, j_* \,\mathrm{Gd}(\underline{\mathbb{Q}})), \tau\right),$$

and $(A_{\mathbb{C}}, W)$ is a cone of cones whose components are global sections of Godement resolutions of sheaves of holomorphic differentials.

We need to show that the finite correspondence Γ_m induces a morphism of Hodge complexes

$$\Gamma^*_m\colon A^{\mathrm{H}}_{\overline{X}_{m-1}}(\log D_{m-1}) \longrightarrow A^{\mathrm{H}}_{\overline{X}_m}(\log D_m)$$

By linearity, we can assume that Γ_m is an irreducible subvariety of $X_m \times X_{m-1}$ which is finite over X_m and dominant over an irreducible component of X_m. The Betti component of Γ^*_m is given by the map $\mathrm{Tr}_{\Gamma_m/X_m} \circ p_2^*$, where $p_2\colon \Gamma_m \to X_{m-1}$ is the restriction to Γ_m of the second projection, and

$$\mathrm{Tr}_{\Gamma_m/X_m}\colon (p_1)_* p_2^* j_* \,\mathrm{Gd}(\underline{\mathbb{Q}}) \to j_* \,\mathrm{Gd}(\underline{\mathbb{Q}})$$

is the trace map from Example A.281, the unique extension of the map given by

$$\operatorname{Tr}_{\Gamma_m/X_m}(s)(x) = \sum_{p_1(y)=x} s(y)$$

on the open subset of X_m where the projection $p_1\colon \Gamma_m \to X_m$ is étale (here, we are considering the sections of $\underline{\mathbb{Q}}$ as locally constant functions).

The construction of the algebraic de Rham component of the morphism, which also uses a variant of the trace map, is more involved. The argument is sketched in [**DG05**, §1.5]. Similar techniques can be adapted to define also a trace at the level of holomorphic de Rham complexes, and hence for $(A_{\mathbb{C}}, W)$. Since all the maps are defined using traces one can check that the constructed morphisms are compatible with the comparison isomorphisms α and β. It is also possible to check that putting together all the different mixed Hodge complexes for different m's and the differential d' obtained from the correspondences, one obtains a dg-mixed Hodge complex as in Definition 2.267. By Proposition 2.270, the associated total complex is a mixed Hodge complex, which thanks to Theorem 2.266 defines an object of the derived category $\mathcal{D}^b(\mathbf{MHS}(k))$

The method we sketched applies in particular to the motive $[\sigma_{\leqslant N}\mathcal{N}(X^\bullet)]$ from the previous section. In this case, things are a little simpler because the maps defining the motive $[\sigma_{\leqslant N}\mathcal{N}(X^\bullet)]$ are morphisms of schemes and not general correspondences, so usual functoriality is enough and we do not need to use traces. Let $X^\bullet$ be a cosimplicial object in $\mathbf{Sm}(k)$. Assume that there is an embedding

$$j_\bullet\colon X^\bullet \longrightarrow \overline{X}^\bullet$$

of cosimplicial smooth varieties over k such that the variety $\overline{X}^n$ is smooth and proper, and $D^n = \overline{X}^n \setminus X^n$ is a simple normal crossing divisor for each n. We obtain a simplicial mixed Hodge complex $A^H_{\overline{X}^\bullet}(\log D^\bullet)$. Taking the normalization in the category of mixed Hodge complexes, we obtain a dg-mixed Hodge complex

$$\mathcal{N}A^H_{\overline{X}^\bullet}(\log D^\bullet).$$

The *bête* truncation $\sigma_{\leqslant N}\mathcal{N}A^H_{\overline{X}^\bullet}(\log D^\bullet)$ is still a dg-mixed Hodge complex, and

$$\operatorname{Tot}\left(\sigma_{\leqslant N}\mathcal{N}A^H_{\overline{X}^\bullet}(\log D^\bullet)\right)$$

is a mixed Hodge complex representing the Hodge realization $R^H([\sigma_{\leqslant N}\mathcal{N}X^\bullet])$.

⋆ ⋆ ⋆

Exercise 4.109. Prove that the composition of the finite correspondences given by the graphs of two morphisms of algebraic varieties $f\colon X \to Y$ and $g\colon Y \to Z$, as defined in (4.81), is the graph of the composition $g \circ f\colon X \to Z$.

Exercise 4.110. Let X be a smooth variety over k and $x\colon \operatorname{Spec}(k) \to X$ a k-point. Consider the composition

$$p\colon X \longrightarrow \operatorname{Spec}(k) \xrightarrow{x} X.$$

i) Show that p is a projector and the class of $(X, 1-p)$ agrees with the reduced motive $\widetilde{M}(X)$ from Definition 4.87. Thus, there is a decomposition

$$M(X) = \mathbb{Q}(0) \oplus \widetilde{M}(X),$$

that does not depend on the choice of the point x.

ii) Show that there is an isomorphism $\widetilde{M}(X) \simeq M(X, x)$.

Exercise 4.111. Let

$$C = [0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow 0]$$

be a complex in $\mathbf{SmCor}(k)$, and write C' for its truncation

$$C' = [0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow 0].$$

Show that there is a distinguished triangle

$$M(X_0) \longrightarrow M(C) \longrightarrow M(C') \longrightarrow M(X_0)[1]$$

in $\mathbf{DM}(k)$. Conclude that, if $\iota\colon Z \to X$ is a closed immersion of smooth schemes, then there is a distinguished triangle

$$M(Z) \longrightarrow M(X) \longrightarrow M(X, Z) \longrightarrow M(Z)[1].$$

4.3. Mixed Tate motives over a number field. As already mentioned, we do not know how to find a t-structure on Voevodsky's triangulated category $\mathbf{DM}(k)$ that would give rise to an equivalence

$$\mathbf{DM}(k) \cong \mathcal{D}^b(\mathbf{MM}(k))$$

with the derived category of the sought-after tannakian category of mixed motives. Instead of working with the whole $\mathbf{DM}(k)$, one can first try to consider the triangulated subcategory $\mathbf{DMT}(k)$ generated by the simplest non-trivial objects: all pure Tate motives $\mathbb{Q}(n)$. The objects of this subcategory are to motives what mixed Tate Hodge structures are to all mixed Hodge structures. When k is a number field, Levine [**Lev93**] figured out how to define a t-structure on $\mathbf{DMT}(k)$; its heart is the abelian category of mixed Tate motives over k. The keystone of the construction is Borel's computation of the K-theory of number fields, which ensures the necessary vanishing of morphism groups for a t-structure to exist.

4.3.1. *The triangulated category of mixed Tate motives.* Recall that Voevodsky's category $\mathbf{DM}(k)$ is a rigid tensor $\mathbb{Q}$-linear category with identity object

$$\mathbb{Q}(0) = M(\mathrm{Spec}(k)),$$

and that it contains the Tate motive

$$\mathbb{Q}(1) = \widetilde{M}(\mathbb{P}^1_k)[-2].$$

The *Lefschetz motive* $\mathbb{Q}(-1)$ is defined as its dual

$$\mathbb{Q}(-1) = \mathbb{Q}(1)^\vee,$$

and the object $\mathbb{Q}(n)$ as the tensor power

$$\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n},$$

with the usual convention of replacing $\mathbb{Q}(1)$ with $\mathbb{Q}(-1)$ for negative n. Therefore, there are canonical isomorphisms:

$$\mathbb{Q}(n) \otimes \mathbb{Q}(m) \cong \mathbb{Q}(n+m), \quad \mathbb{Q}(n)^\vee = \mathbb{Q}(-n).$$

As these are the simplest objects of Voevodsky's category, it is natural to investigate what can be built out of them.

Definition 4.112. The *triangulated category of mixed Tate motives over k* is the smallest triangulated full subcategory of $\mathbf{DM}(k)$ that contains the objects $\mathbb{Q}(n)$ for all $n \in \mathbb{Z}$. We denote it by $\mathbf{DMT}(k)$.

Concretely (see Definition A.70 for the notion of a triangulated subcategory), this means that $\mathbf{DMT}(k)$ contains all shifts $\mathbb{Q}(n)[m]$ and that, if

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

is a distinguished triangle in $\mathbf{DM}(k)$ such that two objects among A, B, C belong to the subcategory $\mathbf{DMT}(k)$, then so does the third. Hence, every object of $\mathbf{DMT}(k)$ is an iterated extension of the objects $\mathbb{Q}(n)[m]$.

EXAMPLE 4.113. Let $C = \mathbb{P}^1 \setminus S$ be the complement of a non-empty finite set of k-points S in the projective line $\mathbb{P}^1$. By the Gysin triangle (4.90), the motive of the curve C sits into a distinguished triangle

$$M(C) \longrightarrow M(\mathbb{P}^1) \longrightarrow M(S)(1)[2] \longrightarrow M(C)[1]$$

in the category $\mathbf{DM}(k)$. The motives $M(\mathbb{P}^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2]$ and $M(S) = \mathbb{Q}(0)^{|S|}$ belong to $\mathbf{DMT}(k)$, and hence so does $M(C)$. Since S is non-empty, we can also write $C = \mathbb{A}^1 \setminus S'$ and consider the Gysin triangle

$$M(C) \longrightarrow M(\mathbb{A}^1) \longrightarrow M(S')(1)[2] \longrightarrow M(C)[1].$$

This has the advantage that the leftmost arrow now factors through the diagram

$$\begin{tikzcd}
M(C) \arrow[rr] \arrow[dr, shift left] & & M(\mathbb{A}^1), \arrow[dl, shift left] \\
& \mathbb{Q}(0) \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

in which the diagonal morphisms are induced by the structure morphisms and the inclusion of a k-rational point of C. The maps between $\mathbb{Q}(0)$ and $M(\mathbb{A}^1)$ are isomorphisms inverse to each other, hence a retraction $M(\mathbb{A}^1) \to M(C)$ making the distinguished triangle split by Proposition A.76. We deduce

$$M(C) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)^{|S|-1}[1].$$

This is to be compared to Example 2.307, in which we computed the mixed Hodge structure $\mathrm{H}^*(C)$.

Let us now discuss the structure of the extension groups between the building blocks of the category $\mathbf{DMT}(k)$. By analogy with the case of the derived category of an abelian category (Proposition A.116), we define

$$\mathrm{Ext}^i_{\mathbf{DMT}(k)}(\mathbb{Q}(n), \mathbb{Q}(m)) = \mathrm{Hom}_{\mathbf{DMT}(k)}(\mathbb{Q}(n), \mathbb{Q}(m)[i])$$

for all integers $i \in \mathbb{Z}$. Using the adjunction (4.28), the fact that $\mathbf{DMT}(k)$ is a full subcategory of $\mathbf{DM}(k)$, and the comparison results between motivic cohomology and K-theory (Theorem 4.102), we find

$$\begin{aligned}
\mathrm{Ext}^i_{\mathbf{DMT}(k)}(\mathbb{Q}(n), \mathbb{Q}(m)) &\cong \mathrm{Ext}^i_{\mathbf{DMT}(k)}(\mathbb{Q}(0), \mathbb{Q}(m-n)) \\
&= \mathrm{Hom}_{\mathbf{DM}(k)}(M(\mathrm{Spec}(k)), \mathbb{Q}(m-n)[i]) \\
&\cong (K_{2(m-n)-i}(k) \otimes \mathbb{Q})^{(m-n)},
\end{aligned}$$

where $(K_{2(m-n)-i}(k) \otimes \mathbb{Q})^{(m-n)}$ denotes the eigenspace with respect to the Adams operations acting on rational K-theory (Remark 4.100).

The K-theory groups of general fields are still largely unknown, but Borel computed their ranks when k is a number field.

THEOREM 4.114 (Borel, [Bor74]). *Let k be a number field with r_1 real embeddings and $2r_2$ complex embeddings. Then:*

$$(K_{2t-i}(k)\otimes\mathbb{Q})^{(m-n)} \cong \begin{cases} \mathbb{Q}, & \text{if } i=0 \text{ and } t=0, \\ k^\times \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } i=1 \text{ and } t=1, \\ \mathbb{Q}^{r_1+r_2}, & \text{if } i=1 \text{ and } t\geqslant 3 \text{ is odd}, \\ \mathbb{Q}^{r_2}, & \text{if } i=1 \text{ and } t\geqslant 2 \text{ is even}, \\ 0, & \text{otherwise}. \end{cases}$$

We summarize the information that we should retain from Borel's theorem in the following straightforward corollary.

COROLLARY 4.115. *Let k be a number field. The extension groups*

$$\mathrm{Ext}^i_{\mathbf{DMT}(k)}(\mathbb{Q}(n),\mathbb{Q}(m))$$

in the triangulated category of mixed Tate motives over k satisfy the following:

i) *the only non-zero extension groups occur for $i=0,1$;*
ii) *the morphism group $\mathrm{Hom}_{\mathbf{DMT}(k)}(\mathbb{Q}(n),\mathbb{Q}(m))$ vanishes unless $m=n$, in which case it is equal to $\mathbb{Q}\,\mathrm{Id}$;*
iii) *the extension group $\mathrm{Ext}^1_{\mathbf{DMT}(k)}(\mathbb{Q}(n),\mathbb{Q}(m))$ vanishes for all $n\geqslant m$;*
iv) *the only infinite-dimensional extension groups are*

$$\mathrm{Ext}^1_{\mathbf{DMT}(k)}(\mathbb{Q}(n),\mathbb{Q}(n+1)).$$

In particular, when $k=\mathbb{Q}$ is the field of rational numbers, there are $r_1=1$ real embeddings and $2r_2=0$ complex embeddings, hence

$$\mathrm{Ext}^1_{\mathbf{DMT}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) \cong \begin{cases} \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } n=1, \\ \mathbb{Q}, & \text{if } n\geqslant 3 \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Along with the fact that $\mathrm{Ext}^i_{\mathbf{DMT}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n))$ vanishes for $i\geqslant 2$, this will determine the structure of the category of mixed Tate motives over $\mathbb{Q}$.

4.3.2. *Kummer motives.* It is now time to give some examples of non-trivial extensions in the category of mixed Tate motives over k.

EXAMPLE 4.116 (Kummer motives). In view of the isomorphism

$$\mathrm{Ext}^1_{\mathbf{DMT}(k)}(\mathbb{Q}(0),\mathbb{Q}(1)) = k^\times \otimes_{\mathbb{Z}} \mathbb{Q},$$

there are plenty of non-trivial extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$ in the category $\mathbf{DMT}(k)$. They are all rational linear combinations of Kummer motives, defined as follows:

DEFINITION 4.117. For each $t\in k^\times$, the *Kummer motive* K_t^{Mot} is the class in $\mathbf{DM}(k)$ of the complex in $\mathbf{SmCor}(k)$ given by

$$[\mathrm{Spec}(k)] \underset{0}{\oplus} [\mathrm{Spec}(k)] \xrightarrow{f_t} \underset{-1}{[\mathbb{G}_m]}, \tag{4.118}$$

where f_t is the finite correspondence $[(*_1,1)]-[(*_2,t)]$.

In order to spell out the definition of f_t, write $\mathrm{Spec}(k)\amalg\mathrm{Spec}(k)=\{*_1,*_2\}$. Since the direct sum $[\mathrm{Spec}(k)]\oplus[\mathrm{Spec}(k)]$ is given by $[\mathrm{Spec}(k)\amalg\mathrm{Spec}(k)]$, each morphism of the shape (4.118) is a linear combination of closed subvarieties of

$$(*_1\times\mathbb{G}_m)\amalg(*_2\times\mathbb{G}_m).$$

This is the meaning of $[(*_1, 1)]$ and $[(*_2, t)]$. The condition that the projection to an irreducible component of $\{*_1, *_2\}$ is finite is in this case automatic.

Recall from formula (4.95) and Exercise 4.110 ii) the decomposition

$$M(\mathbb{G}_m) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1]$$

of the motive of $\mathbb{G}_m$, and that the second summand is isomorphic to the relative motive $M(\mathbb{G}_m, 1)$ given by the complex

$$\underset{1}{[\operatorname{Spec}(k)]} \xrightarrow{1} \underset{0}{[\mathbb{G}_m]}.$$

Therefore, we obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{Q}(0) \arrow[r, "1"] \arrow[d] & M(\mathbb{G}_m) \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathbb{Q}(1)[1] \arrow[r, "0"] & \mathbb{Q}(0)[1] \arrow[d] \\
\mathbb{Q}(0) \oplus \mathbb{Q}(0) \arrow[r, "f_t"] & M(\mathbb{G}_m) \arrow[r] & K_t^{\mathrm{Mot}}[1] \arrow[r] & \big(\mathbb{Q}(0) \oplus \mathbb{Q}(0)\big)[1],
\end{tikzcd}$$

where the rows are distinguished triangles. By axiom **(T3)** of triangulated categories, there is a morphism $\mathbb{Q}(1) \to K_t^{\mathrm{Mot}}$. Using a variant of the construction of Remark A.72, the octahedron axiom implies that this map can be extended to a distinguished triangle

$$\mathbb{Q}(1) \longrightarrow K_t^{\mathrm{Mot}} \longrightarrow \mathbb{Q}(0) \xrightarrow{g_t} \mathbb{Q}(1)[1].$$

Hence, K_t^{Mot} is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. In the exceptional case $t = 1$, the Kummer motive K_1^{Mot} is the trivial extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. Indeed, the map g_t is given by the commutative diagram

$$\begin{tikzcd}
 & \mathbb{Q}(0) \arrow[d, "t"] \arrow[dr, "g_t"] & \\
\mathbb{Q}(0) \arrow[r, "1"] & M(\mathbb{G}_m) \arrow[r] & \mathbb{Q}(1)[1],
\end{tikzcd}$$

where the bottom row is a distinguished triangle. Since the composition of two consecutive arrows in a distinguished triangle is zero (Remark A.71), we get $g_1 = 0$. By Proposition A.76, the motive K_1^{Mot} is a trivial extension.

The Hodge realization of the Kummer motive is the Kummer mixed Hodge structure from Example 2.258 (see also Exercise 4.140).

4.3.3. *The Beilinson–Soulé vanishing conjecture.* We now turn to the question of finding a t-structure on the triangulated category of mixed Tate motives. Let us first assume that it exists, that its heart $\mathbf{MT}(k)$ contains the objects $\mathbb{Q}(n)$, and that $\mathbf{DMT}(k)$ is equivalent to $\mathcal{D}^b(\mathbf{MT}(k))$. Then

$$(K_{2n-i}(k) \otimes \mathbb{Q})^{(n)} \cong \operatorname{Ext}^i_{\mathbf{DMT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) = \operatorname{Hom}_{\mathcal{D}^b(\mathbf{MT}(k))}(\mathbb{Q}(0), \mathbb{Q}(n)[i]).$$

From the fact that there are no non-zero morphisms to a negative shift in a derived category (Example A.140), we deduce that a necessary condition for the existence of a t-structure is that the left-hand side vanishes for $i < 0$. This is precisely the content of the Beilinson–Soulé conjecture.

CONJECTURE 4.119 (Beilinson–Soulé vanishing). *If k is a field, then $K_n(k)^{(r)}_{\mathbb{Q}}$ vanishes for all $n > 2r$.*

Another case where K-theory is well understood is that of finite fields, which was completely computed by Quillen in [**Qui72**, Thm. 8], shortly after he introduced the definition of higher algebraic K-theory:

THEOREM 4.120 (Quillen, [**Qui72**]). *Let $\mathbb{F}_q$ be a finite field with q elements. The K-theory groups of $\mathbb{F}_q$ are equal to*

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ \mathbb{Z}/(q^n - 1), & \text{if } i = 2n-1, \\ 0, & \text{otherwise.} \end{cases}$$

An immediate corollary of Borel and Quillen's theorems is:

COROLLARY 4.121. *The Beilinson–Soulé vanishing conjecture holds when k is either a number field or a finite field.*

4.3.4. *A t-structure on mixed Tate motives (after Levine).* Let k be a field for which the Beilinson–Soulé vanishing conjecture holds. In [**Lev93**], Levine proved that the derived category of mixed Tate motives has a t-structure that allows us to define an abelian category of mixed Tate motives. We sketch here his construction. For more details and proofs of the different steps, we refer the reader to [**Lev93**].

For each pair of integers a and b, let us denote by $\mathcal{T}_{[a,b]}$ the strictly full triangulated subcategory of $\mathbf{DMT}(k)$ generated by the objects $\mathbb{Q}(n)$ for $a \leqslant -2n \leqslant b$. We denote $\mathcal{T}_{[a,a]}$ simply by $\mathcal{T}_a$, and we extend the definition to cover the cases $a = -\infty$ or $b = \infty$ as well. In particular, $\mathcal{T}_{(-\infty,\infty)}$ coincides with the whole $\mathbf{DMT}(k)$.

LEMMA 4.122. *Let $a \leqslant b \leqslant c$ be integers (the cases $a = -\infty$ and $c = \infty$ are also allowed). Then $(\mathcal{T}_{[a,b-1]}, \mathcal{T}_{[b,c]})$ is a t-structure on $\mathcal{T}_{[a,c]}$.*

In particular, the pair $(\mathcal{T}_{(-\infty,b]}, \mathcal{T}_{[b+1,\infty)})$ is a t-structure on $\mathbf{DMT}(k)$ for each integer b. Let us emphasize that this is not the t-structure we are looking for, since its heart is reduced to zero. However, it will allow us to define a *weight structure*.

The truncation functors for the t-structure $(\mathcal{T}_{(-\infty,b]}, \mathcal{T}_{[b+1,\infty)})$ on $\mathbf{DMT}(k)$ will be denoted by

$$W_{\leqslant b} \colon \mathbf{DMT}(k) \longrightarrow \mathcal{T}_{(-\infty,b]}$$
$$W^{>b} \colon \mathbf{DMT}(k) \longrightarrow \mathcal{T}_{[b+1,\infty)}.$$

The reason for the subindex or superindex is that one will give rise to an increasing filtration, whereas the other will give rise to a decreasing filtration.

Let $W^{\geqslant b}$ denote $W^{>b-1}$, and define

$$\mathrm{Gr}_b^W(M) = W^{\geqslant b} W_{\leqslant b}(M).$$

For each even integer a, let $\mathcal{T}_a^{\leqslant 0}$ (resp. $\mathcal{T}_a^{\geqslant 0}$) be the full subcategory of $\mathcal{T}_a$ generated by $\mathbb{Q}(-a/2)[n]$ for $n \leqslant 0$ (resp. $n \geqslant 0$). Finally, let $\mathcal{T}_{[a,b]}^{\leqslant 0}$ (resp. $\mathcal{T}_{[a,b]}^{\geqslant 0}$) be the full subcategory of $\mathcal{T}_{[a,b]}$ generated by the objects M such that $\mathrm{Gr}_c^W(M)$ belongs to $\mathcal{T}_c^{\leqslant 0}$ (resp. $\mathcal{T}_c^{\geqslant 0}$) for all $a \leqslant c \leqslant b$.

THEOREM 4.123 (Levine [**Lev93**]). *Assume that the field k satisfies the Beilinson–Soulé vanishing conjecture. Then the pair of strictly full subcategories*

$$(\mathcal{T}_{(-\infty,\infty)}^{\leqslant 0}, \mathcal{T}_{(-\infty,\infty)}^{\geqslant 0})$$

forms a non-degenerate t-structure on $\mathbf{DMT}(k)$.

DEFINITION 4.124. The category $\mathbf{MT}(k)$ of *mixed Tate motives over* k is the heart of the t-structure of Theorem 4.123.

The category $\mathbf{MT}(k)$ has the following properties:

i) It is a tannakian category with simple objects $\mathbb{Q}(n)$ for $n \in \mathbb{Z}$.
ii) Each object M of $\mathbf{MT}(k)$ has an increasing *weight filtration* W with graded pieces
$$\mathrm{Gr}^W_{-2n} M \simeq \mathbb{Q}(n)^{\oplus k_n} \quad \text{and} \quad \mathrm{Gr}^W_{-2n+1} = 0$$
for some natural numbers k_n.
iii) A fiber functor is given by
$$\omega(M) = \bigoplus_n \mathrm{Hom}(\mathbb{Q}(n), \mathrm{Gr}^W_{-2n} M). \tag{4.125}$$
iv) The extension groups in the category $\mathbf{MT}(k)$ are determined by
$$\mathrm{Ext}^1_{\mathbf{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) = \begin{cases} k^\times \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } n = 1, \\ \mathbb{Q}^{r_2}, & \text{if } n \geqslant 2 \text{ is even}, \\ \mathbb{Q}^{r_1+r_2}, & \text{if } n \geqslant 3 \text{ is odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{4.126}$$
and the vanishing $\mathrm{Ext}^i_{\mathbf{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ for all $i \geqslant 2$.

Moreover, Wildeshaus [**Wil09**, Thm. 1.3] proved that there exists a canonical equivalence of categories
$$F\colon D^b(\mathbf{MT}(k)) \longrightarrow \mathbf{DMT}(k). \tag{4.127}$$
The functor F is t-exact, induces the identity on the heart $\mathbf{MT}(k)$, and has the property that the composition with the cohomology functor H^0 associated with the t-structure as in (A.152) coincides with the canonical cohomology functor
$$D^b(\mathbf{MT}(k)) \to \mathbf{MT}(k).$$
In view of Remark A.150, the main difficulty does not lie in proving that the two categories are equivalent but in constructing a functor between them.

4.3.5. *Examples.* If the motive of a variety X is of mixed Tate type, *i.e.* belongs to the subcategory $\mathbf{DMT}(k)$ of $\mathbf{DM}(k)$, then decomposing the dual of $M(X)$ by means of Levine's t-stucture we obtain the cohomology motives
$$h^i(X) = t_{\leqslant 0} t_{\geqslant 0}(M(X)^\vee[i]) \in \mathrm{Ob}(\mathbf{MT}(k)).$$
Thus, we can isolate the different cohomological degrees, something we do not know how to do for general motives.

EXAMPLE 4.128. By Example 4.92, the motive of the projective space $M(\mathbb{P}^n_k)$ is of mixed Tate type and satisfies
$$h^i(\mathbb{P}^n_k) = \begin{cases} \mathbb{Q}(-m), & \text{if } i = 2m \text{ and } 0 \leqslant m \leqslant n, \\ 0, & \text{otherwise}. \end{cases}$$

Using properties of $\mathbf{DMT}(k)$ such as the homotopy invariance or the Gysin distinguished triangle, we can show that certain motives are mixed Tate. For instance, if a variety X possesses a stratification such that the motive of each locally closed stratum is mixed Tate, then the whole $M(X)$ is a mixed Tate motive.

EXAMPLE 4.129 (Motive of the moduli space $M_{0,n}$). Let $n \geqslant 3$ be an integer. Recall the moduli spaces $M_{0,n}$ from Section 2.5.2. In this example, we show that the motive $M(M_{0,n})$ belongs to $\mathbf{DMT}(\mathbb{Q})$ by mimicking the proof of Proposition 2.309. We proceed by induction on n. The result holds for $n = 3$, since $M_{0,3} = \mathrm{Spec}(k)$, and for $n = 4$ since $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and we saw in Example 4.113 that the motive of a punctured projective line belongs to $\mathbf{DMT}(\mathbb{Q})$. For $n \geqslant 5$, we write

$$M_{0,n} \simeq X \setminus Z, \qquad X = M_{0,4} \times M_{0,n-1}, \qquad Z = \bigsqcup_{i=2}^{n-3} M_{0,n-1}.$$

By the Künneth formula and the induction hypothesis, both the motives $M(X)$ and $M(Z)$ belong to $\mathbf{DMT}(\mathbb{Q})$. Besides, the Gysin triangle reads

$$M(M_{0,n}) \longrightarrow M(X) \longrightarrow M(Z)(1)[2] \longrightarrow M(M_{0,n})[1],$$

and since $M(X)$ and $M(Z)(1)[2]$ belong to $\mathbf{DMT}(\mathbb{Q})$, so does $M(M_{0,n})$.

EXAMPLE 4.130. Let $L = L_1 \cup \cdots \cup L_{n_1}$ and $M = M_1 \cup \cdots \cup M_{n_2}$ be two collections of linear subspaces of the projective space $\mathbb{P}^n_k$. We consider the motive

$$\mathcal{M} = M(\mathbb{P}^n_k \setminus L, M \setminus (M \cap L)) \in \mathrm{Ob}(\mathbf{DM}(k))$$

from Example 4.97. We want to see that it belongs to $\mathbf{DMT}(k)$. If either L_i or M_j is equal to $\mathbb{P}^n_k$, then we obtain the zero motive, so we may assume that the linear subspaces are strict. If $n = 0$, then $\mathbb{P}^n_k = \mathrm{Spec}(k)$ and $L = M = \emptyset$. In this case, the motive $\mathcal{M} = M(\mathrm{Spec}(k))$ belongs to $\mathbf{DMT}(k)$. Assume $n \geqslant 1$ and $n_2 = 0$. If $n_1 = 0$, then $\mathcal{M} = M(\mathbb{P}^n_k)$ belongs to $\mathbf{DMT}(k)$. If $n_1 \geqslant 1$, write

$$L' = L_1 \cup \cdots \cup L_{n_1 - 1}.$$

By the Gysin property from Section 4.2.7, there is a distinguished triangle

$$M(\mathbb{P}^n_k \setminus L) \longrightarrow M(\mathbb{P}^n_k \setminus L') \longrightarrow M(L_{n_1} \setminus L_{n_1} \cap L')(c)[2c] \longrightarrow M(\mathbb{P}^n_k \setminus L)[1]$$

in $\mathbf{DM}(k)$, where c is the codimension of L_{n_1} in $\mathbb{P}^n_k$. By induction (both on n and n_1), the motives $M(\mathbb{P}^n_k \setminus L')$ and $M(L_{n_1} \setminus L_{n_1} \cap L')$ belong to $\mathbf{DMT}(k)$. Since this last category is closed under extensions, it also contains the motive $M(\mathbb{P}^n_k \setminus L)$.

Using the previous case, Exercise 4.111 and the fact that $\mathbf{DMT}(k)$ is closed under extensions, we deduce that $M(M \setminus L)$ lies in $\mathbf{DMT}(k)$. By the distinguished triangle (4.98), the motive $M(\mathbb{P}^n_k \setminus L, M \setminus M \cap L)$ also belongs to $\mathbf{DMT}(k)$.

Applying the t-structure of $\mathbf{DMT}(k)$, we obtain mixed Tate motives

$$h^r(\mathbb{P}^n \setminus L, M \setminus (M \cap L)) \in \mathrm{Ob}(\mathbf{MT}(k)).$$

4.3.6. *Realizations.* The functor R^{H} from Theorem 4.107 restricts to a functor

$$\mathbf{DMT}(\mathbb{Q}) \longrightarrow D^b(\mathbf{MHS}(\mathbb{Q}))$$

which takes values in the derived category of $\mathbf{MHTS}(\mathbb{Q})$, the category of mixed Hodge–Tate structures over $\mathbb{Q}$ from Definition 2.235. As explained in Example A.140, the category on the right-hand side has a canonical t-structure. We have also defined a t-structure on $\mathbf{DMT}(\mathbb{Q})$. In fact, this t-structure has the property that any realization functor is t-exact in the sense of Definition A.137, and hence restricts to a functor on the hearts. This applies in particular to R^{H}, so we obtain a functor from $\mathbf{MT}(\mathbb{Q})$ to $\mathbf{MHS}(\mathbb{Q})$. Since the Hodge realization of a mixed Tate motive is a mixed Hodge–Tate structure, we actually get a functor

$$R^{\mathrm{H}} \colon \mathbf{MT}(\mathbb{Q}) \longrightarrow \mathbf{MHTS}(\mathbb{Q}) \tag{4.131}$$

which respects the weight filtrations.

REMARK 4.132. It is important to keep in mind that the category $\mathbf{MHTS}(\mathbb{Q})$ is much bigger than $\mathbf{MT}(\mathbb{Q})$. For instance, compare the uncountable set of extensions of $\mathbb{Q}(m)$ and $\mathbb{Q}(n)$ in the category $\mathbf{MHTS}(\mathbb{Q})$ given by Theorem 2.256 with the countable set of extensions in $\mathbf{MT}(\mathbb{Q})$ given by Theorem 4.114. For this reason, it is important to know which mixed Hodge structures come from geometry.

This discussion leads to a precise meaning of the word "motivic" when speaking about a mixed Hodge–Tate structure.

DEFINITION 4.133. We say that a mixed Hodge–Tate structure over $\mathbb{Q}$ is *motivic* if it lies in the essential image of the functor R^{H}. The same definition applies to pro-mixed Hodge–Tate structures. More generally, we say that a diagram of pro-mixed Hodge–Tate structures is motivic if it is isomorphic to the image by the functor R^{H} of a diagram of pro-mixed Tate motives.

Even if $\mathbf{MHTS}(\mathbb{Q})$ is much bigger than $\mathbf{MT}(\mathbb{Q})$, the realization functor between these categories is fully faithful and stable by subobjects. This is a very useful result to prove that many mixed Hodge structures have motivic origin. We should mention that to determine whether the Hodge realization functor from the hypotetical category of mixed motives is fully faithful (*i.e.* bijective on Hom sets) would be a extremely difficult problem. For instance, if the realization functor restricted to the category of pure motives is fully faithful, then the Hodge conjecture holds. That we can prove it for $\mathbf{MT}(\mathbb{Q})$ relies once again on Borel's results about the K-theory of number fields.

PROPOSITION 4.134 (Deligne–Goncharov). *The realization functor* (4.131) *is fully faithful and its essential image is stable under subobjects.*

PROOF. The key point of the argument is that the realization functor R^{H} determines injections

$$\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow \mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) \tag{4.135}$$

into the extension groups which were computed in Theorem 2.256. For $n = 1$, this follows from the injectivity of

$$\log\colon \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{C}/2\pi i\mathbb{Q}.$$

For $n > 1$, the injectivity follows by interpreting $\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n))$ as a part of the motivic cohomology of $\mathrm{Spec}(\mathbb{Q})$, which can be computed using K-theory:

$$\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = \mathrm{H}^1_{\mathcal{M}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q}(n)) = K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q},$$

then interpreting $\mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n))$ as Deligne cohomology groups:

$$\mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = \mathrm{H}^1_{\mathcal{D}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q}(n)).$$

Under this interpretation, the realization map (4.135) should correspond to the Borel regulator map mentioned in Digression 1.12, which is known to be injective by the work of Borel.

Consider the fiber functors ω_{dR} on $\mathbf{MHS}(\mathbb{Q})$ and ω on $\mathbf{MT}(\mathbb{Q})$ from Definition 2.227 and equation (4.125). These fiber functors are compatible with the

Hodge realization functor, in the sense that the diagram

$$\begin{tikzcd} \mathbf{MT}(\mathbb{Q}) \arrow[r, "R^{\mathrm{H}}"] \arrow[dr, "\omega"'] & \mathbf{MHTS}(\mathbb{Q}) \arrow[d, "\omega_{\mathrm{dR}}"] \\ & \mathbf{Vec}_{\mathbb{Q}} \end{tikzcd}$$

commutes, and hence induce morphisms of Tannaka groups

$$G^{\mathrm{H}}_{\omega_{\mathrm{dR}}} = \underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MHTS}(\mathbb{Q})}(\omega_{\mathrm{dR}}) \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MT}(\mathbb{Q})}(\omega) = G_{\omega}. \tag{4.136}$$

By the tannakian dictionary (Proposition 4.60), the functor R^H is fully faithful if and only if the morphism (4.136) is surjective.

To prove the latter, we argue as follows: taking into account that both $G^{\mathrm{H}}_{\omega_{\mathrm{dR}}}$ and G_ω can be written as the semidirect product of $\mathbb{G}_m$ and a pro-unipotent group

$$G^{\mathrm{H}}_{\omega_{\mathrm{dR}}} = U^{\mathrm{H}}_{\omega_{\mathrm{dR}}} \rtimes \mathbb{G}_m, \quad G_\omega = U_\omega \rtimes \mathbb{G}_m,$$

the injectivity of (4.135) implies the surjectivity of (4.136) (the precise relationship between the Ext groups and the Lie algebra of U_ω is worked out on the way of proving Theorem 4.180). □

EXAMPLE 4.137. Let $n > 0$ be an even integer and H a mixed Hodge structure over $\mathbb{Q}$ that is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$. If this extension is non-trivial, then H is not motivic over $\mathbb{Q}$, in the sense that it cannot be the Hodge realization of a motive over $\mathbb{Q}$. Indeed, assume that there is a mixed Tate motive over $\mathbb{Q}$ whose Hodge realization is H. Since the functor R^{H} is fully faithful, the exact sequence

$$0 \longrightarrow \mathbb{Q}(n) \longrightarrow H \longrightarrow \mathbb{Q}(0) \longrightarrow 0$$

is the Hodge realization of an exact sequence of mixed Tate motives

$$0 \longrightarrow \mathbb{Q}(n) \longrightarrow M \longrightarrow \mathbb{Q}(0) \longrightarrow 0.$$

The vanishing $\mathrm{Ext}^1_{\mathbf{DMT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ implies that this extension is split, and hence the original extension of mixed Hodge structures is also split.

Of course, in view of (4.126), there exist motivic non-trivial extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ defined over number fields that are not totally real.

⋆ ⋆ ⋆

EXERCISE 4.138. Prove that the pair of subcategories $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ from Example A.140 forms a t-structure.

EXERCISE 4.139. Let $\mathrm{Gr}(d, n)$ be the Grassmanian scheme of d-planes in k^n. Show that the motive $M(\mathrm{Gr}(d, n))$ belongs to $\mathbf{DMT}(k)$.

EXERCISE 4.140. Prove that the Kummer motive K_t^{Mot} of Example 4.116 belongs to $\mathbf{MT}(k)$.

4.4. Mixed Tate motives over $\mathbb{Z}$. From now on, we further specialize the discussion on the category of mixed Tate motives to the case of the field $k = \mathbb{Q}$ of rational numbers. The category $\mathbf{MT}(\mathbb{Q})$ is still too large for our purposes, because of the infinite-dimensional extension group

$$\operatorname{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(1)) \simeq \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \bigoplus_{p \text{ prime}} \mathbb{Q}$$

from (4.126). To remedy this, Goncharov [**Gon01**, § 3] defined the category $\mathbf{MT}(\mathbb{Z})$ of *mixed Tate motives over* $\mathbb{Z}$ as a subcategory of $\mathbf{MT}(\mathbb{Q})$.

4.4.1. *Definition and basic properties.*

DEFINITION 4.141. A motive M in $\mathbf{MT}(\mathbb{Q})$ is said to be *everywhere unramified* if, for each integer n, there is no subquotient E of M which fits into a non-split extension $0 \to \mathbb{Q}(n+1) \to E \to \mathbb{Q}(n) \to 0$.

The full subcategory $\mathbf{MT}(\mathbb{Z})$ of $\mathbf{MT}(\mathbb{Q})$ consisting of everywhere unramified motives is called the category of *mixed Tate motives over* $\mathbb{Z}$.

With a motive M over $\mathbb{Q}$ and a prime number ℓ, we can associate the ℓ-adic realization of M. For instance, with the motive corresponding to a smooth variety X over $\mathbb{Q}$ we associate the dual of the ℓ-adic cohomology $\mathrm{H}^*_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. The ℓ-adic realization of M is a finite-dimensional $\mathbb{Q}_\ell$-vector space, together with a continuous action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let p be a prime number distinct from ℓ. The choice of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and a field embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ allows one to see the local Galois group $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\overline{\mathbb{Q}}_p)$ as a subgroup of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By restriction, we obtain a representation of $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Besides, the Galois group of the maximal unramified extension $\mathbb{Q}_p \subset \mathbb{Q}_p^{\mathrm{ur}} \subset \overline{\mathbb{Q}}_p$ is isomorphic to $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, and the *inertia subgroup* I_p is defined by the exact sequence

$$1 \longrightarrow I_p \longrightarrow \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow 1.$$

DEFINITION 4.142. Let $\rho\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}(V)$ be an ℓ-adic representation, and let p be a prime number distinct from ℓ. We say that ρ is *unramified* at p if its restriction to the inertia subgroup $I_p \subseteq \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is trivial.

We have at our disposal the following criterion to decide whether a mixed Tate motive over $\mathbb{Q}$ belongs to $\mathbf{MT}(\mathbb{Z})$.

PROPOSITION 4.143 (Deligne–Goncharov). *A mixed Tate motive M over $\mathbb{Q}$ belongs to $\mathbf{MT}(\mathbb{Z})$ if and only if, for each prime number p, there exists a prime number $\ell \neq p$ such that the ℓ-adic realization $\omega_\ell(M)$ is unramified at p.*

PROOF. See [**DG05**, Prop. 1.8]. □

EXAMPLE 4.144. Let $t \in \mathbb{Q}^\times$ be a non-zero rational number and K_t^{Mot} the Kummer motive from Example 4.116. For each prime ℓ, the ℓ-adic realization of K_t^{Mot} is the extension

$$0 \longrightarrow \mathbb{Q}_\ell(1) \longrightarrow K_t^\ell \xrightarrow{f} \mathbb{Q}(0) \longrightarrow 0$$

corresponding to the $\mathbb{Q}_\ell(1)$-torsor given by the projective limit of ℓ^n-th roots of unity of t. This is unramified everywhere if and only if $t \in \mathbb{Z}^\times$. Thus, taking into account that $\mathbb{Z}^\times \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, the only Kummer motive that belongs to $\mathbf{MT}(\mathbb{Z})$ is the trivial one K_1^{Mot}. This solves the problem of the extension groups being infinite-dimensional.

The main properties of the category $\mathbf{MT}(\mathbb{Z})$ are summarized as follows:

THEOREM 4.145.

i) $\mathbf{MT}(\mathbb{Z})$ *is a tannakian category generated by* $\mathbb{Q}(n)$ *for all integers* $n \in \mathbb{Z}$.

ii) *Each object* M *of* $\mathbf{MT}(\mathbb{Z})$ *has a canonical increasing weight filtration* W *indexed by even integers satisfying*

$$\operatorname{Gr}^W_{2n} M \simeq \mathbb{Q}(-n)^{\oplus k_n}$$

for some integers $k_n \geqslant 0$.

iii) *The extension groups in the category* $\mathbf{MT}(\mathbb{Z})$ *are given by*

$$\operatorname{Ext}^i_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(n), \mathbb{Q}(m)) = \begin{cases} \mathbb{Q}, & \textit{if } i = 0 \textit{ and } m - n = 0, \\ \mathbb{Q}, & \textit{if } i = 1 \textit{ and } m - n \geqslant 3 \textit{ is odd}, \\ 0, & \textit{otherwise}. \end{cases}$$

Hence, they are all finite-dimensional.

Since $\mathbf{MT}(\mathbb{Z}) \subset \mathbf{MT}(\mathbb{Q})$ is stable under subobjects, we immediately deduce from Proposition 4.134:

COROLLARY 4.146. *The realization functor*

$$R\colon\ \mathbf{MT}(\mathbb{Z}) \longrightarrow \mathbf{MHTS}(\mathbb{Q})$$

is fully faithful with essential image stable under subobjects.

4.4.2. *Fiber functors.* In this section, we introduce various fiber functors on the tannakian category $\mathbf{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$. We will then compute the corresponding Tannaka groups in the next section.

The first fiber functor (see Exercise 4.185) is defined using the filtration on mixed Tate motives given by Theorem 4.145 ii). Namely, we write

$$\omega_n(M) = \operatorname{Hom}_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(-n), \operatorname{Gr}^W_{2n}(M))$$

for each object $M \in \operatorname{Ob}(\mathbf{MT}(\mathbb{Z}))$ and each integer $n \in \mathbb{Z}$, and we define a functor

$$\begin{aligned} \omega\colon\ \mathbf{MT}(\mathbb{Z}) &\longrightarrow \mathbf{Vec}_{\mathbb{Q}} \\ M \quad &\longmapsto \omega(M) = \textstyle\bigoplus_n \omega_n(M). \end{aligned} \tag{4.147}$$

Observe that ω factors through finite-dimensional graded $\mathbb{Q}$-vector spaces.

From the Hodge realization of a motive, we obtain two more fiber functors. The de Rham fiber functor, denoted by ω_{dR}, is the de Rham component of the Hodge structure. For a motive $M \in \operatorname{Ob}(\mathbf{MT}(\mathbb{Z}))$, the vector space $\omega_{\mathrm{dR}}(M)$ is equipped with two filtrations: the decreasing Hodge filtration F, and the increasing weight filtration W. Since $(\omega_{\mathrm{dR}}(M), F, W)$ is part of a mixed Hodge structure of Tate type, these filtrations are opposed to each other. Namely, writing

$$\omega_{\mathrm{dR}}(M)_n = F^n \omega_{\mathrm{dR}}(M) \cap W_{2n} \omega_{\mathrm{dR}}(M),$$

Lemma 2.237 implies the existence of functorial isomorphisms

$$\omega_{\mathrm{dR}}(M) = \bigoplus_n \omega_{\mathrm{dR}}(M)_n,$$
$$F^p\omega_{\mathrm{dR}}(M) = \bigoplus_{m\geqslant p} \omega_{\mathrm{dR}}(M)_m,$$
$$W_{2n}\omega_{\mathrm{dR}}(M) = \bigoplus_{m\leqslant n} \omega_{\mathrm{dR}}(M)_m.$$

Thus, the de Rham fiber functor ω_{dR} also factors through graded vector spaces.

Lemma 4.148. *The de Rham fiber functor ω_{dR} is canonically isomorphic to the fiber functor ω.*

Proof. By Exercise 2.240, there is a canonical isomorphism

$$\omega^{\mathrm{dR}}(M)_n \simeq \mathrm{Hom}_{\mathbf{MHTS}}(\mathbb{Q}(-n), \mathrm{Gr}_n^W(R^{\mathrm{H}}(M))).$$

The fully-faithfulness of the Hodge realization functor (Corollary 4.146) then implies the existence of a canonical isomorphism $\omega_n(M) \simeq \omega^{\mathrm{dR}}(M)_n$. □

There is also a Betti fiber functor ω_{B} given by the Betti component of the Hodge realization. The rational vector space ω_{B} is provided with a weight filtration W, but not a Hodge filtration. Note that ω_{B} does not factor canonically through the category of graded vector spaces. Finally, there is a comparison isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} \colon \omega_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \omega_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.149}$$

Example 4.150. In this example, we explicitly compute the de Rham realization, the Betti realization, and the comparison isomorphism for the motive $\mathbb{Q}(1)$. We begin with the smooth variety

$$X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,\infty\} = \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \mathbb{G}_{m,\mathbb{Q}} = \mathrm{Spec}(\mathbb{Q}[x,x^{-1}]),$$

which by Remark 4.94 has motive

$$M(X) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1].$$

Therefore, the cohomology of $M(X)$ with respect to the t-structure is

$$t_{\leqslant 0}t_{\geqslant 0}(M(X)[-i]) = \begin{cases} \mathbb{Q}(i), & \text{if } i = 0,1, \\ 0, & \text{otherwise.} \end{cases}$$

We work on the compactification $X \subset \mathbb{P}^1_{\mathbb{Q}}$. Explicitly, the complex of differential forms on $\mathbb{P}^1_{\mathbb{Q}}$ with logarithmic poles along $\{0,\infty\}$ is given as follows:

i) $\Omega^0_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0,\infty\}) \simeq \mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}$ is the sheaf of regular functions on $\mathbb{P}^1_{\mathbb{Q}}$.

ii) $\Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0,\infty\})$ is the $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}$-module generated by the differential form

$$\frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}x^{-1}}{x^{-1}}.$$

As a sheaf, $\Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0,\infty\})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}$.

Although $\mathbb{P}^1_{\mathbb{Q}}$ is not affine, thanks to the vanishing

$$\mathrm{H}^i(\mathbb{P}^1_{\mathbb{Q}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}) = 0, \quad \text{for } i > 0,$$

there is no need to search for a resolution of the complex $\Omega^*_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0,\infty\})$: we can use directly the complex of global sections to compute de Rham cohomology as

$$\Gamma(\mathbb{P}^1_{\mathbb{Q}}, \Omega^0_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0,\infty\})) = \mathbb{Q}[x, x^{-1}],$$

$$\Gamma(\mathbb{P}^1_{\mathbb{Q}}, \Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0,\infty\})) = \mathbb{Q}[x, x^{-1}]\frac{\mathrm{d}x}{x}.$$

The differential is given by $\mathrm{d}x^n = nx^{n-1}$. Hence,

$$\mathrm{H}^0_{\mathrm{dR}}(X) = \mathbb{Q}, \qquad \mathrm{H}^1_{\mathrm{dR}}(X) = \mathbb{Q}\frac{\mathrm{d}x}{x}.$$

From this we can find the de Rham realization

$$\omega_{\mathrm{dR}}(\mathbb{Q}(1)) = \left(\mathbb{Q}\frac{\mathrm{d}x}{x}\right)^{\vee}.$$

Thus, $\omega_{\mathrm{dR}}(\mathbb{Q}(1))$ is a one-dimensional vector space with generator $(\mathrm{d}x/x)^{\vee}$.

The Betti realization is given by the singular homology of the complex points

$$\omega_{\mathrm{B}}(\mathbb{Q}(1)) = \mathrm{H}_1(\mathbb{C} \setminus \{0\}, \mathbb{Q})$$

This is again a rational vector space of dimension 1, spanned by the unit circle γ oriented counterclockwise.

The comparison isomorphism is obtained from the integration of differential forms along singular chains. Since

$$\int_\gamma \frac{\mathrm{d}x}{x} = 2\pi i,$$

we deduce that $\mathrm{comp}_{\mathrm{dR,B}}(\gamma) = (\mathrm{d}x/x)^{\vee} \otimes (2\pi i)$.

4.4.3. *Tannaka groups of* $\mathbf{MT}(\mathbb{Z})$*.* We now turn to the description of the affine group schemes associated with the various fiber functors on the category of mixed Tate motives over $\mathbb{Z}$.

NOTATION 4.151. The following notation will be used throughout:

$$G_{\mathrm{dR}} = \underline{\mathrm{Aut}}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{dR}}), \tag{4.152}$$

$$G_{\mathrm{B}} = \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{B}}), \tag{4.153}$$

$$P_{\mathrm{B,dR}} = \underline{\mathrm{Iso}}^{\otimes}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}}), \tag{4.154}$$

$$P_{\mathrm{dR,B}} = \underline{\mathrm{Iso}}^{\otimes}(\omega_{\mathrm{B}}, \omega_{\mathrm{dR}}). \tag{4.155}$$

Observe that $\mathrm{comp}_{\mathrm{B,dR}}$ (resp. $\mathrm{comp}_{\mathrm{dR,B}}$) is a complex point of $P_{\mathrm{B,dR}}$ (resp. $P_{\mathrm{dR,B}}$). Since the spaces $P_{\mathrm{B,dR}}$ and $P_{\mathrm{dR,B}}$ have a complex point, they also have a $\overline{\mathbb{Q}}$-point. This implies that they are both G_{dR}-torsors.

In what follows, we will use the subscript dR/B for properties which are common to G_{dR} and G_{B}.

LEMMA 4.156. *The groups* $G_{\mathrm{dR/B}}$ *fit into an exact sequence*

$$1 \longrightarrow U_{\mathrm{dR/B}} \longrightarrow G_{\mathrm{dR/B}} \longrightarrow \mathbb{G}_m \longrightarrow 1, \tag{4.157}$$

where $U_{\mathrm{dR/B}}$ *is a pro-unipotent group.*

Proof. Recall that the category $\mathbf{MT}(\mathbb{Z})$ contains the object $\mathbb{Q}(-1)$. Since its realization $\omega_{\mathrm{dR/B}}(\mathbb{Q}(-1))$ is a one-dimensional $\mathbb{Q}$-vector space, we get a morphism

$$t_{\mathrm{dR/B}}\colon G_{\mathrm{dR/B}} \longrightarrow \mathrm{GL}(\omega_{\mathrm{dR/B}}(\mathbb{Q}(-1))) = \mathbb{G}_m. \tag{4.158}$$

We define $U_{\mathrm{dR/B}}$ as the kernel of this morphism.

The action of $G_{\mathrm{dR/B}}$ is compatible with the tensor product, which means that an element $g \in G_{\mathrm{dR/B}}$ acts on $\omega_{\mathrm{dR/B}}(\mathbb{Q}(-n))$ as $t_{\mathrm{dR/B}}(g)^n$. Since the weight filtration is a filtration in the category of motives, $G_{\mathrm{dR/B}}$ respects the weight filtration: for all $g \in G_{\mathrm{dR/B}}$ and $X \in \mathrm{Ob}(\mathbf{MT}(\mathbb{Z}))$, the action of g on $\omega_{\mathrm{dR/B}}(X)$ sends $W_{2n}\omega_{\mathrm{dR/B}}(X) = \omega_{\mathrm{dR/B}}(W_{2n}X)$ to $W_{2n}\omega_{\mathrm{dR/B}}(X)$. Therefore, $G_{\mathrm{dR/B}}$ acts on the graded piece $\mathrm{Gr}^W_{2n}\,\omega_{\mathrm{dR/B}}(X)$. Since $\mathrm{Gr}^W_{2n}\,\omega_{\mathrm{dR/B}}(X)$ is a sum of copies of $\omega_{\mathrm{dR/B}}(\mathbb{Q}(-n))$, an element g acts on $\mathrm{Gr}^W_{2n}\,\omega_{\mathrm{dR/B}}(X)$ as $t_{\mathrm{dR/B}}(g)^n$, and the action of an element $u \in U_{\mathrm{dR/B}}$ on the same space is trivial. This implies that $U_{\mathrm{dR/B}}$ is a pro-unipotent affine group scheme. □

At this level, an advantage of using the de Rham fiber functor $\omega = \omega_{\mathrm{dR}}$ instead of the Betti fiber functor ω_{B} is that the exact sequence (4.157) admits a canonical splitting, as the following lemma shows.

Lemma 4.159. *There exists a canonical section $\tau\colon \mathbb{G}_m \to G_{\mathrm{dR}}$ that induces an action of $\mathbb{G}_m$ on U_{dR}, and hence a semi-direct product decomposition*

$$G_{\mathrm{dR}} = U_{\mathrm{dR}} \rtimes \mathbb{G}_m.$$

Proof. The existence of the canonical section comes from the fact that the functor $\omega = \omega_{\mathrm{dR}}$ factors through the category of graded vector spaces. Hence, it decomposes as $\omega = \bigoplus \omega_n$. Given $t \in \mathbb{G}_m$, define $\tau(t) \in G_{\mathrm{dR}}$ as the element that acts as multiplication by t^n on ω_n. This defines a section $\tau\colon \mathbb{G}_m \to G_{\mathrm{dR}}$ of t_{dR}. Hence, G_{dR} is a semidirect product. □

Lemma 4.160. *The action of $\mathbb{G}_m$ on U_{dR} is compatible with the structure of pro-unipotent group. Therefore, U_{dR} is a graded pro-unipotent group.*

Proof. Recall from Lemma 4.64 how the pro-algebraic structure of G_{dR} is defined. For every $Y \in \mathrm{Ob}(\mathbf{MT}(\mathbb{Z}))$, we consider the tensor category $\langle Y\rangle$ generated by Y. Then $G^Y_{\mathrm{dR}} = \mathrm{Aut}^{\otimes}_{\langle Y\rangle}(\omega_{\mathrm{dR}})$ is a closed subgroup of $\mathrm{GL}(\omega_{\mathrm{dR}}(Y))$, which implies that is is algebraic, and

$$G_{\mathrm{dR}} = \varprojlim_{Y} G^Y_{\mathrm{dR}}.$$

Since the objects $Y \oplus \mathbb{Q}(1)$ form a cofinal system, we can also write

$$G_{\mathrm{dR}} = \varprojlim_{Y} G^{Y\oplus\mathbb{Q}(1)}_{\mathrm{dR}}.$$

The restriction of the functor ω_{dR} to $\langle Y \oplus \mathbb{Q}(1)\rangle$ also factors through the category of graded vector spaces. From this, we deduce the existence of two commutative diagrams (with the horizontal arrows to the right or to the left)

$$\begin{tikzcd} G_{\mathrm{dR}} \arrow[r, shift left] \arrow[d] & \mathbb{G}_m \arrow[l, shift left] \arrow[d, equal] \\ G^{Y\oplus\mathbb{Q}(1)}_{\mathrm{dR}} \arrow[r, shift left] & \mathbb{G}_m. \arrow[l, shift left] \end{tikzcd}$$

Writing $U_{\mathrm{dR}}^{Y\oplus\mathbb{Q}(1)} = \mathrm{Ker}(G_{\mathrm{dR}}^{Y\oplus\mathbb{Q}(1)} \to \mathbb{G}_m)$, we identify U_{dR} with the limit

$$U_{\mathrm{dR}} = \varprojlim_{Y} U_{\mathrm{dR}}^{Y\oplus\mathbb{Q}(1)},$$

in which every $U_{\mathrm{dR}}^{Y\oplus\mathbb{Q}(1)}$ is a unipotent algebraic group. Moreover, the action of $\mathbb{G}_m$ on U_{dR} comes from compatible actions of $\mathbb{G}_m$ on each $U_{\mathrm{dR}}^{Y\oplus\mathbb{Q}(1)}$. $\square$

COROLLARY 4.161. *Any G_{dR}-torsor defined over $\mathbb{Q}$ is trivial.*

PROOF. Let us first recall that a G_{dR}-torsor is a scheme P defined over $\mathbb{Q}$, together with an action $G_{\mathrm{dR}} \times P \to P$ such that, after base change to an algebraic closure $\overline{\mathbb{Q}}$, there is an isomorphism $P_{\overline{\mathbb{Q}}} \simeq G_{\mathrm{dR},\overline{\mathbb{Q}}}$ that transforms the action of G_{dR} on P into the left multiplication of G_{dR} on itself. A G_{dR}-torsor is trivial if and only if the isomorphism can be chosen to be defined over $\mathbb{Q}$. In fact, a G_{dR}-torsor P is trivial if and only if $P(\mathbb{Q})$ is non-empty. The set of isomorphism classes of G_{dR}-torsors is in bijection with the Galois cohomology

$$\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), G_{\mathrm{dR}}(\overline{\mathbb{Q}})), \tag{4.162}$$

as explained in [**Wat79**, Thm. 17.6]. Since $G_{\mathrm{dR}}(\overline{\mathbb{Q}})$ is non-abelian, this H^1 is not a group, but only a pointed set.

Thus, the key ingredient in the proof of the corollary is the vanishing of (4.162), that we are going to derive from the vanishing of the Galois cohomology groups

$$\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{G}_m(\overline{\mathbb{Q}})) = \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{G}_a(\overline{\mathbb{Q}})) = 0,$$

proved, for instance, in [**Wat79**, § 18.2] or [**Ser94**, Chap. II, § 1.2, Prop. 1]. In this case, they cohomologies are groups because $\mathbb{G}_m$ and $\mathbb{G}_a$ are abelian. It follows that

$$\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), U(\overline{\mathbb{Q}})) = \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), G(\overline{\mathbb{Q}})) = 0$$

are also trivial for any unipotent algebraic group U or any extension G of $\mathbb{G}_m$ by U (see [**Ser94**, Ch. III, Prop. 6]).

Now, the group G_{dR} can be written as

$$G_{\mathrm{dR}} = \varprojlim_{N} G_{\mathrm{dR}}^N,$$

where each G_{dR}^N is an extension of $\mathbb{G}_m$ by a unipotent algebraic group and all transition maps are surjective. Using a non-abelian version of the Mittag–Leffler theorem (see Proposition A.168 for the abelian version), one can prove that the projective limit commutes with Galois cohomology, hence the vanishing

$$\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), G_{\mathrm{dR}}(\overline{\mathbb{Q}})) = \varprojlim_{N} \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), G_{\mathrm{dR}}^N(\overline{\mathbb{Q}})) = 0,$$

which implies that any G_{dR}-torsor defined over $\mathbb{Q}$ is trivial. $\square$

This corollary has the following important consequence, which will be exploited in the next chapter.

PROPOSITION 4.163. *There exists an element $a \in G_{\mathrm{dR}}(\mathbb{C})$ such that the isomorphism of functors $\mathrm{comp}_{\mathrm{B,dR}} \circ a \colon \omega_{\mathrm{dR}} \otimes \mathbb{C} \to \omega_{\mathrm{B}} \otimes \mathbb{C}$ sends ω_{dR} to ω_{B}. In other words, for every motive M of $\mathbf{MT}(\mathbb{Z})$, the linear map*

$$(\mathrm{comp}_{\mathrm{B,dR}} \circ a)_M \colon \omega_{\mathrm{dR}}(M) \otimes \mathbb{C} \to \omega_{\mathrm{B}}(M) \otimes \mathbb{C}$$

satisfies the following:

$$\omega_{\mathrm{B}}(M) = (\mathrm{comp}_{\mathrm{B,dR}} \circ a)_M(\omega_{\mathrm{dR}}(M)) \subset \omega_{\mathrm{B}}(M) \otimes \mathbb{C}. \tag{4.164}$$

Moreover, a can be chosen of the form $a = u_0 \cdot \tau(2\pi i)^{-1}$ with $u_0 \in U_{\mathrm{dR}}(\mathbb{R})$ and τ the canonical section from Lemma 4.159.

Proof. We follow [**Del89**, § 8.10]. Recall from (4.154) that

$$P_{\mathrm{B,dR}} = \underline{\mathrm{Iso}}^{\otimes}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}})$$

is a G_{dR}-torsor defined over $\mathbb{Q}$. By Corollary 4.161, this torsor is already trivial over $\mathbb{Q}$. Therefore, it has a rational point, and hence there exists an isomorphism of fiber functors $\alpha\colon \omega_{\mathrm{dR}} \xrightarrow{\sim} \omega_{\mathrm{B}}$. Define

$$a = \mathrm{comp}_{\mathrm{dR,B}} \circ \alpha. \tag{4.165}$$

By construction, a is an element of $G_{\mathrm{dR}}(\mathbb{C})$ and $\mathrm{comp}_{\mathrm{B,dR}} \circ a = \alpha$ holds, from which equality (4.164) follows. Note also that any other element of $G_{\mathrm{dR}}(\mathbb{C})$ satisfying this property is of the form $a\gamma$ for some $\gamma \in G_{\mathrm{dR}}(\mathbb{Q})$.

Let us now turn to the assertion that a can be chosen of the form $u_0 \cdot \tau(2\pi i)^{-1}$ for some $u_0 \in U_{\mathrm{dR}}(\mathbb{R})$. This uses in a crucial way the compatibility between the comparison isomorphism and complex conjugation explained in Proposition 2.187. In fact, the morphism ρ in that proposition can be extended to motives to define an automorphism of the functor ω_{B}. Hence, we obtain a rational point $\rho \in G_{\mathrm{B}}(\mathbb{Q})$. The compatibility of complex conjugation with the comparison isomorphism in our context says that the diagram of fiber functors

$$\begin{tikzcd}
\omega_{\mathrm{dR}} \arrow[r, "\alpha"] \arrow[rrr, bend left, "a"] & \omega_{\mathrm{B}} \arrow[r, hook] & \omega_{\mathrm{B}} \otimes \mathbb{C} \arrow[r, "\mathrm{comp}"] & \omega_{\mathrm{dR}} \otimes \mathbb{C} \\
\omega_{\mathrm{dR}} \arrow[r, "\alpha"'] \arrow[rrr, bend right, "a"'] & \omega_{\mathrm{B}} \arrow[r, hook] \arrow[u, "\rho"] & \omega_{\mathrm{B}} \otimes \mathbb{C} \arrow[r, "\mathrm{comp}"] \arrow[u, "\rho\otimes c"] & \omega_{\mathrm{dR}} \otimes \mathbb{C} \arrow[u, "\mathrm{Id}\otimes c"']
\end{tikzcd}$$

is commutative, where c is complex conjugation on the coefficients. The complex conjugate of a is $\overline{a} = \mathrm{Id} \otimes c \circ a$. Define $x = a^{-1}\overline{a}$. By the commutativity of the diagram, $x = \alpha^{-1}\rho\alpha$ holds. Thus, x belongs to $G_{\mathrm{dR}}(\mathbb{Q})$ and has order two.

Let us apply the isomorphism (4.165) to the motive $\mathbb{Q}(-1)$. Recall the homomorphism $t_{\mathrm{dR}}\colon G_{\mathrm{dR}} \to \mathbb{G}_m$ from Lemma 4.156. Since

$$\mathrm{comp}_{\mathrm{dR,B}}\colon \omega_{\mathrm{B}}(\mathbb{Q}(-1)) \longrightarrow \omega_{\mathrm{dR}}(\mathbb{Q}(-1))$$

is multiplication by $(2\pi i)^{-1}$ by Example 4.150, and

$$\alpha|_{\mathbb{Q}(1)}\colon \omega_{\mathrm{dR}}(\mathbb{Q}(1)) \to \omega_{\mathrm{B}}(\mathbb{Q}(1))$$

is an invertible map of one-dimensional $\mathbb{Q}$-vector spaces, $t_{\mathrm{dR}}(a) \in \mathbb{G}_m(\mathbb{C})$ belongs to $(2\pi i)^{-1}\mathbb{Q}^{\times}$. This implies the equality $t_{\mathrm{dR}}(a^{-1}\overline{a}) = -1$. Since τ is a section of t_{dR} defined over $\mathbb{Q}$, up to replacing a with $a\gamma$ for some $\gamma \in G_{\mathrm{dR}}(\mathbb{Q})$, we can assume

$$a^{-1}\overline{a} = \tau(-1). \tag{4.166}$$

Any other element satisfying both the identities (4.164) and (4.166) is of the form $a\gamma$ for some $\gamma \in G_{\mathrm{dR}}(\mathbb{Q})$ such that $\gamma^{-1}\tau(-1)\gamma = \tau(-1)$. In particular, any $\gamma \in \tau(\mathbb{Q}^{\times})$

works. Therefore, replacing a with $a\gamma$ for some $\gamma \in \tau(\mathbb{Q}^\times)$, one can choose a satisfying $t_{\mathrm{dR}}(a) = (2\pi i)^{-1}$. This amounts to $a = u_0 \cdot \tau(2\pi i)^{-1}$ with $u_0 \in U_{\mathrm{dR}}(\mathbb{C})$.

It remains to show that u_0 belongs to $U_{\mathrm{dR}}(\mathbb{R})$. By (4.166), the equality

$$\tau(2\pi i)u_0^{-1}\overline{u_0}\tau(-2\pi i)^{-1} = \tau(-1)$$

holds, and writing $\tau(-1) = \tau(2\pi i)\tau(-2\pi i)^{-1}$, one gets $u_0 = \overline{u_0}$. □

4.4.4. *The period map and the period conjecture.* Recall from the previous sections that $P_{\mathrm{dR,B}}$ denotes the scheme of tensor isomorphisms between ω_{B} and ω_{dR}, which has the structure of a pro-algebraic variety over $\mathbb{Q}$. The ring of regular functions $\mathcal{O}(P_{\mathrm{dR,B}})$ forms an ind-object in the category of $\mathbb{Q}$-algebras of finite type.

Definition 4.167. The *period map* is the ring morphism

$$\mathrm{per}\colon \mathcal{O}(P_{\mathrm{dR,B}}) \longrightarrow \mathbb{C} \tag{4.168}$$

given by evaluation at the point $\mathrm{comp}_{\mathrm{dR,B}}$:

$$\mathrm{per}(f) = f(\mathrm{comp}_{\mathrm{dR,B}}).$$

Similarly, evaluation at the point $\mathrm{comp}_{\mathrm{B,dR}}$ yields a period map

$$\mathcal{O}(P_{\mathrm{B,dR}}) \longrightarrow \mathbb{C}.$$

The following is a variant of Grothendieck's period conjecture for the category of mixed Tate motives over $\mathbb{Z}$ (see also [**And04**, § 25.2]).

Conjecture 4.169 (Grothendieck). *The point* $\mathrm{comp}_{\mathrm{dR,B}}$ *is generic.*

To give a meaning to the word "generic", we observe that the torsor $P_{\mathrm{B,dR}}$ can be written as the projective limit of the torsors $P^Y_{\mathrm{B,dR}}$ for the different mixed Tate motives Y, in analogy with Lemma 4.64. Generic then means that, for every quotient $P_{\mathrm{B,dR}} \to P^Y_{\mathrm{B,dR}}$ the image $\mathrm{comp}^Y_{\mathrm{B,dR}}$ of the point $\mathrm{comp}_{\mathrm{B,dR}}$ in $P^Y_{\mathrm{B,dR}}$ is not contained in any strict subvariety defined over $\mathbb{Q}$. Therefore, $\mathrm{comp}_{\mathrm{B,dR}}$ is generic if and only if, for every mixed Tate motive, the period map

$$\mathrm{per} = \mathrm{ev}_{\mathrm{comp}^Y_{\mathrm{B,dR}}} \colon \mathcal{O}(P^Y_{\mathrm{B,dR}}) \longrightarrow \mathbb{C}$$

is injective. Moreover, if $\mathrm{comp}_{\mathrm{B,dR}}$ is generic, then the transcendence degree of the residue field of $\mathrm{comp}^Y_{\mathrm{B,dR}}$ is equal to the dimension of $P^Y_{\mathrm{B,dR}}$.

From the previous discussion, we see that Grothendieck's period conjecture for mixed Tate motives is equivalent to the following:

Conjecture 4.170. *The period map* (4.168) *is injective.*

4.4.5. *Lie algebras.* By means of the tannakian formalism, the fiber functor ω_{dR} yields an equivalence of categories between $\mathbf{MT}(\mathbb{Z})$ and the finite-dimensional representations of G_{dR}. We can now apply the theory of graded Lie algebras explained in Section 3.3.8 to the semi-direct product decomposition

$$G_{\mathrm{dR}} = U_{\mathrm{dR}} \rtimes \mathbb{G}_m.$$

By Lemma 4.160, the group U_{dR} is a graded pro-unipotent group (Definition 3.211). Let $\mathfrak{u}_{\mathrm{dR}}$ be the Lie algebra of U_{dR} and $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ the associated graded Lie algebra as in Definition 3.213. We want to derive from Theorem 4.145 a structure theorem for the Lie algebras $\mathfrak{u}_{\mathrm{dR}}$ and $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$. To this end, we need to extract some finiteness consequences from Theorem 4.145 that will allow us to use the full force of the theory of graded Lie algebras.

LEMMA 4.171. *The Lie algebra $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is negatively graded. That is,*

$$\bigoplus_{n\geqslant 0}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})_n = 0.$$

PROOF. The proof goes in several steps. First, we write U_{dR} as a limit

$$U_{\mathrm{dR}} = \varprojlim_{\alpha} U_\alpha$$

of unipotent algebraic groups U_α with surjective morphisms $U_{\mathrm{dR}} \to U_\alpha$ surjective (Exercise 3.149) and with an action of $\mathbb{G}_m$ on each U_α that is compatible with the action of $\mathbb{G}_m$ on U_{dR} (Lemma 4.160). For each α, let $\mathfrak{u}_\alpha$ denote the Lie algebra of U_α. It is finite-dimensional and nilpotent.

The second step is to show the implication

$$\bigoplus_{n\geqslant 0}(\mathfrak{u}_\alpha)_n \neq 0 \quad\Longrightarrow\quad \bigoplus_{n\geqslant 0}(\mathfrak{u}_\alpha/[\mathfrak{u}_\alpha,\mathfrak{u}_\alpha])_n \neq 0 \tag{4.172}$$

for each finite-dimensional nilpotent Lie algebra $\mathfrak{u}_\alpha$, Indeed, if the right-hand side does not hold, then

$$\bigoplus_{n\geqslant 0}(\mathfrak{u}_\alpha)_n \subset [\mathfrak{u}_\alpha,\mathfrak{u}_\alpha].$$

Since a bracket of non-negative degree between homogeneous elements should contain at least one non-negative element, we deduce the inclusion

$$\bigoplus_{n\geqslant 0}(\mathfrak{u}_\alpha)_n \subset \left[\mathfrak{u}_\alpha, \bigoplus_{n\geqslant 0}(\mathfrak{u}_\alpha)_n\right].$$

The nilpotency of $\mathfrak{u}_\alpha$ implies the vanishing $\bigoplus_{n\geqslant 0}(\mathfrak{u}_\alpha)_n = 0$, and hence (4.172).

Let now $L_{\mathbb{Q}}(-n)$ be the abelian graded one-dimensional Lie algebra over $\mathbb{Q}$ concentrated in degree n. The third step is the observation that if $(\mathfrak{u}_\alpha/[\mathfrak{u}_\alpha,\mathfrak{u}_\alpha])_n \neq 0$, then one can construct a surjective homomorphism of graded Lie algebras

$$\mathfrak{u}_\alpha \longrightarrow L_{\mathbb{Q}}(-n).$$

The fourth step is the computation

$$\mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{G}_m}(L_{\mathbb{Q}}(-n))}(\mathbb{Q}(n),\mathbb{Q}) \neq 0. \tag{4.173}$$

The notation $\mathbf{Rep}_{\mathbb{G}_m}(L_{\mathbb{Q}}(-n))$ is explained in Definition 3.216. To prove (4.173), we consider the graded vector space $E = \mathbb{Q}v_0 \oplus \mathbb{Q}v_{-n}$ with v_0 in degree zero and v_{-n} in degree $-n$. Let a_n be a generator of $L_{\mathbb{Q}}(-n)$ sitting in degree n. The graded action of $L_{\mathbb{Q}}(-n)$ on E determined by $a_n v_{-n} = v_0$ turns E into a graded representation of $L_{\mathbb{Q}}(-n)$. It is then easy to check that this representation is a non-trivial element of the extension group

$$\mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{G}_m}(L_{\mathbb{Q}}(-n))}(\mathbb{Q}(n),\mathbb{Q}).$$

Finally, assume that there is an integer $n \geqslant 0$ such that $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})_n \neq 0$ holds. Then there is an α satisfying $(\mathfrak{u}^{\mathrm{gr}}_{\alpha})_n \neq 0$. By the second and third steps in the proof, there is an $n' \geqslant 0$ and a surjective graded Lie algebra homomorphism $\mathfrak{u}_\alpha \to L_{\mathbb{Q}}(-n')$. Since the map

$$\mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{G}_m}(L_{\mathbb{Q}}(-n'))}(\mathbb{Q}(n'),\mathbb{Q}) \longrightarrow \mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{G}_m}(\mathfrak{u}_\alpha)}(\mathbb{Q}(n'),\mathbb{Q})$$

is injective, we deduce the non-vanishing

$$\mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{G}_m}(U_\alpha)}(\mathbb{Q}(n'),\mathbb{Q}) = \mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{G}_m}(\mathfrak{u}_\alpha)}(\mathbb{Q}(n'),\mathbb{Q}) \neq 0$$

from the fourth step. Since the map $U_{\mathrm{dR}} \to U_\alpha$ is surjective, the map

$$\mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{G}_m}(U_\alpha)}(\mathbb{Q}(n'),\mathbb{Q}) \longrightarrow \mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{G}_m}(U_{\mathrm{dR}})}(\mathbb{Q}(n'),\mathbb{Q}) = \mathrm{Ext}^1_{\mathbf{Rep}(G_{\mathrm{dR}})}(\mathbb{Q}(n'),\mathbb{Q})$$

is also injective. Therefore, $\mathrm{Ext}^1_{\mathbf{Rep}(G_{\mathrm{dR}})}(\mathbb{Q}(n'),\mathbb{Q}) \neq 0$, in contradiction with Theorem 4.145. From this, we get $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})_n = 0$ for all $n \geqslant 0$. ☐

Lemma 4.174. *The finiteness condition*

$$\dim(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})_n < \infty$$

holds for all $n < 0$.

Proof. The idea is similar to that in the proof of Lemma 4.171. Assume that there exist integers $n < 0$ such that $\dim(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})_n = \infty$, and let n_0 be the maximum among them. Then, $\dim(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}/[\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}},\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}])_{n_0} = \infty$. This implies that we can construct a linearly independent infinite family of graded Lie algebra homomorphism $\omega_i\colon \mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}} \to L_{\mathbb{Q}}(-n_0)$, for all $i \geqslant 0$. Arguing as in the proof of Lemma 4.171, we can construct a linearly independent infinite family of extensions in $\mathrm{Ext}^1_{\mathbf{Rep}(G_{\mathrm{dR}})}(\mathbb{Q}(n_0),\mathbb{Q})$, which contradicts the finite-dimensionality. ☐

Corollary 4.175. *The graded Lie algebra* $\mathcal{O}(U_{\mathrm{dR}})$ *is connected and satisfies*

$$\dim \mathcal{O}(U_{\mathrm{dR}})_n < \infty$$

for all $n \geqslant 0$.

Proof. The ideal $I \subset \mathcal{O}(U_{\mathrm{dR}})$ corresponding to the unit $e \in U_{\mathrm{dR}}$ is homogeneous, and there is a decomposition as $\mathbb{Q}$-vector spaces

$$\mathcal{O}(U_{\mathrm{dR}}) = \mathbb{Q} \oplus I$$

with $\mathbb{Q}$ in degree zero. It is then enough to show that I has only positive degrees. We use the inductive limit

$$\mathcal{O}(U_{\mathrm{dR}}) = \varinjlim_{\alpha}(\mathcal{O}(U_\alpha)) \tag{4.176}$$

Let $I_\alpha \subset \mathcal{O}(U_\alpha)$ be the restriction of I to $\mathcal{O}(U_\alpha)$, which is still a homogeneous ideal. As a consequence of the proof of Lemma 4.171, the condition $(\mathfrak{u}_\alpha)_n = 0$ holds for all $n \geqslant 0$. From this, we deduce the vanishing

$$(I_\alpha/I_\alpha^2)_n = 0$$

for all $n \leqslant 0$. Since $\mathcal{O}(U_\alpha)$ is finitely generated, this implies $(I_\alpha)_n = 0$ for all $n \leqslant 0$. From the fact that the inductive limit is compatible with the grading we deduce that $I_n = 0$ for all $n \leqslant 0$ and $\mathcal{O}(U_{\mathrm{dR}})$ is connected.

By Lemma 4.174, $\dim(I/I^2)_n < \infty$ holds for all $n > 0$. Since the grading of I is positive, this implies $\dim I_n < \infty$ for all $n > 0$ and ends the proof. ☐

After Corollary 4.175, Theorem 3.217 yields an equivalence of categories

$$\mathbf{Rep}(G_{\mathrm{dR}}) = \mathbf{Rep}_{\mathbb{G}_m}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}).$$

Let $\mathbf{GrMod}^{\mathrm{fd}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}$ be the category of left $U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$-modules whose underlying $\mathbb{Q}$-vector spaces are finite-dimensional. There is a natural equivalence of categories

$$\mathbf{Rep}_{\mathbb{G}_m}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}) = \mathbf{GrMod}^{\mathrm{fd}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}.$$

Let $\mathbf{GrMod}^{\mathrm{fg}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}$ be the category of finitely generated left $U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$-modules.

LEMMA 4.177. *There is an equality of groups of Yoneda extensions*

$$\mathrm{Ext}^i_{\mathbf{GrMod}^{\mathrm{fd}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}}(\mathbb{Q}(n),\mathbb{Q}) = \mathrm{Ext}^i_{\mathbf{GrMod}^{\mathrm{fg}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}}(\mathbb{Q}(n),\mathbb{Q}).$$

PROOF. There is a canonical morphism

$$\mathrm{Ext}^i_{\mathbf{GrMod}^{\mathrm{fd}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}}(\mathbb{Q}(n),\mathbb{Q}) \longrightarrow \mathrm{Ext}^i_{\mathbf{GrMod}^{\mathrm{fg}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}}(\mathbb{Q}(n),\mathbb{Q}). \tag{4.178}$$

We first show that this map is surjective. Let $\xi \in \mathrm{Ext}^i_{\mathbf{GrMod}^{\mathrm{fg}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}}(\mathbb{Q}(n),\mathbb{Q})$ be an extension, written as

$$\xi\colon \quad 0 \longrightarrow \mathbb{Q} \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_i \longrightarrow \mathbb{Q}(n) \longrightarrow 0$$

For any graded left $U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$-module E we write

$$F_nE = \bigoplus_{n' \leqslant n} E_{n'}.$$

Since $U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$ is non-positively graded, F_nE is a submodule of E. Set $a = \max(-n,0)$ and $b = \min(-n,0)$. Then there is a diagram of equivalences of Yoneda extensions

$$\begin{tikzcd} & F_a\xi \arrow[dl] \arrow[dr] & \\ F_a\xi/F_b\xi & & \xi. \end{tikzcd}$$

Since each E_i is finitely generated and $U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})_n$ is finite-dimensional for each n, the extension $F_a\xi/F_b\xi$ belongs to $\mathrm{Ext}^i_{\mathbf{GrMod}^{\mathrm{fd}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}}(\mathbb{Q}(n),\mathbb{Q})$, thus showing that the morphism (4.178) is surjective.

The proof of the injectivity of the map (4.178) follows the same principle. The trivial extension is

$$\xi_0\colon \quad 0 \longrightarrow \mathbb{Q} \xrightarrow{\mathrm{Id}} \mathbb{Q} \longrightarrow 0 \dots 0 \longrightarrow \mathbb{Q}(n) \xrightarrow{\mathrm{Id}} \mathbb{Q}(n) \longrightarrow 0,$$

with the obvious variants for $i = 1, 2$. An extension $\xi \in \mathrm{Ext}^i_{\mathbf{GrMod}^{\mathrm{fd}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}}(\mathbb{Q}(n),\mathbb{Q})$ is sent to zero in $\mathrm{Ext}^i_{\mathbf{GrMod}^{\mathrm{fg}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}}(\mathbb{Q}(n),\mathbb{Q})$ if there exists an extension ξ_1 of finitely generated left $U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$-modules and a diagram of equivalences

$$\begin{tikzcd} & \xi_1 \arrow[dl] \arrow[dr] & \\ \xi_0 & & \xi. \end{tikzcd}$$

Now $F_a\xi_1/F_b\xi_1$ is an extension of finite-dimensional modules and the diagram

$$\begin{tikzcd} & F_a\xi_1/F_b\xi_1 \arrow[dl] \arrow[dr] & \\ \xi_0 & & \xi. \end{tikzcd}$$

shows that ξ was already equivalent to the trivial extension. $\square$

Since the category $\mathbf{GrMod}^{\mathrm{fg}}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}$ has enough projectives, the Yoneda extension groups can be computed using projective resolutions. By Theorem A.354,

$$\mathrm{Ext}^i_{\mathbf{GrMod}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{fg}}}}(\mathbb{Q}(n),\mathbb{Q}) = \mathrm{Ext}^i_{\mathbf{GrMod}_{U(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}}(\mathbb{Q}(n),\mathbb{Q}) = \mathrm{H}^i(K^*(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})_n),$$

where K^* stands for the Koszul complex from Definition A.349. Combining this with the previous equalities, we obtain

$$\mathrm{Ext}^i_{\mathbf{Rep}(G_{\mathrm{dR}})}(\mathbb{Q}(n),\mathbb{Q}) = \mathrm{H}^i(K^*(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})_n). \tag{4.179}$$

4.4.6. *The structure of* $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$. The main result of this section is the following:

THEOREM 4.180. *The graded Lie algebra* $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ *is free with one generator in each odd degree* $n \leqslant -3$, *and* $\mathfrak{u}_{\mathrm{dR}}$ *is the completion of* $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ *with respect to the grading.*

PROOF. Applying (4.179), Proposition A.353, and Theorem 4.145, we deduce that $\mathrm{H}_1(K_*(\mathfrak{u}^{\mathrm{gr}}))_n$ is one-dimensional for $n \leqslant -3$ odd, and is zero otherwise, and that $\mathrm{H}_2(K_*(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}))$ is zero. Then Proposition A.355 implies that $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is free with one generator in each odd degree $\leqslant -3$. The second statement follows from Lemma 3.214 and Corollary 4.175. □

REMARK 4.181.

i) The grading on $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ that we consider is the one coming from the action of $\mathbb{G}_m$ as the identity on $\mathbb{Q}(-1)$. This is the opposite of the grading used in [**DG05**] and [**Del13**], but agrees with that of [**And04**] or [**Bro12**].
ii) Consider the abelianization
$$(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}} = \mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}/[\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}},\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}],$$
which is a graded vector space. The proof of Theorem 4.180 yields a canonical identification
$$(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}_n = (\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(0),\mathbb{Q}(n)))^\vee.$$
Moreover, $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is isomorphic to the free Lie algebra generated by $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}$. Nevertheless, there is no canonical section from $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}$ to $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$, and hence no canonical isomorphism between $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ and the free Lie algebra generated by $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}$.
iii) Note also that $\mathfrak{u}_{\mathrm{dR}}$ and $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ are not isomorphic. In fact, $\mathfrak{u}_{\mathrm{dR}}$ is the completion of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ with respect to the grading, which implies that $\mathfrak{u}_{\mathrm{dR}}$ is not a free Lie algebra.

4.4.7. *The Hilbert–Poincaré series.* From Theorem 4.180, we deduce that the universal enveloping algebra $\mathcal{U}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$ of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is the free associative graded algebra with one generator in each odd degree $\leqslant -3$. The algebra of regular functions $\mathcal{O}(U_{\mathrm{dR}})$ is also graded and is the dual of the completed universal enveloping algebra $\widehat{\mathcal{U}}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$ in the graded sense (therefore, it is positively graded). In fact, its Hilbert–Poincaré series is equal to

$$H_{\mathcal{O}(U_{\mathrm{dR}})}(t) = \sum_{n\geqslant 0} \dim_{\mathbb{Q}}(\mathcal{O}(U_{\mathrm{dR}})_n)t^n = \sum_{n\leqslant 0} \dim_{\mathbb{Q}}(\mathcal{U}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})_n)t^{-n} = H_{\mathcal{U}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})}(t^{-1}).$$

Since $\mathcal{U}(\mathfrak{u}_{\mathrm{dR}}^{\mathrm{gr}})$ is the free associative graded algebra with one generator in each odd degree $n \leqslant -3$, the rightmost Hilbert–Poincaré series is equal to

$$H_{\mathcal{U}(\mathfrak{u}_{\mathrm{dR}}^{\mathrm{gr}})}(t^{-1}) = \frac{1}{1-t^3-t^5-t^7-\cdots} = \sum_{k\geqslant 0}(t^3+t^5+t^7+\cdots)^k,$$

as follows from the fact that the number of words of degree n made out of the letters $a_3, a_5, a_7, \ldots$ with a_r in degree r is exactly the coefficient of t^n in the above series. (Compare this with Lemma 1.77, where we computed the Hilbert–Poincaré series of a free commutative algebra.) Therefore, we obtain

$$H_{\mathcal{O}(U_{\mathrm{dR}})}(t) = \frac{1}{1-t^3-t^5-t^7-\cdots} = \frac{1-t^2}{1-t^2-t^3}. \tag{4.182}$$

Let us now, somehow artificially, introduce the algebra

$$\mathcal{H}^{\mathcal{MT}} = \mathcal{O}(U_{\mathrm{dR}}) \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \tag{4.183}$$

where f_2 is given degree 2. Note that $\mathcal{H}^{\mathcal{MT}}$ is isomorphic to the space of functions on $U_{\mathrm{dR}} \times \mathbb{A}^1$. Therefore, it can be seen as a relative of $\mathcal{O}(G_{\mathrm{dR}}) = \mathcal{O}(U_{\mathrm{dR}} \times \mathbb{G}_m)$ of "just the right size". From (4.182), we immediately deduce:

Lemma 4.184. *The Hilbert–Poincaré series of $\mathcal{H}^{\mathcal{MT}}$ is given by*

$$H_{\mathcal{H}^{\mathcal{MT}}}(t) = \frac{1}{1-t^2-t^3} = \sum_{k\geqslant 0} d_k t^k,$$

where the integers d_k are the same as in Zagier's Conjecture 1.71.

Following Deligne–Goncharov and Terasoma, in order to prove the upper bound $\dim \mathcal{Z}_k \leqslant d_k$ of Theorem A, we will construct in Chapter 5 a $\mathbb{Q}$-algebra $\mathcal{H}$, which injects into $\mathcal{H}^{\mathcal{MT}}$, and comes together with a surjective graded map

$$\mathcal{H} \longrightarrow \bigoplus \mathcal{Z}_k.$$

This will imply immediately the bound. The reason we have changed the grading of $\mathcal{O}(U_{\mathrm{dR}})$ is precisely to make this map compatible with the degree. We have already seen that multiple zeta values appear as periods of the pro-unipotent completion of the fundamental group of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. The motivic interpretation of this pro-unipotent completion will give the link between $\mathcal{H}$ and $\bigoplus \mathcal{Z}_k$. We discuss this interpretation in the next section.

⋆ ⋆ ⋆

Exercise 4.185. Use Theorem 4.145 to prove that the functor ω from (4.147) is a fibre functor on the tannakian category $\mathbf{MT}(\mathbb{Z})$.

Exercise 4.186. In this exercise, we study the dimensions of the graded pieces of the Lie algebra $\mathfrak{u}_{\mathrm{dR}}^{\mathrm{gr}}$.

i) Compute $\dim \mathfrak{u}_{\mathrm{dR}}^{n}$ for $n \leqslant 15$.

ii) Deduce that there are many possible sections of the map $\mathfrak{u}_{\mathrm{dR}}^{\mathrm{gr}} \to (\mathfrak{u}_{\mathrm{dR}}^{\mathrm{gr}})^{\mathrm{ab}}$. In particular, there is no canonical choice.

iii) Prove that the dimensions are given by the formula

$$\dim \mathfrak{u}_{\mathrm{dR}}^{n} = \sum_{d|n} \frac{\mu(d)}{d} \sum_{\substack{s_3, s_5, \ldots \in \mathbb{Z}_{\geqslant 0} \\ n/d = 3s_3 + 5s_5 + \cdots}} \frac{(s_3+s_5+\cdots-1)!}{s_3!s_5!\cdots},$$

where $\mu(d)$ denotes the Möbius function.

EXERCISE 4.187. In this exercise, we compare the representations of G_{dR} and U_{dR} using the functors of induction and restriction of representations. We denote by $\mathbf{Rep}_{\mathbb{Q}}^{\infty}(G_{\mathrm{dR}})$ and $\mathbf{Rep}_{\mathbb{Q}}^{\infty}(U_{\mathrm{dR}})$ the categories of possibly infinite-dimensional representations of these affine group schemes. The inclusion $U_{\mathrm{dR}} \to G_{\mathrm{dR}}$ gives rise to a restriction functor

$$\mathrm{Res}_{U_{\mathrm{dR}}}^{G_{\mathrm{dR}}} \colon \mathbf{Rep}_{\mathbb{Q}}^{\infty}(G_{\mathrm{dR}}) \longrightarrow \mathbf{Rep}_{\mathbb{Q}}^{\infty}(U_{\mathrm{dR}}).$$

This functor admits a left adjoint functor denoted by $\mathrm{Ind}_{U_{\mathrm{dR}}}^{G_{\mathrm{dR}}}$ (see Section A.1.4). By definition of a left adjoint functor, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{Rep}_{\mathbb{Q}}^{\infty}(U_{\mathrm{dR}})}(R, \mathrm{Res}_{U_{\mathrm{dR}}}^{G_{\mathrm{dR}}}(S)) = \mathrm{Hom}_{\mathbf{Rep}_{\mathbb{Q}}^{\infty}(G_{\mathrm{dR}})}(\mathrm{Ind}_{U_{\mathrm{dR}}}^{G_{\mathrm{dR}}}(R), S)$$

for all representations R of U_{dR} and S of G_{dR}.

i) Prove the equalities

$$\mathrm{Res}_{U_{\mathrm{dR}}}^{G_{\mathrm{dR}}}(\mathbb{Q}) = \mathbb{Q} \quad \text{and} \quad \mathrm{Ind}_{U_{\mathrm{dR}}}^{G_{\mathrm{dR}}}(\mathbb{Q}) = \prod_{n \in \mathbb{Z}} \mathbb{Q}(n).$$

ii) Prove that the adjoint property can be extended to the groups of extensions. Namely, there is a natural bijection

$$\mathrm{Ext}^i_{\mathbf{Rep}_{\mathbb{Q}}^{\infty}(U_{\mathrm{dR}})}(\mathbb{Q}, \mathrm{Res}_{U_{\mathrm{dR}}}^{G_{\mathrm{dR}}}(\mathbb{Q})) = \mathrm{Ext}^i_{\mathbf{Rep}_{\mathbb{Q}}^{\infty}(G_{\mathrm{dR}})}(\mathrm{Ind}_{U_{\mathrm{dR}}}^{G_{\mathrm{dR}}}(\mathbb{Q}), \mathbb{Q}).$$

4.5. The motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We continue considering the algebraic variety

$$X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$$

over $\mathbb{Q}$, and the complex manifold

$$M = X(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}.$$

As in Section 3.10, we use the following notation:

- $\mathbf{0}$ is the tangential base point $(0, 1)$, *i.e.* the tangent vector 1 at 0,
- $\mathbf{1}$ is the tangential base point $(1, -1)$, *i.e.* the tangent vector -1 at 1.

Let $\boldsymbol{x}, \boldsymbol{y} \in X(\mathbb{Q}) \cup \{\mathbf{0}, \mathbf{1}\}$ be rational or tangential base points. The aim of this section is to explain that the pro-unipotent completion of the torsor of paths from $\boldsymbol{x}$ to $\boldsymbol{y}$, as well as the extra structures given by composition of paths and local monodromy, are motivic in the sense of Definition 4.133. In fact, we want to add to Summary 3.408 a motivic side whose Betti and de Rham realizations give the Betti and de Rham sides of that summary. To exhibit the motivic nature of the affine group schemes and torsors in that summary, it seems necessary to use the language of algebraic geometry over a tannakian category [**Del89**, § 6]. In order to avoid this language, we will only consider the motivic analogues of ${}_{\bullet}U^{?}_{\bullet}$ and ${}_{\bullet}\mathcal{L}^{?}_{\bullet}$.

4.5.1. *The pro-mixed Tate motive* ${}_yU_x^{\mathrm{Mot}}$. We start with the case of two rational base points $x, y \in X(\mathbb{Q}) \subseteq M$. Recall the cosimplicial manifold ${}_yM_x^{\bullet}$ from Construction 3.281. As we already used in Section 3.7.1, when endowing the fundamental group with a mixed Hodge structure over $\mathbb{Q}$, all the maps involved in ${}_yM_x^{\bullet}$ are algebraic and defined over $\mathbb{Q}$, because the points x, y are rational. We will denote by ${}_yX_x^{\bullet}$ the corresponding cosimplicial object in the category $\mathbf{Sm}(\mathbb{Q})$.

As explained in Section 4.2.9, to ${}_yX_x^{\bullet}$ one associates a family of motives

$$\{[\sigma_{\leqslant N}\mathcal{N}\,{}_yX_x^{\bullet}]\}_{N \geqslant 0}.$$

By construction, given integers $M \geqslant N \geqslant 0$, there is a morphism

$$\sigma_{\leqslant M}\mathcal{N}_y X_x^\bullet \longrightarrow \sigma_{\leqslant N}\mathcal{N}_y X_x^\bullet$$

making $\{[\sigma_{\leqslant N}\mathcal{N}_y X_x^\bullet]\}_{N\geqslant 0}$ into a projective system of motives.

LEMMA 4.188. *The object* $[\sigma_{\leqslant N}\mathcal{N}_y X_x^\bullet]$ *belongs to* $\mathbf{DMT}(\mathbb{Q})$.

PROOF. Exercise 4.216. □

We can therefore consider the cohomology of this object with respect to the t-structure of the triangulated category $\mathbf{DMT}(\mathbb{Q})$.

DEFINITION 4.189. For each $N \geqslant 0$, we define a mixed Tate motive

$${}_yU_x^{\mathrm{Mot},N} = \mathrm{H}_0([\sigma_{\leqslant N}\mathcal{N}_y X_x^\bullet]) \in \mathbf{MT}(\mathbb{Q}).$$

As N varies, these motives fit into a pro-mixed Tate motive ${}_yU_x^{\mathrm{Mot}}$.

We also consider the constant cosimplicial variety $\mathrm{Spec}(\mathbb{Q})^\bullet$ given by $\mathrm{Spec}(\mathbb{Q})$ in all degrees, with coface and codegeneracy maps all equal to the identity. Applying the previous construction to $\mathrm{Spec}(\mathbb{Q})^\bullet$, one finds

$$\mathrm{H}_0([\sigma_{\leqslant N}\mathcal{N}\,\mathrm{Spec}(\mathbb{Q})^\bullet]) = \mathbb{Q}(0).$$

for all $N \geqslant 0$ (see Exercise 4.217).

4.5.2. *The structures of* ${}_yU_x^{\mathrm{Mot}}$. We next introduce some extra structures carried by ${}_yU_x^{\mathrm{Mot}}$: the unit and counit, the completed coproduct, the composition of paths, and the antipode. The idea is to give a geometric analogue of the constructions in the reduced bar complex of a connected dg-algebra (Definition 3.257), in such a way that they are compatible with the isomorphism from Lemma 3.284.

We start with the unit and counit. Each point $x \in X(\mathbb{Q})$ determines a morphism of cosimplicial varieties

$$\eta_x^\vee \colon \mathrm{Spec}(\mathbb{Q})^\bullet \longrightarrow {}_xX_x^\bullet \tag{4.190}$$

which sends $\mathrm{Spec}(\mathbb{Q})^n = \mathrm{Spec}(\mathbb{Q})$ to the point $(x, \dots, x) \in {}_xX_x^n$. Besides, we have for each pair of points $x, y \in X(\mathbb{Q})$ a map of cosimplicial varieties

$$\epsilon^\vee \colon {}_yX_x^\bullet \longrightarrow \mathrm{Spec}(\mathbb{Q})^\bullet \tag{4.191}$$

given by the structural map in all degrees. They induce morphisms

$$\eta_x^\vee \colon \mathbb{Q}(0) \longrightarrow {}_xU_x^{\mathrm{Mot}},$$
$$\epsilon^\vee \colon {}_yU_x^{\mathrm{Mot}} \longrightarrow \mathbb{Q}(0),$$

which are called *unit* and *counit* respectively.

REMARK 4.192. To understand the notation we will use in the following constructions, recall from Section 4.2.3 that the direct sum in the category $\mathbf{SmCor}(\mathbb{Q})$ corresponds to the disjoint union of varieties, whereas the tensor product is given by the cartesian product of varieties. Note also that all descriptions of morphisms should be understood in terms of correspondences. For instance, the antipode we write below is the cycle $(-1)^{\frac{n(n+1)}{2}}\Gamma$ in $X^n \times X^n$, where Γ is the graph of the morphism $(x_1, \dots, x_n) \mapsto (x_n, \dots, x_1)$.

For any rational points $x, y \in X(\mathbb{Q})$, consider the unbounded complex $\mathcal{C}^*({}_yX_x^\bullet)$ in the category $\mathbf{SmCor}(\mathbb{Q})$ given by

$$\mathcal{C}^n({}_yX_x^\bullet) = {}_yX_x^n,$$

together with the differential

$$d = \sum_{i=0}^{n+1}(-1)^i\delta^i\colon\ \mathcal{C}^n({}_yX_x^\bullet) \longrightarrow \mathcal{C}^{n+1}({}_yX_x^\bullet).$$

We consider the morphism

$$[X]^{\otimes n} \longrightarrow \bigoplus_{p+q=n} [X]^{\otimes p} \otimes [X]^{\otimes q}$$

in $\mathbf{SmCor}(\mathbb{Q})$ that sends the point $(x_1, \dots, x_n)$ to

$$\sum_{p+q=n} \sum_{\sigma \in Ш(p,q)} (-1)^\sigma (x_{\sigma(1)}, \dots, x_{\sigma(p)}) \otimes (x_{\sigma(p+1)}, \dots, x_{\sigma(n)}), \tag{4.193}$$

where $(-1)^\sigma$ is the sign of the permutation σ.

Remark 4.194. Notice that what appears in this expression is the permutation σ instead of σ^{-1} as in Proposition 1.160 or Definition 3.257. This is due to the contravariant nature of differential forms.

One can check that this map induces a morphism of complexes

$$\nabla^\vee \colon \mathcal{C}^*({}_yX_x^\bullet) \longrightarrow \mathcal{C}^*({}_yX_x^\bullet) \otimes \mathcal{C}^*({}_yX_x^\bullet).$$

Now, for points $x, y, z \in X(\mathbb{Q})$, and integers $p, q \geqslant 0$, we consider the map

$$\begin{aligned} [X]^{\otimes p} \otimes [X]^{\otimes q} &\longrightarrow \qquad [X]^{\otimes (p+q)} \\ (x_1, \dots, x_p) \otimes (y_1, \dots, y_q) &\longmapsto (x_1, \dots, x_p, y_1, \dots, y_q). \end{aligned} \tag{4.195}$$

Varying p and q, we obtain a morphism of complexes

$$\Delta^\vee \colon \mathcal{C}^*({}_zX_y^\bullet) \otimes \mathcal{C}^*({}_yX_x^\bullet) \longrightarrow \mathcal{C}^*({}_zX_x^\bullet).$$

Finally, the correspondence $[X]^{\otimes n} \to [X]^{\otimes n}$ given by

$$(x_1, \dots, x_n) \longmapsto (-1)^{\frac{n(n+1)}{2}}(x_n, \dots, x_1) \tag{4.196}$$

defines a morphism of complexes, called the dual antipode,

$$S^\vee \colon \mathcal{C}^*({}_yX_x^\bullet) \longrightarrow \mathcal{C}^*({}_xX_y^\bullet).$$

The next step consists in inducing morphisms at the level of the normalized complexes $\mathcal{N}({}_yX_x^\bullet)$. For this, one needs to check that the chain morphisms commute with the projector p_n of Lemma 4.103 and take care of the truncations. The precise statement is the following lemma, whose proof is elementary.

Lemma 4.197. *Let $N, M \geqslant 0$ be integers.*

i) *If $N \geqslant 2M$, then the map $\nabla^\vee$ induces a morphism of complexes*

$$\nabla^\vee \colon \sigma_{\leqslant N}\mathcal{N}({}_yX_x^\bullet) \longrightarrow \sigma_{\leqslant M}\mathcal{N}({}_yX_x^\bullet) \otimes \sigma_{\leqslant M}\mathcal{N}({}_yX_x^\bullet).$$

ii) *If $N \geqslant M$, then the map $\Delta^\vee$ induces a morphism of complexes*

$$\Delta^\vee \colon \sigma_{\leqslant N}\mathcal{N}({}_zX_y^\bullet) \otimes \sigma_{\leqslant N}\mathcal{N}({}_yX_x^\bullet) \longrightarrow \sigma_{\leqslant M}\mathcal{N}({}_zX_x^\bullet).$$

iii) *If $N \geqslant M$, then the map $S^\vee$ induces a morphism of complexes*

$$S^\vee : \sigma_{\leqslant N}\mathcal{N}({}_yX_x^\bullet) \longrightarrow \sigma_{\leqslant M}\mathcal{N}({}_xX_y^\bullet).$$

Moreover, when N and M vary within the above constraints, the three morphisms yield maps of projective systems.

As a consequence of Lemma 4.197, we obtain the following result:

PROPOSITION 4.198. *Given any three points $x, y, z \in X(\mathbb{Q})$, there are morphisms of pro-mixed Tate motives:*

i) *a composition of paths*

$$\Delta^\vee : {}_zU_y^{\mathrm{Mot}} \otimes {}_yU_x^{\mathrm{Mot}} \longrightarrow {}_zU_x^{\mathrm{Mot}};$$

ii) *a unit*

$$\eta_x^\vee : \mathbb{Q}(0) \longrightarrow {}_xU_x^{\mathrm{Mot}};$$

iii) *a completed coproduct*

$$\nabla^\vee : {}_yU_x^{\mathrm{Mot}} \longrightarrow {}_yU_x^{\mathrm{Mot}} \widehat{\otimes} {}_yU_x^{\mathrm{Mot}};$$

iv) *a counit*

$$\epsilon^\vee : {}_yU_x^{\mathrm{Mot}} \longrightarrow \mathbb{Q}(0);$$

v) *a dual antipode*

$$S^\vee : {}_yU_x^{\mathrm{Mot}} \longrightarrow {}_xU_y^{\mathrm{Mot}}.$$

4.5.3. *The motivic nature of the fundamental groupoid of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$.*

THEOREM 4.199 (Deligne–Goncharov [**DG05**]). *For $x, y \in X(\mathbb{Q})$, the Hodge realization of ${}_yU_x^{\mathrm{Mot}}$ agrees with the pro-mixed Hodge structure ${}_yU_x^{\mathrm{H}}$ described in Summary 3.408. In symbols,*

$$R^{\mathrm{H}}({}_yU_x^{\mathrm{Mot}}) = {}_yU_x^{\mathrm{H}}.$$

Moreover, R^{H} is compatible with the composition of paths, the unit, the completed coproduct, the counit, and the dual antipode. In particular, the diagram ${}_\bullet U_*^{\mathrm{H}}$ for $\bullet, *$ varying among rational base points, is motivic.

PROOF. Let A^* be the differential graded algebra from Example 2.284, that is,

$$A^0 = \mathbb{Q}, \quad A^1 = \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1$$

with zero differential. The product in this algebra satisfies $\omega_0 \wedge \omega_1 = 0$. The Hodge filtration is given by

$$F^0 = A^* \supset F^1 = A^1 \supset F^2 = 0,$$

and the weight filtration by

$$W_{-1} = 0 \subset W_0 = A^0 \subset W_1 = A^*.$$

As we saw in Proposition 2.287, the dg-algebra A^* allows us to compute the de Rham cohomology of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ with its weight and Hodge filtration. We also saw in Section 3.7.2 that it can be used to compute the de Rham side of ${}_\bullet U_*^{\mathrm{H}}$.

We will now use this algebra to compute the de Rham side of $R^{\mathrm{H}}({}_yU_x^{\mathrm{Mot}})$. Consider the variety $(\mathbb{P}^1_{\mathbb{Q}})^n$ and the divisor D_n consisting of all points with one coordinate equal to $0, 1$, or ∞. This is a simple normal crossing divisor. For all points $x, y \in X(\mathbb{Q})$, the n-th component of the cosimplicial scheme ${}_yX_x^\bullet$ is given by

$${}_yX_x^n = (\mathbb{P}^1_{\mathbb{Q}})^n \setminus D_n.$$

Let $(E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n), F, W)$ be the de Rham algebra of complex-valued smooth differential forms on $(\mathbb{P}^1(\mathbb{C}))^n$ with logarithmic poles along D_n with its Hodge and weight filtration (see Section 2.8.5). We set

$$A^*({}_yX^n_x) = A^* \otimes \overset{n}{\cdots} \otimes A^*.$$

The Hodge and weight filtrations of A^* induce Hodge and weight filtrations on $A^*({}_yX^n_x)$. For all rational points x, y and integer $n \geqslant 0$, there is an inclusion

$$\begin{aligned} A^*({}_yX^n_x) &\hookrightarrow E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n) \\ 1 \otimes \cdots \otimes \omega_{\varepsilon_i} \otimes \cdots \otimes 1 &\longmapsto \qquad \omega_{\varepsilon_i}(t_i), \end{aligned}$$

where $\varepsilon_i = 0, 1$, the 1-form ω_{ε_i} is in i-th position, and t_i is the i-th coordinate of $\mathbb{A}^n_{\mathbb{C}} \subset (\mathbb{P}^1_{\mathbb{C}})^n$. From the fact that

$$A^* \otimes \mathbb{C} \longrightarrow E^*_{\mathbb{P}^1(\mathbb{C})}(\log D_n)$$

is a bifiltered quasi-isomorphism (see the end of Example 2.284), we deduce that

$$A^*({}_yX^n_x) \otimes \mathbb{C} \hookrightarrow E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n)$$

is also a bifiltered quasi-isomorphism. Thus, $A^*({}_yX^n_x)$ determine the Hodge and weight filtration of the de Rham cohomology of ${}_yX^n_x$, even with its $\mathbb{Q}$-structure. Moreover, all the above inclusions are functorial with respect to any morphism involved in the structures of ${}_yX^n_x$. More precisely, the following holds:

Lemma 4.200. *The family of inclusions*

$$A^*({}_yX^n_x) \hookrightarrow E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n), \tag{4.201}$$

for $x, y \in X(\mathbb{Q})$ and $n \geqslant 0$ is functorial with respect to

i) *the coface and codegeneracy maps of the cosimplicial schemes ${}_yX^\bullet_x$;*
ii) *the maps (4.190) and (4.191), where we identify* $\operatorname{Spec}(\mathbb{Q})$ *with ${}_yX^0_x$ through the structure map of $\mathbb{Q}$-schemes;*
iii) *the maps (4.193), (4.195) and (4.196) that will induce the product, the coproduct, and the antipode.*

Moreover, each map in the family is a filtered quasi-isomorphism.

Proof. The fact that each map in the family is a quasi-isomorphism has already been discussed. To be precise about what functoriality means in this lemma, we spell out the case of a coface. All the other maps are treated in a similar way. Consider the coface

$$\delta^0 \colon {}_yX^n_x \longrightarrow {}_yX^{n+1}_x$$

given by $\delta^0(x_1, \dots, x_n) = (y, x_1, \dots, x_n)$, and the diagram

$$\begin{array}{ccc} A^*({}_yX^{n+1}_x) & \longrightarrow & E^*_{\mathbb{P}^1(\mathbb{C})^{n+1}}(\log D_{n+1}) \\ & & \big\downarrow {\scriptstyle (\delta^0)^*} \\ A^*({}_yX^n_x) & \longrightarrow & E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n). \end{array}$$

The statement of the lemma means that there is a unique morphism

$$A^*({}_yX^{n+1}_x) \longrightarrow A^*({}_yX^n_x)$$

completing the diagram to a commutative square. We also denote it by $(\delta^0)^*$. From the fact that the horizontal arrows are injective, the unicity is clear and it suffices to prove the existence. The sought after map is obviously given by

$$(\delta^0)^*(a_1 \otimes \cdots \otimes a_{n+1}) = \varepsilon(a_1)a_2 \otimes \cdots \otimes a_{n+1},$$

where ε is the augmentation of A^* given by (3.344). All the remaining maps are defined in a similar way. The compatibility of all the morphims with the composition of maps is just a consequence of the injectivity of the morphisms (4.201). $\square$

The main consequence of Lemma 4.200 is that to compute the de Rham realization functor of ${}_yU_x^{\mathrm{Mot}}$ as explained in Section 4.2.10, we can use the algebras $A^*({}_yX_x^n)$. From this, we derive an isomorphism

$$R^{\mathrm{dR}}({}_yU_x^{\mathrm{Mot}})^\vee = \varinjlim_N \mathrm{H}_0\left(\mathrm{Tot}\,\sigma_{\leqslant N}\mathcal{N}A^*({}_yM_x^\bullet)\right).$$

By Lemma 3.284, there is a canonical isomorphism

$$\mathrm{Tot}\,\mathcal{N}A^*_\bullet \xrightarrow{\sim} B^*(A^*).$$

Taking the truncation, the cohomological functor H^0, and the inductive limit, we deduce that $R^{\mathrm{dR}}({}_yU_x^{\mathrm{Mot}})^\vee = {}_yA_x^{\mathrm{dR}}$. By duality we get

$$R^{\mathrm{dR}}({}_yU_x^{\mathrm{Mot}}) = {}_yU_x^{\mathrm{dR}}.$$

The next step is to check the compatibility with the structures on both sides. This is the content of next lemma.

LEMMA 4.202. *The morphism ψ of Lemma 3.284 is compatible with the shuffle product, the coproduct, and the antipude.*

PROOF. Since the various structures do not depend on the rational points x, y, we omit them from the notation. We begin by proving the compatibility with the shuffle product. For non-negative integers p, q, r, s, the map (4.193) induces a map

$$\nabla\colon A^r(X^p) \otimes A^s(X^q) \longrightarrow A^{r+s}(X^{p+q})$$

given by the formula

$$\nabla\left((\omega_1(x_1) \wedge \cdots \wedge \omega_p(x_p)) \otimes (\omega_{p+1}(x_{p+1}) \wedge \cdots \wedge \omega_{p+q}(x_{p+q}))\right) = \\ \sum_{\sigma \in Ш(p,q)} (-1)^\sigma (-1)^{ps} \omega_1(x_{\sigma(1)}) \wedge \cdots \wedge \omega_{p+q}(x_{\sigma(p+q)}).$$

The sign $(-1)^\sigma$ comes from the definition of the map (4.193), while the sign $(-1)^{ps}$ comes from the fact that we have to swap the simplicial degree p and the differential degree s. We now compute

$$\nabla(\psi([\omega_1|\dots|\omega_p]) \otimes \psi([\omega_{p+1}|\dots|\omega_{p+q}])) = \\ \sum_{\sigma \in Ш(p,q)} (-1)^{\sum_{i=1}^{p+q} i \deg(\omega_i)} (-1)^\sigma \omega_1(x_{\sigma(1)}) \wedge \cdots \wedge \omega_{p+q}(x_{\sigma(p+q)}).$$

Here, we have used the equality

$$\sum_{i=1}^{p} i \deg(\omega_i) + \sum_{j=1}^{q} j \deg(\omega_{p+j}) + p \sum_{j=1}^{q} \deg(\omega_{p+j}) = \sum_{i=1}^{p+q} i \deg(\omega_i). \tag{4.203}$$

We also compute

$$\psi(\nabla([\omega_1|\dots|\omega_p]\otimes[\omega_{p+1}|\dots|\omega_{p+q}])) = \sum_{\sigma\in Ш(p,q)} \eta(\sigma)(-1)^{\sum_{i=1}^{p+q} i\deg(\omega_{\sigma^{-1}(i)})}\omega_{\sigma^{-1}(1)}(x_1)\wedge\dots\wedge\omega_{\sigma^{-1}(p+q)}(x_{p+q}),$$

where $\eta(\sigma)$ is the sign determined by equation (3.251). To see that the signs in both expressions agree, let us introduce formal variables $a_1\dots a_{p+q}$ of degree -1, and put $a=a_1\wedge\dots\wedge a_{p+q}$. On the one hand,

$$\begin{aligned}\eta(\sigma)&(-1)^{\sum_{i=1}^{p+q} i\deg(\omega_{\sigma^{-1}(i)})}\omega_{\sigma^{-1}(1)}(x_1)\wedge\dots\wedge\omega_{\sigma^{-1}(p+q)}(x_{p+q})\wedge a\\ &=\eta(\sigma)a_1\wedge\omega_{\sigma^{-1}(1)}(x_1)\wedge\dots\wedge a_{p+q}\wedge\omega_{\sigma^{-1}(p+q)}(x_{p+q})\\ &=a_{\sigma(1)}\wedge\omega_1(x_{\sigma(1)})\wedge\dots\wedge a_{\sigma(p+q)}\wedge\omega_{p+q}(x_{\sigma(p+q)}),\end{aligned}$$

while, on the other hand,

$$\begin{aligned}&(-1)^{\sum_{i=1}^{p+q} i\deg(\omega_i)}(-1)^{\sigma}\omega_1(x_{\sigma(1)})\wedge\dots\wedge\omega_{p+q}(x_{\sigma(p+q)})\wedge a\\ &=(-1)^{\sum_{i=1}^{p+q} i\deg(\omega_i)}\omega_1(x_{\sigma(1)})\wedge\dots\wedge\omega_{p+q}(x_{\sigma(p+q)})\wedge a_{\sigma(1)}\wedge\dots\wedge a_{\sigma(p+q)}\\ &=a_{\sigma(1)}\wedge\omega_1(x_{\sigma(1)})\wedge\dots\wedge a_{\sigma(p+q)}\wedge\omega_{p+q}(x_{\sigma(p+q)}).\end{aligned}$$

This proves the compatibility with the shuffle product.

We next prove the compatibility with the coproduct. The maps (4.195) induce morphisms

$$\Delta\colon A^t(X^n)\longrightarrow\bigoplus_{r+s=t}\bigoplus_{p+q=n}A^r(X^p)\otimes A^s(X^q)$$

given by

$$\begin{aligned}&\Delta(\omega_1(x_1)\wedge\dots\wedge\omega_n(x_n))\\ &\quad=\sum_{p=0}^{n}(-1)^{p\sum_{i=p+1}^{n}\deg(\omega_i)}\omega_1(x_1)\wedge\dots\wedge\omega_p(x_p)\otimes\omega_{p+1}(x_{p+1})\wedge\dots\wedge\omega_n(x_n),\end{aligned}$$

where the sign comes again from the swap of a simplicial degree and a differential degree. The equality $\Delta\circ\psi=\psi\circ\Delta$ is easily checked using equation (4.203).

Finally, the map (4.196) induces a morphisms

$$S\colon A^*(X^n)\longrightarrow A^*(X^n)$$

given by the formula

$$S(\omega_1(x_1)\wedge\dots\wedge\omega_n(x_n))=(-1)^{\frac{n(n+1)}{2}}\omega_1(x_n)\wedge\dots\wedge\omega_n(x_1).$$

The proof of the compatibility of the antipode S with the map ψ follows the same method as the previous compatibilities. □

It follows from this lemma that the de Rham realization $R^{\mathrm{dR}}({}_yU_x^{\mathrm{Mot}})$ is isomorphic to ${}_yU_x^{\mathrm{dR}}$ in a compatible way with all the additional structures. To conclude,

$$R^{\mathrm{B}}({}_yU_x^{\mathrm{Mot}})={}_yU_x^{\mathrm{B}}$$

follows from Theorem 3.316, Lemma 3.298, Proposition A.238, and the description of the Betti realization functor in Section 4.2.10. □

4.5.4. *The case of tangential base points.* It remains to prove that the space of paths with tangential base points is also motivic. We start with the particular case of $\mathbb{G}_m = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, \infty\}$ and the tangential base point $\mathbf{0} = (0, 1)$. Recall from Variant 3.412 that the method used to study $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ also works for $\mathbb{G}_m$. In this case, we use the dg-algebra $A(\mathbb{G}_m) = \mathbb{Q} \oplus \mathbb{Q}\,\omega_0$ to obtain ${}_{\boldsymbol{y}}U(\mathbb{G}_m)^{\mathrm{dR}}_{\boldsymbol{x}} = \mathbb{Q}\llbracket e_0 \rrbracket$

Proposition 4.204. *There is an isomorphism*

$$ {}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}} \xrightarrow{\sim} {}_{1}U(\mathbb{G}_m)^{\mathrm{H}}_{1}. $$

Moreover, if $x \in \mathbb{G}_m(\mathbb{Q})$, then there is an isomorphism

$$ {}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{x} \xrightarrow{\sim} {}_{1}U(\mathbb{G}_m)^{\mathrm{H}}_{x}. $$

Proof. We only prove the second statement. The proof of the first one is similar. We define the de Rham component of the sought isomorphism as the identity, so it is clearly compatible with the Hodge and the weight filtrations. As was the case of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$, the de Rham side is independent of the base points.

Recall the straight path $\mathbf{dch}$ between 0 and 1, given by $\mathbf{dch}(t) = t$ for $t \in [0, 1]$. We define the Betti part of the isomorphism as the map induced by the composition of paths which sends a path $\gamma \in \pi_1(\mathbb{G}_m; \mathbf{0}, x)$ to $\mathbf{dch} \cdot \gamma \in \pi_1(\mathbb{G}_m; 1, x)$. We need to prove that both isomorphisms are compatible with the comparison isomorphism. Since $\mathrm{comp} = \mathrm{comp}_{\mathrm{dR,B}}$ is given by the iterated integral map

$$ \mathrm{comp}(\gamma) = \sum_{n \geqslant 0} e_0^n \int_\gamma \omega_0 \overset{n}{\cdots} \omega_0 $$

and satisfies $\mathrm{comp}(\gamma \cdot \gamma') = \mathrm{comp}(\gamma)\,\mathrm{comp}(\gamma')$, we only need to check

$$ \mathrm{comp}(\mathbf{dch}) = 1. $$

This last equality follows by taking the limit $z \to 1$ in Example 3.387.

That the Betti part of the isomorphism is compatible with the weight filtration is now a consequence of the fact that the de Rham side is. $\square$

From the proposition we immediately deduce:

Corollary 4.205. *The pro-mixed Hodge structures ${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{x}$ and ${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}}$ are motivic (i.e. they belong to the essential image of R^H).*

The next lemma describes the structure of ${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}}$.

Lemma 4.206. *The pro-mixed Hodge structure ${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}}$ is split and agrees with*

$$ \prod_{n \geqslant 0} \mathbb{Q}(n). $$

In particular, ${}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}} = \mathbb{Q}(1)$.

Proof. Let f_n and b_n be generators of $\mathbb{Q}(n)_{\mathrm{dR}}$ and $\mathbb{Q}(n)_{\mathrm{B}}$ respectively; they satisfy $\mathrm{comp}(b_n) = (2\pi i)^n f_n$. Let γ_0 be the generator of $\pi_1(\mathbb{G}_m, \mathbf{0})$ introduced in Section 3.9.1. By Example 3.389, we know that $\mathrm{comp}_{\mathrm{dR,B}}(\gamma_0) = \exp(2\pi i e_0)$ holds. Consider the power series

$$ \log(\gamma_0) = \log(1 + (\gamma_0 - 1)) \in \mathbb{Q}[\pi_1(\mathbb{G}_m, \mathbf{0})]^\wedge. $$

For each n, we define a map

$$ \varphi_n \colon \mathbb{Q}(n) \to {}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}} \tag{4.207} $$

which sends f_n to $e_0^n \in \mathbb{Q}\langle\langle e_0 \rangle\rangle$, and b_n to $\log(\gamma_0)^n \in \mathbb{Q}[\pi_1(\mathbb{G}_m, \mathbf{0})]^\wedge$. This map is compatible with the comparison isomorphism:

$$\begin{aligned}\mathrm{comp}_{\mathrm{dR,B}}(\varphi_n(b_n)) &= \mathrm{comp}_{\mathrm{dR,B}}(\log(\gamma_0)^n)\\ &= (2\pi i)^n e_0^n\\ &= \varphi_n(\mathrm{comp}_{\mathrm{dR,B}}(b_n)).\end{aligned}$$

Moreover, taking into account that

$$\log(\gamma_0)^n \in J^n\mathbb{Q}[\pi_1(\mathbb{G}_m, \mathbf{0})]^\wedge = W_{-2n}\mathbb{Q}[\pi_1(\mathbb{G}_m, \mathbf{0})]^\wedge$$

and $e_0^n \in F^{-n} \cap W_{-2n}\mathbb{Q}\langle\langle e_0 \rangle\rangle$, the map (4.207) is a morphism of mixed Hodge structures. The maps φ_n induce the sought isomorphism of pro-mixed Hodge structures. The second statement follows immediately from the first one. □

We next reduce the question of showing that the mixed Hodge structure of the universal enveloping algebra is motivic to that of the Lie algebra.

LEMMA 4.208. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be base points of M (tangential or not). Then the pro-mixed Hodge structure ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$ is motivic if and only if the structure ${}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}}$ is.*

PROOF. Recall that ${}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}}$ is a sub-mixed Hodge structure of ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$. By Proposition 4.134, if ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$ is motivic, then ${}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}}$ is also motivic. Conversely, assume that ${}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}}$ is motivic. Recall that ${}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}}$ is a projective limit

$${}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}} = \varprojlim_N {}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}}/({}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}})_{\geqslant N+1}.$$

By Proposition 4.134, each quotient in this limit is motivic. From

$${}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{H}} = \varinjlim_N \mathrm{Sym}^*({}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}}/({}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{H}})_{\geqslant N+1})^\vee,$$

we deduce that ${}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{H}}$ is motivic. By duality, the same holds for ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$. □

Now let $x \in X(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ be a rational point and $\mathbf{0}$ the tangential base point $(0, 1)$. By Lemma 4.208, to show that ${}_{\mathbf{0}}U_x^{\mathrm{H}}$ is motivic, it is enough to show that ${}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}}$ is. To show that ${}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}}$ is motivic, we will embed it into a motivic mixed Hodge structure. Once this is proved, that ${}_{\mathbf{1}}U_x^{\mathrm{H}}$ is motivic follows from the symmetry of X that sends x to $1 - x$.

Let $f\colon X \to \mathbb{G}_m$ be the natural inclusion. Then f induces a morphism of mixed Hodge structures

$$\varphi_1\colon {}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}} \longrightarrow {}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_x^{\mathrm{H}}. \tag{4.209}$$

The map f also induces a local monodromy map

$$f^*\colon {}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} \longrightarrow {}_{\mathbf{0}}U_{\mathbf{0}}^{\mathrm{H}}.$$

Consider the composition of morphisms of mixed Hodge structures

$$\begin{aligned}{}_{\mathbf{0}}U_x^{\mathrm{H}} \otimes {}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} &\xrightarrow{\nabla^\vee \otimes \mathrm{Id}} {}_{\mathbf{0}}U_x^{\mathrm{H}} \widehat{\otimes} {}_{\mathbf{0}}U_x^{\mathrm{H}} \otimes {}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}}\\ &\xrightarrow{S^\vee \otimes \mathrm{Id} \otimes \mathrm{Id}} {}_xU_{\mathbf{0}}^{\mathrm{H}} \widehat{\otimes} {}_{\mathbf{0}}U_x^{\mathrm{H}} \otimes {}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} \longrightarrow {}_xU_x^{\mathrm{H}},\end{aligned}$$

where the last morphism is induced by the composition of paths

$$\gamma_1 \otimes \gamma_2 \otimes \gamma_3 \longmapsto \gamma_1 \cdot f^*(\gamma_3) \cdot \gamma_2.$$

Restricting to Lie type elements, we obtain a map

$$ {}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}} \otimes {}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} \longrightarrow {}_x\mathcal{L}_x^{\mathrm{H}}, \tag{4.210} $$

which taking the identification ${}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} = \mathbb{Q}(1)$ into account induces a morphism of pro-mixed Hodge structures

$$ \varphi_2 \colon {}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}} \longrightarrow {}_x\mathcal{L}_x^{\mathrm{H}}(-1). \tag{4.211} $$

LEMMA 4.212. *The morphism of pro-mixed Hodge structures*

$$ \varphi_1 + \varphi_2 \colon {}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}} \longrightarrow {}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_{x'}^{\mathrm{H}} \oplus {}_x\mathcal{L}_x^{\mathrm{H}}(-1) $$

is injective.

PROOF. It is enough to check the injectivity on the de Rham side. Let $\mathcal{L}$ be the free Lie algebra with generators e_0 and e_1 on degree -1. Let $\widehat{\mathcal{L}}$ be the completion of $\mathcal{L}$ with respect to this grading. Then we have ${}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{dR}} = \widehat{\mathcal{L}}$ and ${}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_x^{\mathrm{H}} = \mathbb{Q}\, e_0$. Clearly, the map φ_1 is the projection to the e_0 component. By construction, the map (4.210), is given by $a \otimes e_0 \mapsto [e_0, a]$. Therefore, the map $\varphi_2 \colon \widehat{\mathcal{L}} \to \widehat{\mathcal{L}}$ is given by $a \mapsto [e_0, a]$. Let $\varphi_2' \colon \mathcal{L} \to \mathcal{L}$ denote the map given by the same formula. The kernel of φ_2' is equal to $\mathbb{Q}\, e_0$ by [**Reu93**, Thm. 2.10]. It is then an easy exercise in projective limits to show that this implies that the kernel of φ_2 is also $\mathbb{Q}\, e_0$. Since φ_1 does not vanish on the kernel of φ_2, we deduce the lemma. □

Combining Proposition 4.204 and Theorem 4.199, we know that the pro-mixed Hodge structure ${}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_x^{\mathrm{H}} \oplus {}_x\mathcal{L}_x^{\mathrm{H}}(-1)$ is motivic. By Proposition 4.134, we deduce that ${}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}}$ is motivic and, by Lemma 4.208, that ${}_{\mathbf{0}}U_x^{\mathrm{H}}$ is motivic.

Let us now consider the case of two tangential base points $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$ of X. Let $z \in X(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ be a rational point. The composition of paths gives us a surjection

$$ {}_{\boldsymbol{y}}U_z^{\mathrm{H}} \otimes {}_zU_{\boldsymbol{x}}^{\mathrm{H}} \longrightarrow {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}. $$

Since we already know that the structures on the left-hand side are motivic, we deduce that ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$ is also motivic. Once we know that, for all $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$, the mixed Hodge structure ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$ is motivic, the realization functor R^{H} being fully faithful, any morphism among them is also motivic. Therefore, the composition of paths, the completed coproduct, the antipode, the unit and the counit, and the local monodromy maps are all motivic.

4.5.5. *The main theorem.* From the previous discussion, we deduce

THEOREM 4.213 (Deligne–Goncharov [**DG05**]). *For each pair of tangential base points $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$ of X, there is a pro-mixed Tate motive ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{Mot}}$ whose Hodge realization is isomorphic to ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$. By the fully faithfulness of the realization functor, ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{Mot}}$ is unique up to unique isomorphism. Moreover, the unit and the counit, the composition of paths, the completed coproduct, the antipode, and the local monodromy maps are motivic.*

In fact, more is true:

THEOREM 4.214 (Deligne–Goncharov [**DG05**]). *For each pair of tangential base points $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$, the pro-mixed Tate motive ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{Mot}}$ defines a pro-object in the category $\mathbf{MT}(\mathbb{Z})$. The motive ${}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{Mot}}$ belongs to $\mathbf{MT}(\mathbb{Z})$.*

PROOF. The proof relies on showing that the ℓ-adic realizations of these motives are unramified (see [**DG05**, Prop. 4.17]), so that Proposition 4.143 applies. □

COROLLARY 4.215. *The diagram D_U^{H} of Definition 3.414 is motivic over $\mathbb{Z}$.*

The importance of this result is that it connects a very abstract and non-explicit group $G_{\mathrm{dR}} = \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{dR}})$, but with known structure (see Sections 4.4.3 and 4.4.5), with a very concrete combinatorial group $\mathrm{Aut}(D_U^{\mathrm{dR}})$ (see Section 3.10.3). The group G_{dR} is the group of symmetries of the category $\mathbf{MT}(\mathbb{Z})$ and the fiber functor ω_{dR}, so it acts on the de Rham realization of every mixed Tate motive over $\mathbb{Z}$. By Exercise 4.75, it also acts on the de Rham realization of any pro-mixed Tate motive over $\mathbb{Z}$ or even of any diagram of pro-mixed Tate motives over $\mathbb{Z}$. By Theorem 4.214, the group G_{dR} acts on the diagram D_U^{dR}, which gives rise to a group homomorphism

$$G_{\mathrm{dR}} \longrightarrow \mathrm{Aut}(D_U^{\mathrm{dR}}) = \mathrm{Aut}(D^{\mathrm{dR}}).$$

The subgroup $U_{\mathrm{dR}} \subset G_{\mathrm{dR}}$ acts trivially on the motive $\mathbb{Q}(1)$, which implies that its image acts trivially on ${}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{dR}}$, and hence on ${}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}}$. Therefore, the image of U_{dR} is contained in $\mathrm{Aut}^0(D^{\mathrm{dR}})$, and we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & U_{\mathrm{dR}} \arrow[r] \arrow[d] & G_{\mathrm{dR}} \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Aut}^0(D^{\mathrm{dR}}) \arrow[r] & \mathrm{Aut}(D^{\mathrm{dR}}) \arrow[r] & \mathbb{G}_m \arrow[r] & 0.
\end{tikzcd}$$

The next chapter will be mainly devoted to extract consequences of this diagram.

⋆⋆⋆

EXERCISE 4.216. Use that $[X]$ belongs to $\mathbf{DMT}(\mathbb{Q})$ and the fact that the category $\mathbf{DMT}(\mathbb{Q})$ is closed under products and extensions to prove by induction that the object $[\sigma_{\leqslant N}\mathcal{N}_y X_x^{\bullet}]$ belongs to $\mathbf{DMT}(\mathbb{Q})$.

EXERCISE 4.217. Show that the complex $\mathcal{N}\operatorname{Spec}(\mathbb{Q})^{\bullet}$ in $C(\mathbf{SmCor}(\mathbb{Q})_{\mathrm{pa}})$ is isomorphic to the complex $\operatorname{Spec}(\mathbb{Q})$ concentrated in degree zero, and deduce that

$$\mathrm{H}_0(\sigma_{\leqslant N}\mathcal{N}\operatorname{Spec}(\mathbb{Q})^{\bullet}) = \mathbb{Q}(0)$$

holds for all integers $N \geqslant 0$.

5. Motivic multiple zeta values (after Brown, Deligne, and Goncharov)

The end is nigh! In this final chapter, we pull together all the techniques developed so far to prove Theorems A and B from the preface. The strategy is to upgrade multiple zeta values, which are real numbers, to motivic multiple zeta values, which are functions on a certain subscheme of the de Rham fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. As such, they carry an action of the motivic fundamental group G_{dR} of the category of mixed Tate motives $\mathbf{MT}(\mathbb{Z})$. This will be a powerful tool for understanding the relations among multiple zeta values, in the same way that usual Galois theory is for algebraic numbers. More precisely, there is a scheme ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ defined over $\mathbb{Q}$ whose algebra of functions is the Hoffman algebra $\mathfrak{H}$ and that contains a real point $\mathbf{dch}^{\mathrm{dR}}$, the de Rham counterpart of the straight path from 0 to 1. Evaluation at this point yields a map from $\mathfrak{H}$ to multiple zeta values that agrees with the shuffle regularization. Every closed subscheme of ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ that is defined over $\mathbb{Q}$ and contains the point $\mathbf{dch}^{\mathrm{dR}}$ gives rises to an ideal of rational polynomial relations between multiple zeta values. Hence, the problem of finding all of them amounts to finding the smallest such subscheme. A natural candidate from the point of view of the theory of motives is the closure $\mathcal{Y}$ of the G_{dR}-orbit of $\mathbf{dch}^{\mathrm{dR}}$. According to the period conjecture, it should be the smallest one. Motivic multiple zeta values are functions on $\mathcal{Y}$. Theorem A is proved in Section 5.1. We begin by summarizing the main properties of the category $\mathbf{MT}(\mathbb{Z})$ and the graded algebra $\mathcal{H}^{\mathcal{MT}}$ that were established in the last chapter. To relate the known dimensions of the graded pieces of $\mathcal{H}^{\mathcal{MT}}$ to multiple zeta values, we perform a close study of the algebra $\mathcal{H} = \mathcal{O}(\mathcal{Y})$. By construction, $\mathcal{H}$ surjects onto the algebra $\mathcal{Z}$ of multiple zeta values. Besides, using the geometric of $\mathcal{Y}$ we will show that $\mathcal{H}$ injects into $\mathcal{H}^{\mathcal{MT}}$. The upper bound in Theorem A follows from the combination of these results. In Section 5.2, we define motivic multiple zeta values $\zeta^{\mathfrak{m}}(s_1, \ldots, s_\ell)$ as certain elements of $\mathcal{H}$, and we recast Goncharov's coproduct on iterated integrals as a formula for the action of G_{dR} on these elements. Working modulo products gives a simpler expression to deal with, the so-called infinitesimal coaction. The main result of this section is that simple motivic multiple zeta values $\zeta^{\mathfrak{m}}(n)$ span the kernel of this infinitesimal coaction. Using this result, we prove in Section 5.3 that the motivic multiple zeta values of the form $\zeta^{\mathfrak{m}}(2, \ldots, 2, 3, 2, \ldots, 2)$ are rational linear combinations of products of $\zeta^{\mathfrak{m}}(2n+1)$ and $\zeta^{\mathfrak{m}}(2, \ldots, 2)$. The computation of the actual coefficients of the linear combination is due to Zagier. Interestingly, the existence of such a relation among motivic zeta values is predicted by the infinitesimal coaction, but its precise shape can only be obtained working with numbers and is lifted to a motivic relation afterwards. In Section 5.4, we start analyzing the subspace $\mathcal{H}^{2,3} \subset \mathcal{H}$ spanned by motivic multiple zeta values with exponents $s_i \in \{2, 3\}$. A basic tool is the so-called level filtration by the number of 3s and a level lowering operator defined in terms of the infinitesimal coaction. This will be used in Section 5.5 to prove by induction that the generators of $\mathcal{H}^{2,3}$ are linearly independent and that $\mathcal{H}^{2,3}$ is equal to the whole $\mathcal{H}$. Theorem B follows at once. At the end of the chapter, we state some remarkable consequences of both theorems, including the facts that the category of mixed Tate motives over $\mathbb{Z}$ is tensor generated by the motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, all periods of mixed Tate motives over $\mathbb{Z}$ are $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of multiple zeta values, and Zagier's conjecture implies the algebraic independence of π, $\zeta(3)$, $\zeta(5)$, ...

5.1. The upper bound. In this section, we prove Theorem A. That is, we establish the upper bound d_k for the dimension of the $\mathbb{Q}$-vector space $\mathcal{Z}_k$ generated by multiple zeta values of weight k. Recall that the sequence $(d_k)_{k\geqslant 0}$ is given by

$$\sum_{k\geqslant 0} d_k t^k = \frac{1}{1-t^2-t^3}.$$

5.1.1. *Setting.* Recall from Section 4.4 that the category $\mathbf{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$ has as simple objects the Tate motives $\mathbb{Q}(n)$, one for each $n \in \mathbb{Z}$. They satisfy the tensor relation

$$\mathbb{Q}(n+m) = \mathbb{Q}(n) \otimes \mathbb{Q}(m).$$

The structure of the category is determined by the extension groups

$$\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) \simeq \begin{cases} \mathbb{Q}, & \text{if } n \geqslant 3 \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases} \tag{5.1}$$

and the vanishing of all higher extension groups:

$$\mathrm{Ext}^i_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0 \quad \text{for} \quad i \geqslant 2.$$

The fiber functor

$$\begin{aligned} \omega\colon\ \mathbf{MT}(\mathbb{Z}) &\longrightarrow \mathbf{Vec}_{\mathbb{Q}} \\ M \quad &\longmapsto \omega(M) = \textstyle\bigoplus_n \omega_n(M) \end{aligned}$$

from (4.147) makes $\mathbf{MT}(\mathbb{Z})$ into a tannakian category. Namely, $\mathbf{MT}(\mathbb{Z})$ is equivalent to the category of representations of the pro-algebraic $\mathbb{Q}$-group

$$G_{\mathrm{dR}} = \underline{\mathrm{Aut}}^{\otimes}(\omega).$$

We already determined the structure of G_{dR} using the computation of the extension groups (Lemma 4.159). It is a semidirect product

$$G_{\mathrm{dR}} \simeq U_{\mathrm{dR}} \rtimes \mathbb{G}_m, \tag{5.2}$$

where U_{dR} is a pro-unipotent affine group scheme over $\mathbb{Q}$. The action of $\mathbb{G}_m$ on U_{dR} induces an action of $\mathbb{G}_m$ on the Lie algebra

$$\mathfrak{u}_{\mathrm{dR}} = \mathrm{Lie}(U_{\mathrm{dR}}),$$

and the associated graded Lie algebra $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}} \subset \mathfrak{u}_{\mathrm{dR}}$ is non-canonically isomorphic to the free Lie algebra with one generator in each odd degree $\leqslant -3$ (Theorem 4.180). The whole Lie algebra $\mathfrak{u}_{\mathrm{dR}}$ is the completion of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$.

Besides, in Section 3.10 we introduced the algebraic group $\mathrm{Aut}(D^{\mathrm{dR}})$ of symmetries of the de Rham fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and its pro-unipotent part $\mathrm{Aut}^0(D^{\mathrm{dR}})$. Among other things, this fundamental groupoid consists of the pro-algebraic scheme ${}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}} = \mathrm{Spec}(\mathfrak{H})$, where $\mathfrak{H} = (\mathbb{Q}\langle x_0, x_1\rangle, ⧢)$ is the Hoffman algebra. It is thought of as the de Rham counterpart of the space of paths from 0 to 1. We showed in Lemma 3.415 that there is an isomorphism of $\mathbb{Q}$-schemes

$$\mathrm{Aut}^0(D^{\mathrm{dR}}) \simeq {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}.$$

This led us to define (Definition 3.418) a pro-algebraic group $(\Pi, \circ)$ with underlying scheme ${}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}$ and multiplication induced by the composition $\circ$ in $\mathrm{Aut}(D^{\mathrm{dR}})$. Recall from Remark 3.410 the canonical de Rham path ${}_{\mathbf{1}}1_{\mathbf{0}}^{\mathrm{dR}} \in {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}(\mathbb{Q})$, corresponding to the counit map $\mathfrak{H} \to \mathbb{Q}$. The group $(\Pi, \circ)$ acts on ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$, and the map

$$\begin{aligned} \Pi &\longrightarrow {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}} \\ v &\longmapsto v({}_{\mathbf{1}}1_{\mathbf{0}}^{\mathrm{dR}}) \end{aligned}$$

is an isomorphism of schemes. Thus, ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ is a trivial torsor under $(\Pi, \circ)$.

As explained at the end of last chapter, it follows from Theorem 4.213 that there exists a commutative diagram of morphisms of affine group schemes

$$\begin{tikzcd} 0 \arrow[r] & U_{\mathrm{dR}} \arrow[r] \arrow[d] & G_{\mathrm{dR}} \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Pi \arrow[r] & \mathrm{Aut}(D^{\mathrm{dR}}) \arrow[r] & \mathbb{G}_m \arrow[r] & 0. \end{tikzcd} \tag{5.3}$$

We shall denote the first vertical arrow by

$$I \colon U_{\mathrm{dR}} \longrightarrow \Pi. \tag{5.4}$$

In particular, the group G_{dR} acts on the pro-scheme ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$. The action of the subgroup U_{dR} factors through the map I and the action of Π on ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ discussed above. Recall that the decomposition (5.2) is induced by a canonical section

$$\tau \colon \mathbb{G}_m \to G_{\mathrm{dR}}.$$

For each $z \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$, the element $\tau(z)$ acts on the graded piece

$$\omega_n({}_{\mathbf{1}}\Pi_{\mathbf{0}}) = \mathrm{Gr}_{2n}^W \, {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$$

as multiplication by z^n.

The image $I(U_{\mathrm{dR}})$ of U_{dR} is a closed subgroup of Π. We introduce the notation

$$\mathcal{A}^{\mathcal{MT}} = \mathcal{O}(U_{\mathrm{dR}}) \quad \text{and} \quad \mathcal{A} = \mathcal{O}(I(U_{\mathrm{dR}})). \tag{5.5}$$

Note that there is an injective morphism of Hopf algebras $\mathcal{A} \hookrightarrow \mathcal{A}^{\mathcal{MT}}$.

In (4.183), we introduced the algebra

$$\mathcal{H}^{\mathcal{MT}} = \mathcal{A}^{\mathcal{MT}} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2].$$

It is a Hopf module over $\mathcal{A}^{\mathcal{MT}}$, with f_2 in degree two, and by Lemma 4.184 its Hilbert–Poincaré series is given by

$$H_{\mathcal{H}^{\mathcal{MT}}}(t) = \sum_{k \geqslant 0} d_k t^k.$$

5.1.2. *The algebra of motivic multiple zeta values.* Recall that the affine ring of ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ is equal to the Hoffman algebra (Example 3.64)

$$\mathcal{O}({}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}) = \mathbb{Q}\langle x_0, x_1 \rangle = \mathfrak{H},$$

with the grading induced by $\deg(x_0) = \deg(x_1) = 1$ (see Proposition 3.411). The dual of $\mathfrak{H}$ is the completed Hopf algebra $\mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$ from Example 3.74, with the grading determined by $\deg(e_0) = \deg(e_1) = -1$. For any $\mathbb{Q}$-algebra R, the set of R-points ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}(R)$ is identified with the set of group-like elements of $R\langle\langle e_0, e_1 \rangle\rangle$.

From now on, we will let $\mathbf{dch}^{\mathrm{dR}}$ denote the image by

$$\mathrm{comp}_{\mathrm{dR,B}} \colon {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{B}}(\mathbb{C}) \longrightarrow {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}(\mathbb{C})$$

of the straight path $\mathbf{dch} \in {}_{\mathbf{1}}\Pi^{\mathrm{B}}_{\mathbf{0}}(\mathbb{Q})$ from $\mathbf{0}$ to $\mathbf{1}$. This is nothing other than what was previously denoted by

$$L(\mathbf{dch}) = \sum_{\alpha} \zeta_{\sqcup\!\sqcup}(x_\alpha) e_\alpha$$

in (3.393). Since all regularized multiple zeta values $\zeta_{\sqcup\!\sqcup}(x_\alpha)$ are real numbers, the point $\mathbf{dch}^{\mathrm{dR}}$ actually belongs to ${}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(\mathbb{R}) \subset {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(\mathbb{C})$.

Evaluating an element $f \in \mathcal{O}({}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}})$ at the point $\mathbf{dch}^{\mathrm{dR}}$ yields the map

$$\begin{aligned} \mathrm{dch}\colon\ \mathcal{O}({}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}) &\longrightarrow \mathbb{R} \\ f &\longmapsto f(\mathbf{dch}^{\mathrm{dR}}). \end{aligned} \tag{5.6}$$

Seen as a function on ${}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}$, a word w in the alphabet $\{x_0, x_1\}$ takes the value

$$w(\mathbf{dch}^{\mathrm{dR}}) = \zeta_{\sqcup\!\sqcup}(w)$$

at the point $\mathbf{dch}^{\mathrm{dR}} \in {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(\mathbb{R})$. Thus, by Corollary 1.186, we obtain a surjective map from $\mathcal{O}({}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}})$ to the algebra $\mathcal{Z}$ of multiple zeta values.

Remark 5.7. The map $\mathcal{O}({}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}) \to \mathcal{Z}$ is very far from being injective, as all relations between multiple zeta values belong to its kernel. As a result, the algebra $\mathbb{Q}\langle x_0, x_1\rangle$, which has the advantage of being elementary, is too big for the purpose of proving Theorem A. The algebra $\mathcal{O}(G_{\mathrm{dR}})$ looks more promising but it is still too big. In fact, $\mathcal{O}(\mathbb{G}_m)$ is equal to $\mathbb{Q}[x, x^{-1}]$, with x sitting in degree 1. Using the semi-direct product decomposition (5.2), we derive an isomorphism

$$\mathcal{O}(G_{\mathrm{dR}}) \simeq \mathcal{A}^{\mathcal{MT}} \otimes_{\mathbb{Q}} \mathbb{Q}[x, x^{-1}].$$

The presence of x^{-1}, which has degree -1, implies that each graded piece of $\mathcal{O}(G_{\mathrm{dR}})$ is infinite-dimensional, so this algebra is still not useful for our purposes. Thinking of x as $2\pi i$ and of f_2 as $\zeta(2)$ suggests identifying f_2 with $-x^2/24$, and this will yield an injective map $\mathcal{H}^{\mathcal{MT}} \to \mathcal{O}(G_{\mathrm{dR}})$. The strategy to prove Theorem A consists then in showing that the evaluation map (5.6) factors through $\mathcal{H}^{\mathcal{MT}}$. This can be done either in an ad hoc way, or using a nice geometric interpretation due to Brown.

Following [**Bro12**, § 2.3], we define a closed subscheme $\mathcal{Y} \subseteq {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}$ as the Zariski closure of the orbit of $\mathbf{dch}^{\mathrm{dR}}$ under the action of G_{dR}:

$$\mathcal{Y} = \overline{G_{\mathrm{dR}} \cdot \mathbf{dch}^{\mathrm{dR}}}. \tag{5.8}$$

Lemma 5.9. *The subscheme $\mathcal{Y}$ is defined over $\mathbb{Q}$.*

Proof. To see that $\mathcal{Y}$ is defined over $\mathbb{Q}$ we give another interpretation of it. Recall that $P_{\mathrm{dR,B}}$ is the G_{dR}-torsor of isomorphisms between the fiber functors ω_{B} and ω_{dR}. Thus, there is a morphism of affine $\mathbb{Q}$-schemes

$$P_{\mathrm{dR,B}} \times {}_{\mathbf{1}}\Pi^{\mathrm{B}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}.$$

The point $\mathbf{dch} \in {}_{\mathbf{1}}\Pi^{\mathrm{B}}_{\mathbf{0}}(\mathbb{Q})$ induces a G_{dR}-equivariant map

$$\mathrm{dch}\colon P_{\mathrm{dR,B}} \to {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}},$$

which sends $\mathrm{comp}_{\mathrm{dR,B}}$ to $\mathbf{dch}^{\mathrm{dR}}$. Hence, its image is the G_{dR}-orbit of $\mathbf{dch}^{\mathrm{dR}}$. It follows that $\mathcal{Y}$ is the Zariski closure of the image of the map dch. The point $\mathbf{dch}$ being rational, we deduce that $\mathcal{Y}$ is defined over $\mathbb{Q}$. □

DEFINITION 5.10. We call *algebra of motivic zeta values* the $\mathbb{Q}$-algebra

$$\mathcal{H} = \mathcal{O}(\mathcal{Y})$$

of regular functions on the scheme $\mathcal{Y}$.

In Definition 5.33 below, we will write down a set of generators of $\mathcal{H}$ called *motivic multiple zeta values*. For the time being, we note the properties of $\mathcal{H}$ that are relevant for the proof of Theorem A.

- The action of U_{dR} on ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ induces an action of U_{dR} on $\mathcal{Y}$ that factors through $I(U_{\mathrm{dR}})$. Therefore, there is a coaction

$$\Delta\colon \mathcal{H} \longrightarrow \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H} \tag{5.11}$$

  making $\mathcal{H}$ into a Hopf comodule over $\mathcal{A}$.
- The action of $\mathbb{G}_m$ on $\mathcal{Y}$ induces a grading

$$\mathcal{H} = \bigoplus_{k \geqslant 0} \mathcal{H}_k.$$

- Since $\mathcal{Y}$ contains $\mathbf{dch}^{\mathrm{dR}}$, the map (5.6) factors through $\mathcal{H}$ and gives rise to a so-called *period map*

$$\mathrm{per}\colon \mathcal{H} \longrightarrow \mathbb{R}. \tag{5.12}$$

  Since the map (5.6) surjects onto $\mathcal{Z}$, the image of per is also $\mathcal{Z}$.
- Moreover, since the action of $\mathbb{G}_m$ on $\mathcal{Y}$ is compatible with its action on ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$, and since the grading that this action induces on $\mathcal{O}({}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}})$ agrees with the natural grading of $\mathbb{Q}\langle x_0, x_1\rangle$, we deduce the equality

$$\mathrm{per}(\mathcal{H}_k) = \mathcal{Z}_k.$$

REMARK 5.13. The map (5.12) is called the period map because it is compatible with the period map from Definition 4.167. In fact, taking the equality

$$\mathrm{dch}(\mathrm{comp}_{\mathrm{dR,B}}) = \mathbf{dch}^{\mathrm{dR}}$$

into account, there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}(\mathcal{Y}) \arrow[r, "\mathrm{dch}^*"] \arrow[dr, "\mathrm{per}"'] & \mathcal{O}(P_{\mathrm{dR,B}}). \arrow[d, "\mathrm{per}"] \\ & \mathbb{C} \end{tikzcd} \tag{5.14}$$

REMARK 5.15. We can interpret $\mathcal{H}$ as follows. Let $\mathcal{I} \subset \mathbb{Q}\langle x_0, x_1\rangle$ be the ideal of functions vanishing on $\mathbf{dch}^{\mathrm{dR}}$, *i.e.* the ideal of all rational relations among multiple zeta values. The ideal $\mathcal{J}^{\mathcal{MT}}$ defining $\mathcal{Y}$ inside ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ is the ideal of motivic relations between multiple zeta values, that is, those explained by geometry. We will see that $\mathcal{J}^{\mathcal{MT}} \subseteq \mathcal{I}$ implies the upper bound for the dimension of the space of multiple zeta values, while Zagier's conjecture is equivalent to the equality $\mathcal{J}^{\mathcal{MT}} = \mathcal{I}$, that is, that every rational relation among multiple zeta values comes from geometry. In this vein, Zagier's conjecture is equivalent to saying that $\mathcal{Y}$ is the smallest subvariety of ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ that is defined over $\mathbb{Q}$ and contains $\mathbf{dch}^{\mathrm{dR}}$.

The strategy to prove Theorem A is now to derive the inequality $\dim \mathcal{H}_k \leqslant d_k$ from the existence of an injection $\mathcal{H} \hookrightarrow \mathcal{H}^{\mathcal{MT}}$. This injection will come from the study of the geometry of $\mathcal{Y}$. Theorem B, to be proved later, will actually imply the equality $\dim \mathcal{H}_k = d_k$, and hence that the algebras $\mathcal{H}$ and $\mathcal{H}^{\mathcal{MT}}$ are isomorphic.

5.1.3. *The structure of $\mathcal{Y}$.*

Lemma 5.16. *There exists an element $\gamma \in \Pi(\mathbb{Q})$ such that the equality*

$$\mathbf{dch}^{\mathrm{dR}} = (I(u_0) \circ \tau(2\pi i)^{-1}(\gamma))({}_{\mathbf{1}}1_{\mathbf{0}}^{\mathrm{dR}}) \tag{5.17}$$

holds. Moreover, for any γ satisfying (5.17), the equality $\tau(-1)(\gamma) = \gamma$ holds. In other words, γ only contains monomials of even degree.

Proof. Recall from Proposition 4.163 that there exists an element

$$a = u_0 \cdot \tau(2\pi i)^{-1} \in U_{\mathrm{dR}}(\mathbb{R})\tau(2\pi i)^{-1} \subset G_{\mathrm{dR}}(\mathbb{C}) \tag{5.18}$$

such that the equality $\mathrm{comp}_{\mathrm{dR,B}}(\omega_{\mathrm{B}}(M)) = a(\omega_{\mathrm{dR}}(M))$ of subspaces of $\omega_{\mathrm{dR}}(M) \otimes \mathbb{C}$ holds for all mixed Tate motives M in $\mathbf{MT}(\mathbb{Z})$. In particular, there exists an element $\gamma' \in {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}(\mathbb{Q})$ satisfying

$$\mathbf{dch}^{\mathrm{dR}} = \mathrm{comp}_{\mathrm{dR,B}}(\mathbf{dch}) = a(\gamma').$$

Let $\gamma \in \Pi(\mathbb{Q})$ be such that $\gamma' = \gamma({}_{\mathbf{1}}1_{\mathbf{0}}^{\mathrm{dR}})$. Then,

$$\begin{aligned}
\mathbf{dch}^{\mathrm{dR}} &= (u_0 \cdot \tau(2\pi i)^{-1})(\gamma') \\
&= I(u_0)(\tau(2\pi i)^{-1}(\gamma({}_{\mathbf{1}}1_{\mathbf{0}}^{\mathrm{dR}}))) \\
&= (I(u_0) \circ \tau(2\pi i)^{-1} \circ \gamma)({}_{\mathbf{1}}1_{\mathbf{0}}^{\mathrm{dR}}) \\
&= (I(u_0) \circ \tau(2\pi i)^{-1}(\gamma))({}_{\mathbf{1}}1_{\mathbf{0}}^{\mathrm{dR}}).
\end{aligned}$$

To get the last equality, we use the identity

$$\tau(2\pi i)^{-1} \circ \gamma = \tau(2\pi i)^{-1}(\gamma) \circ \tau(2\pi i)^{-1},$$

where the action on the right-hand side is the one induced by the grading, along with the fact that, being the unit of the graded Hopf algebra $\mathfrak{H}$, the element ${}_{\mathbf{1}}1_{\mathbf{0}}^{\mathrm{dR}}$ has degree zero. This finishes the proof of (5.17).

We now turn to the second statement. Since both $\mathbf{dch}$ and u_0 are real, it follows from (5.17) that $\tau(2\pi i)^{-1}(\gamma)$ is real as well. Writing $\gamma = \sum c_w w$ in $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$, an element $z \in \mathbb{C}^{\times}$ acts through

$$\tau(z)^{-1}(\gamma) = \sum c_w z^{\deg(w)} w.$$

Since $\tau(2\pi i)^{-1}(\gamma)$ is real, it follows that $c_w = 0$ for every word w of odd degree. □

As in the proof of the lemma, we write $\gamma = \sum c_w w$ in $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. Since γ only contains monomials of even degree, the map

$$\begin{aligned}
\mathbb{G}_m &\longrightarrow \Pi \\
t &\longmapsto \tau(t)(\gamma)
\end{aligned}$$

only depends on t^2. Indeed, if one defines a map $\rho\colon \mathbb{G}_m \to \Pi$ as

$$\rho(t) = \sum t^{\frac{\deg(w)}{2}} c_w w, \tag{5.19}$$

then $\tau(t)^{-1}(\gamma) = \rho(t^2)$ holds. Observe that ρ extends to $\mathbb{A}^1$ with $\rho(0) = 1$.

THEOREM 5.20. *The morphism of schemes*

$$\begin{aligned}\psi\colon\ I(U_{\mathrm{dR}})\times\mathbb{A}^1 &\longrightarrow \quad \Pi\\ (u,t)\quad &\longmapsto u\circ\rho(t)\end{aligned}$$

induces an isomorphism $I(U_{\mathrm{dR}})\times\mathbb{A}^1\simeq\mathcal{Y}$ *given by* $(u,t)\mapsto\psi(u,t)({}_{\mathbf{1}}1_{\mathbf{0}})$.

PROOF. Recall that the graded Lie algebra $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is negatively graded and is zero in degree >-3 by Theorem 4.180. Thus, any element $u\in I(U_{\mathrm{dR}})$ can be written as

$$u=1+\sum_{\deg(w)\leqslant -3}u_w w.$$

Therefore, the coefficients of the monomial e_0e_1 in $\rho(t)$ and $u\circ\rho(t)$ agree. Let us compute the former. Recall that

$$\mathbf{dch}^{\mathrm{dR}}=1+\zeta(2)e_0e_1+\text{higher degree}.$$

Taking the equality $\mathbf{dch}^{\mathrm{dR}}=(u\circ\tau(2\pi i)^{-1}(\gamma))({}_{\mathbf{1}}1_{\mathbf{0}})$ from Lemma 5.16 into account, one has $(2\pi i)^2c_{e_0e_1}=\zeta(2)$, which yields the value $c_{e_0e_1}=-1/24$ by Euler's formula for $\zeta(2)$. The coefficient of e_0e_1 in $\rho(t)$ is thus equal to $-t/24$.

This leads naturally to consider the maps

$$\begin{aligned}c\colon\ \Pi &\longrightarrow \qquad\qquad \mathbb{A}^1\\ x &\longmapsto -24\cdot\text{coefficient of } e_0e_1 \text{ in } x,\\[1ex] \varphi\colon\ \Pi &\longrightarrow \qquad \Pi\\ x &\longmapsto x\circ\rho(c(x))^{-1}.\end{aligned}$$

By the previous discussion, we have $c(\psi(u,t))=t$, whence

$$\varphi(\psi(u,t))=\psi(u,t)\circ\rho(c(\psi(u,t)))^{-1}=u\circ\rho(t)\circ\rho(t)^{-1}=u.$$

In particular, the morphism ψ is injective.

Observe that $x\in\Pi$ is in the image of ψ if and only if $\varphi(x)$ belongs to $I(U_{\mathrm{dR}})$. Therefore, $\operatorname{Im}\psi=\varphi^{-1}(I(U_{\mathrm{dR}}))$ holds. Since $I(U_{\mathrm{dR}})$ is closed in Π, this equality implies that $\operatorname{Im}\psi$ is closed in Π. By Lemma 5.16, $(\operatorname{Im}\psi)({}_{\mathbf{1}}1_{\mathbf{0}})$ contains $G_{\mathrm{dR}}\cdot\mathbf{dch}^{\mathrm{dR}}$ as an open dense subset, so it has to be equal to its closure $\mathcal{Y}$. Write $\mathcal{Y}'$ for the preimage of $\mathcal{Y}$ in Π. To conclude, we note that the map

$$\begin{aligned}\mathcal{Y}' &\longrightarrow I(U_{\mathrm{dR}})\times\mathbb{A}^1\\ x &\longmapsto\ (\varphi(x),c(x))\end{aligned}$$

is an inverse of ψ. This concludes the proof. □

COROLLARY 5.21. *The isomorphism* $I(U_{\mathrm{dR}})\times\mathbb{A}^1\simeq\mathcal{Y}$ *from Theorem 5.20 induces an isomorphism of graded algebras*

$$\mathcal{H}\simeq\mathcal{A}\otimes_{\mathbb{Q}}\mathbb{Q}[t],$$

where t *sits in degree two. This isomorphism induces an injection* $\mathcal{H}\hookrightarrow\mathcal{H}^{\mathcal{MT}}$ *that sends* t *to* $-24f_2$*. Moreover, if we provide* $\mathbb{Q}[t]$ *with the trivial* $\mathcal{A}$*-comodule structure, then this is an isomorphism of* $\mathcal{A}$*-comodules.*

PROOF. That it is an isomorphism of algebras follows from the duality between affine schemes and algebras of functions. To see that it is a graded isomorphism, we need to show that the map ψ from Theorem 5.20 is $\mathbb{G}_m$-equivariant provided that one makes $\lambda \in \mathbb{G}_m$ act on $\mathbb{A}^1$ by $t \mapsto \lambda^2 t$. On the one hand, formula (5.19) gives $\rho(\lambda^2 t) = \tau(\lambda)(\rho(t))$. On the other hand, using Proposition 4.159, we get

$$\tau(\lambda)(u \circ \rho(t)) = \tau(\lambda)(u) \circ \tau(\lambda)(\rho(t)),$$

from which the first statement of the theorem follows. If we endow $\mathbb{A}^1$ with the trivial $I(U_{\mathrm{dR}})$ action, then the map ψ becomes $I(U_{\mathrm{dR}})$-equivariant, from which the second statement follows. □

5.1.4. *Proof of Theorem A.* Since the map (5.12) is surjective and respects the weight, it suffices to prove the inequality $\dim \mathcal{H}_k \leqslant d_k$ for each $k \geqslant 2$. But Corollary 5.21 and Lemma 4.184 yield

$$\dim \mathcal{H}_k \leqslant \dim(\mathcal{H}^{\mathcal{MT}})_k = d_k,$$

which is what we wanted to show. □

5.2. Motivic multiple zeta values and the motivic coaction. In this section, we define elements of the algebra $\mathcal{H}$ called *motivic multiple zeta values*. Exploiting the coaction (5.11), we can find many relations among them. Upon application of the period map, they give rise to relations among multiple zeta values.

5.2.1. *The structure of* $\mathcal{A}^{\mathcal{MT}}$. Recall that the Lie algebra $\mathfrak{u}_{\mathrm{dR}} = \mathrm{Lie}(U_{\mathrm{dR}})$ is isomorphic to the completion of the free Lie algebra with one generator sitting in each odd degree $\leqslant -3$. Its dual is hence positively graded. From this, it follows that $\mathcal{A}^{\mathcal{MT}} = \mathcal{O}(U_{\mathrm{dR}})$ is non-canonically isomorphic to the graded Hopf algebra

$$\mathcal{U}' = \mathbb{Q}\langle f_3, f_5, f_7, \dots\rangle \tag{5.22}$$

of non-commutative words in symbols f_{2i+1}, one for each $i \geqslant 1$, in degree $2i+1$, with product given by the shuffle, and coproduct by the deconcatenation

$$\Delta(f_{i_1} f_{i_2} \dots f_{i_r}) = \sum_{k=0}^{r} f_{i_1} \dots f_{i_k} \otimes f_{i_{k+1}} \dots f_{i_r}. \tag{5.23}$$

We introduce the commutative graded algebra

$$\mathcal{U} = \mathcal{U}' \otimes_{\mathbb{Q}} \mathbb{Q}[f_2] = \mathcal{U}'[f_2], \tag{5.24}$$

with f_2 in degree 2. For each integer $N \geqslant 0$, let $\mathcal{U}_N \subset \mathcal{U}$ denote the subspace consisting of elements of degree N of $\mathcal{U}$ (*e.g.* the subspace of homogeneous polynomials of degree N in $f_2, f_3, \dots$). The coproduct (5.23) on $\mathcal{U}'$ extends to a coaction

$$\Delta\colon \mathcal{U} \longrightarrow \mathcal{U}' \otimes_{\mathbb{Q}} \mathcal{U} \tag{5.25}$$

by setting $\Delta f_2 = 1 \otimes f_2$. This coaction turns $\mathcal{U}$ into an $\mathcal{U}'$-comodule. Clearly, $\mathcal{H}^{\mathcal{MT}}$ is non-canonically isomorphic to $\mathcal{U}$.

For later use, it is also convenient to introduce the elements $f_4, f_6, \dots \in \mathcal{U}$ defined, for each integer $n \geqslant 2$, by the formula

$$f_{2n} = b_n f_2^n \quad \text{with} \quad b_n = (-1)^{n-1} \frac{24^n}{2(2n)!} B_{2n}. \tag{5.26}$$

By Euler's Theorem 1.3, the equality $\zeta(2n) = b_n \zeta(2)^n$ holds.

The Hopf algebra $\mathcal{U}'$ and the comodule $\mathcal{U}$ are useful for explicit computations. We will later fix a suitable isomorphism

$$\phi\colon \mathcal{H}^{\mathcal{MT}} \longrightarrow \mathcal{U} \tag{5.27}$$

satisfying certain normalization requirements.

For compatibility with the theory of multiple zeta values, the grading of the algebras $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{H}$, and so on, will be called the *weight*.

We first present the computational tools we will use at the level of $\mathcal{U}'$. As in Definition 3.88, the Lie coalgebra associated with $\mathcal{U}'$ is the quotient

$$L = \mathcal{U}'_{>0}/(\mathcal{U}'_{>0})^2.$$

From the canonical decomposition $\mathcal{U}' = \mathbb{Q} \oplus \mathcal{U}'_{>0}$, we get a projection $q\colon \mathcal{U}' \to L$. The Lie coalgebra L inherits a grading from $\mathcal{U}'$. Let $L_N \subset L$ be the subspace of weight N and $p_N\colon L \to L_N$ the projection. For $r \geqslant 1$, we define the map

$$D_{2r+1}\colon\ \mathcal{U} \longrightarrow L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U} \tag{5.28}$$

as the composition

$$\mathcal{U} \xrightarrow{\Delta} \mathcal{U}' \otimes_{\mathbb{Q}} \mathcal{U} \xrightarrow{q\otimes \mathrm{Id}} L \otimes_{\mathbb{Q}} \mathcal{U} \xrightarrow{p_{2r+1}\otimes \mathrm{Id}} L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U},$$

where Δ is the coaction (5.25). We will see in Exercise 5.51 that the maps D_{2r+1} are derivations. We put

$$D_{<N}\colon \mathcal{U} \longrightarrow \bigoplus_{3\leqslant 2r+1<N} L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U}, \quad D_{<N} = \bigoplus_{3\leqslant 2r+1<N} D_{2r+1} \tag{5.29}$$

Lemma 5.30. *For each integer $N \geqslant 2$, the following equality holds:*

$$(\operatorname{Ker} D_{<N}) \cap \mathcal{U}_N = \mathbb{Q} f_N.$$

Proof. We first show that f_N belongs to $\operatorname{Ker} D_{<N}$. When N is even, we already have $\Delta f_N - 1 \otimes f_N = 0$. If N is odd and $2r+1 < N$, then

$$D_{2r+1} f_N = p_{2r+1}(q(f_N)) \otimes 1 = 0.$$

Thus, $f_N \in \operatorname{Ker} D_{<N}$. Conversely, an element $\xi \in \mathcal{U}_N$ can be uniquely written as

$$\xi = \alpha f_N + \sum_{\substack{3\leqslant 2r+1<N\\ 0\leqslant 2j\leqslant N-2r-1}} f_{2r+1} v_{r,j} \cdot f_2^j$$

with $v_{r,j} \in \mathcal{U}'_{N-2r-1-2j}$ and $\alpha \in \mathbb{Q}$. Writing $v_r = \sum_j v_{r,j} \cdot f_2^j$ and using the explicit expression of the coaction (5.25), we get

$$D_{2r+1}\xi = f_{2r+1} \otimes v_r + \text{other terms},$$

where none of the monomials of $\mathcal{U}'_{2r+1}$ that appear in the extra terms is f_{2r+1}. Hence, $D_{2r+1}\xi = 0$ implies $v_r = 0$. All in all, if $\xi \in \operatorname{Ker} D_{<N}$, then $\xi = \alpha f_N$. $\square$

5.2.2. *Motivic multiple zeta values.* Let α be a binary sequence. Recall from formula (3.427) at the end of Chapter 3 the function on Π denoted by

$$I(1;\alpha;0) = x_\alpha.$$

We now let $I^{\mathfrak{m}}(1;\alpha;0)$ denote the restriction of this function to $\mathcal{Y}$, that is, the projection to the quotient

$$I^{\mathfrak{m}}(1;\alpha;0) \in \mathcal{H} = \mathcal{O}(\Pi)/\mathcal{J}^{MT}.$$

Following (3.427), we set for later use

$$I^{\mathfrak{m}}(0;\alpha;1) = x^*_\alpha|_{\mathcal{Y}},$$

where $x_\alpha^* = S^\vee(x_\alpha)$ and

$$I^{\mathfrak{m}}(0;\alpha;0) = I^{\mathfrak{m}}(1;\alpha;1) = \begin{cases} 1, & \text{if } \alpha = \emptyset, \\ 0, & \text{if } \alpha \neq \emptyset. \end{cases} \tag{5.31}$$

The symbols $I^{\mathfrak{m}}$ are called *motivic iterated integrals.*

We now list some useful properties of motivic iterated integrals.

Lemma 5.32.

i) *If* $N \geqslant 1$ *and* $\varepsilon_1 = \cdots = \varepsilon_N$, *then* $I^{\mathfrak{m}}(\varepsilon_0;\varepsilon_1\cdots\varepsilon_N;\varepsilon_{N+1}) = 0$.

ii) *(*Reflection formula*)*

$$\begin{aligned} I^{\mathfrak{m}}(1;\varepsilon_1\cdots\varepsilon_N;0) &= (-1)^N I^{\mathfrak{m}}(0;\varepsilon_N\cdots\varepsilon_1;1) \\ &= I^{\mathfrak{m}}(1;1-\varepsilon_N\cdots 1-\varepsilon_1;0). \end{aligned}$$

Proof. The reflection formula follows from Theorem 3.367 i) and the change of variables $z \mapsto 1-z$ on $\mathbb{P}^1 \setminus \{0,1,\infty\}$, which reverses the path $\mathbf{dch}$.

We prove property i). Since $I^{\mathfrak{m}}(0;\alpha;0) = I^{\mathfrak{m}}(1;\alpha;1) = 0$ holds for a non-empty binary sequence α and $I^{\mathfrak{m}}(0;\varepsilon^{\{N\}};1) = (-1)^N I^{\mathfrak{m}}(1;\varepsilon^{\{N\}};0)$ holds by ii), it suffices to show the vanishing $I^{\mathfrak{m}}(1;\varepsilon^{\{N\}};0) = 0$. For this, we use the identity

$$I^{\mathfrak{m}}(1;\varepsilon^{\{N\}};0) = \frac{1}{N!} I^{\mathfrak{m}}(1;\varepsilon;0)^N$$

and the fact that $I^{\mathfrak{m}}(1;\varepsilon;0) = 0$ since $\mathcal{H}$ has no elements of degree one. □

Recall the binary sequence $\mathrm{bs}(\boldsymbol{s}) = (0^{\{s_1-1\}}, 1, \cdots, 0^{\{s_r-1\}}, 1)$ associated to a positive multi-index $\boldsymbol{s} = (s_1, \ldots, s_r)$ in Definition 1.132.

Definition 5.33. Let $\boldsymbol{s} = (s_1, \ldots, s_r)$ be a positive multi-index. The *motivic multiple zeta value* $\zeta^{\mathfrak{m}}(\boldsymbol{s})$ is the element of $\mathcal{H}$ defined as

$$\zeta^{\mathfrak{m}}(\boldsymbol{s}) = I^{\mathfrak{m}}(1;0^{\{s_1-1\}}1\cdots 0^{\{s_r-1\}}1;0).$$

The period map $\mathrm{per}\colon \mathcal{H} \longrightarrow \mathbb{R}$ from (5.12) satisfies

$$\mathrm{per}(\zeta^{\mathfrak{m}}(\boldsymbol{s})) = \zeta_{\sqcup\!\sqcup}(\boldsymbol{s}).$$

In other words, the motivic multiple zeta value $\zeta^{\mathfrak{m}}(\boldsymbol{s})$ is a function on the variety $\mathcal{Y}$, and the regularized multiple zeta value $\zeta_{\sqcup\!\sqcup}(\boldsymbol{s})$ is the result of evaluating this function at the point $\mathbf{dch}^{\mathrm{dR}} \in \mathcal{Y}$.

Remark 5.34 (Comparison with Brown's notation). Due to the different convention on the definition of multiple zeta values and iterated integrals, there is a discrepancy between the symbols used here and the symbols used in [**Bro12**]. To ease comparison, we summarize it in this remark. Letting

$$\zeta_B^{\mathfrak{m}}(s_1,\ldots,s_r), \quad I_B^{\mathfrak{m}}(\varepsilon_0;\varepsilon_1,\ldots,\varepsilon_n;\varepsilon_{n+1}), \text{ and } \zeta_B(s_1,\ldots,s_r)$$

denote the motivic multiple zeta values, motivic iterated integrals and multiple zeta values used in [**Bro12**], the following relations hold:

$$\begin{aligned} \zeta_B^{\mathfrak{m}}(s_1,\ldots,s_r) &= \zeta^{\mathfrak{m}}(s_r,\ldots,s_1), \\ I_B^{\mathfrak{m}}(\varepsilon_0;\varepsilon_1,\ldots,\varepsilon_n;\varepsilon_{n+1}) &= I^{\mathfrak{m}}(\varepsilon_{n+1};\varepsilon_n,\ldots,\varepsilon_1;\varepsilon_0), \\ \zeta_B(s_1,\ldots,s_r) &= \zeta(s_r,\ldots,s_1). \end{aligned}$$

The map per is the same in [**Bro12**] and in this book because it is the evaluation morphism at a point. The relation between motivic multiple zeta values and motivic iterated integrals in [**Bro12**] is given by

$$\zeta_B^{\mathfrak{m}}(s_1,\dots,s_r) = I_B^{\mathfrak{m}}(0;10^{\{s_1-1\}}\dots 10^{\{s_r-1\}};1),$$

while here is given by

$$\zeta^{\mathfrak{m}}(s_1,\dots,s_r) = I^{\mathfrak{m}}(1;0^{\{s_1-1\}}1\dots 0^{\{s_r-1\}}1;0).$$

Both equations are compatible via the change of notation.

If $\boldsymbol{s}$ is admissible, then the function $\zeta^{\mathfrak{m}}(\boldsymbol{s})$ is non-zero since its value at $\mathbf{dch}^{\mathrm{dR}}$ is the non-zero real number $\zeta(\boldsymbol{s})$. In particular, $\zeta^{\mathfrak{m}}(2)\neq 0$. In fact, $\zeta^{\mathfrak{m}}(2)$ is the function on $\mathcal{Y}$ that sends an element g of $\mathcal{Y}(\mathbb{Q})\subset \mathbb{Q}\langle\langle e_0,e_1\rangle\rangle$ to the coefficient of e_0e_1 in g. It follows that $\zeta^{\mathfrak{m}}(2)$ is sent to $-t/24$ under the isomorphism $\mathcal{H}\to\mathcal{A}\otimes\mathbb{Q}[t]$ of Corollary 5.21, and hence to the element f_2 under the injection $\mathcal{H}\to\mathcal{H}^{\mathcal{MT}}$.

REMARK 5.35. The fact that $\zeta^{\mathfrak{m}}(2)$ is not zero is an important difference between Brown's and Goncharov's approaches to motivic multiple zeta values. Recall the inclusion $U_{\mathrm{dR}}\subset G_{\mathrm{dR}}$ and the elements $1_{\mathrm{dR}}\in\Pi(\mathbb{Q})$ and $\mathbf{dch}^{\mathrm{dR}}\in\Pi(\mathbb{C})$. Goncharov works with the orbit of 1_{dR} under U_{dR}:

$$\mathcal{X} = U_{\mathrm{dR}}\cdot 1_{\mathrm{dR}}\subset \Pi.$$

As a variety, $\mathcal{X}$ is isomorphic to $I(U_{\mathrm{dR}})$. Hence, its ring of functions $\mathcal{O}(\mathcal{X})$ is isomorphic to $\mathcal{A}$. However, Brown works with the variety $\mathcal{Y}$ defined as the closure of the orbit of $\mathbf{dch}^{\mathrm{dR}}$ under G_{dR}

$$\mathcal{Y} = \overline{G_{\mathrm{dR}}\cdot\mathbf{dch}^{\mathrm{dR}}}\simeq I(U_{\mathrm{dR}})\times\mathbb{A}^1.$$

Since the leading term of $\mathbf{dch}^{\mathrm{dR}}$ is 1_{dR}, we deduce the equality

$$\lim_{t\to 0}\tau(t)\mathbf{dch}^{\mathrm{dR}} = 1_{\mathrm{dR}}.$$

This implies that 1_{dR} belongs to $\mathcal{Y}$, hence an inclusion $\mathcal{X}\subset\mathcal{Y}$. Since the action of U_{dR} on the factor $\mathbb{A}^1$ is trivial, we can identify $\mathcal{X}$ with the subscheme $I(U_{\mathrm{dR}})\times\{0\}$ of $I(U_{\mathrm{dR}})\times\mathbb{A}^1$. That is, the inclusion $\mathcal{X}\hookrightarrow\mathcal{Y}$ corresponds to the algebra morphism

$$\pi\colon \mathcal{H}\longrightarrow \mathcal{H}/\zeta^{\mathfrak{m}}(2)\mathcal{H} = \mathcal{A}. \tag{5.36}$$

5.2.3. *The motivic coaction.* We now give an explicit description of the motivic coaction (5.11). Following Remark 5.35, there is an isomorphism $I(U_{\mathrm{dR}})\simeq\mathcal{X}\subset\Pi$ and the action of $I(U_{\mathrm{dR}})$ on $\mathcal{Y}$ fits into the commutative diagram

$$\begin{tikzcd} \mathcal{X}\times\mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \Pi\times\Pi \arrow[r] & \Pi, \end{tikzcd}$$

where the vertical arrows are the inclusions and the lower horizontal arrow is the group law on Π. Passing to functions, we get the commutative diagram

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{H} & & \mathcal{H} \arrow[ll, "\Delta"'] \\ \mathcal{O}(\Pi)\otimes\mathcal{O}(\Pi) \arrow[u] & & \mathcal{O}(\Pi), \arrow[ll, "\Delta^{\Gamma}"'] \arrow[u] \end{tikzcd}$$

where the lower horizontal arrow is the Goncharov coproduct (Proposition 3.429). Therefore, the coaction (5.11) is given by the formula

(5.37) $$\Delta I^{\mathfrak{m}}(\varepsilon_0;\varepsilon_1\cdots\varepsilon_N;\varepsilon_{N+1}) = \sum_{0<i_1<\cdots<i_k<N+1} \pi\left(\prod_{p=0}^{k} I^{\mathfrak{m}}(\varepsilon_{i_p};\varepsilon_{i_p+1}\cdots\varepsilon_{i_{p+1}-1};\varepsilon_{i_{p+1}})\right)\otimes I^{\mathfrak{m}}(\varepsilon_0;\varepsilon_{i_1}\cdots\varepsilon_{i_k};\varepsilon_{N+1}),$$

where $\pi\colon \mathcal{H}\to\mathcal{A}$ denotes the projection (5.36) and we set $i_0 = 0$ and $i_{k+1} = N+1$.

LEMMA 5.38. *For each integer $N \geqslant 2$, the following equality holds:*

$$\Delta\zeta^{\mathfrak{m}}(N) = 1\otimes\zeta^{\mathfrak{m}}(N) + \pi(\zeta^{\mathfrak{m}}(N))\otimes 1.$$

PROOF. The equality $\zeta^{\mathfrak{m}}(N) = I^{\mathfrak{m}}(1;0^{\{N-1\}}1;0)$ holds by Definition 5.33. Using part i) of Lemma 5.32, we see that the only non-vanishing terms in the coaction formula (5.37) correspond to the indices

$$k=0,\ i_0=0,\ i_1=N+1 \quad\text{ and }\quad k=N,\ i_j=j,\ j=0,\dots,N+1.$$

The first choice yields the term $\pi(\zeta^{\mathfrak{m}}(N))\otimes 1$, and the second one $1\otimes\zeta^{\mathfrak{m}}(N)$, thus proving the result. □

Formula (5.37) is rather complicated, so we will use an infinitesimal version of it, which is the analogue of the derivations D_{2r+1} for the algebra of motivic multiple zeta values $\mathcal{H}$. For this, we consider the Lie coalgebra

$$\mathcal{L} = \mathcal{A}_{>0}/(\mathcal{A}_{>0})^2,$$

which inherits a grading from $\mathcal{A}$. For each integer $n\geqslant 1$, let $\mathcal{L}_n\subset\mathcal{L}$ be the subspace of degree n and let $p_n\colon \mathcal{L}\to\mathcal{L}_N$ be the projection. Since $\mathcal{A}$ is graded, the projection from $\mathcal{A}_{>0}$ to $\mathcal{L}$ extends to a map $q\colon\mathcal{A}\to\mathcal{L}$.

DEFINITION 5.39. For each integer $r\geqslant 1$, we define the map

(5.40) $$D_{2r+1}\colon \mathcal{H}\longrightarrow \mathcal{L}_{2r+1}\otimes_{\mathbb{Q}}\mathcal{H}$$

as the composition

$$\mathcal{H}\xrightarrow{\Delta}\mathcal{A}\otimes_{\mathbb{Q}}\mathcal{H}\xrightarrow{q\otimes\mathrm{Id}}\mathcal{L}\otimes_{\mathbb{Q}}\mathcal{H}\xrightarrow{p_{2r+1}\otimes\mathrm{Id}}\mathcal{L}_{2r+1}\otimes_{\mathbb{Q}}\mathcal{H}.$$

As before, we also introduce the notation

(5.41) $$D_{<N} = \bigoplus_{3\leqslant 2r+1<N} D_{2r+1}.$$

For each $n\geqslant 1$, consider the map

$$\varpi_n = p_n\circ q\circ\pi\colon \mathcal{H}\longrightarrow\mathcal{L}_n.$$

Observe that the map ϖ_1 is identically zero. The projection ϖ_n kills $\zeta^{\mathfrak{m}}(2)$, all products, and all motivic multiple zeta values of weight different from n. For example, Lemma 5.38 implies the equality

(5.42) $$D_{2r+1}\zeta^{\mathfrak{m}}(N) = \varpi_{2r+1}(\zeta^{\mathfrak{m}}(N))\otimes 1,$$

where the right-hand side vanishes for $N\neq 2r+1$.

PROPOSITION 5.43. *Let $N \geqslant 2$ be an integer. For each odd integer $n < N$, the action of D_n is given by*

$$D_n I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_N; \varepsilon_{N+1}) = \sum_{p=0}^{N-n} \varpi_n \left(I^{\mathfrak{m}}(\varepsilon_p; \varepsilon_{p+1} \cdots \varepsilon_{p+n}; \varepsilon_{p+n+1}) \right) \otimes I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_p, \varepsilon_{p+n+1}, \dots, \varepsilon_N; \varepsilon_{N+1}). \tag{5.44}$$

PROOF. The projection q kills all decomposable elements of $\mathcal{A}_{>0}$, and the projection p_n kills all elements of degree different from n. Taking into account that

$$I^{\mathfrak{m}}(\varepsilon; \alpha; \varepsilon') \begin{cases} = 1, & \text{if } \alpha = \emptyset, \\ \in \mathcal{A}_{>0}, & \text{if } \alpha \neq \emptyset, \end{cases}$$

it follows that in the sum (5.37) that runs over partitions

$$0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = N + 1$$

only the terms having exactly one gap of length n can be non-zero. This gives the formula of the statement. □

5.2.4. *The kernel of $D_{<N}$.* A crucial ingredient in the proof of Brown's theorem is the explicit knowledge of the kernel of the infinitesimal coaction $D_{<N}$ from (5.41) that is provided by the following theorem:

THEOREM 5.45. *For each integer $N \geqslant 2$, the following equality holds:*

$$(\operatorname{Ker} D_{<N}) \cap \mathcal{H}_N = \mathbb{Q}\zeta^{\mathfrak{m}}(N).$$

This is the analogue of the result we obtained in Lemma 5.30 for the Hopf algebra $\mathcal{U}'$ from (5.22) and its comodule $\mathcal{U}$ from (5.24), and the strategy of the proof will be to reduce to this case by choosing an appropriate isomorphism.

LEMMA 5.46. *There exists an isomorphism of Hopf algebras*

$$\phi \colon \mathcal{A}^{\mathcal{MT}} \longrightarrow \mathcal{U}'$$

that extends to an isomorphism of Hopf comodules $\phi \colon \mathcal{H}^{\mathcal{MT}} \to \mathcal{U}$ sending $\zeta^{\mathfrak{m}}(N)$ to f_N for all $N \geqslant 2$.

PROOF. Recall from Section 5.2.1 that the Hopf algebras $\mathcal{A}^{\mathcal{MT}}$ and $\mathcal{U}'$ are non-canonically isomorphic. Starting from any isomorphism of Hopf algebras

$$\phi_0 \colon \mathcal{A}^{\mathcal{MT}} \to \mathcal{U}',$$

we obtain an isomorphism of Hopf comodules $\phi_0 \colon \mathcal{H}^{\mathcal{MT}} \to \mathcal{U}$ by sending f_2 to f_2. We will still denote by ϕ_0 the restriction of this map to $\mathcal{H} \subset \mathcal{H}^{\mathcal{MT}}$, as well as the map $\mathcal{L} \to L$ induced by the composition $\mathcal{A} \hookrightarrow \mathcal{A}^{\mathcal{MT}} \xrightarrow{\phi_0} \mathcal{U}'$. The diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\phi_0"] \arrow[d, "D_{2r+1}"'] & \mathcal{U} \arrow[d, "D_{2r+1}"] \\ \mathcal{L}_{2r+1} \otimes \mathcal{H} \arrow[r, "\phi_0 \otimes \phi_0"] & L_{2r+1} \otimes \mathcal{U} \end{tikzcd} \tag{5.47}$$

commutes by construction. Moreover, ϕ_0 sends $\zeta^{\mathfrak{m}}(2)$ to f_2 by the discussion before Remark 5.35. It then follows from Lemma 5.38 that $D_{<N}\zeta^{\mathfrak{m}}(N)$ vanishes. By the

commutativity of the diagram (5.47), we derive $D_{<N}\phi_0(\zeta^{\mathfrak{m}}(N)) = 0$. By Lemma 5.30, there exists a non-zero rational number $\alpha_N \in \mathbb{Q}^\times$ satisfying

$$\phi_0(\zeta^{\mathfrak{m}}(N)) = \alpha_N f_N.$$

For $N = 2r$ even, we get

$$\phi_0(\zeta^{\mathfrak{m}}(2r)) = \alpha_{2r} f_{2r} = \alpha_{2r} b_r f_2^r = \phi_0(\alpha_{2r} b_r \zeta^{\mathfrak{m}}(2)^r),$$

where b_r is the rational number from (5.26). From the injectivity of ϕ_0, we deduce the equality $\zeta^{\mathfrak{m}}(2r) = \alpha_{2r} b_r \zeta^{\mathfrak{m}}(2)^r$ and, applying the period map, $\alpha_{2r} = 1$.

Since $\mathcal{U}'$ is the Hopf algebra of non-commutative words in $f_3, f_5, \ldots$, given any family of non-zero rational numbers $(\alpha_{2r+1})_{r\geqslant 1}$, the $\mathbb{Q}$-linear automorphism ψ of $\mathcal{U}'$ determined by

$$f_{i_1} \cdots f_{i_n} \longmapsto (\alpha_{i_i} \cdots \alpha_{i_n})^{-1} f_{i_1} \cdots f_{i_n}$$

is an automorphism of Hopf algebras that sends f_{2r+1} to $\alpha_{2r+1}^{-1} f_{2r+1}$. Therefore, the automorphism $\phi = \psi \circ \phi_0$ sends $\zeta^{\mathfrak{m}}(N)$ to f_N for all $N \geqslant 2$, as wanted. □

Remark 5.48. As a byproduct of the proof, we see that the relation from Euler's theorem $\zeta(2r) = b_r \zeta(2)^r$ lifts to a relation

$$\zeta^{\mathfrak{m}}(2r) = b_r \zeta^{\mathfrak{m}}(2)^r \tag{5.49}$$

between motivic multiple zeta values.

Proof of Theorem 5.45. After choosing a normalized isomorphism ϕ as in Lemma 5.46, the result follows from the combination of Lemma 5.30 and the commutativity of diagram (5.47). □

The theorem has the following useful corollary:

Corollary 5.50. *Let $N \geqslant 2$ be an integer and $a^{\mathfrak{m}}$ an element of $\mathcal{H}_N$. Assume that $D_{<N}(a^{\mathfrak{m}}) = 0$ and $\operatorname{per}(a^{\mathfrak{m}}) = \alpha\zeta(N)$ for some rational number α. Then*

$$a^{\mathfrak{m}} = \alpha \zeta^{\mathfrak{m}}(N).$$

Proof. Since $a^{\mathfrak{m}}$ belongs to $(\operatorname{Ker} D_{<N}) \cap \mathcal{H}_N$, Theorem 5.45 gives the existence of a rational number β satisfying $a^{\mathfrak{m}} = \beta\zeta^{\mathfrak{m}}(N)$. Upon application of the period map, one gets $\beta\zeta(N) = \operatorname{per}(a^{\mathfrak{m}}) = \alpha\zeta(N)$, and hence $\beta = \alpha$. □

The importance of this corollary is that it allows one to lift relations among classical multiple zeta values to their motivic counterparts. This will be exploited in the next sections (for a first application, see Exercise 5.53).

⋆⋆⋆

Exercise 5.51. Show that the maps $D_{2r+1} \colon \mathcal{U} \to L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U}$ from (5.28) are derivations, that is, they satisfy

$$D_{2r+1}(\xi_1\xi_2) = (1 \otimes \xi_1) D_{2r+1}(\xi_2) + (1 \otimes \xi_2) D_{2r+1}(\xi_1)$$

for all $\xi_1, \xi_2 \in \mathcal{U}$. The same holds for the maps $D_{2r+1} \colon \mathcal{H} \to \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} \mathcal{H}$ introduced in Definition 5.39.

Exercise 5.52 (Linear independence of $\zeta^{\mathfrak{m}}(2,3)$ and $\zeta^{\mathfrak{m}}(3,2)$). In this exercise, we prove the linear independence of the motivic multiple zeta values $\zeta^{\mathfrak{m}}(2,3)$ and $\zeta^{\mathfrak{m}}(3,2)$ by exploiting the derivation D_3. Since $\mathcal{H}_5$ has dimension at most $d_5 = 2$ by Theorem A, it will follow that they form a basis. This is the first non-trivial case of Brown's theorem.

i) Prove the equalities $I^{\mathfrak{m}}(1;010;0) = -2\zeta^{\mathfrak{m}}(3)$ and $I^{\mathfrak{m}}(0;100;1) = -\zeta^{\mathfrak{m}}(3)$.

ii) Use the general formula (5.44) for the action of the derivation D_3 and the identities from part i) of the exercise to compute

$$D_3\zeta^{\mathfrak{m}}(2,3) = -2\varpi_3(\zeta^{\mathfrak{m}}(3)) \otimes \zeta^{\mathfrak{m}}(2),$$
$$D_3\zeta^{\mathfrak{m}}(3,2) = 3\varpi_3(\zeta^{\mathfrak{m}}(3)) \otimes \zeta^{\mathfrak{m}}(2).$$

iii) Now assume that the equality $\zeta^{\mathfrak{m}}(2,3) = \lambda\zeta^{\mathfrak{m}}(3,2)$ holds for some rational number λ. By the formulas from part ii), its value is necessarily $\lambda = -2/3$. Get a contradiction using the period map.

Exercise 5.53 (Brown's proof in weight 5). The trick from the previous exercise cannot be generalized to higher weight. Here we present an alternative argument which can be seen as a toy case of Brown's proof.

i) Prove the equality $D_3(\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2)) = \varpi_3(\zeta^{\mathfrak{m}}(3)) \otimes \zeta^{\mathfrak{m}}(2)$. Together with the computations in Exercise 5.52 and Theorem 5.45, this formula implies that there exist rational numbers $\alpha, \beta \in \mathbb{Q}$ satisfying

$$\zeta^{\mathfrak{m}}(2,3) + 2\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) = \alpha\zeta^{\mathfrak{m}}(5),$$
$$\zeta^{\mathfrak{m}}(3,2) - 3\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) = \beta\zeta^{\mathfrak{m}}(5).$$

ii) Combining Corollary 5.50, the stuffle product, and the first identity in the list (1.67), derive the values $\alpha = 9/2$ and $\beta = -11/2$. In particular, the stuffle relation lifts to motivic zeta values:

$$\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) = \zeta^{\mathfrak{m}}(2,3) + \zeta^{\mathfrak{m}}(3,2) + \zeta^{\mathfrak{m}}(5). \tag{5.54}$$

iii) Let $\mathrm{gr}_1^F\mathcal{H}_5^{2,3} \subset \mathcal{H}_5$ be the subspace spanned by $\zeta^{\mathfrak{m}}(2,3)$ and $\zeta^{\mathfrak{m}}(3,2)$ (the reason for this notation will become apparent later). We define a linear map $(f,g)\colon \mathrm{gr}_1^F\mathcal{H}_5^{2,3} \to \mathbb{Q}^2$ by requiring

$$D_3(a) = f(a)\varpi_3(\zeta^{\mathfrak{m}}(3)) \otimes \zeta^{\mathfrak{m}}(2),$$
$$D_5(a) = g(a)\varpi_5(\zeta^{\mathfrak{m}}(5)) \otimes 1$$

for all $a \in \mathrm{gr}_1^F\mathcal{H}_5^{2,3}$. Use parts i) and ii) to show that this map has rank two, and hence $\zeta^{\mathfrak{m}}(2,3)$ and $\zeta^{\mathfrak{m}}(3,2)$ form a basis of $\mathcal{H}_5$.

5.3. A family of motivic multiple zeta values and Zagier's theorem. In this section, we study certain relations involving motivic multiple zeta values with only 2s as entries, or with one entry equal to 3 and the remaining entries equal to 2. The key result is Theorem 5.80. Although the existence of a linear relation of the shape (5.66) is predicted by the motivic coaction, the computation of its actual coefficients relies on a theorem of Zagier about multiple zeta values. Finally, we study the 2-adic properties of the leading coefficient of (5.66).

5.3.1. *Certain relations among motivic multiple zeta values.* From now on, we follow Notation 1.154 to identify the set of words in the alphabet $\{1, 2, \dots\}$ with the set of positive multi-indices. For instance, we make the identification

$$\text{the word } 2^{\{a\}}32^{\{b\}} \longleftrightarrow \text{the multi-index } (\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{b}).$$

Lemma 5.55. *For each $n \geqslant 1$, the following equality holds:*

$$\zeta^{\mathfrak{m}}(2^{\{n\}}) = \frac{6^n}{(2n+1)!}\zeta^{\mathfrak{m}}(2)^n.$$

Proof. Recall that the left-hand side of the equality is defined as

$$\zeta^{\mathfrak{m}}(2^{\{n\}}) = I^{\mathfrak{m}}(1; 01.\overset{n}{\cdots}.01; 0).$$

We first observe the vanishing

$$D_{2r+1}\zeta^{\mathfrak{m}}(2^{\{n\}}) = 0 \tag{5.56}$$

for all $3 \leqslant 2r+1 < 2n$. Indeed, in (5.44) every sequence of the form $\varepsilon_p, \ldots, \varepsilon_{p+2r+2}$ starts and ends with the same value, and hence the corresponding motivic iterated integral is zero by (5.31). It follows that $\zeta^{\mathfrak{m}}(2^{\{n\}})$ belongs to $\operatorname{Ker} D_{<2n}$. By Theorem 5.45 and equation (5.49), we deduce that $\zeta^{\mathfrak{m}}(2^{\{n\}})$ is a rational multiple of $\zeta^{\mathfrak{m}}(2)^n$. For the precise multiple, we use the period map and Example 1.27. $\square$

In order to simplify notation, we write

$$\zeta_1^{\mathfrak{m}}(\boldsymbol{s}) = I^{\mathfrak{m}}(1; 0^{\{s_1-1\}}1 \cdots 0^{\{s_r-1\}}10; 0)$$

for a multi-index $\boldsymbol{s} = (s_1, \ldots, s_r)$.

Lemma 5.57. *For $n \geqslant 1$, the following equalities hold:*

$$\zeta_1^{\mathfrak{m}}(2^{\{n\}}) = -2\sum_{i=0}^{n-1} \zeta^{\mathfrak{m}}(2^{\{i\}}32^{\{n-i-1\}}), \tag{5.58}$$

$$\zeta_1^{\mathfrak{m}}(2^{\{n\}}) = 2\sum_{i=1}^{n} (-1)^i \zeta^{\mathfrak{m}}(2i+1)\zeta^{\mathfrak{m}}(2^{\{n-i\}}). \tag{5.59}$$

Proof. Recall from (5.31) that $I^{\mathfrak{m}}(1;0;1) = 0$. Since the multiplication in $\mathcal{H}$ is given by the shuffle product, we have

$$0 = I^{\mathfrak{m}}(1; 01.\overset{n}{\cdots}.01; 1) I^{\mathfrak{m}}(1;0;1) = \zeta_1^{\mathfrak{m}}(2^{\{n\}}) + 2\sum_{i=0}^{n-1} \zeta^{\mathfrak{m}}(2^{\{i\}}32^{\{n-i-1\}}),$$

from which identity (5.58) follows.

To prove (5.59), we first show the equality of multiple zeta values

$$-\sum_{i=0}^{n-1} \zeta(2^{\{i\}}32^{\{n-i-1\}}) = \sum_{i=1}^{n} (-1)^i \zeta(2i+1)\zeta(2^{\{n-i\}}) \tag{5.60}$$

using the stuffle product. Indeed, the equalities

$$\begin{aligned}
\zeta(3)\zeta(2^{\{n-1\}}) &= \sum_{i=0}^{n-1} \zeta(2^{\{i\}}32^{\{n-1-i\}}) + \sum_{i=0}^{n-2} \zeta(2^{\{i\}}52^{\{n-2-i\}}) \\
\zeta(5)\zeta(2^{\{n-2\}}) &= \sum_{i=0}^{n-2} \zeta(2^{\{i\}}52^{\{n-2-i\}}) + \sum_{i=0}^{n-3} \zeta(2^{\{i\}}72^{\{n-3-i\}}) \\
&\vdots \\
\zeta(2n-1)\zeta(2) &= \zeta(2n-1,2) + \zeta(2,2n-1) + \zeta(2n+1) \\
\zeta(2n+1) &= \zeta(2n+1)
\end{aligned}$$

hold by Exercise 1.45, and taking the alternating sum we then obtain (5.60).

We now prove equation (5.59) by induction on n. The case $n = 1$ is given by (5.58), and the case $n = 2$ follows from the identity (5.54) in Exercise 5.53. Besides, the equality

$$D_{2r+1}\zeta_1^{\mathfrak{m}}(2^{\{n\}}) = \varpi_{2r+1}\big(\zeta_1^{\mathfrak{m}}(2^{\{r\}})\big)\otimes\zeta^{\mathfrak{m}}(2^{n-r}) \tag{5.61}$$

holds for all $3 \leqslant 2r+1 < 2n$ (Exercise 5.89). By the induction hypothesis and the fact that ϖ_{2r+1} kills products, we then get

$$D_{2r+1}\zeta_1^{\mathfrak{m}}(2^{\{n\}}) = 2(-1)^r\varpi_{2r+1}\big(\zeta^{\mathfrak{m}}(2r+1)\big)\otimes\zeta^{\mathfrak{m}}(2^{n-r}).$$

Moreover, combining the fact that D_{2r+1} is a derivation (Exercise 5.51) with equations (5.42) and (5.56), we get

$$D_{2r+1}\big(\zeta^{\mathfrak{m}}(2i+1)\zeta^{\mathfrak{m}}(2^{\{n-i\}})\big) = \begin{cases}\varpi_{2r+1}\big(\zeta^{\mathfrak{m}}(2r+1)\big)\otimes\zeta^{\mathfrak{m}}(2^{\{n-r\}}), & \text{if } r = i,\\ 0, & \text{if } r \neq i.\end{cases} \tag{5.62}$$

Therefore, if Θ denotes the difference of the left-hand side and the right-hand side terms of equation (5.59), then

$$D_{2r+1}\Theta = 0$$

for all $3 \leqslant 2r+1 < 2n$. Hence, Θ is a multiple of $\zeta^{\mathfrak{m}}(2n+1)$ by Theorem 5.45, and formula (5.59) follows from Corollary 5.50 and equations (5.60) and (5.58). $\square$

Given integers r and s, we let $\mathbb{I}(r \geqslant s)$ denote the indicator function

$$\mathbb{I}(r \geqslant s) = \begin{cases}1, & \text{if } r \geqslant s,\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 5.63. *Let $a, b \geqslant 0$ be integers. For each $1 \leqslant r \leqslant a+b$, the equality*

$$D_{2r+1}\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) = \varpi_{2r+1}(\xi^r_{a,b})\otimes\zeta^{\mathfrak{m}}(2^{\{a+b+1-r\}})$$

holds, where $\xi^r_{a,b} \in \mathcal{H}$ is the element given by

$$\xi^r_{a,b} = \sum_{\substack{\alpha\leqslant a\\ \beta\leqslant b\\ \alpha+\beta=r-1}} \zeta^{\mathfrak{m}}(2^{\{\beta\}}32^{\{\alpha\}}) - \sum_{\substack{\alpha\leqslant a\\ \beta\leqslant b-1\\ \alpha+\beta=r-1}} \zeta^{\mathfrak{m}}(2^{\{\alpha\}}32^{\{\beta\}}) + \Big(\mathbb{I}(b\geqslant r) - \mathbb{I}(a \geqslant r)\Big)\zeta_1^{\mathfrak{m}}(2^{\{r\}}). \tag{5.64}$$

Proof. To prove the result, it is enough to check which non-zero terms appear in formula (5.44) for the coaction. These terms are given by consecutive subsequences of $2r+1$ entries and can be of the following types:

i) Subsequences containing 001 and starting with 1; these contribute to the first sum.

ii) Subsequences containing 001 and starting with 0; after applying the reflection formula of Lemma 5.32, these contribute to the second sum.

iii) For $b \geqslant r$, there is exactly one sequence ending with 00; this gives rise to the term $\mathbb{I}(b \geqslant r)\zeta_1^{\mathfrak{m}}(2^{\{r\}})$.

iv) For $a \geqslant r$, there is exactly one sequence starting with 00; after applying the reflection formula, we obtain the term $-\mathbb{I}(a \geqslant r)\zeta_1^{\mathfrak{m}}(2^{\{r\}})$.

Using equation (5.31), it is easy to check that all the other subsequences do not contribute to the result. $\square$

PROPOSITION 5.65. *Given $a, b \geqslant 0$, write $n = a+b+1$. There exists a unique n-tuple of rational numbers $(\gamma^r_{a,b})_{r=1,\dots,n}$ satisfying*

$$\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) = \sum_{r=1}^{n} \gamma^r_{a,b}\zeta^{\mathfrak{m}}(2r+1)\zeta^{\mathfrak{m}}(2^{\{n-r\}}). \tag{5.66}$$

PROOF. We argue by induction on n. The case $n = 1$ is obvious, with $\gamma^1_{0,0} = 1$. Assume that the result holds for all integers smaller than n. In particular, all the numbers $\gamma^r_{a,b}$ are defined for $a+b+1 < n$. Now let a and b satisfy $a+b+1 = n$, and recall the element $\xi^r_{a,b}$ from (5.64). For each $r < n$, we define $\gamma^r_{a,b}$ as the unique rational number satisfying

$$\xi^r_{a,b} \equiv \gamma^r_{a,b}\zeta^{\mathfrak{m}}(2r+1) \text{ mod products},$$

which exists by the induction hypothesis and equation (5.59). Therefore, the projection of $\xi^r_{a,b}$ to $\mathcal{L}_{2r+1}$ is given by

$$\varpi_{2r+1}(\xi^r_{a,b}) = \gamma^r_{a,b}\,\varpi_{2r+1}\big(\zeta^{\mathfrak{m}}(2r+1)\big), \tag{5.67}$$

and from Lemma 5.63 we find

$$D_{2r+1}\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) = \gamma^r_{a,b}\varpi_{2r+1}\big(\zeta^{\mathfrak{m}}(2r+1)\big) \otimes \zeta^{\mathfrak{m}}(2^{\{n-r\}}). \tag{5.68}$$

Using equation (5.62), we deduce that the element

$$\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) - \sum_{r=1}^{n-1} \gamma^r_{a,b}\zeta^{\mathfrak{m}}(2r+1)\zeta^{\mathfrak{m}}(2^{\{n-r\}}) \tag{5.69}$$

belongs to the kernel of the derivation $D_{<2n+1}$. By Theorem 5.45, it is a rational multiple of $\zeta^{\mathfrak{m}}(2n+1)$, and we define the remaining $\gamma^n_{a,b}$ as this rational factor. □

This is a remarkable example of both the strength and the limits of the motivic formalism. Applying the period map (5.12), the motivic identity (5.66) implies the same kind of relations for usual multiple zeta values, something which would have been difficult to predict working only with numbers, where the coaction is invisible. However, the motivic formalism alone does not allow us to compute the precise value of the constants $\gamma^r_{a,b}$. For this, one needs to prove the corresponding identity of numbers first. In fact, for given a, b, and n with $a+b+1 = n$, the numbers $\gamma^r_{a,b}$ for $r < n$ are determined by induction, but the last term $\gamma^n_{a,b}$ can only be computed by means of the corresponding equality of numbers. Zagier [**Zag12**] has been able to prove an equality with the desired shape between multiple zeta values. To prove that Zagier's identity is motivic, we will need to show that its coefficients are compatible with the induction process in the proof of Proposition 5.65.

5.3.2. *Zagier's theorem.* Define, for each $a, b, r \geqslant 0$, rational numbers

$$A^r_{a,b} = \binom{2r}{2a+2} \quad \text{and} \quad B^r_{a,b} = (1-2^{-2r})\binom{2r}{2b+1}. \tag{5.70}$$

As in the previous paragraph, we set $n = a+b+1$.

THEOREM 5.71 (Zagier, [**Zag12**]). *The following equality holds:*

$$\zeta(2^{\{b\}}32^{\{a\}}) = 2\sum_{r=1}^{n}(-1)^r\left(A^r_{a,b} - B^r_{a,b}\right)\zeta(2r+1)\zeta(2^{\{n-r\}}). \tag{5.72}$$

SKETCH OF PROOF. Let $\widehat{H}(a,b)$ denote the right-hand side of (5.72). The strategy of the proof consists in showing that the generating series

$$F(x,y) = \sum_{a,b\geqslant 0} (-1)^{a+b+1}\zeta(2^{\{b\}}32^{\{a\}})x^{2a+2}y^{2b+1},$$
$$\widehat{F}(x,y) = \sum_{a,b\geqslant 0} (-1)^{a+b+1}\widehat{H}(a,b)x^{2a+2}y^{2b+1}$$

are equal. Using a similar technique to that of Example 1.27 (see Exercise 5.91), the first series is seen to be equal to

$$F(x,y) = \frac{\sin \pi y}{\pi}\cdot\frac{\partial}{\partial z}\,{}_3F_2\left(\left.\begin{matrix} x, -x, z \\ 1+y, 1-y\end{matrix}\,\right|\, 1\right)\Bigg|_{z=0}, \tag{5.73}$$

where the second factor involves the hypergeometric function

$${}_3F_2\left(\left.\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2\end{matrix}\,\right|\, t\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n}\frac{t^n}{n!}$$

(in this formula, $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ is the so-called Pocchammer symbol). It follows from the bound

$$0 < \zeta(2^{\{b\}}32^{\{a\}}) < \frac{1}{a+1}\zeta(2^{\{n\}}) = \frac{1}{a+1}\frac{\pi^n}{(2n+1)!} \qquad (n = a+b+1)$$

that the series $F(x,y)$ converges absolutely for all $x,y\in\mathbb{C}$ (and hence defines a holomorphic function on $\mathbb{C}\times\mathbb{C}$) and satisfies the bound

$$\max_{|x|,|y|\leqslant M} |F(x,y)| = O(e^{\pi M}\log M) \qquad \text{as } M\longrightarrow\infty. \tag{5.74}$$

The expression for the second generating series is more involved: $\widehat{F}(x,y)$ is an integral linear combination of fourteen terms of the form

$$\psi\left(1+\frac{u}{2}\right)\frac{\sin \pi v}{2\pi}, \qquad u\in\{\pm x, \pm y, \pm 2x, \pm 2y\}, \quad v\in\{x,y\},$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the digamma function (see Exercise 5.92). Using this explicit expression and standard properties of $\psi(s)$, one sees that $\widehat{F}(x,y)$ is also a holomorphic function on $\mathbb{C}\times\mathbb{C}$ satisfying

$$\max_{|x|,|y|\leqslant M} |\widehat{F}(x,y)| = O(e^{\pi M}\log M) \qquad \text{as } M\longrightarrow\infty. \tag{5.75}$$

At this point, Zagier observes that the expressions for the functions $F(x,y)$ and $\widehat{F}(x,y)$ allow him to prove that they are equal for certain values of x and y. Namely, it is not hard to prove that

$$F(x,x) = \widehat{F}(x,x), \qquad \text{for all } x\in\mathbb{C}. \tag{5.76}$$

Much trickier is the equality

$$F(x,n) = \widehat{F}(x,n), \qquad \text{for all } x\in\mathbb{C} \text{ and } n\in\mathbb{Z}, \tag{5.77}$$

from [**Zag12**, Prop. 6]. One is then led to the question of whether the partial information provided by (5.76) and (5.77) suffices to derive the equality of the functions. It turns out that the growth conditions (5.74) and (5.75) do the trick. Indeed, we can invoke the following result proved in [**Boa54**, Cor. 9.4.2].

THEOREM 5.78. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function that vanishes at all integers and satisfies the estimate $f(z) = O(e^{\pi|z|}\log|z|)$ for $|z| \to +\infty$. Then f is a constant multiple of* $\sin(\pi z)$.

Together with the estimates (5.74) and (5.75) and the equalities (5.77), this theorem implies the existence of a function $c\colon \mathbb{C} \to \mathbb{C}$ satisfying

$$F(x,y) - \widehat{F}(x,y) = c(x)\sin \pi y.$$

The equality (5.76) implies the vanishing $c(x) = 0$ for all $x \in \mathbb{C} \setminus \mathbb{Z}$, and hence the equality $F(x,y) = \widehat{F}(x,y)$ for all $x, y \in \mathbb{C}$ since these are holomorphic functions. $\square$

REMARK 5.79. After the original proof of Zagier's theorem, Z. H. Li in [**Li13**] has shown directly the equality $F(x,y) = \widehat{F}(x,y)$ using the transformation relations of hypergeometric functions ${}_3F_2$ that involve the gamma function.

5.3.3. *Lifting Zagier's theorem to a motivic identity.* The first non-trivial case of Zagier's theorem are the identities

$$\begin{aligned}\zeta(2,3) &= -2\zeta(3)\zeta(2) + \tfrac{9}{2}\zeta(5),\\ \zeta(3,2) &= 3\zeta(3)\zeta(2) - \tfrac{11}{2}\zeta(5).\end{aligned}$$

In Exercise 5.53, we proved that they lift to motivic equalities.

THEOREM 5.80. *For $a, b \geqslant 0$ and $1 \leqslant r \leqslant a+b+1$, the numbers $\gamma^r_{a,b}$ from the statement of Proposition 5.65 are equal to*

$$\gamma^r_{a,b} = (-1)^r 2\left(A^r_{a,b} - B^r_{a,b}\right). \tag{5.81}$$

In other words, writing $n = a+b+1$, the identity

$$\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) = 2\sum_{r=1}^{n}(-1)^r\left(A^r_{a,b} - B^r_{a,b}\right)\zeta^{\mathfrak{m}}(2r+1)\zeta^{\mathfrak{m}}(2^{\{n-r\}}) \tag{5.82}$$

of motivic multiple zeta values holds.

PROOF. We first note that, for any $a, b \geqslant 0$ and $1 \leqslant r \leqslant a+b+1$, the following identities are satisfied:

$$A^r_{a,b} = \sum_{\substack{\alpha\leqslant a\\ \beta\leqslant b\\ \alpha+\beta=r-1}} A^r_{\alpha,\beta} - \sum_{\substack{\alpha\leqslant a\\ \beta\leqslant b-1\\ \alpha+\beta=r-1}} A^r_{\beta,\alpha} + \mathbb{I}(b\geqslant r) - \mathbb{I}(a \geqslant r), \tag{5.83}$$

$$B^r_{a,b} = \sum_{\substack{\alpha\leqslant a\\ \beta\leqslant b\\ \alpha+\beta=r-1}} B^r_{\alpha,\beta} - \sum_{\substack{\alpha\leqslant a\\ \beta\leqslant b-1\\ \alpha+\beta=r-1}} B^r_{\beta,\alpha}. \tag{5.84}$$

This can be proved using that $A^r_{a,b}$ does not depend on b, that $B^r_{a,b}$ does not depend on a, and the symmetries

$$A^{\alpha+\beta+1}_{\alpha,\beta} = A^{\alpha+\beta+1}_{\beta-1,\alpha+1} \quad \text{and} \quad B^{\alpha+\beta+1}_{\alpha,\beta} = B^{\alpha+\beta+1}_{\beta,\alpha}.$$

For instance, the second equality is clear because by symmetry each term of the second sum cancels one term of the first sum; the only remaining term in the first sum is $B^r_{r-1-b,b}$, that agrees with $B^r_{a,b}$ because it is independent of a. To prove the first equality we may distinguish different cases according to whether a and b are bigger than or equal to r or not. For instance, if $a < r$ and $b \geqslant r$, then the term $A^r_{a,b}$ is different from zero. In these cases, both sums range from $(\alpha,\beta) = (a, r-1-a)$

to $(0, r-1)$. By the symmetry of the A's, all terms cancel except $A^r_{a,r-1-a}$ from the first sum, that agrees with $A^r_{a,b}$, and $-A^r_{r-1,0} = -1$, that cancels with $\mathbb{I}(b \geqslant r)$. The remaining cases are similar.

We now prove the theorem by induction on $n = a + b + 1$. For $n = 1$, the right-hand side of (5.81) is equal to 1, which is also the value $\gamma^1_{0,0}$, as we noted in the proof of Proposition 5.65. We then assume that equation (5.81) is true for all a', b' with $a' + b' < n - 1$ and all $1 \leqslant r' \leqslant a' + b' + 1$, and we fix a and b with $a+b+1 = n$. We compute $D_{2r+1}\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}})$ in two ways and compare the results. The first way is equation (5.68), while the second is to apply Lemma 5.63, then use Lemma 5.57 to get rid of the terms $\zeta^{\mathfrak{m}}_1(2^{\{r\}})$, and finally apply (5.67) to the terms $\varpi_{2r+1}\big(\zeta^{\mathfrak{m}}(2^{\{?\}}32^{\{?\}})\big)$. Comparing both results we obtain

$$\gamma^r_{a,b} = \sum_{\substack{\alpha \leqslant a \\ \beta \leqslant b \\ \alpha+\beta = r-1}} \gamma^r_{\alpha,\beta} - \sum_{\substack{\alpha \leqslant a \\ \beta \leqslant b-1 \\ \alpha+\beta = r-1}} \gamma^r_{\beta,\alpha} + 2(-1)^r \big(\mathbb{I}(b \geqslant r) - \mathbb{I}(a \geqslant r)\big).$$

Using the induction hypothesis and the identities (5.83) and (5.84), we deduce the equality (5.81) for $1 \leqslant r \leqslant a + b$.

To treat the remaining case $r = a + b + 1$, set

$$\Theta = \zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) - 2\sum_{r=1}^{n}(-1)^r \left(A^r_{a,b} - B^r_{a,b}\right)\zeta^{\mathfrak{m}}(2r+1)\zeta^{\mathfrak{m}}(2^{\{n-r\}}),$$

which is a motivic zeta value of weight $2a + 2b + 3$. The identities we already proved and equation (5.66) yield $D_{<2a+2b+3}(\Theta) = 0$. By Zagier's Theorem 5.71, we obtain $\mathrm{per}(\Theta) = 0$. Finally, Corollary 5.50 implies $\Theta = 0$, and hence the result. $\square$

5.3.4. *The coefficients $c_{\boldsymbol{s}}$.* Among the coefficients $\gamma^r_{a,b}$, the leading one $\gamma^{a+b+1}_{a,b}$ will play a special role, so we single it out.

DEFINITION 5.85. Let $\boldsymbol{s} = 2^{\{b\}}32^{\{a\}}$ be a word in the alphabet $2, 3$ with only one 3 and all the remaining entries equal to 2. We set

$$c_{\boldsymbol{s}} = \gamma^{a+b+1}_{a,b}.$$

We will also write

$$c_{12^{\{n\}}} = 2(-1)^n.$$

With this notation, Lemma 5.57 and Proposition 5.65 imply the following:

COROLLARY 5.86. *For all integers $n, a, b \geqslant 0$ with $n = a + b + 1$, the following equalities hold:*

i) $\varpi_{2n+1}\big(\zeta^{\mathfrak{m}}_1(2^{\{n\}})\big) = c_{12^{\{n\}}}\zeta^{\mathfrak{m}}(2n+1)$,

ii) $\varpi_{2n+1}\big(\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}})\big) = c_{2^{\{b\}}32^{\{a\}}}\zeta^{\mathfrak{m}}(2n+1)$.

Moreover, the following equality holds:

$$c_{12^{\{n\}}} = -2\sum_{i=0}^{n-1} c_{2^{\{i\}}32^{\{n-i-1\}}}. \tag{5.87}$$

Recall that, given a prime number p, the p-adic valuation of a non-zero rational number x is the only integer $v_p(x)$ such that x can be written as $x = p^{v_p(x)}a/b$, where a and b are integers relatively prime to p. We also set $v_p(0) = \infty$. As a consequence of Theorem 5.80, the coefficients c_w have the following 2-adic properties.

LEMMA 5.88. *Let $\boldsymbol{s}$ be a word of the form $\boldsymbol{s} = 2^{\{b\}}32^{\{a\}}$. Denote by $\boldsymbol{s}^{\mathrm{r}}$ the word written in reverse order (i.e. $\boldsymbol{s}^{\mathrm{r}} = 2^{\{a\}}32^{\{b\}}$), and set $n = a + b + 1$. Then,*

i) $c_{\boldsymbol{s}} \in \mathbb{Z}[\frac{1}{2}]$;

ii) $c_{\boldsymbol{s}} - c_{\boldsymbol{s}^{\mathrm{r}}}$ *is an even integer;*

iii) $v_2(c_{2^{\{n-1\}}3}) = v_2(c_{32^{\{n-1\}}}) \leqslant v_2(c_{\boldsymbol{s}}) \leqslant 0$.

PROOF. Recall the formula

$$c_{\boldsymbol{s}} = (-1)^n 2\left(A^n_{a,b} - B^n_{a,b}\right)$$

from Theorem 5.80. Since $A^n_{a,b}$ is an integer and $B^n_{a,b}$ belongs to $\mathbb{Z}[\frac{1}{2}]$, the first statement follows. Property ii) is obtained from the symmetry $B^n_{a,b} = B^n_{b,a}$. Indeed,

$$c_{\boldsymbol{s}} - c_{\boldsymbol{s}^{\mathrm{r}}} = (-1)^n 2(A^n_{a,b} - A^n_{b,a}) \in 2\mathbb{Z}.$$

To prove iii), we first observe that the inequality $v_2((2n)!) < 2n$ implies

$$v_2(2^{-2n}\tbinom{2n}{2b+1}) < 0.$$

Using the triangle inequality, we get

$$v_2(c_{\boldsymbol{s}}) = v_2(2 \cdot 2^{-2n}\tbinom{2n}{2b+1}) = 1 + v_2(2^{-2n}\tbinom{2n}{2b+1}) \leqslant 0.$$

For the remaining inequality, we write

$$\tbinom{2n}{2b+1} = \frac{2n}{2b+1}\tbinom{2n-1}{2b},$$

so that the 2-adic valuation is given by

$$v_2(c_{\boldsymbol{s}}) = 2 - 2n + v_2(n) + v_2(\tbinom{2n-1}{2b}).$$

Since $v_2(\binom{2n-1}{2b}) \geqslant 0$, the right-hand side of this equality attains its minimum for $b = n-1$ and $b = 0$, which correspond to $\boldsymbol{s} = 2^{\{n-1\}}3$ and $\boldsymbol{s} = 32^{\{n-1\}}$. □

⋆ ⋆ ⋆

EXERCISE 5.89. Prove equation (5.61).

EXERCISE 5.90. Show that one may replace the multiple zeta value $\zeta(2^{\{n-r\}})$ with either $\zeta(2n-2r)$ or $\zeta(2)^{n-r}$ in the right-hand side of Zagier's theorem 5.71 without losing the rationality of the coefficients $\gamma^r_{a,b}$.

EXERCISE 5.91. The goal of this exercise is to prove equation (5.73) in the sketch of proof of Zagier's theorem.

i) Prove that the equalities

$$\prod_{0<k<m}\left(1 - \frac{x^2}{k^2}\right) = \sum_{j=0}^{m-1}\sum_{0<k_1<\cdots<k_j<m}\frac{(-1)^j x^{2j}}{k_1^2\ldots k_j^2}$$
$$\prod_{\ell>m}\left(1 - \frac{y^2}{\ell^2}\right) = \sum_{j\geqslant 0}\sum_{m<\ell_1<\cdots<\ell_j}\frac{(-1)^j y^{2j}}{\ell_1^2\ldots \ell_j^2}$$

hold for each integer $m \geqslant 1$. Deduce that the first generating series in the proof is given by

$$F(x,y) = -x^2 y \sum_{m\geqslant 1}\prod_{0<k<m}\left(1 - \frac{x^2}{k^2}\right)\cdot\frac{1}{m^3}\cdot\prod_{\ell>m}\left(1 - \frac{y^2}{\ell^2}\right).$$

ii) Prove the equality

$$-x^2 \prod_{0<k<m} \left(1 - \frac{x^2}{k^2}\right) = \frac{(-x)_m (x)_m}{(m-1)!^2},$$

where $(x)_m$ stands for the Pocchammer symbol. Similarly, prove

$$\frac{y}{m^2} \prod_{\ell > m} \left(1 - \frac{y^2}{\ell^2}\right) = \frac{\sin \pi y}{\pi} \cdot \frac{(m-1)!^2}{(1-y)_m (1+y)_m}$$

using the product expansion of $\sin(\pi y)/\pi$. Conclude that

$$F(x,y) = \frac{\sin \pi y}{\pi} \sum_{m \geqslant 1} \frac{1}{m} \frac{(-x)_m (x_m)}{(1-y)_m (1+y)_m}.$$

iii) Prove equation (5.73).

EXERCISE 5.92. Consider the meromorphic functions

$$A(z) = \sum_{n=1}^{\infty} \frac{z^2}{n(n^2 - z^2)} \quad \text{and} \quad B(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^2}{n(n^2 - z^2)}$$

defined on the complex plane, with simple poles at all non-zero integers.

i) Prove the equality $B(z) = A(z) - A(z/2)$.

ii) Prove the equalities of formal power series

$$A(z) = \sum_{r=1}^{\infty} \zeta(2r+1) z^{2r} \quad \text{and} \quad B(z) = \sum_{r=1}^{\infty} (1 - 2^{-2r}) \zeta(2r+1) z^{2r}.$$

(The right-hand sides of both equalities only converge for $|z| < 1$. This is the reason to work with $A(z)$ and $B(z)$ instead.)

iii) Use the Taylor expansion of the logarithm of $\Gamma(1+z)$ and $\Gamma(1+z)$ from Exercise 1.18 to get the equality

$$A(z) = \psi(1) - \frac{1}{2}(\psi(1+z) + \psi(1-z)).$$

iv) Using Example 1.27, the binomial theorem, and the above results, derive the expression

$$\begin{aligned} \widehat{F}(x,y) = \frac{\sin \pi y}{\pi} [A(x+y) + A(x-y) - 2A(y)] \\ - \frac{\sin \pi x}{\pi} [B(x+y) - B(x-y)] \end{aligned}$$

for the second generating series in the proof of Zagier's theorem.

5.4. The subspaces $\mathcal{H}^{2,3}$. In this section, we initiate the study of the subspaces $\widetilde{\mathcal{H}}^{2,3} \subset \mathfrak{H}$ and $\mathcal{H}^{2,3} \subset \mathcal{H}$ spanned by admissible words containing only 2s and 3s, and by motivic multiple zeta values with only 2s and 3s as entries respectively. Clearly, there is a surjective map $\widetilde{\mathcal{H}}^{2,3} \to \mathcal{H}^{2,3}$. A key step to prove Theorem B in the next section is to show that this map is an isomorphism. To set the stage, we define a level filtration on both spaces by counting the number of 3s and, using the infinitesimal coaction, a level lowering operator that sends an element of level ℓ and weight N to a linear combination of elements of level $\ell - 1$ and smaller weight. The operators for $\widetilde{\mathcal{H}}^{2,3}$ and $\mathcal{H}^{2,3}$ are compatible. Finally, we exhibit some explicit bases of the source and the target of the level lowering operator that will help us to prove, in the next section, that it is an isomorphism.

5.4.1. *The level filtration.*

DEFINITION 5.93. We denote by $\widetilde{\mathcal{H}}^{2,3} \subset \mathcal{O}(\Pi) = \mathfrak{H}$ the subspace generated by the functions $I(1;\alpha;0)$, where α is the binary sequence associated with an admissible multi-index containing only 2 and 3 as entries, and by $\mathcal{H}^{2,3} \subseteq \mathcal{H}$ the image of $\widetilde{\mathcal{H}}^{2,3}$ under the restriction map

$$\mathrm{res}\colon \mathcal{O}(\Pi) \longrightarrow \mathcal{H}.$$

Clearly, $\mathcal{H}^{2,3}$ is the $\mathbb{Q}$-vector space spanned by the motivic multiple zeta values

$$\zeta^{\mathfrak{m}}(s_1, \ldots, s_r) \text{ with } s_i \in \{2,3\}.$$

We filter $\widetilde{\mathcal{H}}^{2,3}$ by the number of entries equal to 3 in the admissible multi-index. Precisely, for each integer $\ell \geqslant 0$, consider

$$F_\ell \widetilde{\mathcal{H}}^{2,3} = \langle I(1;\mathrm{bs}(\boldsymbol{s});0) \mid \boldsymbol{s} \text{ contains} \leqslant \ell \text{ entries equal to } 3\rangle_{\mathbb{Q}}.$$

This defines an increasing *level filtration*

$$0 \subseteq F_0\widetilde{\mathcal{H}}^{2,3} \subseteq F_1\widetilde{\mathcal{H}}^{2,3} \subseteq \cdots$$

We deduce an increasing filtration on $\mathcal{H}^{2,3}$ with

$$F_\ell \mathcal{H}^{2,3} = \langle \zeta^{\mathfrak{m}}(s_1, \ldots, s_r) \in \mathcal{H}^{2,3} \mid \text{number of } s_i = 3 \leqslant \ell\rangle_{\mathbb{Q}}.$$

The associated graded pieces $\mathrm{gr}^F_\ell \mathcal{H}^{2,3} = F_\ell\mathcal{H}^{2,3}/F_{\ell-1}\mathcal{H}^{2,3}$ are spanned by the projections of motivic multiple zeta values with exactly ℓ entries equal to 3, which will be denoted in the same way. In particular,

$$\begin{aligned}
\mathrm{gr}^F_0 \mathcal{H}^{2,3} &= \langle \zeta^{\mathfrak{m}}(2^{\{n\}}) \mid n \geqslant 1\rangle_{\mathbb{Q}},\\
\mathrm{gr}^F_1 \mathcal{H}^{2,3} &= \langle \zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) \mid a, b \geqslant 0\rangle_{\mathbb{Q}}.
\end{aligned}$$

Note that these are precisely the two families of motivic multiple zeta values that we studied in the previous section.

REMARK 5.94. The $\mathbb{Q}$-vector space $\mathrm{gr}^F_\ell \widetilde{\mathcal{H}}^{2,3}_N$ is non-zero if and only if the weight N and the level ℓ have the same parity and $N \geqslant 3\ell$. When this is the case, writing $N = 2m + 3\ell$, the dimensions are given by

$$\dim_{\mathbb{Q}} \mathrm{gr}^F_\ell \widetilde{\mathcal{H}}^{2,3}_N = \binom{m+\ell}{\ell},$$

hence the inequality

$$\dim_{\mathbb{Q}} \mathrm{gr}^F_\ell \mathcal{H}^{2,3}_N \leqslant \binom{m+\ell}{\ell} \tag{5.95}$$

because we do not know yet that the generators of $\mathrm{gr}^F_\ell \mathcal{H}^{2,3}_N$ discused previously are linearly independent. Indeed, in Theorem 5.111 we will see that this is the case, so that inequality (5.95) turns out to be an equality.

5.4.2. *The level lowering operator.* Recall from Section 3.10.6 that we introduced the Goncharov coproduct as a morphism

$$\Delta^{\Gamma}\colon \mathcal{O}(\Pi) \longrightarrow \mathcal{O}(\Pi) \otimes_{\mathbb{Q}} \mathcal{O}(\Pi). \tag{5.96}$$

From this, we obtained the motivic coaction (5.11)

$$\Delta\colon \mathcal{H} \longrightarrow \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$$

that we have been using in the last pages. In what follows, we will also use an intermediate version

$$\Delta\colon \mathcal{O}(\Pi) \longrightarrow \mathcal{A} \otimes \mathcal{O}(\Pi) \tag{5.97}$$

which is simply obtained from (5.96) via the projection $\mathcal{O}(\Pi) \to \mathcal{A}$ (recall that this corresponds to restricting a function on Π to the subvariety $\mathcal{X}$ of Remark 5.35). This is nothing else but the coaction associated with the action of $I(U_{\mathrm{dR}})$ on Π. As in Definition 5.39, there are maps

$$D_{2r+1}\colon \mathcal{O}(\Pi) \longrightarrow \mathcal{L}_{2r+1} \otimes \mathcal{O}(\Pi).$$

Following the proof of Proposition 5.43 we see that the analogue of (5.44), namely

$$\begin{aligned} D_n I(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_N; \varepsilon_{N+1}) = \\ \sum_{p=0}^{N-n} \varpi_n \left(I^{\mathfrak{m}}(\varepsilon_p; \varepsilon_{p+1} \cdots \varepsilon_{p+n}; \varepsilon_{p+n+1})\right) \\ \otimes I(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_p, \varepsilon_{p+n+1}, \ldots, \varepsilon_N; \varepsilon_{N+1}), \end{aligned} \tag{5.98}$$

also holds for all odd integers $n < N$.

We now study how the filtered subspace $\widetilde{\mathcal{H}}^{2,3} \subset \mathcal{O}(\Pi)$ behaves with respect to the coaction and its infinitesimal version.

Lemma 5.99. *The subspace $\widetilde{\mathcal{H}}^{2,3}$ is stable under the coaction (5.97). That is, the subspace $\Delta(\widetilde{\mathcal{H}}^{2,3}) \subset \mathcal{A} \otimes \mathcal{O}(\Pi)$ is contained in $\mathcal{A} \otimes_{\mathbb{Q}} \widetilde{\mathcal{H}}^{2,3}$. It follows that (5.97) restricts to a coaction*

$$\Delta\colon \widetilde{\mathcal{H}}^{2,3} \longrightarrow \mathcal{A} \otimes_{\mathbb{Q}} \widetilde{\mathcal{H}}^{2,3}.$$

Proof. Let $I(1; \alpha; 0)$ be an element of $\widetilde{\mathcal{H}}^{2,3}$. Then α is a binary sequence obtained by successive concatenation of the subsequences 01 and 001. From the explicit formula for the coaction (5.37) and the vanishing of the iterated integrals $I(\varepsilon; \alpha'; \varepsilon')$ for $\varepsilon = \varepsilon'$ and $\alpha' \neq \emptyset$, we deduce that each non-trivial term appearing in $\Delta I(1; \alpha; 0)$ has a factor of the form $I(1; \beta; 0)$ in the right-hand side of the coaction, where β is a concatenation of the subsequences 01 and 001. □

Remark 5.100. In [**Del13**, § 6.3], the above result is rephrased by saying that the subspace $\widetilde{\mathcal{H}}^{2,3}$ is "motivic", and hence invariant under the action of U_{dR}.

From this, we immediately deduce:

Corollary 5.101. *For each $r \geqslant 1$, the derivation D_{2r+1} restricts to a map*

$$D_{2r+1}\colon \widetilde{\mathcal{H}}^{2,3} \longrightarrow \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} \widetilde{\mathcal{H}}^{2,3}.$$

In fact, more is true:

LEMMA 5.102. *For each $r \geqslant 1$, the derivation D_{2r+1} is compatible with the level filtration, in the sense that it induces a map*

$$D_{2r+1}\colon F_\ell \widetilde{\mathcal{H}}^{2,3} \longrightarrow \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} F_{\ell-1}\widetilde{\mathcal{H}}^{2,3}.$$

PROOF. Let $\boldsymbol{s}$ be a word of level ℓ in the alphabet $\{2,3\}$. The binary sequence $\mathrm{bs}(\boldsymbol{s})$ contains at most ℓ subsequences 00. Any subsequence of odd length of $(1;\mathrm{bs}(\boldsymbol{s});0)$ that begins and ends with the same symbol will be killed by $I^{\mathfrak{m}}$ and will not contribute to D_{2r+1}. Otherwise it must contain at least a subsequence 00. Thus, the complementary quotient sequence will contain at most $\ell-1$ subsequences equal to 00, and hence will have level at most $\ell-1$. □

The above lemma yields a map

$$\mathrm{gr}^F_\ell D_{2r+1}\colon \mathrm{gr}^F_\ell \widetilde{\mathcal{H}}^{2,3} \longrightarrow \mathcal{L}_{2r+1} \otimes \mathrm{gr}^F_{\ell-1}\widetilde{\mathcal{H}}^{2,3}. \tag{5.103}$$

LEMMA 5.104. *For all $r, \ell \geqslant 1$, there is an inclusion*

$$\mathrm{gr}^F_\ell D_{2r+1}(\mathrm{gr}^F_\ell \widetilde{\mathcal{H}}^{2,3}) \ \subseteq \mathbb{Q}\varpi_{2r+1}(\zeta^{\mathfrak{m}}(2r+1)) \otimes_{\mathbb{Q}} \mathrm{gr}^F_{\ell-1}\widetilde{\mathcal{H}}^{2,3}.$$

PROOF. Let $\boldsymbol{s}$ be a word of level ℓ in the alphabet $\{2,3\}$ and $I^{\mathfrak{m}}(1;\mathrm{bs}(\boldsymbol{s});0)$ the corresponding motivic iterated integral. From the definition of D_{2r+1}, we get

$$\mathrm{gr}^F_\ell D_{2r+1}(\zeta^{\mathfrak{m}}(\boldsymbol{s})) = \sum_\gamma \varpi_{2r+1}(I^{\mathfrak{m}}(\gamma)) \otimes \zeta^{\mathfrak{m}}(\boldsymbol{s}_\gamma), \tag{5.105}$$

where the sum runs over all subsequences γ of $(1;\mathrm{bs}(\boldsymbol{s});0)$ of length $2r+1$, and $\boldsymbol{s}_\gamma$ is obtained by removing the internal part of γ.

If γ contains more than one subsequence 00, then $\boldsymbol{s}_\gamma$ has level $< \ell-1$, and hence does not contribute. If γ begins and ends in the same symbol, then $I^{\mathfrak{m}}(\gamma)$ is zero. One checks that $I^{\mathfrak{m}}(\gamma)$ can be of four remaining types:

i) $I^{\mathfrak{m}}(1;01\dots01001\dots01;0) = \zeta^{\mathfrak{m}}(2^{\{\beta\}}32^{\{\alpha\}})$;
ii) $I^{\mathfrak{m}}(0;10\dots10010\dots10;1) = -\zeta^{\mathfrak{m}}(2^{\{\beta\}}32^{\{\alpha\}})$;
iii) $I^{\mathfrak{m}}(1;01\dots10;0) = \zeta^{\mathfrak{m}}_1(2^{\{r\}})$;
iv) $I^{\mathfrak{m}}(0;01\dots10;1) = -\zeta^{\mathfrak{m}}_1(2^{\{r\}})$.

By Corollary 5.86, the term $\varpi_{2r+1}(I^{\mathfrak{m}}(\gamma))$ belongs to $\mathbb{Q}\zeta^{\mathfrak{m}}(2r+1)$ in all cases. □

The above lemma justifies the following definition:

DEFINITION 5.106. For all integers $N, \ell \geqslant 1$, the *level lowering operator* $\widetilde{\partial}_{N,\ell}$ is the $\mathbb{Q}$-linear map

$$\widetilde{\partial}_{N,\ell}\colon \mathrm{gr}^F_\ell \widetilde{\mathcal{H}}^{2,3}_N \longrightarrow \bigoplus_{3\leqslant 2r+1\leqslant N} \mathrm{gr}^F_{\ell-1}\widetilde{\mathcal{H}}^{2,3}_{N-2r-1} \tag{5.107}$$

obtained by first applying

$$\bigoplus_{3\leqslant 2r+1\leqslant N} \mathrm{gr}^F_\ell D_{2r+1}\big|_{\mathrm{gr}^F_\ell \widetilde{\mathcal{H}}^{2,3}_N}$$

and then sending $\varpi_{2r+1}(\zeta^{\mathfrak{m}}(2r+1))$ to 1.

The same construction gives rise to operators

$$\partial_{N,\ell}\colon \mathrm{gr}^F_\ell \mathcal{H}^{2,3}_N \longrightarrow \bigoplus_{3\leqslant 2r+1\leqslant N} \mathrm{gr}^F_{\ell-1}\mathcal{H}^{2,3}_{N-2r-1} \tag{5.108}$$

that fit in the commutative diagrams

$$
\begin{tikzcd}
\mathrm{gr}^F_\ell \widetilde{\mathcal{H}}^{2,3}_N \arrow[r, "\widetilde{\partial}_{N,\ell}"] \arrow[d] & \bigoplus_{3\leqslant 2r+1\leqslant N} \mathrm{gr}^F_{\ell-1} \widetilde{\mathcal{H}}^{2,3}_{N-2r-1} \arrow[d] \\
\mathrm{gr}^F_\ell \mathcal{H}^{2,3} \arrow[r, "\partial_{N,\ell}"] & \bigoplus_{3\leqslant 2r+1\leqslant N} \mathrm{gr}^F_{\ell-1} \mathcal{H}^{2,3}_{N-2r-1}.
\end{tikzcd}
\tag{5.109}
$$

5.4.3. *A pair of bases.* We next describe bases of the source and the target of the map (5.107). For $\ell \geqslant 1$ and $N \geqslant 3$, we define:

$B_{N,\ell} =$ set of words in the alphabet $\{2,3\}$ of weight N and level ℓ;

$B'_{N,\ell} =$ set of words in the alphabet $\{2,3\}$ of weight $\leqslant N-3$ and level $\ell-1$ (this includes the empty word if $\ell = 1$).

Clearly, $B_{N,\ell}$ gives a basis $\mathcal{B}_{N,\ell}$ of $\mathrm{gr}^F_\ell \widetilde{\mathcal{H}}^{2,3}_N$, while $B'_{N,\ell}$ gives a basis $\mathcal{B}'_{N,\ell}$ of

$$\bigoplus_{3\leqslant 2r+1\leqslant N} \mathrm{gr}^F_{\ell-1} \widetilde{\mathcal{H}}^{2,3}_{N-2r-1}.$$

Write $N = 3\ell + 2m$, so that m is the number of 2s in an element of $B_{N,\ell}$. Then

$$|B_{N,\ell}| = \binom{\ell+m}{\ell},$$

$$|B'_{N,\ell}| = \sum_{m'=0}^{m} \binom{\ell-1+m'}{\ell-1}.$$

From the identity of binomial coefficients

$$\binom{\ell+m}{\ell} = \sum_{m'=0}^{m} \binom{\ell-1+m'}{\ell-1},$$

we deduce the equality $|B_{N,\ell}| = |B'_{N,\ell}|$.

We provide $B_{N,\ell}$ with the lexicographic order induced by $2 < 3$ and $B'_{N,\ell}$ with the order in which $\boldsymbol{s} \leqslant \boldsymbol{s}'$ if and only if either $\mathrm{wt}(\boldsymbol{s}) < \mathrm{wt}(\boldsymbol{s}')$, or $\mathrm{wt}(\boldsymbol{s}) = \mathrm{wt}(\boldsymbol{s}')$ and $\boldsymbol{s}$ is smaller than or equal to $\boldsymbol{s}'$ in the lexicographic order.

LEMMA 5.110. *There is an order-preserving bijection $B'_{N,\ell} \to B_{N,\ell}$ that sends an element $\boldsymbol{s} \in B'_{N,\ell}$ to $2^{\{r-1\}}3\boldsymbol{s} \in B_{N,\ell}$, where $2r = N - 1 - \mathrm{wt}(\boldsymbol{s})$.*

PROOF. Let us denote by υ the map in the statement. Given $\boldsymbol{s}, \boldsymbol{s}' \in B'_{N,\ell}$, write $2r = N - 1 - \mathrm{wt}(\boldsymbol{s})$ and $2r' = N - 1 - \mathrm{wt}(\boldsymbol{s}')$.

If $\mathrm{wt}(\boldsymbol{s}) < \mathrm{wt}(\boldsymbol{s}')$, then $r > r'$, and hence

$$\upsilon(\boldsymbol{s}) = 2^{\{r-1\}}3\boldsymbol{s} < 2^{\{r'-1\}}3\boldsymbol{s}' = \upsilon(\boldsymbol{s}').$$

If $\mathrm{wt}(\boldsymbol{s}) = \mathrm{wt}(\boldsymbol{s}')$ but $\boldsymbol{s}$ is smaller than $\boldsymbol{s}'$ in the lexicographic order, then

$$\upsilon(\boldsymbol{s}) = 2^{\{r-1\}}3\boldsymbol{s} < 2^{\{r-1\}}3\boldsymbol{s}' = \upsilon(\boldsymbol{s}').$$

Therefore, υ is injective and order-preserving. Since the sets $B'_{N,\ell}$ and $B_{N,\ell}$ have the same cardinality, υ is a bijection. □

5.5. Brown's theorem. In this final section, we prove Brown's theorem and deduce some consequences concerning mixed Tate motives over $\mathbb{Z}$ and their periods.

5.5.1. *Statement.* Our goal is to prove the following result:

THEOREM 5.111 (Brown). *The set of elements*

$$\{\zeta^{\mathfrak{m}}(s_1, \dots, s_r) \mid s_i \in \{2,3\}\}$$

forms a basis of the $\mathbb{Q}$*-vector space of motivic multiple zeta values.*

Before going into the proof, let us mention the immediate corollary:

COROLLARY 5.112 (Theorem B). *Every multiple zeta value is a* $\mathbb{Q}$*-linear combination of multiple zeta values with only* 2 *and* 3 *as entries.*

PROOF. Apply the period map (5.12). □

REMARK 5.113.

i) The proof does not give an algorithm to compute the linear combination.
ii) The missing information to deduce that such multiple zeta values furnish a basis, as it is conjectured, is to know that all relations among multiple zeta values have motivic origin.

5.5.2. *Strategy of the proof.* The key point to prove Theorem 5.111 is:

LEMMA 5.114. *For all* $N, \ell \geqslant 1$*, the level lowering operator* $\widetilde{\partial}_{N,\ell}$ *is an isomorphism of* $\mathbb{Q}$*-vector spaces.*

We will first show how to deduce Theorem 5.111 from Lemma 5.114. The first step is the following:

LEMMA 5.115. *The map* $\widetilde{\mathcal{H}}_N^{2,3} \to \mathcal{H}_N^{2,3}$ *is an isomorphism.*

PROOF. We first prove by induction on the level that, for every weight N and level ℓ, the restriction map $\mathrm{gr}_\ell^F \widetilde{\mathcal{H}}_N^{2,3} \to \mathrm{gr}_\ell^F \mathcal{H}_N^{2,3}$ is an isomorphism.

The initial step is $\ell = 0$. If $N = 2r$ is even, then $\mathrm{gr}_0^F \widetilde{\mathcal{H}}_N^{2,3}$ is a one-dimensional space generated by $I(1; \mathrm{bs}(2^{\{r\}}); 0)$, while $\mathrm{gr}_0^F \mathcal{H}_N^{2,3}$ is generated by $\zeta^{\mathfrak{m}}(2^{\{r\}}) \neq 0$. Thus, the restriction map

$$\mathrm{gr}_0^F \widetilde{\mathcal{H}}_N^{2,3} \longrightarrow \mathrm{gr}_0^F \mathcal{H}_{N,0}^{2,3} \tag{5.116}$$

is an isomorphism. If N is odd, then both spaces are zero.

We now consider the commutative diagram (5.109). By definition, the left vertical arrow is an epimorphism. By the induction hypothesis, the right vertical map is an isomorphism, and by Lemma 5.114 the upper horizontal map is injective. Hence, the left vertical arrow is an isomorphism.

Once we now that all the restriction maps $\mathrm{gr}_\ell^F \widetilde{\mathcal{H}}_N^{2,3} \to \mathrm{gr}_\ell^F \mathcal{H}_N^{2,3}$ are isomorphisms, we deduce that $\widetilde{\mathcal{H}}_N^{2,3} \to \mathcal{H}_N^{2,3}$ is an isomorphism by using the fact that the filtration by the level is bounded and the five lemma (see Exercise A.215). □

By equation (1.72), the dimension of $\widetilde{\mathcal{H}}_N^{2,3}$ is d_N. By Lemma 5.115, d_N is also the dimension of $\mathcal{H}_N^{2,3}$. Hence, the elements $\zeta^{\mathfrak{m}}(s_1, \dots, s_r)$ of weight N with entries $s_i \in \{2,3\}$ form a basis of $\mathcal{H}_N^{2,3}$. There are injections

$$\mathcal{H}_N^{2,3} \subset \mathcal{H}_N \hookrightarrow \mathcal{H}_N^{\mathcal{MT}}.$$

Since the dimensions of the left and right vector spaces are the same we deduce that the three spaces are isomorphic, and hence Theorem 5.111 holds.

5.5.3. *Proof of Lemma 5.114*. The proof is based on the study of the 2-adic valuation of the coefficients of the matrix of $\widetilde{\partial}_{N,\ell}$ with respect to the bases introduced in Section 5.4.3. We shall use the following lemma:

LEMMA 5.117. *Let $A = (a_{ij})_{i,j}$ be a square matrix of size n with rational coefficients. Assume that there exists a prime number p such that the following conditions hold:*

i) $v_p(a_{ij}) \geqslant 1$ *for all* $i > j$*;*

ii) $v_p(a_{ii}) = \min_j\{v_p(a_{ij})\} \leqslant 0$ *for all* i*.*

Then A is invertible.

PROOF. Consider the matrix A' obtained by multiplying the i-th row of A by $p^{-v_p(a_{ii})}$. By condition ii), the p-adic valuation of the coefficients of A' is non-negative, so it makes sense to reduce the matrix A' modulo p. Since the inequality $v_p(a'_{ij}) \geqslant 1$ still holds for $i > j$ but now $v_p(a'_{ii}) = 0$, the reduction is upper triangular with non-zero elements in the diagonal. It follows that the determinant of A', and hence the determinant of A, is non-zero. $\square$

We next see that, up to terms with even coefficients, the map $\widetilde{\partial}_{N,\ell}$ acts by deconcatenation.

PROPOSITION 5.118. *Let $\boldsymbol{s}$ be a word of weight N and level ℓ in the alphabet $\{2,3\}$. Then the equality*

$$\widetilde{\partial}_{N,\ell} I(1; \mathrm{bs}(\boldsymbol{s}); 0)) = \sum_{\substack{\boldsymbol{s}=\boldsymbol{u}\boldsymbol{v} \\ \deg_3 \boldsymbol{u}=1}} c_{\boldsymbol{u}} I(1; \mathrm{bs}(\boldsymbol{v}); 0) \\ + \textit{terms with } 2\mathbb{Z} \textit{ coefficients}$$

holds, where $\deg_3 \boldsymbol{u}$ is the number of entries equal to 3 in the word $\boldsymbol{u}$, and $c_{\boldsymbol{u}}$ is the coefficient introduced in Definition 5.85.

PROOF. Following the proof of Lemma 5.104, there are four types of terms in $\widetilde{\partial}_{N,\ell} I(1; \mathrm{bs}(\boldsymbol{s}); 0))$. We start with types iii) and iv). Taking $c_{12^{\{n\}}} = 2(-1)^n$ into account, these terms contribute with even coefficients. Besides, almost all terms of types i) and ii) can be grouped in pairs. Indeed, choose four positions as follows

$$I(\ldots 0\underset{a}{1}\underset{b}{0} \ldots 01001 \ldots 0\underset{c}{1}\underset{d}{0} \ldots),$$

that is, a and b (resp. c and d) are consecutive, a (resp. d) contains a 0 and b (resp. c) contains a 1. Combining Lemma 5.32 ii) and Lemma 5.88 ii), the sum of the contributions of the subsequences ac and bd has again coefficients in $2\mathbb{Z}$. The only terms that cannot be paired this way are the leftmost subsequences appearing in the sum of the statement. $\square$

COROLLARY 5.119. *With respect to the bases $\mathcal{B}_{N,\ell}$ and $\mathcal{B}'_{N,\ell}$, ordered as in Section 5.4.3, the matrix $M_{N,\ell}$ of the operator $\widetilde{\partial}_{N,\ell}$ satisfies the assumptions of Lemma 5.117 for the prime $p = 2$. In particular, $M_{N,\ell}$ is invertible.*

PROOF. Let $\boldsymbol{v}$ be a word with only 2 and 3 as entries, of weight $\leqslant N - 3$ and level $\ell - 1$. Set $2r = N - 1 - \mathrm{wt}(\boldsymbol{v})$ and $\boldsymbol{s} = 2^{\{r-1\}}3\boldsymbol{v}$. Then $\boldsymbol{s}$ is the multi-index corresponding to $\boldsymbol{v}$ under the order-preserving bijection from Lemma 5.110. Consider any admissible multi-index with only 2 and 3 as entries, of weight N and level ℓ, that can be written as $\boldsymbol{u}\boldsymbol{v}$ with $\deg_3 \boldsymbol{u} = 1$. If $\boldsymbol{s} \neq \boldsymbol{u}\boldsymbol{v}$, then the number of occurences of 2 before the first 3 in $\boldsymbol{u}$ is smaller than $r-1$, hence the inequality $\boldsymbol{uv} > \boldsymbol{s}$. By Proposition 5.118, this implies that any term in $M_{N,\ell}$ that is not an even integer is above the diagonal. Moreover, by the same proposition and Lemma 5.88 iii), the coefficient of $\boldsymbol{v}$ in $\widetilde{\partial}_{N,\ell}\boldsymbol{s}$ sitting at the diagonal of $M_{N,\ell}$ has a non-positive 2-adic valuation, and it realizes the minimum of this valuation within its row. Therefore, the assumptions of Lemma 5.117 are satisfied. □

Clearly, Lemma 5.114 is a consequence of Corollary 5.119 and Lemma 5.117, thus finishing the proof of Theorem 5.111.

5.5.4. *Some consequences of Brown's theorem.* We conclude this chapter with some corollaries of Brown's theorem. Recall (5.4).

Corollary 5.120. *The map $U_{\mathrm{dR}} \to I(U_{\mathrm{dR}})$ is a group isomorphism.*

Proof. Recall the algebras $\mathcal{A}^{\mathcal{MT}} = \mathcal{O}(U_{\mathrm{dR}})$ and $\mathcal{A} = \mathcal{O}(I(U_{\mathrm{dR}}))$ from (5.5). We want to show that the injective map $\mathcal{A} \hookrightarrow \mathcal{A}^{\mathcal{MT}}$ induced by $U_{\mathrm{dR}} \to I(U_{\mathrm{dR}})$ is also surjective. In Corollary 5.21, we proved that this map extends to an injection $\mathcal{H} \hookrightarrow \mathcal{H}^{\mathcal{MT}}$ compatible with the gradings on both sides. In the course of the proof of Brown's theorem, we saw that the graded pieces of $\mathcal{H}$ are isomorphic to those of $\mathcal{H}^{\mathcal{MT}}$, and hence the algebras are isomorphic. □

Let $\mathbf{MT}'(\mathbb{Z})$ be the full tannakian subcategory of $\mathbf{MT}(\mathbb{Z})$ generated by the objects ${}_{\boldsymbol{x}}U^{\mathrm{Mot},N}_{\boldsymbol{y}}$ for $N \geqslant 0$ and $\boldsymbol{x},\boldsymbol{y} \in \{\mathbf{0},\mathbf{1}\}$ and let ω'_{dR} be the restriction of the fiber functor ω_{dR} to $\mathbf{MT}'(\mathbb{Z})$.

Corollary 5.121. *The map*

$$\underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MT}(\mathbb{Z})}(\omega_{\mathrm{dR}}) \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MT}'(\mathbb{Z})}(\omega'_{\mathrm{dR}}) \tag{5.122}$$

induced by the inclusion of $\mathbf{MT}'(\mathbb{Z})$ in $\mathbf{MT}(\mathbb{Z})$ is an isomorphism of affine group schemes. Therefore, this inclusion is an equivalence of tannakian categories: every mixed Tate motive over $\mathbb{Z}$ is a subquotient of a tensor construction on one of the finite-dimensional pieces of the motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0,1,\infty\}$.

Proof. The Tannaka group $\mathrm{Aut}^{\otimes}(\omega'_{\mathrm{dR}})$ is $I(U_{\mathrm{dR}}) \rtimes \mathbb{G}_m$. Thus, the fact that (5.122) is an isomorphism follows from Corollary 5.120. We deduce that both $\mathbf{MT}(\mathbb{Z})$ and $\mathbf{MT}'(\mathbb{Z})$ are equivalent to the category of finite-dimensional representations of the affine group scheme G_{dR}. □

Corollary 5.123. *The periods of every mixed Tate motive over $\mathbb{Z}$ are linear combinations with $\mathbb{Q}[\frac{1}{2\pi i}]$-coefficients of multiple zeta values. In other words, the ring of periods of mixed Tate motives over $\mathbb{Z}$ is $\mathcal{Z}[\frac{1}{2\pi i}]$.*

Proof. Recall the canonical section $\tau\colon \mathbb{G}_m \to G_{\mathrm{dR}}$ from Lemma 4.159, the element $a \in G_{\mathrm{dR}}(\mathbb{C})$ from Proposition 4.163, and the isomorphism of schemes $\psi\colon I(U_{dR}) \times \mathbb{A}^1 \to \mathcal{Y}$ from Theorem 5.20. Consider the diagram

$$\begin{tikzcd}
U_{\mathrm{dR}} \times \mathbb{G}_m \arrow[r, "f_1"] \arrow[d, "g_1"'] & G_{\mathrm{dR}} \arrow[r, "f_2"] & P_{\mathrm{dR,B}}, \arrow[d, "g_2"] \\
U_{\mathrm{dR}} \times \mathbb{A}^1 \arrow[rr, "f_3"] & & \mathcal{Y}
\end{tikzcd}$$

in which the maps are defined as follows:

$$f_1(u,s) = u\cdot\tau(s)^{-1}, \qquad g_1(u,s)=(u,s^2),$$
$$f_2(g) = g\cdot a^{-1}\cdot \mathrm{comp}_{\mathrm{dR,B}},$$
$$g_2(p) = p\cdot \mathbf{dch}, \qquad f_3(u,t)=\psi(I(u),t)({}_{\mathbf{1}}1_{\mathbf{0}}).$$

This diagram is commutative by the definition of ψ and Lemma 5.16. The upper horizontal arrows are clearly isomorphisms, and the lower horizontal arrow is an isomorphism by Theorem 5.20 and Corollary 5.120.

Let us also recall from (5.18) that a can be written as $a = u_0\cdot\tau(2\pi i)^{-1}$ for some $u_0\in U_{\mathrm{dR}}(\mathbb{R})$, so that $f_1(u_0,2\pi i)=a$. Clearly, the equalities

$$f_2(a)=\mathrm{comp}_{\mathrm{dR,B}},\quad g_2(\mathrm{comp}_{\mathrm{dR,B}})=\mathbf{dch}^{\mathrm{dR}},\quad g_1(u_0,2\pi i)=(u_0,(2\pi i)^2)$$

hold. By the commutativity of the diagram, we get $f_3(u_0,(2\pi i)^2)=\mathbf{dch}^{\mathrm{dR}}$. All the morphisms on the diagram are defined over $\mathbb{Q}$.

The algebra of periods of $\mathbf{MT}(\mathbb{Z})$ is

$$\mathrm{ev}_{\mathrm{comp}_{\mathrm{dR,B}}}(\mathcal{O}(P_{\mathrm{dR,B}}))=\mathrm{ev}_{(u_0,2\pi i)}(\mathcal{O}(U_{\mathrm{dR}}\times\mathbb{G}_m)).$$

The algebra of multiple zeta values is

$$\mathrm{ev}_{\mathbf{dch}^{\mathrm{dR}}}(\mathcal{O}(\mathcal{Y}))=\mathrm{ev}_{(u_0,(2\pi i)^2)}(\mathcal{O}(U_{\mathrm{dR}}\times\mathbb{A}^1)).$$

Finally, the result follows from the fact that g_1 induces an isomorphism

$$\mathcal{O}(U_{\mathrm{dR}}\times\mathbb{G}_m)\simeq\mathcal{O}(U_{\mathrm{dR}}\times\mathbb{A}^1)[s^{-1}],$$

where s is the coordinate of $\mathbb{G}_m$, so that $s(u_0,2\pi i)=2\pi i$ holds. ☐

COROLLARY 5.124. *Zagier's conjecture 1.71 implies that the numbers*

$$\pi,\zeta(3),\zeta(5),\ldots$$

are algebraically independent over $\mathbb{Q}$.

PROOF. The key ingredient is a structure theorem for Hopf algebras due to Milnor and Moore [**MM65**] (see also [**Car07**, Thm. 8.1.3] and its proof).

Let V be a vector space over a field of characteristic zero. The symmetric algebra $\mathrm{Sym}(V)$ is the quotient of the tensor algebra $T(V)=\bigoplus_{n\geqslant 0}V^{\otimes n}$ by the bilateral ideal generated by the elements of the form $x\otimes y - y\otimes x$. This is a commutative algebra. Moreover, if S is a basis of V, then $\mathrm{Sym}(V)$ is the free commutative algebra generated by S. Said differently, $\mathrm{Sym}(V)$ is the polynomial algebra in a basis of V. If V is graded with strictly positive degrees, then $\mathrm{Sym}(V)$ has the structure of a connected graded algebra.

THEOREM 5.125 (Milnor–Moore). *Let k be a field of characteristic zero and let $A=\bigoplus_{n\geqslant 0}A_n$ be a graded connected Hopf algebra over k with commutative multiplication, such that A_n is finite-dimensional for all n. Then there is an isomorphism of graded algebras*

$$A=\mathrm{Sym}[A_{>0}/(A_{>0})^2].$$

We will use the theorem through the following straightforward consequence.

COROLLARY 5.126. *If the classes of $x_1,x_2,\ldots\in A_{>0}$ in $A_{>0}/(A_{>0})^2$ are linearly independent, then $x_1,x_2,\ldots$ are algebraically independent.*

We apply this corollary to the Hopf algebra $\mathcal{A} = \mathcal{O}(U_{\mathrm{dR}})$ and the motivic zeta values $\zeta^{\mathfrak{m}}(3)$, $\zeta^{\mathfrak{m}}(5)$, ... These elements lie in different degrees and their images in the quotient $\mathcal{L} = \mathcal{A}_{>0}/(\mathcal{A}_{>0})^2$ are non-zero, so they are linearly independent. Hence, $\zeta^{\mathfrak{m}}(3)$, $\zeta^{\mathfrak{m}}(5)$, ... are algebraically independent in $\mathcal{A}$. From $\mathcal{H} = \mathcal{A}[\zeta^{\mathfrak{m}}(2)]$, we deduce that the motivic zeta values $\zeta^{\mathfrak{m}}(2)$, $\zeta^{\mathfrak{m}}(3)$, $\zeta^{\mathfrak{m}}(5)$, ... are algebraically independent in $\mathcal{H}$. Now, if one assumes Zagier's conjecture, the map $\mathrm{per}\colon \mathcal{H} \to \mathcal{Z}$ is an isomorphism. Since $\mathrm{per}(\zeta^{\mathfrak{m}}(n)) = \zeta(n)$ and $\zeta(2) = \frac{\pi^2}{6}$, it follows that the numbers $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$. $\square$

Corollary 5.127. *Zagier's conjecture 1.71 is equivalent to Grothendieck's period conjecture for mixed Tate motives 4.170.*

Proof. Zagier's conjecture is equivalent to the injectivity of $\mathrm{per}\colon \mathcal{H} \to \mathbb{C}$. Since $\mathcal{O}(P_{\mathrm{dR,B}}) = \mathcal{H}[s^{-1}]$ with $s^2 = -24\zeta^{\mathfrak{m}}(2)$, this is equivalent to the injectivity of the period map $\mathrm{per}\colon \mathcal{O}(P_{\mathrm{dR,B}}) \to \mathbb{C}$, which is precisely the content of Grothendieck's period conjecture for mixed Tate motives (Conjecture 4.170). $\square$

Appendix A. Some results from homological algebra

In this appendix, we gather some notions and results from category theory, homological algebra, and sheaf theory that are used through the main text. We assume that the reader is familiar with the definitions of category, functor, and natural transformation between functors. Unless otherwise specified, by a functor we mean a covariant functor. A category is called *small* if its objects and morphisms form a set, and *essentially small* if it is equivalent to a small category. All the categories we will work with are essentially small; in order to avoid set-theoretic issues, we will always replace such categories with small equivalent ones. Some standard references for the material we cover without proof are the books by Kashiwara and Schapira [**KS06**], Gelfand and Manin [**GM03**], and Weibel [**Wei94**].

A.1. Abelian categories, complexes, and cohomology.

A.1.1. *The definition of an abelian category.*

Definition A.1. An *additive* category is a category $\mathcal{A}$ in which morphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, Y)$$

are endowed, for all objects $X, Y \in \mathrm{Ob}(\mathcal{A})$, with the structure of an abelian group (in particular, with a group law $+$ that we call *addition* and a zero morphism 0) such that the following conditions hold:

i) Composition of morphisms is distributive with respect to addition:

$$f \circ (g + h) = f \circ g + f \circ h,$$
$$(g + h) \circ f = g \circ f + h \circ g.$$

ii) There exists a *zero object*, that is, an object $0 \in \mathrm{Ob}(\mathcal{A})$ such that there exist unique morphisms $0 \to X$ and $X \to 0$ for every object $X \in \mathrm{Ob}(\mathcal{A})$. The zero morphism $0 \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ is the composition $X \to 0 \to Y$.

iii) Given objects $X, Y \in \mathrm{Ob}(\mathcal{A})$, there exists a *direct sum* object (also known as a *coproduct*; see Exercise A.49), *i.e.* an object

$$X \oplus Y \in \mathrm{Ob}(\mathcal{A})$$

together with morphisms

$$X \longrightarrow X \oplus Y \longleftarrow Y$$

satisfying the following universal property: for each object $Z \in \mathrm{Ob}(\mathcal{A})$ with morphisms $X \to Z$ and $Y \to Z$, there exists a unique morphism

$$X \oplus Y \longrightarrow Z$$

such that the diagram below commutes:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X \oplus Y \arrow[d] & Y. \arrow[l] \arrow[dl] \\ & Z & \end{tikzcd}$$

In the literature, the word *preadditive* sometimes refers to a category in which the morphisms are abelian groups and condition i) above holds.

Definition A.2. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *additive* if it sends a zero object of $\mathcal{A}$ to a zero object of $\mathcal{B}$ and, for all objects $X, Y \in \mathrm{Ob}(\mathcal{A})$, the morphism

$$F(X) \oplus F(Y) \longrightarrow F(X \oplus Y)$$

given by the universal property of the direct sum (Definition A.1 iii)) applied to the objects $F(X), F(Y), F(X \oplus Y) \in \mathrm{Ob}(\mathcal{B})$ is an isomorphism.

An equivalent definition of additive functors is presented in Exercise A.50.

Definition A.3. Let $f\colon X \to Y$ be a morphism in an additive category $\mathcal{A}$.

i) A *kernel* of f is a pair consisting of an object $\mathrm{Ker}(f) \in \mathrm{Ob}(\mathcal{A})$ and a morphism $\iota\colon \mathrm{Ker}(f) \to X$ that satisfies

$$f \circ \iota = 0$$

and is universal for this property. That is, for each morphism $g\colon Z \to X$ with $f \circ g = 0$, there is a unique morphism $\varphi\colon Z \to \mathrm{Ker}(f)$ making the following diagram commutative:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "\varphi"'] & X. \\ \mathrm{Ker}(f) \arrow[ur, "\iota"'] & \end{tikzcd}$$

ii) A *cokernel* of f is a pair consisting of an object $\mathrm{Coker}(f) \in \mathrm{Ob}(\mathcal{A})$ and a morphism $p\colon Y \to \mathrm{Coker}(f)$ that satisfies

$$p \circ f = 0$$

and is universal for this property. That is, for each morphism $g\colon Y \to Z$ with $g \circ f = 0$, there is a unique morphism $\varphi\colon \mathrm{Coker}(f) \to Z$ making the following diagram commutative:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "p"'] & Z. \\ \mathrm{Coker}(f) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

These notions are dual to each other (Exercise A.51). The kernel and the cokernel of a morphism may or may not exist, as Exercise A.52 illustrates. Whenever they do, $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are not unique but unique up to a unique isomorphism, as is the case for all objects defined by means of a universal property. In practice, we will identify all possible choices through the unique isomorphisms and pretend that $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are unique. For simplicity, $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ will denote both the objects and the morphisms to X and from Y respectively.

Definition A.4. Let $f\colon X \to Y$ be a morphism in an additive category $\mathcal{A}$. Whenever they exist, the *image* and the *coimage* of f are defined as

$$\mathrm{Im}(f) = \mathrm{Ker}(\mathrm{Coker}(f)), \quad \mathrm{Coim}(f) = \mathrm{Coker}(\mathrm{Ker}(f)).$$

Thanks to the universal property of the kernel and the cokernel (Definition A.3), there is a canonical morphism (Exercise A.53)

$$\mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f).$$

DEFINITION A.5. An *abelian category* is an additive category $\mathcal{A}$ satisfying the following two conditions:

i) Every morphism f in $\mathcal{A}$ has a kernel and a cokernel. Therefore, every morphism has an image and a coimage.
ii) For every morphism f in $\mathcal{A}$, the map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

See Exercise A.214 for a typical example of an additive category that has all kernels and cokernels but is not abelian since condition ii) fails.

A.1.2. *Categories of modules.* The category $\mathbf{Mod}_R$ of left modules over a ring R is abelian. The kernel and the image of a morphism $f\colon A \to B$ are given by

$$\mathrm{Ker}(f) = \{a \in A \mid f(a) = 0\}, \quad \mathrm{Im}(f) = \{b \in B \mid b = f(a) \text{ for some } a \in A\},$$

as submodules of A and B respectively, and the cokernel and the coimage by

$$\mathrm{Coker}(f) = B/\,\mathrm{Im}(f), \quad \mathrm{Coim}(f) = A/\ker(f)$$

together with the quotient module structure. That the map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism is the content of the first isomorphism theorem of R-modules. In particular, the category $\mathbf{Ab} = \mathbf{Mod}_{\mathbb{Z}}$ of abelian groups is abelian.

Conversely, the following result allows us to work in any small abelian category as if it were the category of modules over a ring. It is particularly useful for performing operations such as "diagram chasing" that require picking elements of the objects of the category, and hence do not make sense if they are not sets.

THEOREM A.6 (Freyd–Mitchell). *Let $\mathcal{A}$ be a small abelian category. There exists a ring R and an exact and fully faithful functor from $\mathcal{A}$ to the category of left R-modules. In particular, $\mathcal{A}$ can be viewed as a full subcategory of $\mathbf{Mod}_R$.*

Let us explain the terms appearing in the statement. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is said to be *faithful* (resp. *full*, resp. *fully faithful*) if the map

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{B}}\big(F(X), F(Y)\big)$$

is injective (resp. surjective, resp. bijective) for all objects $X, Y \in \mathrm{Ob}(\mathcal{A})$. For example, given a group G, the functor $\mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Vec}_{\mathbb{Q}}$ from the category of finite-dimensional $\mathbb{Q}$-linear representations of G to that of vector spaces that forgets the action of G is faithful but not full in general, since there are linear maps between the underlying vector spaces of two representations that are not G-equivariant.

A subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is called *full* if the inclusion functor is full, and hence fully faithful; that is, if all morphisms in $\mathcal{B}$ between objects of $\mathcal{A}$ are already morphisms in $\mathcal{A}$. The notion of exact functor will be introduced in Definition A.22 below. The Freyd–Mitchell theorem is proved, for example, in [**Wei94**, Thm. 1.6.1].

A.1.3. *Subobjects and quotients.* Let $\mathcal{A}$ be a category.

DEFINITION A.7. A morphism $f\colon X \to Y$ in $\mathcal{A}$ is called:

i) a *monomorphism* if the equality $f \circ g = f \circ h$ implies $g = h$ for all morphisms $g, h\colon Z \to X$;
ii) an *epimorphism* if the equality $g \circ f = h \circ f$ implies $g = h$ for all morphisms $g, h\colon Y \to Z$.

In the category of sets, monomorphisms and epimorphisms are injective and surjective maps respectively. This intuition fails for other categories. For example, the inclusion $\mathbb{Z} \to \mathbb{Q}$ in the category of rings or the inclusion of a dense open subset in the category of topological spaces are epimorphisms, and the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ in the category of divisible groups is a monomorphism (see Exercise A.55).

In an abelian category, kernels are monomorphisms and cokernels are epimorphisms; more generally, f is a monomorphism if and only if $\mathrm{Ker}(f)$ is a zero object, and f is an epimorphism if and only if $\mathrm{Coker}(f)$ is a zero object (see Exercise A.54).

Definition A.8. Let $X \in \mathrm{Ob}(\mathcal{A})$ be an object of a category $\mathcal{A}$.

i) A *subobject* of X is an object $Y \in \mathrm{Ob}(\mathcal{A})$ with a monomorphism $Y \to X$.
ii) A *quotient* of X is an object $Z \in \mathrm{Ob}(\mathcal{A})$ with an epimorphism $X \to Z$.
iii) A *subquotient* of X is a quotient of a subobject of X.

Usually, we will denote subobjects by $Y \subset X$. If the category $\mathcal{A}$ is abelian, then to each subobject $Y \subset X$ corresponds a quotient object X/Y defined as the cokernel of the monomorphism $Y \to X$. Moreover, a subquotient is also a subobject of a quotient, and the property of being a subquotient is transitive (Exercise A.58).

Example A.9. In an additive category $\mathcal{A}$, the objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ are both subobjects and quotients of the direct sum object $X \oplus Y$. Indeed, applied to $Z = X$, the identity map $X \to X$ and the zero map $Y \to X$, the universal property of the direct sum (Definition A.1 iii)) gives rise to morphisms $X \oplus Y \to X$ and $X \oplus Y \to Y$ such that the composition with $X \to X \oplus Y$ are the identity and the zero map respectively; from this it follows that $X \to X \oplus Y$ is a monomorphism. The associated quotient object is Y.

A.1.4. *Adjoint functors.* Many universal constructions are more conveniently phrased using the notion of an adjoint pair of functors.

Definition A.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ functors. We say that F is *left adjoint* to G (or that G is *right adjoint* to F) if, for all objects $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$, there is a natural bijection

$$\Phi_{X,Y}\colon\ \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(X), Y).$$

By "natural", we mean that these bijections give an isomorphism of bifunctors

$$\Phi_{\cdot,\cdot}\colon\ \mathrm{Hom}_{\mathcal{C}}(\cdot, G(\cdot)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(\cdot), \cdot).$$

That is, for all morphisms $f\colon X' \to X$ in $\mathcal{C}$ and $g\colon Y \to Y'$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\mathrm{Hom}(f,G(g))"'] & \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \arrow[d, "\mathrm{Hom}(F(f),g)"] \\
\mathrm{Hom}_{\mathcal{C}}(X', G(Y')) \arrow[r, "\Phi_{X',Y'}"'] & \mathrm{Hom}_{\mathcal{D}}(F(X'), Y')
\end{tikzcd}$$

commutes. In this diagram, the map $\mathrm{Hom}(f, G(g))$ sends $\varphi\colon X \to G(Y)$ to the morphism $G(g) \circ \varphi \circ f\colon X' \to G(Y')$, and similarly for $\mathrm{Hom}(F(f), g)$.

If F is left adjoint to G, then we say that (F, G) is an *adjoint pair*. Note that the adjectives "left" and "right" refer to the position inside each Hom set in the bijection Φ, which is the same as in the adjoint pair.

Given an adjoint par (F, G), there are natural transformations of functors

$$\varepsilon\colon F \circ G \longrightarrow \mathrm{Id}_{\mathcal{D}}, \qquad \eta\colon\ \mathrm{Id}_{\mathcal{C}} \longrightarrow G \circ F \tag{A.11}$$

called *adjunction morphisms*. The natural transformation $\varepsilon_Y \colon F(G(Y)) \to Y$ is the image of the identity morphism $\mathrm{Id}_{G(Y)}$ under the bijection

$$\Phi_{G(Y),Y} \colon \mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(G(Y)), Y),$$

and $\eta_X \colon X \to G(F(X))$ is the preimage of $\mathrm{Id}_{F(X)}$ under the bijection

$$\Phi_{X,F(X)} \colon \mathrm{Hom}_{\mathcal{C}}(X, G(F(X))) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(X)).$$

The adjunction morphisms satisfy that the compositions

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F \quad \text{and} \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

are the identity transformations of F and G. In fact, the existence of adjunction morphisms with this property amounts to saying that (F, G) is an adjoint pair.

Let (F, G) be an adjoint pair. For each $Y \in \mathrm{Ob}(\mathcal{D})$, the object $G(Y)$ satisfies the following universal property. There exists a morphism $\varepsilon_Y \colon F(G(Y)) \to Y$ and, for every object $X \in \mathrm{Ob}(\mathcal{C})$ along with a morphism $f \colon F(X) \to Y$, there exists a unique morphism $g \colon X \to G(Y)$ such that the diagram

$$\begin{tikzcd} F(X) \arrow[d, dashed, "F(g)"'] \arrow[dr, "f"] & \\ F(G(Y)) \arrow[r, "\varepsilon_Y"'] & Y \end{tikzcd}$$

is commutative. Similarly, for each $X \in \mathrm{Ob}(\mathcal{C})$, the object $F(X)$ satisfies the dual universal property. That is, there exists a morphism $\eta_X \colon X \to G(F(X))$ and for every object $Y \in \mathrm{Ob}(\mathcal{D})$ along with a morphism $g \colon X \to G(Y)$, there is a unique morphism $f \colon F(X) \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "\eta_X"'] \arrow[dr, "g"] & \\ G(F(X)) \arrow[r, dashed, "G(f)"'] & G(Y). \end{tikzcd}$$

Example A.12. We illustrate the relationship between adjoint pairs and universal properties with the example of vector spaces. Let k be a field and $\mathbf{Vec}_{\mathrm{k}}^{\infty}$ the category of k-vector spaces. There are two functors

$$\begin{tikzcd} \mathbf{Vec}_{\mathrm{k}}^{\infty} \arrow[r, shift left, "\mathrm{For}"] & \mathbf{Set} \arrow[l, shift left, "\mathrm{Free}"] \end{tikzcd}.$$

The functor For sends a vector space to its underlying set, forgetting the vector space structure. The functor Free sends a set S to the vector space with basis S, that is, the space of finite formal sums

$$\mathrm{Free}(S) = \{\sum_{s \in S} a_s[s] \mid a_s = 0 \text{ for almost all } s\}.$$

Clearly, there is a bijection

$$\mathrm{Hom}_{\mathbf{Set}}(S, \mathrm{For}(V)) = \mathrm{Hom}_{\mathbf{Vec}_{\mathrm{k}}^{\infty}}(\mathrm{Free}(S), V),$$

so that Free is left adjoint to For. Moreover, $\mathrm{Free}(S)$ satisfies the following universal property. There exists a map $\eta_S \colon S \to \mathrm{For}(\mathrm{Free}(S))$ that sends the element s to $1[s]$, and for every vector space V together with a map $g\colon S \to \mathrm{For}(V)$, there exists a unique k-linear map $f\colon \mathrm{Free}(S) \to V$ extending g. That is, $g = \mathrm{For}(f) \circ \eta_S$.

A.1.5. *Complexes and cohomology.*

DEFINITION A.13. Let $\mathcal{A}$ be an additive category.

i) A *cochain complex* (or simply a *complex*) $A = (A^*, \mathrm{d}^*)$ is a sequence of objects $A^n \in \mathrm{Ob}(\mathcal{A})$ and morphisms

$$\cdots \longrightarrow A^{n-1} \xrightarrow{\mathrm{d}^{n-1}} A^n \xrightarrow{\mathrm{d}^n} A^{n+1} \longrightarrow \cdots$$

called *differentials* such that the equality

$$\mathrm{d}^n \circ \mathrm{d}^{n-1} = 0 \tag{A.14}$$

holds for all $n \in \mathbb{Z}$.

ii) A *morphism of cochain complexes* $f\colon (A^*, \mathrm{d}^*) \to (B^*, \mathrm{d}^*)$ is a sequence of morphisms $f^n\colon A^n \to B^n$ commuting with the differentials, *i.e.* satisfying

$$f^n \circ \mathrm{d}^{n-1} = \mathrm{d}^{n-1} \circ f^{n-1}$$

for each $n \in \mathbb{Z}$. We picture it as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "\mathrm{d}^{n-1}"] \arrow[d, "f^{n-1}"'] & A^n \arrow[r, "\mathrm{d}^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"'] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "\mathrm{d}^{n-1}"] & B^n \arrow[r, "\mathrm{d}^n"] & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

iii) For a cochain complex $A = (A^*, \mathrm{d}^*)$ and an integer r, the *shifted* complex

$$A[r] = (A[r]^*, \mathrm{d}[r]^*)$$

is the cochain complex with

$$A[r]^n = A^{n+r} \quad \text{and} \quad \mathrm{d}[r] = (-1)^r \mathrm{d}.$$

(One reason why changing the sign of the differential is convenient will be explained when discussing the cone in Definition A.25 below.)

iv) A cochain complex is called *bounded* if there exists an integer M such that $A^n = 0$ holds for all $|n| \geqslant M$. Similarly, one defines the notion of *bounded below* and *bounded above* cochain complex.

v) The notion of *chain complex* in an additive category is dual to the notion of cochain complex; that is, the differentials lower the degree instead of increasing it. Therefore, a chain complex (A_*, d_*) is a sequence of objects $A_n \in \mathrm{Ob}(\mathcal{A})$ and morphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\mathrm{d}_{n+1}} A_n \xrightarrow{\mathrm{d}_n} A_{n-1} \longrightarrow \cdots$$

also called *differentials* such that the equality

$$\mathrm{d}_n \circ \mathrm{d}_{n+1} = 0$$

holds for all $n \in \mathbb{Z}$. A *morphism of chain complexes* $f\colon (A_*, d_*) \to (B_*, d_*)$ is a sequence of morphisms $f_n\colon A_n \to B_n$ satisfying

$$f_{n-1} \circ \mathrm{d}_n = \mathrm{d}_n \circ f_n.$$

vi) A chain complex (A_*, d_*) may be turned into a cochain complex (A^*, d^*) by setting

$$A^n = A_{-n} \quad \text{and} \quad \mathrm{d}^n = \mathrm{d}_{-n}. \tag{A.15}$$

The position of the index will usually be enough to indicate that we have performed this operation. If needed, "raising the index" will be denoted by r, so that the first equality in (A.15) becomes $\mathrm{r}(A)^n = A_{-n}$.

vii) For a chain complex $A = (A_*, \mathrm{d}_*)$ and an integer r, the shifted complex

$$A[r] = (A[r]_*, \mathrm{d}[r]_*)$$

is the chain complex with

$$A[r]_n = A_{n-r} \quad \text{and} \quad \mathrm{d}[r] = (-1)^r \mathrm{d}.$$

Hence, the raising and the shift operators are related by the equality

$$\mathrm{r}(A[r]) = \mathrm{r}(A)[r].$$

Since we will mainly use cochain complexes, we will often simply call them "complexes". It is also convenient to think of a complex as a graded object

$$A^* = \bigoplus_{n \in \mathbb{Z}} A^n$$

in the category $\mathcal{A}$ together with a morphism $\mathrm{d}\colon A^* \to A^*$ that is homogeneous of degree 1, *i.e.* maps the subobject $A^n \subset A^*$ to the subobject $A^{n+1} \subset A^*$ for all integers $n \in \mathbb{Z}$. With this convention, the morphisms

$$\mathrm{d}^n\colon A^n \to A^{n+1}$$

are the restrictions of d to the various A^n, and condition (A.14) simply reads

$$\mathrm{d} \circ \mathrm{d} = 0.$$

For this to make sense, we need to assume either that the complex A^* is bounded or that the abelian category admits *infinite sums*. This is the obvious generalization of the direct sum of two objects from Definition A.1: given a sequence $(X_n)_{n \in \mathbb{Z}}$ of objects of $\mathcal{A}$, there exists $\bigoplus_{n \in \mathbb{Z}} X_n \in \mathrm{Ob}(\mathcal{A})$ and morphisms $f_i\colon X_i \to \bigoplus_{n \in \mathbb{Z}} X_n$ for all $i \in \mathbb{Z}$ such that, for each object Z along with morphisms $g_i\colon X_i \to Z$, there is a unique morphism $h\colon \bigoplus_{n \in \mathbb{Z}} X_n \to Z$ satisfying $h \circ f_i = g_i$.

REMARK A.16. If $\mathcal{A}$ is an abelian category, then condition (A.14) can be rephrased by saying that the morphism $\mathrm{Im}(\mathrm{d}^{n-1}) \to A^n$ factors uniquely through the morphism $\mathrm{Ker}(\mathrm{d}^n) \to A^n$ for each integer n. That is, there exists a unique morphism φ_n making the diagram

$$\begin{tikzcd} \mathrm{Im}(\mathrm{d}^{n-1}) \arrow[r] \arrow[d, "\varphi_n"'] & A^n \\ \mathrm{Ker}(\mathrm{d}^n) \arrow[ur] & \end{tikzcd}$$

commutative. Moreover, φ_n is a monomorphism, so that $\mathrm{Ker}(\varphi_n)$ is a zero object and $\mathrm{Im}(\mathrm{d}^{n-1})$ is a subobject of $\mathrm{Ker}(\mathrm{d}^n)$ (use Exercise A.57).

DEFINITION A.17. Let $\mathcal{A}$ be an abelian category, and let $A = (A^*, \mathrm{d})$ be a cochain complex in $\mathcal{A}$. For each $n \in \mathbb{Z}$, the cohomology in degree n of A is the object

$$\mathrm{H}^n(A) = \mathrm{Coker}(\varphi_n) \in \mathrm{Ob}(\mathcal{A}).$$

If the category $\mathcal{A}$ admits infinite sums or the complex A is bounded, then the *total cohomology* of A is defined as the graded object

$$\mathrm{H}^*(A) = \bigoplus_{n \in \mathbb{Z}} \mathrm{H}^n(A).$$

The *homology* of a chain complex is defined similarly.

By construction, the cohomology in degree n of the shifted complex $A[r]$ is the cohomology in degree $n + r$ of A. In symbols,

$$\mathrm{H}^n(A[r]) = \mathrm{H}^{n+r}(A).$$

EXAMPLE A.18. Let $\mathcal{A} = \mathbf{Mod}_R$ be the abelian category of left modules over a ring R. Let $A = (A^*, \mathrm{d})$ be a complex in $\mathcal{A}$. In this case, the maps φ_n are the inclusions of the submodule $\mathrm{Im}(\mathrm{d}^{n-1})$ into $\mathrm{Ker}(\mathrm{d}^n)$, and the cohomology objects are the quotient modules

$$\mathrm{H}^n(A) = \mathrm{Ker}(\mathrm{d}^n)/\,\mathrm{Im}(\mathrm{d}^{n-1}).$$

This setting (with $R = \mathbb{Z}$ or a field) will be mostly sufficient for our purposes.

Formation of cohomology is functorial: a morphism of complexes $f\colon A^* \to B^*$ induces a morphism of cohomology objects

$$\mathrm{H}(f)\colon\ \mathrm{H}^*(A^*) \longrightarrow \mathrm{H}^*(B^*)$$

that is homogeneous of degree 0 and satisfies

$$\mathrm{H}(\mathrm{Id}) = \mathrm{Id} \quad \text{and} \quad \mathrm{H}(f \circ g) = H(f) \circ H(g)$$

for composable morphisms f and g. In other words, $A \mapsto \mathrm{H}^*(A)$ defines a functor from the category of complexes of $\mathcal{A}$ to the category of graded objects of $\mathcal{A}$.

DEFINITION A.19. A morphism of complexes f is called a *quasi-isomorphism* if the induced morphism on cohomology objects $\mathrm{H}(f)$ is an isomorphism.

A.1.6. *Categories of complexes.* Given an additive category $\mathcal{A}$, we will denote by $\mathcal{C}(\mathcal{A})$ the category whose objects are cochain complexes and whose morphisms are morphisms of complexes. When we want to stress the fact that we are working with cochain complexes, we will write $\mathcal{C}^*(\mathcal{A})$ instead. We also denote by

$$\mathcal{C}^+(\mathcal{A}),\ \mathcal{C}^-(\mathcal{A}),\ \mathcal{C}^b(\mathcal{A}),\ \mathcal{C}^{\geqslant 0}(\mathcal{A})$$

the full subcategories of bounded below complexes, bounded above complexes, bounded complexes, and complexes concentrated in non-negative degrees.

Similarly, we will denote by $\mathcal{C}_*(\mathcal{A})$, $\mathcal{C}_-(\mathcal{A})$, $\mathcal{C}_+(\mathcal{A})$ $\mathcal{C}_b(\mathcal{A})$, and $\mathcal{C}_{\geqslant 0}(\mathcal{A})$ the corresponding categories of chain complexes.

PROPOSITION A.20. *If $\mathcal{A}$ is abelian, then all these categories of complexes are abelian as well.*

Proof. The categories $\mathcal{C}^?(\mathcal{A})$ and $\mathcal{C}_?(\mathcal{A})$ are clearly additive. We need to show the existence of kernels and cokernels, and that the image and the coimage of morphisms agree. To fix ideas, we do this for the category $\mathcal{C}^*(\mathcal{A})$. Let $f\colon A^* \to B^*$ be a morphism of complexes. Then one can check that the complexes

$$\operatorname{Ker}(f)^n = \operatorname{Ker}(f^n), \qquad \operatorname{Coker}(f)^n = \operatorname{Coker}(f^n)$$

with the induced differentials satisfy the universal property of a kernel and a cokernel for f in the category $\mathcal{C}^*(\mathcal{A})$ (Exercise A.56). Hence, kernels and cokernels exist. Since they can be computed component-wise, the same is true for the image and the coimage. Since $\mathcal{A}$ is abelian, we deduce the identity $\operatorname{Im}(f)^n = \operatorname{Coim}(f)^n$ for all $n \in \mathbb{Z}$. Since f is a morphism of complexes, and hence commutes with the differential, the differentials induced on $\operatorname{Im}(f)$ and $\operatorname{Coim}(f)$ agree. □

A.1.7. *Exact sequences and exact functors.*

Definition A.21. Let $\mathcal{A}$ be an abelian category. An *exact sequence* is a complex $A = (A^*, \mathrm{d})$ in $\mathcal{A}$ with vanishing cohomology. In other words, A is an exact sequence if the maps $\varphi_n\colon \operatorname{Im}(\mathrm{d}^{n-1}) \to \operatorname{Ker}(\mathrm{d}^n)$ are isomorphisms for all n. A *short exact sequence* is an exact sequence in which all but three consecutive terms are zero. We will often call A a *long exact sequence* when we want to emphasize that it is not a short exact sequence.

Definition A.22. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. We say that F is *exact* if the sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

in $\mathcal{B}$ is exact for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$.

Many widely used functors are not exact because either $F(A) \to F(B)$ fails to be a monomorphism or $F(B) \to F(C)$ fails to be an epimorphism. This motivates the introduction of the following weaker definitions:

Definition A.23. The functor $F\colon \mathcal{A} \to \mathcal{B}$ is *right exact* if, for all short exact sequences as above, the sequence

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is exact. Similarly, F is *left exact* if, for all exact sequences as above, the sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is exact. There are analogous definitions for contravariant functors.

Remark A.24. If a functor F admits a right adjoint, then F is right exact, and similarly for left adjoints and left exactness (see Exercise A.60).

A.1.8. *The cone of a morphism of complexes.* Another important tool of homological algebra is the *cone* of a morphism of complexes. For the sake of readability, we define it by picking elements of the objects in the complexes (recall that this is justified by the Freyd–Mitchell theorem A.6). The zealous reader will have no trouble in replacing all morphisms below with their abstract definitions.

Definition A.25. Let $\mathcal{A}$ be an additive category, and let $f\colon A^* \to B^*$ be a morphism of complexes in $\mathcal{A}$. The *cone* of f is the complex defined as

$$\operatorname{cone}(f)^n = A^{n+1} \oplus B^n,$$

together with the differential

$$\mathrm{d}(a,b) = (-\mathrm{d}a, \mathrm{d}b + f(a)).$$

By construction, the cone is equipped with two morphisms of complexes

$$\begin{aligned} &\mathrm{b}\colon B^* \longrightarrow \mathrm{cone}(f), && \mathrm{b}(b) = (0,b) \\ &\mathrm{a}\colon \mathrm{cone}(f) \longrightarrow A[1]^*, && \mathrm{a}(a,b) = a, \end{aligned} \tag{A.26}$$

that induce a long exact sequence of cohomology objects

$$\cdots \longrightarrow \mathrm{H}^n(A^*) \xrightarrow{\mathrm{H}(f)} \mathrm{H}^n(B^*) \xrightarrow{\mathrm{H(b)}} \mathrm{H}^n(\mathrm{cone}(f)) \xrightarrow{\mathrm{H(a)}} \mathrm{H}^{n+1}(A^*) \longrightarrow \cdots \tag{A.27}$$

taking the equality $\mathrm{H}^n(A[1]^*) = \mathrm{H}^{n+1}(A^*)$ into account.

Remark A.28. Dually, the cone of a morphism of chain complexes $f\colon A_* \to B_*$ is defined as follows. The raising operator from Definition A.13 vi) yields a morphism of cochain complexes $\mathrm{r}(f)\colon \mathrm{r}(A) \to \mathrm{r}(B)$, and $\mathrm{cone}(f)$ is determined by

$$\mathrm{r}(\mathrm{cone}(f)) = \mathrm{cone}(\mathrm{r}(f)). \tag{A.29}$$

Concretely, $\mathrm{cone}(f)$ is the chain complex given by

$$\mathrm{cone}(f)_n = A_{n-1} \oplus B_n \quad \text{and} \quad \mathrm{d}(a,b) = (-\mathrm{d}a, \mathrm{d}b + f(a)).$$

It is also equipped with two morphisms of chain complexes

$$\begin{aligned} &\mathrm{b}\colon B_* \longrightarrow \mathrm{cone}(f), && \mathrm{b}(b) = (0,b), \\ &\mathrm{a}\colon \mathrm{cone}(f) \longrightarrow A[-1]_*, && \mathrm{a}(a,b) = a, \end{aligned} \tag{A.30}$$

that induce a long exact sequence of homology objects

$$\cdots \longrightarrow \mathrm{H}_n(A_*) \xrightarrow{\mathrm{H}(f)} \mathrm{H}_n(B_*) \xrightarrow{\mathrm{H(b)}} \mathrm{H}_n(\mathrm{cone}(f)) \xrightarrow{\mathrm{H(a)}} \mathrm{H}_{n-1}(A_*) \longrightarrow \cdots \tag{A.31}$$

taking the equality $\mathrm{H}_n(A[-1]_*) = \mathrm{H}_{n-1}(A_*)$ into account.

Remark A.32. A choice of sign in the differential of the cone is needed for the equality $\mathrm{d} \circ \mathrm{d} = 0$ to hold. With the current choice, which appears to be the most standard, a is only a morphism of complexes if the differential of $A[1]^*$ carries a minus sign. With the choice $\mathrm{d}(a,b) = (\mathrm{d}a, -\mathrm{d}b + f(a))$, the map b would not be a morphism of complexes. Finally, the choice $\mathrm{d}(a,b) = (\mathrm{d}a, \mathrm{d}b + (-1)^n f(a))$ would make both a and b morphisms of complexes without changing the sign in the shifted complex, but at the cost of changing the sign of f.

A.1.9. *Double complexes.*

Definition A.33. Let $\mathcal{A}$ be an additive category. A *double complex*

$$C = (C^{*,*}, \mathrm{d}^{\mathrm{hor}}, \mathrm{d}^{\mathrm{ver}})$$

is a collection of objects $C^{p,q} \in \mathrm{Ob}(\mathcal{A})$, one for each $(p,q) \in \mathbb{Z}^2$, and morphisms

$$\mathrm{d}^{\mathrm{hor}}\colon C^{p,q} \longrightarrow C^{p+1,q}, \quad \mathrm{d}^{\mathrm{ver}}\colon C^{p,q} \longrightarrow C^{p,q+1}$$

called the *horizontal* and the *vertical* differentials satisfying

$$(\mathrm{d}^{\mathrm{hor}})^2 = 0, \quad (\mathrm{d}^{\mathrm{ver}})^2 = 0, \quad \mathrm{d}^{\mathrm{hor}} \circ \mathrm{d}^{\mathrm{ver}} = \mathrm{d}^{ver} \circ \mathrm{d}^{\mathrm{hor}}.$$

In other words, $(C^{*,q}, \mathrm{d}^{\mathrm{hor}})$ and $(C^{p,*}, \mathrm{d}^{\mathrm{ver}})$ are cochain complexes for each fixed p and q, and all the diagrams

$$\begin{tikzcd} C^{p,q+1} \arrow[r, "\mathrm{d}^{\mathrm{hor}}"] & C^{p+1,q+1} \\ C^{p,q} \arrow[u, "\mathrm{d}^{\mathrm{ver}}"] \arrow[r, "\mathrm{d}^{\mathrm{hor}}"] & C^{p+1,q} \arrow[u, "\mathrm{d}^{\mathrm{ver}}"'] \end{tikzcd}$$

commute. We say that a double complex C is *bounded* if there exists an integer M such that $C^{p,q} = 0$ holds for all $|p|, |q| \geqslant M$, with the obvious variants for bounded below and above double complexes.

Associated with a bounded below double complex C as above is a usual cochain complex $(\mathrm{Tot}(C), \mathrm{d})$ called the *total complex* and defined as

$$\mathrm{Tot}^n(C) = \bigoplus_{p+q=n} C^{p,q}$$

in degree n, with differential

$$\mathrm{d}x = \mathrm{d}^{\mathrm{hor}}x + (-1)^p \mathrm{d}^{\mathrm{ver}}x \quad \text{for } x \in C^{p,q}.$$

The assumption that C is bounded below ensures that the direct sum in $\mathrm{Tot}^n(C)$ only has a finite number of non-zero objects.

Remark A.34. For $d \circ d = 0$ to hold and give rise to a complex, one needs to change the sign of either the vertical or the horizontal differential in the double complex; this choice is arbitrary and varies from one reference to another.

Example A.35. A morphism of complexes $f\colon A^* \to B^*$ can be viewed as a double complex C with non-trivial terms

$$C^{0,q} = A^q \quad \text{and} \quad C^{1,q} = B^q,$$

and differentials $\mathrm{d}^{\mathrm{hor}} = f$ and $\mathrm{d}^{\mathrm{ver}} = 0$. Its total complex $\mathrm{Tot}(f)$ is then given by

$$\mathrm{Tot}(f)^n = A^n \oplus B^{n-1}, \quad \mathrm{d}(a,b) = (\mathrm{d}a, -\mathrm{d}b + f(a)).$$

Comparing with Definition A.25 of the cone of a morphism, one finds

$$\mathrm{cone}(f) = \mathrm{Tot}(-f)[1].$$

Example A.36. The tensor product of complexes is another instance of a total complex associated with a double complex. Indeed, assume that the additive category $\mathcal{A}$ is equipped with a tensor product $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. This is the case, for example, for the category $\mathbf{Mod}_R$ of left R-modules over some ring R. The *tensor product* of bounded below complexes $A = (A^*, \mathrm{d}_A)$ and $B = (B^*, \mathrm{d}_B)$ in $\mathcal{A}$ is then defined as the complex $A \otimes B$ with degree n terms

$$(A \otimes B)^n = \bigoplus_{p+q=n} A^p \otimes B^q,$$

and differential given on the $A^p \otimes B^q$ component by

$$\mathrm{d}_{A\otimes B} = \mathrm{d}_A \otimes \mathrm{Id}_B + (-1)^p \mathrm{Id}_A \otimes \mathrm{d}_B.$$

This is the total complex associated with the double complex

$$C^{p,q} = A^p \otimes B^q$$

with horizontal differential $\mathrm{d}^{\mathrm{hor}} = \mathrm{d}_A \otimes \mathrm{Id}_B$ and vertical differential $\mathrm{d}^{\mathrm{ver}} = \mathrm{Id}_A \otimes \mathrm{d}_B$.

A.1.10. *Cohomological functors.* Let $\mathcal{A}$ be an abelian category. Recall that the category $\mathcal{C}(\mathcal{A})$ of complexes in $\mathcal{A}$ is abelian, and that bounded below complexes and bounded complexes form abelian subcategories $\mathcal{C}^+(\mathcal{A})$ and $\mathcal{C}^b(\mathcal{A})$. In particular, the notion of exact sequence of complexes makes sense. The cohomology functors

$$\mathrm{H}^n : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A} \qquad (n \in \mathbb{Z})$$

satisfy the property that, for every short exact sequence of complexes

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0, \tag{A.37}$$

there are morphisms $\partial^n \colon \mathrm{H}^n(C) \to \mathrm{H}^{n+1}(A)$ such that the sequence

$$\cdots \longrightarrow \mathrm{H}^n(A) \longrightarrow \mathrm{H}^n(B) \longrightarrow \mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \longrightarrow \mathrm{H}^{n+1}(B) \longrightarrow \cdots \tag{A.38}$$

is exact. Indeed, assuming that $\mathcal{A}$ is a full subcategory of $\mathbf{Mod}_R$ for some ring R, we may define ∂^n by a diagram chase in

$$\begin{tikzcd}
 & A^{n+2} \arrow[r] & B^{n+2} & & \\
0 \arrow[r] & A^{n+1} \arrow[r] \arrow[u] & B^{n+1} \arrow[r] \arrow[u] & C^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & A^{n} \arrow[r] \arrow[u] & B^{n} \arrow[r] \arrow[u] & C^{n} \arrow[r] \arrow[u] & 0. \\
 & & B^{n-1} \arrow[r] \arrow[u] & C^{n-1} \arrow[u] &
\end{tikzcd}$$

Let $c \in C^n$ be an element satisfying $\mathrm{d}^n c = 0$. Since the map $B^n \to C^n$ is surjective and the right square in the middle commutes, c is the image of an element $b \in B^n$ such that $\mathrm{d}^n b \in B^{n+1}$ maps to $0 \in C^{n+1}$. By exactness of the second row, $\mathrm{d}^n b$ is the image of a unique $a \in A^{n+1}$, which satisfies $d^{n+1}a = 0$ since the left upper square commutes and $d^{n+1} \circ d^n = 0$. Moreover, different choices of b give rise to elements a that differ by an element in the image of d^n. Finally, if c is of the form $\mathrm{d}^{n-1}c'$ for some $c' \in C^{n-1}$, then writing c' as the image of some $b' \in B^{n-1}$ and using the commutativity of the right lower square, one finds that $\mathrm{d}^n b$, and hence a, vanishes. All in all, the assignment $c \mapsto a$ is a well defined map $\partial^n \colon \mathrm{H}^n(C) \to \mathrm{H}^{n+1}(A)$ that by design makes the sequence (A.38) exact.

This property leads to the definition of a cohomological δ-functor and is also the inspiration for the definition of a derived functor (see Section A.4).

Definition A.39. A *cohomological δ-functor* from an abelian category $\mathcal{A}$ to an abelian category $\mathcal{B}$ is the data of

i) a sequence of additive functors $F^n : \mathcal{A} \to \mathcal{B}$ indexed by $n \in \mathbb{Z}$;

ii) for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, a sequence of *connection morphisms*

$$\partial^n \colon F^n(C) \to F^{n+1}(A);$$

subject to the conditions:

i) the following sequence is exact:

$$\cdots \longrightarrow F^n(A) \longrightarrow F^n(B) \longrightarrow F^n(C) \xrightarrow{\partial^n} F^{n+1}(A) \longrightarrow F^{n+1}(B) \longrightarrow \cdots$$

ii) for every morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0, \end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd} F^n(C) \arrow[r, "\partial^n"] \arrow[d] & F^{n+1}(A) \arrow[d] \\ F^n(C') \arrow[r, "\partial^n"] & F^{n+1}(A'). \end{tikzcd}$$

In Definition A.77 below, we will introduce the notions of triangulated and cohomological functors in the setting of triangulated categories. The relation with cohomological δ-functors is explained in Exercise A.96.

REMARK A.40. Given any collection of signs $\varepsilon_n \in \{-1, 1\}$, if $(F^n, \partial^n)_{n\in\mathbb{Z}}$ is a cohomological δ-functor, then $(F^n, \varepsilon_n\partial^n)_{n\in\mathbb{Z}}$ is also a cohomological δ-functor. This ambiguity of signs of connection morphisms in the definition of a cohomological δ-functor is already visible in the construction of ∂^n by diagram chasing.

Similarly, associated with a short exact sequence $0 \to A \to B \to C \to 0$ of chain complexes there is a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(B) \longrightarrow \mathrm{H}_n(C) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A) \longrightarrow \mathrm{H}_{n-1}(B) \longrightarrow \cdots \tag{A.41}$$

where the connecting morphisms now decrease the degree.

A.1.11. *Homotopy equivalences.* The algebraic counterpart of a homotopy between topological spaces is the notion of homotopy between morphisms of complexes in an additive category $\mathcal{A}$.

DEFINITION A.42. Let $f, g\colon A^* \to B^*$ be morphisms of complexes in $\mathcal{C}(\mathcal{A})$. A *homotopy* between f and g is a collection of maps $s^n\colon A^n \to B^{n-1}$ such that

$$f^n - g^n = \mathrm{d}^{n-1} \circ s^n + s^{n+1} \circ \mathrm{d}^n$$

holds for all n. We picture it as follows:

$$\begin{tikzcd} \cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] \arrow[dl, "s^n"'] & A^{n+1} \arrow[r] \arrow[d] \arrow[dl, "s^{n+1}"'] & \cdots \\ \cdots \arrow[r] & B^{n-1} \arrow[r] & B^n \arrow[r] & B^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

When such a homotopy exists, we say that f and g are *homotopically equivalent.*

Given two complexes A and B, being homotopically equivalent defines an equivalence relation on the set of morphisms of complexes between A and B.

A basic property of homotopically equivalent morphisms is that they induce the same morphism on cohomology objects.

PROPOSITION A.43. *Let $\mathcal{A}$ be an abelian category. If $f, g\colon A^* \to B^*$ are homotopically equivalent morphisms of complexes in $\mathcal{C}(\mathcal{A})$, then* $\mathrm{H}^*(f) = \mathrm{H}^*(g)$.

Proof. It suffices to prove that $f-g$ induces the zero map from $\mathrm{H}^*(A^*)$ to $\mathrm{H}^*(B^*)$. To see this, let $[x] \in \mathrm{H}^n(A^*)$ be a cohomology class. Then

$$[(f^n - g^n)(x)] = [d^{n-1}(s^n(x)) + s^{n+1}(d^n(x))] = [s^{n+1}(d^n(x))] = 0,$$

since $d^{n-1}(s^n(x))$ is a coboundary and $d^n(x) = 0$. $\square$

Definition A.44. A morphism of complexes $f\colon A^* \to B^*$ is called a *homotopy equivalence* if there exists a morphism of complexes $g\colon B^* \to A^*$, called a *homotopy inverse*, such that $g \circ f$ and $f \circ g$ are homotopically equivalent to Id_{A^*} and Id_{B^*} respectively. In this case, we say that A^* and B^* are homotopically equivalent.

A direct consequence of Proposition A.43 is the next result.

Corollary A.45. *If the category $\mathcal{A}$ is abelian, then any homotopy equivalence f is a quasi-isomorphism*

Proof. Let $f\colon A^* \to B^*$ be a homotopy equivalence, and let $g\colon B^* \to A^*$ be a homotopy inverse of f as in Definition A.44. From the equalities

$$\mathrm{H}(g) \circ \mathrm{H}(f) = \mathrm{H}(g \circ f) = \mathrm{H}(\mathrm{Id}_{A^*}) = \mathrm{Id}_{\mathrm{H}(A^*)},$$

and similarly for $\mathrm{H}(f) \circ \mathrm{H}(g)$, we deduce that $\mathrm{H}(f)$ and $\mathrm{H}(g)$ are inverse to each other, and in particular that $\mathrm{H}(f)$ is an isomorphism. $\square$

A.1.12. *The acyclic models theorem.* The method of acyclic models is an abstract way to construct homotopies which can be seen as a precursor of the theory of derived categories and of model category theory; see [**Spa66**, Chap. 4, § 2]. We used it to prove the Eilenberg–Zilber Theorem 2.24 when constructing the external product in singular homology. A *category with models* is a category $\mathcal{C}$ together with a collection of objects $\mathcal{M} \subset \mathrm{Ob}(\mathcal{C})$.

Definition A.46. Let $(\mathcal{C}, \mathcal{M})$ be a category with models and R a ring.

i) A functor $F\colon \mathcal{C} \to \mathcal{C}_{\geqslant 0}(\mathbf{Mod}_R)$ is called *acyclic* if
$$\mathrm{H}_n(F(M)) = 0$$
holds for all $n \geqslant 1$ and $M \in \mathcal{M}$.

ii) A functor $F\colon \mathcal{C} \to \mathcal{C}_{\geqslant 0}(\mathbf{Mod}_R)$ is called *free* if there exists a collection $(M_\alpha)_{\alpha \in J}$ of objects of $\mathcal{M}$ and elements $m_\alpha \in F(M_\alpha)$, for $\alpha \in J$, such that the set
$$\{F(f)(m_\alpha) \mid \alpha \in J \text{ and } f \in \mathrm{Hom}(M_\alpha, X)\}$$
forms a basis of the R-module $F(X)$ for every object $X \in \mathrm{Ob}(\mathcal{C})$.

These notions are tailored to study the basic example of a category with models, namely that of topological spaces with models the standard simplexes $\{\Delta^n_{\mathrm{st}}\}$. An example of acyclic and free functor is the singular chain functor from Section 2.1.

Theorem A.47 (Acyclic models theorem). *Let $\mathcal{C}$ be a category with models $\mathcal{M}$, and let $F, G\colon \mathcal{C} \to \mathcal{C}_{\geqslant 0}(\mathbf{Mod}_R)$ be functors with F free and G acyclic. Then:*

i) *any natural transformation $\mathrm{H}_0(F) \to \mathrm{H}_0(G)$ is induced by a natural transformation $\tau\colon F \to G$;*

ii) *two natural transformations $\tau, \tau'\colon F \to G$ inducing the same natural transformation $\mathrm{H}_0(F) \to \mathrm{H}_0(G)$ are naturally homotopic.*

⋆⋆⋆

Exercise A.48. Let $\mathcal{A}$ be an additive category. Show that $\mathrm{Hom}_{\mathcal{A}}(X,X)$ is endowed with a ring structure for each object $X \in \mathrm{Ob}(\mathcal{A})$.

Exercise A.49. Let $\mathcal{A}$ be a category. A *product* of objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ is an object $X \times Y \in \mathrm{Ob}(\mathcal{A})$ endowed with morphisms

$$X \longleftarrow X \times Y \longrightarrow Y$$

such that, for each object $Z \in \mathrm{Ob}(\mathcal{A})$ with morphisms $Z \to X$ and $Z \to Y$, there exist a unique morphism $Z \to X \times Y$ making the diagram

$$\begin{tikzcd} X & X \times Y \arrow[l] \arrow[r] & Y \\ & Z \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

commutative. Show that the product $X \times Y$ exists in every additive category $\mathcal{A}$, and it is equal to the direct sum $X \oplus Y$. Conclude that, in an additive category, finite products exist and they agree with finite direct sums. (Since the notion of direct sum objects in Definition A.1 iii) is dual to the above, $X \oplus Y$ is also called a *coproduct* in the categorial sense.)

Exercise A.50. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. Prove that F is additive if and only if, for all objects $X, Y \in \mathrm{Ob}(\mathcal{A})$, the map

$$F\colon\ \mathrm{Hom}_{\mathcal{A}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{B}}\big(F(X), F(Y)\big)$$

is a group homomorphism. [Hint: the sum $f+g$ of morphisms $f, g \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$ is given by the composition

$$X \longrightarrow X \oplus X \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)} Y \oplus Y \longrightarrow Y,$$

where the diagonal $X \to X \oplus X$ and the codiagonal $Y \oplus Y \to Y$ are the maps obtained from the universal property of the product and the coproduct.]

Exercise A.51. Given an additive category $\mathcal{A}$, let $\mathcal{A}^{\mathrm{op}}$ denote the *opposite category*, which has the same objects as $\mathcal{A}$ but reversed morphisms

$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{A}}(Y,X)$$

for all objects $X, Y \in \mathrm{Ob}(\mathcal{A})$. Prove that $\mathcal{A}^{\mathrm{op}}$ is an additive category and that a cokernel for $f\colon X \to Y$ in $\mathcal{A}$ is a kernel for the corresponding morphism $Y \to X$ in $\mathcal{A}^{\mathrm{op}}$, and vice versa.

Exercise A.52. Let $\mathcal{A}$ be the category whose objects are pairs (V, W) consisting of vector spaces of the same dimension over some field k, and whose morphisms are pairs of linear maps. Show that $\mathcal{A}$ is an additive category in which some morphisms do not have a kernel. This is a toy example of the category of vector bundles, which is typically not abelian by contrast with that of coherent sheaves.

Exercise A.53. Let $\mathcal{A}$ be an additive category and let f be a morphism in $\mathcal{A}$. Assume that the image $\mathrm{Im}(f)$ and the coimage $\mathrm{Coim}(f)$ exist. Show that there is a canonical morphism

$$\mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f).$$

Exercise A.54. Let $\mathcal{A}$ be an additive category and $f\colon X \to Y$ a morphism in $\mathcal{A}$ that has a kernel and a cokernel.

i) Prove that $\mathrm{Ker}(f) \to X$ is a monomorphism, and $Y \to \mathrm{Coker}(f)$ is an epimorphism.

ii) Show that f is a monomorphism if and only if $\mathrm{Ker}(f)$ is a zero object, and an epimorphism if and only if $\mathrm{Coker}(f)$ is a zero object.

Exercise A.55. Prove that the projection map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism in the category of abelian divisible groups, *i.e.* those groups G such that, for each element $x \in G$ and each integer $n \geqslant 1$, there exists $y \in G$ satisfying $ny = x$. Thus, the full subcategory of **Ab** consisting of abelian divisible groups is not abelian.

Exercise A.56. Let $\mathcal{A}$ be an abelian category and $f\colon A^* \to B^*$ a morphism in $\mathcal{C}^*(\mathcal{A})$. Show that the complexes $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ defined in the proof of Proposition A.20 are a kernel and a cokernel in the category $\mathcal{C}^*(\mathcal{A})$.

Exercise A.57. Let $\mathcal{A}$ be an additive category, and let f be a morphism in $\mathcal{A}$. Prove the following statements:

i) If $\mathrm{Ker}(f)$ exists, then $\mathrm{Ker}(\mathrm{Ker}(f))$ exists and is a zero object of $\mathcal{A}$.

ii) If $\mathrm{Coker}(f)$ exists, then $\mathrm{Coker}(\mathrm{Coker}(f))$ exists and is a zero object of $\mathcal{A}$.

In other words, $\mathrm{Ker}(f)$ is a monomorphism, and $\mathrm{Coker}(f)$ is an epimorphism.

Exercise A.58. Let $\mathcal{A}$ be an abelian category, and let $X \in \mathrm{Ob}(\mathcal{A})$ be an object. Show that, if there is a sequence

$$X \longleftarrow S_1 \longrightarrow Q_1 \longleftarrow S_2 \longrightarrow Q_2,$$

where the morphisms to the left are monomorphisms and the morphisms to the right are epimorphisms, then Q_2 is a subquotient of X. Deduce that, in an abelian category, the property of being a subquotient (Definition A.8) is transitive, and that a subquotient is also a subobject of a quotient.

Exercise A.59. Let $\mathcal{A}$ be an abelian category and X an object of $\mathcal{A}$. Show that $\mathrm{Hom}_{\mathcal{A}}(X, -)$ is a left exact functor from $\mathcal{A}$ to the category of abelian groups.

Exercise A.60. In this exercise, we show that a functor with a left (resp. right) adjoint is left (resp. right) exact. We start with the converse of Exercise A.59.

i) Let $\mathcal{A}$ be an abelian category. Show that a sequence $0 \to A \to B \to C$ is exact if the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, A) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, B) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, C)$$

of abelian groups is exact for every object $X \in \mathrm{Ob}(\mathcal{A})$.

ii) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and (F, G) an adjoint pair of functors. Let $0 \to A \to B \to C$ be an exact sequence in $\mathcal{B}$. Show that the sequence

$$0 \longrightarrow G(A) \longrightarrow G(B) \longrightarrow G(C)$$

in $\mathcal{A}$ is exact. Conclude that the functor G is left exact.

iii) Dually, show that F is right exact.

A.2. Yoneda extensions.

A.2.1. *Definition.*

DEFINITION A.61. Let $\mathcal{A}$ be an abelian category, let $A, B \in \mathrm{Ob}(\mathcal{A})$ be objects, and $n \geqslant 1$ an integer. An *extension of degree n of A by B* is an exact sequence

$$E\colon \quad 0 \longrightarrow B \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow A \longrightarrow 0.$$

Given extensions of the same degree E and E', we say that E is equivalent to E' if there exists a commutative diagram

$$\begin{tikzcd}
E\colon & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
E'\colon & 0 \arrow[r] & B \arrow[r] & C'_{n-1} \arrow[r] & \cdots \arrow[r] & C'_0 \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

For $n = 1$, this notion agrees with that from Definition 2.247 in the particular case of mixed Hodge structures. In contrast to that case, the binary relation defined by the existence of the above diagram may not be symmetric for $n \geqslant 2$. We consider instead the equivalence relation *generated* by such relations, which means that we force symmetry and transitivity by adding the missing relations.

DEFINITION A.62. For $n \geqslant 1$, we denote by $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ the set of equivalence classes of degree n extensions of A by B. To unify notation, we also set

$$\mathrm{Ext}^0_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)$$

and $\mathrm{Ext}^n_{\mathcal{A}}(A, B) = 0$ for all negative n.

A.2.2. *Some properties of Yoneda extensions.*

The Baer sum. The sets $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ are endowed with a group structure, given by the so-called *Baer sum* of extensions. We describe it by picking elements of the objects of $\mathcal{A}$, starting with the case $n \geqslant 2$. Let

$$E\colon \quad 0 \longrightarrow B \xrightarrow{\iota} C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \xrightarrow{\pi} A \longrightarrow 0,$$

$$E'\colon \quad 0 \longrightarrow B \xrightarrow{\iota'} C'_{n-1} \longrightarrow \cdots \longrightarrow C'_0 \xrightarrow{\pi'} A \longrightarrow 0$$

be degree n extensions of A by B. The *pull-back* of C_0 and C'_0 over A is the object

$$C''_0 = \mathrm{Ker}\left(C_0 \oplus C'_0 \xrightarrow{\phi} A\right), \qquad \phi(c, c') = \pi'(c') - \pi(c).$$

By construction, π and π' agree on C''_0, hence a well defined morphism

$$\pi''\colon C''_0 \longrightarrow A, \qquad \pi''(c, c') = \pi(c) = \pi'(c').$$

Dually, the *push-out* of C_{n-1} and C'_{n-1} under B is the object

$$C''_{n-1} = \mathrm{Coker}\left(B \xrightarrow{\psi} C_{n-1} \oplus C'_{n-1}\right), \qquad \psi(b) = (\iota(b), -\iota'(b)).$$

By construction, the maps $b \mapsto (\iota(b), 0)$ and $b \mapsto (0, \iota'(b))$ from B to $C_{n-1} \oplus C'_{n-1}$ induce the same morphism

$$\iota''\colon B \longrightarrow C''_{n-1}.$$

DEFINITION A.63. For $n \geqslant 2$, the *Baer sum* of the degree n extensions E and E' is the degree n extension given by

$$0 \longrightarrow B \xrightarrow{\iota''} C''_{n-1} \longrightarrow C_{n-2} \oplus C'_{n-2} \longrightarrow \cdots \longrightarrow C_1 \oplus C'_1 \longrightarrow C''_0 \xrightarrow{\pi''} A \longrightarrow 0.$$

For $n = 1$, a variant of this construction is needed to merge the pull-back C_0'' and the push-out C_{n-1}''. Let

$$0 \longrightarrow B \xrightarrow{\iota} E \xrightarrow{\pi} A \longrightarrow 0,$$
$$0 \longrightarrow B \xrightarrow{\iota'} E' \xrightarrow{\pi'} A \longrightarrow 0$$

be extensions of degree 1 of A by B. Write

$$\begin{aligned} E'' &= \mathrm{Coker}(B \xrightarrow{\psi} \mathrm{Ker}(E \oplus E' \xrightarrow{\phi} A)) \\ &= \mathrm{Ker}(\mathrm{Coker}(B \xrightarrow{\psi} E \oplus E') \xrightarrow{\phi} A). \end{aligned}$$

As before, there are induced morphisms $B \to E''$ and $E'' \to A$.

Definition A.64. The *Baer sum* of degree 1 extensions E and E' is the degree 1 extension given by

$$0 \longrightarrow B \longrightarrow E'' \longrightarrow A \longrightarrow 0.$$

Endowed with the Baer sum, the set of equivalence classes $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ forms a group, in which the neutral element is the extension

$$0 \longrightarrow B \longrightarrow A \oplus B \longrightarrow A \longrightarrow 0$$

for $n = 1$, and the extension

$$0 \longrightarrow B = B \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A = A \longrightarrow 0$$

for $n \geqslant 2$ (see Exercise A.67). An extension is said to be *split* if it is equivalent to the neutral element of the extension group.

Functoriality. Formation of extensions is functorial in the following sense: associated with objects $A, B, B' \in \mathrm{Ob}(\mathcal{A})$, a morphism $f\colon B \to B'$, and an extension

$$E\colon \quad 0 \longrightarrow B \xrightarrow{\mathrm{d}_n} C_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \cdots \xrightarrow{\mathrm{d}_1} C_0 \xrightarrow{\mathrm{d}_0} A \longrightarrow 0,$$

there is an extension

$$\mathrm{Ext}^n_{\mathcal{A}}(A, f)(E) \in \mathrm{Ext}^n_{\mathcal{A}}(A, B')$$

constructed as follows. First define C'_{n-1} as the pushout of C_{n-1} and B' under B, so that there are morphisms $f_{n-1}\colon C_{n-1} \to C'_{n-1}$ and $\mathrm{d}'_n\colon B' \to C'_{n-1}$. For notational convenience, write $C_n = B$, $C'_n = B'$, and $f_n = f$. Assume that we have defined the groups $C'_n, \dots, C'_k$, and the maps f_i, for $i = n, \dots, k$, and d'_i, for $i = n, \dots, k+1$, where $1 \leqslant k \leqslant n-1$. Then the group C'_{k-1} is defined as

$$C'_{k-1} = \mathrm{Coker}(\psi\colon C'_{k+1} \oplus C_k \longrightarrow C'_k \oplus C_{k-1}),$$

where the morphism ψ is given by

$$\psi(c', c) = (\mathrm{d}'_{k+1}(c') + f_k(c), -\mathrm{d}_k(c)).$$

There are induced morphisms $f_{k-1}\colon C_{k-1} \to C'_{k-1}$ and $\mathrm{d}'_k\colon C'_k \to C'_{k-1}$. From the universal property of the cokernel, one derives a morphism $\mathrm{d}'_0\colon C'_0 \to A$ such that the resulting sequence

$$0 \longrightarrow B' \xrightarrow{\mathrm{d}'_n} C'_{n-1} \xrightarrow{\mathrm{d}'_{n-1}} \cdots \xrightarrow{\mathrm{d}'_1} C'_0 \xrightarrow{\mathrm{d}'_0} A' \longrightarrow 0$$

is exact and defines the extension $E' = \mathrm{Ext}^n_{\mathcal{A}}(A, f)(E) \in \mathrm{Ext}^n_{\mathcal{A}}(A, B')$.

Connecting morphisms. Let $0 \to B_1 \to B_2 \to B_3 \to 0$ be a short exact sequence. There is a map

$$\partial^0\colon \operatorname{Hom}_{\mathcal{A}}(A, B_3) \to \operatorname{Ext}^1_{\mathcal{A}}(A, B_1)$$

sending a morphism f to the sequence $0 \to B_1 \to E \to A \to 0$, where E is the pull-back of B_2 and A over B_3. There are also maps

$$\partial^n\colon \operatorname{Ext}^n_{\mathcal{A}}(A, B_3) \longrightarrow \operatorname{Ext}^{n+1}_{\mathcal{A}}(A, B_1) \tag{A.65}$$

that send the extension

$$0 \longrightarrow B_3 \xrightarrow{\mathrm{d}_n} C_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \cdots \xrightarrow{\mathrm{d}_1} C_0 \xrightarrow{\mathrm{d}_0} A \longrightarrow 0$$

to the extension

$$0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow C_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \cdots \xrightarrow{\mathrm{d}_1} C_0 \xrightarrow{\mathrm{d}_0} A \longrightarrow 0.$$

In Exercise A.68, you will prove that the extension groups together with the connection morphisms defined above form a cohomological δ- functor (Definition A.39).

A vanishing criterion:

Lemma A.66. *Let $\mathcal{A}$ be an abelian category. Assume that there exists an integer $n_0 \geqslant 0$ such that, for all objects $A \in \operatorname{Ob}(\mathcal{A})$, the functor $\operatorname{Ext}^{n_0}_{\mathcal{A}}(A, -)$ is right exact (Definition A.23). Then*

$$\operatorname{Ext}^n_{\mathcal{A}}(A, B) = 0$$

holds for all objects $A, B \in \operatorname{Ob}(\mathcal{A})$ and all integers $n > n_0$.

Proof. It is enough to prove that, if the functor $\operatorname{Ext}^n_{\mathcal{A}}(A, -)$ is right exact for all objects $A \in \operatorname{Ob}(\mathcal{A})$, then $\operatorname{Ext}^{n+1}_{\mathcal{A}}(A, B) = 0$ for all objects $A, B \in \operatorname{Ob}(\mathcal{A})$. Indeed, the zero functor is right exact, and we can then proceed by induction.

We start with the case $n = 0$. Assume that $\operatorname{Ext}^0_{\mathcal{A}}(A, -) = \operatorname{Hom}_{\mathcal{A}}(A, -)$ is right exact for all $A \in \operatorname{Ob}(\mathcal{A})$, and let

$$0 \longrightarrow B \longrightarrow E \xrightarrow{\pi} A \longrightarrow 0$$

be an extension. Since $\operatorname{Hom}_{\mathcal{A}}(A, -)$ is right exact, there is a morphism $f\colon A \to E$ such that $\pi \circ f = \mathrm{Id}_A$. This means that the extension is split: $\operatorname{Ext}^1_{\mathcal{A}}(A, B) = 0$.

Let now $n \geqslant 1$, and assume that $\operatorname{Ext}^n_{\mathcal{A}}(A, -)$ is right exact. Let

$$E\colon \quad 0 \longrightarrow B \xrightarrow{\mathrm{d}_{n+1}} C_n \xrightarrow{\mathrm{d}_n} \cdots \xrightarrow{\mathrm{d}_1} C_0 \xrightarrow{\mathrm{d}_0} A \longrightarrow 0$$

be an extension in $\operatorname{Ext}^{n+1}_{\mathcal{A}}(A, B)$. Consider the exact sequence

$$0 \longrightarrow B \longrightarrow C_n \longrightarrow C \longrightarrow 0,$$

where we set $C = \operatorname{Coker}(d_{n+1})$, and the extension

$$E'\colon \quad 0 \longrightarrow C \xrightarrow{\mathrm{d}_n} C_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \cdots \xrightarrow{\mathrm{d}_1} C_0 \xrightarrow{\mathrm{d}_0} A \longrightarrow 0,$$

so that $E' \in \operatorname{Ext}^n_{\mathcal{A}}(A, C)$ and $E = \partial^n(E')$. Since $\operatorname{Ext}^n_{\mathcal{A}}(A, -)$ is right exact, one necessarily has $\partial^n(E') = 0$, and hence $E = 0$. $\square$

⋆⋆⋆

Exercise A.67. Prove that the Baer sum is well defined on equivalence classes, and hence induces a group structure on $\operatorname{Ext}^n_{\mathcal{A}}(A, B)$ for each $n \geqslant 1$.

EXERCISE A.68. Let $\mathcal{A}$ be an abelian category, and let $A \in \mathrm{Ob}(\mathcal{A})$ be an object. Prove that the functors $\mathrm{Ext}^n_{\mathcal{A}}(A, -)$ together with the connection morphisms ∂^n from (A.65) form a cohomological δ-functor.

A.3. Triangulated and derived categories.

A.3.1. *The definition of a triangulated category.*

DEFINITION A.69 (Verdier). A *triangulated category* $\mathcal{T}$ is an additive category, together with the following extra data:

i) A self-equivalence of categories

$$\begin{aligned} [1]\colon\ \mathcal{T} &\longrightarrow \mathcal{T} \\ X &\longmapsto X[1]. \end{aligned}$$

We denote by $f[1]$ the image of a morphism f by this functor. Once the self-equivalence $[1]$ is given, we call *triangles* all sequences of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

A morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r, "w"] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]. \end{tikzcd}$$

We will use the convention that arrows decorated with $[1]$ such as $A \xrightarrow{[1]} B$ represent morphisms $A \to B[1]$. A triangle is then pictured as

$$\begin{tikzcd} X \arrow[rr, "u"] & & Y. \arrow[dl, "v"] \\ & Z \arrow[ul, "w[1]"] & \end{tikzcd}$$

ii) A class of triangles called *distinguished triangles*.

These data are required to satisfy the following axioms:

(T1): i) For any $X \in \mathrm{Ob}(\mathcal{T})$, the following triangle is distinguished:

$$X \xrightarrow{\mathrm{Id}} X \longrightarrow 0 \longrightarrow X[1].$$

ii) Any triangle isomorphic to a distinguished one is distinguished.

iii) Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

(T2): The triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished.

(T3): Given distinguished triangles

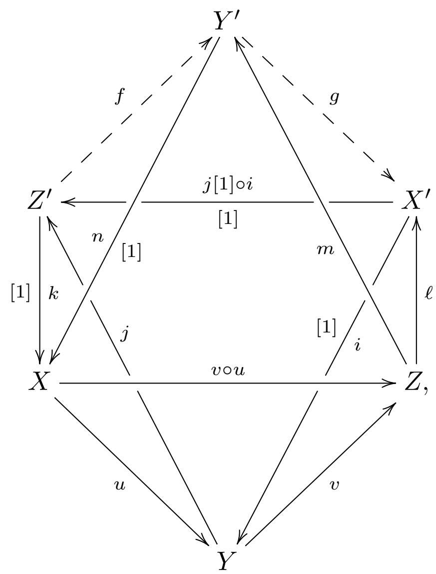
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1], \qquad X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1],$$

and morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ satisfying $g \circ u = u' \circ f$, there exists a (not necessarily unique) morphism $h\colon Z \to Z'$ such that

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

is a morphism of triangles.

(T4): (Octahedron axiom) Given a diagram of solid arrows



if the three triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k} X[1]$$

$$Y \xrightarrow{v} Z \xrightarrow{\ell} X' \xrightarrow{i} Y[1]$$

$$X \xrightarrow{v\circ u} Z \xrightarrow{m} Y' \xrightarrow{n} X[1]$$

are distinguished, then there exist morphisms $f\colon Z' \to Y'$ and $g\colon Y' \to X'$ (the dashed arrows above) such that the triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{j[1]\circ i} Z'[1]$$

is distinguished and the following equalities hold:

$$\begin{aligned} k &= n \circ f, & \ell &= g \circ m, \\ m \circ v &= f \circ j, & u[1] \circ n &= i \circ g. \end{aligned}$$

Definition A.70. Let $\mathcal{T}$ be a triangulated category. A *triangulated subcategory* of $\mathcal{T}$ is a full additive subcategory $\mathcal{S}$ of $\mathcal{T}$ such that:

i) the functor $[1]$ restricts to a self-equivalence on $\mathcal{S}$;
ii) if two out of three objects of a distinguished triangle in $\mathcal{T}$ belong to $\mathcal{S}$, then so does the third.

REMARK A.71. The first basic property of distinguished triangles is that the composition of two consecutive morphisms is zero. That is, if

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is a distinguished triangle in a triangulated category $\mathcal{T}$, then

$$v \circ u = 0, \qquad w \circ v = 0, \qquad u[1] \circ w = 0.$$

Indeed, by axiom **(T2)** it is enough to prove $v \circ u = 0$. Consider now the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{Id}"'] \arrow[d, "\mathrm{Id}"] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dotted] & X[1] \arrow[d, "\mathrm{Id}"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1].
\end{tikzcd}$$

Since a dotted arrow making the diagram commutative exists by axiom **(T3)**, we deduce the vanishing of $v \circ u = 0$.

REMARK A.72. The octahedron axiom can be "flattened" by repeating some vertices. Namely, if in the commutative diagram (without the dotted arrows)

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "u"] & X \arrow[d, "v\circ u"] & & \\
Y \arrow[r, "v"] \arrow[d, "j"] & Z \arrow[r, "\ell"] \arrow[d, "m"] & X' \arrow[r, "i"] \arrow[d, equal] & Y[1] \arrow[d, "j[1]"] \\
Z' \arrow[r, dotted, "f"] \arrow[d, "k"] & Y' \arrow[r, dotted, "g"] \arrow[d, "n"] & X' \arrow[r, "j[1]\circ i"] & Z'[1] \\
X[1] \arrow[r, equal] & X[1] & &
\end{tikzcd}$$

the two long columns and the long row are distinguished triangles, then there exist morphisms $f\colon Z' \to Y'$ and $g\colon Y' \to X'$ making the whole diagram commutative, the second long row a distinguished triangle, and such that the equality

$$i \circ g = u[1] \circ n$$

of morphisms from Y' to $Y[1]$ holds.

PROPOSITION A.73. *Let $\mathcal{T}$ be a triangulated category, and let*

$$X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Z_i \xrightarrow{w_i} X_i[1] \qquad (i = 1, 2)$$

be distinguished triangles. Then the triangle

$$X_1 \oplus X_2 \xrightarrow{u_1 \oplus u_2} Y_1 \oplus Y_2 \xrightarrow{v_1 \oplus v_2} Z_1 \oplus Z_2 \xrightarrow{w_1 \oplus w_2} X_1[1] \oplus X_2[1]$$

is distinguished as well.

See [**Nee01**, Prop. 1.2.1] for a proof of this proposition, which is trickier than one might think at first sight. It follows that the triangle

$$X \longrightarrow X \oplus Z \longrightarrow Z \xrightarrow{0} X[1] \tag{A.74}$$

is distinguished for all objects $X, Z \in \mathrm{Ob}(\mathcal{T})$.

Definition A.75. A distinguished triangle is called *split* if it is isomorphic to a triangle of the form (A.74).

Proposition A.76. *Let $\mathcal{T}$ be a triangulated category and let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be a distinguished triangle. The following statements are equivalent.

i) *The triangle is split.*
ii) *The equality $w = 0$ holds.*
iii) *There exists a morphism $s\colon Z \to Y$ satisfying $v \circ s = \mathrm{Id}_Z$.*
iv) *There exists a morphism $r\colon Y \to X$ satisfying $r \circ u = \mathrm{Id}_X$.*

Proof. Clearly, statement i) implies statement ii). By Exercise A.49, there are morphisms

$$\begin{aligned} s_X &\colon X \longrightarrow X \oplus Z, & s_Z &\colon Z \longrightarrow X \oplus Z \\ r_X &\colon X \oplus Z \longrightarrow X & r_Z &\colon X \oplus Z \longrightarrow Z \end{aligned}$$

satisfying $r_X \circ s_X = \mathrm{Id}_X$ and $r_Z \circ s_Z = \mathrm{Id}_Z$. This shows that i) also implies iii) and iv). Assume that there is a morphism $s\colon Z \to Y$ satisfying $v \circ s = \mathrm{Id}_Z$. Then the equalities $w = w \circ \mathrm{Id}_Z = w \circ v \circ s = 0$ hold by Remark A.71, and hence iii) implies ii). Similarly, iv) implies ii). Therefore, the main content of the proposition is that ii) implies i), which is proved in [**Nee01**, Cor. 1.2.7]. □

Definition A.77. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories and let $\mathcal{A}$ be an abelian category.

i) A *triangulated functor* $F\colon \mathcal{T} \to \mathcal{T}'$ is an additive functor that sends distinguished triangles to distinguished triangles and is compatible with the self-equivalence $[1]$, in that the equalities

$$F(X[1]) = F(X)[1] \quad \text{and} \quad F(f[1]) = F(f)[1]$$

hold for each object X and each morphism f of $\mathcal{T}$.

ii) A *cohomological functor* $\mathrm{H}\colon \mathcal{T} \to \mathcal{A}$ is an additive functor such that each distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

in $\mathcal{T}$ gives rise to an exact sequence

$$\cdots \longrightarrow \mathrm{H}(X[n]) \xrightarrow{\mathrm{H}(u[n])} \mathrm{H}(Y[n]) \xrightarrow{\mathrm{H}(v[n])} \mathrm{H}(Z[n]) \xrightarrow{\mathrm{H}(w[n])} \\ \mathrm{H}(X[n+1]) \xrightarrow{\mathrm{H}(u[n+1])} \mathrm{H}(Y[n+1]) \xrightarrow{\mathrm{H}(v[n+1])} \mathrm{H}(Z[n+1]) \longrightarrow \cdots$$

in $\mathcal{A}$. (Here, $[n]$ denotes the n-th iteration of the self-equivalence $[1]$ if $n \geqslant 0$ and the $(-n)$-th iteration of its inverse $[-1]$ if $n \leqslant 0$.)

A.3.2. *Example: the homotopy and derived categories.* The first example of a triangulated category is that of complexes up to homotopy, whose construction we now sketch. Let $\mathcal{A}$ be an additive category. Recall the notions of shift of a complex and cone of a morphism of complexes from Section A.1.5, as well as the notion of homotopy and of homotopic equivalence from Section A.1.11.

Definition A.78. Let $\mathcal{A}$ be an additive category. The *homotopy category* of $\mathcal{A}$ is the category $\mathcal{K}(\mathcal{A})$ whose objects are the same as those of $\mathcal{C}(\mathcal{A})$ but whose morphisms are equivalence classes with respect to the homotopy equivalence of morphisms in $\mathcal{C}(\mathcal{A})$. The *shift functor*

$$[1]\colon \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$$

sends a complex A to the shifted complex $A[1]$. A distinguished triangle is a triangle isomorphic to one of the form

$$A^* \xrightarrow{f} B^* \xrightarrow{\mathrm{b}} \mathrm{cone}(f) \xrightarrow{\mathrm{a}} A[1]$$

for some morphism of complexes $f\colon A^* \to B^*$ and a and b defined as in (A.26). Together with this data, $\mathcal{K}(\mathcal{A})$ forms a triangulated category.

Remark A.79. There are analogous notions of bounded and bounded below homotopy categories, denoted by $\mathcal{K}^b(\mathcal{A})$ and $\mathcal{K}^+(\mathcal{A})$ respectively, obtained by starting with the categories $\mathcal{C}^b(\mathcal{A})$ and $\mathcal{C}^+(\mathcal{A})$ respectively. Nevertheless, the analogous category $\mathcal{K}^{\geqslant 0}(\mathcal{A})$ obtained starting with the category $\mathcal{C}^{\geqslant 0}(\mathcal{A})$ is not triangulated, as the shift functor [1] fails to be an equivalence of categories and the negative shift functor $[-1]$ is not defined.

Example A.80. Let $\mathcal{A}$ be an abelian category. The exactness of the sequence (A.27) implies that the functor

$$\mathrm{H}\colon \mathcal{K}(\mathcal{A}) \longrightarrow \mathcal{A}, \quad A^* \longmapsto \mathrm{H}^0(A^*)$$

is a cohomological functor in the sense of Definition A.77.

Example A.81. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. The functor $F\colon \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ that sends a complex (A^*, d) to the complex $F((A^*, \mathrm{d}))$ with terms $F((A^*, \mathrm{d}))^n = F(A)^n$ and differential $F(\mathrm{d})$ is a triangulated functor (see Exercise A.95). There are also induced functors from the homotopy categories $\mathcal{K}^b(\mathcal{A})$ and $\mathcal{K}^+(\mathcal{A})$ to $\mathcal{K}^b(\mathcal{B})$ and $\mathcal{K}^+(\mathcal{B})$ respectively.

The second example of a triangulated category, and for our purposes the main one, is the *derived category* of an abelian category $\mathcal{A}$, which is obtained by inverting the quasi-isomorphisms in the homotopy category $\mathcal{K}(\mathcal{A})$. The construction is done in two steps. The first step is the construction of the homotopy category $\mathcal{K}(\mathcal{A})$ that we have already sketched. In the second step, one constructs $\mathcal{D}(\mathcal{A})$ by inverting all quasi-isomorphisms. The objects of $\mathcal{D}(\mathcal{A})$ are the same as the objects of $\mathcal{K}(\mathcal{A})$ (which are the same as the ones of $\mathcal{C}(\mathcal{A})$), while the morphisms between objects A^* and B^* of $\mathcal{D}(\mathcal{A})$ are equivalence classes of diagrams of the form

$$\begin{tikzcd} & C^* \arrow[dl, "\simeq"'] \arrow[dr] & \\ A^* & & B^*, \end{tikzcd} \tag{A.82}$$

where the arrow to the left is a quasi-isomorphism. The diagrams

$$\begin{tikzcd} & C_1^* \arrow[dl, "\simeq"'] \arrow[dr] & \\ A^* & & B^* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C_2^* \arrow[dl, "\simeq"'] \arrow[dr] & \\ A^* & & B^* \end{tikzcd}$$

are considered to be equivalent if there exists a diagram of the same type and morphisms $C_3^* \to C_1^*$ and $C_3^* \to C_2^*$ such that the diagram

$$\begin{tikzcd} & C_1^* \arrow[dl, "\simeq"'] \arrow[dr] & \\ A^* & C_3^* \arrow[l, "\simeq"'] \arrow[u] \arrow[d] \arrow[r] & B^* \\ & C_2^* \arrow[ul, "\simeq"] \arrow[ur] & \end{tikzcd} \tag{A.83}$$

commutes in $\mathcal{K}(\mathcal{A})$. This means that all the triangles in (A.83) are commutative *up to homotopy* (that is, different ways of composing arrows give rise to homotopically equivalent morphisms), although not necessarily commutative.

REMARK A.84. One reason to construct the derived category by inverting quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$ rather than directly in $\mathcal{C}(\mathcal{A})$ is that this procedure allows for a simpler description of the morphisms. Otherwise, morphisms would be given by chains of the form

$$\begin{tikzcd} & C_1^* \arrow[dl, "\simeq"'] \arrow[dr] & & \cdots \arrow[dl, "\simeq"'] \arrow[dr] & & C_k^* \arrow[dl, "\simeq"'] \arrow[dr] & \\ A^* & & C_2^* & & C_{k-1}^* & & B^*, \end{tikzcd}$$

where all the arrows in the left direction are quasi-isomorphisms.

REMARK A.85. There are analogous notions of bounded and bounded below derived categories, denoted by $\mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}^+(\mathcal{A})$ respectively.

The categories $\mathcal{D}(\mathcal{A})$, $\mathcal{D}^b(\mathcal{A})$, and $\mathcal{D}^+(\mathcal{A})$ are triangulated categories, with the self-equivalence [1] defined by the shift of complexes, and the class of distinguished triangles given by those triangles that are isomorphic to one of the form

$$A^* \xrightarrow{f} B^* \xrightarrow{\mathrm{b}} \operatorname{cone}(f) \xrightarrow{\mathrm{a}} A[1]^*.$$

That is, a triangle $X \to Y \to Z \to X[1]$ is distinguished if there exists a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\ A^* \arrow[r] & B^* \arrow[r] & \operatorname{cone}(f) \arrow[r] & A[1]^* \end{tikzcd}$$

whose vertical maps are quasi-isomorphisms and that commutes up to homotopy.

REMARK A.86. In contrast with the ambiguity of signs of connection morphisms in the definition of cohomological δ-functors (see Definition A.39 and Remark A.40), there is no ambiguity of signs in the definition of triangulated functors in Definition A.77. This implies that the choice of the class of distinguished triangles allows us to fix the sign of many cohomological δ-functors, at least all those coming from a triangulated functor between derived categories. We illustrate the choice of signs with the following example. Let $\mathcal{A}$ be an abelian category and let $f\colon A^* \to B^*$ be a morphism of complexes such that $f^n\colon A^n \to B^n$ is a monomorphism for all n.

As in Exercise A.93, let C^* denote the complex assembling the cokernels of the morphisms f^n, so that there is a short exact sequence of complexes

$$0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0.$$

By Exercise A.93, there is an isomorphism

$$\varphi\colon \operatorname{cone}(f) \xrightarrow{\sim} C^*$$

in the derived category $\mathcal{D}(\mathcal{A})$, and hence the connection morphism

$$\partial^n\colon \mathrm{H}^n(C^*) \to \mathrm{H}^{n+1}(A^*)$$

in the associated long exact sequence is given by the composite

$$\mathrm{H}^n(C^*) \xrightarrow{\mathrm{H}^n(\varphi)^{-1}} \mathrm{H}^n(\operatorname{cone}(f)) \longrightarrow \mathrm{H}^n(A[1]^*) = \mathrm{H}^{n+1}(A^*).$$

This yields a long exact sequence

$$\text{(A.87)} \qquad \cdots \longrightarrow \mathrm{H}^n(A^*) \xrightarrow{f} \mathrm{H}^n(B^*) \xrightarrow{\mathrm{b}} \mathrm{H}^n(C^*) \xrightarrow{\mathrm{a}} \mathrm{H}^{n+1}(A^*) \longrightarrow \cdots,$$

where a and b denote the maps (A.26) composed with $\mathrm{H}^n(\varphi)^{-1}$. Besides, the sign in Axiom **(T2)** of Definition A.69 implies that the short exact sequence of complexes

$$0 \longrightarrow B^* \longrightarrow \operatorname{cone}(f) \longrightarrow A[1]^* \longrightarrow 0$$

gives rise to the long exact sequence

$$\text{(A.88)} \qquad \cdots \longrightarrow \mathrm{H}^n(B^*) \xrightarrow{\mathrm{b}} \mathrm{H}^n(C^*) \xrightarrow{\mathrm{a}} \mathrm{H}^{n+1}(A^*) \xrightarrow{-f} \mathrm{H}^{n+1}(B^*) \longrightarrow \cdots.$$

This apparent contradiction in the signs is explained by the fact that the exact sequence (A.88) is isomorphic to (A.87) shifted by 1. Note also that the connection morphism obtained from Definition A.77 is minus the connection morphism of the exact sequence (A.38) that we constructed by hand (see Exercise A.97).

Definition A.89. Let $\mathcal{A}$ be an abelian category. The *derived category* of $\mathcal{A}$ is the category $\mathcal{D}(\mathcal{A})$ we just described. The *localization* functor is the functor

$$Q\colon \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$$

that is the identity on objects and sends a morphism to its equivalence class. There are analogous localization functors from $\mathcal{K}^b(\mathcal{A})$ and $\mathcal{K}^+(\mathcal{A})$ to $\mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}^+(\mathcal{A})$.

A.3.3. *Verdier localization.* The construction of the derived category is an example of a more general process called *Verdier localization*. For more details and a proof of the main result below, the reader is referred to [**Nee01**, § 2.1].

Proposition A.90. *Let $\mathcal{T}$ be a triangulated category and let $\mathcal{E}$ be a triangulated subcategory of $\mathcal{T}$ (Definition A.70). There exists a triangulated category $\mathcal{T}/\mathcal{E}$ that is universal for the following two properties:*

i) *there is a triangulated functor $\mathcal{T} \to \mathcal{T}/\mathcal{E}$ that is the identity on objects;*
ii) *every object $X \in \mathrm{Ob}(\mathcal{E})$ is isomorphic to the zero object 0 in $\mathcal{T}/\mathcal{E}$.*

For example, the derived category $\mathcal{D}^?(\mathcal{A})$ is obtained from the homotopy category $\mathcal{K}^?(\mathcal{A})$ by taking the Verdier localization with respect to the triangulated subcategory of complexes that are quasi-isomorphic to zero.

The Verdier localization is closely related to the notion of thick subcategory.

Definition A.91. A *thick* subcategory of a triangulated category $\mathcal{T}$ is a triangulated subcategory $\mathcal{E}$ that contains all the direct summands of its objects.

Remark A.92. In the statement of Proposition A.90, it is not necessary to assume that the triangulated subcategory $\mathcal{E}$ is thick. Nevertheless, we can always reduce to this case. Indeed, the kernel $\mathcal{E}'$ of the functor $\mathcal{T} \to \mathcal{T}/\mathcal{E}$, defined as the full subcategory of $\mathcal{T}$ consisting of objects that are sent to a zero object, is a thick subcategory by [**Nee01**, Rmk. 2.1.7]. In fact, it is the smallest thick subcategory containing $\mathcal{E}$, and the categories $\mathcal{T}/\mathcal{E}$ and $\mathcal{T}/\mathcal{E}'$ are canonically equivalent.

⋆⋆⋆

Exercise A.93. Let $\mathcal{A}$ be an abelian category and let $f\colon A^* \to B^*$ be a morphism of complexes in $\mathcal{A}$.

i) Assume that $f^n\colon A^n \to B^n$ is a monomorphism for each $n \in \mathbb{Z}$. Let C^* be the complex with $C^n = \operatorname{Coker}(f^n)$ and differential induced by that of B^*. Let $p\colon B^* \to C^*$ be the projection map. Prove that the map

$$\operatorname{cone}(f) \longrightarrow C^*, \quad (a, b) \longmapsto p(b)$$

is a morphism of complexes, which becomes an isomorphism in $\mathcal{D}(\mathcal{A})$.

ii) Assume that the map $f^n\colon A^n \to B^n$ is an epimorphism for each $n \in \mathbb{Z}$. Let C^* be the complex with $C^n = \operatorname{Ker}(f^n)$ and differential induced by that of A^*. Let $\iota\colon C^* \to A^*$ denote the inclusion map. Prove that

$$C^* \longrightarrow \operatorname{Tot}(f), \quad c \longmapsto (\iota(c), 0)$$

is a morphism of complexes, which becomes an isomorphism in $\mathcal{D}(\mathcal{A})$.

iii) Assume that $\mathcal{A}$ has the property that every short exact sequence is split (for instance, the category of vector spaces over a field has this property). Show that the previous quasi-isomorphisms are homotopy equivalences. Give an example showing that the assumption on $\mathcal{A}$ is necessary.

Exercise A.94. Let $f\colon A^* \to B^*$ and $g\colon A^* \to C^*$ be morphisms of complexes in an abelian category $\mathcal{A}$. Show that, if g is a quasi-isomorphism, then the composition $B \to B \oplus C \to \operatorname{cone}(f+g)$ is a quasi-isomorphism.

Exercise A.95. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories.

i) If (A^*, d) is a complex in $\mathcal{A}$, then $(F(A^*), F(\mathrm{d}))$ is a complex in $\mathcal{B}$. Prove that, if f and g are homotopically equivalent morphisms in $\mathcal{A}$, then $F(f)$ and $F(g)$ are also homotopically equivalent in $\mathcal{B}$, so F induces a functor

$$F\colon \mathcal{K}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{B}).$$

Prove also that this functor restricts to functors between the bounded and bounded below homotopy categories.

ii) Prove that the induced functor F is compatible with the shift functor and with formation of the cone. Conclude that F is a triangulated functor.

Exercise A.96. As we saw in Example A.80, taking cohomology in degree zero gives rise to a cohomological functor. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories.

i) Show that the cohomological functor H^0 from $\mathcal{K}(\mathcal{B})$ to $\mathcal{B}$ descends to a cohomological functor $\mathcal{D}(\mathcal{B}) \to \mathcal{B}$.

ii) Starting from a triangulated functor $F\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$, define for $n \in \mathbb{Z}$ the functors $F^n\colon \mathcal{A} \to \mathcal{B}$ as the composition

$$\mathcal{A} \longrightarrow \mathcal{D}(\mathcal{A}) \xrightarrow{F} \mathcal{D}(\mathcal{B}) \xrightarrow{[n]} \mathcal{D}(\mathcal{B}) \xrightarrow{\mathrm{H}^0} \mathcal{B},$$

where the first map sends an object in $\mathcal{A}$ to this object seen as a complex concentrated in degree zero. Show that the functors F^n with the appropriate connection morphisms form a cohomological δ-functor.

iii) Starting from a triangulated functor $F\colon \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$, define for $n \in \mathbb{Z}$ the functors $F^n\colon \mathcal{A} \to \mathcal{B}$ as the composition

$$\mathcal{A} \longrightarrow \mathcal{K}(\mathcal{A}) \xrightarrow{F} \mathcal{K}(\mathcal{B}) \xrightarrow{[n]} \mathcal{D}(\mathcal{B}) \xrightarrow{\mathrm{H}^0} \mathcal{B}.$$

Show by means of an example that the functors F^n do not form a cohomological δ-functor in general.

Exercise A.97. Prove that the connection morphisms appearing in the exact sequence (A.38) and the connection morphism of Remark A.86 have opposite signs.

Exercise A.98. Let $\mathcal{A}$ be an abelian category. In this exercise, we show that in general $\mathcal{K}(\mathcal{A})$ is not abelian, and $\mathcal{C}(\mathcal{A})$ is not triangulated with distinguished triangles given again by $A \to B \to \operatorname{cone}(f) \to A[1]$.

i) Use Proposition A.73 to prove that if

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is a distinguished triangle in a triangulated category and u is a monomorphism, then the triangle is split.

ii) Show that every distinguished triangle in a category that is both abelian and triangulated is isomorphic to one of the form

$$U[-1] \oplus V \xrightarrow{\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{smallmatrix}\right)} V \oplus W \xrightarrow{\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{smallmatrix}\right)} W \oplus U \xrightarrow{\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{smallmatrix}\right)} U \oplus V[1]. \tag{A.99}$$

iii) Consider $\mathbb{Z}$ as a complex in $\mathcal{C}(\mathbf{Ab})$ concentrated in degree zero. Show that, if the triangle

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \operatorname{cone}(2) \longrightarrow \mathbb{Z}[1]$$

could be written in the form (A.99) in either $\mathcal{C}(\mathbf{Ab})$ or in $\mathcal{K}(\mathbf{Ab})$, then the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

would be split in $\mathbf{Ab}$, which is not true.

A.4. Derived functors. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. We know from Example A.81 that F induces a triangulated functor, also denoted by F, between the homotopy categories $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{B})$. If F is exact, then this functor sends quasi-isomorphisms to quasi-isomorphisms, and hence gives rise to a functor between the derived categories $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$. By contrast, non-exact functors do not extend naively to the derived categories. Whenever it exists, the derived functor RF is the triangulated functor between the derived categories that best approximates F.

We will discuss derived functors in the context of bounded below derived categories. This technically easier case is the only one we will use. For instance, the proof of Theorem A.109 that we sketch below uses in a essential way that we are working with bounded below complexes. For more details about derived functors in the unbounded case, the reader may consult [**KS06**, Chap. 14].

A.4.1. *Definition of derived functors.* Recall the localization functor

$$Q\colon \mathcal{K}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{A})$$

from the homotopy to the derived category introduced in Definition A.89.

Definition A.100. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ an additive functor. We also denote by F the induced triangulated functor

$$F\colon \mathcal{K}^+(\mathcal{A}) \longrightarrow \mathcal{K}^+(\mathcal{B})$$

between the corresponding homotopy categories. A *(total) right derived functor* of F is a triangulated functor

$$RF\colon \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{B})$$

together with a natural transformation

$$\xi\colon Q\circ F \longrightarrow RF\circ Q$$

of functors from $\mathcal{K}^+(\mathcal{A})$ to $\mathcal{D}^+(\mathcal{B})$ that satisfies the following universal property: for each triangulated functor $G\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and each natural transformation $\zeta\colon Q\circ F \to G\circ Q$, there exists a unique natural transformation $\eta\colon RF \to G$ with

$$\zeta = (\eta\circ Q)\circ\xi.$$

We give more details on what this definition means. Since the functor Q is the identity on objects, we will denote $Q(A)$ simply by A. The natural transformation ξ provides, for every object A of $\mathcal{K}^+(\mathcal{A})$, a morphism

$$\xi(A)\colon F(A) \longrightarrow RF(A)$$

such that, for every morphism $f \in \operatorname{Hom}_{\mathcal{K}^+(\mathcal{A})}(A,B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\xi(A)"'] & F(B) \arrow[d, "\xi(B)"] \\ RF(A) \arrow[r, "RF(f)"'] & RF(B) \end{tikzcd}$$

commutes in $\mathcal{D}^+(\mathcal{B})$. If G is a functor as in the definition, provided with a natural transformation ζ, then for every object A of $\mathcal{K}^+(\mathcal{A})$ there is also a morphism

$$\zeta(A)\colon F(A) \longrightarrow G(A)$$

such that, for every morphism $f \in \operatorname{Hom}_{\mathcal{K}^+(\mathcal{A})}(A,B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\zeta(A)"'] & F(B) \arrow[d, "\zeta(B)"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes in $\mathcal{D}^+(\mathcal{B})$. The universality of RF means that there exist morphisms

$$\eta(A)\colon RF(A) \longrightarrow G(A), \quad \text{for each} \quad A \in \operatorname{Ob}(\mathcal{D}^+(\mathcal{A})) = \operatorname{Ob}(\mathcal{K}^+(\mathcal{A})),$$

such that the equality $\eta(A)\circ\xi(A)=\zeta(A)$ holds. This η is a natural transformation: for every morphism f as above, the equality $G(f)\circ\eta(A)=\eta(B)\circ RF(f)$ holds.

Remark A.101.

i) Similarly, there is a notion of bounded derived functor
$$RF\colon \mathcal{D}^b(\mathcal{A})\to\mathcal{D}^b(\mathcal{B}).$$
ii) The definition of right derived functor of a contravariant functor is the same as above, thinking of a contravariant functor as a covariant functor on the opposite category.
iii) The definition of the *left derived functor* LF is similar, with the direction of the natural transformations ξ and η reversed.

From the total right derived functor we define the cohomological derived functors by taking cohomology in a given degree.

Definition A.102. Let $F\colon\mathcal{A}\to\mathcal{B}$ be an additive functor between abelian categories, and let $F\colon\mathcal{K}^+(\mathcal{A})\to\mathcal{K}^+(\mathcal{B})$ also denote the extension to the homotopy category. If the total right derived functor RF exists, then the *cohomological derived functors* R^nF are defined, for each $n\in\mathbb{Z}$, as the composition
$$\mathcal{A}\to\mathcal{D}^+(\mathcal{A})\xrightarrow{RF}\mathcal{D}^+(\mathcal{B})\xrightarrow{[n]}\mathcal{D}^+(\mathcal{B})\xrightarrow{\mathrm{H}^0}\mathcal{B}.$$
In other words, for each object $A\in\mathrm{Ob}(\mathcal{A})$, the object $R^nF(A)\in\mathrm{Ob}(\mathcal{B})$ is the cohomology in degree n of the complex $RF(A)$.

Note that the cohomological derived functors $R^nF\colon\mathcal{A}\to\mathcal{B}$ form a cohomological δ-functor in the sense of Definition A.39.

A.4.2. *Categories with enough injectives.* The standard situation in which one can show the existence of the right derived functor is when $F\colon\mathcal{A}\to\mathcal{B}$ is a left exact additive functor and the category $\mathcal{A}$ has enough injectives.

Definition A.103. Let $\mathcal{A}$ be an abelian category. An object I of $\mathcal{A}$ is called *injective* if, for each monomorphism $f\colon A\to B$ and each morphism $\alpha\colon A\to I$, there exists a morphism $\beta\colon B\to I$ satisfying $\alpha=\beta\circ f$. In other words, the map
$$\mathrm{Hom}_{\mathcal{A}}(B,I)\longrightarrow\mathrm{Hom}_{\mathcal{A}}(A,I)$$
given by precomposition with f is surjective.

Injective objects satisfy the following properties:

Lemma A.104. *Let $\mathcal{A}$ be an abelian category and I an injective object of $\mathcal{A}$.*

i) *The functor* $\mathrm{Hom}_{\mathcal{A}}(-,I)$ *is exact.*
ii) *Every short exact sequence* $0\to I\to E\to A\to 0$ *is split.*
iii) *For every object $A\in\mathrm{Ob}(\mathcal{A})$ and every integer $n\geqslant 1$, the Yoneda extension group* $\mathrm{Ext}^n_{\mathcal{A}}(A,I)$ *vanishes.*

Proof. The first statement is just a reformulation of the definition of an injective object because the functor $\mathrm{Hom}_{\mathcal{A}}(-,I)$ is left exact for all objects I. Let
$$0\longrightarrow I\longrightarrow E\longrightarrow A\longrightarrow 0$$

be a short exact sequence. By the definition of an injective object, there is a morphism $E \to I$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & E \arrow[dl] \\ & I & \end{tikzcd}$$

commutative. This precisely means that the sequence is split. The third statement follows either from the first one and Lemma A.66, or from the second statement using the connection morphism. □

Lemma A.105. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor that admits an exact left adjoint functor (Definition A.10). Then F preserves injective objects.*

Proof. Let I be an injective object of $\mathcal{A}$ and let $f\colon A \to B$ be a monomorphism in $\mathcal{B}$. Consider the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{A}}(G(B), I) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Hom}_{\mathcal{A}}(G(A), I) \arrow[d, "\simeq"] \\ \operatorname{Hom}_{\mathcal{B}}(B, F(I)) \arrow[r] & \operatorname{Hom}_{\mathcal{B}}(A, F(I)), \end{tikzcd}$$

where G is a left exact adjoint functor. Since G is exact, $G(f)\colon G(A) \to G(B)$ is a monomorphism and, the object I being injective, the top horizontal arrow is surjective. Hence, so is the bottom one, which means that $F(I)$ is injective. □

Intuitively, on an abelian category that only contains injective objects any additive functor would be exact, since any short exact sequence would be split and additive functors preserve direct sums. The idea to correct the lack of exactness of a functor is then to replace any object with an injective one.

Definition A.106. An abelian category $\mathcal{A}$ is said to have *enough injectives* if, for each $A \in \operatorname{Ob}(\mathcal{A})$, there is an injective object I and a monomorphism $A \to I$.

Example A.107. The injective objects of the category $\mathbf{Ab}$ of abelian groups are the *divisible* abelian groups (see Exercices A.55 and A.128). For example, the group $I = \mathbb{Q}/\mathbb{Z}$ is divisible, and hence injective. For an abelian group A, let $I(A)$ denote the product of copies of $\mathbb{Q}/\mathbb{Z}$ indexed by the set $\operatorname{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})$. By Exercise A.129, the abelian group $I(A)$ is injective. By Excercise A.127, the map

$$\begin{aligned} e_A\colon\ A &\longrightarrow I(A) \\ a &\longmapsto (f(a))_{f \in \operatorname{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \end{aligned}$$

is a monomorphism. From this, it follows that the $\mathbf{Ab}$ has enough injectives.

Example A.108. The category $\mathbf{Mod}_R$ of left modules over a general ring R has enough injectives as well. For this, we first observe that, given a right (resp. left) R-module M and an abelian group A, the set $\operatorname{Hom}_{\mathbf{Ab}}(M, A)$ has a structure of left (resp. right) R-module, given by $(rf)(a) = f(ar)$ (resp. $(fr)(a) = f(ra)$). In particular, this gives us a functor $\operatorname{Hom}_{\mathbf{Ab}}(R, -)$ from $\mathbf{Ab}$ to $\mathbf{Mod}_R$. This functor is a right adjoint to the forgetful functor from $\mathbf{Mod}_R$ to $\mathbf{Ab}$ that is exact. It then follows from Lemma A.105 that the functor $\operatorname{Hom}_{\mathbf{Ab}}(R, -)$ sends injective objects to injective objects. Let now M be a left R-module, and let $I(M)$ be the product of copies of $\mathbb{Q}/\mathbb{Z}$ indexed by $\mathrm{Hom}_{\mathbf{Ab}}(M, \mathbb{Q}/\mathbb{Z})$. The morphism $M \to I(M)$ of abelian groups discussed in Example A.107 induces a morphism of R-modules

$$\mathrm{Hom}_{\mathbf{Ab}}(R, M) \longrightarrow \mathrm{Hom}_{\mathbf{Ab}}(R, I(M)).$$

Since the functor $\mathrm{Hom}_{\mathbf{Ab}}(R, -)$ left exact, this map is a monomorphism. Composing with the monomorphism $M \to \mathrm{Hom}_{\mathbf{Ab}}(R, M)$ that sends an element $m \in M$ to the unique R-linear map that sends 1 to m, we obtain a monomorphism

$$M \longrightarrow \mathrm{Hom}_{\mathbf{Ab}}(R, I(M)).$$

By Exercise A.129, the abelian group $I(M)$ is injective and, by Lemma A.105, the R-module $\mathrm{Hom}_{\mathbf{Ab}}(R, I(M))$ is injective. Thus, $\mathbf{Mod}_R$ has enough injectives.

The previous examples suggest that it might not be possible to find a monomorphism into an injective object if one imposes some finiteness conditions on the objects. Indeed, the subcategory of $\mathbf{Mod}_R$ consisting of finitely generated left R-modules does not have enough injectives in general. For example, already in the case $R = \mathbb{Z}$, there are no finitely generated divisible abelian groups.

A.4.3. *Existence of derived functors in the presence of enough injectives.*

THEOREM A.109. *Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{I}$ the full subcategory of $\mathcal{A}$ consisting of injective objects. The natural functor*

$$\mathcal{K}^+(\mathcal{I}) \longrightarrow \mathcal{D}^+(\mathcal{A}) \tag{A.110}$$

is an equivalence of categories.

This is proved, for instance, in [**GM03**, Thm. III.5.21], relying on the existence and main properties of injective resolutions.

DEFINITION A.111.

i) Let A^* be a bounded below complex in $\mathcal{C}^+(\mathcal{A})$. A *resolution* of A^* is a bounded below complex I^* with a quasi-isomorphism $A^* \to I^*$.
ii) Let $A \in \mathrm{Ob}(\mathcal{A})$ be an object. An *injective resolution* of A is a complex of injective objects $I^0 \to I^1 \to \cdots$ along with a morphism $A \to I^0$ such that the sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

is exact. Note that this is a particular case of a resolution of a complex concentrated in degree zero.

To show that the functor (A.110) is essentially surjective, one proves that every bounded below cochain complex admits an injective resolution. To show that the functor is fully faithful, one combines the following results:

i) a morphism $f\colon A^* \to B^*$ in $\mathcal{C}^+(\mathcal{A})$ can always be completed to a diagram

$$\begin{tikzcd} A^* \arrow[r] \arrow[d] & B^* \arrow[d] \\ I_A^* \arrow[r] & I_B^*, \end{tikzcd}$$

where I_A^* and I_B^* are injective resolutions of A and B respectively;

ii) if $f\colon A^* \to I^*$ and $g\colon A^* \to J^*$ are quasi-isomorphisms, then there exist a third bounded below complex K^* made up of injective objects and quasi-isomorphisms $\psi\colon I^* \to K^*$ and $\varphi\colon J^* \to K^*$;

iii) if $I_1^* \to I_2^*$ is a quasi-isomorphism between objects of $\mathcal{K}^+(\mathcal{I})$, then it is a homotopy equivalence, and hence already an isomorphism in $\mathcal{K}^+(\mathcal{I})$.

PROPOSITION A.112. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives. Then the composition*

$$\mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{K}^+(\mathcal{I}) \xrightarrow{F} \mathcal{K}^+(\mathcal{B}) \xrightarrow{Q} \mathcal{D}^+(\mathcal{B}),$$

where the first functor is a quasi-inverse of the equivalence of categories of Theorem A.109, satisfies the universal property of Definition A.100, and hence is the total derived functor of F. In concrete terms, RF can be computed as

$$RF(A^*) = F(I^*),$$

where I^* is any injective resolution of A^*.

PROOF. Let $G\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ be a functor along with a natural transformation $Q \circ F \to G \circ Q$. Out of any injective resolution $A^* \to I^*$, we obtain a morphism $RF(A^*) \to G(A^*)$ as the composition

$$RF(A^*) = F(I^*) \longrightarrow G(I^*) \longrightarrow G(A^*),$$

where the last morphism is the result of applying the functor G to the inverse of the isomorphism $A^* \to I^*$ in $\mathcal{D}^+(\mathcal{A})$. Then one verifies that this composition is independent of the choice of the resolution, and that it provides a natural transformation $RF \to G$ satisfying the required properties. This is done in the proof of [**GM03**, Thm. III.6.8]. In this reference, a slightly more general result is proved in terms of *an adapted class of objects*, but as explained in [**GM03**, Thm. III.6.12], under the assumption that there are enough injectives, the class of injective objects is an adapted class in which quasi-isomorphism are homotopy equivalences. □

It follows from Proposition A.112 that the *cohomological derived functors*

$$R^nF\colon \mathcal{A} \longrightarrow \mathcal{B}$$

of Definition A.102 are given by

$$R^nF(A) = \mathrm{H}^n(F(I^*)),$$

for any injective resolution I^* of an object $A \in Ob(\mathcal{A})$.

LEMMA A.113. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives.*

i) *For each object $A \in Ob(\mathcal{A})$, the equality $F(A) = R^0F(A)$ holds.*

ii) *For each injective object I, the vanishing $R^nF(I) = 0$ holds for all $n \geqslant 1$.*

PROOF. Exercise A.126. □

A.4.4. *Yoneda extensions and higher derived functors.*

EXAMPLE A.114. Let $\mathcal{A}$ be an abelian category with enough injectives. Then the Yoneda extension groups can also be obtained by deriving the functor Hom. More precisely, given an object $A \in \mathrm{Ob}(\mathcal{A})$, the functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ is left exact (Exercise A.59) and, for any object $B \in \mathrm{Ob}(\mathcal{A})$, there are functorial isomorphisms

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) \xrightarrow{\sim} R^n \mathrm{Hom}_{\mathcal{A}}(A, -)(B). \tag{A.115}$$

For ease of notation, we write $R^n \mathrm{Hom}_{\mathcal{A}}(A, -)(B) = R^n \mathrm{Hom}_{\mathcal{A}}(A, B)$. This is unambiguous in view of the equality

$$R^n \mathrm{Hom}_{\mathcal{A}}(A, -)(B) = R^n \mathrm{Hom}_{\mathcal{A}}(-, B)(A).$$

To prove (A.115), we start with the equalities

$$\mathrm{Ext}^0_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B) = R^0 \mathrm{Hom}_{\mathcal{A}}(A, B)$$

that hold by definition and Lemma A.113 i). Assume by induction that there are functorial isomorphisms

$$\mathrm{Ext}^m_{\mathcal{A}}(A, B) \xrightarrow{\sim} R^m \mathrm{Hom}_{\mathcal{A}}(A, B)$$

for all integers $0 \leqslant m < n$. Let

$$0 \longrightarrow B \longrightarrow I \longrightarrow C \longrightarrow 0$$

be a short exact sequence with I injective. By induction, there is a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Ext}^{n-1}_{\mathcal{A}}(A, I) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^{n-1}_{\mathcal{A}}(A, C) \arrow[r, "\partial"] \arrow[d, "\simeq"] & \mathrm{Ext}^n_{\mathcal{A}}(A, B) \arrow[r] & 0 \\
R^{n-1} \mathrm{Hom}_{\mathcal{A}}(A, I) \arrow[r] & R^{n-1} \mathrm{Hom}_{\mathcal{A}}(A, C) \arrow[r, "\partial"] & R^n \mathrm{Hom}_{\mathcal{A}}(A, B) \arrow[r] & 0.
\end{tikzcd}$$

Since the two vertical arrows are isomorphisms, there is a unique isomorphism

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) \xrightarrow{\simeq} R^n \mathrm{Hom}_{\mathcal{A}}(A, B)$$

making the diagram commutative. One then checks that this isomorphism is independent of the chosen exact sequence and is functorial.

In the previous example, we have interpreted Yoneda extension groups as the derived functors of the Hom functor. There is yet another interpretation as Hom functors in the derived category.

PROPOSITION A.116. *Let $\mathcal{A}$ be an abelian category, and let $A, B \in \mathrm{Ob}(\mathcal{A})$ be objects of $\mathcal{A}$. We see A and B as objects in $\mathcal{D}^+(\mathcal{A})$ concentrated in degree zero. For each $n \geqslant 0$, there are functorial isomorphisms*

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(A, B[n]).$$

PROOF. We start by constructing the map. Let

$$E\colon \quad 0 \longrightarrow B \xrightarrow{\iota} C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \xrightarrow{\pi} A \longrightarrow 0$$

be a degree n extension of A by B. Let C^* be the complex obtained from E by deleting A and putting the object C_i in degree $-i$, so that B sits in degree $-n$. The map $C_0 \to A$ induces a morphisms of complexes $C^* \to A$, which by the exactness of E is a quasi-isomorphism. The identity of B also defines a morphism of complexes $C^* \to B[n]$. We then map the extension E to the morphism

$$\begin{tikzcd} & C^* \arrow[dl, "\simeq"'] \arrow[dr] & \\ A & & B[n] \end{tikzcd} \tag{A.117}$$

in the derived category. Conversely, for an element of $\mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(A, B[n])$ represented by a diagram (A.117) with (C^*, d) a complex quasi-isomorphic to A, we set

$$\begin{aligned} C_0' &= \mathrm{Ker}(d^0 \colon C^0 \longrightarrow C^1), \\ C_i' &= C^{-i}, \text{ for } i = 1, \dots, n-1, \\ B' &= C^{-n} / \mathrm{Im}(\mathrm{d}^{-n-1}). \end{aligned}$$

We obtain an extension $E' \in \mathrm{Ext}^n_{\mathcal{A}}(A, B')$ and a morphism $B' \to B$. By the functoriality of Yoneda extensions, we deduce an extension $E \in \mathrm{Ext}^n_{\mathcal{A}}(A, B)$.

The following facts are left as an exercise:

i) Equivalent extensions give rise to the same morphism in $\mathcal{D}^+(\mathcal{A})$.
ii) Two representations of the same morphism in the derived category define the same class of extensions.
iii) The two constructions are inverse of each other.

This concludes the proof. □

A.4.5. *Projective objects.* The dual notion of injective object is that of projective object. Most of the discussion in the previous sections explaining how to compute right derived functors using injective objects carries through and allows us to compute left derived functors using projective objects. This is left as an exercise in [**GM03**, Ex. III.5.1] and is developed in detail in [**Wei94**, § 2.4].

Definition A.118. Let $\mathcal{A}$ be an abelian category. An object P of $\mathcal{A}$ is called *projective* if, for each epimorphism $f \colon B \to A$ and each morphism $\alpha \colon P \to A$, there exists a morphism $\beta \colon P \to B$ satisfying $\alpha = f \circ \beta$. In other words, the map

$$\mathrm{Hom}_{\mathcal{A}}(P, B) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, A)$$

given by postcomposition with f is surjective.

The dual of Lemma A.104 is the next result, whose proof is left to the reader.

Lemma A.119. *Let $\mathcal{A}$ be an abelian category and P a projective object of $\mathcal{A}$.*

i) *The functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact.*
ii) *Every short exact sequence $0 \to A \to E \to P \to 0$ is split.*
iii) *For every object $A \in \mathrm{Ob}(\mathcal{A})$ and every integer $n \geqslant 1$, the Yoneda extension group $\mathrm{Ext}^n_{\mathcal{A}}(P, A)$ vanishes.*

There is also the notion of having enough projectives.

Definition A.120. An abelian category $\mathcal{A}$ is said to have *enough projectives* if, for each $A \in \mathrm{Ob}(\mathcal{A})$, there is a projective object P and an epimorphism $P \to A$.

Example A.121. Let R be a ring. By Exercise A.132, the category $\mathbf{Mod}_R$ of left R-modules has enough projectives.

Dually to Theorem A.109, we have:

Theorem A.122. *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{P}$ the full subcategory of $\mathcal{A}$ consisting of projective objects. The natural functor*

$$\mathcal{K}^-(\mathcal{P}) \longrightarrow \mathcal{D}^-(A)$$

is an equivalence of categories.

If A^* is a bounded above complex on an abelian category $\mathcal{A}$, a *projective resolution* of A^* is a bounded above complex P^* consisting of projective objects, together with a quasi-isomorphism $P^* \to A^*$. As for a category with enough injectives, the main ingredient of the proof of Theorem A.122 is that, if $\mathcal{A}$ has enough projectives, then projective resolutions always exist. By the dual argument of the proof of Proposition A.112, we then deduce the following.

Corollary A.123. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor. Assume that $\mathcal{A}$ has enough projectives. Then the total left derived functor LF of F can be computed as*

$$LF(A^*) = F(P^*),$$

where P^* is any projective resolution of A^*.

Example A.124. Let R be a ring and M an R-module. The functor

$$\cdot \otimes M\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$$

is right exact. The Tor-*modules* are defined as the cohomological left derived functors of the tensor product functor. That is,

$$\mathrm{Tor}_k^R(N, M) = L_k(\cdot \otimes M)(N).$$

Remark A.125. Recall that the data of a *contravariant functor* $F\colon \mathcal{A} \to \mathcal{B}$ is equivalent to the data of a covariant functor $F^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$. A contravariant functor is left (resp. right) exact if and only if F^{op} is. An injective object of $\mathcal{A}$ is the same as a projective object of $\mathcal{A}^{\mathrm{op}}$, and vice versa. Therefore, the right derived functors of a left exact contravariant functor are computed using projective resolutions, whereas the left derived functors of a right exact contravariant functor are computed using injective resolutions. See Exercise A.133 for an example.

⋆⋆⋆

Exercise A.126. Prove Lemma A.113.

Exercise A.127. Let A be an abelian group and $a \in A$ a non-neutral element. Show that there is a morphism of groups $f\colon A \to \mathbb{Q}/\mathbb{Z}$ satisfying $f(a) \neq 0$.

Exercise A.128. Prove that an abelian group is an injective object in the category $\mathbf{Ab}$ if and only if it is a divisible abelian group.

Exercise A.129. Let $\mathcal{A}$ be an abelian category and let $(I_\alpha)_\alpha$ be a family of objects of $\mathcal{A}$. Assume that there exist an object I of $\mathcal{A}$ and morphisms $I \to I_\alpha$ for all α such that, for every object X of $\mathcal{A}$ along with morphisms $X \to I_\alpha$ for all α, there exists a unique morphism $X \to I$ making the diagrams

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & I \arrow[dl] \\ & I_\alpha & \end{tikzcd}$$

commutative for all α. We then say that I is the *product* of the objects I_α and write $I = \prod I_\alpha$ (see A.163). Show that a product of injective objects is injective.

Exercise A.130. Fill the details in the proof of Proposition A.116.

Exercise A.131. Let $\mathcal{A}$ be an abelian category that has enough injectives.

i) (Horseshoe lemma) Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$, and let I_A^* and I_C^* be injective resolutions of A and C respectively. Prove that there is an injective resolution I_B^* of B and morphisms of complexes $I_A^* \to I_B^*$ and $I_B^* \to I_C^*$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & I_A^* \arrow[r] & I_B^* \arrow[r] & I_C^* \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

is commutative with exact rows.

ii) Let A^* be a bounded below complex in $\mathcal{A}$. Show that we can find a double complex $I^{*,*}$ of injective objects, fitting in a commutative diagram

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ \cdots \arrow[r] & I^{n-1,1} \arrow[r, "d"] \arrow[u] & I^{n,1} \arrow[r, "d"] \arrow[u] & I^{n+1,1} \arrow[r] \arrow[u] & \cdots \\ \cdots \arrow[r] & I^{n-1,0} \arrow[r, "d"] \arrow[u] & I^{n,0} \arrow[r, "d"] \arrow[u] & I^{n+1,0} \arrow[r] \arrow[u] & \cdots \\ \cdots \arrow[r] & A^{n-1} \arrow[r, "d"] \arrow[u] & A^{n} \arrow[r, "d"] \arrow[u] & A^{n+1} \arrow[r] \arrow[u] & \cdots \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

with exact columns and such that, for each p, the following holds:

a) the complex $B^{p,*} = \operatorname{Im}(d\colon I^{p-1,*} \to I^{p,*})$ is an injective resolution of $\operatorname{Im}(d\colon A^{p-1} \to A^p)$;

b) the complex $Z^{p,*} = \operatorname{Ker}(d\colon I^{p,*} \to I^{p+1,*})$ is an injective resolution of $\operatorname{Ker}(d\colon A^p \to A^{p+1})$;

c) the complex $Z^{p,*}/B^{p,*}$ is an injective resolution of $\mathrm{H}^p(A^*)$.

iii) Let us now assume that $\mathcal{A}$ has enough projectives instead of injectives. State and prove the corresponding statements for projective objects.

Exercise A.132. Let $\mathbf{Mod}_R$ be the category of left modules over a ring R. Show that an object P of $\mathbf{Mod}_R$ is projective if and only if it is a direct summand of a free R-module. Conclude that the category $\mathbf{Mod}_R$ has enough projectives.

Exercise A.133. Let $\mathcal{A}$ be a category with enough injectives and projectives. For any object $A \in \mathrm{Ob}(\mathcal{A})$, the functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ is a left exact covariant functor, while $\mathrm{Hom}_{\mathcal{A}}(-, A)$ is a left exact contravariant functor. Therefore, the notation

$$R^n \operatorname{Hom}_{\mathcal{A}}(A, B)$$

might in principle be ambiguous. Your task in this exercise is to show that there is no such ambiguity. Prove that there are canonical isomorphisms

$$R^n \operatorname{Hom}_{\mathcal{A}}(-, B)(A) \xrightarrow{\sim} R^n \operatorname{Hom}_{\mathcal{A}}(A, -)(B),$$

where the right-hand side is computed using an injective resolution of B, and the left-hand side using a projective resolution of A.

Exercise A.134. By Exercise A.132, every free abelian group is projective.

i) Use that any subgroup of a free abelian group is free to prove that every abelian group G admits a two steps projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow G \longrightarrow 0.$$

ii) Using such a resolution, prove the vanishing $\mathrm{Tor}^k_{\mathbf{Ab}}(A, B) = 0$ for all abelian groups A, B and all $k \geqslant 2$.

iii) Use Exercise A.133 to show the vanishing $\mathrm{Ext}^k_{\mathbf{Ab}}(A, B) = 0$ for all abelian groups A, B and all $k \geqslant 2$.

A.5. t-structures. There are many natural situations in which we are able to construct a triangulated category but we would like to obtain an abelian category instead. In their work on perverse sheaves [**BBD82**], Beilinson, Bernstein, Deligne, and Gabber introduced the notion of a *t-structure* as a way of extracting an abelian category from a triangulated category. This is precisely how the abelian category of mixed Tate motives over a number field is constructed in Section 4.3.

A.5.1. *t-structures and their hearts.*

Definition A.135 (Beilinson–Bernstein–Deligne–Gabber). Let $\mathcal{T}$ be a triangulated category. A *t-structure* on $\mathcal{T}$ is a pair of *strictly full* (that is, full and closed under isomorphism) subcategories

$$(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$$

such that, defining for each integer n the subcategories

$$\mathcal{T}^{\leqslant n} = \mathcal{T}^{\leqslant 0}[-n] \quad \text{and} \quad \mathcal{T}^{\geqslant n} = \mathcal{T}^{\geqslant 0}[-n]$$

of $\mathcal{T}$, the following three conditions hold:

i) $\mathcal{T}^{\leqslant -1} \subseteq \mathcal{T}^{\leqslant 0}$ and $\mathcal{T}^{\geqslant 1} \subseteq \mathcal{T}^{\geqslant 0}$.

ii) (*Orthogonality*) If $X \in \mathrm{Ob}(\mathcal{T}^{\leqslant 0})$ and $Y \in \mathrm{Ob}(\mathcal{T}^{\geqslant 1})$, then

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0.$$

iii) Each object $X \in \mathrm{Ob}(\mathcal{T})$ sits into a distinguished triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y[1] \tag{A.136}$$

with $Y \in \mathrm{Ob}(\mathcal{T}^{\leqslant 0})$ and $Z \in \mathrm{Ob}(\mathcal{T}^{\geqslant 1})$.

We say that a t-structure is *non-degenerate* if, in addition to the previous axioms, the intersections $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\leqslant n}$ and $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\geqslant n}$ are reduced to zero.

Definition A.137. The *heart* of a t-structure is the full subcategory

$$\mathcal{T}^0=\mathcal{T}^{\leqslant 0}\cap\mathcal{T}^{\geqslant 0}.$$

A functor $F\colon \mathcal{T}_1\to\mathcal{T}_2$ of triangulated categories endowed with t-structures is said to be *t-exact* whenever $F(\mathcal{T}_1^{\leqslant 0})\subseteq\mathcal{T}_2^{\leqslant 0}$ and $F(\mathcal{T}_1^{\geqslant 0})\subseteq\mathcal{T}_2^{\geqslant 0}$, and hence F restricts to a functor between the hearts.

Note that the objects Y and Z in the triangle (A.136) are not a priori required to be unique. However, this follows from the other axioms:

Lemma A.138 ([**BBD82**, Prop. 1.3.3]). *Let $\mathcal{T}$ be a triangulated category endowed with a t-structure.*

i) *The inclusion of $\mathcal{T}^{\leqslant n}$ into $\mathcal{T}$ admits a right adjoint functor*

$$t_{\leqslant n}\colon\mathcal{T}\longrightarrow\mathcal{T}^{\leqslant n}$$

and the inclusion $\mathcal{T}^{\geqslant n}$ into $\mathcal{T}$ admits a left adjoint functor

$$t_{\geqslant n}\colon\mathcal{T}\longrightarrow\mathcal{T}^{\geqslant n}.$$

ii) *For each object $X\in\mathrm{Ob}(\mathcal{T})$, there exists a unique morphism*

$$w\in\mathrm{Hom}_{\mathcal{T}}(t_{\geqslant 1}X,t_{\leqslant 0}X[1])$$

such that the following is a distinguished triangle:

$$t_{\leqslant 0}X\longrightarrow X\longrightarrow t_{\geqslant 1}X\xrightarrow{w}t_{\leqslant 0}X[1].$$

Up to unique isomorphism, this triangle is the only one satisfying condition iii) in Definition A.135.

Moreover, for $a\leqslant b$, there is a unique isomorphism

$$t_{\geqslant a}t_{\leqslant b}X\xrightarrow{\sim}t_{\leqslant b}t_{\geqslant a}X. \tag{A.139}$$

The standard example of a t-structure is the following:

Example A.140. Let $\mathcal{A}$ be an abelian category, and let $D^b(\mathcal{A})$ be its bounded derived category as in Section A.3.2. This triangulated category is endowed with a canonical t-structure that measures how far a complex is from having its cohomology concentrated in degree zero. For each integer n, define the full subcategories

$$\begin{aligned}\mathcal{T}^{\leqslant n}&=\{C^*\in D^b(\mathcal{A})\mid \mathrm{H}^m(C^*)=0\text{ for all }m>n\},\\ \mathcal{T}^{\geqslant n}&=\{C^*\in D^b(\mathcal{A})\mid \mathrm{H}^m(C^*)=0\text{ for all }m<n\}.\end{aligned} \tag{A.141}$$

We claim that the pair $(\mathcal{T}^{\leqslant 0},\mathcal{T}^{\geqslant 0})$ forms a non-degenerate t-structure on $\mathcal{D}^b(\mathcal{A})$. Its heart $\mathcal{T}^0$ is the subcategory of complexes whose cohomology is concentrated in degree 0, and the functor $\mathcal{A}\to\mathcal{T}^0$ obtained by viewing an object of $\mathcal{A}$ as a complex concentrated in degree 0 is an equivalence of categories.

First of all, the relations

$$\mathcal{T}^{\leqslant n}=\mathcal{T}^{\leqslant 0}[-n],\quad \mathcal{T}^{\geqslant n}=\mathcal{T}^{\geqslant 0}[-n],\quad \mathcal{T}^{\leqslant -1}\subseteq\mathcal{T}^{\leqslant 0},\quad \mathcal{T}^{\geqslant 1}\subseteq\mathcal{T}^{\geqslant 0}$$

are clear from the formulas (A.141). In particular, axiom i) of Definition A.135 holds. To check the remaining axioms and make the functors from Lemma A.138 explicit, we consider the *canonical truncations*

$$(\tau_{\leqslant n}C)^p = \begin{cases} C^p, & \text{if } p < n, \\ \mathrm{Ker}(\mathrm{d}), & \text{if } p = n, \\ 0, & \text{if } p > n. \end{cases} \qquad (\tau^{\geqslant n}C)^p = \begin{cases} 0, & \text{if } p < n, \\ C^p/\,\mathrm{Im}(\mathrm{d}), & \text{if } p = n, \\ C^p, & \text{if } p > n. \end{cases}$$

For each integer n, the inclusions and projections induce morphisms of complexes

$$\tau_{\leqslant n}C \longrightarrow C \quad \text{and} \quad C \longrightarrow \tau^{\geqslant n}C. \tag{A.142}$$

Moreover, the canonical truncations are functorial and satisfy

$$\mathrm{H}^p(\tau_{\leqslant n}C) = \begin{cases} \mathrm{H}^p(C), & \text{if } p \leqslant n, \\ 0, & \text{if } p > n, \end{cases}$$

$$\mathrm{H}^p(\tau^{\geqslant n}C) = \begin{cases} 0, & \text{if } p < n, \\ \mathrm{H}^p(C), & \text{if } p \geqslant n. \end{cases}$$

In particular, $\tau_{\leqslant n}C$ belongs to the subcategory $\mathcal{T}^{\leqslant n}$, and $\tau^{\geqslant n}C$ belongs to $\mathcal{T}^{\geqslant n}$. In this case, the functors $t_{\leqslant n}$ and $t_{\geqslant n}$ are given by the canonical truncations, namely

$$t_{\leqslant n}C = \tau_{\leqslant n}C \quad \text{and} \quad t_{\geqslant n}C = \tau^{\geqslant n}C,$$

so that the following equality holds:

$$t_{\leqslant 0}t_{\geqslant 0}(C[n]) = \mathrm{H}^n(C).$$

Let us check the orthogonality axiom ii). Let C^* and D^* be complexes satisfying $\mathrm{H}^m(C^*) = 0$ for all $m > n$ and $\mathrm{H}^m(D^*) = 0$ for all $m \leqslant n$, and let $f\colon C^* \to D^*$ be a morphism of complexes. Then we can complete f to a sequence

$$\tau_{\leqslant n}C \xrightarrow{\sim} C^* \xrightarrow{f} D^* \xrightarrow{\sim} \tau^{\leqslant n+1}D, \tag{A.143}$$

where the leftmost and the rightmost morphisms are quasi-isomorphisms by the assumptions on the cohomology of the complexes. Thus, $\tau_{\leqslant n}C$ and $\tau^{\leqslant n+1}D$ represent the same objects as C^* and D^* in the derived category. The composition (A.143) is zero since the source and the target are complexes concentrated in disjoint degrees, and hence f is zero as well. The distinguished triangle (A.136) is given by

$$\tau_{\leqslant 0}C \longrightarrow C \longrightarrow \tau^{\geqslant 1}C \longrightarrow \tau_{\leqslant 0}C[1],$$

where the first two arrows are the morphisms (A.142), and the last arrow is zero. Finally, the t-structure is non-degenerate since $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\leqslant n}$ and $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\geqslant n}$ consist of complexes quasi-isomorphic to zero.

We now copy two definitions from [**BBD82**, § 1.2].

Definition A.144. Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{A}$ be a subcategory of $\mathcal{T}$. Given objects $X, Y, Z \in \mathrm{Ob}(\mathcal{T})$, we say that Y is an *extension* of Z by X if there exists a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

We say that $\mathcal{A}$ is *stable under extensions* if, whenever X and Z are objects of $\mathcal{A}$ and Y is an extension of Z by X, then Y belongs to $\mathcal{A}$ as well.

DEFINITION A.145. Let $\mathcal{T}$ be as above and let $\mathcal{A}$ be a full abelian subcategory of $\mathcal{T}$. We say that $\mathcal{A}$ is *admissible* if it satisfies the following conditions:

i) given objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ and an integer $i < 0$,
$$\mathrm{Hom}_{\mathcal{T}}(X, Y[i]) = 0;$$

ii) given objects $A, B, C \in \mathrm{Ob}(\mathcal{A})$, a sequence
$$0 \longrightarrow B \xrightarrow{u} C \xrightarrow{v} A \longrightarrow 0$$
is a short exact sequence in $\mathcal{A}$ if and only if
$$B \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B[1] \tag{A.146}$$
is a distinguished triangle in $\mathcal{T}$.

REMARK A.147. The definition presented in [**BBD82**, 1.2.5] is apparently weaker than the above Definition A.145, as it only demands that every short exact sequence $B \to C \to A$ can be extended to a distinguished triangle. Nevertheless, it follows from Propositions 1.2.2 and 1.2.4 of loc. cit. that both definitions are in fact equivalent. In [**Lev93**] the definition of admissible subcategory only includes statement ii) and not condition i). In any case, if a subcategory satisfies the conditions of Definition A.145 it is also admissible in the sense of [**Lev93**].

REMARK A.148. The extension (A.146) to a distinguished triangle is unique. Indeed, it follows from axiom **(T3)** in the definition of triangulated categories that, given two extensions as in (A.146), the identity maps $B \to B$ and $C \to C$ can be completed to a morphism of triangles
$$\begin{tikzcd}
B \arrow[r, "u"] \arrow[d, equal] & C \arrow[r, "v"] \arrow[d, equal] & A \arrow[r, "w"] \arrow[d, "h"'] & B[1] \arrow[d, equal] \\
B \arrow[r, "u"] & C \arrow[r, "v"] & A \arrow[r, "w'"] & B[1]
\end{tikzcd}$$
in $\mathcal{T}$. In particular, $w = w' \circ h$ holds, and uniqueness amounts to proving that h is the identity. Since $\mathcal{A}$ is a full subcategory by definition, $h\colon A \to A$ is a morphism in $\mathcal{A}$ satisfying $h \circ v = v$. Since v is an epimorphism, we deduce that $h = \mathrm{Id}_A$.

The following theorem is proved in [**BBD82**, Thm. 1.3.6]:

THEOREM A.149 (Beilinson–Bernstein–Deligne–Gabber). *Let $\mathcal{T}$ be a triangulated category and $(\mathcal{T}^{\geqslant 0}, \mathcal{T}^{\leqslant 0})$ a t-structure. The heart $\mathcal{T}^0$ is a full admissible abelian subcategory of $\mathcal{T}$, which is stable under extensions.*

REMARK A.150. In general, however, $\mathcal{T}$ is not equivalent to the derived category of the heart of the t-structure as a triangulated category (see Exercise A.155).

DEFINITION A.151. Let $\mathcal{T}$ be a triangulated category endowed with a t-structure. For each integer n, the n-th cohomology of an object $X \in \mathrm{Ob}(\mathcal{T})$ with respect to the t-structure is the following object of the heart:
$$h^n(X) = t_{\leqslant 0} t_{\geqslant 0}(X[n]) \in \mathrm{Ob}(\mathcal{T}^0). \tag{A.152}$$

By [**BBD82**, Thm. 1.3.6], the functor h^0 is a *cohomological functor*. Recall from Definition A.77 that this means that a distinguished triangle $X \to Y \to Z \to X[1]$ induces a long exact sequence
$$\cdots \longrightarrow h^n(X) \longrightarrow h^n(Y) \longrightarrow h^n(Z) \longrightarrow h^{n+1}(X) \longrightarrow \cdots .$$

A.5.2. *Extensions.* Recall from Proposition A.116 that the extension groups in an abelian category can be interpreted as morphism groups in the derived category. To some extent, this generalizes to abelian subcategories of a triangulated category. Consider a full admissible abelian subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$. The definition of the map $\mathrm{Ext}^n_{\mathcal{A}}(A,B) \to \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(A,B[n])$ can be adapted to the triangulated category $\mathcal{T}$. Indeed, let $0 \to B \to C \to A \to 0$ be an extension in $\mathcal{A}$. By Remark A.148, there exists a *unique* distinguished triangle $B \to C \to A \to B[1]$, yielding a map $w\colon A \to B[1]$. Moreover, the same argument shows that equivalent extensions give rise to the same morphism w. We thus obtain a homomorphism

$$\varphi_1\colon\ \mathrm{Ext}^1_{\mathcal{A}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A,B[1]).$$

More generally, breaking a degree n extension

$$0 \longrightarrow B \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

into several short exact sequences yields a morphism $A \to B[n]$ which only depends on the equivalence class of the extension. For instance, if $n = 2$, one associates with

$$0 \longrightarrow B \longrightarrow C_1 \xrightarrow{a} C_0 \xrightarrow{b} A \longrightarrow 0$$

the short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & C_1 \arrow[r] & \mathrm{Im}(a) \arrow[r] \arrow[d, equal] & 0 & & \\
 & & 0 \arrow[r] & \mathrm{Ker}(b) \arrow[r] & C_0 \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Setting $D = \mathrm{Im}(a) = \mathrm{Ker}(b)$ and applying φ_1 to the rows of the above diagram, we get maps $\alpha\colon D \to B[1]$ and $\beta\colon A \to D[1]$. Then we form

$$\alpha[1] \circ \beta\colon A \longrightarrow B[2].$$

Proposition A.153. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ a full admissible abelian subcategory that is stable under extensions in $\mathcal{T}$. Then the process we have sketched gives well defined maps*

$$\varphi_n\colon\ \mathrm{Ext}^n_{\mathcal{A}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A,B[n]) \quad (n \geqslant 0).$$

Moreover, φ_1 is an isomorphism and φ_2 is an injection.

Proof. See [**Lev93**, Prop. 1.6]. □

⋆ ⋆ ⋆

Exercise A.154. Show that the distinguished triangle (A.136) in the definition of t-structure is uniquely determined by X up to a unique isomorphism. Thus, it makes sense to write $Y = X^{\leqslant 0}$ and $Z = X^{\geqslant 1}$. Moreover, the assignments $X \mapsto X^{\leqslant 0}$ and $X \mapsto X^{\geqslant 1}$ determine functors $t_{\leqslant 0}$ and $t_{\geqslant 0}$.

Exercise A.155 (A t-structure such that the derived category of the heart is not equivalent to the original triangulated category). Let X be a non-empty connected finite CW-complex and let $\mathbf{Sh}_{\mathbb{Q}}(X)$ be the abelian category of sheaves of $\mathbb{Q}$-vector spaces on X. Consider the full subcategory

$$\mathcal{T} \subseteq D^b(\mathbf{Sh}_{\mathbb{Q}}(X))$$

consisting of complexes of sheaves C whose cohomology sheaves $\mathcal{H}^i(C)$ are all constant. Then $\mathcal{T}$ inherits the structure of a triangulated category. We define

$$\mathcal{T}^{\leqslant 0} = \{C \mid \mathcal{H}^i(C) = 0 \text{ for } i > 0\},$$
$$\mathcal{T}^{\geqslant 0} = \{C \mid \mathcal{H}^i(C) = 0 \text{ for } i < 0\}.$$

i) Show that the pair $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ forms a t-structure on $\mathcal{T}$ and that its heart is equivalent to the category $\mathbf{Vec}_{\mathbb{Q}}$ of finite-dimensional $\mathbb{Q}$-vector spaces.

ii) Let $\underline{\mathbb{Q}}_X$ be the constant sheaf with stalk $\mathbb{Q}$ on X. Prove the equality

$$\operatorname{Hom}_{\mathcal{T}}(\underline{\mathbb{Q}}_X, \underline{\mathbb{Q}}_X[n]) = \mathrm{H}^n(X, \mathbb{Q})$$

for every $n \geqslant 0$. Using the fact that the categories $D^b(\mathcal{T}^0)$ and $D^b(\mathbf{Vec}_{\mathbb{Q}})$ are equivalent, show the vanishing

$$\operatorname{Hom}_{D^b(\mathcal{T}^0)}(\underline{\mathbb{Q}}_X, \underline{\mathbb{Q}}_X[n]) = 0 \text{ for } n > 0.$$

Deduce that, as long as there exists some $n \geqslant 1$ such that $\mathrm{H}^n(X, \mathbb{Q})$ is non-zero, there is no faithful triangulated functor from $D^b(\mathcal{T}^0)$ to $\mathcal{T}$ extending the inclusion $\mathcal{T}^0 \subset \mathcal{T}$.

Exercise A.156 (Weight structures). Let $\mathcal{T}$ be a triangulated category. After Bondarko [Bon10], a *weight structure* on $\mathcal{T}$ is a pair of strictly full subcategories

$$(\mathcal{T}_{w\leqslant 0}, \mathcal{T}_{w\geqslant 0})$$

such that, defining for each integer n the subcategories

$$\mathcal{T}_{w\leqslant n} = \mathcal{T}_{w\leqslant 0}[n] \quad \text{and} \quad \mathcal{T}_{w\geqslant n} = \mathcal{T}_{w\geqslant 0}[n]$$

of $\mathcal{T}$, the following conditions hold:

- The categories $\mathcal{T}_{w\leqslant 0}$ and $\mathcal{T}_{w\geqslant 0}$ are stable under extraction of direct summands.
- $\mathcal{T}_{w\leqslant 0} \subseteq \mathcal{T}_{w\leqslant 1}$ and $\mathcal{T}_{w\geqslant 1} \subseteq \mathcal{T}_{w\geqslant 0}$.
- (*Orthogonality*) If $X \in \operatorname{Ob}(\mathcal{T}_{w\leqslant 0})$ and $Y \in \operatorname{Ob}(\mathcal{T}_{w\geqslant 1})$, then
$$\operatorname{Hom}_{\mathcal{T}}(X, Y) = 0.$$
- Each object $X \in \operatorname{Ob}(\mathcal{T})$ fits into a distinguished triangle
$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y[1]$$
with $Y \in \operatorname{Ob}(\mathcal{T}_{w\leqslant 0})$ and $Z \in \operatorname{Ob}(\mathcal{T}_{\geqslant 1})$.

By analogy with t-structures, the heart of a weight structure is the subcategory

$$\mathcal{T}_{w\leqslant 0} \cap \mathcal{T}_{w\geqslant 0}.$$

i) Spot the differences with the definition of a t-structure.

ii) Let $\mathcal{T}$ be the bounded derived category of the abelian category **MHS** of mixed Hodge structures. Let $\mathcal{T}_{w\leqslant 0}$ (resp. $\mathcal{T}_{w\geqslant 0}$) be the subcategory consisting of complexes A such that the mixed Hodge structure $\mathrm{H}^n(A)$ has weights $\leqslant n$ (resp. $\geqslant n$). Show that the pair $(\mathcal{T}_{w\leqslant 0}, \mathcal{T}_{w\geqslant 0})$ defines a weight structure and compute its heart.

A.6. Ind and pro-objects in a category. Inductive and projective limits are important operations in category theory. Nevertheless, in many interesting categories such limits may not exist. This is the case, for instance, for the category of mixed Hodge structures. To remedy this situation, given a category $\mathcal{C}$, one can define categories $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ of inductive and projective systems in $\mathcal{C}$, where inductive or projective limits exist. We give a concise introduction to ind and pro-categories and refer the reader to [**KS06**, Chap. 6] for more details.

A.6.1. *Limits and colimits.* A *directed set* is a non-empty set I endowed with a partial order $\leqslant$ with the property that, for all elements $a, b \in I$, there exists an element $c \in I$ satisfying $a \leqslant c$ and $b \leqslant c$.

EXAMPLE A.157. Here are examples of directed sets:

i) The set consisting of a single element.
ii) Any totally ordered set.
iii) The set $\mathcal{P}(X)$ of subsets of a given set X, with the order given by inclusion.
iv) The set $\mathbb{N}$ of natural numbers ordered by divisibility: $a \leqslant b$ if a divides b.

The notion of filtered category generalizes that of directed set as follows:

DEFINITION A.158. A *filtered category* D is a category such that:

i) there exists at least one object in D;
ii) given objects $a, b \in \mathrm{Ob}(D)$, there exists an object c and morphisms $a \to c$ and $b \to c$;
iii) given morphisms $f_1, f_2 \colon a \to b$ with the same source and target, there exists a morphism $g \colon b \to c$ satisfying $g \circ f_1 = g \circ f_2$.

EXAMPLE A.159. A directed set I gives rise to a filtered category with set of objects, I and morphisms $\mathrm{Hom}(x, y)$ reduced to a singleton for $x \leqslant y$ and empty otherwise. We write $*$ for the filtered category associated with the singleton.

DEFINITION A.160. Let $\mathcal{C}$ be a category and let D be a small category.

i) A *diagram* in $\mathcal{C}$ with *shape* D is a functor $X \colon D \to \mathcal{C}$.
ii) A diagram is called *constant* if it factors through the category $*$.
iii) Let X and Y be diagrams with shape D. A morphism f from X to Y is a natural transformation from X to Y.
iv) An *inductive system* X in $\mathcal{C}$ indexed by a filtered category D is a diagram with shape D.
v) A *direct system* is an inductive system indexed by the filtered category associated with a directed set as in Example A.159.
vi) A *projective system* X in $\mathcal{C}$ indexed by a filtered category D is a diagram in $\mathcal{C}^{\mathrm{op}}$ of shape D.
vii) An *inverse system* is a projective system indexed by the filtered category associated to a directed set.

DEFINITION A.161. Let $\mathcal{C}$ be a category, D a small category, and $X = (X_d)_{d \in D}$ a diagram in $\mathcal{C}$ with shape D.

i) A *colimit* of X is a universal solution to the following problem: find an object X_0 in $\mathcal{C}$ together with a morphism of diagrams $X \to X_0$. Here, X_0

denotes at the same time the object X_0 and the constant diagram X_0 with shape D. If X_0 is such a universal solution, then we write

$$X_0 = \varinjlim_{d \in D} X_d.$$

If D is a filtered category, colimits are called *inductive limits*. An inductive limit whose index category is a directed set is called a *direct limit*.

ii) A *limit* of X is a colimit of the diagram $X^{\mathrm{op}} \colon D^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. Alternatively, a limit is a universal solution to the following problem: find an object X_0 in $\mathcal{C}$ together with a morphism diagrams $X_0 \to X$. Again, X_0 denotes at the same time the object X_0 and the constant diagram with shape D. If X_0 is such a universal solution, then we write

$$X_0 = \varprojlim_{d \in D} X_d.$$

If D^{op} is a filtered category, limits are called *projective limits*. A projective limit whose index category is a directed set is called an *inverse limit*.

Remark A.162. The distinctions between colimit, inductive limit, and direct limit, as well as between limit, projective limit, and inverse limit are not uniform in the literature. The categorical notion of limit and colimit may look a little awkward as now a direct limit is not a limit but a colimit. It is chosen in this way so that products and kernels are examples of limits, while coproducts and cokernels are examples of colimits (see Example A.163 and Exercise A.185).

Example A.163 (Products and coproducts as limits and colimits). The basic examples of limits and colimits are products and coproducts. Let D be a small category whose only morphisms are the identity morphisms for each object. This is called a *discrete category*. A diagram with shape D in a category $\mathcal{C}$ is a collection $X = \{X_d\}_{d \in \mathrm{Ob}(D)}$ of objects of $\mathcal{C}$ without fixing any morphism between them. The product of this collection is defined as the limit of the discrete diagram, while the coproduct is defined as the colimit of the same diagram:

$$\prod_{d \in D} X_d = \varprojlim_{d \in D} X_d, \qquad \coprod_{d \in D} X_d = \varinjlim_{d \in D} X_d.$$

In the category **Set**, the coproduct agrees with the disjoint union, while the product agrees with the cartesian product. In an abelian category, the coproduct agrees with the direct sum and the product is the usual one.

Remark A.164. In many cases, it is important to know if a functor respects inductive limits. For this, it is enough to check if the functor respects direct limits. Similarly, a functor that respects inverse limits also respects projective limits with respect to any small filtered category.

A.6.2. *Exactness properties of direct and inverse limits.* We now discuss exactness properties of direct and inverse limits. Direct limits are always exact, but inverse limits may fail to be exact. Nevertheless, the so-called *the Mittag–Lefler condition* is sufficient to guarantee that an inverse limit is exact.

Proposition A.165. *Let $\mathcal{A}$ be an abelian category that admits colimits. Let I be a directed set. Let $K = \{K_i\}_{i \in I}$, $L = \{L_i\}_{i \in I}$, and $M = \{M_i\}_{i \in I}$ be directed* *systems indexed by I. Let $f\colon K \to L$ and $g\colon L \to M$ be morphisms of direct systems such that the sequence*

$$K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} M_i$$

is exact for all $i \in I$. Then the following sequence is also exact:

$$\varinjlim_{i\in I} K_i \xrightarrow{\varinjlim f_i} \varinjlim_{i\in I} L_i \xrightarrow{\varinjlim g_i} \varinjlim_{i\in I} M_i.$$

Proof. Writing $\widetilde{f} = \varinjlim f_i$, $\widetilde{g} = \varinjlim g_i$, we need to prove the equality

$$\operatorname{Im} \widetilde{f} = \operatorname{Ker} \widetilde{g}.$$

To ease the exposition, we will assume that $\mathcal{A}$ is the category $\mathbf{Mod}_R$ of left modules over some ring R, and we will use the concrete description of the direct limit from Exercise A.187. Given an element $x \in K_i$ (resp. L_i and M_i), we write $[x, i]$ for its image in $\varinjlim K_i$ (resp. $\varinjlim L_i$ and $\varinjlim M_i$). Then

$$\widetilde{g}(\widetilde{f}([x, i])) = \widetilde{g}([f_i(x_i), i]) = [g_i(f_i(x_i)), i] = 0$$

by the exactness of the sequence at the i-th level, whence $\operatorname{Im} \widetilde{f} \subset \operatorname{Ker} \widetilde{g}$. Let us now assume $\widetilde{g}([x, i]) = 0$. This implies that there is a $j \geqslant i$ such that $g_j(\phi_{ji}(x_i)) = 0$, where $\phi_{ji}\colon L_i \to L_j$ denotes the corresponding map in the direct limit. By exactness at the j-th level, there is an element $y \in K_j$ such that $f_j(y) = \phi_{ji}(x_i)$. Therefore,

$$\widetilde{f}([y, j]) = [x, i],$$

and we deduce that $\operatorname{Ker} \widetilde{g} \subset \operatorname{Im} \widetilde{f}$, proving the equality and the exactness. □

We next introduce the *Mittag–Leffler* condition.

Definition A.166. Let I be a directed set and let $K = \{K_j, \varphi_{ij}\}_{j\geqslant i\in I}$ be an inverse system indexed by I in an abelian category $\mathcal{A}$. We say that K satisfies the *Mittag–Leffler* condition if, for all $i \in I$, the collection of subobjects

$$K_{ij} = \operatorname{Im}(\varphi_{ij}\colon K_j \to K_i)$$

of K_i stabilizes. That is, there exists an index $j \geqslant i$ such that, for all $k \geqslant j$, the natural map $K_{ik} \to K_{ij}$ is an isomorphism.

Example A.167. In the category $\mathbf{Mod}_R$, if all the maps φ_{ij} in an inverse system $K = \{K_j, \varphi_{ij}\}_{j\geqslant i\in I}$ are surjective, then the Mittag–Leffler condition holds.

The importance of the Mittag–Leffler condition stems from the next result, which is a particular case of [**Nee01**, Lem. A.3.15].

Proposition A.168. *Let $\mathcal{A}$ be an abelian category that admits arbitrary products and has enough projectives. Let I be a directed set. Let $K = \{K_i\}_{i\in I}$, $L = \{L_i\}_{i\in I}$, and $L = \{M_i\}_{i\in I}$ be inverse systems indexed by I, and $f\colon K \to L$ and $g\colon L \to M$ morphisms of inverse systems such that*

$$0 \longrightarrow K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} M_i \longrightarrow 0$$

is an exact sequence for all $i \in I$. Then the sequence

$$0 \longrightarrow \varprojlim_{i\in I} K_i \xrightarrow{\varprojlim f_i} \varprojlim_{i\in I} L_i \xrightarrow{\varprojlim g_i} \varprojlim_{i\in I} M_i$$

is exact. If, furthermore, K satisfies the Mittag–Leffler condition, then the following sequence is also exact:

$$0 \longrightarrow \varprojlim_{i\in I} K_i \xrightarrow{\varprojlim f_i} \varprojlim_{i\in I} L_i \xrightarrow{\varprojlim g_i} \varprojlim_{i\in I} M_i \longrightarrow 0.$$

Proof. Since the inverse limit functor is right adjoint to the functor that sends an element X to the constant projective system X, it is left exact by Exercise A.60. Exactness in the presence of the Mittag–Leffler condition and enough projectives is proved in [**Nee01**, Lem. A.3.15]. □

Remark A.169. The Mittag–Leffler condition is not enough to guarantee the exactness of inverse limits in the previous proposition. This fact was overlooked in the literature until Neeman [**Nee02**] found a counterexample.

A.6.3. *The ind and pro categories.* We start with the basic definitions.

Definition A.170. Let $\mathcal{C}$ be any category. The *ind-category* of $\mathcal{C}$ is the universal category that "contains" $\mathcal{C}$ and is closed under inductive limits. More precisely, it is a category $\mathrm{Ind}(\mathcal{C})$ closed under inductive limits, together with a functor

$$h_I : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$$

such that, for any category $\mathcal{A}$ closed under inductive limits, along with a functor $\mathcal{C} \to \mathcal{A}$, there exists a unique functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{A}$ preserving inductive limits and making the following triangle commute:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h_I"] \arrow[dr] & \mathrm{Ind}(\mathcal{C}). \arrow[d] \\ & \mathcal{A} \end{tikzcd}$$

The *pro-category* of $\mathcal{C}$ is a category $\mathrm{Pro}(\mathcal{C})$ closed under projective limits together with a functor $h_P : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ such that, for any category $\mathcal{A}$ closed under projective limits, with a functor $\mathcal{C} \to \mathcal{A}$, there exists a unique functor $\mathrm{Pro}(\mathcal{C}) \to \mathcal{A}$ preserving projective limits and making the following triangle commute:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h_P"] \arrow[dr] & \mathrm{Pro}(\mathcal{C}). \arrow[d] \\ & \mathcal{A} \end{tikzcd}$$

We next give a description of the categories $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ unraveling the construction of [**KS06**, § 6].

Proposition A.171. *Let $\mathcal{C}$ be a category.*

i) *The ind-category $\mathrm{Ind}(\mathcal{C})$ is the category whose objects are inductive systems in $\mathcal{C}$ and whose morphisms are given by*

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, Y) = \varprojlim_{d\in D} \varinjlim_{e\in E} \mathrm{Hom}_{\mathcal{C}}(X_d, Y_e)$$

for inductive systems $X = (X_d)_{d\in D}$ and $Y = (Y_e)_{e\in E}$.

ii) *The pro-category* $\mathrm{Pro}(\mathcal{C})$ *is the category whose objects are projective systems in* $\mathcal{C}$ *and whose morphisms are given by*

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X,Y) = \varprojlim_{e\in E} \varinjlim_{d\in D} \mathrm{Hom}_{\mathcal{C}}(X_d, Y_e)$$

for projective systems $X = (X_d)_{d\in D}$ *and* $Y = (Y_e)_{e\in E}$.

With this description of the ind-category $\mathrm{Ind}(\mathcal{C})$ and the pro-category $\mathrm{Pro}(\mathcal{C})$, the functors $h_I\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ and $h_P\colon \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ are given on an object $X \in \mathrm{Ob}(\mathcal{C})$ by the constant inductive and projective system X.

Once we know that the categories $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ exist, the first useful property is the following.

Proposition A.172. *The functors* h_I *and* h_P *are fully faithful.*

Proof. Denote by $\mathcal{C}^\wedge$ the category of functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathbf{Set}$, and by $\mathcal{C}^\vee$ the category of functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathbf{Set}^{\mathrm{op}}$. There are canonical functors $h_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}^\wedge$ and $k_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}^\vee$ (see [KS06, Def. 1.4.2]) given by

$$h_{\mathcal{C}}(X)(Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X) \quad \text{and} \quad k_{\mathcal{C}}(X)(Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

Since the category $\mathcal{C}^\wedge$ is closed under inductive limits and the category $\mathcal{C}^\vee$ is closed under projective limits (see [KS06, Cor. 2.4.3]), by the universal property of the ind and pro-categories there are commutative diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h_I"] \arrow[dr, "h_{\mathcal{C}}"'] & \mathrm{Ind}(\mathcal{C}) \arrow[d] \\ & \mathcal{C}^\wedge \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "h_P"] \arrow[dr, "k_{\mathcal{C}}"'] & \mathrm{Pro}(\mathcal{C}) \arrow[d] \\ & \mathcal{C}^\vee \end{tikzcd}.$$

Besides, the functors $h_{\mathcal{C}}$ and $k_{\mathcal{C}}$ are fully faithful (see [KS06, Cor. 1.4.4]), and hence h_I and h_P are fully faithful as well. □

Remark A.173. In fact, the functors $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ and $\mathrm{Pro}(\mathcal{C}) \to \mathcal{C}^\vee$ are also fully faithful, and we can alternatively define $\mathrm{Ind}(\mathcal{C})$ as a subcategory of $\mathcal{C}^\wedge$, and $\mathrm{Pro}(\mathcal{C})$ as a subcategory of $\mathcal{C}^\vee$.

One needs to be careful with the fact that the objects in $\mathrm{Ind}(\mathcal{C})$ are "formal" limits and not "true" limits; that is, if $\mathcal{C}$ is already closed under inductive limits, then in general the functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ does not preserve inductive limits (see Exercise A.186). In this case, to understand the difference between $\mathcal{C}$ and $\mathrm{Ind}(\mathcal{C})$, it is convenient to introduce the notion of compact objects.

Definition A.174. Let $\mathcal{C}$ be a category that admits inductive limits. An object X of $\mathcal{C}$ is called *compact* if the canonical map

$$\varinjlim_{d\in D} \mathrm{Hom}(X, Y_d) \longrightarrow \mathrm{Hom}(X, \varinjlim_{d\in D} Y_d)$$

is an isomorphism for every inductive system $(Y_d)_{d\in D}$.

Dually, if $\mathcal{C}$ is a category that admits projective limits, then an object X of $\mathcal{C}$ is called *cocompact* if the canonical map

$$\varinjlim_{d\in D} \mathrm{Hom}(Y_d, X) \longrightarrow \mathrm{Hom}(\varprojlim_{d\in D} Y_d, X)$$

is an isomorphism for every projective system $(Y_d)_{d\in D}$.

The interest of the notion of compact object is the following result, which is proved in [**KS06**, Cor. 6.3.5].

Theorem A.175. *Let $\mathcal{C}$ be a category that admits inductive limits, and let $\mathcal{C}^{\mathrm{cpt}}$ be the full subcategory of $\mathcal{C}$ consisting of compact objects. If every object of $\mathcal{C}$ is an inductive limit of compact objects, then the composition*

$$\mathrm{Ind}(\mathcal{C}^{\mathrm{cpt}}) \longrightarrow \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathcal{C}$$

is an equivalence of categories. Dually, let $\mathcal{C}$ be a category that admits projective limits, and let $\mathcal{C}^{\mathrm{ccpt}}$ be the full subcategory of $\mathcal{C}$ consisting of cocompact objects. If every object of $\mathcal{C}$ is a projective limit of cocompact objects, then the composition

$$\mathrm{Pro}(\mathcal{C}^{\mathrm{ccpt}}) \longrightarrow \mathrm{Pro}(\mathcal{C}) \longrightarrow \mathcal{C}$$

is an equivalence of categories.

Example A.176. In the category $\mathbf{Vec}_{\mathrm{k}}^{\infty}$ of vector spaces over a field k, the compact objects and the cocompact objects are the finite-dimensional vector spaces (see also Exercise A.188).

A.6.4. *Derived functors via ind-objects.* Working on the ind-category allows us to construct derived functors even when the original category does not have enough injectives (see, for instance, [**Wil00**]). In what follows, we explain how this works for the bounded below derived category; a similar construction applies to the bounded derived category. Throughout, $\mathcal{A}$ denotes a small abelian category. Consider the functor

$$(-)^{+} \colon \mathcal{K}^{+}(\mathcal{A}) \longrightarrow \mathrm{Ind}(\mathcal{K}^{+}(\mathcal{A}))$$

that with a complex C associates the ind-complex

$$C^{+} = \varinjlim_{(i,D)} D,$$

where the inductive limit is taken with respect to the small category in which

- objects are pairs (i, D) of $D \in \mathcal{K}^{+}(\mathcal{A})$ and a quasi-isomorphism $i \colon C \to D$,
- morphisms from (i_1, D_1) to (i_2, D_2) are morphisms $f \colon D_1 \to D_2$ such that the equality $i_2 = f \circ i_1$ holds in $\mathcal{K}^{+}(\mathcal{A})$, that is, the morphisms i_2 and $f \circ i_1$ from C to D_2 are homotopically equivalent.

It follows from the definition of the functor $(-)^{+}$ that, if $f \colon E \to C$ is a quasi-isomorphism, then the induced morphism

$$f^{+} \colon E^{+} \longrightarrow C^{+}$$

is an isomorphism. There is a *tautological* natural transformation

$$\tau \colon h_I \longrightarrow (-)^{+} \tag{A.177}$$

from the functor $h_I \colon \mathcal{K}^{+}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{K}^{+}(\mathcal{A}))$ introduced in Definition A.170. The morphism τ_C from $h_I(C)$ (the constant system C) to C^{+} is the one induced by the quasi-isomorphism i in the position (i, D). This natural transformation has the property that $\mathrm{Ind}(Q) \circ \tau$ is an isomorphism. Here, Q stands for the localization functor from $\mathcal{K}^{+}(\mathcal{A})$ to $\mathcal{D}^{+}(\mathcal{A})$, and $\mathrm{Ind}(Q)$ is the induced functor between the ind-categories. In the sequel, we will denote $\mathrm{Ind}(Q)$ simply by Q.

An additive functor of abelian categories $F \colon \mathcal{A} \to \mathcal{B}$ extends to a functor

$$F \colon \mathrm{Ind}(\mathcal{K}^{+}(\mathcal{A})) \longrightarrow \mathrm{Ind}(\mathcal{K}^{+}(\mathcal{B})).$$

Since the functor $(-)^+$ inverts quasi-isomorphisms, the composition

$$\begin{tikzcd} \mathcal{K}^+(\mathcal{A}) \arrow[r, "(-)^+"] & \mathrm{Ind}(\mathcal{K}^+(\mathcal{A})) \arrow[r, "F"] & \mathrm{Ind}(\mathcal{K}^+(\mathcal{B})) \arrow[d, "Q"] \\ & & \mathrm{Ind}(\mathcal{D}^+(\mathcal{B})) \end{tikzcd}$$

factors uniquely through $\mathcal{D}^+(\mathcal{A})$, thus defining a functor

$$RF\colon \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathrm{Ind}(\mathcal{D}^+(\mathcal{B})).$$

Definition A.178 (Deligne [**Del73**, Déf. 1.2.1. (iii)]). The functor F is said to be *right derivable* (in the bounded below derived category) if RF factors through the category $\mathcal{D}^+(\mathcal{B})$. In this case, the functor

$$RF\colon \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{B})$$

is called the *total right derived functor*.

Remark A.179. By Proposition A.172, the functor $\mathcal{D}^+(\mathcal{B}) \to \mathrm{Ind}(\mathcal{D}^+(\mathcal{B}))$ is fully faithful. If it exists, the functor $RF\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ is hence unique up to a unique isomorphism of functors. As it is customary, we will pretend that it is well defined.

There is also a criterion for existence of the total right derived functor: in order to check that RF factors through a functor to $\mathcal{D}^+(\mathcal{B})$, it suffices to check this property on objects of $\mathcal{A}$ considered as complexes concentrated in degree zero.

Proposition A.180 ([**Del73**, Prop. 1.2.2. (ii)]). *The functor F is right derivable (in the bounded below derived category) if and only if the object*

$$RF(A) \in \mathrm{Ob}(\mathrm{Ind}(\mathcal{D}^+(\mathcal{B})))$$

belongs to $h_I(\mathcal{D}^+(\mathcal{B}))$ for any object $A \in \mathrm{Ob}(\mathcal{A})$.

The following result justifies calling RF the total right derived functor.

Theorem A.181. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. If $F\colon \mathcal{A} \to \mathcal{B}$ is a right derivable functor as in Definition A.178, then the total right derived functor RF satisfies the universal property of Definition A.100.*

Proof. Assume that there is a factorization $RF\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ as in Definition A.178. Using the tautological transformation τ from (A.177), we will first prove that there is a natural transformation from $Q \circ F$ to $RF \circ Q$. Since h_I is fully faithful (Proposition A.172), it is enough to construct a natural transformation after composition with h_I, that is, from $h_I \circ Q \circ F$ to $h_I \circ RF \circ Q$. On the one hand, h_I commutes with any additive functor, and hence the equality

$$h_I \circ Q \circ F = Q \circ F \circ h_I$$

holds. On the other hand, by definition of right derivability (Definition A.178),

$$h_I \circ RF \circ Q = Q \circ F \circ (-)^+.$$

Therefore, τ gives rises to a natural transformation

$$h_I \circ Q \circ F = Q \circ F \circ h_I \longrightarrow Q \circ F \circ (-)^+ = h_I \circ RF \circ Q,$$

and hence to a unique natural transformation $\xi\colon Q\circ F\to RF\circ Q$ satisfying

$$h_I\circ\xi=(Q\circ F)\circ\tau.$$

Let us now assume that a functor $G\colon \mathcal{D}^+(\mathcal{A})\to\mathcal{D}^+(\mathcal{B})$ and a natural transformation $\zeta\colon Q\circ F\to G\circ Q$ are given. According to Definition A.100, to show that RF is a total right derived functor, we need to construct a natural transformation $\eta\colon RF\to G$ satisfying

$$\zeta=(\eta\circ Q)\circ\xi. \tag{A.182}$$

We first observe that the natural transformation

$$(G\circ Q)\circ\tau\colon\ G\circ Q\circ h_I\longrightarrow G\circ Q\circ(-)^+$$

is an isomorphism because the localization functor Q inverts quasi-isomorphisms. Thanks to the identifications $h_I\circ G\circ Q=G\circ Q\circ h_I$ and $h_I\circ RF\circ Q=Q\circ F\circ(-)^+$, we obtain a natural transformation

$$\eta'=((G\circ Q)\circ\tau)^{-1}\circ(\zeta\circ(-)^+)\colon\ h_I\circ RF\circ Q\longrightarrow h_I\circ G\circ Q.$$

Using again that h_I is fully faithful, we obtain a natural transformation

$$\eta''\colon RF\circ Q\longrightarrow G\circ Q.$$

Since Q is a localization functor, this yields a natural transformation $\eta\colon RF\to G$.

Let us explain this last step in more detail. As Q is the identity on objects, for each object A of $\mathcal{D}^+(\mathcal{A})$, we define

$$\eta_A\colon RF(A)=RF(Q(A))\longrightarrow G(Q(A))=G(A)$$

as η''_A. We need to check that, given $f\in\operatorname{Hom}_{\mathcal{D}^+(\mathcal{A})}(A,B)$, the diagram

$$\begin{tikzcd} RF(A) \arrow[r, "RF(f)"] \arrow[d, "\eta_A"'] & RF(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd} \tag{A.183}$$

commutes. Since morphisms in the derived category are diagrams of the form (A.82), it suffices to check the commutativity of the diagram for morphisms of the form

- $f=Q(g)$, with $g\in\operatorname{Hom}_{\mathcal{K}^+(\mathcal{A})}(A,B)$;
- $f=Q(g)^{-1}$, for a quasi-isomorphism g.

In the first case, the commutativity of the diagram (A.183) follows from the commutativity of the corresponding diagram for g and η''. In the second case, the fact that η'' is a natural transformation gives a commutative diagram

$$\begin{tikzcd} RF(A) \arrow[d, "\eta_A"'] & RF(B) \arrow[l, "RF(Q(g))"'] \arrow[d, "\eta_B"] \\ G(A) & G(B). \arrow[l, "G(Q(g))"] \end{tikzcd}$$

Since the horizontal arrows are isomorphisms, we deduce the commutativity of diagram (A.183) for f.

Finally, it remains to show that η is the unique natural transformation satisfying the compatibility condition (A.182). Indeed, given such a natural transformation η_0, consider the commutative diagram of natural transformations

$$\begin{tikzcd}
RF \circ Q \circ (-)^+ \arrow[dddd, "\eta_0 \circ (Q \circ (-)^+)"'] & & & \\
& h_I \circ RF \circ Q \arrow[ul, "(RF \circ Q) \circ \tau"', "\simeq"] \arrow[dd, "h_I \circ \eta_0 \circ Q"'] & & \\
& & h_I \circ Q \circ F \arrow[ul, "h_I \circ \xi"'] \arrow[dl, "h_I \circ \zeta"] \arrow[r, "(Q \circ F) \circ \tau"] & Q \circ F \circ (-)^+ . \arrow[uulll, bend right, "\xi \circ (-)^+"', "\simeq"] \arrow[ddlll, bend left, "\zeta \circ (-)^+"] \\
& h_I \circ G \circ Q \arrow[dl, "(G \circ Q) \circ \tau"', "\simeq"] & & \\
G \circ Q \circ (-)^+ & & &
\end{tikzcd}$$

Since h_I is fully faithful and Q is a localization functor, η_0 is univocally determined by $h_I \circ \eta_0 \circ Q$. Since $(G \circ Q) \circ \tau$ and $\xi \circ (-)^+$ are isomorphisms, the natural tranformation $h_I \circ \eta_0 \circ Q$ is univocally determined by the diagram. That is,

$$h_I \circ \eta_0 \circ Q = ((G \circ Q) \circ \tau)^{-1} \circ (\zeta \circ (-)^+) \circ (\xi \circ (-)^+)^{-1} \circ ((RF \circ Q) \circ \tau).$$

From this, we see that η_0 is unique, which concludes the proof. $\square$

Example A.184. Let k be a subfield of $\mathbb{C}$. The abelian category $\mathbf{MHS}(k)$ of mixed Hodge structures over k does not have enough injectives (see Exercise A.189). We will see in Exercise A.190 that if we consider the functor $\mathrm{Hom}_{\mathbf{MHS}(k)}(H, -)$ to take values in finite-dimensional vector spaces, then it is not right derivable, but if we consider it as taking values in arbitrary vector spaces, then it is right derivable.

⋆ ⋆ ⋆

Exercise A.185. Let $\mathcal{A}$ be an abelian category and $f\colon A \to B$ a morphism. Show that $\mathrm{Ker}(f)$ is a limit of the diagram

$$\begin{tikzcd}
& 0 \arrow[d] \\
A \arrow[r, "f"] & B,
\end{tikzcd}$$

and $\mathrm{Coker}(f)$ is a colimit of the diagram

$$\begin{tikzcd}
0 & \\
A \arrow[u] \arrow[r, "f"] & B.
\end{tikzcd}$$

Exercise A.186. In this exercise, we illustrate the fact that the categories $\mathcal{C}$ and $\mathrm{Ind}(\mathcal{C})$ may be non-equivalent even if the category $\mathcal{C}$ admits inductive limits.

Let $\mathbf{Vec}_{\mathbb{Q}}^{\infty}$ be the category of arbitrary $\mathbb{Q}$-vector spaces, not necessarily of finite dimension. Let V be a vector space with a countable basis and write it as

$$V = \varinjlim_{n\in\mathbb{N}} W_n$$

with W_n a vector space of dimension n. Write also $V_n = V$ for all $n \in \mathbb{N}$. Thus,

$$V = \varinjlim_{n\in\mathbb{N}} V_n.$$

Show that the natural map

$$(W_n)_{n\in N} \longrightarrow (V_n)_{n\in\mathbb{N}}$$

in $\mathrm{Ind}(\mathbf{Vec}_{\mathbb{Q}}^{\infty})$ is not an isomorphism. Conclude that the functor

$$\mathrm{Ind}(\mathbf{Vec}_{\mathbb{Q}}^{\infty}) \longrightarrow \mathbf{Vec}_{\mathbb{Q}}^{\infty}$$

is not an equivalence of categories.

Exercise A.187. In this exercise, we see that direct and inverse limits with respect to directed sets admit a very concrete description. Let E be the small filtered category associated with a directed set.

i) If X is an inductive system in **Set** indexed by E, then

$$\varinjlim_{e\in E} X_e = \coprod_{e\in E} X_e \Big/ \sim,$$

where $x \sim y$ is the equivalence relation defined for $x \in X_e$ and $y \in X_{e'}$ by

there exist $f\colon e \longrightarrow e''$ and $g\colon e' \longrightarrow e''$ with $X(f)(x) = X(g)(y)$.

ii) If X is a projective system in **Set** indexed by E, then

$$\varprojlim_{e\in E} X_e \subset \prod_{e\in E} X_e$$

is the subset consisting of elements $(x_e)_{e\in E}$ satisfying the condition

$$\text{for all } f\colon e \longrightarrow e',\ x_e = X(f)(x_{e'}).$$

iii) Prove that the same descriptions are valid if X is an inductive system (resp. projective system) in the category $\mathbf{Mod}_R$ for a ring R.

Exercise A.188. Let $\mathbf{Vec}_k$ be the category of finite-dimensional vector spaces over some field k.

i) Let V be an ind-vector space over k. Prove that its dual

$$V^{\vee} = \mathrm{Hom}(V, k)$$

is a pro-vector space over k.

ii) Let $f\colon V \to W$ be a morphism of ind-vector spaces. Prove that f induces a morphism $f^{\vee}\colon W^{\vee} \to V^{\vee}$ of pro-vector spaces.

iii) Show that $\mathrm{Ind}(\mathbf{Vec}_k)$ is equivalent to the category of arbitrary vector spaces $\mathbf{Vec}_{\mathrm{k}}^{\infty}$.

iv) Prove that an infinite-dimensional pro-vector space over k does not admit a countable basis. In particular, not every vector space has the structure of a pro-vector space.

EXERCISE A.189. Let k be a subfield of $\mathbb{C}$ and $\mathbf{MHS}(k)$ the category of mixed Hodge structures over k. Assume that there is an injective object $I \in \mathbf{MHS}(k)$ with a monomorphism $\mathbb{Q}(0) \to I$.

i) Prove the non-vanishing $\mathrm{Gr}^W_{2n} I \neq 0$ for all $n \geqslant 1$, by considering monomorphisms $\mathbb{Q}(0) \to E$ for a non-split extension $E \in \mathrm{Ext}^1_{\mathbf{MHS}(k)}(\mathbb{Q}(-n), \mathbb{Q}(0))$.
ii) Conclude that I is not an object of $\mathbf{MHS}(k)$, and hence that $\mathbf{MHS}(k)$ does not have enough injectives.

EXERCISE A.190. Let k be a subfield of $\mathbb{C}$ and let H be a mixed Hodge structure over k. Observe that, for every mixed Hodge structure H', the abelian group $\mathrm{Hom}_{\mathbf{MHS}(k)}(H, H')$ has a structure of $\mathbb{Q}$-vector space. Moreover, since it is a subspace of $\mathrm{Hom}_{\mathbf{Vec}_{\mathbb{Q}}}(H_{\mathrm{B}}, H'_{\mathrm{B}})$ it is finite-dimensional. Show that the functor

$$\mathrm{Hom}_{\mathbf{MHS}(k)}(H, -)\colon \mathbf{MHS}(k) \longrightarrow \mathbf{Vec}_{\mathbb{Q}}$$

is not right derivable, but the functor with the same values

$$\mathrm{Hom}_{\mathbf{MHS}(k)}(H, -)\colon \mathbf{MHS}(k) \longrightarrow \mathbf{Vec}^{\infty}_{\mathbb{Q}}$$

is right derivable. Moreover, there is an equality

$$R^i \mathrm{Hom}_{\mathbf{MHS}(k)}(H, -) = \mathrm{Ext}^i_{\mathbf{MHS}(k)}(H, -).$$

A.7. Filtrations and spectral sequences.

A.7.1. *Basic definitions.*

DEFINITION A.191. Let $\mathcal{A}$ be an abelian category and let $V \in \mathrm{Ob}(\mathcal{A})$ be an object. A *decreasing filtration* on V is a collection of subobjects

$$V \supset \cdots \supset F^{p-1}V \supset F^pV \supset F^{p+1}V \supset \cdots \supset 0.$$

This can be seen as a projective system $(F^pV)_p$ or an inductive system $(F^{-p}V)_p$ indexed by the directed set $(\mathbb{Z}, \leqslant)$. If the limits exist, then we will use the notation

$$F^{\infty}V = \varprojlim_p F^pV \quad \text{and} \quad F^{-\infty}V = \varinjlim_p F^{-p}V.$$

For instance, if $\mathcal{A}$ is the category $\mathbf{Mod}_R$ of left modules over a ring R, then

$$F^{\infty}V = \bigcap_p F^pV \quad \text{and} \quad F^{-\infty}V = \bigcup_p F^pV.$$

A filtration is called

i) *separated* if $F^{\infty}V = 0$,
ii) *exhaustive* if $F^{-\infty}V = V$,
iii) *finite* if there are integers p_1 and p_2 such that $F^{p_1}V = 0$ and $F^{p_2}V = V$.

Thus, a finite filtration is separated and exhaustive.

Similarly, an *increasing filtration* on V is a collection of subobjects

$$0 \subset \cdots \subset F_{p-1}V \subset F_pV \subset F_{p+1}V \subset \cdots \subset V.$$

A decreasing filtration F can be made into an increasing filtration by setting

$$F_pV = F^{-p}V.$$

CONVENTION A.192. Unless explicitly indicated, all filtrations will be decreasing from now on. We leave to the reader the task of translating our statements from decreasing to increasing filtrations.

An object V equipped with a filtration F is called a *filtered object*. Let (V, F) be a filtered object of $\mathcal{A}$. The *associated graded object* is

$$\operatorname{Gr}_F^* V = \bigoplus_{p\in\mathbb{Z}} \operatorname{Gr}_F^p V, \quad \text{where} \quad \operatorname{Gr}_F^p V = F^p V / F^{p+1} V.$$

For the object $\operatorname{Gr}_F^* V$ to exist, one needs to assume either that the filtration F is finite or that the abelian category $\mathcal{A}$ admits infinite sums.

If F is a filtration on an object V and $n \in \mathbb{Z}$, then the *shifted* filtration $F[n]$ is the filtration on V defined as

$$F[n]^p V = F^{n+p} V.$$

Definition A.193. Let (V, F) and (V', F) be filtered objects of an abelian category $\mathcal{A}$. A morphism $f\colon V \to V'$ is called *filtered* if $f(F^p V) \subset F^p V'$ holds for all $p \in \mathbb{Z}$ and *strict* (with respect to the filtration F) if, in addition,

$$f(F^p V) = F^p V' \cap \operatorname{Im}(f).$$

A.7.2. *Filtrations and algebraic operations.* Assume that $\mathcal{A}$ is endowed with a tensor product. Given filtered objects (V, F) and (V', F) with finite filtrations, there are induced filtrations on $V \otimes V'$ and $\operatorname{Hom}(V, V')$ that we now describe:

- The subobject $F^n(V \otimes V')$ is the image of the map

$$\bigoplus_{p+q=n} F^p V \otimes F^q V' \longrightarrow V \otimes V'.$$

- The subobject $F^n \operatorname{Hom}(V, W)$ is given by

$$F^n \operatorname{Hom}(V, W) = \{\varphi \in \operatorname{Hom}(V, W) \mid \varphi|_{F^p V} \text{ factors through } F^{p+n} W\}.$$

The last definition means that, for every p, there is a unique morphism f_p that makes the following diagram commutative:

$$\begin{tikzcd} F^p V \arrow[r, "f_p"] \arrow[d] & F^{p+n} V' \arrow[d] \\ V \arrow[r, "\varphi"] & V'. \end{tikzcd}$$

Example A.194. Let k be a field and $\mathcal{A} = \mathbf{Vec}_{\mathrm{k}}^\infty$ the category of k-vector spaces. Then, k viewed as an object of $\mathcal{A}$ has canonical increasing and decreasing filtrations, given by

$$F^0 k = k,\ F^1 k = \{0\}, \qquad F_{-1} k = \{0\},\ F_0 k = k.$$

Therefore, the algebraic dual $V^\vee = \operatorname{Hom}(V, k)$ of a filtered vector space (V, F) has an induced filtration given, when F is decreasing, by

$$F^n V^\vee = (F^{1-n} V)^\perp,$$

and when F is increasing, by

$$F_n V^\vee = (F_{-1-n} V)^\perp.$$

A.7.3. *Filtered complexes.* Let $A = (A^*, \mathrm{d})$ be a cochain complex in $\mathcal{C}^+(\mathcal{A})$. A *filtration* F on A is the data of filtrations F on each A^n that are compatible with the differential in the sense that, for all integers n and p, one has

$$\mathrm{d}(F^p A^n) \subset F^p A^{n+1}.$$

A filtered complex is called *strict* if the differential is strict with respect to the filtration. A filtered complex is called *biregular* if, for every $n \in \mathbb{Z}$, the filtration F on the object A^n is finite.

Definition A.195. Let (A, F) and (B, G) be filtered complexes. A filtered morphism of complexes $f\colon A^* \to B^*$ is called a *filtered quasi-isomorphism* if, for every p, the induced morphism on graded pieces

$$\mathrm{Gr}_F^p A^* \longrightarrow \mathrm{Gr}_G^p B^*$$

is a quasi-isomorphism.

Example A.196. Given a cochain complex A, the following biregular filtrations are widely used:

i) The *decreasing bête filtration* $\sigma^{\geqslant p}$ is given by

$$\sigma^{\geqslant p} A^n = \begin{cases} 0, & \text{if } n < p, \\ A^n, & \text{if } n \geqslant p. \end{cases}$$

ii) The *increasing canonical filtration* $\tau_{\leqslant p}$ is given by

$$(\tau_{\leqslant p} A)^n = \begin{cases} A^n, & \text{if } n < p, \\ \mathrm{Ker}(\mathrm{d}), & \text{if } n = p, \\ 0, & \text{if } n > p. \end{cases}$$

Observe that the canonical truncation $\tau_{\leqslant p} C$ of a complex from Example A.140 is an example of an increasing canonical filtration. The truncation $\tau_{\geqslant p} C$ does not define a filtration since $C^p/\mathrm{Im}(d)$ is not a subobject but a quotient.

A.7.4. *Spectral sequences.* A filtration on a complex can be used to construct successive approximations to its cohomology by means of a tool called *spectral sequence*. This process generalizes the long exact sequence associated with a short exact sequence of complexes, which corresponds to the case of a two-step filtration.

Let $\mathcal{A}$ be an abelian category. For the sake of simplicity, we assume that $\mathcal{A}$ is the category of modules over a ring, so that we can pick elements of objects of $\mathcal{A}$ in the discussion. Let (A^*, F) be a complex in $\mathcal{A}$ endowed with a decreasing filtration. For each integer n, set $ZA^n = \mathrm{Ker}(\mathrm{d}\colon A^n \to A^{n+1})$, so that $\mathrm{H}^n(A) = ZA^n/\mathrm{d}(A^{n-1})$. The filtration F induces a filtration

$$F^p\,\mathrm{H}^n(A) = \mathrm{Im}\left(F^p A^n \cap ZA^n \longrightarrow ZA^n/\mathrm{d}(A^{n-1})\right)$$

on cohomology objects. The spectral sequence will allow us to recover the graded object $\mathrm{Gr}_F^*\,\mathrm{H}^*(A)$ rather than the total cohomology $\mathrm{H}^*(A)$; see Remark A.203. For example, for a complex in the category of mixed Hodge structures, we will recover the split mixed Hodge structure Gr_W but not the extension data.

The basic idea is to approximate $\mathrm{Gr}_F^*\,\mathrm{H}^*(A)$ by first computing the cohomology of the graded complex $\mathrm{Gr}_F^* A$, and then improving this approximation step by step. Each succesive approximation is called a *page*. The first page of the spectral sequence is the collection of objects

$$E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}_F^p A) = \frac{F^p A^{p+q} \cap \mathrm{d}^{-1}(F^{p+1} A^{p+q+1})}{\mathrm{d}(F^p A^{p+q-1}) + F^{p+1} A^{p+q}}$$

for all $(p, q) \in \mathbb{Z}^2$, together with the morphisms

$$\mathrm{d}_1 \colon E_1^{p,q} \longrightarrow E_1^{p+1,q}$$

induced by sending the class of $x \in F^p A^{p+q}$ to the class of $\mathrm{d}x \in F^{p+1} A^{p+q+1}$ (see Exercise A.219). We picture it in Figure 24.

$$\begin{tikzcd}
{} \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & {} \\
{} \arrow[r] & \bullet \arrow[r] & E_1^{p,q} \arrow[r] & E_1^{p+1,q} \arrow[r] & \bullet \arrow[r] & {} \\
{} \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & {}
\end{tikzcd}$$

FIGURE 24. First page of the spectral sequence

There are two possible sources of inaccuracy in approximating $\mathrm{Gr}_F^p \mathrm{H}^{p+q}(A)$ by $E_1^{p,q}$. First, we are taking elements $x \in F^p A^{p+q}$ such that $\mathrm{d}x$ lies in $F^{p+1} A^{p+q+1}$, while those of $\mathrm{Gr}_F^p \mathrm{H}^{p+q}(A)$ are represented by elements with $\mathrm{d}x = 0$. Secondly, we are taking the quotient modulo $\mathrm{d}(F^p A^{p+q-1})$, while all coboundaries should be taken into account. Thus, $\mathrm{Gr}_F^p \mathrm{H}^{p+q}(A)$ will in general only be a subquotient of $E_1^{p,q}$. The second page of the spectral sequence will be a better approximation to $\mathrm{Gr}_F^p \mathrm{H}^{p+q}(A)$ than the first page. It is defined as the collection of objects

$$E_2^{p,q} = \frac{F^p A^{p+q} \cap \mathrm{d}^{-1}(F^{p+2} A^{p+q+1})}{F^p A^{p+q} \cap \mathrm{d}(F^{p-1} A^{p+q-1}) + F^{p+1} A^{p+q} \cap \mathrm{d}^{-1}(F^{p+2} A^{p+q+1})}$$

for all $(p, q) \in \mathbb{Z}^2$, together with the morphisms

$$\mathrm{d}_2 \colon E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$$

induced by $[x] \to [\mathrm{d}x]$. We picture it in Figure 25.

The errors in the approximation now come from the fact that $\mathrm{d}x$ is only required to lie in $F^{p+2} A^{p+q+1}$, rather than being zero, and that we are considering classes modulo $\mathrm{d}(F^{p-1} A^{p+q-1})$ instead of all coboundaries. Since $F^{p+2} A^{p+q+1}$ will in general be smaller than $F^{p+1} A^{p+q+1}$, and $F^{p-1} A^{p+q-1}$ bigger than $F^p A^{p+q-1}$, this is a finer approximation to $\mathrm{Gr}_F^p \mathrm{H}^{p+q}(A)$. Moreover, there are isomorphisms

$$E_2^{p,q} \simeq \frac{\mathrm{Ker}(\mathrm{d}_1 \colon E_1^{p,q} \longrightarrow E_1^{p+1,q})}{\mathrm{Im}(\mathrm{d}_1 \colon E_1^{p-1,q} \longrightarrow E_1^{p,q})},$$

through which we can identify the second page of the spectral sequence with the cohomology of the first page (see Exercice A.219).

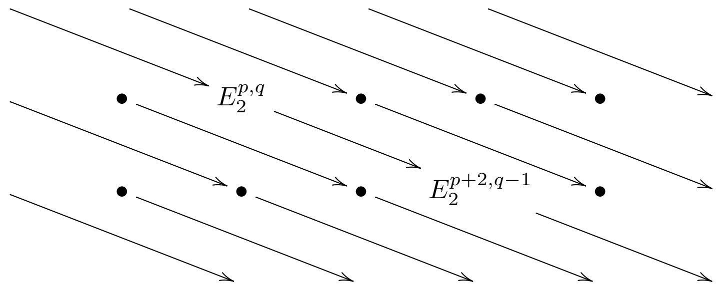


FIGURE 25. Second page of the spectral sequence

This process can be iterated. Explicitly, for each $r \geqslant 1$, we define the r-th page of the spectral sequence as the collection of objects

$$E_r^{p,q} = \frac{F^pA^{p+q} \cap \mathrm{d}^{-1}(F^{p+r}A^{p+q+1})}{F^pA^{p+q} \cap \mathrm{d}(F^{p-r+1}A^{p+q-1}) + F^{p+1}A^{p+q} \cap \mathrm{d}^{-1}(F^{p+r}A^{p+q+1})} \tag{A.197}$$

for all $(p,q) \in \mathbb{Z}^2$, together with the differential

$$\mathrm{d}_r \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

induced by $[x] \mapsto [\mathrm{d}x]$. Again, there is an isomorphism

$$E_{r+1}^{p,q} \simeq \frac{\mathrm{Ker}(\mathrm{d}_r \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1})}{\mathrm{Im}(\mathrm{d}_r \colon E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q})}. \tag{A.198}$$

This is the outcome of Exercise A.219.

The definition (A.197) makes sense for $r = \infty$ as well, and gives

$$E_\infty^{p,q} = \frac{F^pA^{p+q} \cap \mathrm{d}^{-1}(F^{\infty}A^{p+q+1})}{F^pA^{p+q} \cap \mathrm{d}(F^{-\infty}A^{p+q-1}) + F^{p+1}A^{p+q} \cap \mathrm{d}^{-1}(F^{\infty}A^{p+q+1})}.$$

Assume that the filtration F on A^n is separated and exhaustive for every $n \in \mathbb{Z}$. Then $F^{-\infty}A^n = A^n$ and $F^{\infty}A^n = 0$, and hence

$$E_\infty^{p,q} = \frac{F^pA^{p+q} \cap ZA^{p+q}}{F^pA^{p+q} \cap \mathrm{d}(A^{p+q-1}) + F^{p+1}A^{p+q} \cap ZA^{p+q}} = \mathrm{Gr}_F^p \mathrm{H}^{p+q}(A). \tag{A.199}$$

DEFINITION A.200. Let (A^*, F) be a filtered complex. We denote by $E_r^{p,q}$ the objects defined by equation (A.197). The collection of pages $E_r^{*,*}$ and morphisms d_r is called *the spectral sequence associated with the filtration* F. If the conditions for equation (A.199) are met, then we say that the spectral sequence *converges* and, for any $r \geqslant 1$, we use the notation

$$E_r^{p,q} \Longrightarrow \mathrm{H}^{p+q}(A)$$

to indicate that the spectral sequence converges to the cohomology of A. Recall, however, that we only recover the associated graded object. If we want to stress the filtration that gives rise to the spectral sequence, because the original complex may have more than one filtration, then we denote the spectral sequence by ${}_FE_r^{p,q}$.

If the filtration F is biregular, then for every $p, q \in \mathbb{Z}$ there is an integer $r_0 \geqslant 1$ (depending on p and q) such that $E_r^{p,q} = E_\infty^{p,q}$ holds for all $r \geqslant r_0$. Therefore, we can compute each $\mathrm{Gr}_F^p \mathrm{H}^{p+q}(A)$ with a finite number of pages of the spectral sequence. If there is an $r_0 \geqslant 1$ that works for all degrees, that is, such that the equality $E_{r_0}^{p,q} = E_\infty^{p,q}$ holds for all p and q, then we say that the spectral sequence *degenerates* at the term E_{r_0} or that it degenerates at the r_0-th page. This is equivalent to asking that all differentials d_r are zero for $r \geqslant r_0$.

A.7.5. *Degeneracy criteria.* In this section, we discuss several criteria for the degeneration of a spectral sequence.

PROPOSITION A.201. *Let (A^*, F) be a filtered complex such that the associated spectral sequence converges.*

i) *If there exists an integer $r \geqslant 2$ such that the page $E_r^{*,*}$ is reduced to one row, that is*
$$E_r^{p,q} = 0 \quad \text{for all } q \neq q_0,$$
then the spectral sequence degenerates at the r-th page and
$$\mathrm{H}^{p+q_0}(A) = E_r^{p,q_0}.$$
Note that in this case, we recover the full cohomology of the complex and not just the associated graded object.

ii) *Assume that there exists an integer $r_0 \geqslant 1$ such that the page $E_{r_0}^{*,*}$ is reduced to two rows, that is,*
$$E_{r_0}^{p,q} = 0 \quad \text{for all } q \neq q_0, q_1$$
with $q_0 < q_1$. Write $r = q_1 - q_0$. Then the spectral sequence degenerates at the page $E_{\max(r_0,r+2)}^{*,*}$. If $r_0 < r+2$, then $E_{r+1}^{p,q} = E_{r_0}^{p,q}$ holds, and there is a long exact sequence
$$\cdots \longrightarrow \mathrm{H}^{p+q_1}(A) \longrightarrow E_{r_0}^{p-1,q_1} \xrightarrow{\mathrm{d}_{r+1}} E_{r_0}^{p+r,q_0} \longrightarrow \mathrm{H}^{p+q_1}(A) \longrightarrow E_{r_0}^{p,q_1} \longrightarrow \cdots.$$
If $r_0 \geqslant r+2$, then $E_\infty^{p,q} = E_{r_0}^{p,q}$ and there are short exact sequences
$$0 \longrightarrow E_{r_0}^{p+r,q_0} \longrightarrow \mathrm{H}^{p+q_1}(A) \longrightarrow E_{r_0}^{p,q_1} \longrightarrow 0. \tag{A.202}$$

PROOF. Exercise A.221. □

REMARK A.203. This is a good point to understand why spectral sequences do not allow us to recover the whole cohomology but only an associated graded object. Imagine that we are in the situation of Proposition A.201 ii), with $r_0 \geqslant r+2$. Then the spectral sequence only gives us the terms $E_{r_0}^{p,q}$. From the exact sequences (A.202), we cannot recover $\mathrm{H}^{p+q_1}(A)$ unless we know the vanishing
$$\mathrm{Ext}^1(E_{r_0}^{p,q_1}, E_{r_0}^{p+r,q_0}) = 0.$$
We can interpret the exact sequences (A.202) by considering the decreasing filtration of $\mathrm{H}^{p+q_1}(A)$ given by
$$F^s \mathrm{H}^{p+q_1}(A) = \begin{cases} \mathrm{H}^{p+q_1}(A), & \text{if } s \leqslant p, \\ E_{r_0}^{p+r,q_0}, & \text{if } p < s \leqslant p+r, \\ 0, & \text{if } p+r < s. \end{cases}$$

Its graded pieces are equal to

$$\operatorname{Gr}_F^s \mathrm{H}^{p+q_1}(A) = \begin{cases} E_{r_0}^{p,q_1}, & \text{if } s = p, \\ E_{r_0}^{p+r,q_0}, & \text{if } s = p+r, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the graded object $\operatorname{Gr}_F^s \mathrm{H}^{p+q_1}(A)$ is exactly the information we recover from the spectral sequence.

We next explain the relation between degeneracy at the first page of a spectral sequence and strictness of the differential with respect to the filtration. This criterion applies notably to the Hodge filtration in algebraic geometry.

Proposition A.204. *Let (A^*, F) be a biregular filtered complex. The differential* d *is strict with respect to the filtration F if and only if the spectral sequence $E_*^{*,*}$ degenerates at the page E_1. In this case, the following holds:*

$$F^p \mathrm{H}^n(A) = \mathrm{H}^n(F^p A), \quad \operatorname{Gr}_F^p \mathrm{H}^n(A) = \mathrm{H}^n(\operatorname{Gr}_F^p A).$$

Proof. Recall the equality

$$E_1^{p,n-p} = \frac{F^p A^n \cap \mathrm{d}^{-1}(F^{p+1}A^{n+1})}{\mathrm{d}(F^p A^{n-1}) + F^{p+1}A^n}.$$

Since F is biregular, we have in addition

$$E_\infty^{p,n-p} = \frac{F^p A^n \cap \operatorname{Ker} \mathrm{d}}{F^p A^n \cap \operatorname{Im} \mathrm{d} + F^{p+1} A^n \cap \operatorname{Ker} \mathrm{d}}.$$

Therefore, we can consider the maps

$$\begin{tikzcd} \frac{F^p A^n \cap \operatorname{Ker} \mathrm{d}}{\mathrm{d}(F^p A^{n-1}) + F^{p+1}A^n \cap \operatorname{Ker}\mathrm{d}} \arrow[r, "f"] \arrow[d, "g"] & E_1^{p,n-p} \\ E_\infty^{p,n-p} & \end{tikzcd} ,$$

where f is the monomorphism induced by the inclusion $\operatorname{Ker} \mathrm{d} \subseteq \mathrm{d}^{-1}(F^{p+1}A^{n+1})$, and g is the epimorphism induced by the inclusion $\mathrm{d}(F^p A^{n-1}) \subseteq F^p A^n \cap \operatorname{Im} \mathrm{d}$. Assume that d is strict. We want to show that the spectral sequence degenerates at the term E_1. If $x \in F^p A^n$ satisfies $\mathrm{d}x \in F^{p+1}A^{n+1}$, then by strictness of d, there is an element $y \in F^{p+1}A^n$ such that $\mathrm{d}x = \mathrm{d}y$. This implies that the map f is an isomorphism. If $x \in F^p A^n$ satisfies $x \in \operatorname{Im} \mathrm{d}$, again by strictness of d, there is an element $z \in F^p A^{n-1}$ such that $x = \mathrm{d}z$. Hence, the map g is also an isomorphism.

Since the maps f and g are isomorphisms, we deduce that $E_1^{p,n-p}$ and $E_\infty^{p,n-p}$ are equal as subquotients of A^n, and hence that the spectral sequence degenerates at E_1. In particular, the complex being biregular, this implies the equalities

$$\mathrm{H}^n(\operatorname{Gr}_F^p A) = E_1^{p,n-p} = E_\infty^{p,n-p} = \operatorname{Gr}_F^p \mathrm{H}^n(A).$$

The equality $\mathrm{H}^n(F^p A) = F^p \mathrm{H}^n(A)$ follows from

$$\mathrm{d}(F^p A^{n-1}) = \mathrm{d}(A^{n-1}) \cap F^p A^n,$$

which is true by the strictness of d.

Conversely, assume that the spectral sequence degenerates at the term E_1. Given an element $x \in F^p A^n \cap \operatorname{Im}(\mathrm{d})$, we need to show that x lies in $\mathrm{d}(F^p A^{n-1})$. We prove this by inverse induction on p. Since the filtration is biregular for big enough p, the condition $x \in F^pA^n$ implies $x = 0$, and hence $x \in \mathrm{d}(F^pA^{n-1})$. Assume that the result is true for $p+1$. The element $x \in F^pA^n \cap \mathrm{Im}(\mathrm{d})$ satisfies $\mathrm{d}x = 0$, so it determines an element $[x]_1$ in $E_1^{p,n-p}$ with $d_r[x]_1 = 0$ for all $r \geqslant 1$, and hence an element $[x]_\infty \in E_\infty^{p,n-p}$. Since $x \in \mathrm{Im}(\mathrm{d})$ and the filtration is biregular, we deduce that $[x]_1 \in \mathrm{Im}(\mathrm{d}_r)$ for some $r \geqslant 1$, so the element $[x]_\infty$ is zero. Since the spectral sequence degenerates at the term E_1, we get $[x]_1 = 0$ so $x \in F^{p+1}A^n + \mathrm{d}(F^pA^{n-1})$. Therefore, there is a $z \in F^pA^{n-1}$ such that $y = x - \mathrm{d}z \in F^{p+1}A^n \cap \mathrm{Im}(\mathrm{d})$. Applying the induction hypothesis to y, we conclude that x belongs to $\mathrm{d}(F^pA^{n-1})$. □

A.7.6. *Examples of spectral sequences.*

Example A.205. The total complex of a double complex (Definition A.33), together with the horizontal bête filtration

$$\sigma^{\geqslant p}\operatorname{Tot}(A)^n = \bigoplus_{p' \geqslant p} A^{p', n-p'}$$

gives rise to a spectral sequence with first page

$$E_1^{p,q} = \mathrm{H}^q(A^{p,*}) \Longrightarrow \mathrm{H}^{p+q}(A). \tag{A.206}$$

This construction is the source of many spectral sequences in geometry. For instance, the Frölicher or Hodge–de Rham spectral sequence in Section 2.2.4 follows from this construction.

As a first easy application of spectral sequences, we show that to compute a right derived functor, there is no need to always use an injective resolution.

Definition A.207. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a left exact functor. Assume that $\mathcal{A}$ has enough injectives. An object $A \in \mathrm{Ob}(\mathcal{A})$ is called *F-acyclic* if, for all $p > 0$, the condition $R^pF(A) = 0$ holds.

An *F-acyclic resolution* of an object $B \in \mathrm{Ob}(\mathcal{A})$ is a complex $(\bigoplus_{p\geqslant 0} A^p, \mathrm{d})$ with a morphism $B \to A^0$ such that the sequence

$$0 \longrightarrow B \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

is exact and A^p is F-acyclic for every $p \geqslant 0$.

Remark A.208. Although according to Lemma A.113 an injective resolution is F-acyclic for any left exact functor F, in many circumstances it is easier to find an explicit F-acyclic resolution than an injective resolution.

Proposition A.209. *Let us keep the assumptions from Definition A.207. If A^* is an F-acyclic resolution of B, then*

$$R^pF(B) = \mathrm{H}^p(F(A^*)).$$

Proof. Let $(\bigoplus_{p,q\geqslant 0} I^{p,q}, \mathrm{d}^{\mathrm{hor}}, \mathrm{d}^{\mathrm{ver}})$ be a double complex of injective objects such that, for every $p \geqslant 0$ the complex $(I^{p,*}, \mathrm{d}^{\mathrm{ver}})$ is a resolution of A^p, and $\mathrm{d}^{\mathrm{hor}}$ commutes with the differential of A^*. Then $\operatorname{Tot}(I^{*,*})$ is an injective resolution of B, so $R^pF(B) = \mathrm{H}^p(\operatorname{Tot} F(I^{*,*}))$ holds. We apply Example A.205 to the double complex $F(I^{*,*})$ to obtain a spectral sequence

$$E_1^{p,q} = H^q(F(I^{p,*})) = R^qF(A^p) \Longrightarrow \mathrm{H}^{p+q}(\operatorname{Tot} F(I^{*,*})) = R^{p+q}F(B).$$

Since the objects A^p are F-acyclic, we get $R^qF(A^p) = 0$ for $q > 0$. Thus, this spectral sequence is reduced to a single row, so Proposition A.201 i) tells us that this spectral sequence degenerates at the term E_2 and

$$R^pF(B) = E_2^{p,0} = \mathrm{H}^p(F(A^*)),$$

as we wanted to show. □

The second example of a spectral sequence is a variant of Example A.205 that is the second main source of concrete spectral sequences in many applications.

PROPOSITION A.210. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a left exact functor. Let C^* be a bounded below complex in $\mathcal{A}$. Assume that $\mathcal{A}$ has enough injectives. Then there is a spectral sequence with second page*

$$E_2^{p,q} = R^pF(\mathrm{H}^q(C^*)) \Longrightarrow \mathrm{H}^{p+q}(RF(C^*)).$$

PROOF. Let $C^* \to I^{*,*}$ be an injective resolution satisfying the three conditions of Exercise A.131 ii). Then

$$RF(C^*) = F(\mathrm{Tot}(I^{*,*})) = \mathrm{Tot}(F(I^{*,*})).$$

By applying Example A.205 to the double complex $F(I^{*,*})$, we obtain a spectral sequence whose first page is

$$E_1^{p,q} = R^qF(C^p) \Longrightarrow \mathrm{H}^{p+q}(\mathrm{Tot}(F(I^{*,*}))) = \mathrm{H}^{p+q}(RF(C^*)).$$

This is not what we want. The trick is to first interchange the indexes of the double complex $F(I^{*,*})$ to obtain a spectral sequence whose first page is

$$E_1^{p,q} = \mathrm{H}^q(F(I^{*,p})) \Longrightarrow \mathrm{H}^{p+q}(\mathrm{Tot}(F(I^{*,*}))) = \mathrm{H}^{p+q}(RF(C^*)).$$

Since the functor F is exact when restricted to the subcategory of injective objects, the conditions of Exercise A.131 ii) imply that $\mathrm{H}^q(F(I^{*,p})) = F(\mathrm{H}^q(I^{*,p}))$ and that

$$\mathrm{H}^q(C^*) \longrightarrow \mathrm{H}^q(I^{*,0}) \longrightarrow \mathrm{H}^q(I^{*,1}) \longrightarrow \cdots \tag{A.211}$$

is an injective resolution of $\mathrm{H}^q(C^*)$. Moreover, the differential

$$d_1\colon F(\mathrm{H}^q(I^{*,p})) \to F(\mathrm{H}^q(I^{*,p+1}))$$

is induced by the differential of the injective resolution (A.211), so the second page of the spectral sequence is, as claimed,

$$E_2^{p,q} = R^pF(\mathrm{H}^q(C^*)).$$ □

REMARK A.212. There is an analogue of Proposition A.210 for right exact functors using projective resolutions, as well as variants for contravariant functors.

As a particular case of Proposition A.210, we derive the Grothendieck spectral sequence for the composition of functors.

THEOREM A.213 (Grothendieck spectral sequence). *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories. Assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and let $G\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{C}$ be two left exact functors. Assume that G sends injective objects in $\mathcal{A}$ to F-acyclic objects in $\mathcal{B}$. Then*

$$RF(RG(A^*)) = R(F \circ G)(A^*)$$

holds for each bounded below complex $A^* \in \mathcal{C}^+(\mathcal{A})$. Moreover, for all $A \in \mathrm{Ob}(\mathcal{A})$, there is a spectral sequence

$$E_2^{p,q} = R^pFR^qG(A) \Longrightarrow R^{p+q}(F \circ G)(A).$$

Proof. For the first statement, let I^* be a bounded below injective resolution of A^*. Then $G(I^*)$ is a complex in $\mathcal{C}^+(\mathcal{B})$ made of F-acyclic objects, which is isomorphic to $RG(A^*)$ in $\mathcal{D}^+(\mathcal{B})$. Therefore,

$$RF(RG(A^*)) = RF(G(I^*)) = F(G(I^*)) = R(F \circ G)(A^*).$$

For the second statement, let I^* be an injective resolution of A. Since the constituents of $G(I^*)$ are F-acyclic, we have $RF(G(I^*)) = F(G(I^*))$. Applying Proposition A.210 to the complex $G(I^*)$, we then obtain a spectral sequence

$$E_2^{p,q} = R^pF(\mathrm{H}^q(G(I^*))) = R^pF(R^qG(A)) \Longrightarrow \mathrm{H}^{p+q}(F(G(I^*))) = R^{p+q}F \circ G(A),$$

as we wanted to show. □

⋆ ⋆ ⋆

Exercise A.214. Let k be a field and let $\mathbf{FVec}_k$ denote the category of filtered k-vector spaces together with filtered morphisms. This is an additive category.

i) Show that every morphism in $\mathbf{FVec}_k$ has a kernel and a cokernel. More precisely, the kernel of a linear map f agrees with the kernel computed in $\mathbf{Vec}_k$ together with the induced filtration as a subobject. Similarly, the cokernel agrees with the one computed in $\mathbf{Vec}_k$, together with the induced filtration as a quotient.
ii) Let $f\colon (V, F) \to (W, F)$ be a morphism of filtered vector spaces. Show that the map f is strict with respect to the filtration F if and only if the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.
iii) Conclude that $\mathbf{FVec}_k$ is not an abelian category.

Exercise A.215. Let (V, F) and (W, F) be objects of an abelian category equipped with a finite increasing filtration. Let $f\colon (V, F) \to (W, F)$ be a filtered morphism such that $\mathrm{gr}_n^F f\colon \mathrm{gr}_n^F V \to \mathrm{gr}_n^F W$ is an isomorphism for all n. Show that f itself is an isomorphism.

Exercise A.216. Let $f\colon (A^*, F) \to (B^*, G)$ be a filtered quasi-isomorphism of complexes. Assuming that the filtrations F and G are biregular, prove that f is a quasi-isomorphism. Give an example showing that the assumption is needed.

Exercise A.217. Let $\mathcal{A}$ be an abelian category, and let $f\colon (A^*, F) \to (B^*, F)$ and $g\colon (A^*, F) \to (C^*, F)$ be morphisms of filtered complexes in $\mathcal{A}$. The given filtrations induce a filtration on $\mathrm{cone}(f + g)$. Assuming that the map g is a filtered quasi-isomorphism, show that the composition

$$B \longrightarrow B \oplus C \longrightarrow \mathrm{cone}(f + g)$$

is a filtered quasi-isomorphism.

Exercise A.218. Let $f\colon A^* \to B^*$ be a morphism of complexes. Show that the following statements are equivalent:

i) f is a quasi-isomorphism.
ii) f is a filtered quasi-isomorphism with respect to the increasing canonical filtration $\tau_{\leqslant}$.

Exercise A.219. Show that, for each $r \geqslant 1$, the map $E_r^{p,q} \to E_r^{p+r,q-r+1}$ given by $[x] \mapsto [\mathrm{d}x]$ is well defined and construct the isomorphism (A.198).

Exercise A.220. Let B be a complex in an abelian category, and $A \subset B$ a subcomplex. Then B carries the two-step filtration

$$F^0B = B \supset F^1B = A \supset F^2B = 0.$$

i) Show that the spectral sequence associated with this filtration is reduced to two columns, and hence degenerates at the second page.
ii) Show that there exist exact sequences

$$0 \longrightarrow E_2^{1,q-1} \longrightarrow \mathrm{H}^q(B) \longrightarrow E_2^{0,q} \longrightarrow 0,$$
$$\cdots \longrightarrow E_1^{0,q-1} \xrightarrow{\mathrm{d}_1} E_1^{1,q-1} \longrightarrow \mathrm{H}^q(B) \longrightarrow E_1^{0,q} \xrightarrow{\mathrm{d}_1} E_1^{1,q} \longrightarrow \cdots$$

iii) Prove the equalities $E_1^{1,q} = H^{q+1}(A)$ and $E_1^{0,q} = H^q(C)$, where $C = B/A$ and recover from the above the long exact sequence associated with the short exact sequence of complexes $0 \to A \to B \to C \to 0$.

Exercise A.221. Prove Proposition A.201.

Exercise A.222. Let E and E' be spectral sequences. A *morphism of spectral sequences* is a collection of maps $E_r^{p,q} \to E'^{p,q}_r$ that commute with the differentials d_r on both sides. Let $f\colon (A^*, F) \to (B^*, G)$ be a filtered quasi-isomorphism. Show that f induces an isomorphism between the spectral sequences associated with the filtrations F and G.

Exercise A.223. In this exercise, we introduce the shifted filtration. Given a filtered complex (A^*, F), the shifted filtration $\mathrm{Dec}(F)$ is defined as

$$\mathrm{Dec}(F)^pA^n = \{x \in F^{p+n}A^n \mid \mathrm{d}x \in F^{p+n+1}A^{n+1}\}.$$

The notation comes from “filtration décalée” in French.

i) Prove that $(A^*, \mathrm{Dec}(F))$ is a filtered complex.
ii) Prove that there are isomorphisms compatible with the differentials

$${}_{\mathrm{Dec}(F)}E_r^{p,n-p} \xrightarrow{\sim} {}_F E_{r+1}^{p+n,-p}.$$

A.8. Simplicial techniques. In this section, we briefly introduce a few simplicial tools that are very useful in homological algebra.

A.8.1. *Simplicial and cosimplicial objects.* We first review the definition of simplicial and cosimplicial objects in a category. Let $\boldsymbol{\Delta}$ denote the small category whose objects are the finite ordered sets

$$\Delta_n = \{0, \dots, n\} \qquad (n \geqslant 0),$$

and whose morphisms are the non-decreasing maps between the various Δ_n. Any morphism in $\boldsymbol{\Delta}$ can be written as a composition of *cofaces* $\delta^i\colon \Delta_n \to \Delta_{n+1}$, for $i = 0, \dots, n+1$, and *codegeneracies* $\sigma^i\colon \Delta_{n+1} \to \Delta_n$, for $i = 0, \dots, n$, given by

$$\delta^i(j) = \begin{cases} j, & \text{if } j < i, \\ j+1, & \text{if } j \geqslant i, \end{cases} \qquad \sigma^i(j) = \begin{cases} j, & \text{if } j \leqslant i, \\ j-1, & \text{if } j > i. \end{cases}$$

In other words, the coface δ^i is the map that skips i, while the codegeneracy σ^i is the map that repeats i. As it is straightforward to check, cofaces and codegeneracies satisfy the commutativity relations

$$\begin{aligned}
&(a) && \delta^j\delta^i = \delta^i\delta^{j-1}, && \text{for } i < j,\\
&(b) && \sigma^j\sigma^i = \sigma^i\sigma^{j+1}, && \text{for } i \leqslant j,\\
&(c) && \sigma^j\delta^i = \delta^i\sigma^{j-1}, && \text{for } i < j,\\
&(d) && \sigma^j\delta^i = \mathrm{Id}, && \text{for } i = j, j+1,\\
&(e) && \sigma^j\delta^i = \delta^{i-1}\sigma^j, && \text{for } i > j+1.
\end{aligned} \tag{A.224}$$

Moreover, the category $\mathbf{\Delta}$ is generated by these morphisms and relations (see, for instance, [**Mac71**, Chap. VII, §5, Prop. 2]). That is, $\mathbf{\Delta}$ is the universal category with objects Δ_n, for $n \geqslant 0$, and morphisms δ^i and σ^j such that every morphism is a composition of these morphisms, the relations (A.224) hold, and any other relation among the morphisms follows from these relations.

Definition A.225. Let $\mathcal{C}$ be a category.

i) A *simplicial* object in $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

ii) A *cosimplicial* object in $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$.

Using the above characterization of morphisms in $\mathbf{\Delta}$, simplicial and cosimplicial objects admit a very concrete description. For instance, a *cosimplicial object* $X^\bullet$ is a collection $(X^n)_{n\geqslant 0}$ of objects of $\mathcal{C}$, each X^n being the image of Δ_n through the functor $\mathbf{\Delta} \to \mathcal{C}$, together with morphisms

$$\begin{aligned}
\delta^i &\colon X^n \longrightarrow X^{n+1}, && i = 0, \dots, n+1,\\
\sigma^i &\colon X^{n+1} \longrightarrow X^n, && i = 0, \dots, n,
\end{aligned}$$

satisfying the commutativity relations (A.224). The maps δ^i and σ^i are again called *cofaces* and *codegeneracies*, and one usually represents these data by a diagram

$$X^0 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X^1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X^2 \;\cdots .$$

The description of a *simplicial object* is the dual one. It is thus given by a collection of objects $(X_n)_{n\geqslant 0}$, together with morphisms

$$\begin{aligned}
\delta_i &\colon X_{n+1} \longrightarrow X_n, && i = 0, \dots, n+1,\\
\sigma_i &\colon X_n \longrightarrow X_{n+1}, && i = 0, \dots, n,
\end{aligned}$$

called *faces* and *degeneracies*, satisfying the commutativity relations dual to (A.224):

$$\begin{aligned}
&(a) && \delta_i\delta_j = \delta_{j-1}\delta_i, && \text{for } i < j,\\
&(b) && \sigma_i\sigma_j = \sigma_{j+1}\sigma_i, && \text{for } i \leqslant j,\\
&(c) && \delta_i\sigma_j = \sigma_{j-1}\delta_i, && \text{for } i < j,\\
&(d) && \delta_i\sigma_j = \mathrm{Id}, && \text{for } i = j, j+1,\\
&(e) && \delta_i\sigma_j = \sigma_j\delta_{i-1}, && \text{for } i > j+1.
\end{aligned} \tag{A.226}$$

The data of a simplicial object is usually represented by a diagram

$$X_0 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; X_1 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; X_2 \;\cdots .$$

Remark A.227. The category $\mathbf{\Delta}$ is equivalent to the category **FOS** of totally ordered non-empty finite sets. We can also view a simplicial object $X_\bullet = (X_n)_{n\geqslant 0}$ in $\mathcal{C}$ as a functor $\mathbf{FOS}^{\mathrm{op}} \to \mathcal{C}$, by sending a totally ordered non-empty finite set I to the object $X_I = X_{|I|-1}$, where $|I|$ denotes the cardinal of I.

We will try to systematically use the convention that simplicial and cosimplicial objects are denoted with a bullet, while complexes are denoted with an asterisk (see, for example, Definition A.230 below).

EXAMPLE A.228. Recall from Notation 1.114 that the symbol Δ^n was already used for the topological simplex

$$\Delta^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid 1 \geqslant t_1 \geqslant \dots \geqslant t_n \geqslant n\}.$$

As n varies, the simplices Δ^n form a cosimplicial object $\Delta^\bullet$ in the category of topological spaces. The cofaces are given by

$$\delta^i(t_1, \dots, t_n) = \begin{cases} (1, t_1, \dots, t_n), & \text{if } i = 0, \\ (t_1, \dots, t_i, t_i, \dots, t_n), & \text{if } i = 1, \dots, n, \\ (t_1, \dots, t_n, 0), & \text{if } i = n+1, \end{cases}$$

and the codegeneracies by

$$\sigma^i(t_1, \dots, t_{n+1}) = (t_1, \dots, \widehat{t_{i+1}}, \dots, t_{n+1}) \qquad (i = 0 \dots, n),$$

where the symbol $\widehat{t_{i+1}}$ means that the coordinate t_{i+1} is omitted.

What we called the standard simplex in Section 2.1, that is,

$$\Delta^n_{\mathrm{st}} = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geqslant 0 \text{ for all } i = 0, \dots, n\}$$

gives a more symmetric representation of the same cosimplicial topological space (see Exercise A.240). The coface maps $\delta^i \colon \Delta^n_{\mathrm{st}} \to \Delta^{n+1}_{\mathrm{st}}$ are now defined as

$$\delta^i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n) \qquad (i = 0, \dots, n+1),$$

and the codegeneracy maps $\sigma^i \colon \Delta^{n+1}_{\mathrm{st}} \to \Delta^n_{\mathrm{st}}$ as

$$\sigma^i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_i + t_{i+1}, \dots, t_{n+1}) \qquad (i = 0, \dots, n).$$

EXAMPLE A.229. Recall that an n-simplex in a topological space M is a continuous map $S \colon \Delta^n_{\mathrm{st}} \to M$. The i-th face of S is the $(n-1)$-simplex

$$\delta_i S = S \circ \delta^i \colon \Delta^{n-1}_{\mathrm{st}} \to M,$$

and the i-th degeneracy is the $(n+1)$-simplex

$$\sigma_i S = S \circ \sigma^i \colon \Delta^{n+1}_{\mathrm{st}} \to M.$$

Thus, the set of all simplices in M is an example of a simplicial set. The terminology *faces* and *degeneracies* arise from this example.

A.8.2. *Simplicial abelian groups and chain complexes.* Simplicial and cosimplicial objects in an abelian category are closely related to chain and cochain complexes, as introduced in Section A.1. In this paragraph, we review some constructions making this relation precise. To begin with, we associate a chain complex with a simplicial object in an abelian category.

DEFINITION A.230. Given a simplicial object $X_\bullet$ in an abelian category, the *associated chain complex* is the complex CX_* with

$$CX_n = X_n, \qquad \partial_n = \sum_{i=0}^{n} (-1)^i \delta_i \colon CX_n \longrightarrow CX_{n-1}.$$

When there is no need to emphasize the degree, we will denote the differential of this complex simply by ∂. We can also consider a "smaller" complex that, as we will see in the next theorem, has the same homology as CX_* and plays a crucial role in the Dold–Kan correspondance (see [**GJ09**, Chap. III, Sect. 2]).

Definition A.231. Given a simplicial object $X_\bullet$ in an abelian category, the associated *normalized chain complex* is the subcomplex $\mathcal{N}X_* \subset CX_*$ with

$$\mathcal{N}X_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} \delta_i.$$

The *complex of degenerate elements* is the subcomplex $DX_* \subset CX_*$ with

$$DX_n = \sum_{i=0}^{n-1} \sigma_i(X_{n-1}).$$

The reader is encouraged to check in Exercise A.241 that CX_* is indeed a complex and that $\mathcal{N}X_*$ and DX_* form subcomplexes of CX_*. Note that the differential ∂_n of CX_* restricts to $\partial_n = (-1)^n \delta_n$ on the subcomplex $\mathcal{N}X_n$.

Theorem A.232. *Let $X_\bullet$ be a simplicial object in an abelian category. The composition of the inclusion and the quotient maps*

$$\mathcal{N}X_* \longrightarrow CX_* \longrightarrow CX_*/DX_*$$

is an isomorphism. Moreover, $\mathcal{N}X_* \longrightarrow CX_*$ is a quasi-isomorphism.

Proof. For each integer k, consider the subobjects of X_n given by

$$\mathcal{N}_k X_n = \bigcap_{j=0}^{\min(k,n-1)} \operatorname{Ker} \delta_j, \quad D_k X_n = \sum_{i=0}^{\min(k,n-1)} \sigma_i(X_{n-1}).$$

It follows from the simplicial identities (A.226) that $(\mathcal{N}_k X_*, \partial)$ and $(D_k X_*, \partial)$ are subcomplexes of (CX_*, ∂).

We are going to prove that, for all k and $n \geqslant 0$, the composition

$$\mathcal{N}_k X_n \longrightarrow CX_n \longrightarrow CX_n/D_k X_n \tag{A.233}$$

of the inclusions and the quotient maps are isomorphisms and that the inclusions

$$\mathcal{N}_k X_* \longrightarrow \mathcal{N}_{k-1} X_*$$

are quasi-isomorphisms. The theorem follows from this, on noting the equalities $\mathcal{N}_k X_n = CX_n$ for $k \leqslant -1$ and $\mathcal{N}_k X_n = \mathcal{N}X_n$ and $D_k X_n = DX_n$ for $k \geqslant n-1$.

We argue by induction on k. The above statements clearly hold for $k \leqslant -1$, since $\mathcal{N}_k X_n = CX_n$ and $D_k X_n = 0$ in that case. We assume that both statements are true for $j < k$, and we prove them for k. We first prove that the composition (A.233) is an isomorphism. For $n \leqslant k$, this follows directly from the induction hypothesis because, in this case, $\mathcal{N}_k X_n = \mathcal{N}_{k-1} X_n$ and $D_k X_n = D_{k-1} X_n$. Let $x \in \mathcal{N}_{k-1} X_n$ with $n \geqslant k+1$. Then, by the simplicial identities,

$$x - \sigma_k \delta_k x \in \mathcal{N}_k X_n.$$

Since $\sigma_k \delta_k x \in D_k X_n$, using the induction hypothesis, we get the equality

$$CX_n = D_{k-1} X_n + \mathcal{N}_{k-1} X_n = D_{k-1} X_n + D_k X_n + \mathcal{N}_k X_n = D_k X_n + \mathcal{N}_k X_n.$$

Therefore, the composition (A.233) is surjective.

Using again the simplicial identities, we derive

$$\sigma_k(\mathcal{N}_{k-1}X_n) \subset \mathcal{N}_{k-1}X_n, \quad \sigma_k(D_{k-1}X_n) \subset D_{k-1}X_n.$$

In particular, σ_k induces a map $\sigma_k\colon X_n/D_{k-1}X_n \to X_n/D_{k-1}X_n$ and, by the induction hypothesis, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{N}_{k-1}X_n \arrow[r, "\simeq"] \arrow[d, "\sigma_k"'] & X_n/D_{k-1}X_n \arrow[d, "\sigma_k"] \\
\mathcal{N}_{k-1}X_n \arrow[r, "\simeq"'] & X_n/D_{k-1}X_n.
\end{tikzcd}$$

There is a canonical isomorphism

$$\frac{X_n}{D_kX_n} = \frac{X_n}{D_{k-1}X_n + \sigma_k(X_n)} \xrightarrow{\sim} \frac{X_n/D_{k-1}X_n}{\sigma_k(X_n/D_{k-1}X_n)}.$$

From this isomorphism and the previous commutative diagram, we deduce that, if $x \in \mathcal{N}_{k-1}X_n$ belongs to D_kX_n, then $x = \sigma_k y$ with $y \in \mathcal{N}_{k-1}X_n$.

Let $x \in \mathcal{N}_kX_n \cap D_kX_n$. By the previous discussion, $x = \sigma_k y$ with $y \in \mathcal{N}_{k-1}X_n$. Since $x \in \operatorname{Ker}\delta_k$, the equality

$$0 = \delta_k x = \delta_k\sigma_k y = y$$

holds, so $y = 0$, and hence $x = 0$. This shows that $\mathcal{N}_kX_n \cap D_kX_n = 0$ and that the composition (A.233) is injective. It is thus an isomorphism.

Now we prove that the inclusion $\mathcal{N}_kX_* \to \mathcal{N}_{k-1}X_*$ is a quasi-isomorphism. In fact, we will prove that it is a homotopy equivalence. Let $\iota_k\colon \mathcal{N}_kX_* \to \mathcal{N}_{k-1}X_*$ denote the inclusion, and π_k the composition

$$\mathcal{N}_{k-1}X_* \twoheadrightarrow X_*/D_{k-1}X_* \simeq \mathcal{N}_kX_*.$$

One checks that, for $x \in \mathcal{N}_{k-1}X_n$,

$$\pi_k \circ \iota_k(x) = x = \operatorname{Id}(x),$$

$$\iota_k \circ \pi_k(x) = \begin{cases} x, & \text{if } n \leqslant k, \\ x - \sigma_k\delta_k(x), & \text{if } n > k. \end{cases}$$

We need to show that $\iota_k \circ \pi_k$ is homotopy equivalent to the identity. For this, let $s\colon \mathcal{N}_{k-1}X_* \to \mathcal{N}_{k-1}X_*$ be the map that sends $x \in \mathcal{N}_{k-1}X_n$ to

$$s(x) = \begin{cases} 0, & \text{if } n < k, \\ (-1)^k\sigma_k(n), & \text{if } n \geqslant k. \end{cases}$$

Using the simplicial identities it follows that

$$x - \iota_k \circ \pi_k(x) = (\partial s + s\partial)(x),$$

thus showing that $\mathcal{N}_kX_*$ and $\mathcal{N}_{k-1}X_*$ are homotopy equivalent. We are done. □

Example A.234. Let M be a topological space. The cosimplicial structure of $\Delta^\bullet_{\mathrm{st}}$ induces a simplicial group structure $C_\bullet(M)$ on the free abelian groups

$$C_n(M) = C_n(M, \mathbb{Z})$$

generated by all continuous maps $\sigma\colon \Delta^n_{\mathrm{st}} \to M$. Let $C_*(M)$ denote the associated chain complex, and $\widetilde{C}_*(M)$ the normalized chain complex. As explained in

Section 2.1, the complex $(C_*(M), \partial_*)$ computes the singular homology of M. By Theorem A.232, the same is true for the complex $(\widetilde{C}_*(M), \partial_*)$.

Similarly, for a differentiable manifold M, we denote by $S_*(M)$ the chain complex of smooth singular chains, and by $\widetilde{S}_*(M)$ the normalized complex of $S_*(M)$.

Dualizing the construction of the chain complex, we obtain the definition of the associated cochain complex.

Definition A.235. Let $X^\bullet$ be a cosimplicial object in an abelian category. The *associated cochain complex* is the complex CX^* with

$$CX^n = X^n, \qquad \mathrm{d} = \sum_{i=0}^{n+1} (-1)^i \delta^i \colon CX^n \longrightarrow CX^{n+1},$$

and the *normalized cochain complex* is

$$\mathcal{N}X^n = X^n / \sum_{i=0}^{n-1} \operatorname{Im} \delta^i \simeq \bigcap_{i=0}^{n-1} \operatorname{Ker} \sigma^i, \qquad \mathrm{d} = \sum_{i=0}^{n+1} (-1)^i \delta^i.$$

The statements that $X^n / \sum_{i=0}^{n-1} \operatorname{Im} \delta^i$ and $\bigcap_{i=0}^{n-1} \operatorname{Ker} \sigma^i$ are isomorphic and that the inclusion $\mathcal{N}X^* \to CX^*$ is a quasi-isomorphism are dual to the statements in Theorem A.232. They are proved in a similar way.

Example A.236. Let M be a topological space. For any ring R, the groups

$$C^n(M, R) = \operatorname{Hom}(C_n(M), R)$$

form a cosimplicial abelian group. We will also denote by $C^*(M, R)$ the associated cochain complex and by $\widetilde{C}^*(M, R)$ the normalized cochain complex. Similarly, when M is a differentiable manifold, $S^*(M, R)$ and $\widetilde{S}^*(M, R)$ will denote the cochain complex of smooth singular cochains and the corresponding normalized complex. All these complexes compute the singular cohomology of M with coefficients in R.

A.8.3. *A truncated normalized chain complex.* In the course of the proof of Beilinson's Theorem 3.316, one needs to associate with a cosimplicial manifold a variant of the normalized cochain complex from Definition A.235. In fact, it is a complex homotopically equivalent to a truncation of it. For each $N \geqslant 0$ and each simplicial object $X_\bullet$ in an abelian category, we introduce a new complex

$$C_*(\Delta_N, X_\bullet).$$

For each non-empty subset $I \subset \Delta_N$, using the convention of Remark A.227, we have the object $X_I = X_{|I|-1}$. Given $K = \{k_0, \dots, k_p\}$ with the indices k_l in increasing order, and $I = \{k_0, \dots, \hat{k}_i, \dots, k_p\}$, we set the sign $\varepsilon(I, K) = (-1)^i$, which is the same sign appearing in Notation 3.286. We also use the notation

$$\mathrm{d}_{I,K} = \delta_i \colon X_K \longrightarrow X_I.$$

For each $n \geqslant 0$, we define

$$C_n(\Delta_N, X_\bullet) = \bigoplus_{\substack{I \subset \Delta_N \\ |I| = n+1}} X_I, \tag{A.237}$$

with differential $\mathrm{d} \colon C_n(\Delta_N, X_\bullet) \to C_{n-1}(\Delta_N, X_\bullet)$ given by

$$\mathrm{d} = \bigoplus_{I \subset K} \varepsilon(I, K) \mathrm{d}_{I,K}.$$

For a chain complex C_*, let $\sigma_{\leqslant N}$ denote the *bête* filtration

$$\sigma_{\leqslant N} C_n = \begin{cases} C_n, & \text{if } n \leqslant N, \\ 0, & \text{if } n > N. \end{cases}$$

This filtration is dual to that of a cochain complex from Example A.196.

For a conceptual proof of the following result, see [**DG05**, Prop. 3.10]. We propose an elementary proof similar to that of Theorem A.232.

PROPOSITION A.238. *Given a simplicial object $X_\bullet$ in an abelian category and an integer $N \geqslant 0$, the complexes $C_*(\Delta_N, X_\bullet)$ and $\sigma_{\leqslant N}\mathcal{N}X_*$ are functorially homotopically equivalent.*

PROOF. Let $\phi\colon \sigma_{\leqslant N}\mathcal{N}X_* \to C_*(\Delta_N, X_\bullet)$ be the map sending the object $\mathcal{N}X_n$ to the factor $X_{\Delta_n} \subset C_n(\Delta_N, X_\bullet)$ in the direct sum (A.237). This map is a morphism of complexes because for each subset $J \subset \Delta_n$, with $|J| = n$, the restriction to $\mathcal{N}X_n$ of the map $d_{\Delta_n,J}$ vanishes unless $J = \Delta_{n-1}$. Indeed, recall that for $J = \Delta_n \setminus \{j\}$ the map $\mathrm{d}_{\Delta_n,J}$ is equal to δ_j. Since $\mathcal{N}X_n = \bigcap_{j=0}^{n-1} \operatorname{Ker}\delta_j$, the restriction of $\mathrm{d}_{\Delta_n,J}$ to $\mathcal{N}X_n$ is zero unless $J = \Delta_n \setminus \{n\}$. The map ϕ is injective, and we identify $\sigma_{\leqslant N}\mathcal{N}X_*$ with its image inside $C_*(\Delta_N, X_\bullet)$.

We consider the decreasing filtration F on the complex $C_*(\Delta_N, X_\bullet)$ defined, using (A.237) again, as follows. For each factor X_I of $C_n(\Delta_N, X_\bullet)$, we write the subset $I = \{i_0, \dots, i_k, \dots\}$ in increasing order, and set

$$F^p X_I = \begin{cases} \bigcap_{j=0}^{\min(p,|I|-1)} \operatorname{Ker}\delta_j, & \text{if } i_j = j \text{ for } j = 0, \dots, \min(p, |I|-1), \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the simplicial identities that this is a filtration by subcomplexes. Moreover, the following equalities hold:

$$F^{-1}C_*(\Delta_N, X_\bullet) = C_*(\Delta_N, X_\bullet),$$
$$F^N C_*(\Delta_N, X_\bullet) = \sigma_{\leqslant N}\mathcal{N}X_*.$$

Therefore, it is enough to show, for $p \geqslant 0$, that each inclusion

$$F^p C_*(\Delta_N, X_\bullet) \hookrightarrow F^{p-1}C_*(\Delta_N, X_\bullet),$$

is a homotopy equivalence.

For $p \geqslant 0$, let $I \subset \Delta_N$ be a subset of the form

$$I = \{0, 1, \dots, p-1, i_p, \dots, i_n\}, \quad p < i_p < i_{p+1} < \dots < i_n. \tag{A.239}$$

Write $I_p = \{0, 1, \dots, p-1, p, i_p, \dots, i_n\}$. There is an increasing map $\sigma^p\colon I_p \to I$ that sends p to i_p, and the other elements to themselves. Since $X_\bullet$ is a simplicial object, there is a map $\sigma_p\colon X_I \to X_{I_p}$.

For $p \geqslant 0$, we define the degree 1 map

$$s^p\colon F^{p-1}C_*(\Delta_N, X_\bullet) \longrightarrow F^{p-1}C_*(\Delta_N, X_\bullet)$$

given, on the factor X_I, by $(-1)^p\sigma_p\colon X_I \to X_{I_p}$ if I has the shape (A.239), and by zero otherwise. We then look at the map

$$\psi_p\colon F^{p-1}C_*(\Delta_N, X_\bullet) \longrightarrow F^{p-1}C_*(\Delta_N, X_\bullet)$$

given by $\psi_p = \mathrm{Id} - (\mathrm{d}s^p + s^p\mathrm{d})$. The simplicial identities imply the following:

i) if $x \in F^pC_*(\Delta_N, X_\bullet)$, then $\psi_p(x) = x$;

ii) if $x \in F^{p-1}C_*(\Delta_N, X_\bullet)$, then $\psi_p(x) \in F^pC_*(\Delta_N, X_\bullet)$.

Thus, ψ_p induces a morphism of complexes

$$\psi'_p\colon F^{p-1}C_*(\Delta_N, X_\bullet) \longrightarrow F^pC_*(\Delta_N, X_\bullet)$$

that, combined with the inclusion in the opposite direction, yields a homotopy equivalence. Writing $\psi = \psi'_{N-1} \circ \cdots \circ \psi'_0$, we obtain a homotopy inverse of the map ϕ. Since we have only used the simplicial maps, it is clear that the resulting homotopy equivalence is functorial. □

⋆⋆⋆

Exercise A.240. Let $n \geqslant 0$ be an integer. We keep the notation from Example A.228. Show that the map $\Delta^n \to \Delta^n_{\mathrm{st}}$ given by

$$(t_1, \dots, t_n) \longmapsto (1 - t_1, t_1 - t_2, \dots, t_{n-1} - t_n, t_n)$$

is a homeomophism that commutes with the face and degeneracy maps on the topological and the standard simplex.

Exercise A.241. Let $X_\bullet$ be a simplicial object in an abelian category. Use the simplicial identities (A.226) to prove the following statements.

i) The composition $\partial_n \circ \partial_{n+1}$ is zero. Therefore, (CX_*, ∂) is a complex.
ii) The map ∂_n sends $\mathcal{N}X_n$ to $\mathcal{N}X_{n-1}$ and agrees with $(-1)^n\delta_n$ when restricted to $\mathcal{N}X_n$.
iii) The map ∂_n sends DX_n to DX_{n-1}.

Exercise A.242 (The nerve of a category). Let $\mathcal{C}$ be a small category. Let $N(\mathcal{C})_0$ denote the set of objects and $N(\mathcal{C})_1$ the set of morphisms. For each integer $n \geqslant 2$, define $N(\mathcal{C})_n$ as the set of n-tuples of composable morphisms

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n. \tag{A.243}$$

On the one hand, there are maps

$$\delta_i\colon N(C)_n \longrightarrow N(C)_{n-1}, \qquad i = 0, \dots, n,$$

given by composing at the i-th object or removing it whenever $i = 0$ or n. In other words, δ_i sends an n-tuple as in (A.243) to the $(n-1)$-tuple

$$\begin{aligned} &C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} C_n, && \text{if } i = 0,\\ &C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1}\circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} C_n, && \text{if } 0 < i < n,\\ &C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1}, && \text{if } i = 0. \end{aligned}$$

On the other hand, there are maps

$$\sigma_i\colon N(C)_n \longrightarrow N(C)_{n+1}, \qquad i = 0, \dots, n,$$

obtained by inserting an identity morphism at the i-th object, that is, σ_i sends an n-tuple as in (A.243) to the $(n+1)$-tuple

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} C_i \xrightarrow{\mathrm{Id}} C_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} C_n.$$

Prove that $N(\mathcal{C})_\bullet$, together with the maps δ_i as faces and the maps σ_i as degeneracies, has the structure of a simplicial set. In particular, identify the simplicial identity which corresponds to the associativity of the composition of morphisms. This construction is called the *nerve* of the category $\mathcal{C}$.

A.9. Sheaf cohomology. This section contains a brief summary of the main properties of sheaf cohomology. For more detailed accounts, we refer the reader to the books by Bredon [**Bre97**] and Iversen [**Ive86**].

A.9.1. *The definition of a sheaf.*

DEFINITION A.244. Let M be a topological space. A *presheaf of abelian groups* F on M is the data of

- an abelian group $F(U)$ for each open subset $U \subset M$;
- a group homomorphism
$$\rho_{U,V} : F(V) \longrightarrow F(U)$$
for each inclusion $U \subset V$ of open subsets;

satisfying the following properties:

i) $\rho_{U,U} = \mathrm{Id}_{F(U)}$ for all open subsets U;

ii) $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$ for all open subsets $U \subset V \subset W$.

The elements of $F(U)$ are called the *sections* of F on U, and the maps $\rho_{U,V}$ are called *restriction maps.* The notation
$$t\,|_U = \rho_{U,V}(t)$$
is most often used for the restriction of a section $t \in F(V)$. Sometimes, $F(U)$ is also denoted by $\Gamma(U, F)$, especially for $U = M$.

DEFINITION A.245. Let F and G be presheaves of abelian groups on a topological space M. A *morphism of presheaves* $\varphi \colon F \to G$ is the data of a group homomorphism $\varphi_U : F(U) \to G(U)$ for each open subset $U \subset M$ that is compatible with the restriction maps in that the diagram
$$\begin{tikzcd} F(V) \arrow[r, "\varphi_V"] \arrow[d, "\rho_{U,V}"'] & G(V) \arrow[d, "\rho_{U,V}"] \\ F(U) \arrow[r, "\varphi_U"] & G(U) \end{tikzcd}$$
commutes for all open subsets $U \subset V$.

DEFINITION A.246. A presheaf of abelian groups F on M is called a *sheaf* if it satisfies the following two extra conditions:

iii) for each open subset $U \subset M$ and each open cover $U = \bigcup_{i \in I} U_i$, if a section $t \in F(U)$ satisfies $t\,|_{U_i} = 0$ for all $i \in I$, then $t = 0$;

iv) for each open subset $U \subset M$, each open cover $U = \bigcup_{i \in I} U_i$, and each collection of sections $t_i \in F(U_i)$ satisfying
$$t_i\,|_{U_i \cap U_j} = t_j\,|_{U_i \cap U_j}$$
for all $i, j \in I$, there exists a section $t \in F(U)$ satisfying $t\,|_{U_i} = t$.

A *morphism of sheaves* $\varphi \colon F \to G$ is the same as a morphism of presheaves.

Unless otherwise indicated, the word "sheaf" means "sheaf of abelian groups", and we denote by $\mathbf{Sh}(M)$ the category of sheaves.

Remark A.247. Properties i) and ii) in the definition of presheaf can be rephrased as follows: let $\mathbf{Op}(M)$ be the category whose objects are the open subsets of M, and whose sets of morphisms $\mathrm{Hom}(U, V)$ consist of a singleton if U is a subset of V and are empty otherwise; then F is a contravariant functor from $\mathbf{Op}(M)$ to the category $\mathbf{Ab}$ of abelian groups. Property iii) means that the conditions defining $F(U)$ are of *local* nature, *i.e.* can be tested on an open neighbourhood of each point. For example, being a closed differential form is a local property, but being exact is not. Property iv) allows one to *glue* local sections. Moreover, thanks to the locality property, the section t in iv) is unique. In particular, a sheaf F is determined by its sections on a basis of the topology of M and the restriction maps between them. For example, if M is a differentiable manifold, it suffices to work with contractible open subsets (Exercice A.322).

Definition A.248. Let F be a presheaf on a topological space M and $x \in M$ a point. The *stalk* of F at x is the direct limit

$$F_x = \varinjlim_{x \in U} F(U)$$

over the directed set (Section A.6.1) of open neighbourhoods of x in M, with the partial order $U \leqslant V$ if and only if $V \subset U$ and transition maps $\rho_{U,V}$.

An element of the stalk F_x is thus an equivalence class of pairs (U, t), where U is an open neighbourhood of x and $t \in F(U)$ is a section, with respect to the equivalence relation $(U, t) \sim (V, s)$ if there exists an open neighbourhood $W \subset U \cap V$ of x such that $t|_W = s|W$. The class of (U, t) in F_x will be denoted by t_x. Each stalk is an abelian group, and $t \mapsto t_x$ is a group homomorphism $F(U) \to F_x$.

In practice, it is often easier to write down a presheaf than a sheaf, as the locality and the gluing property may fail in natural situations. For this reason, it is very useful to have a canonical way to produce a sheaf starting from a presheaf.

Proposition A.249 (Sheaf associated with a presehaf). *Given a presheaf F on a topological space M, there exists a sheaf F^+ on M and a morphism of presheaves $\theta\colon F \to F^+$ that is universal for this property. That is, for every morphism of presheaves $f\colon F \to G$ with target a sheaf G, there exists a unique morphism of sheaves $\phi\colon F^+ \to G$ satisfying $f = \phi \circ \theta$.*

This is proved, for instance, in [**Har77**, Prop.–Def. II 1.2]. The idea is to define the sections $F^+(U)$ as the group of functions $s\colon U \to \coprod_{x \in M} F_x$ such that $s(x)$ belongs to F_x and is locally given by a section of F, *i.e.* there exists an open neighbourhood $V \subset U$ of x and $t \in F(V)$ such that $t_y = s(y)$ for all $y \in V$.

Definition A.250. We call F^+ the *sheaf associated with the presheaf* F or the *sheafification* of the presheaf F.

By construction, F^+ and F have the same stalks at all points.

Examples A.251. Let M be a topological space and let A be an abelian group.

i) The *constant presheaf* A° is the presheaf with $A^o(U) = A$ for each open subset $U \subset M$, and all restriction maps equal to the identity. This is rarely a sheaf, because sections on two disjoint open subsets do not glue to a section on the union unless they are equal. The *constant sheaf* $\underline{A}$ is defined as the sheaf associated with A°; its sections are equal to

$$\underline{A}(U) = \{\text{locally constant functions } U \to A\}.$$

We will also write $\underline{A}_M$ when we want to emphasise the topological space. A sheaf is called *constant* if it is of the form $\underline{A}$ for some abelian group A.

ii) Given a sheaf F on M and an open subset $U \subset M$, the restriction of F to U is the sheaf $F|_U$ on U with sections $F|_U(W) = F(W)$ for each open subset $W \subset U$, and the same restriction maps as those of F. A sheaf F is said to be *locally constant* if each point of M has an open neighbourhood U such that $F|_U$ is a constant sheaf. For example, by Cauchy's theorem, the holomorphic solutions of a differential equation

$$\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^n f + a_{n-1}(z)\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^{n-1} f + \cdots + a_0(z) f = 0,$$

where $a_i(z)$ are holomorphic functions on a punctured complex plane $\mathbb{C}\setminus S$, form a locally constant sheaf on $\mathbb{C} \setminus S$.

iii) Let A be an abelian group, and let $x \in M$ be a point. The *skyscraper sheaf* A_x is the sheaf with sections

$$\underline{A}_x(U) = \begin{cases} A, & \text{if } x \in U, \\ 0, & \text{if } x \notin U, \end{cases}$$

and restriction maps $\rho_{U,V}$ equal to the identity if both U and V contain x, and the zero map otherwise.

A.9.2. *Sheaf cohomology.* The category $\mathbf{Sh}(M)$ is abelian, so it makes sense to talk about kernels, cokernels, and images of morphisms of sheaves; of complexes of sheaves and their cohomology; of exact sequences; of injective sheaves, and so on.

Let F be a sheaf on a topological space M. The elements of the group

$$F(M) = \Gamma(M, F)$$

are called the *global sections* of F. The assignment $F \mapsto \Gamma(M, F)$ gives rise to the global sections functor

$$\Gamma\colon\ \mathbf{Sh}(M) \longrightarrow \mathbf{Ab}\,.$$

The functor Γ is left exact (Definition A.23): for each short exact sequence of sheaves $0 \to F_1 \to F_2 \to F_3 \to 0$, the sequence of abelian groups

$$0 \longrightarrow \Gamma(M, F_1) \longrightarrow \Gamma(M, F_2) \longrightarrow \Gamma(M, F_3)$$

is exact. However, the rightmost map does not need to be surjective, and hence the functor is not exact. For example, let M be a connected Hausdorff topological space and let $x, y \in M$ be two distinct points. The morphism of sheaves $\underline{\mathbb{Z}}_M \to \mathbb{Z}_x \oplus \mathbb{Z}_y$ that sends a locally constant function to its values at x and y is surjective (check stalk by stalk), but the induced map on global sections $\Gamma(M, \underline{\mathbb{Z}}_M) \to \Gamma(M, \mathbb{Z}_x \oplus \mathbb{Z}_y)$ is the diagonal map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, which is not. This observation is the starting point of the definition of sheaf cohomology. Recall the notion of derived functor from Definition A.102, and the criterion for existence given in Proposition A.112. The functor Γ is left exact, so in order to apply it it remains to prove the following:

Lemma A.252. *The category* $\mathbf{Sh}(M)$ *has enough injectives.*

Proof. Let F be a sheaf. For each point $x \in M$ and each abelian group A, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}}(F_x, A) \simeq \mathrm{Hom}_{\mathbf{Sh}(M)}(F, A_x),$$

which shows that the skyscraper sheaf A_x is an injective object of $\mathbf{Sh}(M)$ if A is an injective abelian group. Using that the category $\mathbf{Ab}$ has enough injectives (Example A.107), there exists an injective homomorphism $F_x \to I_x$ into an injective abelian group for each x. Thinking of the group I_x as a skyscraper sheaf supported at x, the product $\prod_{x\in M} I_x$ is an injective sheaf by Exercise A.129, and the morphism $F \to \prod_{x\in M} I_x$ obtained from $F \to F_x$ is a monomorphism. □

Definition A.253. Let M be a topological space and let F be a sheaf on M. The *cohomology groups* of F are the derived functors

$$\mathrm{H}^n(M,F) = R^n\Gamma(M,F)$$

of the global sections functor.

Sheaf cohomology can be computed by choosing an injective resolution

$$0 \longrightarrow F \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

of F (*i.e.* a long exact sequence where all the I^i are injective sheaves) and considering the cohomology of the complex

$$0 \longrightarrow \Gamma(M,I^0) \longrightarrow \Gamma(M,I^1) \longrightarrow \Gamma(M,I^2) \longrightarrow \cdots$$

obtained by taking global sections. That is, the equality

$$\mathrm{H}^n(M,F) = \mathrm{H}^n(\Gamma(M,I^*))$$

holds. The resulting groups are independent of the choice of the injective resolution.

Although injective resolutions are useful for theoretical purposes, they may not be the best way to compute cohomology explicitly. For this reason, it is useful to have more concrete techniques at disposal.

Definition A.254. Let F be a sheaf on a topological space M.

i) We say that F is *flasque* (or *flabby*) if the restriction maps

$$\rho_{U,V}\colon F(V) \longrightarrow F(U)$$

are surjective for all open subsets $U \subset V \subset M$.

ii) We say that F is *acyclic* if the vanishing $\mathrm{H}^n(X,F) = 0$ holds for all $n \geqslant 1$.

An *acyclic resolution* of a sheaf F is an exact sequence

$$0 \longrightarrow F \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots$$

in which all A^i are acyclic sheaves.

Remark A.255. It is clear from this definition that the notion of an acyclic sheaf coincides with the notion of a Γ-acyclic sheaf from Definition A.207.

In view of this remark, a particular case of Proposition A.209 is the following lemma. For comparison, we give a direct proof that does not use spectral sequences.

Lemma A.256. *Let F be a sheaf and let A^* be an acyclic resolution of F. Then there is a canonical isomorphism*

$$\mathrm{H}^n(M,F) \xrightarrow{\sim} \mathrm{H}^n(\mathrm{H}^0(M,A^*)).$$

Proof. Consider the short exact sequence

$$0 \longrightarrow F \longrightarrow A^0 \longrightarrow A^0/F \longrightarrow 0.$$

Using the acyclicity of A^0, we deduce an exact sequence

$$0 \longrightarrow \mathrm{H}^0(F) \longrightarrow \mathrm{H}^0(A^0) \longrightarrow \mathrm{H}^0(A^0/F) \longrightarrow \mathrm{H}^1(F) \longrightarrow 0$$

and isomorphisms

$$\mathrm{H}^n(A^0/F) \simeq \mathrm{H}^{n+1}(F) \quad \text{for all} \quad n \geqslant 1. \tag{A.257}$$

Besides, the complex

$$0 \longrightarrow A^0/F \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots \tag{A.258}$$

is an acyclic resolution of A^0/F. From this, one deduces the equalities

$$\mathrm{H}^0(F) = \mathrm{Ker}(\mathrm{H}^0(A^0) \longrightarrow \mathrm{H}^0(A^0/F)) = \mathrm{Ker}(\mathrm{H}^0(A^0) \longrightarrow \mathrm{H}^0(A^1)),$$
$$\mathrm{H}^1(F) = \frac{\mathrm{H}^0(A^0/F)}{\mathrm{Im}(\mathrm{H}^0(A^0) \longrightarrow \mathrm{H}(A^0/F))} = \frac{\mathrm{Ker}(\mathrm{H}^0(A^1) \longrightarrow \mathrm{H}^0(A^2))}{\mathrm{Im}(\mathrm{H}^0(A^0) \longrightarrow \mathrm{H}^0(A^1))}.$$

We have thus proved the cases $n = 0$ and $n = 1$ of the lemma. The remaining cases follow inductively from the resolution (A.258) and the isomorphisms (A.257). ☐

Lemma A.259. *Every flasque sheaf is acyclic.*

A proof can be found in [**Har77**, Prop. III 2.5].

There is another class of acyclic sheaves called *fine sheaves* that is very useful in differential geometry. For instance, we used them in the proof of de Rham's Theorem 2.79 to show that the sheaf of smooth differential forms is acyclic.

Definition A.260. A topological space M is said to be *paracompact* if it is Hausdorff and every open cover of M can be refined to a locally finite cover. That is, given an open cover $\mathfrak{U} = (U_i)_{i\in I}$ there exists an open cover $\mathfrak{V} = (V_j)_{j\in J}$ such that each V_j is contained in some U_i and every point $x \in M$ has an open neighborhood which only intersects finitely many V_j.

Definition A.261. Let M be a paracompact space. A sheaf F on M is said to be *fine* if, for every locally finite open cover $\{U_i\}$ of M, there exists a family $\{h_i\}_{i\in I}$ of sheaf endomorphisms $h_i\colon F \to F$ such that

i) the support of h_i is contained in U_i, that is, $h_i(s)_x = 0$ holds for all local sections s of F and all x in some open neighborhood of $X \setminus U_i$;
ii) the sum $\sum_{i\in I} h_i$, that is well defined because of i) and the fact that the covering is locally finite, is the identity endomorphism.

Such a family is called a *partition of unity*.

Example A.262. Let M be a differentiable manifold, and let $\mathcal{E}^0_M$ be the sheaf of differentiable functions on M. For every locally finite open cover $\{U_i\}_{i\in I}$ of M, one can find a family $\{f_i\}_{i\in I}$ of differentiable functions $f_i\colon M \to \mathbb{R}_{\geqslant 0}$ such that

i) $\mathrm{supp}(f_i) \subset U_i$,
ii) $\sum_{i\in I} f_i(x) = 1$ for all $x \in M$.

Any such family $\{f_i\}_{i\in I}$ is called a differentiable partition of unity subordinated to the given open cover. If a sheaf F is an $\mathcal{E}^0_M$-module, then multiplication by the function f_i provides the endomorphism h_i in Definition A.261. It follows that every $\mathcal{E}^0_M$-module is a fine sheaf. This applies in particular to the sheaves of smooth differential forms $\mathcal{E}^p_M$.

LEMMA A.263. *Every fine sheaf is acyclic.*

PROOF. Exercise A.333. □

A.9.3. *Godement's canonical resolution.* A sheaf F on a topological space M admits a canonical resolution by flasque sheaves called *Godement's resolution*. The construction starts as follows: for each open subset U of M, consider the product

$$\mathrm{Gd}^0(F)(U) = \prod_{x\in U} F_x$$

of the stalks of F at all points $x \in U$. Together with the obvious restriction maps, one obtains a presheaf $\mathrm{Gd}^0(F)$ on M, which is readily seen to be a sheaf.

LEMMA A.264. *For every sheaf F, the sheaf $\mathrm{Gd}^0(F)$ is flasque.*

PROOF. It follows immediately from the definition. □

The natural morphism of sheaves $F \to \mathrm{Gd}^0(F)$ is injective, and one defines

$$\mathrm{Gd}^1(F) = \mathrm{Gd}^0(\mathrm{Gd}^0(F)/F).$$

There is an obvious morphism of sheaves $\partial\colon\ \mathrm{Gd}^0(F) \to \mathrm{Gd}^1(F)$. Assume now that we have constructed sheaves $\mathrm{Gd}^i(F)$ with morphisms $\partial\colon\ \mathrm{Gd}^{i-1}(F) \to \mathrm{Gd}^i(F)$, for all $i < k$, satisfying $\partial \circ \partial = 0$. Then one defines

$$\mathrm{Gd}^k(F) = \mathrm{Gd}^0(\mathrm{Gd}^{k-1}(F)/\partial\,\mathrm{Gd}^{k-2}(F)).$$

Clearly, there is a map $\partial\colon\ \mathrm{Gd}^{k-1}(F) \to \mathrm{Gd}^k(F)$ satisfying $\partial(\partial\,\mathrm{Gd}^{k-2}(F)) = 0$.

The following result can be checked directly from the definition.

LEMMA A.265. *For every sheaf F,*

$$0 \longrightarrow F \longrightarrow \mathrm{Gd}^0(F) \longrightarrow \mathrm{Gd}^1(F) \longrightarrow \mathrm{Gd}^2(F) \longrightarrow \cdots$$

is an exact sequence of sheaves.

DEFINITION A.266. The *Godement resolution* of F is the complex

$$\mathrm{Gd}^*(F)\colon \qquad \mathrm{Gd}^0(F) \longrightarrow \mathrm{Gd}^1(F) \longrightarrow \mathrm{Gd}^2(F) \longrightarrow \cdots$$

Moreover, Godement's resolution is *functorial*.

LEMMA A.267. *If $f\colon F \to G$ is a morphism of sheaves, then there is an induced morphism of complexes of sheaves*

$$\mathrm{Gd}(f)\colon\ \mathrm{Gd}(F) \longrightarrow \mathrm{Gd}(G)$$

satisfying $\mathrm{Gd}(f\circ g) = \mathrm{Gd}(f)\circ \mathrm{Gd}(g)$ *and* $\mathrm{Gd}(\mathrm{Id}) = \mathrm{Id}$.

Another important property of Godement's resolution is its exactness.

LEMMA A.268. *The formation of Godement's resolution is an exact functor. That is, for each exact sequence*

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

of sheaves on M, the sequence of sheaves

$$0 \longrightarrow \mathrm{Gd}^n(F) \longrightarrow \mathrm{Gd}^n(G) \longrightarrow \mathrm{Gd}^n(H) \longrightarrow 0$$

is exact for all $n \geqslant 0$.

PROOF. We argue by induction on n, setting $\mathrm{Gd}^{-1}(F) = F$ and $\mathrm{Gd}^{-2}(F) = 0$ for notational convenience. For $n = 0$, it follows from the definition of $\mathrm{Gd}^0(F)$ as the product of stalks that the sequence

$$0 \longrightarrow \mathrm{Gd}^0(F)(U) \longrightarrow \mathrm{Gd}^0(G)(U) \longrightarrow \mathrm{Gd}^0(H)(U) \longrightarrow 0$$

is exact for all open subsets $U \subset M$, and hence that the sequence of sheaves

$$0 \longrightarrow \mathrm{Gd}^0(F) \longrightarrow \mathrm{Gd}^0(G) \longrightarrow \mathrm{Gd}^0(H) \longrightarrow 0$$

is exact. Assume that the functors Gd^{n-1} and $\mathrm{Gd}^{n-2}/\partial\,\mathrm{Gd}^{n-3}$ are exact, and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
(\mathrm{Gd}^{n-2}/\partial\,\mathrm{Gd}^{n-3})(F) \arrow[r] \arrow[d] & (\mathrm{Gd}^{n-2}/\partial\,\mathrm{Gd}^{n-3})(G) \arrow[r] \arrow[d] & (\mathrm{Gd}^{n-2}/\partial\,\mathrm{Gd}^{n-3})(H) \arrow[d] \\
\mathrm{Gd}^{n-1}(F) \arrow[r] \arrow[d] & \mathrm{Gd}^{n-1}(G) \arrow[r] \arrow[d] & \mathrm{Gd}^{n-1}(H) \arrow[d] \\
(\mathrm{Gd}^{n-1}/\partial\,\mathrm{Gd}^{n-2})(F) \arrow[r] \arrow[d] & (\mathrm{Gd}^{n-1}/\partial\,\mathrm{Gd}^{n-2})(G) \arrow[r] \arrow[d] & (\mathrm{Gd}^{n-1}/\partial\,\mathrm{Gd}^{n-2})(H). \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

In this diagram, all three columns are exact by design. For lack of space, we have omitted the zeros from the beginning and the end of each row. The first two rows are exact by the induction hypothesis. It then follows from the 3×3 lemma, also called the nine lemma (see [**Wei94**, Ex. 1.3.2]), that the third row is exact as well. Once that we know that the functors Gd^0 and $\mathrm{Gd}^{n-1}/\partial\,\mathrm{Gd}^{n-2}$ are exact, exactness of Gd^n follows from the formula

$$\mathrm{Gd}^n(F) = \mathrm{Gd}^0(\mathrm{Gd}^{n-1}(F)/\partial\,\mathrm{Gd}^{n-2}(F)).$$

This concludes the proof. $\square$

Thanks to this exactness property, Godement's resolution of a filtered sheaf is canonically endowed with a filtration.

DEFINITION A.269. Let F be a sheaf, and let W be an increasing filtration on F. We define the filtration $\mathrm{Gd}(W)$ on $\mathrm{Gd}^*(F)$ as

$$\mathrm{Gd}(W)_n\,\mathrm{Gd}^p(F) = \mathrm{Gd}^p(W_nF).$$

Note that the exactness of Gd^p implies that $\mathrm{Gd}^p(W_nF)$ is a subsheaf of $\mathrm{Gd}^p(F)$. The definition for decreasing filtrations is similar.

A.9.4. *Hypercohomology.* We now turn to complexes of sheaves. Let F^* be a bounded below complex of sheaves on a topological space M. There are two possible meanings for the cohomology of F^*: either the cohomology of the complex viewed as a cochain complex in the abelian category $\mathbf{Sh}(M)$ of sheaves on M, in which case the cohomology objects will also be sheaves, or the result of applying the derived functor of global sections to F^*, in which case the resulting objects will be abelian groups. To distinguish between these two, the latter is classically called the *hypercohomology* of the complex. This terminology is now a bit outdated.

Definition A.270. The *hypercohomology* groups of F^* are the cohomological right derived functors of the global sections functor applied to F^* as an object of $\mathcal{D}^+(\mathbf{Sh}(M))$. In other words,

$$\mathbb{H}^n(M, F^*) = R^n\Gamma(F^*).$$

Therefore, we may compute them by means of any bounded below acyclic resolution D^* of the complex F^*. In symbols,

$$\mathbb{H}^n(M, F^*) = \mathrm{H}^n(\Gamma(M, D^*)).$$

From the functoriality and exactness of Godement's canonical resolution, we can construct an acyclic resolution of any bounded below complex of sheaves (F^*, d). Indeed, let $\mathrm{Gd}^*(F^n)$ be Godement's canonical resolution of the sheaf F^n. By functoriality (Lemma A.267), the differentials of the complex induce morphisms of sheaves

$$\mathrm{d}^{\mathrm{hor}} \colon \mathrm{Gd}^m(F^n) \longrightarrow \mathrm{Gd}^m(F^{n+1}),$$

that commute with the morphisms

$$\mathrm{d}^{\mathrm{ver}} \colon \mathrm{Gd}^m(F^n) \longrightarrow \mathrm{Gd}^{m+1}(F^n)$$

in Godement's resolution. We thus obtain a double complex $\mathrm{Gd}^*(F^*)$ in the sense of Definition A.33.

Definition A.271. Let F^* be a bounded below complex of sheaves. The *Godement resolution* of F^* is the total complex of the double complex $\mathrm{Gd}^*(F^*)$:

$$\mathrm{Gd}(F^*) = \mathrm{Tot}^*(\mathrm{Gd}^*(F^*)).$$

Recall that this means that $\mathrm{Gd}(F^*)$ is the complex with terms

$$\mathrm{Tot}^n(\mathrm{Gd}^*(F^*)) = \bigoplus_{p+q=n} \mathrm{Gd}^q(F^p)$$

and differential d given, for each $x \in \mathrm{Gd}^q(F^p)$, by

$$\mathrm{d}x = \mathrm{d}^{\mathrm{hor}}x + (-1)^p\mathrm{d}^{\mathrm{ver}}x.$$

The functoriality of the Godement resolution gives us a commutative diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \mathrm{Gd}^1(F^{n-1}) \arrow[r] \arrow[u] & \mathrm{Gd}^1(F^n) \arrow[r] \arrow[u] & \mathrm{Gd}^1(F^{n+1}) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathrm{Gd}^0(F^{n-1}) \arrow[r] \arrow[u] & \mathrm{Gd}^0(F^n) \arrow[r] \arrow[u] & \mathrm{Gd}^0(F^{n+1}) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & F^{n-1} \arrow[r] \arrow[u] & F^n \arrow[r] \arrow[u] & F^{n+1} \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The commutativity of this diagram implies that there is a morphism of complexes of sheaves $F^* \to \mathrm{Gd}(F^*)$. The exactness of the Godement resolution for sheaves (Lemma A.265) means that, in the above diagram, all columns are exact. Hence, this map is a quasi-isomorphism making $\mathrm{Gd}(F^*)$ into an acyclic resolution of F^*. The hypercohomology of $F^\bullet$ is then given by

$$\mathbb{H}^n(M, F^*) = \mathrm{H}^n(\Gamma(M, \mathrm{Gd}(F^*))).$$

Thanks to Godement's resolution, the hypercohomology of a sheaf can be computed as the cohomology of a double complex. In particular, it comes equipped with the spectral sequence (A.206). The first page of this spectral sequence is

$$E_1^{p,q} = \mathrm{H}^q(M, F^p) \Longrightarrow \mathbb{H}^{p+q}(M, F^*). \tag{A.272}$$

As a particular case of Proposition A.210, there is another useful spectral sequence that relates the hypercohomology groups of a complex of sheaves with the cohomology groups of the cohomology sheaves of the complex. Instead of using an injective resolution, one can use Godement's resolution. Roughly speaking, this second spectral sequence is constructed by flipping the indices of Godement's resolution. Indeed, by the exactness of Godement's resolution (Lemma A.268), we have

$$\begin{gathered}
\mathrm{d}^{\mathrm{hor}}(\mathrm{Gd}^p(F^{q-1})) = \mathrm{Gd}^p(\mathrm{d}F^{q-1}),\\
\mathrm{Ker}(\mathrm{d}^{\mathrm{hor}}\colon \mathrm{Gd}^p(F^q) \to \mathrm{Gd}^p(F^{q+1})) = \mathrm{Gd}^p(\mathrm{Ker}(\mathrm{d}\colon F^q \to F^{q+1})),\\
\mathrm{H}^q_{\mathrm{d}^{\mathrm{hor}}}(\mathrm{Gd}^p(F^*)) = \mathrm{Gd}^p(\mathrm{H}^q(F^*)).
\end{gathered}$$

Compare these properties with Exercise A.131 ii). Since the sheaves $\mathrm{Gd}^p(F)$ are flasque for any F (Lemma A.264), this implies the equality

$$\mathrm{H}^q_{\mathrm{d}^{\mathrm{hor}}}(\Gamma(M, \mathrm{Gd}^p(F^*))) = \Gamma(M, \mathrm{Gd}^p(\mathrm{H}^q(F^*))).$$

Therefore, if we endow the complex $\Gamma(M, \mathrm{Gd}^*(F^*))$ with the filtration

$$''F^p\Gamma(M, \mathrm{Gd}(F^*)) = \bigoplus_{p' \geqslant p} \Gamma(M, \mathrm{Gd}^{p'}(F^*)),$$

then we obtain a spectral sequence whose second page is

$$E_2^{p,q} = \mathrm{H}^p(M, \mathrm{H}^q(F^*)) \Longrightarrow \mathbb{H}^{p+q}(M, F^*). \tag{A.273}$$

A.9.5. *Higher direct images.* Higher direct images are a way to encode how sheaf cohomology varies in continuous families of topological spaces.

Definition A.274. Let $f\colon M \to N$ be a continuous map of topological spaces and let F be a sheaf on M. The *direct image* sheaf f_*F is the sheaf on N whose sections on an open subset $U \subset N$ are given by

$$(f_*F)(U) = F(f^{-1}(U)), \tag{A.275}$$

and whose restriction maps for open subsets $U \subset V \subset N$ are those of F for the open subsets $f^{-1}(U) \subset f^{-1}(V) \subset M$.

In fact, the direct image construction (A.275) defines a left exact functor

$$f_*\colon \mathbf{Sh}(M) \longrightarrow \mathbf{Sh}(N).$$

Since the category $\mathbf{Sh}(M)$ has enough injectives (Lemma A.252), the general constructions of Section A.4 apply to this setting, thus giving rise to a total right derived functor Rf_* and to cohomological δ-functors R^if_* for each integer $i \geqslant 0$. These cohomological δ-functors are called *higher direct images*.

Example A.276. The cohomology groups of a sheaf are a particular case of higher direct images. Indeed, let M be a topological space and let F be a sheaf on M. Let $*$ denote the topological space consisting of a single point and let $\pi\colon M \to *$ be the unique map with target $*$. Since a sheaf on $*$ is simply an abelian group, the categories $\mathbf{Sh}(*)$ and $\mathbf{Ab}$ are canonically equivalent. Under this identification, the following equalities hold:

$$f_*F = \Gamma(M, F), \qquad R^if_*F = \mathrm{H}^i(M, F).$$

The stalks of the higher direct image sheaves are related with the cohomology of a small neighborhood of the fiber. For a proof of the next proposition, see for instance [**Har77**, Prop. III.8.1], together with the fact that the stalks of a presheaf agree with the stalks of the associated sheaf.

Proposition A.277. *Let $f\colon M \to N$ be a continuous map of topological spaces, let F be a sheaf on M, and let $y \in N$ be a point. The stalk of the higher direct image sheaf R^if_*F at y is given by*

$$(R^if_*F)_y = \varinjlim_{y\in U} \mathrm{H}^i(f^{-1}(U), F),$$

where the limit runs over all open sets U of N containing y ordered by inclusion.

The *Leray spectral sequence* associated with a continuous map $f\colon M \to N$ allows us to compute the cohomology of a sheaf F on M in terms of the cohomology of its higher direct images sheaves on N. It is a particular case of the Grothendieck spectral sequence from Theorem A.213.

Proposition A.278. *Let $f\colon M \to N$ be a continuous map, F^* a bounded below complex of sheaves on M, and F a sheaf on M. Then*

$$\mathbb{H}^*(M, F^*) = \mathbb{H}^*(N, Rf_*F).$$

Moreover, there is a spectral sequence with second page

$$E_2^{p,q} = \mathrm{H}^p(N, R^qf_*F) \Longrightarrow \mathrm{H}^{p+q}(M, F).$$

It is called the Leray spectral sequence.

PROOF. Consider the functors $\mathcal{G}\colon \mathbf{Sh}(M) \to \mathbf{Sh}(N)$ and $\mathcal{F}\colon \mathbf{Sh}(N) \to \mathbf{Ab}$ given by $\mathcal{G}(F) = f_*F$ and $\mathcal{F}(G) = \Gamma(N, G)$, so that their composition is

$$\mathcal{F} \circ \mathcal{G}(F) = \Gamma(N, f_*F) = \Gamma(M, F).$$

We then get $R^p\mathcal{F}(F) = R^p f_*(F)$ and

$$\mathrm{H}^*(R\mathcal{G}(R\mathcal{F}(F^*))) = \mathbb{H}^*(N, R\mathcal{F}(F^*)), \quad \mathrm{H}^*(R(\mathcal{F} \circ \mathcal{G})(F^*)) = \mathbb{H}^*(M, F^*),$$
$$R^q\mathcal{G}(R^p\mathcal{F}(F)) = \mathrm{H}^q(N, R^p\mathcal{F}(F)), \quad R^{p+q}(\mathcal{F} \circ \mathcal{G})(F) = \mathrm{H}^{p+q}(M, F).$$

By Exercise A.327, every injective sheaf in M is flasque, and it is clear from the definition of a flasque sheaf that, if I is a flasque sheaf then f_*I is again flasque, and hence acyclic. It follows that the functor $\mathcal{G}$ sends injective objects to $\mathcal{F}$-acyclic objects, so we can apply Theorem A.213 to obtain the result. □

We will also use a compatibility between Godement's resolution and direct images that we now explain. Let $f\colon M \to N$ be a continuous map of topological spaces and F a sheaf on M. Let $x \in M$ be a point and $y = f(x)$. Then

$$(f_*F)_y = \varinjlim_{y \in U} F(f^{-1}(U)),$$
$$F_x = \varinjlim_{x \in V} F(V).$$

Since $y \in U$ implies $x \in f^{-1}(U)$, we deduce a map $(f_*F)_y \to F_x$. Putting together all these maps, we deduce a morphism of sheaves

$$\mathrm{Gd}^0(f_*F) \longrightarrow f_* \mathrm{Gd}^0(F)$$

given, on an open set $U \subset N$, by

$$\big(\mathrm{Gd}^0(f_*F)\big)(U) = \prod_{y \in U} (f_*F)_y \longrightarrow \prod_{x \in f^{-1}(U)} F_x = \big(f_* \mathrm{Gd}^0(F)\big)(U).$$

The following result is left as an exercise. It follows easily from the previous construction and the definition of Godement's resolution.

LEMMA A.279. *The above construction induces a morphism of complexes*

$$\mathrm{Gd}^*(f_*F) \longrightarrow f_* \mathrm{Gd}^*(F) \tag{A.280}$$

that represents the morphism $f_*F \to Rf_*F$ in the derived category.

EXAMPLE A.281 (The trace map). Let $f\colon X \to Y$ be a finite morphism of algebraic varieties over the field of complex numbers. Assume that Y is smooth, X is irreducible, and f is dominant over an irreducible component of Y. Let f also denote the induced map of complex analytic spaces $f\colon X(\mathbb{C}) \to Y(\mathbb{C})$. In this example, we construct a *trace map*

$$\mathrm{Tr}_{X/Y}\colon f_*\underline{\mathbb{Q}}_{X(\mathbb{C})} \longrightarrow \underline{\mathbb{Q}}_{Y(\mathbb{C})}.$$

Let $U \subset Y(\mathbb{C})$ be an open subset and $s \in \Gamma(U, f_*\underline{\mathbb{Q}}_{X(\mathbb{C})})$ a section. Then s is a locally constant function $s\colon f^{-1}(U) \to \mathbb{Q}$, and we need to produce a locally constant function $\mathrm{Tr}_{X/Y}\, s\colon U \to \mathbb{Q}$. Let U_0 be a connected component of U, and choose a point $y \in U_0$ with the property that there is a connected neighborhood $W \subset U_0$ of y such that the map $f|_{f^{-1}(W)}$ is étale. For all $z \in U_0$, we define

$$\mathrm{Tr}_{X/Y}\, s(z) = \sum_{f(x)=y} s(x). \tag{A.282}$$

Since the set of points $y \in U_0$ satisfying the above condition is a connected subset U_0' of U_0, and the set of points $y' \in U_0'$ such that

$$\sum_{f(x)=y'} s(x) = \sum_{f(x)=y} s(x)$$

is open and closed, we deduce that (A.282) does not depend on the choice of y. The morphism (A.282) is hence well defined.

A.9.6. *Inverse images.* Given a continuous map $f\colon M \to N$ and a sheaf F on N, there is also a construction of the inverse image of F as a sheaf on M.

Definition A.283. Let $f\colon M \to N$ be a continuous map and F a sheaf on N. The *inverse image sheaf* $f^{-1}F$ is the sheaf on M associated with the presheaf

$$U \longmapsto \varinjlim_{V \supset f(U)} F(V),$$

where the limit runs over all open subsets $V \subset N$ containing $f(U)$ ordered by inclusion.

Remark A.284. If $\iota\colon Z \to M$ is the inclusion of either an open or a closed subset Z of M, and F is a sheaf on M, then we will denote the inverse image $\iota^{-1}F$ by $F|_Z$ and call it the *restriction* of F to Z.

Although the definition of $f^{-1}F$ looks more difficult than the definition of the direct image f_*F, this construction behaves better in some respects. For example, the stalks are easier to compute, as we have

$$(f^{-1}F)_x = F_{f(x)}$$

for all $x \in M$. From this it follows that $f^{-1}\colon \mathbf{Sh}(N) \to \mathbf{Sh}(M)$ is an exact functor.

Moreover, the direct image and the inverse image functors are *adjoint* to each other (see Section A.1.4). This means that, given a continuous map $f\colon M \to N$, a sheaf F on M, and a sheaf G on N, there is a functorial isomorphism

$$\operatorname{Hom}_{\mathbf{Sh}(M)}(f^{-1}G, F) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{Sh}(N)}(G, f_*F). \tag{A.285}$$

The adjunction morphisms (A.11) read

$$G \longrightarrow f_*f^{-1}G, \qquad f^{-1}f_*F \longrightarrow F \tag{A.286}$$

for each sheaf G on N and each sheaf F on M. We picture the situation as follows:

$$M \xrightarrow{f} N \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathbf{Sh}(M) \arrow[r, bend left, "f_*"] & \mathbf{Sh}(N). \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

A.9.7. *Direct image with proper support.* Given a continuous map $f\colon M \to N$, there is a second way to obtain a sheaf on N starting from a sheaf F on M which is called the *direct image with proper support* and denoted by $f_!F$. These symbols are often read as "f lower shriek of F", as "shriek" is a word for "exclamation". Before giving the definition of $f_!$, we recall the notion of support of a section.

Definition A.287. Let M be a topological space, F a sheaf on M, and $U \subset M$ an open subset. The *support* of a section $s \in F(U)$ is the set

$$\operatorname{supp}(s) = \{x \in U \mid s_x \neq 0\},$$

where s_x is the class of s in the stalk F_x (Definition A.248).

By definition, the complement $U \setminus \operatorname{supp}(s)$ of the support of a section s consists of the points $y \in U$ such that there is an open neighborhood $V \subset U$ of y satisfying $s|_V = 0$. It follows that $\operatorname{supp}(s)$ is a closed subset of U.

We also recall the notion of a proper map between topological spaces.

DEFINITION A.288. A continuous map of topological spaces $f\colon M \to N$ is called *proper* if the preimage $f^{-1}(K)$ of every compact subset $K \subset N$ is compact.

REMARK A.289. In algebraic geometry, there is also the notion of a proper morphism, as explained for instance in [**Har77**, § II 4]. One has to be careful that for a morphism of algebraic varieties, being proper is not equivalent to the map of the underlying topological spaces being proper. Nevertheless, if k is a subfield of $\mathbb{C}$ and $f\colon X \to Y$ is a morphism of k-varieties, then f is proper in the algebro-geometric sense if and only if $f^{\mathrm{an}}\colon X(\mathbb{C}) \to Y(\mathbb{C})$ is a proper map of topological spaces (see [**Har77**, App. B]).

Since the definition of proper we are using is pathological when the spaces are not Hausdorff and locally compact, in the remainder of this section we will restrict ourselves to this case.

DEFINITION A.290. Let $f\colon M \to N$ be a continuous map of locally compact Hausdorff topological spaces and let F be a sheaf on M. The *direct image with proper support* $f_!F$ is the sheaf

$$U \longmapsto \{s \in F(f^{-1}(U)) \mid \text{the restriction } \operatorname{supp}(s) \to U \text{ is proper}\}.$$

REMARK A.291. The fact that $f_!F$ is a sheaf is proved in [**Ive86**, VII Prop. 1.2]. Moreover, the formation of the direct image with proper support is functorial in the sense that $(f \circ g)_! = f_! \circ g_!$ holds. It is also clear form the definition that $f_!F$ is a subsheaf of f_*F and $f_!F = f_*F$ if f is proper. The functor $f_!$ is left exact.

When f is the inclusion of an open subset into a locally compact Hausdorff space, $f_!$ coincides with the extension by zero.

LEMMA A.292. *Let N be a locally compact Hausdorff space, let $j\colon M \to N$ be the inclusion of an open subset M of N, and let F be a sheaf on M. Then $j_!F$ is the sheaf on N associated with the presheaf*

$$U \longmapsto \begin{cases} F(U), & \text{if } U \subset M, \\ 0, & \text{otherwise.} \end{cases} \tag{A.293}$$

PROOF. The proof consists in constructing a map from the presheaf (A.293) to the sheaf $j_!F$ and showing that this map is an isomorphism on stacks. To this end, we first compute the stalks of $j_!F$.

If $x \in M$, then $(j_!F)_x = F_x$. Indeed, to compute $(j_!F)_x$ we can use open sets contained in M. If $V \subset M$ is an open subset and $s \in F(V)$ is a section, then $\operatorname{supp}(s)$ is closed in V. Since any closed subset of a compact set is compact, the restriction $\operatorname{supp}(s) \to V$ is proper.

If $y \notin M$, then $(j_!F)_y = 0$. Let U be an open set containing y. Put $V = U \cap M$, and let $s \in F(V)$ be a section such that $\operatorname{supp}(s) \to U$ is proper. Again by the hypothesis on N, this implies that $\operatorname{supp}(s)$ is closed in U. Hence, $W = U \setminus \operatorname{supp}(s)$ is open, contains y, and is such that $s|_W = 0$. We conclude that the class of any section with proper support vanishes in $(j_!F)_y$.

Let F_0 be the presheaf (A.293). In general, F_0 is not a sheaf (Exercise A.325). One readily checks that the stalks of this presheaf are equal to $(F_0)_x = F_x$ if $x \in M$, and to zero otherwise. Moreover, using that a closed immersion is a proper map, for every open subset $U \subset N$, there is an inclusion $F_0(U) \to j_!F(U)$, and all these inclusions glue together to give a morphism of presheaves $F_0 \to j_!F$.

Summing up, we deduce the existence of a morphism from the sheaf associated with the presheaf F_0 to $j_!F$ that is an isomorphism on stalks. Therefore, both sheaves are isomorphic. □

Remark A.294. The analogue of Lemma A.292 is also valid in the context of algebraic geometry.

The functor $f_!$ allows us to define cohomology with compact support.

Definition A.295. Let M be a locally compact Hausdorff topological space and $*$ the topological space consisting of a single point. Let $\pi\colon M \to *$ be the unique map from M to $*$. The functor of *global sections with compact support* is

$$\Gamma_c(M, F) = \Gamma(*, \pi_! F) = \{s \in F(M) \mid \operatorname{supp}(s) \text{ is compact}\},$$

and the *cohomology with compact support* is the derived functor of Γ_c:

$$\mathrm{H}^p_c(M, F) = R^p\Gamma_c(M, F).$$

Remark A.296. We can give a second interpretation of the cohomology with compact support using again that $\Gamma\colon \mathbf{Sh}(*) \to \mathbf{Ab}$ is an equivalence of categories. Through this identification, we have

$$\mathrm{H}^p_c(M, F) = R^p\pi_! F. \tag{A.297}$$

In the special case where $f\colon M \to N$ is the inclusion of an open subset, the direct image with compact support and the inverse image functors are also adjoint to each other: there is a functorial isomorphism

$$\operatorname{Hom}_{\mathbf{Sh}(M)}(F, f^{-1}G) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{Sh}(N)}(f_!F, G) \tag{A.298}$$

for each sheaf F on M and each sheaf G on N. Observe that the position of f^{-1} in this isomorphism and in (A.285) are different. Thus, f_* is a *right adjoint* of f^{-1}, while $f_!$ is a *left adjoint* of f^{-1}. Exercise A.60 implies that f^{-1} is exact. As in the case of f_*, the *adjunction morphisms* (A.11) read

$$f_!f^{-1}G \longrightarrow G, \qquad F \longrightarrow f^{-1}f_!F. \tag{A.299}$$

Remark A.300. The adjunction morphisms (A.286) and (A.299) allow us to interpret cohomology with compact support as relative cohomology. This was used in Section 2.8.7 to construct a mixed Hodge structure on cohomology with compact support. The steps for this identification are spelled out in Exercise A.326.

Formation of cohomology with compact support is contravariant for proper morphisms. Namely, let $f\colon N \to M$ be a proper map of locally compact Hausdorff topological spaces. For a sheaf F in M, there are inverse image maps

$$H^p_c(M, F) \longrightarrow H^p_c(N, f^{-1}F). \tag{A.301}$$

Indeed, using the equality $\pi_N = \pi_M \circ f$, the functoriality of the direct image with compact support implies the equality

$$(\pi_N)_! f^{-1}F = (\pi_M)_! f_! f^{-1}F,$$

where π_N and π_M are the maps to the point $*$. Since f is proper, $f_! = f_*$ holds, and the adjunction morphism

$$F \longrightarrow f_* f^{-1} F = f_! f^{-1} F$$

gives a map $(\pi_M)_! F \to (\pi_N)_! f^{-1} F$, and hence a map $R(\pi_M)_! F \to R(\pi_N)_! f^{-1} F$ that, by means of the identification (A.297), yields the map (A.301).

A.9.8. *Singular cohomology as cohomology of the constant sheaf.* Let M be a topological space. For each open subset U of M, let

$$C_*(U) = (C_*(U), d_*)$$

denote the *singular chain complex* of U, as introduced in (2.3).

Definition A.302. Given an open cover $\mathfrak{U} = (U_i)_{i \in I}$ of U, we say that a singular chain $\sum n_j \sigma_j \in C_n(U)$ is subordinated to $\mathfrak{U}$ if each simplex $\sigma_j \colon \Delta^n_{\mathrm{st}} \to U$ lands in one of the open subsets from the cover.

Singular chains subordinated to a fixed open cover $\mathfrak{U}$ form a subcomplex $C^{\mathfrak{U}}_*(U)$, and the inclusion

$$C^{\mathfrak{U}}_*(U) \hookrightarrow C_*(U)$$

induces an isomorphism in homology by barycentric subdivision (this is proved, for instance, [**Ram05**, Chap. 4, Prop. 4.12]). Let now

$$C^*(U) = \operatorname{Hom}(C_*(U), \mathbb{Z}) \text{ and } C^*_{\mathfrak{U}}(U) = \operatorname{Hom}(C^{\mathfrak{U}}_*(U), \mathbb{Z})$$

denote the complexes of singular cochains and singular cochains subordinated to $\mathfrak{U}$. The assignments $U \mapsto C^*(U)$ and $U \mapsto C^*_{\mathfrak{U}}(U)$ define complexes C^* and $C^*_{\mathfrak{U}}$ of presheaves of abelian groups on M such that the natural map

$$C^* \longrightarrow C^*_{\mathfrak{U}}$$

is a quasi-isomorphism. The presheaves C^n are flasque, which means that the restriction maps $C^n(U) \to C^n(V)$ are surjective: a preimage of $\varphi \colon C_n(V) \to \mathbb{Z}$ is the cochain $C_n(U) \to \mathbb{Z}$ that sends a simplex $\sigma \colon \Delta^n_{\mathrm{st}} \to U$ to $\varphi(\sigma)$ if σ lands in V and to zero otherwise. The presheaves C^n are *not* sheaves but they satisfy the gluing condition iv) from Definition A.246. Let $\tilde{C}^n$ denote the associated sheaves. If one further assumes that all open subsets U of M are paracompact, then the natural maps $C^n(U) \to \tilde{C}^n(U)$ are surjective [**Ram05**, Chap. 1, Prop. 1.4]. It follows that the sheaves $\tilde{C}^n$ are also flasque. Indeed, given open subsets $U \subset V \subset M$, in the commutative diagram

$$\begin{tikzcd} C^n(U) \arrow[r] \arrow[d] & C^n(V) \arrow[d] \\ \tilde{C}^n(U) \arrow[r] & \tilde{C}^n(V) \end{tikzcd}$$

we already know that the upper horizontal map and the vertical maps are surjective, and hence so is the lower horizontal map.

Let now U be an open subset and $\mathfrak{U}$ an open cover of M. The kernel of the map $C^*(U) \to C^*_{\mathfrak{U}}(U)$ consists of the sections that vanish when restricted to $V \cap U$ for all $V \in \mathfrak{U}$. Therefore, any such section is mapped to zero in $\tilde{C}^*(U)$ by the sheaf condition. Thus, the map $C^*(U) \to \tilde{C}^*(U)$ factors through a map $C^*_{\mathfrak{U}}(U) \to \tilde{C}^*(U)$. Varying the cover $\mathfrak{U}$, we obtain a map

$$\varinjlim_{\mathfrak{U}} C^*_{\mathfrak{U}}(U) \longrightarrow \tilde{C}^*(U). \tag{A.303}$$

The surjectivity of (A.303) follows from that of the map $C^*(U) \to \tilde{C}^*(U)$. Moreover, if a section $s \in C^*(U)$ is mapped to zero in $\tilde{C}^*(U)$, then it is mapped to zero in all the stalks at points of U. Hence, there is an open cover $\mathfrak{U}$ of U such that s is mapped to zero in $C^*_{\mathfrak{U}}(U)$. We conclude that (A.303) is also injective.

From the exactness of direct limits (Proposition A.165), we deduce

$$\mathrm{H}^i(\tilde{C}^*(U)) = \mathrm{H}^i\left(\varinjlim_{\mathfrak{U}} C^*_{\mathfrak{U}}(U)\right) = \varinjlim_{\mathfrak{U}} \mathrm{H}^i(C^*_{\mathfrak{U}}(U)) = \mathrm{H}^i(C^*(U)).$$

In other words, the complexes $\tilde{C}^*(U)$ and $C^*(U)$ are quasi-isomorphic.

Let $\mathbb{Z}^\circ \to C^0$ be the morphism of presheaves that sends $1 \in \mathbb{Z}^\circ(U) = \mathbb{Z}$ to the singular cochain $(\sum n_x[x] \mapsto \sum n_x)$ and let

$$\underline{\mathbb{Z}} \longrightarrow \tilde{C}^0$$

be the associated morphism of sheaves.

THEOREM A.304. *Let M be a locally contractible topological space in which all open subsets are paracompact (e.g. the underlying topological space of a complex manifold). The map $\underline{\mathbb{Z}} \to \tilde{C}^0$ induces an isomorphism*

$$\mathrm{H}^*(M, \underline{\mathbb{Z}}) \xrightarrow{\sim} \mathrm{H}^*(M, \mathbb{Z})$$

between sheaf and singular cohomology of M.

PROOF. The sheaves $\tilde{C}^n$ are flasque, and hence acyclic by Lemma A.259. Besides, the sequence of sheaves

$$\underline{\mathbb{Z}} \longrightarrow \tilde{C}^0 \longrightarrow \tilde{C}^1 \longrightarrow \cdots \tag{A.305}$$

is exact. Indeed, each point $x \in M$ has a contractible open neighborhood U_x, and it suffices to check exactness after taking sections on such a U_x. Since the complex $\tilde{C}^*(U_x)$ is quasi-isomorphic to $C^*(U_x)$, exactness follows from the computation $\mathrm{H}^0(U_x, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{H}^i(U_x, \mathbb{Z}) = 0$ for all $i > 0$, which is an instance of homotopy invariance of singular cohomology. Hence, (A.305) is an acyclic resolution, and $\mathrm{H}^*(M, \underline{\mathbb{Z}})$ is isomorphic to the cohomology of the complex $\tilde{C}^*(M)$. We then conclude using again that $\tilde{C}^*(M)$ is quasi-isomorphic to $C^*(M)$. □

REMARK A.306. The preprint [**Sel16**] contains a proof of this comparison result without the paracompactness assumption. A different argument taking advantage of the formalism of ∞-categories appears in [**Pet22**].

REMARK A.307. Let M be a differentiable manifold. As was mentioned in Remark 2.12 ii), singular homology and cohomology can also be computed using smooth chains and cochains. A simplex $\sigma\colon \Delta^n_{\mathrm{st}} \to M$ is called *smooth* if there exists an open neighbourhood V of $\Delta^n_{\mathrm{st}} \subset \mathbb{R}^{n+1}$ and a $\mathcal{C}^\infty$ map $\bar{\sigma}\colon V \to M$ with $\sigma = \bar{\sigma}|_{\Delta^n_{\mathrm{st}}}$. A chain $\sum_j n_j \sigma_j$ is called smooth if all the simplices σ_j are smooth. If $U \subset M$ is an open subset, we denote by $C^{\mathrm{sm}}_n(U) \subset C_n(U)$ the subspace of smooth chains and by

$$S^n(U) = \mathrm{Hom}(C^{\mathrm{sm}}_n(U), \mathbb{Z})$$

the space of smooth singular cochains. The boundary maps ∂ send smooth singular chains to smooth singular chains, making $C^{\mathrm{sm}}_*(U)$ into a chain complex and defining a differential in $S^*(U)$. The maps

$$C^{\mathrm{sm}}_*(U) \longrightarrow C_*(U) \text{ and } C^*(U) \longrightarrow S^*(U)$$

are quasi-isomorphisms, as proved for instance in [**War83**, 5.32] or [**Lee13**, Thm. 18.7]. Therefore, everything we have done in this section can be repeated for smooth singular cochains. In particular, we obtain a complex of flasque sheaves $\tilde{S}^*$ and, by Theorem A.304, a quasi-isomorphism

$$\underline{\mathbb{Z}} \longrightarrow \tilde{S}^*,$$

that gives an isomorphism between the cohomology of the constant sheaf and smooth singular cohomology. In fact, for any any ring Λ, writing

$$S^*(M, \Lambda) = \mathrm{Hom}(C_n^{\mathrm{sm}}(M), \Lambda),$$

there are isomorphisms

$$\mathrm{H}^*(M, \underline{\Lambda}) \xrightarrow{\sim} \mathrm{H}^*(S^*(M, \Lambda)).$$

We will apply this result mainly when Λ is $\mathbb{Q}$ or $\mathbb{R}$.

A.9.9. *Coherent sheaves.* In this section, we briefly summarize the notion of coherent sheaf in algebraic and analytic geometry and some vanishing theorems for higher cohomology groups of coherent sheaves due to Cartan and Serre. Recall that an algebraic variety X comes equipped with a sheaf of rings $\mathcal{O}_X$, the sheaf of regular functions. Similarly, a complex manifold X is endowed with a sheaf of rings, the sheaf of holomorphic functions that is denoted in the same way $\mathcal{O}_X$. In both cases, $\mathcal{O}_X$ is called the *structure sheaf*. This abuse of notation is intentional because we can state similar results in algebraic geometry and in complex geometry with the same words, so from now on X will be either an algebraic variety or a complex manifold. Once we have the structural sheaf $\mathcal{O}_X$, we can talk about $\mathcal{O}_X$-modules.

Definition A.308. Let X be either an algebraic variety or a complex manifold. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf of abelian groups F on the underlying topological space such that, for every open set U of X, the abelian group $F(U)$ carries the structure of an $\mathcal{O}_X(U)$-module. Moreover, the restriction maps are compatible with the $\mathcal{O}_X$-module structures: for each inclusion $V \subset U$, and sections $f \in \mathcal{O}_X(U)$ and $s \in F(U)$, the relation

$$(f \cdot s)|_V = f|_V \cdot s|_V$$

holds. The basic examples of $\mathcal{O}_X$-modules are the sheaf $\mathcal{O}_X$ itself and the direct sums $\mathcal{O}_X^{\oplus I}$, for I a set. A sheaf of $\mathcal{O}_X$-modules F is called *quasi-coherent* if every point has a neighborhood U such that the restriction $F|_U$ sits in an exact sequence

$$\mathcal{O}_U^{\oplus J} \longrightarrow \mathcal{O}_U^{\oplus I} \longrightarrow F|_U \longrightarrow 0,$$

with I and J arbitrary sets. The sheaf F is called *coherent* if the sets I and J can be chosen to be finite.

Remark A.309. The definition of coherent sheaves in algebraic and complex geometry and of quasi-coherent sheaves in algebraic geometry is the standard one. It is subtle question to find a good notion of quasi-coherent sheaf in complex geometry.

Example A.310. A sheaf F is *locally free* if each point has a neighbourhood U such that $F|_U$ is isomorphic to $\mathcal{O}_U^{\oplus I}$ for some set I. It follows directly from the definitions that every locally free sheaf is quasi-coherent. A locally free sheaf has *finite rank* if the sets I are finite. The *rank* of a locally free sheaf of finite rank is the locally constant function that sends a point x as before to the number $|I|$. Again, it follows from the definition that a locally free sheaf of finite rank is coherent.

In algebraic geometry, a scheme is called *affine* if it is of the form $\operatorname{Spec}(A)$ for some ring A. If X is an algebraic variety over a field k, being affine is equivalent to being a closed subvariety of an affine space $\mathbb{A}_k^N$ over k. Similarly, in complex geometry there is the analogous notion of *Stein manifold* (or more generally Stein space). See [**GR65**] for more details about Stein spaces. The embedding theorem for Stein manifolds implies that a complex manifold is Stein if and only if it is a closed holomorphic submanifold of an affine space $\mathbb{C}^n$. The next result is due to Cartan in the case of coherent sheaves over a Stein manifold, and to Serre for quasi-coherent sheaves on a noetherian scheme. For a proof in the algebraic case, see [**Har77**, Thm. III 3.7]. For a counterexample to the validity of this theorem for quasi-coherent sheaves in complex manifolds, see [**Rei**].

THEOREM A.311. *Let X be a noetherian affine scheme (resp. a Stein complex manifold) and let F be a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_X$-modules. Then*

$$\mathrm{H}^i(X, F) = 0 \textit{ in all degrees } i \geqslant 1.$$

The previous theorem has also a relative version, that we now explain. A morphism of schemes $f\colon X \to Y$ is called *affine* if there is a covering $Y = \bigcup_i U_i$ of Y by affine open subsets such that $f^{-1}(U_i)$ is affine for each i. Equivalently, f is affine if the inverse image of any affine open subset of Y is affine.

EXAMPLE A.312. Let X be a scheme, and let D be an effective Cartier divisor on X, *e.g.* any effective divisor on a smooth variety. Let $|D|$ denote the support of D. Then the inclusion $j\colon X \setminus |D| \to X$ is affine. Indeed, since D is an effective Cartier divisor, there exists a covering of X by open affine subsets $U_i = \operatorname{Spec}(A_i)$ and elements $f_i \in A_i$ such that $D|_{U_i}$ is defined by the equation $f_i = 0$. Then

$$j^{-1}(U_i) = \operatorname{Spec}(A_i[f_i^{-1}])$$

is affine. Note that for this property to hold it is crucial that D is a divisor. For instance, the inclusion $\mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{A}^2$ is not affine. Note also that $X \setminus |D|$ need not be affine, as the example where X is the blow-up of $\mathbb{A}^2$ at the origin and D the exceptional divisor shows.

Similarly, a morphism of complex manifolds is called *Stein* if the inverse image of every open Stein subset of the target is Stein.

THEOREM A.313. *Let $f\colon X \to Y$ be an affine morphism of noetherian schemes (resp. a Stein morphism of complex manifolds) and let F be a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_X$-modules. Then*

$$R^i f_* F = 0 \textit{ for all } i \geqslant 1.$$

In other words, quasi-coherent (resp. coherent) sheaves are acyclic for the direct image functor f_*.

PROOF. By Proposition A.277, the stalk of the higher direct image sheaf $R^i f_* F$ at a closed point y of Y is given by

$$(R^i f_* F)_y = \varinjlim_{y \in U} \mathrm{H}^i(f^{-1}(U), F).$$

As one can restrict to affine (resp. Stein) neighborhoods of y to compute the limit, the right-hand vanishes for all $i \geqslant 1$ by Theorem A.311. $\square$

A.9.10. *Local systems as representations of the fundamental group.* In this section, we discuss a class of sheaves on a topological space called *local systems* and we prove that, under mild assumptions, the category they form is equivalent to the category of representations of the fundamental group of the topological space.

Definition A.314. A sheaf F of vector spaces on a topological space M is said to be *locally constant* if there exists an open cover $\{U_i\}_{i\in I}$ of M such that all the restrictions $F|_{U_i}$ are constant sheaves as in Examples A.251 i). A locally constant sheaf is also called a *local system*.

Theorem A.315. *Let M be a Hausdorff, second countable, connected, locally compact and locally contractible topological space. Let $x_0 \in M$ be a point and V a vector space. There is an equivalence of categories*

$$\left\{\begin{array}{c}\textit{local systems } F \textit{ on } M\\ \textit{with fiber } F_{x_0}=V\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\textit{representations of}\\ \pi_1(X,x_0) \textit{ on } V\end{array}\right\}.$$

Proof. We first show how to construct a local system out of a representation of the fundamental group. For every point $x \in M$, we choose once and for all a path $\alpha_x \in {}_x\mathcal{P}(M)_{x_0}$. Thus, $\alpha_x\colon [0,1] \to M$ is a piecewise smooth map with endpoints $\alpha_x(0) = x_0$ and $\alpha_x(1) = x$. Let $\rho\colon \pi_1(M, x_0) \to \mathrm{GL}(V)$ be a representation. We define a sheaf F by describing its sections. For every open set $U \subset M$, let $F(U)$ be the vector space of all functions $f\colon U \to V$ satisfying the following property: for all points $x, y \in U$ and all paths $\gamma \in {}_y\mathcal{P}(U)_x$, the relation

$$\rho([\alpha_y^{-1}\cdot\gamma\cdot\alpha_x])\cdot f(x) = f(y)$$

holds. The fact that F is a sheaf is left as Exercise A.332. We now show that F is locally constant. Since M is assumed to be locally contractible, we can cover M by contractible open subsets. Let U be one of these subsets. We show that $F(U) \simeq V$. Choose a point $x \in U$, and let $\varphi_x\colon F(U) \to V$ be the map $f \mapsto f(x)$, which is injective. Indeed, since U is contractible, it is in particular connected, and hence for each y there is a path $\gamma \in {}_y\mathcal{P}(U)_x$. If $\varphi_x(f) = f(x) = 0$, then

$$f(y) = \rho([\alpha_y^{-1}\cdot\gamma\cdot\alpha_x])\cdot f(x) = 0,$$

and hence $f = 0$. We now show that φ_x is surjective. Given $v \in V$, choose for every point y a path $\gamma_y \in {}_y\mathcal{P}(U)_x$, and define f_v by the rule

$$f_v(y) = \rho([\alpha_y^{-1}\cdot\gamma_y\cdot\alpha_x])\cdot v.$$

The resulting function is independent of the choice of paths γ_y. Indeed, since U is contractible, it is in particular simply connected. Therefore, if γ'_y is another choice of path, then $[\gamma_y] = [\gamma'_y] \in \pi(M; y, x)$. Therefore,

$$\rho([\alpha_y^{-1}\cdot\gamma'_y\cdot\alpha_x]) = \rho([\alpha_y^{-1}\cdot\gamma_y\cdot\alpha_x]).$$

Then f_v is easily seen to be a section of $F(U)$. Moreover, $f_v(x) = v$ holds, showing that φ_x is surjective, and hence an isomorphism. The same argument shows that for each connected open subset $U' \subset U$ the restriction $F(U) \to F(U')$ is an isomorphism, showing that $F|_U$ is isomorphic to the constant sheaf.

The next step is to produce a representation of $\pi_1(M, x_0)$ starting with a locally constant sheaf F with fiber $F_{x_0} = V$. During this proof, an open subset $U \subset M$ will be called good (for this sheaf) if U is connected and $F|_U$ is isomorphic to the constant sheaf $\underline{V}$ on U. Since F is locally constant, M can be covered by good open subsets. Let $x, y \in M$, and let $\gamma \in {}_y\mathcal{P}(X)_x$ be a path between two points. Using that $[0,1]$ is compact, we can choose a finite set of points

$$0 = t_0 < t_1 < \dots < t_k < t_{k+1} = 1$$

and good open sets U_j, for $j = 0, \dots, k$, such that $\gamma([t_j, t_{j+1}])$ is contained in U_j. For each $j = 0, \dots, k$, there is an isomorphism

$$\rho_j \colon F_{\gamma(t_j)} \longrightarrow F_{\gamma(t_{j+1})}$$

given as the composition of the maps to the stalks

$$F_{\gamma(t_j)} \xleftarrow{\sim} F(U_j) \xrightarrow{\sim} F_{\gamma(t_{j+1})}.$$

We denote by $\rho_\gamma \colon F_x \to F_y$ the composition

$$\rho_\gamma = \rho_k \circ \dots \circ \rho_0.$$

The fact that ρ_γ is independent of the choices follows from two properties:

i) the isomorphism ρ_j does not depend on the choice of the good open set U' containing $\gamma([t_j, t_{j+1}])$;

ii) the composition ρ_γ does not change if we add t'_j satisfying $t_j < t'_j < t_{j+1}$.

The next step is to show that the isomorphism ρ_γ only depends on the homotopy class $[\gamma] \in \pi_1(M; y, x)$. Let γ and γ' be two paths and H a homotopy between them. We can find points

$$0 = t_0 < t_1 < \dots < t_k < t_{k+1} = 1, \quad 0 = s_0 < s_1 < \dots < s_\ell < s_{\ell+1} = 1$$

and good open sets $U_{i,j}$ such that $H([t_i, t_{i+1}] \times [s_j, s_{j+1}]) \subset U_{i,j}$. Then one checks that the square

$$\begin{tikzcd}
F_{H(t_i,s_j)} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & F_{H(t_i,s_{j+1})} \arrow[d, "\simeq"] \\
F_{H(t_{i+1},s_j)} \arrow[r, "\simeq"'] & F_{H(t_{i+1},s_{j+1})}
\end{tikzcd}$$

is commutative. From this, the equality $\rho_{\gamma'} = \rho_\gamma$ follows.

As a consequence of this construction, there is a representation

$$\rho \colon \pi_1(X, x_0) \longrightarrow \mathrm{GL}(V)$$

given by $[\gamma] \mapsto \rho_\gamma$. The map ρ_γ is called the *parallel transport* along γ, and the map γ is called the *monodromy representation.*

We leave it to the reader to check that the constructions we have described are inverses of each other and functorial. □

REMARK A.316. It is clear from the proof of Theorem A.315 that the hypotheses on M are much more restrictive than needed, but they are satisfied in the examples we are interested in. For instance, instead of asking the topological space to be locally contractible it is enough to assume that it is locally path-connected and semilocally simply-connected. The second condition means that every point has a neighborhood such that every loop in the neighborhood is null-homotopic in the whole space.

The sheaf cohomology of a local system is related to the group cohomology of the fundamental group with coefficients in the fiber. More precisely, let M, x_0, and V be as in Theorem A.315. Write $\Gamma = \pi_1(M, x_0)$ for the fundamental group of M based at x_0, and let F be a local system on M with fiber $F_{x_0} = V$. This turns V into a Γ-module. We refer the reader to [**Hat02**, Ch. 4] for the definition and properties of the higher homotopy groups appearing in the next result.

Theorem A.317. *Assume that there exists $r \geqslant 1$ such that the higher homotopy groups $\pi_i(X, x_0)$ vanish for $i = 2, \ldots, r$. Then there are canonical isomorphisms*

$$\mathrm{H}^i(M, F) \cong \mathrm{H}^i(\Gamma, V) \quad \text{for } i = 0, \ldots, r, \tag{A.318}$$

where the left-hand side is sheaf cohomology, and the right-hand side is group cohomology. There is also an injective map

$$\mathrm{H}^{r+1}(\Gamma, V) \longrightarrow \mathrm{H}^{r+1}(M, F).$$

Proof. We sketch of two proofs, based on the theory of Postnikov towers and on the universal covering. The key point of both is to construct a spectral sequence.

First proof. By the theory of Postnikov towers [**Hat02**, §4.3], one can construct a diagram of topological spaces

$$\begin{tikzcd} & M_\infty \arrow[d, "\pi"] \\ M \arrow[ur, "\varphi_\infty"] \arrow[r, "\varphi_1"'] & M_1 \end{tikzcd}$$

satisfying the following properties:

i) The map φ_∞ is a weak homotopy equivalence, so that M and M_∞ have the same homotopy groups.

ii) The map π is a fibration, *i.e.* there exists an open covering $\{U_\alpha\}$ of M_1 and isomorphisms $\pi^{-1}(U_\alpha) \simeq U_\alpha \times N$ for a certain space N. Moreover the space N is connected and $\pi_i(N, y_0) = 0$ holds for all points $y_0 \in N$ and all $i = 1, \ldots, r$.

iii) The space M_1 is a $K(\Gamma, 1)$ space, *i.e.* M_1 is connected, $\pi_1(M_1, y_0) = \Gamma$, and all the higher homotopy groups of M_1 vanish.

These properties imply that the universal covering space $\widetilde{M_1}$ of M_1 is weakly contractible (*i.e.* all homotopy groups vanish) and that the map φ_1 induces an isomorphism of fundamental groups.

Let F_∞ and F_1 be the local systems on M_∞ and M_1 induced by the same representation of Γ as F, that is $F_\infty = \pi^* F_1$ and $F = \varphi_1^* F_1 = \varphi_\infty^* F_\infty$. Then:

i) $\mathrm{H}^i(M, F) = \mathrm{H}^i(M_\infty, F_\infty)$, for all $i \geqslant 0$ because M and M_∞ are homotopy equivalent.

ii) $\mathrm{H}^i(\Gamma, V) = \mathrm{H}^i(M_1, F_1)$, for all $i \geqslant 0$ because M_1 is a $K(\Gamma, 1)$ space.

iii) The Leray spectral sequence gives us a convergent spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(M_1, R^q\pi_* F_\infty) \Longrightarrow \mathrm{H}^{p+q}(M_\infty, F_\infty) = \mathrm{H}^{p+q}(M, F). \tag{A.319}$$

iv) Since N is connected and satisfies $\pi_i(N, y_0) = 0$ for $i = 1, \ldots, r$, the Hurewicz theorem (see [**Hat02**, Thm. 4.32]) implies the vanishing

$$\mathrm{H}_i(N, \mathbb{Z}) = 0 \quad \text{for } i = 1, \ldots, r.$$

Since F_∞ is constant along the fibers, Proposition A.277 along with the universal coefficients theorem, in the form of exact sequence (2.17), imply

$$R^0\pi_*F_\infty = F_1, \qquad R^i\pi_*F_\infty = 0, i = 1, \dots, r.$$

Combining these facts, we see that the zeroth row of the spectral sequence (A.319) is given by $\mathrm{H}^i(\Gamma, V)$, while the rows from 1 to r are zero, hence isomorphisms

$$\mathrm{H}^p(\Gamma, V) = E_2^{p,0} = E_\infty^{p,0} = \mathrm{H}^p(M, F) \quad \text{for } p = 0, \dots, r$$

and an exact sequence

$$0 \to \mathrm{H}^{r+1}(\Gamma, V) \longrightarrow \mathrm{H}^{r+1}(M, F) \longrightarrow E_2^{0,r+1} \xrightarrow{\mathrm{d}_{r+2}} E_2^{r+2,0}.$$

Second proof. Let $\widetilde{M}$ be the universal covering space of M and $\pi\colon \widetilde{M} \to M$ the covering map. Since the space $\widetilde{M}$ is connected and satisfies $\pi_i(\widetilde{M}, y_0) = 0$ for all $i = 1, \dots, r$, from the Hurewicz theorem we get $\mathrm{H}_i(\widetilde{M}, \mathbb{Z}) = 0$ for all $i = 1, \dots, r$. The group Γ acts freely on $\widetilde{M}$, and M is homeomorphic to M/Γ as a topological space. An *equivariant sheaf* on $\widetilde{M}$ is a sheaf F on $\widetilde{M}$ together with isomorphisms

$$\rho_\gamma\colon \gamma^*F \longrightarrow F$$

for all $\gamma \in \Gamma$ satisfying the cocycle condition

$$\rho_{\gamma'\gamma} = \rho_\gamma \circ \gamma^*(\rho_{\gamma'})$$

for all $\gamma, \gamma' \in \Gamma$. Let $\mathbf{Sh}_\Gamma(\widetilde{M})$ denote the category of Γ-equivariant sheaves on $\widetilde{M}$. The inverse image functor gives us an equivalence of categories $\mathbf{Sh}(M) \to \mathbf{Sh}_\Gamma(\widetilde{M})$.

Let Γ-**Mod** denote the category of Γ-modules, and V^Γ the abelian group of invariant elements of a Γ-module V. There are functors

$$\mathbf{Sh}_\Gamma(\widetilde{M}) \xrightarrow{\mathcal{G}} \Gamma\text{-}\mathbf{Mod} \xrightarrow{\mathcal{F}} \mathbf{Ab}$$

given by $\mathcal{G}(F) = \mathrm{H}^0(\widetilde{M}, F)$ and $\mathcal{F}(V) = V^\Gamma$. They satisfy:

$$R^q\mathcal{G}(F) = \mathrm{H}^q(\widetilde{M}, F), \quad R^p\mathcal{F}(V) = \mathrm{H}^p(\Gamma, V), \quad R^n(\mathcal{F} \circ \mathcal{G})(\pi^*F) = \mathrm{H}^n(M, F).$$

It is proved in [**Gro57**, p, 198] that the functor $\mathcal{G}$ sends injective equivariant sheaves to injective Γ-modules, which are in particular $\mathcal{F}$-acyclic. We can then apply Theorem A.213 to get a spectral sequence of composition of functors

$$E_2^{p,q} = \mathrm{H}^p(\Gamma, \mathrm{H}^q(\widetilde{M}, \pi^*F)) \Longrightarrow \mathrm{H}^{p+q}(M, F).$$

Since $\widetilde{M}$ is simply connected, the sheaf π^*F is the locally constant sheaf $\underline{V}$. Using the vanishing of the homology of $\widetilde{M}$ and the universal coefficients theorem, we get

$$\mathrm{H}^q(\widetilde{M}, \pi^*F) = \begin{cases} V, & \text{if } q = 0, \\ 0, & \text{if } q = 1, \dots, r, \\ ?, & \text{for } q \geqslant r+1. \end{cases}$$

The spectral sequence has $E_2^{p,0} = \mathrm{H}^p(\Gamma, V)$ and zero on the rows 1 to r, so as before we get isomorphisms

$$\mathrm{H}^p(\Gamma, V) = E_2^{p,0} = E_\infty^{p,0} = \mathrm{H}^p(M, F) \quad \text{for } p = 0, \dots, r$$

and an exact sequence

$$0 \to \mathrm{H}^{r+1}(\Gamma, V) \longrightarrow \mathrm{H}^{r+1}(M, F) \longrightarrow \mathrm{H}^{r+1}(\widetilde{M}, \pi^*F)^\Gamma \xrightarrow{\mathrm{d}_{r+2}} \mathrm{H}^{r+2}(\Gamma, V).$$

This concludes the proof. □

Since the condition of Theorem A.317 always holds for $r = 1$, we get:

Corollary A.320. *Let M be a Hausdorff, second countable, connected, locally compact and locally contractible topological space. For each local system F on M, there are canonical isomorphisms*

$$\mathrm{H}^i(M, F) \cong \mathrm{H}^i(\Gamma, V) \quad \textit{for } i = 0 \textit{ and } i = 1,$$

as well as an injection $\mathrm{H}^2(\Gamma, V) \hookrightarrow \mathrm{H}^2(M, F)$.

⋆ ⋆ ⋆

Exercise A.321. Prove that a sheaf F satisfies $F(\emptyset) = 0$, while this does not necessarily hold for presheaves.

Exercise A.322. Let M be a topological space and let $\mathfrak{B}$ be a *basis* of its topology, that is, a collection of open subsets that cover M and such that, for any two basis elements $U_1, U_2 \in \mathfrak{B}$ and any point $x \in U_1 \cap U_2$, there exists a basis element $U_3 \in \mathfrak{B}$ satisfying $x \in U_3 \subset U_1 \cap U_2$. Show that a sheaf F on M is uniquely determined by an assignment $U \mapsto F(U)$, for each $U \in \mathfrak{B}$, such that the four conditions in Definition A.246 hold.

Exercise A.323. Let M be a topological space and let $\iota\colon N \hookrightarrow M$ be the inclusion of a closed subspace. Show that, given a sheaf F on N, the direct image sheaf $\iota_* F$ from Definition A.274 is the extension by zero of F, that is, the sheaf on N with stalks F_x if $x \in N$ and 0 otherwise. Then prove that ι_* is an exact functor, so that the higher direct image sheaves vanish.

Exercise A.324. Let M be a locally compact Hausdorff space and $j\colon U \to M$ the inclusion of an open subset. Prove that the functor $j_!$ is exact.

Exercise A.325. Let N be the topological space consisting of two points with the discrete topology and $j\colon M \to N$ the inclusion of the open subset M consisting of a single point. Let $\underline{\mathbb{Z}}$ be the constant sheaf on M. Show that $\Gamma(N, j_! \underline{\mathbb{Z}}) = \mathbb{Z}$. Conclude that the presheaf (A.293) is not a sheaf.

Exercise A.326. Let M be a compact Hausdorff topological space, $j\colon U \to M$ the inclusion of an open subset, and $i\colon Z \to M$ the inclusion of the complementary closed subset. Let F be a sheaf on X.

i) Prove that there is an exact sequence
$$0 \longrightarrow j_! j^{-1} F \longrightarrow F \longrightarrow i_* i^{-1} F \longrightarrow 0.$$
ii) Prove that, since M is compact, there is a canonical isomorphism
$$\mathrm{H}^*(M, j_! j^{-1} F) \cong \mathrm{H}^*_c(U, j^{-1} F).$$
iii) Let $*$ be the topological space reduced to a single point, and let π_M, π_U, and π_Z be the unique maps from M, U, and Z to $*$. Using Exercises A.323 and A.324, prove that there is a canonical isomorphism
$$R(\pi_U)_! j^{-1} F \simeq \mathrm{Tot}\big(R(\pi_M)_* F \to R(\pi_Z)_* i^{-1} F\big).$$

iv) Assume that M and Z satisfy the hypothesis of Theorem A.304. Specialize the previous result to $F = \underline{\mathbb{Z}}$ to obtain a canonical isomorphism
$$\mathrm{H}_c^*(U, \underline{\mathbb{Z}}) \simeq \mathrm{H}^*(M, Z; \mathbb{Z}).$$
Conclude that there is a long exact sequence
$$\cdots \longrightarrow \mathrm{H}_c^n(U, \underline{\mathbb{Z}}) \longrightarrow \mathrm{H}^n(M, \mathbb{Z}) \longrightarrow \mathrm{H}^n(Z, \mathbb{Z}) \longrightarrow \cdots$$

v) Generalize the previous result to non-compact M. Namely, if M is a topological Hausdorff space, then there is a long exact sequence
$$\cdots \longrightarrow \mathrm{H}_c^n(U, \underline{\mathbb{Z}}) \longrightarrow \mathrm{H}_c^n(M, \underline{\mathbb{Z}}) \longrightarrow \mathrm{H}_c^n(Z, \underline{\mathbb{Z}}) \longrightarrow \cdots$$

Exercise A.327. In this exercise, we show that every injective sheaf is flasque. Let M be a topological space, F a sheaf on M, and I an injective sheaf. Consider open subsets $U \subset V \subset M$, and let $i\colon U \to M$ and $j\colon V \to M$ denote the inclusions.

i) Show that $F(U) = \mathrm{Hom}_{\mathbf{Sh}(M)}(i_*\underline{\mathbb{Z}}, F)$.
ii) Prove that there is a canonical monomorphism $i_*\underline{\mathbb{Z}} \to j_*\underline{\mathbb{Z}}$.
iii) Use the previous results and the definition of injective sheaf to prove that the restriction map $I(V) \to I(U)$ is surjective.

Exercise A.328. Let M be a topological space and let A^* be a bounded below complex of sheaves of abelian groups. Consider the Godement resolution Gd from Section A.9.3. Show that there is a natural isomorphism of complexes
$$\mathrm{Gd}(A[k]^*) \longrightarrow \mathrm{Gd}(A^*)[k].$$

Exercise A.329. Let F^* be a complex of sheaves. Considering the increasing canonical filtration of F^* and applying Godement's resolution to it, we obtain the filtered complex $(\mathrm{Gd}(F^*), \mathrm{Gd}(\tau_{\leqslant}))$. Alternatively, we can consider directly the canonical filtration of the total complex $(\mathrm{Tot}^*(\mathrm{Gd}^*(F^*)), \tau_{\leqslant})$. In general, both filtrations are different. Prove that the identity map gives a filtered quasi-isomorphism
$$(\mathrm{Tot}^*(\mathrm{Gd}^*(F^*)), \tau_{\leqslant}) \longrightarrow (\mathrm{Gd}(F^*), \mathrm{Gd}(\tau_{\leqslant})).$$

Exercise A.330. Let $M = \mathbb{C}^2$ be the complex affine plane, and let $j\colon U \hookrightarrow M$ be the inclusion of the complement of a point. Compute the higher direct image sheaves $R^i j_*\underline{\mathbb{Z}}$.

Exercise A.331. Prove Lemma A.279.

Exercise A.332. Let M be a connected locally contractible topological space and let ρ be a representation of its fundamental group. Show that the presheaf F constructed in the proof of Theorem A.315 is a sheaf.

Exercise A.333. Let F be a fine sheaf on a paracompact topological space M. Let $\mathrm{Gd}^*(F)$ be Godement's canonical resolution of F. Prove the following:

i) Let $\{U_i\}$ be a locally finite covering of M and h_i a partition of unity subordinated to this covering. Prove that the endomorphisms h_i induce endomorphisms, also denoted by h_i, of the complex $\mathrm{Gd}^*(F)$, and the support of these endomorphisms is still contained in U_i.
ii) Use i) to prove the vanishing $\mathrm{H}^p(M, F) = 0$ for all $p > 0$. [Hint: given a global section $s \in \Gamma(M, \mathrm{Gd}^p(M))$ with $\mathrm{d}s = 0$, use a partition of unity to glue together local primitives into a global primitive of s.]

Exercise A.334. Let $X = \mathbb{C}^\times$ viewed as a Stein complex manifold and consider the constant sheaf $\underline{\mathbb{C}}_X$. Use Theorem A.311 to show that this sheaf does not admit any structure of coherent $\mathcal{O}_X$-module.

A.10. Lie algebra homology and cohomology. In this final section, we gather some properties of Lie algebra homology and cohomology following Weibel's book [**Wei94**, Chap. VII]. Throughout, k denotes a field of characteristic zero.

A.10.1. *Lie algebras and Lie modules.*

Definition A.335. A *Lie algebra* over k is a k-vector space L together with a bilinear map, called the Lie bracket,

$$[\cdot,\cdot]\colon L \times L \to L$$

such that the following equalities hold for all $x, y, z \in L$:

$$[x,y] + [y,x] = 0 \qquad \text{(antisymmetry)}, \tag{A.336}$$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \qquad \text{(Jacobi identity)}. \tag{A.337}$$

Definition A.338. Let L be a Lie algebra over k. A *left L-module* is the data of a k-vector space M and a bilinear map

$$\begin{aligned} L \times M &\longrightarrow M \\ (z,m) &\longmapsto zm \end{aligned}$$

such that the equality

$$[x,y]m = x(ym) - y(xm)$$

holds for all $x, y \in L$ and all $m \in M$. A *morphism of left L-modules* is a k-linear map $f\colon M \to N$ that satisfies $f(xm) = xf(m)$ for all $x \in L$ and all $m \in M$.

The notion of *right L-module* is defined with the obvious modifications.

Note that a left L-module gives rise to a Lie algebra representation

$$\rho\colon L \longrightarrow \mathrm{End}(M)$$

in the sense of Definition 3.126, and conversely every Lie algebra representation defines a left L-module structure on the space of the representation. The notions of Lie algebra representation and left L-module are thus equivalent.

Example A.339. Let L be a Lie algebra and M a k-vector space. The *trivial Lie module structure* on M is defined by the rule

$$xm = 0 \qquad \text{for all } x \in L \text{ and } \ m \in M.$$

This gives rise to a functor from the category of k-vector spaces to the category of left L-modules, which will be called the *trivial L-module functor*.

Recall from Section 3.2.11 that every Lie algebra can be embedded into an associative algebra in such a way that the Lie bracket is given by the commutator.

Definition A.340. Let L be a Lie algebra over k. The *universal enveloping algebra* is an associative k-algebra $U(L)$ along with a morphism of Lie algebras

$$\iota_L\colon L \longrightarrow U(L),$$

that is, a k-linear map satisfying

$$\iota([x,y]) = \iota(x)\iota(y) - \iota(y)\iota(x)$$

for all $x, y \in L$, and that is universal for associative algebras with this property.

It is proved in loc. cit. that

- the universal enveloping algebra always exists (see formula (3.96));
- the map ι_L is injective (Corollary 3.100);
- the category of left L-modules is equivalent to that of left $U(L)$-modules (Proposition 3.128).

A.10.2. *Homology and cohomology.* Given a Lie algebra L, there are two functors from the category of left L-modules to the category of k-vector spaces that play a role in the definition of Lie algebra homology and cohomology.

Definition A.341. Let L be a Lie algebra and M a left L-module.

i) The subspace of *invariant* elements is defined as

$$M^L = \{m \in M \mid xm = 0 \text{ for all } x \in L\}.$$

The assignment $M \mapsto M^L$ defines a functor from the category of left L-modules to the category of k-vector spaces. Moreover, if k is given the trivial left L-module structure, then the invariants are equal to

$$M^L = \operatorname{Hom}_{U(L)}(k, M). \tag{A.342}$$

ii) The quotient of *coinvariants* is defined as

$$M_L = M/LM = k \otimes_{U(L)} M, \tag{A.343}$$

where k is again given the trivial left L-module structure, and we use the identification $k = U(L)/LU(L)$ to derive the second equality.

Remark A.344. The functor $M \mapsto M^L$ is right adjoint to the functor that sends a k-vector space to the trivial left L-module functor from Example A.339. That is, there is a canonical bijection

$$\operatorname{Hom}_{U(L)}(M, N) = \operatorname{Hom}_k(M, N^L)$$

for all k-vector spaces M and all left L-modules N (see Exercise A.356). Since the functor of invariants is a right adjoint, it is left exact by Exercise A.60.

Similarly, the functor $M \mapsto M_L$ is left adjoint to the trivial left L-module functor. That is, there is a canonical bijection

$$\operatorname{Hom}_k(M_L, N) = \operatorname{Hom}_{U(L)}(M, N)$$

for all left L-modules M and all k-vector spaces N. In particular, since the functor of coinvariants is a left adjoint, it is right exact by Exercise A.60.

Being equivalent to the category of left modules over the ring $U(L)$, the category of left L-modules has enough injectives (Example A.107) and projectives (Exercise A.132). The universal enveloping algebra $U(L)$ with the left L-module structure induced by the product in $U(L)$ is a projective left L-module.

Definition A.345. Let L be a Lie algebra over k, let M be a left L-module, and $i \geqslant 0$ an integer.

i) The i-th *cohomology group* $\mathrm{H}^i(L, M)$ is the i-th cohomological right derived functor of the invariants functor $(\cdot)^L$.
ii) The i-th *homology group* $\mathrm{H}_i(L, M)$ is the i-th cohomological left derived functor of the coinvariants functor $(\cdot)_L$.

Taking the equality (A.342) into account, Lie cohomology can be computed as

$$\mathrm{H}^i(L, M) = \mathrm{Ext}^i_L(k, M) = \mathrm{Ext}^i_{U(L)}(k, M).$$

Similarly, using the equality (A.343), Lie homology is given by

$$\mathrm{H}_i(L, M) = \mathrm{Tor}^{U(L)}_i(k, M).$$

REMARK A.346. Be aware that Tor groups were defined only for modules over a commutative ring in Example A.124. Nevertheless, almost the same definition works for non-commutative rings. Namely, let M be a left $U(L)$-module. Then the tensor product $-\otimes M$ is a functor from the category of right $U(L)$-modules to the category **Ab**, and the Tor groups are defined as the left derived functor

$$\mathrm{Tor}^{U(L)}_i = L_i(\cdot \otimes M)(N).$$

A similar remark applies to Ext groups as well.

A.10.3. *The Chevalley–Eilenberg complex.* The main tool to compute homology and cohomology of Lie algebras is the Chevalley–Eilenberg complex.

DEFINITION A.347. Let L be a Lie algebra over k. The *Chevalley–Eilenberg complex* of L is the chain complex of left $U(L)$-modules $(V_*(L), \partial)$ with

$$V_p(L) = U(L) \otimes_k \Lambda^p L$$

in degree p and differential $\partial_p \colon V_p(L) \to V_{p-1}(L)$ given by

$$\begin{aligned}\partial_p(u \otimes x_1 \wedge \cdots \wedge x_p) = &\sum_{i=1}^{p} (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_p \\ &+ \sum_{1 \leqslant i < j \leqslant p} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_p\end{aligned}$$

for all $u \in U(L)$ and $x_i \in L$.

Observe that all terms appearing in the Chevalley–Eilenberg complex are free (and hence projective) $U(L)$-modules. Indeed, they are the tensor product of the free $U(L)$-module $U(L)$ with the k-vector space $\Lambda^p L$.

THEOREM A.348. *Let $\epsilon \colon V_0(L) = U(L) \to k$ be the counit of $U(L)$ that sends the bilateral ideal generated by L to zero and 1 to 1. Then*

$$V_*(L) \xrightarrow{\epsilon} k$$

is a projective resolution of the trivial module k in the category of left $U(L)$-modules.

The proof, for which the reader is referred to [**Wei94**, Cor. 7.7.3], relies on the Poincaré–Birkhoff–Witt theorem (Theorem 3.99). Applying the coinvariants functor to the Chevalley–Eilenberg complex, one gets a chain complex of k-vector spaces called the Koszul complex.

DEFINITION A.349. Let L be a Lie algebra over k. The *Koszul complex* is the chain complex of k-vector spaces $(K_*(L), \partial)$ with

$$K_p(L) = \Lambda^p L$$

in degree p and differential

$$\partial_p(x_1 \wedge \cdots \wedge x_p) = \sum_{1 \leqslant i < j \leqslant p} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_p.$$

In particular, the Koszul complex starts as

$$\cdots \longrightarrow L \wedge L \xrightarrow{[\cdot,\cdot]} L \xrightarrow{0} k.$$

Thanks to Theorem A.348, Lie algebra homology and cohomology with trivial coefficients can be computed using the Koszul complex.

Proposition A.350. *Let L be a Lie algebra over k. The Lie algebra homology groups $\mathrm{H}_*(L,k)$ are the homology of the chain complex $K_*(L)$, and the Lie algebra cohomology groups $\mathrm{H}^*(L,k)$ are the cohomology of the cochain complex $\mathrm{Hom}_k(K_*(L),k)$. That is, there canonical isomorphisms*

$$\mathrm{H}_n(L,k) = \mathrm{H}_n(K_*(L)), \qquad \mathrm{H}^n(L,k) = \mathrm{H}^n(\mathrm{Hom}_k(K_*(L),k)).$$

Proof. By definition, $\mathrm{H}_i(L,k) = \mathrm{Tor}_i^{U(L)}(k,k)$ and $\mathrm{H}^i(L,k) = \mathrm{Ext}^i_{U(L)}(k,k)$. Therefore, to compute Lie algebra homology and cohomology, we start with a projective resolution P_* of k, and then set

$$\mathrm{H}_i(L,k) = \mathrm{H}_i(k \otimes_{U(L)} P_*),$$
$$\mathrm{H}^i(L,k) = \mathrm{H}^i(\mathrm{Hom}_{U(L)}(P_*,k)).$$

By Theorem A.348, the complex $V_*(L)$ is a projective resolution of k, and hence the result follows from the equalities

$$K_*(L) = k \otimes_{U(L)} V_*(L),$$
$$\mathrm{Hom}_k(K_*(L),k) = \mathrm{Hom}_{U(L)}(V_*(L),k). \qquad \square$$

A.10.4. *Homology and cohomology of graded Lie algebras.* Recall that a *graded algebra A* is an algebra A together with a decomposition

$$A = \bigoplus_{n\in\mathbb{Z}} A_n$$

satisfying $A_n \cdot A_m \subset A_{n+m}$. This notion applies both to Lie algebras (as in Definition 3.82) and to associative algebras (as in Definition 1.75).

Definition A.351. Let A be an algebra (either an associative algebra or a Lie algebra). A *graded A-module* M is an A-module together with a decomposition

$$M = \bigoplus_{n\in\mathbb{Z}} M_n$$

such that there are inclusions $A_n \cdot M_m \subset M_{n+m}$. We will denote by $\mathbf{GrMod}_A$ the category of graded left A-modules.

Example A.352. Let n be an integer. The graded left L-module $k(n)$ is the trivial L-module k concentrated in degree $-n$. Given a graded left L-module M, define $M(n) = M \otimes_k k(n)$. That is, $M(n)$ is the same module M with the decomposition given by $M(n)_m = M_{n+m}$.

Let L be a graded Lie algebra. By the Poincaré–Birkhoff–Witt theorem, the associative algebra $U(L)$ inherits the structure of an associative graded algebra. The categories $\mathbf{GrMod}_L$ and $\mathbf{GrMod}_{U(L)}$ are equivalent.

The grading of L induces a grading on the Chevalley–Eilenberg complex turning it into a complex of graded left $U(L)$-modules. It follows that the Koszul complex inherits a structure of complex of graded k-vector spaces. More concretely,

$$K_*(L) = \bigoplus_n K_*(L)_n$$

with the n-th graded piece of the i-th term given by

$$K_i(L)_n = (\Lambda^i L)_n,$$

where $(\Lambda^i L)_n$ is the subspace of $\Lambda^i L$ generated by elements of the form $x_1 \wedge \cdots \wedge x_i$ with each x_i homogeneous and $\sum_i \deg(x_i) = n$. The differential of the Koszul complex is homogeneous of degree zero, *i.e.* satisfies

$$\partial K_i(L)_n \subset K_{i-1}(L)_n.$$

Therefore, the homology of the Koszul complex is again a graded vector space with

$$\mathrm{H}_i(K_*(L))_n = \mathrm{H}_i(K_*(L)_n).$$

From this, we obtain a grading in Lie algebra homology

$$\mathrm{H}_i(L,k) = \bigoplus \mathrm{H}_i(L,k)_n, \qquad \mathrm{H}_i(L,k)_n = \mathrm{H}_i(K_*(L)_n).$$

We define the graded dual of the Koszul complex as

$$K^*(L)_n = \mathrm{Hom}_k(K_*(L)_{-n}, k),$$

which induces the grading in Lie algebra cohomology

$$\mathrm{H}^*(L,k) = \bigoplus_n \mathrm{H}^*(L,k)_n, \quad \mathrm{H}^*(L,k)_n = \mathrm{H}^*(K^*(L)_n).$$

The duality and the grading are compatible with each other.

Proposition A.353. *Let L be a graded Lie algebra. Then the equalities*

$$\mathrm{H}^i(L,k)_n = \mathrm{H}^i(K^*(L)_n) = (\mathrm{H}_i(K_*(L))_{-n})^\vee = (\mathrm{H}_i(L,k)_{-n}))^\vee$$

holds for all integers i, n.

Proof. Since every k-vector space is projective, the universal coefficients theorem of [**Wei13**, Thm. 3.6.5] implies the result. □

Theorem A.354. *Let L be a graded Lie algebra. Then*

$$\mathrm{Ext}^i_{\mathbf{GrMod}_{U(L)}}(k(n), k) = \mathrm{H}^i(K^*(L)_{-n}).$$

Proof. The proof relies on the fact that the category $\mathbf{GrMod}_{U(L)}$ has enough projectives. In fact, the Chevalley–Eilenberg complex is still a projective resolution

$$V_*(L) \xrightarrow{\epsilon} k$$

of the trivial module k in the category $\mathbf{GrMod}_{U(L)}$. Changing the degree, we deduce that $V_*(L)(n)$ is a projective resolution of $k(n)$. Then, the result follows from the chain of equalities

$$\begin{aligned}
\mathrm{Ext}^i_{\mathbf{GrMod}_{U(L)}}(k(n), k) &= \mathrm{H}^i(\mathrm{Hom}_{\mathbf{GrMod}_{U(L)}}(V_*(L)(n), k)) \\
&= \mathrm{H}^i(\mathrm{Hom}_{\mathbf{GrMod}_k}(K_*(L)(n), k)) \\
&= \mathrm{H}^i(\mathrm{Hom}_k(K_*(L)_n, k)) \\
&= \mathrm{H}^i(K^*(L)_{-n}),
\end{aligned}$$

where the second equality results from the adjunction of Exercise A.357. □

A.10.5. *The structure of a Lie algebra from its homology.* In this section, we show one example where the structure of a Lie algebra is determined by its homology. Recall from Definition 3.165 that a Lie algebra is called quasi-nilpotent if its lower central series $L^{(i)}$ satisfies $\bigcap L^{(i)} = \{0\}$.

PROPOSITION A.355. *Let $L = \bigoplus_n L_n$ be a quasi-nilpotent graded Lie algebra over k such that $\mathrm{H}_1(L,k)$ is either concentrated in strictly negative degrees or in strictly positive degrees, and $\mathrm{H}_2(L,k)$ vanishes. Then L is isomorphic to the free Lie algebra generated by $\mathrm{H}_1(L,k)$.*

PROOF. We will only treat the negatively graded case. The Koszul complex of L (see Definition A.349 and Proposition A.350) in lower degrees reads

$$\cdots \longrightarrow L \wedge L \wedge L \longrightarrow L \wedge L \xrightarrow{[\ ,\ ]} L \xrightarrow{0} k,$$

where the last map in the complex is the zero map, and the previous map is given by the Lie bracket. From this complex, we derive the well-known identity

$$\mathrm{H}_1(L,k) = L/[L,L].$$

The map $L \to \mathrm{H}_1(L,k)$ is homogeneous and surjective, so that we can choose a homogeneous section $s\colon \mathrm{H}_1(L,k) \to L$. In general, this section is non-canonical. Let $\mathfrak{F}$ be the free Lie algebra generated by $\mathrm{H}_1(L,k)$. It is also a quasi-nilpotent graded Lie algebra. By the universal property of free Lie algebras (Definition 3.195), the chosen section defines a graded map $\mathfrak{F}(s)\colon \mathfrak{F} \to L$. The rest of the proof consists in showing that this map is an isomorphism.

Let F denote the increasing filtration of the Lie algebras L and $\mathfrak{F}$ given by the (opposite of the) degree, that is,

$$F_n L = \bigoplus_{n' \leqslant -n} L_{n'}, \quad F_n \mathfrak{F} = \bigoplus_{n' \leqslant -n} \mathfrak{F}_{n'}.$$

We prove by induction on $n \geqslant 0$ that the map

$$F_n \mathfrak{F} \longrightarrow F_n L$$

is surjective. Since $\mathrm{H}_1(L,k)$ has only negative degrees, $F_0\mathfrak{F} = 0$ holds by construction. Since L is graded, we deduce that $F_0 L$ is a Lie subalgebra. Since $\mathrm{H}_1(L,k)$ is concentrated in negative degrees, there is an inclusion $F_0 L \subset [L,L]$. Since L is a graded Lie algebra, this implies $F_0 L \subset [L, F_0 L]$. Using that L is quasi-nilpotent, we deduce the vanishing of $F_0 L$, which proves the case $n = 0$.

We now assume that $F_{n'}\mathfrak{F} \to F_{n'} L$ is surjective for all $n' < n$. Writing

$$F_n L / F_{n-1} L = s\,(\mathrm{H}_1(L,k)_n) + [L,L]_n,$$

where $\mathrm{H}_1(L,k)_n$ and $[L,L]_n$ denote the homogeneous components of degree n of $\mathrm{H}_1(L,k)$ and $[L,L]$ respectively. Clearly, $s\,(\mathrm{H}_1(L,k)_n)$ lies in the image of $\mathfrak{F}(s)$. Since $F_0 L = 0$, every element of $[L,L]_n$ is a linear combination of products of terms of lower degree. Therefore, the induction hypothesis implies that $[L,L]_n$ also lies in the image of $\mathfrak{F}(s)$. Hence, the map $F_n\mathfrak{F} \to F_n L$ is surjective. Since L is graded,

$$L = \bigoplus_{n \in \mathbb{Z}} L_n = \bigcup_{n \geqslant 0} F_n L$$

holds, and we derive the surjectivity of $\mathfrak{F} \to L$.

Let $\mathfrak{k} \subset \mathfrak{F}$ be the kernel of the map $\mathfrak{F} \to L$. There is a commutative diagram

$$\begin{tikzcd}
 & \mathfrak{F} \wedge \mathfrak{k} \arrow[r] \arrow[d, hook] & \mathfrak{k} \arrow[d, hook] & \\
\mathfrak{F} \wedge \mathfrak{F} \wedge \mathfrak{F} \arrow[r] \arrow[d, two heads] & \mathfrak{F} \wedge \mathfrak{F} \arrow[r] \arrow[d, two heads] & \mathfrak{F} \arrow[r, two heads] \arrow[d, two heads] & \mathrm{H}_1(L, k) \arrow[d, equal] \\
L \wedge L \wedge L \arrow[r] & L \wedge L \arrow[r] & L \arrow[r, two heads] & \mathrm{H}_1(L, k),
\end{tikzcd}$$

where $\mathfrak{F} \wedge \mathfrak{k}$ is the image of $\mathfrak{F} \otimes \mathfrak{k}$ in $\mathfrak{F} \wedge \mathfrak{F}$. The long vertical sequences are exact by definition, and the upper long horizontal sequence is exact because $\mathfrak{F}$ is the free Lie algebra generated by $\mathrm{H}_1(L, k)$. The lower long sequence is exact because of the assumption $\mathrm{H}_2(L, k) = 0$. Then a diagram chase argument shows that the top horizontal arrow is surjective, and from this we deduce the inclusion

$$\mathfrak{k} \subset [\mathfrak{k}, \mathfrak{F}].$$

Since $\mathfrak{F}$ is quasi-nilpotent, we conclude that $\mathfrak{k} = 0$. This proves the injectivity of the map $\mathfrak{F} \to L$ and concludes the argument. $\square$

⋆⋆⋆

Exercise A.356. Let L be a Lie algebra and M a left L-module. Recall the invariants M^L and the coinvariants M_L from Definition A.341.

i) Prove that M^L is the largest sub-L-module of M on which L acts trivially. Deduce that M^L is right adjoint to the trivial left L-module functor.
ii) Prove that M_L is the largest L-module quotient of M on which L acts trivially. Deduce that M_L is left adjoint to the trivial L-module functor.

Exercise A.357. This exercise is a variant of Exercise A.356 written in the language of universal enveloping algebras. Let L be a graded Lie algebra over a field k and $U(L)$ the universal enveloping algebra. Consider the functors

$$\mathbf{GrMod}_{U(L)} \underset{\mathrm{Triv}}{\overset{k \otimes_{U(L)} -}{\rightleftarrows}} \mathbf{GrMod}_k,$$

where Triv is the functor that sends a k-vector space to the corresponding trivial $U(L)$-module. Show that these functors form an adjoint pair.

Exercise A.358. Let L be a Lie algebra and M a left L-module. Show that the action $mx = -xm$ defines a structure of right L-module on M.

Exercise A.359. Let A be a graded associative algebra with unit. Let $\mathcal{C}$ be the abelian category of graded left A-modules.

i) For each integer $n \in \mathbb{Z}$, let $A(n) = A \otimes_k k(n)$ be the graded A-module with $A(n)_m = A_{n+m}$. Prove that $A(n)$ is a projective object of $\mathcal{C}$.
ii) Conclude that $\mathcal{C}$ has enough projectives.

Exercise A.360. Find examples showing that all the hypothesis of Proposition A.355 are necessary.

References

[AK11] D. Arapura and S-J. Kang. Kähler-de Rham cohomology and Chern classes. *Comm. Algebra*, 39(4):1153–1167, 2011.

[And04] Y. André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2004.

[Apé79] R. Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

[BB03] P. Brosnan and P. Belkale. Matroids motives, and a conjecture of Kontsevich. *Duke Math. J.*, 116(1):147–188, 2003.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BBV10] J. Blümlein, D. J. Broadhurst, and J. A. M. Vermaseren. The multiple zeta value data mine. *Comput. Phys. Comm.*, 181(3):582–625, 2010.

[Beĭ86] A. A. Beĭlinson. Notes on absolute Hodge cohomology. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 35–68. Amer. Math. Soc., Providence, RI, 1986.

[Beĭ12] A. A. Beĭlinson. Remarks on Grothendieck's standard conjectures. In *Regulators*, volume 571 of *Contemp. Math.*, pages 25–32. Amer. Math. Soc., Providence, RI, 2012.

[BEK06] S. Bloch, H. Esnault, and D. Kreimer. On motives associated to graph polynomials. *Comm. Math. Phys.*, 267(1):181–225, 2006.

[Beu79] F. Beukers. A note on the irrationality of $\zeta(2)$ and $\zeta(3)$. *Bull. London Math. Soc.*, 11(3):268–272, 1979.

[Beu87] F. Beukers. Irrationality proofs using modular forms. *Astérisque*, 147-148:271–283, 1987. Journées arithmétiques de Besançon (Besançon, 1985).

[BG02] J. I. Burgos Gil. *The regulators of Beilinson and Borel*, volume 15 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2002.

[BGSV90] A. A. Beĭlinson, A. A. Goncharov, V. V. Schechtman, and A. N. Varchenko. Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of a pair of triangles in the plane. In *The Grothendieck Festschrift*, number 86 in Progress in Mathematics, pages 135–171. Birkhäuser, 1990.

[BK94] S. Bloch and I. Kříž. Mixed Tate motives. *Ann. of Math.*, 140(3):557–605, 1994.

[BK97] D. J. Broadhurst and D. Kreimer. Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. *Phys. Lett. B*, 393:403–412, 1997.

[Blo86] S. Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986.

[Boa54] R. P. Boas. *Entire functions.* Academic Press Inc., New York, 1954.

[Bon10] M. V. Bondarko. Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general). *J. K-Theory*, 6(3):387–504, 2010.

[Bor74] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[Bor77] A. Borel. Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 4(4):612–636, 1977.

[Bou60] N. Bourbaki. *Éléments de mathématique. XXVI. Groupes et algèbres de Lie. Chapitre 1: Algèbres de Lie.* Hermann, Paris, 1960. Actualités Sci. Indust., No. 1285.

[BR01] K. Ball and T. Rivoal. Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs. *Invent. Math.*, 146(1):193–207, 2001.

[Bre97] G. E. Bredon. *Sheaf Theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[Bri17] M. Brion. Some structure theorems for algebraic groups. In *Algebraic groups: structure and actions*, volume 94 of *Proc. Sympos. Pure Math.*, pages 53–126. Amer. Math. Soc., Providence, RI, 2017.

[Bro96] D. Broadhurst. On the enumeration of irreducible k-fold Euler sums and their roles in knot theory and field theory. hep-th/9604128, 1996.

[Bro09] F. Brown. Multiple zeta values and periods of moduli spaces $\overline{M}_{0,n}$. *Ann. Sci. Éc. Norm. Supér.*, 42(3):371–489, 2009.

[Bro12] F. Brown. Mixed Tate motives over $\mathbb{Z}$. *Ann. of Math.*, 175(2):949–976, 2012.

[Bro13] F. Brown. Iterated integrals in quantum field theory. In *Geometric and topological methods for quantum field theory*, pages 188–240. Cambridge Univ. Press, Cambridge, 2013.

[Bro16] F. Brown. Irrationality proofs for zeta values, moduli spaces and dinner parties. *Mosc. J. Comb. Number Theory*, 6(2–3):102–165, 2016.

[Bro17] F. Brown. Notes on motivic periods. *Commun. Number Theory Phys.*, 11(3):557–655, 2017.

[Bro21] F. Brown. Depth-graded motivic multiple zeta values. *Compos. Math.*, 157:529–572, 2021.

[BS01] P. Balmer and M. Schlichting. Idempotent completion of triangulated categories. *J. Algebra*, 236(2):819–834, 2001.

[BS12] F. Brown and O. Schnetz. A K3 in ϕ^4. *Duke Math. J.*, 161(10):1817–1862, 2012.

[BT82] R. Bott and L. W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[Car80] J. A. Carlson. Extensions of mixed Hodge structures. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 107–127. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[Car88] P. Cartier. Jacobiennes généralisées, monodromie unipotente et intégrales itérées. *Astérisque*, 161-162:Exp. No. 687, 3, 31–52 (1989), 1988. Séminaire Bourbaki, Vol. 1987/88.

[Car02] P. Cartier. Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents. In *Séminaire Bourbaki*, number 282 in Astérisque, pages 137–173. Société Mathématique de France, 2002. Exp. no. 885.

[Car07] P. Cartier. A primer of Hopf algebras. In *Frontiers in number theory, physics, and geometry. II*, pages 537–615. Springer, Berlin, 2007.

[CEOP21] K. Coulembier, P. Etingof, V. Ostrik, and B. Pauwels. Monoidal abelian envelopes with a quotient property. arXiv:2103.00094, 2021.

[CF06] C. Consani and C. Faber. On the cusp form motives in genus 1 and level 1. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 297–314. Math. Soc. Japan, Tokyo, 2006.

[Che77] K-T. Chen. Iterated path integrals. *Bull. Amer. Math. Soc.*, 83(5):831–879, 1977.

[CK98] A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Comm. Math. Phys.*, 199(1):203–242, 1998.

[CLNS18] A. Chambert-Loir, J. Nicaise, and J. Sebag. *Motivic integration*, volume 325 of *Progress in Mathematics*. Birkhäuser/Springer, New York, 2018.

[Col03] P. Colmez. Arithmétique de la fonction zêta. In *La fonction zêta*, pages 37–164. Ed. Éc. Polytech., Palaiseau, 2003.

[Con07] B. Conrad. Deligne's notes on Nagata compactifications. *J. Ramanujan Math. Soc.*, 22(3):205–257, 2007.

[Del70] P. Deligne. *Equations differentielles a points singuliers reguliers*. Number 163 in Lecture Notes in Mathematics. Springer-Verlag, 1970.

[Del71] P. Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, 40:5–57, 1971.

[Del73] P. Deligne. *SGA 4 III*, volume 305 of *Lecture Notes in Mathematics*, chapter Cohomologie à supports propres. Springer, 1973.

[Del74] P. Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, 44:5–77, 1974.

[Del89] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.

[Del90] P. Deligne. Catégories tannakiennes. In *Grothendieck Festschrift, Vol II*, volume 87 of *Progr. Math.* Birkhäuser, 1990.

[Del13] P. Deligne. Multizêtas, d'après Francis Brown. *Astérisque*, 352:Exp. No. 1048, viii, 161–185, 2013. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058.

[DF24] C. Dupont and J. Fresán. A construction of the polylogarithm motive. *Épijournal de Géométrie Algébrique*, to appear, 2024.

[DG70] M. Demazure and P. Gabriel. *Groupes Algébriques.* North Holland Publishing Company, Paris, 1970.

[DG05] P. Deligne and A. B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup. (4)*, 38(1):1–56, 2005.

[DM82] P. Deligne and J. S. Milne. Tannakian categories. In *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*, pages 101–228. Springer-Verlag, 1982.

[Dri90] V. G. Drinfel'd. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Leningrad Math. J.*, 2(4):829–860, 1990.

[Dup17] C. Dupont. Relative cohomology of bi-arrangements. *Trans. Amer. Math. Soc.*, 369(11):8105–8160, 2017.

[Dup18] C. Dupont. Odd zeta motive and linear forms in odd zeta values. *Compos. Math.*, 154:342–379, 2018.

[Dur83] A. H. Durfee. A naive guide to mixed Hodge theory. In *Singularities, Part 1 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 313–320. Amer. Math. Soc., Providence, RI, 1983.

[EM53] S. Eilenberg and S. MacLane. On the groups $H(\Pi, n)$. I. *Ann. of Math.*, 58:55–106, 1953.

[ES02] P. Etingof and O. Schiffmann. *Lectures on quantum groups*. Lectures in Mathematical Physics. International Press, Somerville, MA, second edition, 2002.

[Eul76] L. Euler. Meditationes circa singulare serierum genus. *Novi Comm. Acad. Sci. Petropol.*, 20:140–186, 1776. Reprinted in Opera Omnia ser. I, vol. 15, Teubner, Berlin (1927), p. 217–267. Downloadable from the website http://eulerarchive.maa.org/pages/E477.html.

[EZ53] S. Eilenberg and J. A. Zilber. On products of complexes. *Amer. J. Math.*, 75:200–204, 1953.

[Fis04] S. Fischler. Irrationalité de valeurs de zêta (d'après Apéry, Rivoal, ...). *Astérisque*, 294:vii, 27–62, 2004.

[Fis21] S. Fischler. Linear independence of odd zeta values using Siegel's lemma. https://arxiv.org/abs/2109.10136, 2021.

[Fre24] J. Fresán. Une introduction aux périodes. In Éditions de l'École polytechnique, editor, *Périodes et transcendance. Journées mathématiques X-UPS 2019*, pages 1–147, 2024.

[FSZ19] S. Fischler, J. Sprang, and W. Zudilin. Many odd zeta values are irrational. *Compos. Math.*, 155(5):938–952, 2019.

[Fur10] H. Furusho. Pentagon and hexagon equations. *Ann. of Math. (2)*, 171(1):545–556, 2010.

[Fur11] H. Furusho. Double shuffle relation for associators. *Ann. of Math. (2)*, 174(1):341–360, 2011.

[GJ09] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition.

[GKZ06] H. Gangl, M. Kaneko, and D. Zagier. Double zeta values and modular forms. In *Automorphic forms and zeta functions*, pages 71–106. World Sci. Publ., Hackensack, NJ, 2006.

[GM03] S. I. Gelfand and Y. I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[GM04] A. B. Goncharov and Yu. I. Manin. Multiple ζ-motives and moduli spaces $\overline{\mathcal{M}}_{0,n}$. *Compos. Math.*, 140(1):1–14, 2004.

[Gon98] A. B. Goncharov. Multiple polylogarithms, cyclotomy and modular complexes. *Math. Res. Lett.*, 5(4):497–516, 1998.

[Gon01] A. Goncharov. Multiple polylogarithms and mixed Tate motives. Preprint, https://arxiv.org/abs/math/0103059, 2001.

[Gon05] A. B. Goncharov. Galois symmetries of fundamental groupoids and noncommutative geometry. *Duke Math. J.*, 128(2):209–284, 2005.

[GR65] R. C. Gunning and H. Rossi. *Analytic functions of several complex variables*. Prentice-Hall, Inc., 1965.

[Gra58] H. Grauert. On Levi's problem and the imbedding of real-analytic manifolds. *Ann. of Math. (2)*, 68:460–472, 1958.

[Gro57] A. Grothendieck. Sur quelques points d'algèbre homologique. *Tohoku Math. J. (2)*, 9:119–221, 1957.

[Gro61] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, 8:222, 1961.

[Gro66] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

[Gro03] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 3 of *Documents Mathématiques*. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck, with two papers by M. Raynaud.

[Hai87a] R. M. Hain. The de Rham homotopy theory of complex algebraic varieties. I. *K-Theory*, 1(3):271–324, 1987.

[Hai87b] R. M. Hain. The geometry of the mixed Hodge structure on the fundamental group. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46, pages 247–282. Amer. Math. Soc., Providence, RI, 1987.

[Hai93] R. M. Hain. Completions of mapping class groups and the cycle $C - C^{-}$. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, volume 150 of *Contemp. Math.*, pages 75–105. Amer. Math. Soc., Providence, RI, 1993.

[Han99] M. Hanamura. Mixed motives and algebraic cycles. III. *Math. Res. Lett.*, 6(1):61–82, 1999.

[Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Har18] R. Harada. On Euler's formulae for double zeta values. *Kyushu J. Math.*, 72(1):15–24, 2018.

[Hat02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HMS17] A. Huber and S. Müller-Stach. *Periods and Nori motives*, volume 65 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer, Cham, 2017. With contributions of Benjamin Friedrich and Jonas von Wangenheim.

[Hof97] M. E. Hoffman. The algebra of multiple harmonic series. *J. Algebra*, 194(2):477–495, 1997.

[Hof00] M. E. Hoffman. Quasi-shuffle products. *J. Algebraic Combin.*, 11:49–68, 2000.

[HRV08] T. Hausel and F. Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. *Invent. math.*, 174(3):555–624, 2008. with an appendix by Nicholas M. Katz.

[IK04] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[IKZ06] K. Ihara, M. Kaneko, and D. Zagier. Derivation and double shuffle relations for multiple zeta values. *Compos. Math.*, 142(2):307–338, 2006.

[Ive86] B. Iversen. *Cohomology of sheaves*. Universitext. Springer-Verlag, Berlin, 1986.

[Jac62] N. Jacobson. *Lie Algebras*. Interscience Publishers, 1962.

[Jac75] N. Jacobson. *Lectures in abstract algebra II*, volume 31 of *GTM*. Springer Verlag, 1975.

[Jan92] U. Jannsen. Motives, numerical equivalence and semi-simplicity. *Invent. Math.*, 107:447–452, 1992.

[JS91] A. Joyal and R. Street. An introduction to Tannaka duality and quantum groups. In *Category Theory*, volume 1488 of *Lecture Notes in Mathematics*. Springer, 1991.

[KN96] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[Knu83] F. Knudsen. The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.*, 52(2), 161–199 1983.

[Kol13] J. Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 2013.

[Kow14] E. Kowalski. *An introduction to the representation theory of groups*, volume 155 of *Graduate Studies in Mathematics*. AMS, 2014.

[KS06] M. Kashiwara and P. Schapira. *Categories and Sheaves*, volume 332 of *Grundelehren der mathematischen Wissenschaften*. Springer Verlag, Berlin, 2006.

[KV07] J. Kock and I. Vainsencher. *An invitation to quantum cohomology. Kontsevich's formula for rational plane curves*, volume 249 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2007.

[KZ01] M. Kontsevich and D. Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[Lee13] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[Lev93] M. Levine. Tate motives and the vanishing conjectures for algebraic K-theory. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 167–188. Kluwer Acad. Publ., Dordrecht, 1993.

[Lev94] M. Levine. Bloch's higher Chow groups revisited. *Astérisque*, 226:10, 235–320, 1994. K-theory (Strasbourg, 1992).

[Li13] Z-H. Li. Another proof of Zagier's evaluation formula of the multiple zeta values $\zeta(2,\dots,2,3,2,\dots,2)$. *Math. Res. Lett.*, 20(5):947–950, 2013.

[Lin82] F. Lindemann. Über die zahl π. *Math. Ann.*, 20(2):213–225, 1882.

[Loj64] S. Lojasiewicz. Triangulation of semi-analytic sets. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 18:449–474, 1964.

[Mac71] S. MacLane. *Categories for the working mathematician*. Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, 1971.

[Mal49] A. I. Malcev. Nilpotent torsion-free groups. *Izvestiya Akad. Nauk. SSSR. Ser. Mat.*, 13:201–212, 1949.

[Mil17] J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field.

[MM65] J. W. Milnor and J. C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[Mor78] J. Morgan. The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, 48:137–204, 1978.

[MVW06] C. Mazza, V. Voevodsky, and C. Weibel. *Lectures notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.

[Nee01] A. Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[Nee02] A. Neeman. A counterexample to a 1961 "theorem" in homological algebra. *Invent. Math.*, 148:397–420, 2002.

[Nes96] Yu. V. Nesterenko. A few remarks on $\zeta(3)$. *Math. Notes 59*, 59(5-6):625–636, 1996.

[Neu99] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[Nie65] N. Nielsen. *Die Gammafunktion. Band I. Handbuch der Theorie der Gammafunktion. Band II. Theorie des Integrallogarithmus und verwandter Transzendenten, Teubner 1906*. Reprint by Chelsea Publishing Co., New York, 1965.

[Pet22] D. Petersen. A remark on singular cohomology and sheaf cohomology. *Math. Scand.*, 128(2):229–238, 2022.

[Pré96] M. Prévost. A new proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ using Padé approximants. *J. Comput. Appl. Math.*, 67(2):219–235, 1996.

[PS08] C. A. M. Peters and J. H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, 2008.

[Qui69] D. Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

[Qui72] D. Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math.*, 96:552–586, 1972.

[Qui73] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, volume 341 of *Lecture Notes in Mathematics*, pages 85–147. Springer, Berlin, 1973.

[Ram05] S. Ramanan. *Global calculus*, volume 65 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[Rei] K. Reiser. Gaga and quasicoherent sheaf. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/2841087 (version: 2018-07-04).

[Reu93] C. Reutenauer. *Free Lie algebras*, volume 7 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[Rie59] B. H. Riemann. Über die anzahl der primzahlen unter einer gegebenen. *Monatsberichte der Berliner Akademie*, 1859.

[Riv00] T. Rivoal. La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(4):267–270, 2000.

[Sel16] Y. Sella. Comparison of sheaf cohomology and singular cohomology. https://arxiv.org/abs/1602.06674, 2016.

[Ser56] J-P. Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.

[Ser88] J-P. Serre. *Algebraic groups and class fields*, volume 117 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.

[Ser94] J-P. Serre. *Cohomologie Galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer Verlag, fifth edition, 1994.

[Sor98] V. N. Sorokin. On Apéry's theorem. *Moscow Univ. Math. Bull.*, 53(3):48–52, 1998.

[Sou10] C. Soulé. Higher K-theory of algebraic integers and the cohomology of arithmetic groups. In *Cohomology of groups and algebraic K-theory*, volume 12 of *Adv. Lect. Math. (ALM)*, pages 503–517. Int. Press, Somerville, MA, 2010. Notes by Marco Varisco.

[Spa66] E. H. Spanier. *Algebraic Topology*. Springer Verlag, 1966.

[Spr18] J. Sprang. Infinitely many odd zeta values are irrational. By elementary means. Preprint, https://arxiv.org/abs/1802.09410, 2018.

[SR72] N. Saavedra Rivano. *Catégories Tannakiennes*. Number 265 in Lecture Notes in Mathematics. Springer Verlag, 1972.

[Ste98] J. R. Stembridge. Counting points on varieties over finite fields related to a conjecture of Kontsevich,. *Ann. Combin.*, 2(365–385), 1998.

[Swe69] M. E. Sweedler. *Hopf Algebras*. W. A. Benjamin, Inc., 1969.

[Ter02] T. Terasoma. Mixed Tate motives and multiple zeta values. *Invent. Math.*, 149(2):339–369, 2002.

[Tit86] E. C. Titchmarsh. *The theory of the Riemann zeta-function*. The Clarendon Press, Oxford University Press, New York, second edition, 1986.

[Tor50] L. Tornheim. Harmonic double series. *Amer. J. Math.*, 72:303–314, 1950.

[vdP79] A. van der Poorten. A proof that Euler missed. . . Apéry's proof of the irrationality of $\zeta(3)$. *Math. Intelligencer*, 1(4):195–203, 1978/79.

[Voe00] V. Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*, pages 188–238. Princeton University Press, Princeton, NJ, 2000.

[Voe02] V. Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. *Int. Math. Res. Not.*, 2002(7):351–355, 2002.

[Voi02] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[Wal12] M. Waldschmidt. Lectures on Multiple Zeta Values, imsc 2011. http://www.math.jussieu.fr/~miw/articles/pdf/MZV2011IMSc.pdf, 2012.

[War83] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[Wat79] W.C. Waterhouse. *Introduction to Affine Group Schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979.

[Wei89] A. Weil. Prehistory of the zeta-function. In *Number theory, trace formulas and discrete groups (Oslo, 1987)*, pages 1–9. Academic Press, Boston, MA, 1989.

[Wei94] C. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Wei13] C. Weibel. *The K-book. An introduction to algebraic K-theory*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013.

[Wil00] J. Wildeshaus. On derived functors on categories without enough injectives. *J. Pure Appl. Algebra*, 150(2):207–213, 2000.

[Wil09] J. Wildeshaus. f-catégories, tours et motifs de Tate. *C. R. Math. Acad. Sci. Paris*, 347(23-24):1337–1342, 2009. Cf. also Erratum à la note "f-catégories, tours et motifs de Tate", avec un appendice de Q. Liu et de J. Vitória. *C. R. Math. Acad. Sci. Paris*, 350(3-4):129–131, 2012.

[Zag93] D. Zagier. Periods of modular forms, traces of Hecke operators, and multiple zeta values. *Sūrikaisekikenkyūsho Kōkyūroku*, 843:162–170, 1993. Research into automorphic forms and L functions (Japanese) (Kyoto, 1992).

[Zag94] D. Zagier. Values of zeta functions and their applications. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, volume 120 of *Progr. Math.*, pages 497–512. Birkhäuser, Basel, 1994.

[Zag08] D. Zagier. Elliptic modular forms and their applications. In *The 1-2-3 of modular forms*, Universitext, pages 1–103. Springer, Berlin, 2008.

[Zag12] D. Zagier. Evaluation of the multiple zeta values $\zeta(2,\dots,3,2,\dots,2)$. *Ann. of Math. (2)*, 175:977–1000, 2012.

[Zha16] J. Zhao. *Multiple zeta functions, multiple polylogarithms and their special values*, volume 12 of *Series on Number Theory and its Applications*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.

[Zud01] V. V. Zudilin. One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. *Uspekhi Mat. Nauk*, 56(4(340)):149–150, 2001.

[Zud03] V. V. Zudilin. Algebraic relations for multiple zeta values. *Uspekhi Mat. Nauk*, 58(1(349)):3–32, 2003.

[Zud18] W. Zudilin. One of the odd zeta values from $\zeta(5)$ to $\zeta(25)$ is irrational. By elementary means. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 14:Paper No. 028, 8, 2018.

Index

List of symbols

$\mathbf{\Delta}$	the simplicial category, page 477
$\mathbf{Sm}(k)$	the category of smooth varieties over a field k, page 339
$s^{\{n\}}$	the multi-index $(s,\dots,s)$ of length n, page 21
$\mathbf{0},\mathbf{1}$	tangential base points, page 282
Tot	total complex, page 424
U_{dR}	The pro-unipotent part of G_{dR}, page 361
$\widehat{U}(L)$	completed universal enveloping algebra, page 207
$U(L)$	universal enveloping algebra, page 205
$\mathrm{wt}(\boldsymbol{s})$	weight of a multi-index, page 17
X^*	the set of words in the alphabet X, page 33
$\mathcal{Z}$	the $\mathbb{Q}$-algebra of multiple zeta values, page 18
$F_\ell\mathcal{Z}$	the vector subspace of $\mathcal{Z}$ spanned by MZVs of length $\leqslant \ell$, page 18
$F_\ell\mathcal{Z}_k$	the vector subspace of $\mathcal{Z}$ spanned by multiple zeta values of weight k and length $\leqslant \ell$, page 18
$\mathcal{Z}_k$	the vector subspace of $\mathcal{Z}$ spanned by MZVs of weight k, page 18
$\zeta(s)$	zeta value, page 10

(J. I. Burgos Gil) Instituto de Ciencias Matemáticas (CSIC-UAM-UCM-UC3M), Calle Nicolás Cabrera 15, Campus UAM, Cantoblanco, 28049 Madrid, Spain.
Email address: burgos@icmat.es
URL: http://www.icmat.es/miembros/burgos/

(J. Fresán) CMLS, École Polytechnique, 91128 Palaiseau, France.
Email address: javier.fresan@polytechnique.edu
URL: http://javier.fresan.perso.math.cnrs.fr

(U. Kühn) Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany.
Email address: kuehn@math.uni-hamburg.de
URL: http://www.math.uni-hamburg.de/home/kuehn/