

Conjectures on the logarithmic derivatives of Artin L -functions II

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Abstract. We formulate a general conjecture relating Chern classes of subbundles of Hodge bundles in Arakelov geometry to logarithmic derivatives of Artin L -functions of number fields. This conjecture may be viewed as a far-reaching generalisation of the (Lerch–)Chowla–Selberg formula computing logarithms of periods of elliptic curves in terms of special values of the Γ -function. We prove several special cases of this conjecture in the situation where the involved Artin characters are Dirichlet characters. This article contains the computations promised in [42], where our conjecture was announced. We also give a quick introduction to the Grothendieck–Riemann–Roch theorem and to the geometric fixed point formula, which form the geometric backbone of our conjecture.

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1. Introduction

The main aim of this text is to provide the computations missing (and promised...) in the text [42]. In that article, we formulated a conjecture, which relates the logarithmic derivatives of Artin L -functions at negative integers to certain Chern classes in Arakelov theory. This conjecture (see Conjectures 6.20 and Conjecture 6.32 below) can be viewed as a far-reaching generalisation of the Lerch–Chowla–Selberg formula, which computes the periods of CM elliptic curves in terms of special values of the Γ -function.¹ A secondary aim of this article is to provide a quick introduction to the main geometric ideas that lie behind our approach to Conjecture 6.32. These geometric ideas, although classical in some ways, are unfortunately not very well-known and we felt that to include a discussion of them here would make the computational part of the article (Section 6) more palatable.

This text is an expanded version of some notes prepared by the second author for lectures given during two instructional conferences: the conference ‘Advanced Courses on Arakelov Geometry and Shimura Varieties’ which took place at the Centre de Recerca Matemàtica in Barcelona in February 2006 and the summer school ‘Motives and complex multiplication’, which took place in Ascona (Switzerland) in August 2016.

Very loosely speaking, the conjecture made in [42] says the following. Suppose that you are provided with a homogenous polarised “semistable relative motive \mathcal{M} ” over an arithmetic base B (which may have large dimension). For example, one could consider a generically smooth family of projective arithmetic varieties with semistable degeneration and consider the direct summand of the corresponding semistable relative motive cut out by the relative correspondence (conjecturally) giving the projection on one of the relative cohomology sheaves with logarithmic singularities. Note that as usual, we cannot define the term “semistable relative motive” precisely since no theory of relative mixed motivic sheaves is available but it can be given a precise definition in certain situations, for instance when looking at abelian schemes. Suppose now also that \mathcal{M} carries the action of a number field K , with some compatibility with the polarisation. Then the Hodge realisation of this motive is a vector bundle H on B , which is endowed with a (possibly mildly singular) hermitian metric coming from the polarisation. Furthermore, the vector bundle H comes with an orthogonal decomposition

$$H \simeq \bigoplus_{\sigma \in \text{Gal}(K|\mathbb{Q})} H_{\sigma}.$$

Call \tilde{H}_{σ} the vector bundle H_{σ} together with its hermitian metric. Arakelov theory associates with each \tilde{H}_{σ} its arithmetic Chern character $\widehat{\text{ch}}(\tilde{H}_{\sigma})$, which lives in the arithmetic Chow group $\widehat{\text{CH}}_{\mathbb{Q}}^{\bullet}(B)$ of B .

Let now $\chi: \text{Gal}(K|\mathbb{Q}) \rightarrow \mathbb{C}$ be an irreducible Artin character and $l \geq 1$.

¹This formula was initially discovered by Lerch (see [41]) but his discovery was largely forgotten. Chowla and Selberg rediscovered it more than fifty years later (see [54]).

Conjecture: *the quantity*

$$\sum_{\sigma \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\bar{H}_\sigma) \bar{\chi}(\sigma)$$

is equal to the quantity

$$2 \frac{L'(\chi, 1-l)}{L(\chi, 1-l)} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{l-1}\right)$$

multiplied by an explicit $\bar{\mathbb{Q}}$ -linear combination of ordinary Chern classes of subbundles of H .

Here $\widehat{\text{ch}}^{[l]}(\cdot)$ is the degree l part of the arithmetic Chern character. See Conjecture 6.32 below for a slightly more technical (but still vague) formulation. For abelian schemes, we can make a completely precise conjecture: this is Conjecture 6.20. It should be possible to make a precise conjecture for semiabelian but generically abelian schemes but this seems difficult to do at the present time for lack of a sufficiently general theory of arithmetic Chern classes for singular hermitian metrics. Part of this theory has been built in the articles [15] and [14]. The issue is that for some automorphic Hodge bundles (those that are not ‘totally decomposable’) the singularities are not known to be logarithmic and their nature is not well understood (private communication by J.-I. Burgos).

For abelian varieties with complex multiplication by a CM field, the quantities $\widehat{\text{ch}}^{[1]}(\bar{H}_\sigma)$ can be computed in terms of periods. Thus in this case the equality above computes some linear combination of logarithms of periods in terms of the logarithmic derivatives $L'(\chi, 0)/L(\chi, 0)$ of the irreducible Artin characters χ of the CM field. When the abelian variety is an elliptic curve, one recovers (a slight variant of) the formula of Lerch–Chowla–Selberg (see [54]). If $l = 1$ and K is an abelian extension of \mathbb{Q} one recovers a variant of the period conjecture of Gross–Deligne [33, p. 205] (not to be confused with the conjecture of Deligne [19] relating periods and values of L -functions). For $l > 1$, the invariant $\widehat{\text{ch}}^{[l]}(\bar{H}_\sigma)$ cannot be interpreted in terms of classical invariants anymore. Section 7 collects examples of computations in the literature, which fall in the framework of our conjecture (up to some finite factors which depend on the choices of models).

Remark 1.1. It is important to see that our conjecture falls outside the grid of the conjectures of Beilinson, Deligne, Stark, Gross and others (see e.g. [50]) on the values of L -functions of motives. This can be seen from the fact that we are concerned here with the quotient between the second and the first coefficient of the Taylor series of an L -function at a non negative integer. This quotient in particular concerns the second coefficient of the Taylor series of the L -function at a non negative integer, about which the conjectures of Beilinson and Deligne do not make any prediction. The case of CM abelian

varieties is somewhat confusing in this context, because in this case (as explained above), the conjecture computes some linear combinations of logarithms of periods. On the other hand, periods appear in Deligne’s conjecture (see [19]) relating the values of the L -function of a motive to its periods. For CM abelian varieties, this conjecture was proven by Blasius (see [10] and the bibliography therein): the L -function of a CM abelian variety is an L -function associated with an algebraic Hecke character of a CM field and the values of this L -function can be related to the periods of the abelian variety. The L -functions associated with algebraic Hecke characters are very different from Artin L -functions though, as is witnessed by the fact that in this case our conjecture relates the logarithmic derivatives of Artin L -functions to the *logarithms* of the periods of the abelian variety, whereas the result of Blasius relates values of certain Hecke L -functions to the the periods themselves (not their logarithms). For a concrete example, see [34, formulae (1) and (2) on p. 18 and middle of p. 20].

Our main contribution in this paper is a proof of a stabilised form of our conjecture in the situation where the number field K is an abelian extension of \mathbb{Q} (so that the Artin characters become Dirichlet characters) and where the motive is smooth and arises from a finite group action on a Hodge bundle of geometric origin. See Theorem 6.12 below. We also prove a stronger form of the conjecture in the situation where the number field K is an abelian extension of \mathbb{Q} and the motive is the motive of an abelian scheme. See Theorem 6.27 below. In both cases, we derive our results from the equivariant Grothendieck–Riemann–Roch theorem in Arakelov geometry, applied to the relative de Rham complex. This theorem was first proven in degree one in [37] and in full generality in [58] and [27] (put together). More details on the history of this theorem (whose main contributors are Bismut, Gillet, Soulé and Faltings) are given in subsection 5.2.

The structure of the text is as follows. Sections 2 to 5 do not contain any original material and have been included for pedagogical reasons. In Section 2, we give a very quick introduction to the Grothendieck–Riemann–Roch formula. This theorem, although quite famous, is not as well-known as it should be and is rarely part of a standard course on algebraic geometry. In Section 3, we explain the content of Thomason’s geometric fixed point formula for the action of a diagonalisable group. This formula (and its forerunners) is also a central result of algebraic geometry, which is not widely known. These two theorems can be formally combined to obtain an equivariant extension of the Grothendieck–Riemann–Roch formula, which we formulate in Section 4. We also examine there what statement one obtains when this theorem is applied to the relative de Rham complex. The resulting statement is a relative equivariant form of the Gauss–Bonnet formula. This statement is the geometric heart of our approach to Conjecture 6.32. In Section 5, we give

a precise formulation of the equivariant Grothendieck–Riemann–Roch formula in Arakelov geometry (also called equivariant arithmetic Grothendieck–Riemann–Roch formula), starting with a historical snapshot of Arakelov theory. The equivariant Grothendieck–Riemann–Roch formula in Arakelov geometry will be our central tool. In Section 6, we apply this formula to the relative de Rham complex and we obtain a lifting to Arakelov theory of the relative equivariant form of the Gauss–Bonnet formula (see equation (6.2)); applying finite Fourier theory and some elementary results of analytic number theory, we transform this formula in an equality between a linear combination of logarithmic derivatives of Dirichlet L -functions evaluated at negative integers on the one hand and a linear combination of Chern classes of subbundles of Hodge bundles on the other hand. The final formula naturally suggests the general Conjecture 6.20, which is also included in Section 6. In Section 7, we examine various results on logarithmic derivatives of L -functions that have appeared in the literature and we show that they are all compatible with our general conjecture. We also explain there what part of these results are an actual consequence of (6.2).

2. The Grothendieck–Riemann–Roch formula

In this section, ‘scheme’ will be short for ‘noetherian and separated scheme’.

Let C be a smooth projective curve over \mathbb{C} . Let $D := \sum_i n_i D_i$ be a divisor on C . The simplest instance of the Grothendieck–Riemann–Roch formula is probably the well-known equality

$$\begin{aligned} \chi(\mathcal{O}(D)) &:= \dim_{\mathbb{C}} H^0(C, \mathcal{O}(D)) - \dim_{\mathbb{C}} H^1(C, \mathcal{O}(D)) \\ &= \deg(D) + 1 - g \end{aligned} \tag{2.1}$$

where

$$\deg(D) := \sum_i n_i$$

is the degree of D and

$$g := \dim_{\mathbb{C}} H^0(C, \Omega_C)$$

is the genus of C . One can show that

$$\deg(D) = \int_{C(\mathbb{C})} c_1(\mathcal{O}(D))$$

where $c_1(\mathcal{O}(D))$ is the first Chern class of D (see e.g. [35, Appendix A.3]). Thus (2.1) can be construed as a formula for the Euler characteristic $\chi(\mathcal{O}(D))$ in terms of integrals of cohomology classes.

The Grothendieck–Riemann–Roch formula provides a similar formula for the Euler characteristic of any vector bundle, on any scheme satisfying certain conditions and in a relative setting. Furthermore, the Grothendieck–Riemann–Roch formula is universal in the sense that it is independent of the cohomology theory. The remainder of this section is dedicated to the formulation of this theorem (in a slightly restricted setting).

First a definition.

Definition 2.1. Let X be a scheme. The group $K_0(X)$ (resp. $K'_0(X)$) is the free abelian group generated by the isomorphism classes of coherent locally free sheaves (resp. coherent sheaves) on X , with relations $E = E' + E''$ if there is a short exact sequence

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0.$$

We shall also call a coherent locally free sheaf a *vector bundle*.

The group $K_0(X)$ (resp. $K'_0(X)$) is called the Grothendieck group of coherent locally free sheaves (resp. coherent sheaves) on X . If $f : X \rightarrow Y$ is a proper morphism of schemes, we define the map of abelian groups $\mathbb{R}f_* : K'_0(X) \rightarrow K'_0(Y)$ by the formula

$$\mathbb{R}f_*(E) := \sum_{k \geq 0} (-1)^k \mathbb{R}^k f_*(E).$$

This is well-defined, because the existence of the long exact sequence in cohomology implies that we have $\mathbb{R}f_* E = \mathbb{R}f_* E' + \mathbb{R}f_* E''$ in $K'_0(Y)$, for E, E' and E'' as in Definition 2.1. The group $K_0(X)$ is a commutative ring under the tensor product \otimes and $K'_0(X)$ has a natural $K_0(X)$ -module structure. The obvious map $K_0(X) \rightarrow K'_0(X)$ is an isomorphism if X is regular (see [47, Th. I.9] if X carries an ample line bundle and [60, Lemme 3.3] for the general case). Via this isomorphism, we obtain a map $\mathbb{R}f_* : K_0(X) \rightarrow K_0(Y)$, if both X and Y are regular. For any morphism $f : X \rightarrow Y$ of schemes, there is a pull-back map $f^* : K_0(Y) \rightarrow K_0(X)$, defined in the obvious way, which is a map of rings.

A theory kindred to K_0 -theory is Chow theory. We first need a definition.

Definition 2.2. Let R be a one-dimensional domain. Let $K := \text{Frac}(R)$ and let $f \in K^*$. Define the order of f by the formula

$$\text{ord}_R(f) := \text{length}_R(R/aR) - \text{length}_R(R/bR)$$

where $a \in R, b \in R \setminus \{0\}$ are such that $f = a/b$.

One can show that the definition of $\text{ord}_R(f)$ does not depend on the choice of a, b . Here the symbol $\text{length}_R(\cdot)$ refers to the length of an R -module. See [26, Appendix A.1 & A.3] for more details.

Suppose for the time of the present paragraph that X is an integral scheme. If $f \in \kappa(X)^*$ is a non-zero rational function on X , we may define a formal \mathbb{Z} -linear combination of codimension one closed integral subschemes of X by the formula

$$\text{div}(f) := \sum_{x \in X, \text{cod}(\bar{x})=1} \text{ord}_{\mathcal{O}_x}(f)\bar{x}.$$

Here \bar{x} is the Zariski closure of x . For $p \geq 0$, we let $Z^p(X)$ be the free abelian group on all integral closed subschemes of codimension p of X . An element

of $Z^p(X)$ is called a p -cycle. We let $\text{tot}^p(X) \subseteq Z^p(X)$ be the subgroup of elements of the form $\text{div}(f)$, where $f \in k(Z)^*$ is a rational function on a closed integral subscheme Z of codimension $p - 1$ of X .

Definition 2.3. $\text{CH}^p(X) := Z^p(X)/\text{tot}^p(X)$.

The group $\text{CH}^p(X)$ is called the p -th *Chow group* of X and we shall write

$$\text{CH}^\bullet(X) := \bigoplus_{p \geq 0} \text{CH}^p(X).$$

If V is a closed subscheme of X , we write $[V]$ for the cycle

$$\sum_{v \in V, \bar{v} \text{ irred. comp. of } V} \text{length}_{\mathcal{O}_{V,v}}(\mathcal{O}_{V,v})\bar{v}$$

in X . Here v runs through the generic points of the irreducible components of V .

By work of Gillet and Soulé, if X is also regular, the group

$$\text{CH}^\bullet(X)_{\mathbb{Q}} := \text{CH}^\bullet(X) \otimes \mathbb{Q}$$

can be made into a commutative \mathbb{N} -graded ring. If we denote by \cdot the multiplication in this ring, then we have $[W] \cdot [Z] = [W \cap Z]$, if W and Z are closed integral subschemes of X intersecting transversally (see [55, I.2] for more details and references).

If $f : X \rightarrow Y$ is a proper morphism of schemes, there is a unique push-forward map $f_* : \text{CH}^\bullet(X) \rightarrow \text{CH}^\bullet(Y)$ such that

$$f_*([Z]) = [\kappa(Z) : \kappa(f(Z))] \cdot [f(Z)]$$

if Z is a closed integral subscheme Z of X such that $\dim(f(Z)) = \dim(Z)$ and such that

$$f_*([Z]) = 0$$

otherwise. Here the set $f(Z)$ (which is closed since f is proper) is endowed with its induced reduced closed subscheme structure. See [26, Ex. 20.1.3, p.396] for details. If f is a flat morphism, there is a pull-back map

$$f^* : \text{CH}^\bullet(Y) \rightarrow \text{CH}^\bullet(X)$$

such that $f^*([Z]) = [f^*(Z)]$. Again see [26, p.394] for details.

Suppose now that X is regular. There is a unique ring homomorphism

$$\text{ch} : K_0(X) \rightarrow \text{CH}^\bullet(X)_{\mathbb{Q}}$$

called the *Chern character*, with the following properties:

- $\text{ch}(\cdot)$ is compatible with pull-back by flat morphisms;
- if Z is an integral closed subscheme of codimension one of X , then

$$\text{ch}(\mathcal{O}(Z)) = \exp([Z]) := 1 + [Z] + \frac{1}{2!}[Z] \cdot [Z] + \dots$$

There is also a unique map

$$\mathrm{Td} : K_0(X) \rightarrow \mathrm{CH}^\bullet(X)_{\mathbb{Q}}^*$$

called the *Todd class*, with the following properties:

- $\mathrm{Td}(\cdot)$ is compatible with pull-back by flat morphisms;
- $\mathrm{Td}(x + x') = \mathrm{Td}(x) \cdot \mathrm{Td}(x')$;
- if Z is an integral closed subscheme of codimension one of X , then

$$\mathrm{Td}(\mathcal{O}(Z)) = \frac{[Z]}{1 - \exp(-[Z])}.$$

Finally there is a unique map $c : K_0(X) \rightarrow \mathrm{CH}^\bullet(X)_{\mathbb{Q}}^*$, called the *total Chern class*, such that

- $c(\cdot)$ is compatible with pull-back by flat morphisms;
- $c(x + x') = c(x) \cdot c(x')$;
- if Z is an integral closed subscheme of codimension one of X , then $c(\mathcal{O}(Z)) = 1 + [Z]$.

The element $c_p(x) := c(x)^{[p]}(x)$ (where $(\cdot)^{[p]}$ takes the p -th graded part) is called the p -th Chern class of $x \in K_0(X)$. For a vector bundle E on X , we have

$$\begin{aligned} \mathrm{ch}(E) &= 1 + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) \\ &\quad + \frac{1}{6}(c_1(E)^3 - 3c_1(E) \cdot c_2(E) + 3c_3(E)) + \dots \end{aligned}$$

and

$$\mathrm{Td}(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1(E)^2 + c_2(E)) + \frac{1}{24}c_1(E) \cdot c_2(E) + \dots$$

We can now formulate the Grothendieck–Riemann–Roch theorem for smooth morphisms:

Theorem 2.4. *Let X, Y be regular schemes. Let $f : X \rightarrow Y$ be a smooth and strongly projective S -morphism. Then*

$$\mathrm{ch}(\mathbb{R}f_*(x)) = f_*(\mathrm{Td}(\mathrm{T}_f) \cdot \mathrm{ch}(x))$$

for any $x \in K_0(X)$.

Here the vector bundle $\mathrm{T}_f := \Omega_f^\vee$ is the dual of the sheaf of differentials of f . The vector bundle $\mathrm{T}_f := \Omega_f^\vee$ is also called the relative tangent bundle of f .

Example 2.5. Let $X := C$ be a smooth and projective curve of genus g over \mathbb{C} , as at the beginning of this section. Let $Y := \mathrm{Spec} \mathbb{C}$. Notice that $\mathrm{CH}^\bullet(Y) = \mathrm{CH}^0(Y) \simeq \mathbb{Z}$ and that the Chern character of a vector bundle over S is simply its rank under this identification. If we apply Theorem 2.4

to $E := \mathcal{O}(D)$, we obtain,

$$\begin{aligned} \text{ch}(\mathbb{R}f_*(\mathcal{O}(D))) &= \chi(\mathcal{O}(D)) = f_*\left(\left(1 + \frac{1}{2}c_1(\mathbb{T}_C)\right)(1 + c_1(\mathcal{O}(D)))\right) \\ &= f_*(c_1(\mathcal{O}(D)) - \frac{1}{2}c_1(\Omega_C)) = \deg(D) - \frac{1}{2}(2g - 2) \\ &= \deg(D) + 1 - g \end{aligned}$$

and thus we have recovered formula (2.1).

The smoothness assumption on f in Theorem 2.4 can be relaxed. Suppose that f is a strongly projective (but not necessarily smooth) S -morphism. Then by definition $f : X \rightarrow Y$ has a factorisation

$$f : X \xrightarrow{j} \mathbf{P}^r \times Y \xrightarrow{\pi} Y,$$

where j is a closed immersion and π is the natural projection. Since X and $\mathbf{P}^r \times Y \xrightarrow{\pi} Y$ are regular, the immersion j is regular and thus its conormal sheaf is locally free. Theorem 2.4 still holds as stated if one replaces $\text{Td}(\mathbb{T}_f)$ by $j^*(\text{Td}(\mathbb{T}_\pi)) \cdot \text{Td}(N)^{-1}$, where N is the normal bundle of the closed immersion j (which is locally free, being the dual of the conormal sheaf). The expression $(j^*\text{Td}(\mathbb{T}_\pi))\text{Td}(N)^{-1}$ can be shown to be independent of the factorisation of f into j and π . See [2, VIII, 2] for this. The fact that Theorem 2.4 holds in this generality is a fundamental insight of Grothendieck; it shows that the theorem can be proved by reduction to the case of immersions and to the case of the structural morphism of ordinary projective space.

The Riemann–Roch theorem for curves was discovered by B. Riemann and his student G. Roch in the middle of the nineteenth century. In the 1950s, F. Hirzebruch generalised the theorem to higher-dimensional manifolds (but not to a relative situation). See his book [36] for this, where more historical references are given and the genesis of the Todd class is also described. The general relative case was first treated in the seminar [2] (see also [11]). The presentation of the Grothendieck–Riemann–Roch theorem given here follows W. Fulton’s book [26, chap. 15].

3. Thomason’s fixed point formula

In this section, ‘scheme’ will be short for ‘noetherian and separated scheme’. We fix a (noetherian) base scheme S . ‘ S -scheme’ will be short for ‘ S -scheme of finite type over S ’.

We shall review a special case of Thomason’s fixed point formula [60, Th. 3.5].

In the next paragraph, we give a list of definitions and basic results. These can be found at the beginning of [60].

Let $\mu_n := \text{Spec } \mathbb{Z}[T]/(1 - T^n)$ be the diagonalisable group scheme over $\text{Spec } \mathbb{Z}$ which corresponds to the finite group $\mathbb{Z}/(n)$. It is the unique affine

group scheme such that for any affine scheme $\text{Spec } R$ we have

$$\mu_n(\text{Spec } R) = \{r \in R \mid r^n = 1\},$$

functorially in R .

In the following, since we shall always be working over the base scheme S , we shall use the short-hand μ_n for $\mu_{n,S} := \mu_n \times_{\text{Spec } \mathbb{Z}} S$.

Let X be an S -scheme. Suppose that X carries an action by $\mu_n (= \mu_{n,S})$ over S . We shall write $K_0^{\mu_n}(X) = K_0^{\mu_n}(X)$ for the Grothendieck group of coherent locally free sheaves on X which carry a μ_n -equivariant structure. The definition of this group is completely similar to the definition of the ordinary Grothendieck group of locally free sheaves (see Definition 2.1). Replacing locally free sheaves by coherent sheaves in the definition of $K_0^{\mu_n}(X)$ leads to the group $K_0^{\prime\mu_n}(X)$ and there is an obvious $K_0^{\mu_n}(X)$ -module structure on the group $K_0^{\prime\mu_n}(X)$. If X is regular, the natural morphism $K_0^{\mu_n}(X) \rightarrow K_0^{\prime\mu_n}(X)$ is an isomorphism (see [60, Lemme 3.3]). If the μ_n -equivariant structure of X is trivial, then the datum of a μ_n -equivariant structure on a locally free sheaf E on X is equivalent to the datum of a $\mathbb{Z}/(n)$ -grading of E . For any μ_n -equivariant locally free sheaf E on X , we write

$$\Lambda_{-1}(E) := \sum_{k=0}^{\text{rk}(E)} (-1)^k \Lambda^k(E) \in K_0^{\mu_n}(X),$$

where $\Lambda^k(E)$ is the k -th exterior power of E , endowed with its natural μ_n -equivariant structure.

The functor of fixed points associated with X is by definition the functor

$$\mathbf{Schemes}/S \rightarrow \mathbf{Sets}$$

described by the rule

$$T \mapsto X(T)_{\mu_n(T)}.$$

Here $X(T)_{\mu_n(T)}$ is the set of elements of $X(T)$ which are fixed under each element of $\mu_n(T)$. The functor of fixed points is representable by an S -scheme X_{μ_n} and the canonical morphism $X_{\mu_n} \rightarrow X$ is a closed immersion (see [1, VIII, 6.5 e]). Furthermore, if X is regular then the scheme X_{μ_n} is regular (see [60, Prop. 3.1]). We shall denote by i the immersion $X_{\mu_n} \hookrightarrow X$. If X is regular, we shall write N^\vee for the dual of the normal sheaf of the closed immersion i (this is also called the conormal sheaf of i). It is a locally free sheaf on X_{μ_n} and carries a natural μ_n -equivariant structure. This structure corresponds to a μ_n -grading, since X_{μ_n} carries the trivial μ_n -equivariant structure and it can be shown that the weight 0 term of this grading vanishes (see [60, Prop. 3.1]).

Let $f : X \rightarrow Y$ be an S -morphism between μ_n -equivariant schemes which respects the μ_n -actions. If f is proper then the morphism f induces a direct image map

$$\mathbb{R}f_* : K_0^{\prime\mu_n}(X) \rightarrow K_0^{\prime\mu_n}(Y),$$

which is a homomorphism of groups and is uniquely determined by the fact that

$$\mathbb{R}f_*(E) := \sum_{k \geq 0} (-1)^k \mathbb{R}^k f_*(E)$$

for any μ_n -equivariant coherent sheaf E on X . Here, as before, $\mathbb{R}^k f_*(E)$ refers to the k -th higher direct image sheaf of E under f ; the sheaves $\mathbb{R}^k f_*(E)$ are coherent and carry a natural μ_n -equivariant structure. If X and Y are regular, the direct image map $\mathbb{R}f_*$ induces a map $K_0^{\mu_n}(X) \rightarrow K_0^{\mu_n}(Y)$ that we shall also denote by the symbol $\mathbb{R}f_*$.

The morphism f also induces a pull-back map

$$f^* : K_0^{\mu_n}(Y) \rightarrow K_0^{\mu_n}(X);$$

this is a ring morphism which sends a μ_n -equivariant locally free sheaf E on Y to the class of the locally free sheaf $f^*(E)$ on X , endowed with its natural μ_n -equivariant structure. For any elements $z \in K_0^{\mu_n}(X)$ and $w \in K_0^{\mu_n}(Y)$, the projection formula

$$\mathbb{R}f_*(z \cdot f^*(w)) = w \cdot \mathbb{R}f_*(z)$$

holds (provided f is proper).

Fix $\zeta_n \in \mathbb{C}$ a primitive n -root of unity. Let $\mathcal{R} := \mathbb{Z}[T]/(1 - T^n)$. In the following theorem, we shall view $\mathbb{Q}(\mu_n)$ as an \mathcal{R} -algebra via the homomorphism sending T to ζ_n .

Theorem 3.1. *Let X, Y be schemes with μ_n -actions and let $f : X \rightarrow Y$ be an S -morphism compatible with the μ_n -actions. Suppose that X and Y are regular and that f is proper. Suppose that the μ_n -action on Y is trivial. Let $f^{\mu_n} : X_{\mu_n} \rightarrow Y$ be the natural morphism. Then*

- (1) *The element $\lambda_{-1}(N^\vee)$ is a unit in the ring $K_0^{\mu_n}(X_{\mu_n}) \otimes_{\mathcal{R}} \mathbb{Q}(\mu_n)$.*
- (2) *For any element $x \in K_0^{\mu_n}(X)$, the equality*

$$\mathbb{R}f_*(x) = \mathbb{R}f_*^{\mu_n}(\Lambda_{-1}(N^\vee)^{-1} \cdot i^*(x))$$

holds in $K_0^{\mu_n}(Y) \otimes_{\mathcal{R}} \mathbb{Q}(\mu_n)$.

Here $K_0^{\mu_n}(X_{\mu_n})$ is viewed as an \mathcal{R} -algebra via the morphism of rings $\mathcal{R} \rightarrow K_0^{\mu_n}(X_{\mu_n})$ sending T to the structure sheaf $\mathcal{O}_{X_{\mu_n}}$ of X_{μ_n} , endowed with the $\mathbb{Z}/(n)$ -grading of weight 1. Similarly, $K_0^{\mu_n}(Y)$ is viewed as an \mathcal{R} -algebra via the morphism of rings $\mathcal{R} \rightarrow K_0^{\mu_n}(Y)$ sending T to the structure sheaf \mathcal{O}_Y of Y , endowed with the $\mathbb{Z}/(n)$ -grading of weight 1.

Notice the formal analogy between Theorem 3.1 and Theorem 2.4: i^* takes the place of the Chern character and $\Lambda_{-1}(N^\vee)^{-1}$ takes the place of the Todd class.

Example 3.2. Let X and Y be as in Theorem 3.1, $S = Y = \text{Spec } \mathbb{C}$ and suppose that X_{μ_n} is finite over Y . Let $g : X \rightarrow X$ be the automorphism corresponding to $\zeta_n \in \mu_n(\mathbb{C})$. Note that, in this case, we have $K_0^{\mu_n}(Y) \simeq \mathcal{R}$

via the algebra structure described right after Theorem 3.1 so that there is an isomorphism

$$I : K_0^{\mu_n}(Y) \otimes_{\mathcal{R}} \mathbb{Q}(\mu_n) \simeq \mathbb{Q}(\mu_n).$$

We leave it to the reader to verify that if V is a μ_n -equivariant vector space over \mathbb{C} , then $I(V \otimes 1) = \text{Trace}(g_* : V \rightarrow V)$. If $x = E$ is a μ_n -equivariant vector bundle on X , Theorem 3.1 gives the equality

$$\begin{aligned} & \sum_{k \geq 0} (-1)^k \text{Trace}(g_* : H^k(X, E) \rightarrow H^k(X, E)) \\ &= \sum_{P \in X_{\mu_n}(\mathbb{C})} \frac{\text{Trace}(E_P)}{\sum_{t=0}^{\text{rk}(\Omega_{X,P})} (-1)^t \text{Trace}(g_* : \Lambda^t(\Omega_{X,P}) \rightarrow \Lambda^t(\Omega_{X,P}))}. \end{aligned}$$

It is an exercise of linear algebra to show that

$$\begin{aligned} & \sum_{t=0}^{\text{rk}(\Omega_{X,P})} (-1)^t \text{Trace}(g_* : \Lambda^t(\Omega_{X,P}) \rightarrow \Lambda^t(\Omega_{X,P})) \\ &= \det(\text{Id} - g_* : \Omega_{X,P} \rightarrow \Omega_{X,P}) \end{aligned}$$

so that

$$\begin{aligned} & \sum_{k \geq 0} (-1)^k \text{Trace}(g_* : H^k(X, E) \rightarrow H^k(X, E)) \\ &= \sum_{P \in X_{\mu_n}(\mathbb{C})} \frac{\text{Trace}(E_P)}{\det(\text{Id} - g_* : \Omega_{X,P} \rightarrow \Omega_{X,P})} \end{aligned}$$

This formula is a special case of the so-called ‘Woods Hole’ fixed point formula (see [18, Letter 2-3 August 1964]).

4. An equivariant extension of the Grothendieck–Riemann–Roch theorem

The assumptions made at the beginning of the last section are still in force.

If we formally combine the Grothendieck–Riemann–Roch theorem and the fixed point theorem of Thomason, we obtain the following theorem. Fix a primitive root of unity $\zeta_n \in \mathbb{C}$.

Theorem 4.1. *Let X and Y be regular S -schemes. Suppose that X and Y are equipped with a μ_n -action. Suppose also that the μ_n -structure of Y is trivial. Let $f : X \rightarrow Y$ be a μ_n -equivariant proper morphism and suppose that f^{μ_n} is smooth and strongly projective. Then for any $x \in K_0^{\mu_n}(X)$, the formula*

$$\text{ch}_{\mu_n}(\mathbb{R}f_*(x)) = f_*(\text{ch}_{\mu_n}(\Lambda_{-1}(N^\vee))^{-1} \text{Td}(\text{T}_{f^{\mu_n}}) \text{ch}_{\mu_n}(x))$$

holds in $\text{CH}^\bullet(Y)_{\mathbb{Q}(\mu_n)}$.

See after Example 2.5 for the definition of ‘strongly projective’ (this is what Hartshorne calls ‘projective’ in his book [35]). Here again, N refers to the normal bundle of the immersion $X_{\mu_n} \rightarrow X$. If E is a μ_n -equivariant

vector bundle on X , writing E_k for the k -th graded piece of the restriction of E to X_{μ_n} , we define

$$\mathrm{ch}_{\mu_n}(E) := \sum_{k \in \mathbb{Z}/n} \zeta_n^k \cdot \mathrm{ch}(E_k) \in \mathrm{CH}^\bullet(X_{\mu_n})_{\mathbb{Q}(\mu_n)}.$$

The element $\mathrm{ch}_{\mu_n}(E)$ is called the *equivariant Chern character* of E .

Example 4.2 (The generalised Gauss–Bonnet formula). Suppose that the assumptions of Theorem 4.1 hold and that, in addition, f is smooth. We shall apply Theorem 4.1 to the image in $K_0^{\mu_n}(X)$ of the relative de Rham complex of f , i.e. to the element

$$\Lambda_{-1}(\Omega_f) = 1 - \Omega_f + \Lambda^2(\Omega_f) - \Lambda^3(\Omega_f) + \cdots + (-1)^{\mathrm{rk}(\Omega_f)} \Lambda^{\mathrm{rk}(\Omega_f)}(\Omega_f).$$

Recall that we have the exact sequence

$$0 \rightarrow N^\vee \rightarrow \Omega_f|_{X_{\mu_n}} \rightarrow \Omega_{f\mu_n} \rightarrow 0$$

on X_{μ_n} . One can show that the symbol $\Lambda_{-1}(\cdot)$ is multiplicative on short exact sequences of vector bundles (exercise! Use the splitting principle). In particular, we have

$$\Lambda_{-1}(N^\vee) \cdot \Lambda_{-1}(\Omega_{f\mu_n}) = \Lambda_{-1}(\Omega_f|_{X_{\mu_n}})$$

in $K_0^{\mu_n}(X_{\mu_n})$. A last point is that for any vector bundle E , there is an identity of characteristic classes

$$\mathrm{ch}(\Lambda_{-1}(E))\mathrm{Td}(E^\vee) = c_{\mathrm{top}}(E^\vee).$$

This identity is called the Borel–Serre identity, see [26, Example 3.2.4, 3.2.5] for a proof. With a view to simplifying the right hand side of the equality in Theorem 4.1, we now compute

$$\begin{aligned} \mathrm{ch}_{\mu_n}(\Lambda_{-1}(N^\vee))^{-1} \mathrm{Td}(\mathrm{T}_{f\mu_n}) \mathrm{ch}_{\mu_n}(\Lambda_{-1}(\Omega_f|_{X_{\mu_n}})) &= \mathrm{ch}(\Lambda_{-1}(\Omega_{f\mu_n})) \mathrm{Td}(\Omega_{f\mu_n}^\vee) \\ &= c_{\mathrm{top}}(\mathrm{T}_{f\mu_n}). \end{aligned}$$

Thus, by Theorem 4.1 we have

$$\sum_{p,q \geq 0} (-1)^{p+q} \mathrm{ch}_{\mu_n}(\mathbb{R}^p f(\Lambda^q(\Omega_f))) = f_*^{\mu_n}(c_{\mathrm{top}}(\mathrm{T}_{f\mu_n})). \quad (4.1)$$

Formula (4.1) is an equivariant extension of the Gauss–Bonnet formula (see e.g. [62, chap. III, ex. 3.8] for the non-equivariant formula in a cohomological setting) and we shall see further below that the lifting of formula (4.1) to Arakelov theory carries deep arithmetic information.

Suppose that $S = Y = \mathrm{Spec} \mathbb{C}$. Let $g : X \rightarrow X$ be the automorphism corresponding to a primitive n -th root of unity $\zeta_n \in \mu_n(\mathbb{C})$, as in the example given in the previous section. Formula (4.1) together with the existence of

the Hodge decomposition gives the identities

$$\begin{aligned} & \sum_{i,j \geq 0} (-1)^{i+j} \operatorname{Tr}(g_* : H^i(X, \Omega_X^j) \rightarrow H^i(X, \Omega_X^j)) \\ &= \sum_{k \geq 0} (-1)^k \operatorname{Tr}(g_* : H^k(X(\mathbb{C}), \mathbb{C}) \rightarrow H^k(X(\mathbb{C}), \mathbb{C})) = \int_{X_{\mu_n}} c_{\text{top}}(\mathbb{T}_{f^{\mu_n}}) \end{aligned}$$

where $H^k(X(\mathbb{C}), \mathbb{C})$ is the k -th singular cohomology group of $X(\mathbb{C})$ with coefficients in \mathbb{C} . In particular, if X_{μ_n} consists of a finite set of points, we have

$$\sum_{k \geq 0} (-1)^k \operatorname{Tr}(g_* : H^k(X(\mathbb{C}), \mathbb{C}) \rightarrow H^k(X(\mathbb{C}), \mathbb{C})) = \#X_{\mu_n}(\mathbb{C}). \quad (4.2)$$

Formula (4.2) is just the classical topological Lefschetz fixed point formula applied to $X(\mathbb{C})$ and the endomorphism g .

5. An equivariant Riemann–Roch theorem in Arakelov geometry

5.1. Arakelov geometry

Arakelov geometry is an extension of scheme-theoretic algebraic geometry, where one tries to treat the places at infinity of a number field (corresponding to the archimedean valuations) on the same footing as the finite ones. To be more precise, consider a scheme S which is proper over $\operatorname{Spec} \mathbb{Z}$ and generically smooth. For each prime $p \in \operatorname{Spec} \mathbb{Z}$, we then obtain by base-change a scheme $S_{\mathbb{Z}_p}$ on the spectrum of the ring of p -adic integers \mathbb{Z}_p . The set $S(\mathbb{Q}_p)$ is then endowed with the following natural notion of distance. Let $P, R \in S(\mathbb{Q}_p)$; by the valuative criterion of properness, we can uniquely extend P and R to elements \tilde{P}, \tilde{R} of $S(\mathbb{Z}_p)$. We can then define a distance $d(P, R)$ by the formula

$$d(P, R) := p^{-\sup\{k \in \mathbb{Z} \mid \tilde{P} \pmod{p^k} = \tilde{R} \pmod{p^k}\}}.$$

This distance arises naturally from the scheme structure of S . No such distance is available for the set $S(\mathbb{C})$ and the strategy of Arakelov geometry is to equip $S(\mathbb{C})$, as well as the vector bundles thereon with a hermitian metric in order to make up for that lack. The scheme S together with a metric on $S(\mathbb{C})$ is then understood as a ‘compactification’ of S , in the sense that it is supposed to live on the ‘compactification’ of $\operatorname{Spec} \mathbb{Z}$ obtained by formally adding the archimedean valuation. The introduction of hermitian metrics, which are purely analytic data, implies that Arakelov geometry will rely on a lot of analysis to define direct images, intersection numbers, Chern classes etc. Here is the beginning of a list of extensions of classical scheme-theoretic objects that have been worked out in the literature:

S	S with a Kähler metric on $S(\mathbb{C})$
E a vector bundle on S	E a vector bundle on S with a hermitian metric on $E(\mathbb{C})$
cycle Z on S	a cycle Z on S with a Green current for $Z(\mathbb{C})$
the degree of a variety	the height of a variety over a number field
the determinant of cohomology	the determinant of cohomology equipped with its Quillen metric
the Todd class of f	the arithmetic Todd class of f multiplied by $(1-R(T_f))$
\vdots	\vdots

Here f is the morphism $S \rightarrow \text{Spec } \mathbb{Z}$.

Many theorems of classical algebraic geometry have been extended to Arakelov theory. In particular, there are analogs of the Hilbert–Samuel theorem (see [30] and [3]), of the Nakai–Moishezon criterion for ampleness (see [64]), of the Grothendieck–Riemann–Roch theorem (see [30]) and finally there is an analog of the equivariant Grothendieck–Riemann–Roch theorem, whose description is one of the main aims of this text.

Arakelov geometry started officially in S. Arakelov’s paper [4], who developed an intersection theory for surfaces in the compactified setting. G. Faltings (see [21]) then proved a Riemann–Roch theorem in the framework of Arakelov’s theory. After that L. Szpiro and his students proved many other results in the Arakelov theory of surfaces. See [57] and also Lang’s book [40] for this. The theory was then vastly generalised by H. Gillet and C. Soulé, who defined compactified Chow rings, Grothendieck groups and characteristic classes in all dimensions (see [28] and [29]). For an introduction to Arakelov geometry, see the book [55].

5.2. The equivariant Riemann–Roch theorem

The aim of this section is to formulate the analog in Arakelov geometry of Theorem 4.1. With the exception of the relative equivariant analytic torsion form, we shall define precisely all the objects that we need but the presentation will be very compact and this section should not be used as a partial introduction to higher-dimensional Arakelov theory. For this, we recommend reading the first few chapters of the book [55].

Let D be a regular arithmetic ring. By this we mean a regular, excellent, noetherian domain, together with a finite set \mathcal{S} of injective ring homomorphisms $D \rightarrow \mathbb{C}$, which is invariant under complex conjugation. We fix a primitive root of unity $\zeta_n \in \mathbb{C}$. In this subsection, we shall use the short-hand μ_n for $\mu_{n,D}$.

Let $n \geq 1$. We shall call *equivariant arithmetic variety* a scheme X of finite type over $\text{Spec } D$, endowed with a μ_n -equivariant structure over D and

such that there is an ample μ_n -equivariant line bundle on X . We also require the fibre of X over the generic point of D to be smooth.

We shall denote by $X(\mathbb{C})$ the set of complex points of the variety $\coprod_{\sigma \in \mathcal{S}} X \times_{D, \sigma} \mathbb{C}$, which naturally carries the structure of a complex manifold. The group $\mu_n(\mathbb{C})$ acts on $X(\mathbb{C})$ by holomorphic automorphisms and we shall write g for the automorphism corresponding to ζ_n . As we have seen in Section 3, the fixed point scheme X_{μ_n} is regular and there are natural isomorphisms of complex manifolds $X_{\mu_n}(\mathbb{C}) \simeq (X(\mathbb{C}))_g$, where $(X(\mathbb{C}))_g$ is the set of fixed points of $X(\mathbb{C})$ under the action of $\mu_n(\mathbb{C})$. Complex conjugation induces an antiholomorphic automorphism of $X(\mathbb{C})$ and $X_{\mu_n}(\mathbb{C})$, both of which we denote by F_∞ .

If M is a complex manifold, we shall write $\mathfrak{A}^{p,q}(M)$ for the \mathbb{C} -vector space of smooth complex differential forms ω of type (p, q) on M and $\mathfrak{A}^k(M)$ for the set of smooth complex differential forms ω of degree k . Recall that we have a natural direct sum decomposition

$$\mathfrak{A}^k(M) = \bigoplus_{p+q=k} \mathfrak{A}^{p,q}(M).$$

The differential operators ∂ and $\bar{\partial}$ induce endomorphisms of the vector space $\mathfrak{A}^{\bullet,\bullet}(M)$, making it into a double complex

$$(\mathfrak{A}^{\bullet,\bullet}(M); \partial, \bar{\partial}). \quad (5.1)$$

Here are some cohomology spaces associated with this double complex. Each of them inherits a grading or a bigrading from the bigrading of the complex (5.1). We write $d := \partial + \bar{\partial}$ as usual.

- de Rham cohomology

$$H_{\text{dR}}^{\bullet}(M) := \frac{\ker d}{\text{im } d};$$

- ∂ -cohomology and Dolbeault cohomology (also called Hodge cohomology)

$$H_{\partial}^{\bullet,\bullet}(M) := \frac{\ker \partial}{\text{im } \partial}, \quad H_{\bar{\partial}}^{\bullet,\bullet}(X) := \frac{\ker \bar{\partial}}{\text{im } \bar{\partial}};$$

- Bott–Chern cohomology and Aepli cohomology

$$H_{\text{BC}}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im } \partial \bar{\partial}}, \quad H_{\text{Aep}}^{\bullet,\bullet}(X) := \frac{\ker \partial \bar{\partial}}{\text{im } \partial + \text{im } \bar{\partial}}.$$

It can be checked that the wedge product of differential forms induces a bigraded \mathbb{C} -algebra structure on each of these cohomology spaces. Furthermore, from the definition we see that there are natural maps of \mathbb{C} -vector

spaces between them, as follows:

$$\begin{array}{ccccc}
 & & H_{\text{BC}}^{\bullet,\bullet}(M) & & \\
 & \swarrow & \downarrow & \searrow & \\
 H_{\partial}^{\bullet,\bullet}(M) & & H_{\text{dR}}^{\bullet,\bullet}(M) & & H_{\bar{\partial}}^{\bullet,\bullet}(M) \\
 & \searrow & \downarrow & \swarrow & \\
 & & H_{\text{Aep}}^{\bullet,\bullet}(M) & &
 \end{array} \tag{5.2}$$

See e.g. [63, par. 2] for more details about all this.

Now we define

$$\tilde{\mathfrak{A}}^p(M) := \mathfrak{A}^{p,p}(M)/(\text{Im } \partial + \text{Im } \bar{\partial})$$

so that there is a natural inclusion

$$H_{\text{Aep}}^{p,p}(M) \subseteq \tilde{\mathfrak{A}}^p(M).$$

In this text, as in the foundational articles on higher-dimensional Arakelov theory, we shall use the short-hand $H^{p,p}(M)$ for $H_{\text{Aep}}^{p,p}(M)$.

Going back to arithmetic varieties, we now write $\mathfrak{A}^{p,p}(X_{\mu_n})$ for the subspace of $\mathfrak{A}^{p,p}(X_{\mu_n}(\mathbb{C}))$ consisting of smooth complex differential forms ω of type (p,p) , such that $F_{\infty}^* \omega = (-1)^p \omega$ and

$$\tilde{\mathfrak{A}}^p(X_{\mu_n}) := \mathfrak{A}^{p,p}(X_{\mu_n})/(\text{Im } \partial + \text{Im } \bar{\partial}).$$

Finally we shall write $H^{p,p}(X_{\mu_n})$ for the kernel of $\bar{\partial}\partial$ acting on $\tilde{\mathfrak{A}}^p(X_{\mu_n})$. Note that $H^{p,p}(X_{\mu_n})$ (resp. $\tilde{\mathfrak{A}}^p(X_{\mu_n})$) is a subspace of $H^{p,p}(X_{\mu_n}(\mathbb{C}))$ (resp. $\tilde{\mathfrak{A}}^p(X_{\mu_n}(\mathbb{C}))$).

A *hermitian equivariant sheaf* (resp. *vector bundle*) on X is a coherent sheaf (resp. a vector bundle) E on X , assumed locally free on $X(\mathbb{C})$, equipped with a μ_n -action which lifts the action of μ_n on X and a hermitian metric h on the vector bundle $E(\mathbb{C})$, which is invariant under F_{∞} and μ_n . We shall write (E, h) or \bar{E} for an hermitian equivariant sheaf (resp. vector bundle). There is a natural $\mathbb{Z}/(n)$ -grading $E|_{X_{\mu_n}} \simeq \bigoplus_{k \in \mathbb{Z}/(n)} E_k$ on the restriction of E to X_{μ_n} , whose terms are orthogonal, because of the assumed g -invariance of the metric. For $k \in \mathbb{Z}/(n)$, we write \bar{E}_k for E_k endowed with the induced metric. We also often write \bar{E}_{μ_n} for \bar{E}_0 .

If $\bar{V} = (V, h_V)$ is a hermitian vector bundle on X_{μ_n} we write $\text{ch}(\bar{V})$ for the differential form $\text{Tr}(\exp(-\frac{1}{2i\pi}\Omega_{h_V}))$. Here Ω_{h_V} is the curvature form associated with the unique connection on $V(\mathbb{C})$ whose matrix is given locally by $\partial H \cdot H^{-1}$, where H is the matrix of functions representing h_V in a local holomorphic frame. The differential form $\text{ch}(\bar{V})$ is both ∂ - and $\bar{\partial}$ -closed and its class in Bott–Chern cohomology represents the Chern character of $V(\mathbb{C})$ in the Bott–Chern cohomology of $X_{\mu_n}(\mathbb{C})$. Recall also that there is a natural map from Bott–Chern cohomology to Aeppli cohomology (see (5.2)) so that

$\text{ch}(\bar{V})$ may also be viewed as a differential form representative for the Chern character of $V(\mathbb{C})$ in the Aeppli cohomology of $X_{\mu_n}(\mathbb{C})$. From the differential form $\text{ch}(\bar{V})$, using the fundamental theorem on symmetric functions, we may define differential form representatives in Bott–Chern cohomology of other linear combinations of Chern classes, like the Todd class $\text{Td}(\bar{V})$ or the total Chern class $c(\bar{V})$.

If (E, h) is a hermitian equivariant sheaf, we write $\text{ch}_g(\bar{E})$ for the equivariant Chern character form

$$\text{ch}_g(\bar{E}) := \text{ch}_g((E(\mathbb{C}), h)) := \sum_{k \in \mathbb{Z}/(n)} \zeta_n^k \text{ch}(\bar{E}_k).$$

The symbol $\text{Td}_g(\bar{E})$ refers to the differential form

$$\text{Td}(\bar{E}_{\mu_n}) \left(\sum_{i \geq 0} (-1)^i \text{ch}_g(\Lambda^i(\bar{E})) \right)^{-1}.$$

If

$$\mathcal{E} : 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

is an exact sequence of equivariant sheaves (resp. vector bundles), we shall write $\bar{\mathcal{E}}$ for the sequence \mathcal{E} together with a datum of $\mu_n(\mathbb{C})$ - and F_∞ -invariant hermitian metrics on $E'(\mathbb{C})$, $E(\mathbb{C})$ and $E''(\mathbb{C})$. With $\bar{\mathcal{E}}$ and ch_g is associated an *equivariant Bott–Chern secondary class* $\tilde{\text{ch}}_g(\bar{\mathcal{E}}) \in \tilde{\mathfrak{A}}^\bullet(X_{\mu_n})$, which satisfies the equation

$$\frac{i}{2\pi} \bar{\partial} \partial (\tilde{\text{ch}}_g(\bar{\mathcal{E}})) = \text{ch}_g(\bar{E}') + \text{ch}_g(\bar{E}'') - \text{ch}_g(\bar{E}).$$

This class is functorial for any morphism of arithmetic varieties and vanishes if the sequence $\bar{\mathcal{E}}$ splits isometrically. See [37, par. 3.3] for all this.

Definition 5.1. The arithmetic equivariant Grothendieck group $\widehat{K}_0^{\prime\mu_n}(X)$ (resp. $\widehat{K}_0^{\mu_n}(X)$) of X is the free abelian group generated by the elements of $\tilde{\mathfrak{A}}^\bullet(X_{\mu_n})$ and by the equivariant isometry classes of hermitian equivariant sheaves (resp. vector bundles), together with the relations

- (a) for every exact sequence $\bar{\mathcal{E}}$ as above, we have $\tilde{\text{ch}}_g(\bar{\mathcal{E}}) = \bar{E}' - \bar{E} + \bar{E}''$;
- (b) if $\eta \in \tilde{\mathfrak{A}}^\bullet(X_{\mu_n})$ is the sum in $\tilde{\mathfrak{A}}^\bullet(X_{\mu_n})$ of two elements η' and η'' , then $\eta = \eta' + \eta''$ in $\widehat{K}_0^{\prime\mu_n}(X)$ (resp. $\widehat{K}_0^{\mu_n}(X)$).

We shall now define a ring structure on $\widehat{K}_0^{\mu_n}(X)$. Let \bar{V}, \bar{V}' be hermitian equivariant vector bundles. Let η, η' be elements of $\tilde{\mathfrak{A}}^\bullet(X_{\mu_n})$. We define a product \cdot on $\widehat{K}_0^{\mu_n}(X)$ by the rules

$$\begin{aligned} \bar{V} \cdot \bar{V}' &:= \bar{V} \otimes \bar{V}', \\ \bar{V} \cdot \eta &= \eta \cdot \bar{V} := \text{ch}_g(\bar{V}) \wedge \eta \end{aligned}$$

and

$$\eta \cdot \eta' := \frac{i}{2\pi} \bar{\partial} \partial \eta \wedge \eta'$$

and we extend it by linearity. We omit the proof that it is well-defined (see [37, par. 4] for this). Notice that the definition of $\widehat{K}_0^{\prime\mu_n}(X)$ (resp. $\widehat{K}_0^{\mu_n}(X)$) implies that there is an exact sequence of abelian groups

$$\widetilde{\mathfrak{A}}^\bullet(X_{\mu_n}) \rightarrow \widehat{K}_0^{\prime\mu_n}(X) \rightarrow K_0^{\prime\mu_n}(X) \rightarrow 0 \quad (5.3)$$

(resp.

$$\widetilde{\mathfrak{A}}^\bullet(X_{\mu_n}) \rightarrow \widehat{K}_0^{\mu_n}(X) \rightarrow K_0^{\mu_n}(X) \rightarrow 0 \quad),$$

where $K_0^{\prime\mu_n}(X)$ (resp. $K_0^{\mu_n}(X)$) is the Grothendieck group of μ_n -equivariant coherent sheaves (resp. locally free sheaves) considered in Section 3. Notice finally that there is a map from $\widehat{K}_0^{\prime\mu_n}(X)$ to the space of complex closed differential forms, which is defined by the formula

$$\mathrm{ch}_g(\overline{E} + \kappa) := \mathrm{ch}_g(\overline{E}) + \frac{i}{2\pi} \bar{\partial} \partial \kappa$$

(where \overline{E} is an hermitian equivariant sheaf and $\kappa \in \widetilde{\mathfrak{A}}^\bullet(X_{\mu_n})$). This map is well-defined and we shall denote it by $\mathrm{ch}_g(\cdot)$ as well. We have as before: if X is regular then the natural morphism $\widehat{K}_0^{\mu_n}(X) \rightarrow \widehat{K}_0^{\prime\mu_n}(X)$ is an isomorphism. See [37, Prop. 4.2] for this.

Now let $f : X \rightarrow Y$ be an equivariant projective morphism of relative dimension d over D of equivariant regular arithmetic varieties. We suppose that f is smooth over the generic point of D . We suppose that $X(\mathbb{C})$ is endowed with a Kähler fibration structure with respect to $f(\mathbb{C})$. This structure is encoded in a real closed $(1, 1)$ -form ω_f on $X(\mathbb{C})$, with the property that the restriction of ω_f to each fibre of $f(\mathbb{C})$ is a Kähler form on that fibre (see [8, par. 1] for details). In particular, the datum of ω_f induces a hermitian metric on the relative tangent bundle $T_f(\mathbb{C})$. We shall see an example of such a structure in the applications. We suppose that ω_f is g -invariant. Suppose also that the action of μ_n on Y is trivial and finally suppose that there is a μ_n -equivariant line bundle over X , which is very ample relatively to f .

Let now $\overline{E} := (E, h)$ be an equivariant hermitian sheaf on X and suppose that $\mathbb{R}^k f_*(E)_{\mathbb{C}}$ is locally free for all $k \geq 0$. Let $\eta \in \widetilde{\mathfrak{A}}^\bullet(X_{\mu_n})$.

We let $\mathbb{R}^\bullet f_* \overline{E} := \sum_{k \geq 0} (-1)^k R^k f_* \overline{E}$ be the alternating sum of the higher direct image sheaves, endowed with their natural equivariant structures and L_2 -metrics. For each $y \in Y(\mathbb{C})$, the L^2 -metric on

$$\mathbb{R}^i f_* E(\mathbb{C})_y \simeq H_{\bar{\partial}}^i(X(\mathbb{C})_y, E(\mathbb{C})|_{X(\mathbb{C})_y})$$

is defined by the formula

$$\frac{1}{(2\pi)^d} \int_{X(\mathbb{C})_y} h(s, t) \omega_f^d \quad (5.4)$$

where s and t are harmonic sections (i.e. in the kernel of the Kodaira Laplacian $\bar{\partial} \partial^* + \bar{\partial}^* \bar{\partial}$) of $\Lambda^i(T^{*(0,1)} X(\mathbb{C})_y) \otimes E(\mathbb{C})|_{X(\mathbb{C})_y}$. This definition is meaningful because, by Hodge theory, there is exactly one harmonic representative in each cohomology class.

Consider the rule, which associates the element $\mathbb{R}f_*^\bullet \bar{E} - T_g(X, \bar{E})$ of $\widehat{K}_0^{\prime\mu_n}(Y)$ to \bar{E} and the element $\int_{X(\mathbb{C})_g} \text{Td}_g(\overline{T_f}(\mathbb{C}))\eta$ of $\widehat{K}_0^{\prime\mu_n}(Y)$ to η . Here $T_g(\bar{E}) \in \widetilde{\mathfrak{A}}(Y)$ is the equivariant analytic torsion form defined at the beginning of [9]. Its definition is too involved to be given in its entirety here but we shall define below its component of degree 0.

Proposition 5.2. *The above rule extends to a well-defined group homomorphism*

$$f_* : \widehat{K}_0^{\prime\mu_n}(X) \rightarrow \widehat{K}_0^{\prime\mu_n}(Y).$$

Proof. See [58, Th. 6.2]. □

Now for the definition of the component of degree 0 of $T_g(X, \bar{E})$. Let

$$\square_q^E := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

be the Kodaira Laplacian, which acts on the C^∞ -sections of the C^∞ -vector bundle $\Lambda^q T^{*(0,1)}X(\mathbb{C})_y \otimes E(\mathbb{C})|_{X(\mathbb{C})_y}$ on $X(\mathbb{C})_y$. This space of sections is equipped with the L^2 -metric as above and the operator $\square_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}$ is symmetric for that metric; we let $\text{Sp}(\square_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}) \subseteq \mathbb{R}$ be the set of eigenvalues of $\square_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}$. The set $\text{Sp}(\square_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}})$ is a discrete subset of $\mathbb{R}_{\geq 0}$ and it grows according to the Weyl law, see [6, chap. 2, Prop. 2.36]. We let $\text{Eig}_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}(\lambda)$ be the eigenspace associated with an eigenvalue λ (which is finite-dimensional, see [6, chap. 2, Prop. 2.36]). For $s \in \mathbb{C}$ with $\Re(s)$ sufficiently large, we define

$$Z(\bar{E}|_{X(\mathbb{C})_y}, g, s) := \sum_{q \geq 1} (-1)^{q+1} q \sum_{\lambda \in \text{Sp}(\square_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}) \setminus \{0\}} \text{Tr}(g^*|_{\text{Eig}_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}(\lambda)}) \lambda^{-s}.$$

As a function of s , the function $Z(\bar{E}|_{X(\mathbb{C})_y}, g, s)$ has a meromorphic continuation to the whole complex plane, which is holomorphic around 0. This is a byproduct of the theory of heat kernels. The degree 0-part of the equivariant analytic torsion form $T_g(\bar{E})$ is then the complex number $Z'(\bar{E}|_{X(\mathbb{C})_y}, g, 0)$. If $\mathbb{R}^k f_*(E)_\mathbb{C}$ is locally free for all $k \geq 0$ (which is our assumption) then it can be shown that $Z'(\bar{E}|_{X(\mathbb{C})_y}, g, 0)$ is a C^∞ -function of y .

We shall need the definition (due to Gillet and Soulé) of compactified' Chow theory. Let X be a regular arithmetic variety over D . Let $p \geq 0$. We shall write $D^{p,p}(X)$ for the space of complex currents of type (p, p) on $X(\mathbb{C})$ on which F_∞^* acts by multiplication by $(-1)^p$. Now let A be a subring of \mathbb{C} and suppose that $\mathbb{Q} \subseteq A$. If Z is a cycle of codimension p with coefficients in A on X (in other words, a formal linear combination of integral closed subschemes of codimension p with coefficients in A), a *Green current* g_Z for Z is an element of $D^{p,p}(X)$, which satisfies the equation

$$\frac{i}{2\pi} \partial \bar{\partial} g_Z + \delta_{Z(\mathbb{C})} = \omega_Z$$

where ω_Z is a differential form and $\delta_{Z(\mathbb{C})}$ is the Dirac current associated with $Z(\mathbb{C})$. See the beginning of [28] for this.

Definition 5.3. Let $p \geq 0$. The arithmetic Chow group $\widehat{\text{CH}}_A^p(X)$ is the A -module generated by the ordered pairs (Z, g_Z) , where Z is a cycle of codimension p with coefficients in A on X and g_Z is a Green current for Z , with the relations

- (a) $\lambda \cdot (Z, g_Z) + (Z', g_{Z'}) = (\lambda \cdot Z + Z', \lambda \cdot g_Z + g_{Z'})$ for any $\lambda \in A$;
- (b) $(\text{div}(f), -\log |f|^2 + \partial u + \bar{\partial} v) = 0$;

where f is a non-zero rational function defined on a closed integral subscheme of codimension $p - 1$ of X and u (resp. v) is a complex current of type $(p - 2, p - 1)$ (resp. $(p - 1, p - 2)$) such that $F_\infty^*(\partial u + \bar{\partial} v) = (-1)^{p-1}(\partial u + \bar{\partial} v)$.

We shall write $\widehat{\text{CH}}_A^\bullet(X) := \bigoplus_{p \geq 0} \widehat{\text{CH}}_A^p(X)$.

Remark 5.4. The arithmetic Chow group with coefficients in A defined in Definition 5.3 is a formal variant of the arithmetic Chow group introduced by Gillet and Soulé in [28]. In [31] they also consider a group, which is similar to ours in the case $A = \mathbb{R}$ (but not identical). The properties of the arithmetic Chow group with coefficients in A listed below are similar to the properties of the arithmetic Chow group introduced in [28] (with the same proofs going through verbatim) and we shall always refer to [28] for properties of our group, although strictly speaking a different group is treated there.

The group $\widehat{\text{CH}}_A^\bullet(X)$ is equipped with a natural A -algebra structure, such that

$$(Z, g_Z) \cdot (Z', g_{Z'}) = (Z \cap Z', g_Z * g_{Z'})$$

if Z, Z' are integral and intersect transversally. Here the symbol $*$ refers to the star product, whose definition is too involved to be given here. See [28, par. 2.1] for this. A special case of the star product is described in the next example. If $Z(\mathbb{C})$ and $Z'(\mathbb{C})$ intersect transversally then one has the formula

$$g_Z * g_{Z'} = g_Z \wedge \delta_{Z'} + \left[\frac{i}{2\pi} \bar{\partial} \partial (g_Z) + \delta_{Z(\mathbb{C})} \right] \wedge g_{Z'}.$$

Here the wedge product can be defined because the wave-front sets of the involved currents are disjoint. If $f : X \rightarrow Y$ is a projective and generically smooth morphism over D between regular arithmetic varieties, there is a push-forward map

$$f_* : \widehat{\text{CH}}_A^\bullet(X) \rightarrow \widehat{\text{CH}}_A^\bullet(Y),$$

such that

$$f_*(Z, g_Z) = (\text{deg}(Z/f(Z)) \cdot f(Z), f(\mathbb{C})_*(g_Z))$$

for every integral closed subscheme Z of X and Green current g_Z of Z . Here we set $\text{deg}(Z/f(Z)) = [\kappa(Z) : \kappa(f(Z))]$ if $\dim(f(Z)) = \dim(Z)$ and $\text{deg}(Z/f(Z)) = 0$ otherwise. The expression $f(\mathbb{C})_*(g_Z)$ refers to the push-forward of currents. See [28, par. 3.6] for more details. For any morphism $f : X \rightarrow Y$ over D between regular arithmetic varieties, there is a pull-back

map $f^* : \widehat{\text{CH}}_A^\bullet(Y) \rightarrow \widehat{\text{CH}}_A^\bullet(X)$, whose definition presents the same difficulties as the definition of the ring structure on $\widehat{\text{CH}}_A^\bullet(\cdot)$. See [28, par. 4.4] for details.

It is an easy exercise to show that the map of complex vector spaces $\mathbb{C} \rightarrow \widehat{\text{CH}}_{\mathbb{C}}^1(\mathbb{Z})$, defined by the recipe $z \mapsto (0, z)$ is an isomorphism.

If X is a regular arithmetic variety, there is a unique ring morphism

$$\widehat{\text{ch}} : \widehat{K}_0(X) \rightarrow \widehat{\text{CH}}_{\mathbb{Q}}^\bullet(X)$$

called the arithmetic Chern character, such that

- $\widehat{\text{ch}}$ is compatible with pull-backs by D -morphisms;
- $\widehat{\text{ch}}(\eta) = (0, \eta)$ if $\eta \in \widetilde{\mathfrak{A}}^\bullet(X)$;
- if $\overline{L} = (L, h)$ is a hermitian line bundle on X and s a rational section of L then

$$\widehat{\text{ch}}(\overline{L}) = \exp((\text{div } s, -\log h(s, s))).$$

See the beginning of [29] for this.

Example 5.5. Suppose in this example that X is regular and projective and flat of relative dimension 1 over $D = \mathbb{Z}$. Suppose also that Z and Z' are two closed subschemes of codimension 1 of X , which are flat with geometrically integral fibres over $\text{Spec } \mathbb{Z}$, which intersect transversally and do not intersect on the generic fiber. As $Z(\mathbb{C})$ (resp. $Z'(\mathbb{C})$) consists of one point P (resp. P'), the last condition just says that $P \neq P'$ in $X(\mathbb{C})$. Now equip $\mathcal{O}(Z)$ (resp. $\mathcal{O}(Z')$) with a conjugation invariant hermitian metric h (resp. h') and let s be a section of $\mathcal{O}(Z)$ (resp. s' be a section of $\mathcal{O}(Z')$) vanishing exactly on Z (resp. Z'). In this case, we have

$$\begin{aligned} (Z, -\log h(s, s)) \cdot (Z', -\log h'(s', s')) \\ = (Z \cap Z', -\log h(s, s)\delta_{Z'(\mathbb{C})} - c_1(\overline{\mathcal{O}(Z)}) \log h'(s', s')) \end{aligned}$$

in $\widehat{\text{CH}}_{\mathbb{Q}}^\bullet(X)$ and hence, if f is the morphism $X \rightarrow \text{Spec } \mathbb{Z}$,

$$\begin{aligned} f_*(\widehat{c}^1(\overline{\mathcal{O}(Z)}) \cdot \widehat{c}^1(\overline{\mathcal{O}(Z')})) \\ = \left(0, \sum_{p \in f_*(Z \cap Z')} \# \text{length}(Z_{\mathbb{F}_p} \cap Z'_{\mathbb{F}_p}) \log p \right. \\ \left. - \log h(s(P), s(P')) - \int_{X(\mathbb{C})} c_1(\overline{\mathcal{O}(Z)}) \log h'(s', s') \right). \end{aligned}$$

From the arithmetic Chern character, using the fundamental theorem on symmetric functions, we may also define an arithmetic Todd class

$$\widehat{\text{Td}} : \widehat{K}_0(X) \rightarrow \widehat{\text{CH}}_{\mathbb{Q}}^\bullet(X)^*$$

and an arithmetic total Chern class

$$\widehat{c} : \widehat{K}_0(X) \rightarrow \widehat{\text{CH}}_{\mathbb{Q}}^\bullet(X)^*.$$

If \overline{E} is an equivariant hermitian vector bundle on a regular equivariant arithmetic variety X , we define the *equivariant arithmetic Chern character* by the formula

$$\widehat{\text{ch}}_{\mu_n}(\overline{E}) = \widehat{\text{ch}}_{\mu_n, \zeta}(\overline{E}) := \sum_{k \in \mathbb{Z}/n} \zeta_n^k \widehat{\text{ch}}(\overline{E}_k) \in \widehat{\text{CH}}_{\mathbb{Q}(\mu_n)}^\bullet(X_{\mu_n}).$$

We write as before $\Lambda_{-1}(\overline{E}) := \sum_{k=0}^{\text{rk}(E)} (-1)^k \Lambda^k(\overline{E}) \in \widehat{K}_0^{\mu_n}(X)$, where $\Lambda^k(\overline{E})$ is the k -th exterior power of \overline{E} , endowed with its natural hermitian and equivariant structure.

Finally, to formulate the equivariant Grothendieck–Riemann–Roch theorem in Arakelov geometry, we shall need the following exotic characteristic class. Let X be a regular arithmetic variety.

Recall that for any $z \in \mathbb{C}$ with $|z| = 1$, the Lerch zeta function $\zeta_L(z, s)$ is defined by the formula

$$\zeta_L(z, s) := \sum_{k \geq 1} \frac{z^k}{k^s}$$

which is naturally defined for $\Re(s) > 1$ and can be meromorphically continued to the whole plane. For n any positive integer, define the n -th *harmonic number* \mathcal{H}_n by the formula

$$\mathcal{H}_0 := 0$$

and

$$\mathcal{H}_n := \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

when $n > 0$. For any $z \in \mathbb{C}$ we now define the formal complex power series

$$\widetilde{R}(z, x) := \sum_{k \geq 0} \left(2\zeta_L'(z, -k) + \mathcal{H}_k \cdot \zeta_L(z, -k)\right) \frac{x^k}{k!}.$$

(for those $z \in \mathbb{C}$ where it makes sense) and

$$R(z, x) := \frac{1}{2}(\widetilde{R}(z, x) - \widetilde{R}(\bar{z}, -x)).$$

For any fixed $z \in \mathbb{C}$, we identify $R(z, x)$ (resp. $\widetilde{R}(z, x)$) with the unique additive cohomology class it defines in Aepli cohomology. For a $\mu_n(\mathbb{C})$ -equivariant vector bundle E on $X(\mathbb{C})$, where $X(\mathbb{C})$ is endowed with the trivial $\mu_n(\mathbb{C})$ -equivariant structure, we now define the cohomology class $R_g(E)$ (resp. $\widetilde{R}_g(E)$) on $X(\mathbb{C})_g$ by the formula

$$R_g(E) := \sum_{u \in \mathbb{Z}/n} R(\zeta_n^u, E_u)$$

(resp.

$$\widetilde{R}_g(E) := \sum_{u \in \mathbb{Z}/n} \widetilde{R}(\zeta_n^u, E_u)).$$

The class $R_g(E)$ is often called the R_g -genus of E . Note that by construction we have

$$\tilde{R}_g(E) = \sum_{k \geq 0} \sum_{u \in \mathbb{Z}/n} \left(2\zeta'_L(\zeta_n^u, -k) + \mathcal{H}_k \cdot \zeta_L(\zeta_n^u, -k) \right) \text{ch}^{[k]}(E_u)$$

(resp.

$$R_g(E) = \sum_{k \geq 0} \sum_{u \in \mathbb{Z}/n} \left((\zeta'_L(\zeta_n^u, -k) - (-1)^k \zeta'_L(\bar{\zeta}_n^u, -k)) + \frac{1}{2} \mathcal{H}_k \cdot (\zeta_L(\zeta_n^u, -k) - (-1)^k \zeta_L(\bar{\zeta}_n^u, -k)) \right) \text{ch}^{[k]}(E_u).$$

Let now again $f : X \rightarrow Y$ be an equivariant projective morphism over D between regular equivariant arithmetic varieties. Suppose that there is an equivariant relatively ample line bundle on X and that the equivariant structure of Y is trivial. Suppose also that $f^{\mu_n} : X_{\mu_n} \rightarrow Y$ is smooth.

Let $N = N_{X/X_{\mu_n}}$ be the normal bundle of X_{μ_n} in X , which has a natural μ_n -equivariant structure. The bundle $N(\mathbb{C})$ is by construction a quotient of the restriction to $X(\mathbb{C})_g$ of the relative tangent bundle $T_f(\mathbb{C})$ and we thus endow it with the corresponding quotient metric structure (which is F_∞ -invariant). We refer to the resulting μ_n -equivariant hermitian vector bundle as $\bar{N} = \bar{N}_{X/X_{\mu_n}}$.

Theorem 5.6 (equivariant arithmetic Riemann–Roch theorem). *Suppose that f^{μ_n} is smooth. Then the equality*

$$\widehat{\text{ch}}_{\mu_n}(f_*(x)) = f_*^{\mu_n}(\widehat{\text{ch}}_{\mu_n}(\Lambda_{-1}(\bar{N}^\vee))^{-1} \text{Td}(\bar{T}_f^{\mu_n})(1 - R_g(T_f)) \widehat{\text{ch}}_{\mu_n}(x))$$

holds in $\widehat{\text{CH}}_{\mathbb{Q}(\mu_n)}^\bullet(Y)$, for any $x \in \widehat{K}_0^{\mu_n}(X)$.

Remark 5.7. If f is smooth then f^{μ_n} is smooth. We leave the proof of this statement as an exercise for the reader.

Theorem 5.6 results from a formal combination of the main results of [58] and [27]. It is important to underline that the most difficult part of the proof is analytic in nature and is contained in J.-M. Bismut's article [9]. A proof of the degree one part of Theorem 5.6 is given in [37].

6. Logarithmic derivatives of Dirichlet L -functions and arithmetic Chern classes of Hodge bundles

In this section, we shall apply Theorem 5.6 to the relative de Rham complex of a smooth and projective morphism between regular equivariant arithmetic varieties (satisfying certain conditions) and interpret the resulting equality in terms of logarithmic derivatives of Dirichlet L -functions.

As usual, fix a primitive n -th root of unity $\zeta_n \in \mathbb{C}$. For convenience, we shall write $\zeta = \zeta_n$ in this section. If $\sigma \in (\mathbb{Z}/n)^*$, we shall often write $\sigma(\zeta)$ for

ζ^σ . If χ is a primitive Dirichlet character modulo n (see e.g. [61, chap. 4] for an introduction to Dirichlet characters) we shall write

$$\tau(\chi) = \tau_\zeta(\chi) := \sum_{\sigma \in (\mathbb{Z}/n)^*} \sigma(\zeta)\chi(\sigma)$$

for the corresponding Gauss sum. We shall also write

$$L(\chi, s) = \sum_{n=0}^{\infty} \frac{\chi(n)}{n^s}$$

for the L -function associated with χ . This function is defined for $\Re(s) > 1$ but can be meromorphically continued to the whole complex plane. The resulting function is holomorphic everywhere if χ is not the trivial character.

The following combinatorial lemma will be needed in the proof.

Lemma 6.1. *Let M be a complex projective manifold and let E be a vector bundle on M together with an automorphism $g : E \rightarrow E$ of finite order (acting fiberwise). Let κ be the class*

$$\kappa := \text{Td}(E_0) \frac{\sum_{p \geq 0} (-1)^p p \cdot \text{ch}_g(\Lambda^p(E^\vee))}{\sum_{p \geq 0} (-1)^p \text{ch}_g(\Lambda^p(E^\vee))}$$

in the Aeppli cohomology of M . Then the equality

$$\kappa^{[l+\text{rk}(E_0)]} = -c_{\text{top}}(E_0) \sum_{z \in \mathbb{C}} \zeta_L(z, -l) \text{ch}^{[l]}((E^\vee)_z)$$

holds for all $l \geq 0$.

Here E_z is the largest subbundle of E on which g acts by multiplication by z .

Proof. See [43, Lemma 3.1]. □

Let M be a complex projective manifold and let (L, h_L) be an ample line bundle on M , endowed with a positive metric h_L . It is interesting (and it will be necessary later) to have an explicit formula for the L^2 -metric carried by the vector spaces $H^p(M, \Omega_M^q)$ ($p, q \geq 0$), where the L^2 -metric is computed using the Kähler metric coming from $c_1((L, h_L))$ and the metric on Ω_M^q is induced by $c_1((L, h_L))$.

Let us denote by $\omega \in H^2(M, \mathbb{C})$ the first Chern class of L and, for each $k \leq \dim(M)$, let us write $P^k(M, \mathbb{C}) \subseteq H^k(M, \mathbb{C})$ for the primitive cohomology associated with ω ; this is a Hodge substructure of $H^k(M, \mathbb{C})$. Recall that for any $k \geq 0$, the primitive decomposition theorem of Lefschetz establishes an isomorphism

$$H^k(M, \mathbb{C}) \simeq \bigoplus_{r \geq \max(k-d, 0)} \omega^r \wedge P^{k-2r}(M, \mathbb{C}).$$

Define the cohomological star operator

$$* : H^k(M, \mathbb{C}) \rightarrow H^{2d-k}(M, \mathbb{C})$$

by the rule

$$*\omega^r \wedge \phi := i^{p-q}(-1)^{(p+q)(p+q+1)/2} \frac{r!}{(d-p-q-r)!} \omega^{d-p-q-r} \wedge \phi$$

if ϕ is a primitive element of pure Hodge type (p, q) and extend it by additivity. We can now define a pairing on $H^k(M, \mathbb{C})$ by the formula

$$(\nu, \eta)_L := \frac{1}{(2\pi)^d} \int_M \nu \wedge *\bar{\eta}$$

for any $\nu, \eta \in H^k(M, \mathbb{C})$. This pairing turns out to be a hermitian metric, which is sometimes called the *Hodge metric*. See [32] for all this.

Lemma 6.2. *The Hodge-de Rham isomorphism*

$$H^k(M, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(M, \Omega_M^p)$$

is an isometry if the right-hand side is endowed with the Hodge metric and the left-hand side with the L^2 -metric.

Proof. See [43, Lemma 2.7]. □

Corollary 6.3. *Let $h : M \rightarrow N$ be a projective and smooth morphism between quasi-projective complex manifolds. Let g be a finite automorphism of M over N (i.e. g acts fiberwise). Let $h_0 : M_g \rightarrow N$ be the induced morphism (which is smooth). The equality of characteristic classes*

$$\begin{aligned} \sum_{p,q} p \cdot (-1)^{p+q} \cdot \text{ch}_g(\mathbb{R}^q h_* (\Omega_h^p)) \\ = - \int_{M_g/N} c_{\text{top}}(\mathbb{T}_{h_0}) \left[\sum_{l \geq 0} \sum_{z \in \mathbb{C}} \zeta_L(z, -l) \text{ch}^{[l]}((\Omega_h|_{M_g})_z) \right] \end{aligned}$$

in Aeppli cohomology holds.

Proof. This is an immediate consequence of Lemma 6.1, the main result of [5, Th. 2.12] and the fact that there is a natural map from Hodge cohomology (also called $\bar{\partial}$ -cohomology) to Aeppli cohomology (see (5.2)). □

Lemma 6.4. *For any primitive Dirichlet character modulo n and any $u \in \mathbb{Z}$, the equality*

$$\sum_{\sigma \in (\mathbb{Z}/n)^*} \sigma(\zeta^u) \chi(\sigma) = \bar{\chi}(u) \tau(\chi)$$

holds.

Proof. See [61, chap. 4, lemma 4.7]. □

Lemma 6.4 implies the following two lemmata:

Lemma 6.5. *Let \bar{V} be a μ_n -equivariant hermitian vector bundle on a regular arithmetic variety Z . Suppose that μ_n acts trivially on Z . Then for any primitive character modulo n we have*

$$\sum_{\sigma \in (\mathbb{Z}/n)^*} \widehat{\text{ch}}_{\mu_n, \sigma(\zeta)}(\bar{V})\chi(\sigma) = \tau(\chi) \sum_{u \in \mathbb{Z}/n} \widehat{\text{ch}}(\bar{V}_u)\bar{\chi}(u).$$

Lemma 6.6. *For any primitive character modulo n we have*

$$\sum_{\sigma \in (\mathbb{Z}/n)^*} \zeta_L(\sigma(\zeta^u), s)\chi(\sigma) = \tau(\chi)\bar{\chi}(u)L(\bar{\chi}, s).$$

Remark 6.7. (Important). Let us call the number $\chi(-1) \in \{1, -1\}$ the parity of the Dirichlet character χ and for any (positive) integer $l \in \mathbb{Z}$ let us define the parity of l to be $(-1)^l \in \{1, -1\}$. By classical results of analytic number theory, we have $L(\chi_{\text{Prim}}, 1-l) \neq 0$ if χ and l have the same parity (see [61, before Th. 4.2]). More generally, if χ is now an Artin character attached to any finite-dimensional complex irreducible representation R_χ of the Galois group of a finite Galois extension of \mathbb{Q} , we will say that χ is even (resp. odd) if $R_\chi(F_\infty) = \text{Id}$ (resp. $R_\chi(F_\infty) = -\text{Id}$), where F_∞ is acting as the complex conjugation. Let's then denote by $L(\chi, s)$ the Artin L -function associated with χ (cf. [59] or [49, § 7.10-12] for an introduction). The function $L(\chi, s)$ is non-vanishing for $\Re(s) > 1$ and by Brauer admits a functional equation and a meromorphic continuation to the whole complex plane. One easily deduces from this the zeroes of $L(\chi, s)$ lying on the real negative line (cf. for instance [49, p.541]). We get again $L(\chi, 1-l) \neq 0$ when χ and l have the same parity.

We shall also need the following deep vanishing statement, due to J.-M. Bismut. This statement is what makes the calculations below possible and it would be very interesting to have a better conceptual understanding of it. Its proof relies on the comparison between two completely different types of analytic torsion (holomorphic torsion and flat torsion).

Theorem 6.8. *Let $h : M \rightarrow N$ be a proper and smooth morphism of complex manifolds. Let g be a finite automorphism of M over N (i.e. g acts fiberwise). Suppose that h is endowed with a g -invariant Kähler fibration structure ω_h . Then the element*

$$\sum_{k \geq 0} (-1)^k T_g(\bar{\Omega}_h^k) \in \tilde{\mathfrak{A}}^\bullet(N)$$

vanishes.

Proof. See [7]. □

Remark 6.9. In the paper [13] it is shown that in the non-equivariant setting the vanishing property stated in Theorem 6.8 can be used to characterise the holomorphic torsion form axiomatically.

Let $f : X \rightarrow Y$ be a μ_n -equivariant smooth and projective morphism of regular arithmetic varieties over an arithmetic ring D . Let g be the automorphism of X corresponding to $\zeta \in \mu_n(\mathbb{C})$. Suppose that $X(\mathbb{C})$ is endowed with a g -invariant Kähler fibration structure ω_f with respect to $f(\mathbb{C})$ and suppose that the μ_n -structure of Y is trivial. Suppose also that there is an equivariant line bundle on X , which is ample relatively to f .

We shall apply Theorem 5.6 to the elements of the relative de Rham complex of f . To ease notation, let us write $\overline{H}_{\text{Dlb}}^k(X/Y)$ for the hermitian equivariant vector bundle

$$\overline{H}_{\text{Dlb}}^k(X/Y) := \bigoplus_{p+q=k} \mathbb{R}^q f_* (\overline{\Omega}_f^p)$$

and $H^{p,q}(X/Y)$ for the vector bundle

$$H^{p,q}(X/Y) := \mathbb{R}^q f_* (\Omega_f^p).$$

Theorem 5.6 together with the Borel–Serre identity (see the end of Section 4) now gives the identity

$$\sum_k (-1)^k \widehat{\text{ch}}_{\mu_n}(\overline{H}_{\text{Dlb}}^k(X/Y)) = f_*^{\mu_n}(c_{\text{top}}(\overline{T}_f)) - \int_{X(\mathbb{C})_g/Y(\mathbb{C})} c_{\text{top}}(T_f) R_g(T_f) \quad (6.1)$$

in $\widehat{\text{CH}}_{\mathbb{Q}(\mu_n)}^\bullet(Y)$. In particular, for any $l \geq 1$,

$$\boxed{\sum_k (-1)^k \widehat{\text{ch}}_{\mu_n}^{[l]}(\overline{H}_{\text{Dlb}}^k(X/Y)) = - \int_{X(\mathbb{C})_g/Y(\mathbb{C})} c_{\text{top}}(T_f) R_g^{[l-1]}(T_f)} \quad (6.2)$$

We shall see below that equation (6.2) carries astonishingly deep arithmetic information. It should be viewed as a lifting to Arakelov theory of the relative equivariant form of the Gauss–Bonnet formula.

We shall now translate equation (6.2) into a statement about logarithmic derivatives of Dirichlet L -functions at negative integers. That this kind of translation should be possible is suggested by Lemma 6.6 and the definition of the R_g -genus.

We compute:

Lemma 6.10. *For any $l \geq 1$ and for any primitive Dirichlet character χ modulo n we have*

$$\begin{aligned} & \sum_{\sigma \in (\mathbb{Z}/n)^*} \widetilde{R}_{g^\sigma}^{[l-1]}(T_f) \chi(\sigma) \\ &= \tau(\chi) \left[2L'(\bar{\chi}, 1-l) + \mathcal{H}_{l-1} \cdot L(\bar{\chi}, 1-l) \right] \sum_{u \in \mathbb{Z}/n} \text{ch}^{[l-1]}(T_{f_u}) \bar{\chi}(u). \end{aligned}$$

Proof. This is an immediate consequence of Lemma 6.6 and the definition of the class \widetilde{R}_g . \square

Lemma 6.11. *Let $h : M \rightarrow N$ be a projective and smooth morphism between quasi-projective complex manifolds. Let g be a finite automorphism of M over N (i.e. g acts fiberwise). Let $h_0 : M_g \rightarrow N$ be the induced morphism (which is smooth). For any primitive Dirichlet character χ modulo n the equality of characteristic classes in Aeppli cohomology*

$$\begin{aligned} \sum_{u \in \mathbb{Z}/n} \sum_{p,q} (-1)^{p+q} p \cdot \text{ch}^{[l]}(H^{p,q}(M/N)_u) \bar{\chi}(u) \\ = -L(\bar{\chi}, -l) \int_{M_g/N} \sum_{u \in \mathbb{Z}/n} c_{\text{top}}(\mathbb{T}_{h_0}) \text{ch}^{[l]}((\Omega_h|_{M_g})_u) \bar{\chi}(u) \end{aligned}$$

holds.

Here $H^{p,q}(M/N)_u$ (resp. $(\Omega_h|_{M_g})_u$) denotes the largest subbundle of $H^{p,q}(M/N)$ (resp. $\Omega_h|_{M_g}$) where g acts by multiplication by ζ^u .

Proof. This is an immediate consequence of Lemma 6.6 and Corollary 6.3. \square

If $K \subseteq \mathbb{C}$ is a subfield and χ is a Dirichlet character, we shall write $K(\chi)$ for the subfield of \mathbb{C} obtained by adjoining all the values of χ to K .

Combining Lemma 6.10 with equality (6.2), we get the following. For any primitive Dirichlet character modulo n , the equality

$$\begin{aligned} \sum_k (-1)^k \sum_{u \in \mathbb{Z}/n} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dib}}^k(X/Y)_u) \bar{\chi}(u) \\ = -\frac{1}{2} \left[2L'(\bar{\chi}, 1-l) - 2L'(\bar{\chi}, 1-l) (-1)^{\chi(-1)+l-1} \right. \\ \left. + \mathcal{H}_{l-1} \cdot \left(L(\bar{\chi}, 1-l) - L(\bar{\chi}, 1-l) (-1)^{\chi(-1)+l-1} \right) \right] \\ \cdot \int_{X(\mathbb{C})_g/Y(\mathbb{C})} c_{\text{top}}(\mathbb{T}_f) \sum_{u \in \mathbb{Z}/n} \text{ch}^{[l-1]}(\mathbb{T}_{f_u}) \bar{\chi}(u) \end{aligned} \quad (6.3)$$

holds in $\widehat{\text{CH}}_{\mathbb{Q}(\mu_n)(\chi)}^l(Y)$.

Notice that if χ and l have the same parity, we have

$$L(\bar{\chi}, s) - L(\bar{\chi}, s) (-1)^{\chi(-1)+l-1} = 2L(\bar{\chi}, s)$$

whereas if χ and l do not have the same parity then

$$L(\bar{\chi}, s) - L(\bar{\chi}, s) (-1)^{\chi(-1)+l-1} = 0.$$

So we obtain from the equation (6.3): if χ and l have the same parity, then

$$\begin{aligned}
& \sum_k (-1)^k \sum_{u \in \mathbb{Z}/n} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^k(X/Y)_u) \bar{\chi}(u) \\
&= - \left[2L'(\bar{\chi}, 1-l) + \mathcal{H}_{l-1} \cdot L(\bar{\chi}, 1-l) \right] \\
&\quad \cdot \int_{X_g(\mathbb{C})/Y(\mathbb{C})} c_{\text{top}}(\mathbb{T}_f) \sum_{u \in \mathbb{Z}/n} \text{ch}^{[l-1]}(\mathbb{T}_{f_u}) \bar{\chi}(u) \\
&= - \left[2L'(\bar{\chi}, 1-l) + \mathcal{H}_{l-1} \cdot L(\bar{\chi}, 1-l) \right] \\
&\quad \cdot \int_{X_g(\mathbb{C})/Y(\mathbb{C})} c_{\text{top}}(\mathbb{T}_f) \sum_{u \in \mathbb{Z}/n} \text{ch}^{[l-1]}(\mathbb{T}_{f_{-u}}) \bar{\chi}(-u) \\
&= - \left[2L'(\bar{\chi}, 1-l) + \mathcal{H}_{l-1} \cdot L(\bar{\chi}, 1-l) \right] \\
&\quad \cdot \int_{X_g(\mathbb{C})/Y(\mathbb{C})} c_{\text{top}}(\mathbb{T}_f) \sum_{u \in \mathbb{Z}/n} \text{ch}^{[l-1]}((\Omega_{f,u})^\vee) \bar{\chi}(-u) \\
&= -(-1)^{l-1} \bar{\chi}(-1) \left[2L'(\bar{\chi}, 1-l) + \mathcal{H}_{l-1} \cdot L(\bar{\chi}, 1-l) \right] \\
&\quad \cdot \int_{X_g(\mathbb{C})/Y(\mathbb{C})} c_{\text{top}}(\mathbb{T}_f) \sum_{u \in \mathbb{Z}/n} \text{ch}^{[l-1]}(\Omega_{f,u}) \bar{\chi}(u) \\
&= \left[2L'(\bar{\chi}, 1-l) + \mathcal{H}_{l-1} \cdot L(\bar{\chi}, 1-l) \right] \\
&\quad \cdot \int_{X_g(\mathbb{C})/Y(\mathbb{C})} c_{\text{top}}(\mathbb{T}_f) \sum_{u \in \mathbb{Z}/n} \text{ch}^{[l-1]}(\Omega_{f,u}) \bar{\chi}(u)
\end{aligned}$$

and if χ and l do not have the same parity, then

$$\sum_k (-1)^k \sum_{u \in \mathbb{Z}/n} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^k(X/Y)_u) \bar{\chi}(u) = 0. \quad (6.4)$$

Finally, if χ and l have the same parity (hence $L(\bar{\chi}, 1-l) \neq 0$ by Remark 6.7) then we obtain using Lemma 6.11 that

$$\begin{aligned}
& \sum_k (-1)^k \sum_{u \in \mathbb{Z}/n} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^k(X/Y)_u) \bar{\chi}(u) \\
&= - \sum_k (-1)^k \left[2 \frac{L'(\bar{\chi}, 1-l)}{L(\bar{\chi}, 1-l)} + \mathcal{H}_{l-1} \right] \\
&\quad \cdot \sum_{u \in \mathbb{Z}/n} \sum_{p+q=k} p \cdot \text{ch}^{[l-1]}(H^{p,q}(X/Y)_u) \bar{\chi}(u). \quad (6.5)
\end{aligned}$$

Note that this equality does not depend on the initial choice of root of unity ζ_n anymore.

Now suppose that χ is not primitive. Then we apply (6.5) again, but replace the action of μ_n by the action of its subgroup scheme μ_{f_χ} , where

f_χ is the conductor of χ . We shall write χ_{Prim} for the primitive character modulo f_χ associated with χ . Replacing χ by $\bar{\chi}$ for convenience, we finally get the following basic formula. We shall encase it in a theorem to underline its importance.

Theorem 6.12. *Let $f : X \rightarrow Y$ be a μ_n -equivariant smooth and projective morphism of equivariant regular arithmetic varieties. Suppose that the μ_n -action on Y is trivial. Fix a $\mu_n(\mathbb{C})$ -invariant Kähler fibration structure ω_f for f on X and suppose that there is a μ_n -equivariant line bundle on X , which is ample relatively to f . Let χ be a Dirichlet character modulo n . Then the equation*

$$\begin{aligned} \sum_k (-1)^k \sum_{u \in \mathbb{Z}/n} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^k(X/Y)_u) \chi_{\text{Prim}}(u) \\ = - \sum_k (-1)^k \left[2 \frac{L'(\chi_{\text{Prim}}, 1-l)}{L(\chi_{\text{Prim}}, 1-l)} + \mathcal{H}_{l-1} \right] \cdot \\ \cdot \sum_{u \in \mathbb{Z}/n} \sum_{p+q=k} p \cdot \text{ch}^{[l-1]}(H^{p,q}(X/Y)_u) \chi_{\text{Prim}}(u) \end{aligned} \quad (6.6)$$

holds in $\widehat{\text{CH}}_{\mathbb{Q}(\mu_{f_\chi})(\chi_{\text{Prim}})}(Y)$, if χ and l have the same parity (and hence $L(\chi_{\text{Prim}}, 1-l) \neq 0$ by Remark 6.7). If χ and l do not have the same parity then

$$\sum_k (-1)^k \sum_{u \in \mathbb{Z}/n} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^k(X/Y)_u) \chi_{\text{Prim}}(u) = 0.$$

This is the promised translation of formula (6.2).

Now notice that in Theorem 6.12, it is very natural to wonder whether the equality holds before the alternating sum $\sum_k (-1)^k$ is taken on both sides. It is difficult to make a meaningful conjecture about this ‘separation of weights’ (in particular because the Kähler fibration structure ω_f is defined on X and not only on the Hodge bundles). It nevertheless makes sense to conjecture the following purely geometric statement.

Conjecture 6.13. *Let $f : X \rightarrow Y$ be a μ_n -equivariant smooth and projective morphism of equivariant regular arithmetic varieties. Suppose that the μ_n -action on Y is trivial. Let χ be a Dirichlet character modulo n . Then for any $k \geq 1$ the equation*

$$\sum_{u \in \mathbb{Z}/n} \text{ch}^{[l]}(H_{\text{Dlb}}^k(X/Y)_u) \chi_{\text{Prim}}(u) = 0$$

holds in $\text{CH}^l(Y)_{\mathbb{Q}(\mu_{f_\chi})(\chi_{\text{Prim}})}$ if χ and l have the same parity.

When $D = \mathbb{C}$ and $\chi = 1$, this conjecture was studied and refined in [45, Conj. 1.1]. See also [20] for this conjecture.

In the direction of ‘separation of weights’ in the context of Arakelov geometry, we can nevertheless prove the following result.

Proposition 6.14. *Let $f : \mathcal{A} = X \rightarrow Y$ be a μ_n -equivariant smooth and projective morphism of equivariant regular arithmetic varieties and suppose that \mathcal{A} is an abelian scheme. Suppose that $\mu_n(\mathbb{C})$ acts on $\mathcal{A}(\mathbb{C})$ by automorphisms respecting the group scheme structure. Suppose that there is a μ_n -equivariant line bundle on \mathcal{A} , which is ample relatively to f . Suppose that the μ_n -action on Y is trivial. Suppose also that for any subgroup scheme $\mu_t \subseteq \mu_n$ such that $t \neq 1$ the closed subscheme \mathcal{A}_{μ_t} is finite over Y .*

Fix a $\mu_n(\mathbb{C})$ -invariant Kähler fibration structure ω_f and suppose also that ω_f is translation invariant on the fibres of $f(\mathbb{C})$ and that

$$\frac{1}{\dim(\mathcal{A}/Y)!} \int_{\mathcal{A}(\mathbb{C})/Y(\mathbb{C})} \omega_f^{\dim(\mathcal{A}/Y)} = 1.$$

Let χ be a Dirichlet character modulo n . If χ is the trivial character, suppose as well that 2 is invertible in D and that $[-1]_{\mathcal{A}(\mathbb{C})}^(\omega_f) = \omega_f$.*

Then the equation

$$\begin{aligned} \sum_{u \in \mathbb{Z}/n} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^k(\mathcal{A}/Y)_u) \chi_{\text{Prim}}(u) \\ = - \left[2 \frac{L'(\chi_{\text{Prim}}, 1-l)}{L(\chi_{\text{Prim}}, 1-l)} + \mathcal{H}_{l-1} \right] \cdot \\ \cdot \sum_{u \in \mathbb{Z}/n} \sum_{p+q=k} p \cdot \text{ch}^{[l-1]}(H^{p,q}(\mathcal{A}/Y)_u) \chi_{\text{Prim}}(u) \end{aligned} \quad (6.7)$$

holds in $\widehat{\text{CH}}_{\mathbb{Q}(\mu_{f_\chi})(\chi_{\text{Prim}})}^l(Y)$ if χ and l have the same parity (and hence $L(\chi_{\text{Prim}}, 1-l) \neq 0$ by Remark 6.7).

Consider the following formal power series:

$$\exp(x) := \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

and

$$\log(1+x) := \sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^j}{j}.$$

Lemma 6.15. *Let $\lambda \in \mu_n(\mathbb{C})$, $\lambda \neq 1$. Then the equality*

$$\log \left(\frac{1 - \lambda \cdot \exp(x)}{1 - \lambda} \right) = - \sum_{j \geq 1} \zeta_L(\lambda, 1-j) \frac{x^j}{j!}$$

holds in $\mathbb{C}[[x]]$.

Proof. See [44, Lemma 4]. □

Lemma 6.16. *Let \overline{V} be a μ_n -equivariant hermitian bundle on an arithmetic variety Z . Suppose that the μ_n -action on Z is trivial. Then we have the*

equality

$$\begin{aligned} \log \left[\left(\prod_{u=0}^{n-1} (1 - \zeta^u)^{\text{rk}(V_u)} \right)^{-1} \sum_{r \geq 0} (-1)^r \widehat{\text{ch}}_{\mu_n}(\Lambda^r(\overline{V})) \right] \\ = - \sum_{l \geq 1} \sum_{u=0}^{n-1} \zeta_L(\zeta^u, 1-l) \widehat{\text{ch}}^{[l]}(\overline{V}_u) \end{aligned} \quad (6.8)$$

in $\widehat{\text{CH}}_{\mathbb{Q}(\mu_n)}(Z)$.

Proof. Notice first that if \overline{V} and \overline{W} are μ_n -equivariant vector bundles, then we have

$$\sum_{r \geq 0} (-1)^r \widehat{\text{ch}}_{\mu_n}(\Lambda^r(\overline{V} \oplus \overline{W})) = \sum_{r \geq 0} (-1)^r \widehat{\text{ch}}_{\mu_n}(\Lambda^r(\overline{V})) \cdot \sum_{r \geq 0} (-1)^r \widehat{\text{ch}}_{\mu_n}(\Lambda^r(\overline{W})).$$

Thus

$$\begin{aligned} \log \left[\left(\prod_{u=0}^{n-1} (1 - \zeta^u)^{\text{rk}((V \oplus W)_u)} \right)^{-1} \sum_{r \geq 0} (-1)^r \widehat{\text{ch}}_{\mu_n}(\Lambda^r(\overline{V} \oplus \overline{W})) \right] \\ = \log \left[\left(\prod_{u=0}^{n-1} (1 - \zeta^u)^{\text{rk}(V_u)} \right)^{-1} \sum_{r \geq 0} (-1)^r \widehat{\text{ch}}_{\mu_n}(\Lambda^r(\overline{V})) \right] \\ + \log \left[\left(\prod_{u=0}^{n-1} (1 - \zeta^u)^{\text{rk}(W_u)} \right)^{-1} \sum_{r \geq 0} (-1)^r \widehat{\text{ch}}_{\mu_n}(\Lambda^r(\overline{W})) \right] \end{aligned}$$

and in particular both sides of the equality (6.8) are additive in \overline{V} . On the other hand if V is a line bundle then equality 6.8 holds by Lemma 6.15. Thus we have proven equality 6.8 in the situation where \overline{V} is an orthogonal direct sum of hermitian line bundles. The general case is now a consequence of a version of the splitting principle. For details about this last step, see the proof of Lemma 6.11 in [51], which proceeds along entirely similar lines. \square

Proof of Proposition 6.14. Suppose first that $n \neq 1$. Notice that from the definition of the L^2 -metric and the assumption that

$$\frac{1}{\dim(\mathcal{A}/Y)!} \int_{\mathcal{A}(\mathbb{C})/Y(\mathbb{C})} \omega_f^{\dim(\mathcal{A}/Y)} = 1,$$

there exists an isometric isomorphism

$$\Lambda^r(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)) \simeq \overline{H}_{\text{Dlb}}^r(\mathcal{A}/Y)$$

for all $r \geq 0$. We now apply equality (6.1) to $X = \mathcal{A}$ over Y . We obtain

$$\begin{aligned} \sum_{r \geq 0} (-1)^r \widehat{\text{ch}}_{\mu_n}(\Lambda^r(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y))) \\ = \left(\prod_{u=0}^{n-1} (1 - \zeta^u)^{\text{rk}(H^1(\mathcal{A}/Y)_u)} \right) (1 - R_g(f_*(T_f))). \end{aligned}$$

Applying Lemma 6.16, this shows that

$$\sum_{l \geq 1} \sum_{u=0}^{n-1} \zeta_L(\zeta^u, 1-l) \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_u) = R_g(f_*(\mathbb{T}_f)). \quad (6.9)$$

Let χ be any primitive Dirichlet character modulo n . There is an equivariant isomorphism $f_*(\mathbb{T}_f(\mathbb{C})) \simeq H^{1,0}(\mathcal{A}/Y)(\mathbb{C})^\vee$ (given by the polarisation induced by a μ_n -equivariant relatively ample line bundle) and applying Lemma 6.6, we finally obtain that

$$\begin{aligned} & \sum_{l \geq 1} \sum_{u=0}^{n-1} L(\bar{\chi}, 1-l) \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_u) \bar{\chi}(u) \\ &= (-1)^{l-1} \bar{\chi}(-1) \sum_{l \geq 1} \frac{1}{2} \left[2L'(\bar{\chi}, 1-l) - 2L'(\bar{\chi}, 1-l)(-1)^{\text{sign}(\chi)+l-1} \right. \\ & \quad \left. + \mathcal{H}_{l-1} \cdot \left(L(\bar{\chi}, 1-l) - L(\bar{\chi}, 1-l)(-1)^{\text{sign}(\chi)+l-1} \right) \right] \\ & \quad \cdot \text{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)) \bar{\chi}(u) \end{aligned}$$

so that if χ and l have the same parity we get

$$\begin{aligned} & \sum_{l \geq 1} \sum_{u=0}^{n-1} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_u) \bar{\chi}(u) \\ &= - \sum_{l \geq 1} \left[2 \cdot \frac{L'(\bar{\chi}, 1-l)}{L(\bar{\chi}, 1-l)} + \mathcal{H}_{l-1} \right] \cdot \text{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)) \bar{\chi}(u). \end{aligned}$$

In particular, the theorem is proven if χ is primitive and $n \neq 1$.

Now suppose that $n = 1$, that 2 is invertible in D and that $[-1]_{\mathcal{A}(\mathbb{C})}^*(\omega_f)$ is equal to ω_f . In this case χ is necessarily the trivial character. Furthermore, we then have a natural isomorphism between μ_2 and the constant group scheme $\mathbb{Z}/(2)_D$. Notice that there is a natural action of $\mathbb{Z}/(2)_D$ on \mathcal{A} given by the automorphism $[-1]_{\mathcal{A}}$. We shall apply equality (6.1) to the corresponding group scheme action of μ_2 . Equation (6.9) holds for this action and it gives

$$\sum_{l \geq 1} \zeta_L(-1, 1-l) \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)) = R_{[-1]_{\mathcal{A}(\mathbb{C})}}(f_*(\mathbb{T}_f)).$$

By definition, we have

$$\begin{aligned} & R_{[-1]_{\mathcal{A}(\mathbb{C})}}(f_*(\mathbb{T}_f)) \\ &= \sum_{l \geq 1} \left((\zeta'_L(-1, 1-l) - (-1)^{1-l} \zeta'_L(-1, 1-l)) \right. \\ & \quad \left. + \frac{1}{2} \mathcal{H}_k \cdot (\zeta_L(-1, 1-l) - (-1)^{1-l} \zeta_L(-1, 1-l)) \right) \text{ch}^{[l-1]}(f_*(\mathbb{T}_f)) \end{aligned}$$

so if l is even we have

$$\begin{aligned} \zeta_L(-1, 1-l) \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)) \\ = \left(2\zeta'_L(-1, 1-l) + \mathcal{H}_k \cdot \zeta_L(-1, 1-l) \right) \text{ch}^{[l-1]}(f_*(\mathbf{T}_f)) \end{aligned}$$

i.e.

$$\widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)) = \left(2\frac{\zeta'_L(-1, 1-l)}{\zeta_L(-1, 1-l)} + \mathcal{H}_k \right) \text{ch}^{[l-1]}(f_*(\mathbf{T}_f)).$$

Now recall that we have the elementary identity

$$\zeta_L(-1, s) = (1 - 2^{1-s})\zeta_{\mathbb{Q}}(s). \quad (6.10)$$

Using (6.10) we see that

$$\frac{\zeta'_L(-1, 1-l)}{\zeta_L(-1, 1-l)} = \frac{\zeta'_{\mathbb{Q}}(1-l)}{\zeta_{\mathbb{Q}}(1-l)} + \log(2) \frac{2^l}{1-2^l}.$$

Note further that $\log(2) = 0$ in $\widehat{\text{CH}}_{\mathbb{Q}}^1(Y)$ since 2 is invertible in Y so we finally get

$$\widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)) = -\left(2\frac{\zeta'_{\mathbb{Q}}(-1, 1-l)}{\zeta_{\mathbb{Q}}(-1, 1-l)} + \mathcal{H}_k \right) \text{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y))$$

if l is even. In particular, the theorem is proven if $n = 1$. We can now conclude the proof following the same line of argument as at the end of the proof of Theorem 6.12 (see before the statement of Theorem 6.12). \square

Corollary 6.17. *Conjecture 6.13 holds if X is an abelian scheme over Y , $k = 1$ and X_G is finite over Y for any non-trivial closed subgroup scheme G of μ_n over D .*

We now wish to translate Proposition 6.14 into the language of complex multiplication of abelian schemes. For this and later applications, we shall need the following

Lemma 6.18. *Suppose that L, K are number fields and that all the embeddings of K into \mathbb{C} factor through an embedding of L into \mathbb{C} . Let d_K be the discriminant of K . Then there is a canonical isomorphism of \mathcal{O}_L -algebras*

$$(\mathcal{O}_L \otimes \mathcal{O}_K) \left[\frac{1}{d_K} \right] \simeq \bigoplus_{\sigma: K \hookrightarrow L} \mathcal{O}_L \left[\frac{1}{d_K} \right]$$

such that $l \otimes k \mapsto \bigoplus_{\sigma} l \cdot \sigma(k)$.

Proof. Notice to begin with that we have an isomorphism of L -algebras

$$L \otimes_{\mathbb{Q}} K \simeq \bigoplus_{\sigma: K \hookrightarrow L} L \quad (6.11)$$

such that $l \otimes k \mapsto \bigoplus_{\sigma} l \cdot \sigma(k)$. This can be seen by writing $K \simeq \mathbb{Q}[t]/(P(t))$ for some monic irreducible polynomial $P(t)$ and noticing that by assumption, $P(t)$ splits in L .

Notice now that $\mathrm{Spec} \mathcal{O}_K[\frac{1}{d_K}] \rightarrow \mathrm{Spec} \mathbb{Z}[\frac{1}{d_K}]$ is by construction a finite and étale morphism. Hence the morphism

$$\mathrm{Spec} \mathcal{O}_L \otimes \mathcal{O}_K \left[\frac{1}{d_K} \right] \rightarrow \mathrm{Spec} \mathcal{O}_L \left[\frac{1}{d_K} \right]$$

is also finite and étale. Thus $\mathrm{Spec} \mathcal{O}_L \otimes \mathcal{O}_K[\frac{1}{d_K}]$ is the disjoint union of its irreducible components and any of these components, say C , is integral, finite and étale over $\mathrm{Spec} \mathcal{O}_L[\frac{1}{d_K}]$. On the other hand, notice that the morphism $C_L \rightarrow \mathrm{Spec} L$ is an isomorphism because of the existence of the decomposition (6.11). Thus there is a section $\mathrm{Spec} L \rightarrow C_L$, which extends uniquely to a section

$$\mathrm{Spec} \mathcal{O}_L \left[\frac{1}{d_K} \right] \rightarrow C$$

by the valuative criterion of properness and this section is an open immersion because $C \rightarrow \mathrm{Spec} \mathcal{O}_L[\frac{1}{d_K}]$ is étale. Hence this section is an isomorphism, since C is integral. To summarise, the irreducible components of $\mathrm{Spec} \mathcal{O}_L \otimes \mathcal{O}_K[\frac{1}{d_K}]$ are all images of sections of the morphism $\mathrm{Spec} \mathcal{O}_L \otimes \mathcal{O}_K[\frac{1}{d_K}] \rightarrow \mathrm{Spec} \mathcal{O}_L[\frac{1}{d_K}]$. Furthermore, every section of the morphism $\mathrm{Spec} L \otimes_{\mathbb{Q}} K \rightarrow \mathrm{Spec} L$ extends uniquely to a section over $\mathrm{Spec} \mathcal{O}_L[\frac{1}{d_K}]$, which is an open immersion and whose image is an irreducible component. Translating these two statements back into the language of rings gives the lemma. \square

Suppose now that $X = \mathcal{A}$ is an abelian scheme over Y and that there is an injection $\mathbb{Z}[\mu_n] = \mathcal{O}_{\mathbb{Q}(\mu_n)} \hookrightarrow \mathrm{End}_Y(\mathcal{A})$ for some even integer $n > 1$. Suppose also that n is invertible in the arithmetic base ring D , that there is a primitive n -th root of unity in D and that D is a localisation of the rings of integers of a number field. Then $\mu_n = \mu_{n,D}$ (see our conventions at the beginning of subsection 5.2) is isomorphic to the constant group scheme over D associated with $\mathbb{Z}/(n)$. We fix such an isomorphism; this is equivalent to choosing a primitive root of unity in D , or in other words to choosing an embedding $\iota : \mathbb{Z}[\mu_n] \hookrightarrow D$. We are now given a μ_n -action on \mathcal{A} over Y . Note that $n = 2$ is allowed; in that case the μ_2 -action given by the injection $\mathcal{O}_{\mathbb{Q}(\mu_2)} = \mathbb{Z} \hookrightarrow \mathrm{End}_Y(\mathcal{A})$ is given by the action of the automorphism $[-1]_{\mathcal{A}}$. By Lemma 6.18, we have $\overline{H}_{\mathrm{Dib}}^1(\mathcal{A}/Y)_u = 0$ if u is not prime to n . In particular, for any subgroup scheme $\mu_t \subseteq \mu_n$ such that $t \neq 1$ the closed subscheme \mathcal{A}_{μ_t} is finite over Y . Hence, for any Dirichlet character modulo n , Proposition 6.14 gives us the equality

$$\begin{aligned} & \sum_{u \in (\mathbb{Z}/n)^*} \widehat{\mathrm{ch}}^{[l]}(\overline{H}_{\mathrm{Dib}}^1(\mathcal{A}/Y)_u) \chi(u) \\ &= - \left[2 \frac{L'(\chi_{\mathrm{Prim}}, 1-l)}{L(\chi_{\mathrm{Prim}}, 1-l)} + \mathcal{H}_{l-1} \right] \sum_{u \in (\mathbb{Z}/n)^*} \mathrm{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)_u) \chi(u) \end{aligned}$$

in $\widehat{\text{CH}}_{\mathbb{Q}(\mu_n)}^l(\chi_{\text{Prim}})(Y)$, if χ and l have the same parity (so $L(\chi_{\text{Prim}}, 1-l) \neq 0$, see Remark 6.7.) This can be rewritten as:

$$\begin{aligned} & \sum_{\tau \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau) \\ &= - \left[2 \frac{L'(\chi_{\text{Prim}}, 1-l)}{L(\chi_{\text{Prim}}, 1-l)} + \mathcal{H}_{l-1} \right] \sum_{\tau \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \text{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau) \end{aligned} \quad (6.12)$$

where $\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\iota \circ \tau}$ is the subsheaf on which $\mathbb{Z}[\mu_n]$ acts via $\iota \circ \tau$. Notice that since n is invertible in D , Lemma 6.18 implies that there is an inner direct sum

$$\bigoplus_{\tau \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} H_{\text{Dlb}}^1(\mathcal{A}/Y)_{\iota \circ \tau} \simeq H_{\text{Dlb}}^1(\mathcal{A}/Y).$$

Remark 6.19. Notice the interesting fact that the truth of equation (6.12) is independent of the embedding $\iota : \mathbb{Z}[\mu_n] \hookrightarrow D$. Indeed if $\iota_1 : \mathbb{Z}[\mu_n] \hookrightarrow D$ is another embedding then there exist $\tau_1 \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})$ such that $\iota_1 = \iota \circ \tau_1$ (because $\mathbb{Q}(\mu_n)$ is a Galois extension of \mathbb{Q}). Thus

$$\begin{aligned} & \sum_{\tau \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\iota_1 \circ \tau})\chi(\tau) \\ &= \sum_{\tau \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau_1^{-1} \circ \tau) \\ &= \chi(\tau_1^{-1}) \left(\sum_{\tau \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau) \right) \end{aligned}$$

and similarly

$$\begin{aligned} & - \left[2 \frac{L'(\chi_{\text{Prim}}, 1-l)}{L(\chi_{\text{Prim}}, 1-l)} + \mathcal{H}_{l-1} \right] \sum_{\tau \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \text{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)_{\iota_1 \circ \tau})\chi(\tau) \\ &= - \left[2 \frac{L'(\chi_{\text{Prim}}, 1-l)}{L(\chi_{\text{Prim}}, 1-l)} + \mathcal{H}_{l-1} \right] \sum_{\tau \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \text{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau_1^{-1} \circ \tau) \\ &= - \chi(\tau_1^{-1}) \left(\left[2 \frac{L'(\chi_{\text{Prim}}, 1-l)}{L(\chi_{\text{Prim}}, 1-l)} + \mathcal{H}_{l-1} \right] \cdot \sum_{\tau \in \text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \text{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau) \right) \end{aligned}$$

and we can thus conclude that if equality (6.12) is true for a certain embedding ι then it is true for any such embedding. This might seem a moot point since we know anyway that equality (6.12) is true but it seemed worth underlining in view of Remark 6.24 below.

Equality (6.12) suggests the following conjecture:

Conjecture 6.20. *Suppose that K is a finite Galois extension of \mathbb{Q} . Suppose that there is an element $c \in \text{Gal}(K|\mathbb{Q})$ in the center of $\text{Gal}(K|\mathbb{Q})$ such that for all embeddings $\iota : K \rightarrow \mathbb{C}$ and all $k \in K$, we have $\iota(c(k)) = \overline{\iota(k)}$ (where $\overline{(\cdot)}$ refers to complex conjugation). Suppose that all the embeddings of K into \mathbb{C} factor through an embedding of $\text{Frac}(D)$ into \mathbb{C} . Suppose finally that $2\text{-disc}(K)$ is invertible in D and that D is a localisation of the ring of integers of a number field.*

Let $f : \mathcal{A} \rightarrow Y$ be an abelian scheme and suppose that we are given an embedding of rings $\rho : \mathcal{O}_K \hookrightarrow \text{End}_Y(\mathcal{A})$.

Let $\chi : \text{Gal}(K|\mathbb{Q}) \rightarrow \mathbb{C}$ be an irreducible Artin character and let $l \geq 1$.

Suppose given a Kähler fibration structure ν_f such that

- ν_f represents the first Chern class of a relatively ample line bundle;
- for any $x \in \mathcal{O}_K$, the endomorphism $\rho(x)^*$ of $H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})$ is adjoint to the endomorphism $\rho(c(x))^*$ of $H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})$, with respect to the metric coming from ν_f .

Suppose that χ and l have the same parity (hence $L(\chi, 1-l) \neq 0$ by Remark 6.7).

Then for any embedding $\iota : \mathcal{O}_K \hookrightarrow D$ we have:

$$\begin{aligned} & \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\iota \circ \tau}) \chi(\tau) \\ &= - \left[2 \frac{L'(\chi, 1-l)}{L(\chi, 1-l)} + \mathcal{H}_{l-1} \right] \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \text{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)_{\iota \circ \tau}) \chi(\tau) \end{aligned}$$

in $\widehat{\text{CH}}_{\mathbb{Q}}^l(Y)$.

Here $\text{disc}(K)$ is the discriminant of the number field K . The endomorphism $\rho(x)^*$ is the endomorphism of $H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})$ obtained by pull-back. The sheaves $\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)$ are understood to carry the L^2 -metric induced by the Kähler fibration structure ν_f . The notation $L(\chi, s)$ refers to the Artin L -function associated with χ (see [59] or [49, § 7.10-12] for an introduction). Note that if $K = \mathbb{Q}(\mu_n)$ then χ can be identified with a Dirichlet character χ_0 via the canonical isomorphism $\text{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q}) \simeq (\mathbb{Z}/(n))^*$ and then one has $L(\chi, s) = L(\chi_0, \text{Prim}, s)$.

In the following remarks, we keep the notation of Conjecture 6.20.

Remark 6.21. If $Y = \text{Spec } D$, K is a CM field and the generic fibre $f : \mathcal{A} \rightarrow Y$ is an abelian variety of dimension $[K : \mathbb{Q}]/2$ (in particular the generic fibre of \mathcal{A} has CM by \mathcal{O}_K), then there always exists a Kähler fibration structure of the type described in Conjecture 6.20. See [52, after Th. A].

Remark 6.22. If $K = \mathbb{Q}(\mu_n)$ then a polarisation with the properties required in Conjecture 6.20 can be constructed as follows. Choose first a $\mu_n(\mathbb{C})$ -equivariant relatively ample line bundle \mathcal{L} on $\mathcal{A}(\mathbb{C})$. Such a line bundle can be obtained in the following way. Let \mathcal{M} be a relatively ample line bundle

on $\mathcal{A}(\mathbb{C})$ (without equivariant structure). The line bundle $\otimes_{a \in \mu_n(\mathbb{C})} a^* \mathcal{M}$ then carries a $\mu_n(\mathbb{C})$ -equivariant structure and is also relatively ample. Suppose without loss of generality that the restriction of \mathcal{L} to the 0-section of $\mathcal{A}(\mathbb{C})$ is an equivariantly trivial line bundle and choose a trivialisation. There is then a unique hermitian metric on \mathcal{L} , whose first Chern character form is translation invariant on the fibres of $\mathcal{A}(\mathbb{C}) \rightarrow Y(\mathbb{C})$ and such that the trivialising map has norm 1 (see e.g. [48, II, 2.1] for all this). Let $\overline{\mathcal{L}}$ be the resulting hermitian line bundle. The first Chern character form $c_1(\overline{\mathcal{L}})$ is then a $\mu_n(\mathbb{C})$ -invariant Kähler fibration structure for $\mathcal{A}(\mathbb{C}) \rightarrow Y(\mathbb{C})$ and it satisfies the properties required in Conjecture 6.20 because for any $\zeta \in \mu_n(\mathbb{C})$ the adjoint of $\rho(\zeta)^*$ is then $\rho(\zeta)^{*, -1} = \rho(\zeta^{-1})^* = \rho(\overline{\zeta})$ and $\mu_n(\mathbb{C})$ generates $\mathbb{Z}[\mu_n]$ as a \mathbb{Z} -module.

Remark 6.23. Notice that the assumptions of Conjecture 6.20 imply that the subbundles $\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)_{\iota \circ \tau}$ of $\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)$ are orthogonal to each other. This follows from the fact that by construction the pull-back endomorphisms $\rho(x)^*$ commute with their adjoints for any $x \in \mathcal{O}_K$.

Remark 6.24. Note that if K is not an abelian extension of \mathbb{Q} , it is not clear that Conjecture 6.20 is independent of the embedding ι . To state this more precisely: we do not know how to show that if Conjecture 6.20 is true for one embedding ι then it is true for any such embedding.

One might wonder how much Conjecture 6.20 depends on the polarisation. We shall show:

Proposition 6.25. *Suppose that K is a finite Galois extension of \mathbb{Q} . Suppose that there is an element $c \in \text{Gal}(K|\mathbb{Q})$ such that for all embeddings $\iota : K \rightarrow \mathbb{C}$ and all $k \in K$, we have $\iota(c(k)) = \iota(k)$ (where $\overline{(\cdot)}$ refers to complex conjugation). Suppose that all the embeddings of K into \mathbb{C} factor through an embedding of $\text{Frac}(D)$ into \mathbb{C} . Suppose finally that $\text{disc}(K)$ is invertible in D and that D is a localisation of the ring of integers of a number field.*

Let $f : \mathcal{A} \rightarrow Y$ be an abelian scheme and suppose that we are given an embedding of rings $\rho : \mathcal{O}_K \hookrightarrow \text{End}_Y(\mathcal{A})$.

Suppose given a Kähler fibration structure ν_f (resp. κ_f) such that

- ν_f (resp. κ_f) represents the first Chern class of a relatively ample line bundle;
- for any $x \in \mathcal{O}_K$, the pull-back endomorphism $\rho(x)^*$ of $H_{\text{Dib}}^1(\mathcal{A}/Y)_{\mathbb{C}}$ is adjoint to the pull-back endomorphism $\rho(c(x))^*$ of $H_{\text{Dib}}^1(\mathcal{A}/Y)_{\mathbb{C}}$, with respect to the metric coming from ν_f (resp. κ_f).

Suppose that l and χ have the same parity. Then for any embedding $\iota : \mathcal{O}_K \hookrightarrow D$, we have

$$\begin{aligned} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)_{\nu_f, \iota \circ \tau}) \chi(\tau) \\ = \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)_{\kappa_f, \iota \circ \tau}) \chi(\tau) \end{aligned}$$

in $\widehat{\text{CH}}_{\mathbb{Q}}^l(Y)$.

Here we write $\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)_{\nu_f, \iota \circ \tau}$ (resp. $\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)_{\kappa_f, \iota \circ \tau}$) for the bundle $H_{\text{Dib}}^1(\mathcal{A}/Y)_{\iota \circ \tau}$ endowed with the L^2 -metric induced by ν_f (resp. κ_f).

Proposition 6.25 in particular says that the truth of Conjecture 6.20 does not depend on the choice of the polarisation. To prove Proposition 6.25, we shall need the following lemma.

Lemma 6.26. *Let M be a complex manifold and let (V, h_0) be a holomorphic vector bundle V on M , endowed with a hermitian metric h_1 . Let $\phi : V \rightarrow V$ be an automorphism of vector bundles and suppose that ϕ is positive definite with respect to h_0 (on each fibre of V). Let h_2 be the hermitian metric on V defined by the formula $h_2(v, w) := h_1(\phi(v), w)$ for any elements $v, w \in V$ which lie in the same fibre.*

Let $\widetilde{\text{ch}}(V, h_1, h_2) \in \widetilde{\mathfrak{A}}(M)$ be the Bott–Chern secondary class of the exact sequence

$$\bar{\mathcal{E}} : 0 \rightarrow V \xrightarrow{\text{Id}} V \rightarrow 0$$

where the first non-zero term from the right carries the metric h_1 and the second non-zero term from the right carries the metric h_2 .

Then the eigenvalues of ϕ are locally constant on M and we have

$$\widetilde{\text{ch}}(V, h_1, h_2) = \sum_{t \in \mathbb{R}_{>0}} \log(t) \text{ch}((V_t, h_1|_{V_t}))$$

where V_t is the kernel of $\phi - t \cdot \text{Id}$.

Proof. Since ϕ is self adjoint on the fibres of V with respect to h_1 , the coefficients of the polynomial $\det(\text{Id} - x \cdot \phi) \in \mathcal{O}(M)[x]$ are real valued holomorphic functions and they are thus locally constant. Furthermore, the automorphism ϕ is diagonalisable on each fibre of V and thus we have a decomposition

$$V \simeq \bigoplus_{t \in \mathbb{R}_{>0}} V_t$$

as an orthogonal direct sum of vector bundles. Furthermore, we have by construction

$$h_2(v, w) = h_1(t \cdot v, w)$$

for any elements of V_t that lie in the same fibre. Hence we have

$$\widetilde{\text{ch}}(V, h_1, h_2) = \sum_{t \in \mathbb{R}_{>0}} \widetilde{\text{ch}}(V_t, h_1|_{V_t}, t \cdot h_1|_{V_t}).$$

Now we have

$$\widetilde{\text{ch}}(V_t, h_1|_{V_t}, t \cdot h_1|_{V_t}) = \log(t) \cdot \text{ch}((V_t, h_1|_{V_t})).$$

See e.g. [22, ex. on p. 22] for this. □

Proof of Proposition 6.25. We start with some preliminary considerations. Let N be a projective complex manifold and let L be an ample line bundle on N . Let $\omega \in H^2(N, \mathbb{C})$ be the first Chern class of L in complex Betti cohomology. Let $v, w \in H^1(N, \mathbb{C})$. We shall write \bar{v} for the complex conjugate of v and $v_{0,1}$ (resp. $v_{1,0}$) for the Hodge components of v (and similarly for w). By the discussion preceding Lemma 6.2, we have the formula

$$\langle v, w \rangle_{\text{Hodge}, L} = \int_N v \wedge * \bar{w} = \frac{i}{(\dim(N) - 1)!} \int_N v \wedge \omega^{\dim(N)-1} \wedge (\bar{w}_{0,1} - \bar{w}_{1,0})$$

for the Hodge metric on $H^1(N, \mathbb{C})$. Now choose another ample line bundle J , with first Chern class $\eta \in H^2(N, \mathbb{C})$ say. The maps

$$\bullet \wedge \omega^{\dim(N)-1} : H^1(N, \mathbb{C}) \rightarrow H^{2 \dim(N)-1}(N, \mathbb{C})$$

and

$$\bullet \wedge \eta^{\dim(N)-1} : H^1(N, \mathbb{C}) \rightarrow H^{2 \dim(N)-1}(N, \mathbb{C})$$

are both isomorphisms by the Hard Lefschetz theorem for singular cohomology. These isomorphisms also respect the underlying \mathbb{Q} -rational Hodge structures. Hence there is a unique isomorphism of \mathbb{Q} -rational Hodge structures

$$M = M(L, J) : H^1(N, \mathbb{Q}) \rightarrow H^1(N, \mathbb{Q})$$

such that

$$\langle M(v), w \rangle_{\text{Hodge}, L} = \langle v, w \rangle_{\text{Hodge}, J}$$

for all $v, w \in H^1(N, \mathbb{C})$. Since both $\langle \bullet, \bullet \rangle_{\text{Hodge}, L}$ and $\langle \bullet, \bullet \rangle_{\text{Hodge}, J}$ are hermitian metrics, the isomorphism M is necessarily positive definite for the metric $\langle \bullet, \bullet \rangle_{\text{Hodge}, L}$. Now suppose furthermore that we are given endomorphisms $e, d : H^1(N, \mathbb{C}) \rightarrow H^1(N, \mathbb{C})$ of \mathbb{C} -vector spaces and suppose that e and d commute and that d is the adjoint of e with respect to $\langle \bullet, \bullet \rangle_{\text{Hodge}, L}$ and with respect to $\langle \bullet, \bullet \rangle_{\text{Hodge}, J}$. Then we contend that e commutes with M . Indeed from the assumptions on d and e we may compute

$$\langle e(M(v)), w \rangle_{\text{Hodge}, L} = \langle M(v), d(w) \rangle_{\text{Hodge}, L}$$

$$\langle M(e(v)), w \rangle_{\text{Hodge}, L} = \langle M(v), d(w) \rangle_{\text{Hodge}, L}$$

and since v, w are arbitrary we conclude that $e \circ M = M \circ e$.

Now let us return to the matter at hand. A straightforward generalisation of the preceding calculation to a relative setting shows that there is an automorphism of vector bundles

$$M = M(\nu_f, \kappa_f) : H_{\text{Dbl}}^1(\mathcal{A}/Y)(\mathbb{C}) \rightarrow H_{\text{Dbl}}^1(\mathcal{A}/Y)(\mathbb{C})$$

which is self adjoint with respect to the L^2 -metric induced by ν_f and such that

$$\langle M(\bullet), \bullet \rangle_{L^2, \nu_f} = \langle \bullet, \bullet \rangle_{L^2, \kappa_f}$$

Furthermore, for any $x \in \mathcal{O}_K$, in view of the assumptions on $\rho(x)^*$, we see that $\rho(x)^*$ commutes with M . Thus M respects the decomposition

$$H_{\text{Dbl}}^1(\mathcal{A}/Y)(\mathbb{C}) \simeq \bigoplus_{\tau \in \text{Gal}(K|\mathbb{Q})} H_{\text{Dbl}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota \circ \tau}$$

Thus, using Lemma 6.26 we may compute

$$\begin{aligned} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\nu_f, \iota\circ\tau}) - \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\kappa_f, \iota\circ\tau}) \\ = \sum_{t \in \mathbb{R}_{>0}} \log(t) \text{ch}^{[l-1]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t}) \end{aligned}$$

where $H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t}$ is the subbundle of $H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau}$ corresponding to the eigenvalue t of M . Now notice that $H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t}$ is isomorphic as a C^∞ -vector bundle to a flat bundle via the comparison isomorphism with the corresponding relative Betti cohomology sheaves. Hence $\text{ch}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t})$ is d -exact in positive degrees and in particular the positive degree part of the expression

$$\sum_{t \in \mathbb{R}_{>0}} \log(t) \text{ch}^{[l-1]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t})$$

vanishes in Aeppli cohomology. We conclude that the difference

$$\sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\nu_f, \iota\circ\tau})\chi(\tau) - \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\kappa_f, \iota\circ\tau})\chi(\tau)$$

vanishes if $l > 1$. This settles the proposition for $l > 1$. If $l = 1$, the difference is

$$\begin{aligned} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^1(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\nu_f, \iota\circ\tau})\chi(\tau) - \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^1(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\kappa_f, \iota\circ\tau})\chi(\tau) \\ = \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \sum_{t \in \mathbb{R}_{>0}} \log(t) \text{rk}(H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t})\chi(\tau) \\ = \sum_{t \in \mathbb{R}_{>0}} \log(t) \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \text{rk}(H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t})\chi(\tau). \end{aligned}$$

We shall now show that $\sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \text{rk}(H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t})\chi(\tau) = 0$ if χ is an odd character. This will conclude the proof of the proposition. To show this, we may suppose that $Y(\mathbb{C})$ is a finite set of points, so suppose for simplicity that $Y = \text{Spec } \mathbb{Z}$. In that case, $H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau}$ reduces to a complex vector space. Via the comparison isomorphism, this vector space has a \mathbb{Q} -rational structure and the automorphism M respects this structure. Thus

$$\text{rk}(H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t}) = \text{rk}(H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\sigma\circ\tau, t})$$

so that the function $\text{rk}(H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t})$ is an even function on $\text{Gal}(K|\mathbb{Q})$. We conclude that

$$\sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \text{rk}(H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})_{\iota\circ\tau, t})\chi(\tau) = 0.$$

□

We shall now prove:

Theorem 6.27. *Conjecture 6.20 holds if K is an abelian extension of \mathbb{Q} .*

Proof. We first record the following elementary construction. Let R and T be two commutative rings and suppose that we are given a ring homomorphism $\phi : R \rightarrow T$. Suppose furthermore that T is free as an R -module and let t_1, \dots, t_r be a basis of T as an R -module. Then there is by definition an isomorphism of R -modules

$$T \simeq \bigoplus_{k=1}^r R$$

and the T -module structure of T is described by a morphism of R -algebras $\psi : T \rightarrow \text{Mat}_{r \times r}(R)$ on the right-hand side of this isomorphism. Here $\text{Mat}_{r \times r}(R)$ is the ring of $r \times r$ matrices with coefficients in R . In particular, if M is an R -module, then there is an isomorphism of R -modules

$$M \otimes_R T \simeq \bigoplus_{k=1}^r M$$

and the T -module structure of $M \otimes_R T$ is again described by ψ via the natural action of $\text{Mat}_{r \times r}(R)$ on $\bigoplus_{k=1}^r M$.

Recall that we now suppose that all assumptions of Conjecture 6.20 are satisfied and that K is an abelian extension of \mathbb{Q} . Let $f = 2 \cdot (\text{conductor of } K)$ (where we mean the conductor in the sense of class field theory). We may replace D by a finite extension and so we may also suppose that D contains some primitive f -th root of unity. By class field theory, there now exists an embedding $\rho : \mathcal{O}_K \hookrightarrow \mathbb{Z}[\mu_f]$ and by assumption there is an embedding $\lambda : \mathbb{Z}[\mu_f] \hookrightarrow D$. We also see that every embedding of K (resp. $\mathbb{Q}(\mu_f)$) into \mathbb{C} factors through an embedding of $\text{Frac}(D)$ into \mathbb{C} and similarly every embedding of K into \mathbb{C} factors through an embedding of $\mathbb{Q}(\mu_f)$ into \mathbb{C} .

Now notice that the ring $\mathbb{Z}[\mu_f]$ is a free module over \mathcal{O}_K via ρ . Indeed, $\mathbb{Z}[\mu_f]$ is generated by a primitive root of unity z as an \mathcal{O}_K -algebra. The minimal polynomial $P(X)$ of z over K divides $X^n - 1$ and hence by Gauss's lemma, we have $P(X) \in \mathcal{O}_K[X]$ and $P(X)$ is a prime element in $\mathcal{O}_K[X]$. Hence there is a surjection $\mathcal{O}_K[X]/(P(X)) \rightarrow \mathbb{Z}[\mu_f]$, which is also injective since $\mathcal{O}_K[X]/(P(X))$ is a domain and $\mathcal{O}_K[X]/(P(X)) \otimes \mathbb{Q} \simeq \mathbb{Q}(\mu_n)$. Thus the elements $1, X, \dots, X^{\deg(P)-1}$ form a basis for $\mathbb{Z}[\mu_f]$ over \mathcal{O}_K via that isomorphism.

So choose a basis b_1, \dots, b_r of $\mathbb{Z}[\mu_f]$ over \mathcal{O}_K . The $\mathbb{Z}[\mu_n]$ -module structure of $\mathbb{Z}[\mu_n]$ viewed as an \mathcal{O}_K -module is then described by a morphism of \mathcal{O}_K -algebras $\psi : \mathbb{Z}[\mu_n] \rightarrow \text{Mat}_{r \times r}(\mathcal{O}_K)$ (see the above elementary construction). We let $\mathcal{B} := \times_{j=1, Y}^r \mathcal{A}$ be the fibre product of \mathcal{A} , r -times with itself over Y . The abelian scheme \mathcal{B} carries an action of $\mathbb{Z}[\mu_f]$ via ψ and we have an isomorphism of $\mathbb{Z}[\mu_f]$ -modules

$$H_{\text{Dib}}^1(\mathcal{B}/Y) \simeq H_{\text{Dib}}^1(\mathcal{A}/Y) \otimes_{\mathcal{O}_K} \mathbb{Z}[\mu_f].$$

Recall that the conductor of a finite abelian extension of \mathbb{Q} has the same support as its discriminant. Thus by Lemma 6.18, there are decompositions into direct sums of \mathcal{O}_Y -modules

$$H_{\text{Dib}}^1(\mathcal{B}/Y) \simeq \bigoplus_{\sigma : \mathbb{Z}[\mu_f] \hookrightarrow D} H_{\text{Dib}}^1(\mathcal{B}/Y)_\sigma \tag{6.13}$$

and

$$H_{\mathrm{Dlb}}^1(\mathcal{A}/Y) \simeq \bigoplus_{\tau: \mathcal{O}_K \hookrightarrow D} H_{\mathrm{Dlb}}^1(\mathcal{A}/Y)_\tau.$$

There is a natural compatibility

$$\bigoplus_{\sigma: \mathbb{Z}[\mu_f] \hookrightarrow D, \sigma|_{\mathcal{O}_K} = \tau} H_{\mathrm{Dlb}}^1(\mathcal{B}/Y)_\sigma \simeq \bigoplus_{j=1}^{[\mathbb{Q}(\mu_f):K]} H_{\mathrm{Dlb}}^1(\mathcal{A}/Y)_\tau \quad (6.14)$$

where $\mathbb{Z}[\mu_f]$ (resp. \mathcal{O}_K) acts on $H_{\mathrm{Dlb}}^1(\mathcal{B}/Y)_\sigma$ (resp. $H_{\mathrm{Dlb}}^1(\mathcal{A}/Y)_\tau$) via σ (resp. τ).

Now choose a $\mu_f(\mathbb{C})$ -invariant Kähler fibration κ on $\mathcal{B}(\mathbb{C})$ associated with a relatively ample line bundle on $\mathcal{A}(\mathbb{C})$. See Remark 6.22 for this.

The character χ of $\mathrm{Gal}(K|\mathbb{Q})$ induces by composition a character of $\mathrm{Gal}(\mathbb{Q}(\mu_f)|\mathbb{Q})$, which we shall also refer to as χ . Choose an extension of the embedding $\iota: K \hookrightarrow D$ to $\mathbb{Q}(\mu_f)$ and also refer to it as ι .

Applying (6.12) to \mathcal{B} and κ and using (6.14), we obtain

$$\begin{aligned} & \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \widehat{\mathrm{ch}}^{[l]}(\overline{H}_{\mathrm{Dlb}}^1(\mathcal{B}/Y)_{\iota \circ \sigma}) \chi(\sigma) \\ &= - \left[2 \frac{L'(\chi_{\mathrm{Prim}}, 1-l)}{L(\chi_{\mathrm{Prim}}, 1-l)} + \mathcal{H}_{l-1} \right] \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \mathrm{ch}^{[l-1]}(H^{1,0}(\mathcal{B}/Y)_{\iota \circ \sigma}) \chi(\sigma) \\ &= -[\mathbb{Q}(\mu_f):K] \left[2 \frac{L'(\chi_{\mathrm{Prim}}, 1-l)}{L(\chi_{\mathrm{Prim}}, 1-l)} + \mathcal{H}_{l-1} \right] \cdot \\ & \quad \cdot \sum_{\tau \in \mathrm{Gal}(K|\mathbb{Q})} \mathrm{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)_{\iota \circ \tau}) \chi(\tau) \quad (6.15) \end{aligned}$$

Here $\overline{H}_{\mathrm{Dlb}}^1(\mathcal{A}/Y)_{\iota \circ \sigma}$ is by assumption equipped with the L^2 -metric induced by the Kähler fibration structure κ . By Proposition 6.25 and (6.14), the element

$$\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \widehat{\mathrm{ch}}^{[l]}(\overline{H}_{\mathrm{Dlb}}^1(\mathcal{B}/Y)_{\iota \circ \sigma}) \chi(\sigma) = \sum_{\tau \in \mathrm{Gal}(K|\mathbb{Q})} \widehat{\mathrm{ch}}^{[l]}(\overline{H}_{\mathrm{Dlb}}^1(\mathcal{B}/Y)_{\iota \circ \tau}) \chi(\tau)$$

does not change if we replace κ by the Kähler fibration structure $\times_{j=1}^r \nu_j$. Hence we have

$$\begin{aligned} & \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \widehat{\mathrm{ch}}^{[l]}(\overline{H}_{\mathrm{Dlb}}^1(\mathcal{B}/Y)_{\iota \circ \sigma}) \chi(\sigma) \\ &= [\mathbb{Q}(\mu_f):K] \sum_{\tau \in \mathrm{Gal}(K|\mathbb{Q})} \widehat{\mathrm{ch}}^{[l]}(\overline{H}_{\mathrm{Dlb}}^1(\mathcal{A}/Y)_{\iota \circ \tau}) \chi(\tau) \end{aligned}$$

which concludes the proof. \square

Remark 6.28. For $l = 1$, Theorem 6.27 proves a weak form of the conjecture of Gross–Deligne for certain linear combinations of Hodge structures cut out in the cohomology of $\mathcal{A}(\mathbb{C})$. See [43] and also [56] for details. A different

approach to this special case is described in the paper [25] which relies on a deep result of Saito and Terasoma (see [53]). It would be very interesting if Fresán's approach [25] could be generalised to include the case $l > 1$.

Complement 6.29. We work under the assumptions of Conjecture 6.20 and we suppose that K is an abelian extension of \mathbb{Q} . Then the proof of Theorem 6.27 shows that the equality

$$\begin{aligned} & \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau) \\ &= - \left[2 \frac{L'(\chi, 1-l)}{L(\chi, 1-l)} + \mathcal{H}_{l-1} \right] \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \text{ch}^{[l-1]}(H^{1,0}(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau) \end{aligned}$$

actually holds in $\widehat{\text{CH}}_{\mathbb{Q}(\mu_f)(\chi)}^l(Y)$ (and not just in $\widehat{\text{CH}}_{\mathbb{Q}}^l(Y)$).

Corollary 6.30. *Suppose that the assumptions of Conjecture 6.20 hold and suppose that K is an abelian extension of \mathbb{Q} . Then we have*

$$\sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \text{ch}^{[l]}(H_{\text{Dib}}^1(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau) = 0$$

in $\text{CH}^l(Y)_{\mathbb{Q}}$ for any character χ of $\text{Gal}(K|\mathbb{Q})$ of the same parity as l and any embedding $\iota : \mathcal{O}_K \hookrightarrow D$.

Again, it makes sense to ask whether Corollary 6.30 might hold in a more general situation. This leads to the purely geometric

Conjecture 6.31. *Suppose that the assumptions of Conjecture 6.20 hold. Then*

$$\sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \text{ch}^{[l]}(H_{\text{Dib}}^1(\mathcal{A}/Y)_{\iota \circ \tau})\chi(\tau) = 0$$

in $\text{CH}^l(Y)_{\mathbb{Q}}$ for any character χ of $\text{Gal}(K|\mathbb{Q})$ of the same parity as l and any embedding $\iota : \mathcal{O}_K \hookrightarrow D$.

See [44, Prop. 3] for more about this conjecture in a slightly more restrictive setting.

We now indulge in some wilder speculation. The fact that the formula in Conjecture 6.20 looks very ‘motivic’ suggests the following vague conjecture, which seems to be a good computational thumb rule in many examples.

Vague conjecture 6.32. *Suppose that K is a finite Galois extension of \mathbb{Q} . Suppose that all the embedding of K into \mathbb{C} factor through an embedding of $\text{Frac}(D)$ into \mathbb{C} . Suppose also that $2 \cdot \text{disc}(K)$ is invertible in D and that D is a localisation of the ring of integers of a number field.*

Let $f : \mathcal{M} \rightarrow Y$ be a ‘log smooth relative motive’ over Y and suppose that we are given an embedding of rings $\mathcal{O}_K \hookrightarrow \text{End}_Y(\mathcal{M})$.

Let $\chi : \text{Gal}(K|\mathbb{Q}) \rightarrow \mathbb{C}$ be an irreducible Artin character. Let $l \geq 1$. Suppose that χ and l have the same parity (hence $L(\chi, 1-l) \neq 0$ by Remark 6.7).

Then there exists a ‘polarisation’ on \mathcal{M}/Y , which is compatible with the action of \mathcal{O}_K in some sense and such that for any embedding $\iota : \mathcal{O}_K \hookrightarrow D$ we have:

$$\begin{aligned} & \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[l]}(\overline{H}_{\text{Dlb}}^k(\mathcal{M}/Y)_{\iota \circ \tau}(\log))\chi(\tau) \\ &= - \left[2 \frac{L'(\chi, 1-l)}{L(\chi, 1-l)} + \mathcal{H}_{l-1} \right] \cdot \\ & \quad \cdot \sum_{p+q=k} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} p \cdot \text{ch}^{[l-1]}(H^{p,q}(\mathcal{M}/Y)_{\iota \circ \tau}(\log))\chi(\tau) \end{aligned}$$

in $\widehat{\text{CH}}_{K(\chi)}^l(Y)(\log)$.

Here $H_{\text{Dlb}}^k(X/Y)(\log)$ and $H^{p,q}(X/Y)(\log)$ refer to logarithmic cohomology and the metric on $H_{\text{Dlb}}^k(X/Y)(\log)$ is induced by the polarisation, which is general mildly singular. The ring $\widehat{\text{CH}}_{K(\chi)}^l(Y)(\log)$ is a generalised arithmetic intersection ring, as in [15]. Note that in this vague conjecture, if \mathcal{M} is smooth over Y , then one may remove the ‘(log)’ symbols from the formula.

In particular, this ‘conjecture’ should apply to generically abelian semi-abelian schemes, where it should be possible to make a precise conjecture, extending Conjecture 6.20. We refrain from trying to do this here because the generalised arithmetic intersection theory that would be necessary for this has apparently not yet been fully defined (see also the discussion in the introduction). In some of the examples drawn from the literature that we shall consider in section 7 below, the corresponding articles produce generalised arithmetic intersection theories tailor-made for the geometric situation under consideration.

7. Examples

We shall now show that various formulae proven in the literature are formally compatible with Conjecture 6.20 and in some cases are partially consequences of Theorem 6.27.

We use the notation of Conjecture 6.20. We repeat them for the convenience of the reader.

Let K be a finite Galois extension of \mathbb{Q} . Suppose that there is an element $c \in \text{Gal}(K|\mathbb{Q})$ in the center of $\text{Gal}(K|\mathbb{Q})$ such that for all embeddings $\iota : K \rightarrow \mathbb{C}$ and all $k \in K$, we have $\iota(c(k)) = \iota(\overline{k})$ (where $\overline{(\cdot)}$ refers to complex conjugation). Suppose that all the embeddings of K into \mathbb{C} factor through an embedding of $\text{Frac}(D)$ into \mathbb{C} . Suppose finally that $2 \cdot \text{disc}(K)$ is invertible in D and that D is a localisation of the ring of integers of a number field. Let $f : \mathcal{A} \rightarrow Y$ be an abelian scheme and suppose that we are given an embedding of rings $\rho : \mathcal{O}_K \hookrightarrow \text{End}_Y(\mathcal{A})$. Let $\chi : \text{Gal}(K|\mathbb{Q}) \rightarrow \mathbb{C}$ be an irreducible Artin

character and let $l \geq 1$. Finally suppose given a Kähler fibration structure ν_f such that

- ν_f represents the first Chern class of a relatively ample line bundle;
- for any $x \in \mathcal{O}_K$, the endomorphism $\rho(x)^*$ of $H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})$ is adjoint to the endomorphism $\rho(c(x))^*$ of $H_{\text{Dlb}}^1(\mathcal{A}/Y)(\mathbb{C})$, with respect to the metric coming from ν_f .

We shall also make the assumption that the polarisation has been chosen in such a way that for each $\tau \in \text{Gal}(K|\mathbb{Q})$ we have an isometric isomorphism

$$f_*(\Omega_{\mathcal{A}/Y})_{\iota \circ \tau} \simeq \mathbb{R}^1 f_*(\mathcal{O}_{\mathcal{A}})_{\iota \circ c \circ \tau}^{\vee}.$$

This is a compatibility with duality that is often verified in practice.

Example 7.1 (the formula of Colmez). See [16] and [17]. Suppose that $Y = \text{Spec } D$ and that K is a CM field of degree $2 \cdot \dim(\mathcal{A}/Y)$. Suppose that K is an abelian extension of \mathbb{Q} . Let $\Phi : \text{Hom}(K, D) \rightarrow \{0, 1\}$ be the associated CM type. By definition,

$$\Phi(\iota \circ \tau) = \text{rk}(H^{1,0}(\mathcal{A}/Y)_{\iota \circ c \circ \tau}).$$

We identify Φ with a function $\text{Gal}(K|\mathbb{Q}) \rightarrow \{0, 1\}$ via ι . From now until the end of the computation, we shall drop the embedding ι from the notation. Theorem 6.27 gives the equality:

$$\begin{aligned} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{c}_1(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\tau}) \chi(\tau) &= -2 \frac{L'(\chi, 0)}{L(\chi, 0)} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \Phi(\tau) \chi(c \circ \tau) \\ &= 2 \frac{L'(\chi, 0)}{L(\chi, 0)} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \Phi(\tau) \chi(\tau) \end{aligned}$$

in $\widehat{\text{CH}}_{\mathbb{Q}}^1(D)$ for any odd one-dimensional character. By assumption, we have

$$\widehat{c}_1(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\tau}) = -\widehat{c}_1(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{c \circ \tau})$$

so that for even characters χ , we have

$$\sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{c}_1(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\tau}) \chi(\tau) = 0.$$

We recall the definition of the scalar product

$$\langle f, g \rangle := \frac{1}{[K : \mathbb{Q}]} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} f(\tau) \overline{g(\tau)}$$

and of the convolution product

$$(f * g)(\sigma) := \frac{1}{[K : \mathbb{Q}]} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} g(\tau) f(\tau^{-1} \sigma)$$

of two functions $f, g : \text{Gal}(K|\mathbb{Q}) \rightarrow \mathbb{C}$. Recall that if h is a one-dimensional character then we have

$$\langle f * g, h \rangle = \langle f, h \rangle \cdot \langle g, h \rangle$$

for any two functions $f, g : \text{Gal}(K|\mathbb{Q}) \rightarrow \mathbb{C}$.

Define the function $\Phi^\vee : \text{Gal}(K|\mathbb{Q}) \rightarrow \{0, 1\}$ by $\Phi^\vee(\tau) := \Phi(\tau^{-1})$. Using the fact that the one-dimensional characters $\text{Gal}(K|\mathbb{Q}) \rightarrow \mathbb{C}$ form an orthogonal basis of the space of complex valued functions on $\text{Gal}(K|\mathbb{Q})$ we get that

$$\sum_{\tau} \widehat{c}_1(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\tau}) = \frac{1}{[K : \mathbb{Q}]} \sum_{\chi \text{ odd}} \bar{\chi}(\tau) \left[2 \frac{L'(\chi, 0)}{L(\chi, 0)} \sum_{\sigma \in \text{Gal}(K|\mathbb{Q})} \Phi(\sigma) \chi(\sigma) \right] \quad (7.1)$$

From (7.1) we obtain the equality

$$\begin{aligned} & \widehat{c}_1(f_*(\Omega_{\mathcal{A}/Y})) \\ &= \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{c}_1(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\tau}) \Phi(c \circ \tau) \\ &= - \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{c}_1(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_{\tau}) \Phi(\tau) \\ &= - \frac{1}{[K : \mathbb{Q}]} \sum_{\chi \text{ odd}} 2 \frac{L'(\chi, 0)}{L(\chi, 0)} \left(\sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \Phi(\tau) \bar{\chi}(\tau) \right) \left(\sum_{\sigma \in \text{Gal}(K|\mathbb{Q})} \Phi(\sigma) \chi(\sigma) \right) \\ &= - \frac{1}{[K : \mathbb{Q}]} \sum_{\chi \text{ odd}} 2 \frac{L'(\chi, 0)}{L(\chi, 0)} \left(\sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \Phi(\tau) \bar{\chi}(\tau) \right) \left(\sum_{\sigma \in \text{Gal}(K|\mathbb{Q})} \Phi^\vee(\sigma) \bar{\chi}(\sigma) \right) \\ &= -[K : \mathbb{Q}] \cdot \sum_{\chi \text{ odd}} 2 \frac{L'(\chi, 0)}{L(\chi, 0)} \langle \Phi * \Phi^\vee, \chi \rangle. \end{aligned} \quad (7.2)$$

The formula (7.2) implies the formula of Colmez (see [17, Conjecture 3] and the discussion after the statement) up to a term of the form

$$\sum_{p|D_K} r_p \log(p)$$

where $r_p \in \bar{\mathbb{Q}}$.

Example 7.2 (the formula of Bost and Khn). See [39] and also an unpublished manuscript by J.-B. Bost. In that case, \mathcal{A}/Y is an elliptic scheme and $K = \mathbb{Q}$. Applying Theorem 6.27, we obtain

$$\widehat{\text{ch}}^{[2]}(\overline{H}^1(\mathcal{A}/Y)) = - \left[2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 1 \right] \cdot c_1(H^{1,0}(\mathcal{A}/Y))$$

i.e.

$$\widehat{c}_1(\bar{\omega})^2 = - \left[2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 1 \right] \cdot c_1(\omega) \quad (7.3)$$

where $\bar{\omega}$ is the Hodge bundle of \mathcal{A} (this is the restriction of the sheaf of differentials of \mathcal{A}/Y by the unit section) endowed with the Petersson metric. Note that equality (7.3) is of little interest because if $\mathcal{A}_{\mathbb{C}}$ has non-zero Kodaira–Spencer class then Y cannot be chosen to be proper over D (this follows from the structure of the moduli spaces of elliptic curves) so that one

always has $c_1(\omega) = 0$. The formula of Bost and Khn has the same shape as (7.3) but is valid for some generically abelian semiabelian schemes over Y (for which $c_1(\omega) \neq 0$). It allows mild singularities and can thus be understood as a ‘special case’ of the vague Conjecture 6.32.

Note that G. Freixas gives in [23] a proof of a weak form of the formula of Bost–Kühn by relating it to the formula obtained in Example 7.3 below, which has almost the same form. The link comes from the Jacquet–Langlands correspondence. A conceptually similar method is used in a different context in the article [46], where the Fourier–Mukai transformation takes the place of the Jacquet–Langlands correspondence.

Example 7.3 (families of abelian surfaces with complex multiplication by a quadratic imaginary extension of \mathbb{Q} ; the formula of Kudla, Rapoport and Yang). See [38, T. 1.05].

In that case, $\dim(\mathcal{A}/Y) = 2$ and K is a quadratic imaginary extension of \mathbb{Q} . In particular the group $\text{Gal}(K|\mathbb{Q})$ has precisely one non-trivial character χ and this character is odd. Theorem 6.27 gives

$$\begin{aligned} & \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[1]}(\overline{H}_{\text{Dlb}}^1(\mathcal{A}/Y)_\tau) \chi(\tau) \\ &= -2 \frac{L'(\chi, 0)}{L(\chi, 0)} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \text{rk}(H^{1,0}(\mathcal{A}/Y)_\tau) \chi(\tau). \end{aligned} \quad (7.4)$$

Write $\bar{\tau} := c \circ \tau$ in the following computations. From now on until the end of the computation, we shall drop the embedding ι from the notation. We have a decomposition

$$f_*(\Omega_{\mathcal{A}/Y}) \simeq f_*(\Omega_{\mathcal{A}/Y})_\tau \oplus f_*(\Omega_{\mathcal{A}/Y})_{\bar{\tau}}$$

and

$$\mathbb{R}^1 f_*(\mathcal{O}_{\mathcal{A}}) \simeq (f_*(\Omega_{\mathcal{A}/Y})_\tau)^\vee \oplus (f_*(\Omega_{\mathcal{A}/Y})_{\bar{\tau}})^\vee$$

so that we may rewrite (7.4) as

$$\begin{aligned} & 2 \left(\widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_\tau) - \widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_{\bar{\tau}}) \right) \\ &= -2 \frac{L'(\chi, 0)}{L(\chi, 0)} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \text{rk}(f_*(\Omega_{\mathcal{A}/Y})_\tau) \chi(\tau). \end{aligned}$$

Squaring the preceding equality, we see that

$$\begin{aligned} & \left(\widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_\tau) - \widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_{\bar{\tau}}) \right)^2 \\ &= \widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_\tau)^2 + \widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_{\bar{\tau}})^2 - 2 \cdot \widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_\tau) \cdot \widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_{\bar{\tau}}) \\ &= 0 \end{aligned}$$

so that

$$\widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_\tau)^2 + \widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_{\bar{\tau}})^2 = 2 \cdot \widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_\tau) \cdot \widehat{c}_1(f_*(\bar{\Omega}_{\mathcal{A}/Y})_{\bar{\tau}}).$$

Now since \mathcal{A}/Y can also be viewed as carrying an action of $\mathbb{Q} = \mathbb{Q}(\mu_2)$, Theorem 6.27 also gives

$$\begin{aligned} \widehat{\text{ch}}^{[2]}(\overline{H}^1(\mathcal{A}/Y)) &= -\left[2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 1\right] \cdot c_1(H^{1,0}(\mathcal{A}/Y)) \\ &= -\left[2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 1\right] \cdot c_1(f_*(\Omega_{\mathcal{A}/Y})) \end{aligned}$$

where now

$$\widehat{\text{ch}}^{[2]}(\overline{H}^1(\mathcal{A}/Y)) = \widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y})_{\tau})^2 + \widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y})_{\bar{\tau}})^2$$

so that

$$\begin{aligned} \widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y}))^2 &= (\widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y})_{\tau}) + \widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y})_{\bar{\tau}}))^2 \\ &= 2 \cdot \widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y})_{\tau})^2 + 2 \cdot \widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y})_{\bar{\tau}})^2 \end{aligned}$$

and

$$\widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y}))^2 = -2 \cdot \left[2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 1\right] \cdot c_1(f_*(\Omega_{\mathcal{A}/Y})). \quad (7.5)$$

Formula (7.5) implies the formula [38, Th. 1.0.5] up to a factor of the form

$$\sum_{p|D_K} r_p \log(p)$$

where $r_p \in \overline{\mathbb{Q}}$.

Example 7.4 (families of abelian surfaces with an action by a real quadratic extension of \mathbb{Q} ; the formula of Bruiner, Burgos and Kühn). See [12, Th. B]. In this case $\dim(\mathcal{A}/Y) = 2$ and K is a real quadratic extension of \mathbb{Q} . All the one-dimensional characters of $\text{Gal}(K|\mathbb{Q}) = \{\text{Id}, \tau_0\}$ are even and there is only one non-trivial one-dimensional character χ_0 . We again drop the embedding $\iota: K \hookrightarrow D$ from the notation. For any one-dimensional character, Theorem 6.27 gives:

$$\begin{aligned} \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[2]}(\overline{H}_{\text{Dib}}^1(\mathcal{A}/Y)_{\tau}) \chi(\tau) \\ = -\left[2 \frac{L'(\chi, -1)}{L(\chi, -1)} + \mathcal{H}_{l-1}\right] \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} c_1(H^{1,0}(\mathcal{A}/Y)_{\tau}) \chi(\tau). \end{aligned}$$

By assumption, this translates to

$$\begin{aligned} 2 \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} \widehat{\text{ch}}^{[2]}((f_*(\overline{\Omega}_{\mathcal{A}/Y}))_{\tau}) \chi(\tau) \\ = -\left[2 \frac{L'(\chi, -1)}{L(\chi, -1)} + 1\right] \sum_{\tau \in \text{Gal}(K|\mathbb{Q})} c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau}) \chi(\tau). \end{aligned}$$

Specialising this to each character, we obtain:

$$\begin{aligned} & \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}})^2 + \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})^2 \\ &= -\left[2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 1\right] c_1(f_*(\Omega_{\mathcal{A}/Y})) \end{aligned}$$

and

$$\begin{aligned} & \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}})^2 - \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})^2 \\ &= -\left[2 \frac{L'(\chi_0, -1)}{L(\chi_0, -1)} + 1\right] \left(c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}}) - c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})\right). \end{aligned}$$

Note that this implies that

$$c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})^2 = c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}})^2 = 0.$$

Now we may compute

$$\begin{aligned} & \widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y}))^3 \\ &= \left(\widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}}) + \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})\right)^3 \\ &= \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}})^3 + \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})^3 \\ &\quad + 3 \cdot \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0}) \cdot \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}})^2 \\ &\quad + 3 \cdot \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}}) \cdot \widehat{c}_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})^2 \\ &= -\left(c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}}) + 3 \cdot c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})\right) \cdot \frac{1}{2} \left[2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 1\right] \\ &\quad \cdot c_1(f_*(\Omega_{\mathcal{A}/Y})) + \left[2 \frac{L'(\chi_0, -1)}{L(\chi_0, -1)} + 1\right] (c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}}) - c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})) \\ &\quad - \left(c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0}) + 3 \cdot c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}})\right) \cdot \frac{1}{2} \left[2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 1\right] \\ &\quad \cdot c_1(f_*(\Omega_{\mathcal{A}/Y})) - \left[2 \frac{L'(\chi_0, -1)}{L(\chi_0, -1)} + 1\right] (c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\text{Id}}) - c_1((f_*(\Omega_{\mathcal{A}/Y}))_{\tau_0})) \\ &= -\left(2 \cdot \left[2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 1\right] + \left[2 \frac{L'(\chi_0, -1)}{L(\chi_0, -1)} + 1\right]\right) \cdot c_1(\Omega_{\mathcal{A}/Y})^2. \end{aligned}$$

This may be rewritten in terms of the zeta function of K . Recall that we have

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi_0, s).$$

We finally obtain the equality

$$\widehat{c}_1(f_*(\overline{\Omega}_{\mathcal{A}/Y}))^3 = -\left(2 \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + 2 \frac{\zeta'_K(-1)}{\zeta_K(-1)} + 3\right) \cdot c_1(\Omega_{\mathcal{A}/Y})^2, \quad (7.6)$$

which should be compared with [12, Th. B]. As for the formula of Bost and Kühn, equality (7.6) is not very interesting because \mathcal{A} is not allowed to be semiabelian. The formula in [12, Th. B] has the same shape as (7.6) but allows semiabelian schemes and allows the metric to have mild singularities. It can thus again be understood as a ‘special case’ of the vague Conjecture 6.32.

Another result that it would be very interesting to relate to our vague Conjecture 6.32 is Th. 1.1 in the article [24], which concerns twisted Hilbert modular surfaces. We do not know yet whether Th. 1.1 can be related to Conjecture 6.20, or even to the vague Conjecture 6.32.

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