HODGE THEORY OF KLOOSTERMAN CONNECTIONS

JAVIER FRESÁN, CLAUDE SABBAH, and JENG-DAW YU

Abstract

We construct motives over the rational numbers associated with symmetric power moments of Kloosterman sums, and prove that their L-functions extend meromorphically to the complex plane and satisfy a functional equation conjectured by Broadhurst and Roberts. Although the motives in question turn out to be "classical," we compute their Hodge numbers by means of the irregular Hodge filtration on their realizations as exponential mixed Hodge structures. We show that all Hodge numbers are either zero or one, which implies potential automorphy thanks to recent results of Patrikis and Taylor.

Contents

1. Introduction

This article is devoted to the study of a family of global L-functions built up by assembling symmetric power moments of Kloosterman sums over finite fields. We prove that they arise from potentially automorphic motives over the rational numbers, and hence admit a meromorphic extension to the complex plane that satisfies the expected functional equation. The exact shape of the latter was conjectured by Broadhurst and Roberts.

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1.1. L-Functions of symmetric power moments of Kloosterman sums

Let p be a prime number, let $\mathbf{F}_p$ be the finite field with p elements, and let $\overline{\mathbf{F}}_p$ be an algebraic closure of $\mathbf{F}_p$. If q is a power of p, then we denote by $\mathbf{F}_q$ the subfield of $\overline{\mathbf{F}}_p$ with q elements and by $\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}\colon \mathbf{F}_q \to \mathbf{F}_p$ its trace map. Let $\psi\colon \mathbf{F}_p \to \mathbf{C}^\times$ be a nontrivial additive character. For each $a \in \mathbf{F}_q^\times$, the *Kloosterman sum* is the real number

$$\mathrm{Kl}_2(a;q) = \sum_{x \in \mathbf{F}_q^\times} \psi\big(\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}(x + a/x)\big). \tag{1.1}$$

As an application of the Riemann hypothesis for curves over finite fields, Weil [62] proved that there exist algebraic integers α_a, β_a of absolute value $\sqrt{q}$ satisfying $\mathrm{Kl}_2(a;q) = -(\alpha_a + \beta_a)$ and $\alpha_a\beta_a = q$. For each integer $k \geq 1$, we define the kth symmetric powers of Kloosterman sums as

$$\mathrm{Kl}_2^{\mathrm{Sym}^k}(a;q) = \sum_{i=0}^{k} \alpha_a^i \beta_a^{k-i}$$

and, summing over all a, we form the *moments*

$$m_2^k(q) = \sum_{a \in \mathbf{F}_q^\times} \mathrm{Kl}_2^{\mathrm{Sym}^k}(a;q).$$

Note that this convention makes $\mathrm{Kl}_2^{\mathrm{Sym}^1}(a;q)$ opposite to $\mathrm{Kl}_2(a;q)$. Contrary to Kloosterman sums and their symmetric powers, the moments $m_2^k(q)$ are rational integers that do not depend on the choice of the additive character. We pack them into the generating series

$$Z_k(p;T) = \exp\Big(\sum_{n=1}^{\infty} m_2^k(p^n)\frac{T^n}{n}\Big),$$

which in fact turns out to be a polynomial with integer coefficients. The first few cases are easy to compute: both $Z_1(p;T)$ and $Z_2(p;T)$ are equal to $1-T$, and the equalities

$$Z_3(p;T) = (1-T)\Big(1 - \Big(\frac{p}{3}\Big)p^2T\Big),$$

$$Z_4(p;T) = \begin{cases} 1-T & \text{if } p = 2, \\ (1-T)(1-p^2T) & \text{if } p > 2, \end{cases}$$

hold, where $(p/3)$ stands for the Legendre symbol. From this one may already infer that the polynomial $Z_k(p;T)$ is always divisible by $1-T$. Other so-called *trivial* *factors* appear when k is a multiple of 4 or when k is even and p is small compared with k (see Section 5.1.1 below). Better behaved than $Z_k(p;T)$ is the polynomial $M_k(p;T)$ obtained by removing these trivial factors, since all its roots then have the same absolute value $p^{-(k+1)/2}$.

We shall now build a global L-function over $\mathbf{Q}$ with the above polynomials as local Euler factors. We first discuss the case of odd symmetric powers, say of the form $k = 2m + 1$. Let S be the set of odd prime numbers smaller than or equal to k. For all $p \notin S$, define the local factor at p as $L_k(p;s) = M_k(p;p^{-s})^{-1}$, and consider the Euler product

$$L_k(s) = \prod_{p \notin S} L_k(p;s),$$

which by the previous remark about the roots of $M_k(p;T)$ converges absolutely on the half-plane $\mathrm{Re}(s) > (k+3)/2$. This function is expected to have meromorphic continuation to the entire complex plane and satisfy a functional equation relating its values at s and $k+2-s$. As usual, the functional equation has a neat formulation after completing the L-function by adding local factors at $p \in S$, as we shall do in (5.13) below, and gamma factors at infinity. We set $\mathfrak{N}_k = 1_{\mathrm{s}} 3_{\mathrm{s}} 5_{\mathrm{s}} \cdots k_{\mathrm{s}}$, where n_{s} denotes the square-free part of an integer n (i.e., the product of all prime numbers p such that the p-adic valuation $v_p(n)$ is odd), and we consider the complete L-function

$$\Lambda_k(s) = \Big(\frac{\mathfrak{N}_k}{\pi^m}\Big)^{s/2} \prod_{j=1}^{m} \Gamma\Big(\frac{s-j}{2}\Big) \prod_{p \text{ prime}} L_k(p;s).$$

THEOREM 1.2
Assume that k is odd. The function $L_k(s)$ admits a meromorphic continuation to the complex plane and satisfies the functional equation

$$\Lambda_k(s) = \Lambda_k(k+2-s).$$

A similar result holds for even symmetric powers, except that we were unable to make the local invariants explicit at $p = 2$. To formulate the statement, we write either $k = 2m + 4$ or $k = 2m + 2$ with m an even integer, and we define S as the set of all prime numbers smaller than or equal to $k/2$. The local factors at odd primes in S are described in (5.22) below. Besides, we set $\mathfrak{N}'_k = 2_{\mathrm{u}} 4_{\mathrm{u}} 6_{\mathrm{u}} \cdots k_{\mathrm{u}}$, where n_{u} denotes the odd part of the radical of an integer n (i.e., the product of all odd primes dividing n). We then complete the L-function outside the prime 2 as follows:

$$\Lambda'_k(s) = \Big(\frac{\mathfrak{N}'_k}{\pi^m}\Big)^{s/2} \prod_{j=1}^{m} \Gamma\Big(\frac{s-j}{2}\Big) \prod_{p \neq 2} L_k(p;s).$$

THEOREM 1.3

Assume that k is even. The function $L_k(s)$ meromorphically extends to the complex plane. Moreover, there exists a sign $\varepsilon_k \in \{\pm 1\}$, an integer $r_k \geq 0$, and a reciprocal of a polynomial with rational coefficients $L_k(2;T)$ such that, setting

$$\Lambda_k(s) = 2^{r_k s/2} L_k(2; 2^{-s}) \Lambda'_k(s),$$

the following functional equation holds:

$$\Lambda_k(s) = \varepsilon_k \Lambda_k(k+2-s).$$

The above formulas for $\Lambda_k(s)$ match the numerical observations made by Broadhurst and Roberts in [4, (128)], [5], and [47], up to replacing their variable s with $s-2$ due to a Tate twist. For even k, they also conjecture that the elusive invariants take the values $r_k = \lfloor k/6 \rfloor$ and $\varepsilon_k = (-1)^{t_k}$, with t_k given by the formula

$$t_k = \lfloor k/8 \rfloor + \sum_{p \equiv 1 \,(\mathrm{mod}\, 4)} \lfloor k/2p \rfloor + \sum_{p \equiv 3 \,(\mathrm{mod}\, 4)} \lfloor k/4p \rfloor + \delta_{8\mathbf{Z}}(k),$$

where $\delta_{8\mathbf{Z}}$ is the characteristic function of multiples of 8. We explain in Section 5.3.1 below how the last three terms above fit with the local computations at odd primes in S and at infinity.

Theorems 1.2 and 1.3 were previously only known for $k \leq 8$. The first four cases are straightforward, as the L-function is trivial for $k = 1, 2, 4$ and agrees, for $k = 3$, with the shifted Dirichlet L-function $L(\chi_3, s-2)$ of the nontrivial quadratic character modulo 3. In the next four cases, the L-function can be expressed in terms of a Hecke cusp form for some congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbf{Z})$, as indicated in Table 1.

Table 1. Modularity for $k = 5, 6, 7, 8$.

k	$L_k(s)$	Modular form	References
5	$L(f_3, s-2)$	$f_3 \in S_3(\Gamma_0(15), (\cdot/15))$	Livné [39], Peters, Top, and van der Vlugt [45]
6	$L(f_4, s-2)$	$f_4 \in S_4(\Gamma_0(6))$	Hulek et al. [29]
7	$L(\mathrm{Ad}(g), s-2)$	$g \in S_3(\Gamma_0(125), (\cdot/21)\chi_5)$, χ_5 Dirichlet character modulo 5 with $\chi_5(2) = -i$	conjectured by Evans [17]; proved by Yun [64]
8	$L(f_6, s-2)$	$f_6 \in S_6(\Gamma_0(6))$	conjectured by Evans [16]; proved by Yun [64]

1.2. Cohomological interpretation

After Deligne [8], Kloosterman sums arise as traces of Frobenius acting on an étale local system on the torus $\mathbb{G}_{\mathrm{m},\mathbf{F}_p}$. Let ℓ be a prime number distinct from p, and let $\overline{\mathbf{Q}}_\ell$ be an algebraic closure of the field of ℓ-adic numbers. Once we view the character

ψ as taking values in $\overline{\mathbf{Q}}_\ell$ by choosing a primitive pth root of unity in this field, there is a rank-1 ℓ-adic local system $\mathscr{L}_\psi$ on the affine line $\mathbb{A}^1_{\mathbf{F}_p}$ with trace function $z \mapsto \psi(\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}(z))$, the so-called *Artin–Schreier sheaf*. The *Kloosterman sheaf* Kl_2 is then defined by pulling back and pushing out $\mathscr{L}_\psi$ through the diagram

$$\begin{tikzcd} & \mathbb{G}^2_{\mathrm{m},\mathbf{F}_p} \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ \mathbb{G}_{\mathrm{m},\mathbf{F}_p} & & \mathbb{A}^1_{\mathbf{F}_p} \end{tikzcd} \tag{1.4}$$

where, if x and z are coordinates on $\mathbb{G}^2_{\mathrm{m},\mathbf{F}_p}$, the function f is given by $x + z/x$ and π stands for the projection to the z-coordinate. That is, we set

$$\mathrm{Kl}_2 = \mathrm{R}\pi_! f^* \mathscr{L}_\psi[1].$$

Deligne showed that the object Kl_2 is concentrated in degree 0, and that it is a rank-2 lisse sheaf on $\mathbb{G}_{\mathrm{m},\mathbf{F}_p}$ tamely ramified at zero, wildly ramified at infinity, and pure of weight 1. Indeed, the "forget supports" map $\mathrm{R}\pi_! f^*\mathscr{L}_\psi \to \mathrm{R}\pi_* f^*\mathscr{L}_\psi$ is an isomorphism. Grothendieck's trace formula and base change yield the equalities

$$\mathrm{Kl}_2(a, q) = -\,\mathrm{tr}(\mathrm{Frob}_a \mid \mathrm{Kl}_2) = -(\alpha_a + \beta_a),$$

where α_a and β_a are the eigenvalues of Frobenius acting on a geometric fiber of Kl_2 above a. In the same vein, symmetric powers of Kloosterman sums are local traces of Frobenius on the symmetric powers $\mathrm{Sym}^k \mathrm{Kl}_2$. To obtain the moments, we consider the action of the geometric Frobenius F_p on the étale cohomology with compact support of $\mathrm{Sym}^k \mathrm{Kl}_2$. Since it is concentrated in degree 1, invoking the trace formula again we get the equality

$$Z_k(p;T) = \det\bigl(1 - F_p T \bigm| \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)\bigr). \tag{1.5}$$

It follows that $Z_k(p;T)$ is a polynomial with integer coefficients of degree minus the Euler characteristic of $\mathrm{Sym}^k \mathrm{Kl}_2$ which, by the Grothendieck–Ogg–Shafarevich formula, is equal to its Swan conductor at infinity. Fu and Wan computed it for odd primes p in [21], completing partial results by Robba [46]:

$$\deg Z_k(p;T) = \mathrm{Sw}_\infty(\mathrm{Sym}^k \mathrm{Kl}_2) = \begin{cases} \frac{k+1}{2} - \lfloor \frac{k}{2p} + \frac{1}{2} \rfloor & \text{if } k \text{ odd}, \\ \frac{k}{2} - \lfloor \frac{k}{2p} \rfloor & \text{if } k \text{ even}. \end{cases}$$

The remaining case $p = 2$ was treated by Yun [64], who proved that the Swan conductor is equal to $(k+1)/2$ if k is odd and to $\lfloor (k+2)/4 \rfloor$ if k is even. Observe that, when p is large compared with k, the degree takes the uniform value $\lfloor (k+1)/2 \rfloor$.

The sets S from Section 1.1 consist exactly of those prime numbers p at which the degree drops.

From this perspective, the trivial factors of the polynomial $Z_k(p;T)$ are accounted for by the invariants and the coinvariants of the inertia action at zero and infinity. By "removing them," we mean replacing étale cohomology with compact support on the right-hand side of (1.5) with *middle extension* cohomology, which is defined as

$$\begin{aligned}&\mathrm{H}^1_{\text{ét},\mathrm{mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)\\ &\qquad = \mathrm{im}\big[\mathrm{H}^1_{\text{ét},\mathrm{c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \longrightarrow \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})\big].\end{aligned}$$

The terminology is coherent with the fact that, letting $j : \mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{P}^1$ denote the inclusion, the above image agrees with the cohomology on $\mathbb{P}^1$ of the intermediate (i.e., middle) extension sheaf $j_{!*}\,\mathrm{Sym}^k \mathrm{Kl}_2$. By definition, $M_k(p;T)$ is the polynomial

$$M_k(p;T) = \det\big(1 - F_p T \mid \mathrm{H}^1_{\text{ét},\mathrm{mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)\big).$$

Since the étale cohomology and the étale cohomology with compact support of $\mathrm{Sym}^k \mathrm{Kl}_2$ have weights at least $k+1$ and at most $k+1$, respectively, by the main theorem of Weil II [10], the middle extension cohomology is pure of weight $k+1$. What was called m in Section 1.1 above is the degree of $M_k(p;T)$, that is, the dimension of the middle extension cohomology for all $p \notin S$.

1.3. Exponential Hodge structures and irregular Hodge filtration

Theorems 1.2 and 1.3 are proved by constructing a compatible system of potentially automorphic Galois representations

$$r_{k,\ell} \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_m(\mathbf{Q}_\ell), \tag{1.6}$$

with ℓ running over all prime numbers, such that $r_{k,\ell}$ is unramified at primes $p \neq \ell$ outside S and has traces of Frobenius

$$\mathrm{tr}\big(r_{k,\ell}(\mathrm{Frob}_p)\big) = \begin{cases} -m_2^k(p) - 1 & \text{if } 4 \nmid k \\ -m_2^k(p) - 1 - p^{k/2} & \text{if } 4 \mid k \end{cases} \quad \big(p \notin S \cup \{\ell\}\big). \tag{1.7}$$

The search for such Galois representations was initiated by Fu and Wan, who showed in [22] that L-functions of symmetric power moments of Kloosterman sums can be realized as Hasse–Weil zeta functions of *virtual* schemes over $\mathrm{Spec}\,\mathbf{Z}$. An actual Galois representation with traces (1.7) was first constructed by Yun [64] as a subquotient of the étale cohomology of a smooth projective variety over $\mathbf{Q}$ cut off the affine Grassmannian of GL_2.

Our construction is instead inspired by the theory of exponential motives, as developed by the first author and Jossen [19]. In a nutshell, this is a theory of motives for pairs (X, f) consisting of a smooth variety X over $\mathbf{Q}$ and a regular function f on X that enriches the de Rham cohomology of the vector bundle with connection $E^f = (\mathscr{O}_X, \mathrm{d} + \mathrm{d}f)$; that is,

$$\mathrm{H}^n_{\mathrm{dR}}(X, E^f) = \mathbf{H}^n(X, \mathscr{O}_X \xrightarrow{\mathrm{d}+\mathrm{d}f} \Omega^1_X \xrightarrow{\mathrm{d}+\mathrm{d}f} \Omega^2_X \longrightarrow \cdots).$$

We shall also consider the de Rham cohomology with compact support $\mathrm{H}^n_{\mathrm{dR,c}}(X, E^f)$ and the middle de Rham cohomology $\mathrm{H}^n_{\mathrm{dR,mid}}(X, E^f)$, which is defined as the image of the latter in the former under the "forget supports" map. When f is the zero function, $\mathrm{H}^n_{\mathrm{dR}}(X, E^f)$ is the usual de Rham cohomology of X and one can indeed prove that Nori motives, one of the candidates for the abelian category of mixed motives, form a full subcategory of exponential motives. However, the function does not need to be identically zero for an a priori exponential motive to be classical. For instance, this is always the case for an exponential motive of the form $\mathrm{H}^n(X \times \mathbb{A}^1, tf)$. If the zero locus $Z = \{f = 0\}$ is smooth, then it is isomorphic to $\mathrm{H}^{n-2}(Z)(-1)$, which should be thought of as a cohomological shadow of the identity

$$\int_0^\infty \int_{T(\gamma)} e^{-tf}\omega\, \mathrm{d}t = 2\pi \mathrm{i} \int_\gamma \mathrm{Res}_Z(\omega/f),$$

where ω is a differential form on the complement of Z and $T(\gamma)$ is the tube of a chain γ in Z. In general, the existence of square roots of the Tate motive $\mathbf{Q}(-1)$ prevents exponential motives from having realizations in mixed Hodge structures, but they do realize into certain mixed Hodge modules over the affine line that Kontsevich and Soibelman [36] call *exponential mixed Hodge structures*. It will be enough for our purposes to work in this category, whose main properties are summarized in the appendix.

In analogy with the ℓ-adic setting, the *Kloosterman connection* Kl_2 on $\mathbb{G}_{\mathrm{m}}$ over a field of characteristic 0 is defined by keeping the same diagram (1.4) but replacing the Artin–Schreier sheaf with the differential equation of the exponential. Then the pullback $f^*\mathscr{L}_\psi$ becomes $E^f = (\mathscr{O}_{\mathbb{G}^2_{\mathrm{m}}}, \mathrm{d} + \mathrm{d}f)$ and one sets

$$\mathrm{Kl}_2 = \pi_+ E^f,$$

which can be thought of as the family of exponential motives $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, x + z/x)$ parameterized by $z \in \mathbb{G}_{\mathrm{m}}$. Over the complex numbers, Kl_2 is the rank-2 vector bundle with connection associated with the modified Bessel differential equation

$$\mathrm{d}^2 y/\mathrm{d}z^2 + (1/z)\,\mathrm{d}y/\mathrm{d}z - y = 0,$$

which is indeed the one the exponential periods of $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, x+z/x)$ satisfy. Note that this equation has a regular singularity at zero and an irregular singularity at infinity. We can then form the symmetric powers $\mathrm{Sym}^k \mathrm{Kl}_2$ and consider the various flavors of de Rham cohomology

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2), \qquad \mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2), \qquad \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2),$$

where the last space agrees again with the cohomology of the intermediate extension computed on $\mathbb{P}^1$ (see Corollary 2.15). These vector spaces admit an exponential Hodge-theoretic interpretation as de Rham fibers of exponential mixed Hodge structures, respectively denoted by

$$\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2), \qquad \mathrm{H}^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2), \qquad \mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2).$$

As such, they carry an *irregular Hodge filtration*, constructed in [50, Section 6] by extending an idea of Deligne [12]. On the other hand, we shall prove in Theorem 3.2 that these exponential mixed Hodge structures are indeed classical mixed Hodge structures. When this is the case, the irregular Hodge filtration agrees with the usual Hodge filtration, as we show in Proposition A.13, so we can rely on the geometric interpretation of the former to compute the latter.

THEOREM 1.8
The mixed Hodge structure $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *has weights at least* $k+1$ *and the following numerical data.*

(1) *For odd* k, *it is mixed of weights* $k+1$ *and* $2k+2$, *with*

$$\dim \mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)^{p,q} = \begin{cases} 1 & \text{if } p+q=k+1 \text{ and } p \in \{2,4,\dots,k-1\}, \\ 1 & \text{if } p=q=k+1, \\ 0 & \text{otherwise}. \end{cases}$$

(2) *For even* k, *it is mixed of weights* $k+1$ *and* $2k+2$ *if* $k \equiv 2 \bmod 4$, *and of weights* $k+1$, $k+2$, *and* $2k+2$ *if* $k \equiv 0 \bmod 4$, *with*

$$\dim \mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)^{p,q} = \begin{cases} 1 & \text{if } p+q=k+1 \text{ and } \min\{p,q\} \in \{2,4,\dots,2\lfloor (k-1)/4 \rfloor\}, \\ 1 & \text{if } p=q=k/2+1 \text{ and } k \equiv 0 \bmod 4, \\ 1 & \text{if } p=q=k+1, \\ 0 & \text{otherwise}. \end{cases}$$

Furthermore, the mixed Hodge structure $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *is pure of weight* $k+1$ *and is equal to* $W_{k+1}\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$.

To prove that $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ carries a mixed Hodge structure and compute its Hodge numbers, we first establish the analogous result for the pullback $\widetilde{\mathrm{Kl}}_2$ of Kl_2 by the double cover of $\mathbb{G}_{\mathrm{m}}$. The symmetric power $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ turns out to be the restriction of the Fourier transform of a $\mathscr{D}$-module on $\mathbb{A}^1$ that underlies a pure Hodge module and, as we explain in Section A.7, the theory of mixed Hodge modules endows its cohomology with a mixed Hodge structure whose numerical invariants can be computed in terms of nearby cycles. On the other hand, seeing the cohomology of symmetric powers as the alternating part of the cohomology of tensor powers and using a refined form of the Künneth formula, we get an isomorphism

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \simeq \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_2^{\otimes k})^{\mathrm{sign}} \simeq \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})^{\mathrm{sign}}, \tag{1.9}$$

where f_k is the function $x_1 + \cdots + x_k + z(1/x_1 + \cdots + 1/x_k)$ and sign denotes the eigenspace on which the symmetric group $\mathfrak{S}_k$ acts through the sign. After pullback by the cover $t \mapsto z = t^2$ and the change of coordinates $x_i = ty_i$, the function f_k takes the form $\widetilde{f_k} = tg^{\boxplus k}$, where $g^{\boxplus k}$ is the k-fold Thom–Sebastiani sum $g(y_1) + \cdots + g(y_k)$ of the function $g(y) = y + 1/y$ with itself. The cohomology group (1.9) is hence also given by the invariants of

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) \simeq \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{tg^{\boxplus k}})^{\mathrm{sign}} \tag{1.10}$$

under the action of μ_2 coming from the cover. On toric varieties such as a compactification of $\mathbb{G}_{\mathrm{m}}^k$ adapted to the function f_k, the work of Adolphson and Sperber [1] and the results of [15] and [63] lead to a geometric interpretation of the irregular Hodge filtration that, once we know the Hodge numbers of (1.10), enables us to complete the proof of Theorem 1.8.

From this circle of ideas we also see that the Hodge structure $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ has motivic origin, in the sense that it is cut out of the cohomology of an algebraic variety. Indeed, replacing with $\mathbb{A}^1$ the copy of $\mathbb{G}_{\mathrm{m}}$ with coordinate t in $\mathbb{G}_{\mathrm{m}}^{k+1}$ and combining the Gysin and the localization long exact sequences, we obtain the following description in Theorem 3.8. Let $\mathscr{K} \subset \mathbb{G}_{\mathrm{m}}^k$ be the zero locus of $g^{\boxplus k}$, on which the group $\mathfrak{S}_k \times \mu_2$ acts by permuting the coordinates and sending y_i to $-y_i$. Then there is an isomorphism of pure Hodge structures

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \cong \mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K})^{\mathrm{sign} \times \mu_2}(-1). \tag{1.11}$$

For odd k, the hypersurface $\mathscr{K}$ is smooth and we also obtain an isomorphism

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \cong \mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}(\mathscr{K})^{\mathrm{sign} \times \mu_2}(-1).$$

The right-hand side of (1.11) is the Hodge realization of a pure motive M_k over $\mathbf{Q}$ and the Galois representations $r_{k,\ell}$ from (1.6) arise as its ℓ-adic realizations.

That a paper seemingly about L-functions bears the title "Hodge theory of Kloosterman connections" may come as a surprise. The reason for this choice is that we see Theorem 1.8 as the crux of our contribution. Once we know that all Hodge numbers are either zero or one (equivalently, that the Galois representations $r_{k,\ell}$ are *regular*), a recent theorem of Patrikis and Taylor [44], building on previous work of Barnet-Lamb et al. [2], implies that the $r_{k,\ell}$ are potentially automorphic, and hence that their L-functions meromorphically extend to the complex plane and satisfy a functional equation. As expounded in the sequel paper [20], our approach also explains the relation, numerically checked to high precision by Broadhurst and Roberts in many examples, between special values of the L-functions at critical integers and certain determinants of *Bessel moments*

$$\int_0^\infty I_0(z)^a K_0(z)^{k-a} z^b \, dz/z,$$

where $I_0(z)$ and $K_0(z)$ are the modified Bessel functions of the first and the second kind.

1.4. Overview

Briefly, the paper is organized as follows. In the preparatory Section 2, we gather the main properties of Kloosterman connections and their symmetric powers. The mixed Hodge structures are constructed in Section 3, where we also exhibit their avatars over finite fields. Section 4 is devoted to the proof of Theorem 1.8. Finally, in Section 5 we compute the étale realizations of the motives and pull everything together to prove Theorems 1.2 and 1.3. The paper is supplemented by an appendix concerning exponential mixed Hodge structures and the irregular Hodge filtration.

2. Symmetric powers of Kloosterman connections

In this section, we gather the properties of Kloosterman connections and their symmetric powers that are relevant for the construction of the mixed Hodge structures in the next sections. We refer the reader to Appendix A.1 for the notation and results from the theories of $\mathscr{D}$-modules and mixed Hodge modules that are used in what follows, and to [32, Section II] for the notion of slopes of a meromorphic connection at an irregular singularity.

2.1. Structure of Kloosterman connections

Let $n \geq 1$ be an integer. We first define Kloosterman connections Kl_{n+1} generalizing Kl_2 from the introduction. For simplicity, we work over the base field $\mathbf{C}$, although all results remain valid over a field of characteristic 0. Let $\mathbb{G}_{\mathrm{m}}$ denote the 1-dimensional torus. We endow the product $\mathbb{G}_{\mathrm{m}}^{n+1}$ with coordinates $(z,x)=(z,x_1,\dots,x_n)$ and consider the diagram

$$\begin{tikzcd} & \mathbb{G}_{\mathrm{m}}^{n+1} \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ \mathbb{G}_{\mathrm{m}} & & \mathbb{A}^1 \end{tikzcd}$$

where π is the projection $(z,x)\mapsto z$ to the first factor and f is the function

$$f(z,x)=x_1+\cdots+x_n+\frac{z}{x_1\cdots x_n}. \tag{2.1}$$

We define Kl_{n+1} as the bounded complex of $\mathscr{D}_{\mathbb{G}_{\mathrm{m}}}$-modules

$$\mathrm{Kl}_{n+1}=\pi_+E^f, \tag{2.2}$$

where $E^f=(\mathscr{O}_{\mathbb{G}_{\mathrm{m}}^{n+1}},\mathrm{d}+\mathrm{d}f)$ is the connection attached to the exponential of f. After identifying the global sections $\Gamma(\mathbb{G}_{\mathrm{m},z},\mathscr{O}_{\mathbb{G}_{\mathrm{m}}})$ with $\mathbf{C}[z,z^{-1}]$, we can see Kl_{n+1} as a complex of $\mathbf{C}[z,z^{-1}]\langle z\partial_z\rangle$-modules.

Besides, we consider the cyclic Galois cover

$$[n+1]\colon \mathbb{G}_{\mathrm{m},t}\longrightarrow \mathbb{G}_{\mathrm{m},z}$$

induced by $z\mapsto t^{n+1}$, which has Galois group μ_{n+1}, and we set

$$\widetilde{\mathrm{Kl}}_{n+1}=[n+1]^+\,\mathrm{Kl}_{n+1}\simeq\widetilde{\pi}_+E^{\widetilde{f}},$$

with $\widetilde{\pi}(t,x)=t$ and $\widetilde{f}(t,x)=f(t^{n+1},x)$. The group μ_{n+1} acts on $\widetilde{\mathrm{Kl}}_{n+1}$, and hence on the pushforward $[n+1]_+\widetilde{\mathrm{Kl}}_{n+1}$, and the original complex can be retrieved as the invariants

$$\mathrm{Kl}_{n+1}=\big([n+1]_+\widetilde{\mathrm{Kl}}_{n+1}\big)^{\mu_{n+1}}.$$

Let $g\colon\mathbb{G}_{\mathrm{m}}^n\to\mathbb{A}^1$ be the function defined as

$$g(y_1,\dots,y_n)=f(1,y_1,\dots,y_n)=y_1+\cdots+y_n+\frac{1}{y_1\cdots y_n}. \tag{2.3}$$

Since the change of variables $(t,x)\mapsto(t,y)=(t,x/t)$ on $\mathbb{G}_{\mathrm{m}}^{n+1}$ turns $\widetilde{f}$ into $tg(y)$ and is compatible with projections to the first factor, we also get

$$\widetilde{\mathrm{Kl}}_{n+1}=\widetilde{\pi}_+E^{tg(y)}.$$

PROPOSITION 2.4

The complex Kl_{n+1} *is concentrated in degree* 0*; that is, the equality*

$$\mathrm{Kl}_{n+1} = \mathscr{H}^0 \pi_+ E^f$$

holds. Moreover, Kl_{n+1} *has the following properties:*

(1) Kl_{n+1} *is the irreducible free* $\mathscr{O}_{\mathbb{G}_\mathrm{m}}$*-module of rank* $n+1$ *with connection associated with the hypergeometric differential operator* $(z\partial_z)^{n+1} - z$.

(2) Kl_{n+1} *has a regular singularity at* $z = 0$ *with unipotent monodromy and a single Jordan block, and an irregular singularity of pure slope* $1/(n+1)$ *at* $z = \infty$.

(3) *Let* $\mathrm{Kl}_{n+1}^\vee$ *be the* $\mathscr{O}_{\mathbb{G}_\mathrm{m}}$*-module dual to* Kl_{n+1} *endowed with the dual connection, and let* ι_r *denote the involution* $z \mapsto (-1)^r z$*. There is an isomorphism*

$$\mathrm{Kl}_{n+1}^\vee \simeq \iota_{n+1}^+ \mathrm{Kl}_{n+1}\,.$$

(4) $\widetilde{\mathrm{Kl}}_{n+1}$ *is the restriction to* $\mathbb{G}_\mathrm{m}$ *of the Fourier transform of a regular holonomic* $\mathscr{D}_{\mathbb{A}^1}$*-module.*

Proof

The arguments of [8, Theorems 7.4, 7.8] can readily be transposed to the complex setting to show that Kl_{n+1} is a free $\mathscr{O}_{\mathbb{G}_\mathrm{m}}$-module with connection sitting in degree 0. Instead, we give a proof based on the following recursive description. Let inv be the involution $z \mapsto 1/z$ on $\mathbb{G}_\mathrm{m}$, and consider the localized Fourier transformation

$$\mathscr{F} = j_0^+ \,\mathrm{FT}\, j_{0+} \colon \mathsf{D}^\mathrm{b}_\mathrm{hol}(\mathscr{D}_{\mathbb{G}_\mathrm{m}}) \longrightarrow \mathsf{D}^\mathrm{b}_\mathrm{hol}(\mathscr{D}_{\mathbb{G}_\mathrm{m}}).$$

Recall that this functor preserves holonomic modules, and sends regular holonomic modules to smooth holonomic modules on $\mathbb{G}_\mathrm{m}$ with a regular singularity at the origin and a possibly irregular singularity with slopes in $\{0,1\}$ at infinity. For the sake of the induction, it will be convenient to set $\mathrm{Kl}_1 = j_0^+ E^y$ and start with $n = 0$.

LEMMA 2.5

For $n \geq 0$*, there is an isomorphism* $\mathrm{Kl}_{n+2} \simeq \mathscr{F}\,\mathrm{inv}^+ \mathrm{Kl}_{n+1}$.

Proof

We set $z' = 1/z$ and $x_0 = z/x_1 \cdots x_n$, so that $p_{z'} = 1/x_0 \cdots x_n$ and

$$\mathrm{inv}^+ \mathrm{Kl}_{n+1} = p_{z'+} E^{x_0 + \cdots + x_n}.$$

Hence, denoting by ζ a coordinate on a new factor $\mathbb{G}_\mathrm{m}$ and by π_ζ the corresponding projection,

$$\mathscr{F}\,\mathrm{inv}^+ \mathrm{Kl}_{n+1} = \pi_{\zeta+} E^{x_0 + \cdots + x_n + \zeta z'} = \pi_{\zeta+} E^{x_0 + \cdots + x_n + \zeta/x_0 \cdots x_n} \simeq \mathrm{Kl}_{n+2}\,. \qquad \square$$

Let us now show (1) and (2). Since Kl_1 is a holonomic $\mathscr{D}_{\mathbb{G}_\mathrm{m}}$-module, the same goes for each Kl_{n+1} by induction. That Kl_{n+1} is isomorphic to the free $\mathscr{O}_{\mathbb{G}_\mathrm{m}}$-module of rank $n+1$ with connection defined by $L=(z\partial_z)^{n+1}-z$ is clear for $n=0$. Assuming it true for Kl_{n+1}, Lemma 2.5 implies that Kl_{n+2} is isomorphic to

$$\begin{aligned}\mathscr{F}\,\mathrm{inv}^+(\mathscr{D}_{\mathbb{G}_\mathrm{m}}/\mathscr{D}_{\mathbb{G}_\mathrm{m}}L) &\simeq \mathscr{F}\big(\mathscr{D}_{\mathbb{G}_\mathrm{m}}/\mathscr{D}_{\mathbb{G}_\mathrm{m}}\big((-z\partial_z)^{n+1}z-1\big)\big)\\ &\simeq \mathscr{D}_{\mathbb{G}_\mathrm{m}}/\mathscr{D}_{\mathbb{G}_\mathrm{m}}\big((z\partial_z)^{n+2}-z\big),\end{aligned}$$

where the first isomorphism is given by inversion and multiplication by z on the right. With respect to a suitable basis, the matrix of $z\partial_z$ acting on Kl_{n+1} is thus given by

$$\begin{pmatrix}0&0&\cdots&0&z\\1&0&\cdots&0&0\\0&1&\cdots&0&0\\0&0&\cdots&1&0\end{pmatrix},$$

which shows that $z=0$ is a regular singularity with unipotent monodromy and a single Jordan block. It also follows from this description that $z=\infty$ is an irregular singularity of slope $1/(n+1)$, and that Kl_{n+1} is irreducible. Indeed, $j_{0+}\,\mathrm{inv}^+\,\mathrm{Kl}_{n+1}$ agrees with the intermediate extension $j_{0\dagger+}\,\mathrm{inv}^+\,\mathrm{Kl}_{n+1}$ because $\mathrm{inv}^+\,\mathrm{Kl}_{n+1}$ has slope $1/(n+1)$ at 0; it is thus an irreducible $\mathscr{D}_{\mathbb{A}^1}$-module, and Fourier transformation preserves this property.

We note that property (3) is true for $n=0$. Assuming it holds for Kl_{n+1}, we prove it for Kl_{n+2}. Fourier transformation commutes with duality of $\mathscr{D}_{\mathbb{A}^1}$-modules up to ι_1, as follows from [40, Lemme V.3.6], and the latter duality corresponds to the duality of free $\mathscr{O}_{\mathbb{G}_\mathrm{m}}$-modules with connection through the pair of functors j_0^+ and $j_{0\dagger+}$. Thus Lemma 2.5 yields (3).

Finally, the pullback $\widetilde{\mathrm{Kl}}_{n+1}$ of Kl_{n+1} by the finite morphism $[n+1]$ is also concentrated in degree 0 and is smooth of rank $n+1$ on $\mathbb{G}_\mathrm{m}$. Consider the $\mathscr{D}_{\mathbb{A}^1}$-module

$$M_{n+1}=\mathscr{H}^0 g_+\mathscr{O}_{\mathbb{G}_\mathrm{m}^n}.$$

Letting $j_0\colon \mathbb{G}_\mathrm{m}\hookrightarrow\mathbb{A}^1$ denote the inclusion, we claim that there is an isomorphism

$$\widetilde{\mathrm{Kl}}_{n+1}\simeq j_0^+\,\mathrm{FT}\,M_{n+1} \tag{2.6}$$

of $\mathbf{C}[t,t^{-1}]\langle t\partial_t\rangle$-modules. This will yield (4). We first observe that $\widetilde{\mathrm{Kl}}_{n+1}$ is isomorphic to $j_0^+\,\mathrm{FT}(g_+\mathscr{O}_{\mathbb{G}_\mathrm{m}^n})$. Indeed, denoting the projections by $p_t\colon \mathbb{G}_{\mathrm{m},t}\times\mathbb{A}^1_\tau\to\mathbb{G}_{\mathrm{m},t}$ and $p_\tau\colon \mathbb{G}_{\mathrm{m},t}\times\mathbb{A}^1_\tau\to\mathbb{A}^1_\tau$ and writing $\widetilde{\pi}$ as $\mathrm{Id}\times g\colon \mathbb{G}_{\mathrm{m},t}\times\mathbb{G}_\mathrm{m}^n\to\mathbb{G}_{\mathrm{m},t}\times\mathbb{A}^1$ composed with p_t, the projection formula gives isomorphisms

$$\widetilde{\pi}_+E^{tg}\simeq p_{t+}\big((\mathrm{Id}\times g)_+E^{tg}\big)\simeq p_{t+}(p_\tau^+ g_+\mathscr{O}_{\mathbb{G}_\mathrm{m}^n}\otimes E^{t\tau})\simeq j_0^+\,\mathrm{FT}(g_+\mathscr{O}_{\mathbb{G}_\mathrm{m}^n}).$$

The claim then follows from the fact that $\widetilde{\mathrm{Kl}}_{n+1}$ is concentrated in degree 0. □

2.2. *Cohomology of* $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$

In what follows, we work in the abelian tensor category of vector bundles with connection on $\mathbb{G}_{\mathrm{m}}$. For each $k \geq 1$, we define the kth symmetric power $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ (resp., $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$) as the invariants of the tensor product $\mathrm{Kl}_{n+1}^{\otimes k}$ (resp., $\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}$) under the action of the symmetric group $\mathfrak{S}_k$. There are isomorphisms

$$\begin{aligned}
\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k} &\simeq [n+1]^+ \mathrm{Kl}_{n+1}^{\otimes k}, \\
\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1} &\simeq [n+1]^+ \mathrm{Sym}^k \mathrm{Kl}_{n+1}, \\
\mathrm{Kl}_{n+1}^{\otimes k} &\simeq \big([n+1]_+ \widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}\big)^{\mu_{n+1}}, \\
\mathrm{Sym}^k \mathrm{Kl}_{n+1} &\simeq \big([n+1]_+ \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}\big)^{\mu_{n+1}}, \\
[n+1]_+ \widetilde{\mathrm{Kl}}_{n+1}^{\otimes k} &\simeq \bigoplus_{i=0}^{n} z^{i/(n+1)} \mathrm{Kl}_{n+1}^{\otimes k}, \\
[n+1]_+ \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1} &\simeq \bigoplus_{i=0}^{n} z^{i/(n+1)} \mathrm{Sym}^k \mathrm{Kl}_{n+1},
\end{aligned}$$

where $z^{i/(n+1)}\mathscr{E}$ denotes the Kummer twist $(\mathscr{O}_{\mathbb{G}_{\mathrm{m}}}, \mathrm{d} + \frac{i}{n+1}\,\mathrm{d}z/z) \otimes \mathscr{E}$ of a vector bundle with connection $\mathscr{E}$. This follows from the decomposition

$$[n+1]_+ \mathscr{O}_{\mathbb{G}_{\mathrm{m}}} = \bigoplus_{i=0}^{n} \Big(\mathscr{O}_{\mathbb{G}_{\mathrm{m}}}, \mathrm{d} + \frac{i}{n+1}\,\mathrm{d}z/z\Big)$$

into eigenspaces for the action of the Galois group μ_{n+1} of the cover $[n+1]$. Note the equality

$$\mathrm{rk}\,\mathrm{Sym}^k \mathrm{Kl}_{n+1} = \mathrm{rk}\,\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1} = \binom{n+k}{k}.$$

PROPOSITION 2.7

The connections $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ *and* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ *are irreducible.*

Proof

For each integer $n \geq 1$, the differential Galois group of Kl_{n+1} and $\widetilde{\mathrm{Kl}}_{n+1}$ is equal to $\mathrm{SL}_{n+1}(\mathbf{C})$ if n is even and to $\mathrm{Sp}_{n+1}(\mathbf{C})$ if n is odd (see [32, Corollary 4.4.8] taking Proposition 2.4(1) into account). Since any symmetric power of the standard representation of these groups is irreducible (see, e.g., [24, Sections 15.3, 17.3]), it follows that $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ are irreducible. $\square$

Let us consider the Laurent polynomial

$$f_k = \sum_{j=1}^{k}\Big(\sum_{i=1}^{n} x_{ji} + z\prod_{i=1}^{n}\frac{1}{x_{ji}}\Big)\colon \mathbb{G}_{\mathrm{m}}^{kn+1} \longrightarrow \mathbb{A}^1, \tag{2.8}$$

and let $\pi_k\colon \mathbb{G}_{\mathrm{m}}^{kn+1} \to \mathbb{G}_{\mathrm{m},z}$ denote the projection to the coordinate z. With the above notation, the equalities $f_1 = f$ and $\pi_1 = \pi$ hold. Let us consider the Cartesian square

$$\begin{tikzcd} \mathbb{G}_{\mathrm{m}}^{kn+1} \arrow[r, "{[n+1]}"] \arrow[d, "\widetilde{\pi}_k"'] & \mathbb{G}_{\mathrm{m}}^{kn+1} \arrow[d, "\pi_k"] \\ \mathbb{G}_{\mathrm{m},t} \arrow[r, "{[n+1]}"] & \mathbb{G}_{\mathrm{m},z} \end{tikzcd}$$

and set $\widetilde{f_k} = [n+1]^* f_k$, so that we have again $\widetilde{f_1} = \widetilde{f}$. Making the change of variables $(t, y_{ji}) = (t, x_{ji}/t)$ as in the previous section, we can write $\widetilde{f_k}$ as

$$\widetilde{f_k} = t\cdot g^{\boxplus k}, \tag{2.9}$$

where $g^{\boxplus k}$ is the k-fold Thom–Sebastiani sum of the Laurent polynomial g given by (2.3) with itself. There is a natural action of μ_{n+1} on $\mathscr{H}^0[n+1]_+E^{\widetilde{f_k}}$ such that

$$E^{f_k} = \big(\mathscr{H}^0[n+1]_+E^{\widetilde{f_k}}\big)^{\mu_{n+1}}.$$

PROPOSITION 2.10
There are isomorphisms

$$\mathrm{Kl}_{n+1}^{\otimes k} \simeq \mathscr{H}^0\pi_{k+}E^{f_k} = \pi_{k+}E^{f_k} \qquad \textit{and} \qquad \widetilde{\mathrm{Kl}}_{n+1}^{\otimes k} \simeq \mathscr{H}^0\widetilde{\pi}_{k+}E^{\widetilde{f_k}} = \widetilde{\pi}_{k+}E^{\widetilde{f_k}}.$$

Proof
We first prove the statement about $\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}$. Recall the equality $\widetilde{f_k} = t\cdot g^{\boxplus k}$, and consider the complex of $\mathscr{D}_{\mathbb{A}^1_\tau}$-modules with regular holonomic cohomology

$$M_{n+1}^{(k)} = g_+^{\boxplus k}\mathscr{O}_{\mathbb{G}_{\mathrm{m}}^{kn}}.$$

Arguing as in the proof of Proposition 2.4, we get an isomorphism

$$\widetilde{\pi}_{k+}E^{\widetilde{f_k}} \simeq j_0^+\,\mathrm{FT}\,M_{n+1}^{(k)}.$$

Let $s_k\colon \mathbb{A}^1\times\cdots\times\mathbb{A}^1 \to \mathbb{A}^1$ be the sum map. Writing $g^{\boxplus k} = s_k\circ(g\times\cdots\times g)$, we can identify $M_{n+1}^{(k)}$ with the k-fold additive $*$-convolution of $M_{n+1}^{(1)}$ with itself. Since Fourier transformation exchanges additive $*$-convolution and derived tensor product on $\mathbb{A}^1_t$ (as recalled in Section A.1), we deduce upon localization an isomorphism of complexes of $\mathbf{C}[t,t^{-1}]$-modules with connection

$$j_0^+ \operatorname{FT} M_{n+1}^{(k)} \simeq (j_0^+ \operatorname{FT} M_{n+1}^{(1)})^{\overset{\mathrm{L}}{\otimes} k},$$

where the tensor product is taken over the coordinate ring $\mathbf{C}[t,t^{-1}]$ of $\mathbb{G}_{\mathrm{m},t}$. Now, recall from the proof of Proposition 2.4 that the complex $j_0^+ \operatorname{FT} M_{n+1}^{(1)}$ is concentrated in degree 0 and is a free $\mathbf{C}[t,t^{-1}]$-module isomorphic to $\widetilde{\mathrm{Kl}}_{n+1}$. It follows that the above derived tensor product is an ordinary tensor product, isomorphic to $\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}$.

Let us now consider the case of $\mathrm{Kl}_{n+1}^{\otimes k}$. From the equality

$$\pi_k \circ (\mathrm{Id} \times [n+1]) = [n+1] \circ \widetilde{\pi}_k,$$

along with the first part, we deduce μ_{n+1}-equivariant isomorphisms

$$\begin{aligned}[n+1]_+ \widetilde{\mathrm{Kl}}_{n+1}^{\otimes k} \simeq [n+1]_+ \widetilde{\pi}_{k+} E^{\widetilde{f}_k} &\simeq \pi_{k+}\big((\mathrm{Id} \times [n+1])_+ E^{\widetilde{f}_k}\big) \\ &= \mathscr{H}^0 \pi_{k+}\big((\mathrm{Id} \times [n+1])_+ E^{\widetilde{f}_k}\big).\end{aligned}$$

Moreover, the action on the rightmost term comes from that on $(\mathrm{Id} \times [n+1])_+ E^{\widetilde{f}_k}$, whose invariant submodule is E^{f_k}. Taking μ_{n+1}-invariants on the extreme terms, we deduce the first isomorphism of the statement. □

COROLLARY 2.11
There are isomorphisms

$$\begin{aligned}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_{n+1}^{\otimes k}) &\simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{f_k}), \\ \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}) &\simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{\widetilde{f}_k}).\end{aligned}$$

Proof
Letting q and $\widetilde{q}$ denote the structure morphisms of $\mathbb{G}_{\mathrm{m},z}$ and $\mathbb{G}_{\mathrm{m},t}$, respectively, we deduce isomorphisms of complexes

$$q_+ \mathrm{Kl}_{n+1}^{\otimes k} \simeq (q \circ \pi_k)_+ E^{f_k}, \qquad q_+ \widetilde{\mathrm{Kl}}_{n+1}^{\otimes k} \simeq (\widetilde{q} \circ \widetilde{\pi}_k)_+ E^{\widetilde{f}_k}$$

from Proposition 2.10 along with the isomorphisms of functors $(q \circ \pi_k)_+ \simeq q_+ \circ \pi_{k+}$ and $(\widetilde{q} \circ \widetilde{\pi}_k)_+ \simeq \widetilde{q}_+ \circ \widetilde{\pi}_{k+}$. The statement then follows by taking cohomology in degree 0. □

Since the dual of E^{f_k} is E^{-f_k}, Poincaré–Verdier duality yields a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^{kn+1}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{f_k}) \otimes \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{-f_k}) \arrow[r] \arrow[d] & \mathrm{H}^{2kn+2}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1}) = \mathbf{C} \arrow[d, equal] \\
\mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{f_k}) \otimes \mathrm{H}^{kn+1}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{-f_k}) \arrow[r] \arrow[u] & \mathrm{H}^{2kn+2}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1}) = \mathbf{C}
\end{tikzcd}$$

in which the horizontal arrows are nondegenerate pairings. Denote the image of the first vertical arrow by $\mathrm{H}^{kn+1}_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{f_k})$. Consider the involution ι on $\mathbb{G}_{\mathrm{m}}^{kn+1}$ given by

$$(z, x_{ji}) \longmapsto \big((-1)^{n+1} z, -x_{ji}\big).$$

Since $\iota_+ E^{f_k} = \iota^+ E^{f_k} = E^{-f_k}$, it induces an isomorphism from the de Rham cohomology and the de Rham cohomology with compact support of E^{f_k} to those of E^{-f_k}, from which we deduce a self-nondegenerate pairing

$$\mathrm{H}^{kn+1}_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{f_k}) \otimes \mathrm{H}^{kn+1}_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{f_k}) \longrightarrow \mathbf{C}. \tag{2.12}$$

This pairing is $(-1)^{kn+1}$-symmetric since ι acts trivially on $\mathrm{H}^{2kn+2}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1})$. There is a similarly defined pairing on $\mathrm{H}^{kn+1}_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{\widetilde{f}_k})$, which induces (2.12) by taking μ_{n+1}-invariants.

In what follows, we consider objects acted upon by one of the groups $\mathfrak{S}_k$ and $\mathfrak{S}_k \times \mu_{n+1}$, and we use the uniform notation $\chi\colon G \to \{\pm 1\}$ for the sign character if $G = \mathfrak{S}_k$ and the product of the sign character on $\mathfrak{S}_k$ and the trivial character on μ_{n+1} if $G = \mathfrak{S}_k \times \mu_{n+1}$. We also set $\chi_n = \chi^n$ for $n \geq 1$. Given a representation V of G over a field K of characteristic 0, we let V^{G,χ_n} denote the χ_n-isotypic part of V, defined as the image of the idempotent

$$\frac{1}{|G|} \sum_{\sigma \in G} \chi_n(\sigma)\sigma$$

in the group ring $K[G]$ acting on V.

PROPOSITION 2.13
The de Rham cohomology of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ *and* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ *is concentrated in degree* 1. *Moreover, there are isomorphisms*

$$\begin{aligned}
\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) &\simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{f_k})^{\mathfrak{S}_k,\chi_n} \\
&\simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{\widetilde{f}_k})^{\mathfrak{S}_k \times \mu_{n+1},\chi_n}, \\
\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}) &\simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{\widetilde{f}_k})^{\mathfrak{S}_k,\chi_n}.
\end{aligned}$$

Proof
Since $\mathbb{G}_{\mathrm{m}}$ is an affine curve, there is no cohomology in degree greater than 1 and it suffices to prove that $\mathrm{H}^0_{\mathrm{dR}}$ vanishes. This cohomology space being given by the invariants under the action of the differential Galois group, the vanishing follows from the fact that $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ are nontrivial irreducible representations (see Proposition 2.7).

Notice, however, that the actions of $\sigma \in \mathfrak{S}_k$ on the left- and right-hand sides of each of the isomorphisms in Proposition 2.10 commute up to the sign $\chi_n(\sigma)$, which can be seen, for example, by representing elements as differential forms (this is similar to the discussion on the determinant of H^1 in [19, Section 12.3.1]). By taking the G-action on the isomorphisms of Corollary 2.11 into account, the desired isomorphisms are then clear. □

Let $j: \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{P}^1$ denote the inclusion. In addition to the meromorphic extension $j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1}$, we consider the following $\mathscr{D}_{\mathbb{P}^1}$-modules that also extend the $\mathscr{D}_{\mathbb{G}_{\mathrm{m}}}$-module $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ to $\mathbb{P}^1$:

$$\begin{aligned} j_\dagger \mathrm{Sym}^k \mathrm{Kl}_{n+1} &= \boldsymbol{D}\, j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1}^\vee \\ &\simeq \iota_{n+1}^+ \boldsymbol{D}\, j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1} \quad \text{(after Proposition 2.4(3))}, \qquad (2.14) \\ j_{\dagger+} \mathrm{Sym}^k \mathrm{Kl}_{n+1} &= \mathrm{im}[j_\dagger \mathrm{Sym}^k \mathrm{Kl}_{n+1} \longrightarrow j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1}], \end{aligned}$$

where $\boldsymbol{D}$ denotes the duality of $\mathscr{D}$-modules. The equality

$$\mathrm{H}^r_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) = \mathrm{H}^r_{\mathrm{dR}}(\mathbb{P}^1, j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1})$$

holds, and we set

$$\begin{aligned} &\mathrm{H}^r_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) = \mathrm{H}^r_{\mathrm{dR}}(\mathbb{P}^1, j_\dagger \mathrm{Sym}^k \mathrm{Kl}_{n+1}), \\ &\mathrm{H}^r_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \\ &\qquad = \mathrm{im}\big[\mathrm{H}^r_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \longrightarrow \mathrm{H}^r_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})\big]. \end{aligned}$$

COROLLARY 2.15
The de Rham cohomology with compact support $\mathrm{H}^r_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ *and the middle de Rham cohomology* $\mathrm{H}^r_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ *vanish for* $r \neq 1$. *Moreover, there are isomorphisms*

$$\begin{aligned} \mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) &\simeq \mathrm{H}^{kn+1}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{f_k})^{\mathfrak{S}_k, \chi_n} \\ &\simeq \mathrm{H}^{kn+1}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{\widetilde{f}_k})^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}, \qquad (2.16) \\ \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) &= \mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1, j_{\dagger+} \mathrm{Sym}^k \mathrm{Kl}_{n+1}), \end{aligned}$$

and a perfect $(-1)^{kn+1}$*-symmetric pairing*

$$\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_{n+1}) \otimes \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_{n+1}) \longrightarrow \mathbf{C}$$

induced by Poincaré duality. The same statement holds for $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$.

Proof

Combining Poincaré duality for $\mathscr{D}_{\mathbb{P}^1}$-modules, the isomorphism (2.14), and the fact that ι^+_{n+1} induces an isomorphism on de Rham cohomology, we get

$$\begin{aligned}\mathrm{H}^r_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_{n+1}) &\simeq \mathrm{H}^r_{\mathrm{dR}}(\mathbb{P}^1, \iota^+_{n+1} \boldsymbol{D}\, j_+ \operatorname{Sym}^k \mathrm{Kl}_{n+1}) \\ &\simeq \mathrm{H}^{2-r}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_{n+1})^\vee,\end{aligned}$$

and hence the vanishing of the left-hand side for $r \neq 1$ by Proposition 2.13. This immediately implies the vanishing of $\mathrm{H}^r_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_{n+1})$ for $r \neq 1$. The first two isomorphisms are proved in the same way as those of Proposition 2.13.

To shorten notation, we write $\mathrm{H}^r = \mathrm{H}^r_{\mathrm{dR}}(\mathbb{P}^1, j_{\dagger+} \operatorname{Sym}^k \mathrm{Kl}_{n+1})$. From the fact that the intermediate extension $j_{\dagger+} \operatorname{Sym}^k \mathrm{Kl}_{n+1}$ is self-dual up to ι_{n+1}, we get an isomorphism $\mathrm{H}^r \simeq (\mathrm{H}^{2-r})^\vee$ similarly as above. Since the morphism

$$j_\dagger \operatorname{Sym}^k \mathrm{Kl}_{n+1} \longrightarrow j_+ \operatorname{Sym}^k \mathrm{Kl}_{n+1}$$

factors through the module $j_{\dagger+} \operatorname{Sym}^k \mathrm{Kl}_{n+1}$, to prove (2.16) it suffices to show that the map $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_{n+1}) \to \mathrm{H}^1$ is surjective, which then implies the injectivity of $\mathrm{H}^1 \to \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_{n+1})$ by duality. The former appears in the long exact sequence associated with

$$\begin{aligned}0 \longrightarrow \ker[j_\dagger \operatorname{Sym}^k \mathrm{Kl}_{n+1} &\longrightarrow j_{\dagger+} \operatorname{Sym}^k \mathrm{Kl}_{n+1}] \\ &\longrightarrow j_\dagger \operatorname{Sym}^k \mathrm{Kl}_{n+1} \longrightarrow j_{\dagger+} \operatorname{Sym}^k \mathrm{Kl}_{n+1} \longrightarrow 0\end{aligned}$$

and, since the kernel has punctual support by [34, Proposition 2.9.8], we get the surjectivity.

Finally, we observe that the pairing (2.12) is compatible with the induced actions of the symmetric group $\mathfrak{S}_k$, which moreover acts trivially on the target $\mathbf{C}$. Taking the χ_n-isotypic parts yields the desired self-duality for $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_{n+1})$. The proof for $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ is completely parallel. □

2.3. The inverse Fourier transform of $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$

Let $j_0 \colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1$ denote the inclusion. Recall that $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ is the restriction to $\mathbb{G}_{\mathrm{m}}$ of the Fourier transform of a regular holonomic $\mathscr{D}_{\mathbb{A}^1_t}$-module. Applying inverse Fourier transformation to the exact sequence

$$0 \longrightarrow j_{0\dagger+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1} \longrightarrow j_{0+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1} \longrightarrow \widetilde{C}_{k,n} \longrightarrow 0, \tag{2.17}$$

where $\widetilde{C}_{k,n}$ is supported at the origin, we thus get an exact sequence of regular holonomic $\mathscr{D}$-modules on the dual affine line $\mathbb{A}^1_\tau$. Set

$$\widetilde{M} = \mathrm{FT}^{-1}(j_{0\dagger+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}). \tag{2.18}$$

The endofunctor $\Pi\colon N \mapsto N \star j_{0\dagger}\mathscr{O}_{\mathbb{G}_\mathrm{m}}$ on the category of regular holonomic $\mathscr{D}_{\mathbb{A}^1}$-modules, where "$\star$" stands for additive $*$-convolution (see Section A.1), is a projector onto the full subcategory of those with vanishing global de Rham cohomology, and the functors $\Pi \circ \mathrm{FT}^{-1}$ and $\mathrm{FT}^{-1} \circ j_{0+} j_0^+$ are canonically isomorphic (see [34, Proposition 12.3.5]). Therefore, the equality

$$\Pi(\widetilde{M}) = \mathrm{FT}^{-1}(j_{0+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}) \tag{2.19}$$

holds, and we get an exact sequence of regular holonomic $\mathscr{D}_{\mathbb{A}^1}$-modules

$$0 \longrightarrow \widetilde{M} \longrightarrow \Pi(\widetilde{M}) \longrightarrow \widetilde{M}' \longrightarrow 0, \tag{2.20}$$

where $\widetilde{M}'$ is constant (i.e., a sum of copies of the trivial $\mathscr{D}_{\mathbb{A}^1}$-module $\mathscr{O}_{\mathbb{A}^1}$).

As recalled in Section A.3 of the appendix, the projector Π lifts to a projector on the category $\mathsf{MHM}(\mathbb{A}^1)$ of mixed Hodge modules on the affine line $\mathbb{A}^1$, denoted in the same way.

PROPOSITION 2.21
The exact sequence (2.20) underlies an exact sequence

$$0 \longrightarrow \widetilde{M}^{\mathrm{H}} \longrightarrow \Pi(\widetilde{M}^{\mathrm{H}}) \longrightarrow \widetilde{M}'^{\mathrm{H}} \longrightarrow 0$$

in the category $\mathsf{MHM}(\mathbb{A}^1)$. *More precisely,* $\widetilde{M}^{\mathrm{H}}$ *is the pure Hodge module* $W_{kn}\Pi(\widetilde{M}^{\mathrm{H}})$ *of weight* kn *and* $\widetilde{M}'^{\mathrm{H}}$ *is a mixed Hodge module of weights at least* $kn+1$, *which is pure of weight* $2k+1$ *if* $n=1$.

Proof
Set $U = \mathbb{G}_\mathrm{m}^{kn}$, and recall the Laurent polynomial $g^{\boxplus k}\colon U \to \mathbb{A}^1_\tau$. We consider the associated Gauss–Manin system $\mathscr{H}^0 g_+^{\boxplus k}\mathscr{O}_U$ and its localized Fourier transform $j_0^+ \mathrm{FT}(\mathscr{H}^0 g_+^{\boxplus k}\mathscr{O}_U)$. The second isomorphism of Proposition 2.10 reads

$$\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k} \simeq j_0^+ \mathrm{FT}(\mathscr{H}^0 g_+^{\boxplus k}\mathscr{O}_U).$$

The Laurent polynomial g is convenient and nondegenerate with respect to its Newton polytope, and hence so is $g^{\boxplus k}$. The argument of [13, (3.6), (3.9)(c)] extends to the complex setting and shows that the cone of the natural morphism $g_\dagger^{\boxplus k}\mathscr{O}_U \to g_+^{\boxplus k}\mathscr{O}_U$

has constant cohomology,[1] and hence the same holds for the kernel and cokernel of the induced morphism

$$\mathscr{H}^0 g_{\dagger}^{\boxplus k} \mathscr{O}_U \longrightarrow \mathscr{H}^0 g_{+}^{\boxplus k} \mathscr{O}_U.$$

Letting $\mathscr{H}^0 g_{\dagger+}^{\boxplus k} \mathscr{O}_U$ denote its image, it follows that the induced morphisms

$$(\mathscr{H}^0 g_{\dagger}^{\boxplus k} \mathscr{O}_U)[\partial_\tau^{-1}] \longrightarrow (\mathscr{H}^0 g_{\dagger+}^{\boxplus k} \mathscr{O}_U)[\partial_\tau^{-1}] \longrightarrow (\mathscr{H}^0 g_{+}^{\boxplus k} \mathscr{O}_U)[\partial_\tau^{-1}],$$

after inverting ∂_τ, are isomorphisms. We finally obtain a morphism

$$\mathscr{H}^0 g_{\dagger+}^{\boxplus k} \mathscr{O}_U \longrightarrow \mathrm{FT}^{-1}(j_{0+} \widetilde{\mathrm{Kl}}_{n+1}^{\otimes k})$$

whose kernel and cokernel are constant $\mathscr{D}_{\mathbb{A}^1}$-modules.

Recall from Section A.2 that the mixed Hodge module ${}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}$ is pure of weight $kn = \dim U$ and that the associated perverse sheaf and filtered $\mathscr{D}_U$-module are $\mathbf{Q}_U[kn]$ and $\mathscr{O}_U$ together with the trivial filtration jumping at the index 0. As mixed Hodge modules on $\mathbb{A}^1_\tau$, the proper pushforward $\mathscr{H}^0{}_{\mathrm{H}}g_!^{\boxplus k}\,{}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}$ has weights at most kn, the pushforward $\mathscr{H}^0{}_{\mathrm{H}}g_*^{\boxplus k}\,{}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}$ has weights at least kn, and the image $\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k}\,{}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}$ of the former in the latter is pure of weight kn, by [54, (4.5.2)]. Away from its singularities, $\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k}\,{}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}$ corresponds to a polarizable variation of pure Hodge structures of weight $kn-1$.

The symmetric group $\mathfrak{S}_k$ acts on $\mathbb{G}_{\mathrm{m}}^{kn}$ by permuting the index j in the coordinates y_{ji} and this action preserves $g^{\boxplus k}$. Therefore, $\mathfrak{S}_k$ acts on $\mathscr{H}^0 g_{+}^{\boxplus k} \mathscr{O}_U$ and hence on its Fourier transform $\mathrm{FT}\,\mathscr{H}^0 g_{+}^{\boxplus k} \mathscr{O}_U$ and its localized Fourier transform $j_0^+\,\mathrm{FT}\,\mathscr{H}^0 g_{+}^{\boxplus k} \mathscr{O}_U$. Through the identification

$$\mathrm{Sym}^k\,\widetilde{\mathrm{Kl}}_{n+1} \simeq j_0^+\,\mathrm{FT}(\mathscr{H}^0 g_{+}^{\boxplus k} \mathscr{O}_U)^{\mathfrak{S}_k,\chi_n},$$

we obtain a morphism

$$(\mathscr{H}^0 g_{\dagger+}^{\boxplus k} \mathscr{O}_U)^{\mathfrak{S}_k,\chi_n} \longrightarrow \mathrm{FT}^{-1}(j_{0+}\,\mathrm{Sym}^k\,\widetilde{\mathrm{Kl}}_{n+1}) = \Pi(\widetilde{M}) \tag{2.22}$$

inducing an isomorphism after inverting ∂_τ, a property which implies the isomorphisms

$$\Pi(\widetilde{M}) \simeq \Pi\big((\mathscr{H}^0 g_{\dagger+}^{\boxplus k} \mathscr{O}_U)^{\mathfrak{S}_k,\chi_n}\big) \simeq \Pi(\mathscr{H}^0 g_{\dagger+}^{\boxplus k} \mathscr{O}_U)^{\mathfrak{S}_k,\chi_n}. \tag{2.23}$$

[1] Indeed, since $h = g^{\boxplus k}$ is convenient and nondegenerate with respect to its Newton polytope, it can be extended to a proper morphism $h_\Sigma\colon Y_\Sigma \to \mathbb{A}^1$ on a smooth toroidal variety Y_Σ such that $R^i(h_{\Sigma|\overline{Y}_\Sigma^\sigma})_*\mathbf{C}$ is constant for each i and each cone σ of the regular fan Σ. Setting $D = Y_\Sigma \smallsetminus \mathbb{G}_{\mathrm{m}}^{kn}$, it follows that $R^i(h_{\Sigma|D})_*\mathbf{C}$ is constant for each i, and thus the assertion for h reduces to that for h_Σ, which holds because Y_Σ is smooth and h_Σ is proper and locally acyclic except at a finite number of points.

LEMMA 2.24
The $\mathscr{D}_{\mathbb{A}^1}$-module $\widetilde{M}$ is irreducible and is equal to the image of (2.22).

Proof
Since $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ is irreducible by Proposition 2.7, so are $j_{0\dagger+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ and its inverse Fourier transform $\widetilde{M}$. Besides, the left-hand side $\widetilde{M}_0$ of (2.22) underlies a pure Hodge module $\widetilde{M}_0^{\mathrm{H}}$ on the affine line, which decomposes as a direct sum $\widetilde{M}_1^{\mathrm{H}} \oplus \widetilde{M}_2^{\mathrm{H}}$ in which the $\mathscr{D}$-module $\widetilde{M}_1$ underlying $\widetilde{M}_1^{\mathrm{H}}$ is the maximal constant submodule of $\widetilde{M}_0$. The image of (2.22) is thus isomorphic to the $\mathscr{D}$-module $\widetilde{M}_2$ underlying $\widetilde{M}_2^{\mathrm{H}}$. Since $\widetilde{M}_2^{\mathrm{H}}$ is pure, $\widetilde{M}_2$ is a direct sum of irreducible $\mathscr{D}$-modules, and hence $\mathrm{FT}(\widetilde{M}_2)$ as well. It follows that $\mathrm{FT}(\widetilde{M}_2)$ is an intermediate extension at the origin, in the sense that the equality $\mathrm{FT}(\widetilde{M}_2) = j_{0\dagger+} j_0^+ \mathrm{FT}(\widetilde{M}_2)$ holds. On the other hand, (2.23) gives the equality $j_0^+ \mathrm{FT}(\widetilde{M}_2) = \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$, from which we get

$$\mathrm{FT}(\widetilde{M}_2) = j_{0\dagger+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1} = \mathrm{FT}(\widetilde{M}),$$

as we wanted to show. □

The χ_n-isotypic component $(\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k\,\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}})^{\mathfrak{S}_k,\chi_n}$ is a pure object of weight kn of $\mathsf{MHM}(\mathbb{A}^1)$. Upon application of the projector Π, we obtain an object

$$\Pi(\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k\,\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}})^{\mathfrak{S}_k,\chi_n}$$

of $\mathsf{MHM}(\mathbb{A}^1)$ whose underlying $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module is $\Pi(\widetilde{M})$. The image of the lift of (2.22) to $\mathsf{MHM}(\mathbb{A}^1)$ is a pure Hodge module of weight kn that lifts $\widetilde{M}$ to $\mathsf{MHM}(\mathbb{A}^1)$ and is denoted by $\widetilde{M}^{\mathrm{H}}$. Therefore, $\Pi(\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k\,\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}})^{\mathfrak{S}_k,\chi_n} = \Pi(\widetilde{M}^{\mathrm{H}})$ holds. We denote by $\widetilde{M}'^{\mathrm{H}}$ the quotient object in $\mathsf{MHM}(\mathbb{A}^1)$.

It remains to check the weight properties. The equality $\widetilde{M}^{\mathrm{H}} = W_{kn}\Pi(\widetilde{M}^{\mathrm{H}})$ holds. Otherwise, the nonzero quotient object $W_{kn}\Pi(\widetilde{M}^{\mathrm{H}})/\widetilde{M}^{\mathrm{H}}$, which is constant, would be a direct summand of $W_{kn}\Pi(\widetilde{M}^{\mathrm{H}})$, and hence $\Pi(\widetilde{M})$ would have a nonzero constant submodule, which contradicts the vanishing of its global de Rham cohomology. It follows that $\widetilde{M}'^{\mathrm{H}}$ has weights at least $kn+1$ and it remains to show that it is pure of weight $2k+1$ if $n=1$. This is equivalent to showing that the nearby cycles at infinity $\psi_{1/\tau}\widetilde{M}'^{\mathrm{H}}$ have weight $2k$ (note that $\widetilde{M}'^{\mathrm{H}}$ extends smoothly at infinity). By [41, Proposition A.1], although $\Pi(\widetilde{M}^{\mathrm{H}})$ is mixed, the weight filtration W on $\psi_{1/\tau,1}\Pi(\widetilde{M}^{\mathrm{H}})$ is nevertheless identified with a shifted monodromy filtration, namely the monodromy filtration centered at k. Moreover, $\psi_{1/\tau}\widetilde{M}'^{\mathrm{H}}$ is identified with the primitive part $\mathrm{P}_k\,\mathrm{gr}_{2k}^W\,\psi_{1/\tau,1}\Pi(\widetilde{M}^{\mathrm{H}})$ (see [41, proof of Proposition A.3]), and hence is pure of weight $2k$, since the monodromy of $\psi_{1/\tau,1}\Pi(\widetilde{M}^{\mathrm{H}})$ has only one Jordan block, by Corollary 4.10 below. □

3. Motives of symmetric power moments of Kloosterman connections

The main goal of this section is to prove that the middle de Rham cohomology of the connection $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ is the de Rham realization of a Nori motive over $\mathbf{Q}$ (see [28]). The expressions of the various cohomology spaces obtained in Section 2.2 naturally exhibit them as the de Rham fibers of certain exponential mixed Hodge structures, and we first show in Theorem 3.2 that they are in fact usual mixed Hodge structures. Building on this result, we then obtain in Theorem 3.8 a geometric description of the middle mixed Hodge structure in terms of a hypersurface $\mathscr{K} \subset \mathbb{G}_{\mathrm{m}}^{kn}$ acted upon by the group $\mathfrak{S}_k \times \mu_{n+1}$. This leads us to *define* the middle motive M_k of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ as

$$\mathrm{M}_k = \mathrm{gr}^W_{kn+1}\big[\mathrm{H}_{\mathrm{c}}^{kn-1}(\mathscr{K})(-1)\big]^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}, \tag{3.1}$$

where on the right-hand side $W_\bullet$ stands for the weight filtration, which exists at the motivic level by [28, Theorem 10.2.5]. In the second part, we establish the counterpart of these results for the étale cohomology of symmetric powers of the ℓ-adic Kloosterman sheaf and the rigid cohomology of symmetric powers of the Kloosterman F-isocrystal on $\mathbb{G}_{\mathrm{m}}$ over a finite field. They turn out to be easier to prove because a full formalism of weights is at hand regardless of the ramification of the coefficients, whereas in the Hodge setting we need to resort to the interpretation of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ as the localized Fourier transform of a mixed Hodge module in order to remain within the framework of exponential mixed Hodge structures.

3.1. *The mixed Hodge structures attached to* $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$

Throughout this section, we work in the abelian category EMHS of exponential mixed Hodge structures, which is a full subcategory of the category of mixed Hodge modules over the affine line $\mathbb{A}^1$ (see Section A.3 for a review). It contains the category of mixed Hodge structure as a full subcategory (see Lemma A.12) and objects $\mathrm{H}^r(U, f)$, $\mathrm{H}^r_{\mathrm{c}}(U, f)$, and $\mathrm{H}^r_{\mathrm{mid}}(U, f)$ associated with a regular function $f : U \to \mathbb{A}^1$ and an integer r, whose de Rham fibers are the cohomology spaces $\mathrm{H}^r_{\mathrm{dR}}(U, E^f)$, $\mathrm{H}^r_{\mathrm{dR,c}}(U, E^f)$, and $\mathrm{H}^r_{\mathrm{dR,mid}}(U, E^f)$, respectively (see Section A.5).

Recall from (2.8) the Laurent polynomial f_k on the torus $\mathbb{G}_{\mathrm{m}}^{kn+1}$ with coordinates (z, x_{ji}) and its pullback $\widetilde{f_k}$ to the torus $\mathbb{G}_{\mathrm{m}}^{kn+1}$ with coordinates (t, x_{ji}) by the map $z \mapsto t^{n+1}$. The group $\mathfrak{S}_k \times \mu_{n+1}$ acts on $\mathbb{G}_{\mathrm{m}}^{kn+1} = \mathbb{G}_{\mathrm{m},t} \times (\mathbb{G}_{\mathrm{m}}^n)^k$ by permutation of the copies of $\mathbb{G}_{\mathrm{m}}^n$ and multiplication on the coordinate t. As this action leaves the function $\widetilde{f_k}$ invariant, the objects $\mathrm{H}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f_k})$ and $\mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f_k})$ of the category EMHS inherit an action of $\mathfrak{S}_k \times \mu_{n+1}$.

For $? = \varnothing, \mathrm{c}, \mathrm{mid}$, we define the objects

$$\begin{aligned}\mathrm{H}^1_?(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) &= \mathrm{H}_?^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, f_k)^{\mathfrak{S}_k, \chi_n} \\ &= \mathrm{H}_?^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f_k})^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n},\end{aligned}$$

$$\mathrm{H}^1_?(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}) = \mathrm{H}^{kn+1}_?(\mathbb{G}_\mathrm{m}^{kn+1}, \widetilde{f_k})^{\mathfrak{S}_k, \chi_n}$$

of the category EMHS, the de Rham fibers of which are $\mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ and $\mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1})$ in view of Proposition 2.13 and Corollary 2.15.

THEOREM 3.2
The exponential mixed Hodge structures

$$\mathrm{H}^1(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}), \qquad \mathrm{H}^1_\mathrm{c}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}), \qquad \textit{and}$$
$$\mathrm{H}^1_\mathrm{mid}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$$

are usual mixed Hodge structures of weights at least $kn+1$, at most $kn+1$, and $kn+1$, respectively, and the natural morphisms between them are morphisms of mixed Hodge structures.

Moreover, the induced map

$$\mathrm{gr}^W_{kn+1} \mathrm{H}^1_\mathrm{c}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \longrightarrow \mathrm{gr}^W_{kn+1} \mathrm{H}^1(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \tag{3.3}$$

is an isomorphism, and $\mathrm{H}^1_\mathrm{mid}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ is equal to its image. This pure Hodge structure of weight $kn+1$ is equipped with a $(-1)^{kn+1}$-symmetric pairing

$$\mathrm{H}^1_\mathrm{mid}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \otimes \mathrm{H}^1_\mathrm{mid}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \longrightarrow \mathbf{Q}(-kn-1). \tag{3.4}$$

The same results hold for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$.

Proof
Once we know that $\mathrm{H}^1_\mathrm{c}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ and $\mathrm{H}^1(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ are both mixed Hodge structures, the natural morphism between them is a morphism of Hodge structures since MHS is a full subcategory of EMHS by Lemma A.12. Its image $\mathrm{H}^1_\mathrm{mid}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ is therefore a usual mixed Hodge structure as well. The pairing (3.4) is induced by Poincaré duality and the involution ι_{n+1} as in Corollary 2.15 for the de Rham fiber.

Since the exponential mixed Hodge structures attached to $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ are defined as the χ_n-isotypic components of $\mathrm{H}^{kn+1}_?(\mathbb{G}_\mathrm{m}^{kn+1}, \widetilde{f_k})$ for the action of $\mathfrak{S}_k \times \mu_{n+1}$ and $\mathfrak{S}_k$, respectively, it suffices to prove the result for the latter. For $\mathrm{H}^{kn+1}(\mathbb{G}_\mathrm{m}^{kn+1}, \widetilde{f_k})$ and $\mathrm{H}^{kn+1}_\mathrm{c}(\mathbb{G}_\mathrm{m}^{kn+1}, \widetilde{f_k})$, we apply Theorems A.24(2) and A.24(3), respectively, with $V = \mathbb{G}_\mathrm{m}^{kn}$, $M^\mathrm{H}_V = \mathscr{O}^\mathrm{H}_V$, $f = \widetilde{f_k}$, and $r = kn+1$. The weight properties follow from Proposition A.19, taking Lemma A.12 into account.

We will prove that the analogue of the map (3.3) for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ is an isomorphism and deduce the result for $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ by taking μ_{n+1}-invariants. According to Proposition A.19, it suffices to establish the equality

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}) = W_{kn+1}\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}). \tag{3.5}$$

The inclusion $\subset$ follows from the purity of $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1})$. Since we only deal with weight properties, we may as well work with the de Rham fibers of the corresponding exponential mixed Hodge structures. Recall that the $\mathscr{D}_{\mathbb{A}^1}$-module $\widetilde{M}$ defined in (2.18) underlies a pure Hodge module $\widetilde{M}^{\mathrm{H}}$ of weight kn by Proposition 2.21. Letting $j_\infty \colon \mathbb{A}^1_t \hookrightarrow \mathbb{P}^1_t$ denote the inclusion, the map

$$\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}) \hookrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1})$$

decomposes as

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1_t, j_{\infty\dagger+}\,\mathrm{FT}\,\widetilde{M}) \hookrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}) \hookrightarrow \mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{A}^1_t, \mathrm{FT}\,\Pi(\widetilde{M})\big).$$

From Proposition 2.21 and Corollary A.31(3), we deduce that the second inclusion induces an equality between the subspaces W_{kn+1}. Corollary A.31(1) identifies the weight filtration of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M})$ with that of

$$\mathrm{coker}\big[\widetilde{\mathrm{N}}_\tau \colon \psi_{\tau,1}\widetilde{M}^{\mathrm{H}} \longrightarrow \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}(-1)\big].$$

Recall also that the weight filtration on $\psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ is the monodromy filtration associated with $\widetilde{\mathrm{N}}_\tau$ centered at $kn-1$ (kn is the pure weight of $\widetilde{M}^{\mathrm{H}}$). Then the graded space $\mathrm{gr}^W \mathrm{coker}\,\widetilde{\mathrm{N}}_\tau$ is the direct sum of the primitive parts of the Lefschetz decomposition of $\mathrm{gr}^W \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}(-1)$. The weight of the component corresponding to the primitive part P_0 (Jordan blocks of $\widetilde{\mathrm{N}}_\tau$ of size 1) is $kn+1$, and the weight for the component corresponding to P_i (for $i \geq 1$) is $kn+1+i$.

Proving the equality (3.5) amounts therefore to proving

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1_t, j_{\infty\dagger+}\,\mathrm{FT}\,\widetilde{M}) = \dim \mathrm{P}_0.$$

It will be easier to deal with codimensions. Note that the cokernel of the morphism $j_{\infty\dagger+}\,\mathrm{FT}\,\widetilde{M} \hookrightarrow j_{\infty+}\,\mathrm{FT}\,\widetilde{M}$ is a $\mathscr{D}_{\mathbb{P}^1_t}$-module supported at $t=\infty$, and hence takes the form $C[\partial_{t'}]$ for some finite-dimensional vector space C, where t' denotes the coordinate $1/t$ at infinity. Then $\dim C$ is the codimension of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1_t, j_{\infty\dagger+}\,\mathrm{FT}\,\widetilde{M})$ in $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1_t, j_{\infty+}\,\mathrm{FT}\,\widetilde{M})$, and we aim at proving the equality

$$\dim C = \sum_{i \geq 1} \dim \mathrm{P}_i. \tag{3.6}$$

Since $\widetilde{M}$ is irreducible (see Lemma 2.24), it is an intermediate extension at the origin. Therefore, $\sum_{i\geq 1} \dim \mathrm{P}_i$ is the number of Jordan blocks of $\widetilde{\mathrm{N}}_\tau$ acting on the vanishing cycle space

$$\phi_{\tau,1}\widetilde{M} = \mathrm{im}[\widetilde{\mathrm{N}}_\tau \colon \psi_{\tau,1}\widetilde{M} \longrightarrow \psi_{\tau,1}\widetilde{M}]$$

and, with an argument already used for $\psi_{\tau,1}\widetilde{M}$, the following equality holds:

$$\sum_{i\geq 1}\dim \mathrm{P}_i = \dim\operatorname{coker}[\widetilde{\mathrm{N}}_\tau\colon \phi_{\tau,1}\widetilde{M} \longrightarrow \phi_{\tau,1}\widetilde{M}].$$

On the other hand, the module $C[\partial_{t'}]$ can be computed by replacing $j_{\infty+}\operatorname{FT}\widetilde{M}$ with its formalization at infinity $(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge = \mathbf{C}[\![t']\!]\otimes_{\mathbf{C}[t']} j_{\infty+}\operatorname{FT}\widetilde{M}$. Decomposing the latter as the direct sum of its regular and irregular parts and taking the equalities

$$(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{irr}} = (j_{\infty\dagger}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{irr}} = (j_{\infty\dagger+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{irr}}$$

into account, we see that the irregular part does not contribute to $C[\partial_{t'}]$, hence an isomorphism

$$C \simeq \operatorname{coker}\bigl[\phi_{t',1}(j_{\infty\dagger+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{reg}} \hookrightarrow \phi_{t',1}(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{reg}}\bigr].$$

The target of this morphism is isomorphic to $\psi_{t',1}(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{reg}}$ and, by definition of the intermediate extension, the source is isomorphic to the image of

$$\widehat{\mathrm{N}}_{t'}\colon \psi_{t',1}(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{reg}} \longrightarrow \psi_{t',1}(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{reg}},$$

the morphism above being the inclusion. Therefore, we get the equality

$$\dim C = \dim\operatorname{coker}\bigl[\widehat{\mathrm{N}}_{t'}\colon \psi_{t',1}(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{reg}} \longrightarrow \psi_{t',1}(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{reg}}\bigr].$$

Finally, the stationary phase formula identifies the pair $(\phi_{\tau,1}\widetilde{M}, \widetilde{\mathrm{N}}_\tau)$ with

$$\bigl(\psi_{t',1}(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{reg}}, \widehat{\mathrm{N}}_{t'}\bigr)$$

since $(j_{\infty+}\operatorname{FT}\widetilde{M})^\wedge_{\mathrm{reg}}$ is the microlocalization of $\widetilde{M}$ at the origin (see, e.g., [48, Proposition 2.3]). This implies the equality (3.6), thus concluding the proof. ☐

Remark 3.7
For $k = n = 1$, there are isomorphisms

$$\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_2) \simeq \mathrm{H}^2(\mathbb{G}_{\mathrm{m}}^2, x+y) \simeq \mathbf{Q}(-2),$$
$$\mathrm{H}^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_2) \simeq \mathrm{H}^2_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}}^2, x+y) \simeq \mathbf{Q}(0)$$

by the change of variables $(x,z)\mapsto(x,y)=(x,z/x)$ and Example A.22. Hence, the middle Hodge structure $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_2)$ vanishes.

THEOREM 3.8
Assume that $kn \geq 2$. Consider the hypersurface $\mathscr{K} \subset \mathbb{G}_{\mathrm{m}}^{kn}$ over $\mathbf{Q}$ defined as the zero locus of the Laurent polynomial

$$g^{\boxplus k}(y) = \sum_{j=1}^{k}\Big(\sum_{i=1}^{n} y_{ji} + \prod_{i=1}^{n} 1/y_{ji}\Big), \tag{3.9}$$

on which the group $\mathfrak{S}_k \times \mu_{n+1}$ *acts by permuting the indices* j *in* y_{ji} *and by the map* $y_{ji} \mapsto \zeta^{-1} y_{ji}$ *for each* $\zeta \in \mu_{n+1}$.

There is a canonical isomorphism of pure Hodge structures

$$\begin{aligned}\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) &\cong \mathrm{gr}^W_{kn+1}\big[\mathrm{H}^{kn-1}_{\mathrm{c}}(\mathscr{K})(-1)\big]^{\mathfrak{S}_k\times\mu_{n+1},\chi_n}\\ &\cong \mathrm{gr}^W_{kn+1}\,\mathrm{H}^{kn+1}_{\mathscr{K}}(\mathbb{G}_{\mathrm{m}}^{kn})^{\mathfrak{S}_k\times\mu_{n+1},\chi_n}.\end{aligned}$$

If $\mathscr{K}$ *is smooth, then the last term is also isomorphic to*

$$\mathrm{gr}^W_{kn+1}\big[\mathrm{H}^{kn-1}(\mathscr{K})(-1)\big]^{\mathfrak{S}_k\times\mu_{n+1},\chi_n}.$$

Proof

Consider the open immersion $(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f_k}) \subset (\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f_k})$ of pairs compatible with the action of $\mathfrak{S}_k \times \mu_{n+1}$. On noting the isomorphism

$$(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f_k}) \smallsetminus (\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f_k}) \simeq (\mathbb{G}_{\mathrm{m}}^{kn}, 0),$$

we obtain a diagram with exact rows and a commutative square

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \mathrm{H}^{kn-1}(\mathbb{G}_{\mathrm{m}}^{kn})(-1) \arrow[r] & \mathrm{H}^{kn+1}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f_k}) \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{H}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f_k}) \arrow[r] & \mathrm{H}^{kn}(\mathbb{G}_{\mathrm{m}}^{kn})(-1) \arrow[r] & 0\\
\cdots & \mathrm{H}^{kn+1}_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}}^{kn}) \arrow[l] & \mathrm{H}^{kn+1}_{\mathrm{c}}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f_k}) \arrow[l] \arrow[u] & \mathrm{H}^{kn+1}_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f_k}) \arrow[l] \arrow[u] & \mathrm{H}^{kn}_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}}^{kn}) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

as in Example A.20 from the appendix. Exactness at the rightmost term of both rows follows from the fact that $\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}$ is an affine variety of dimension $kn+1$. In the above diagram, the four terms involving $\mathbb{G}_{\mathrm{m}}^{kn}$ on the corners are pure of weights $2kn$ (top left), $2kn+2$ (top right), 2 (bottom left), and 0 (bottom right). Moreover, by Proposition A.19 the two middle upper terms have weights at least $kn+1$ and the middle lower terms have weights at most $kn+1$. Since $W_\bullet$ is an exact functor, the commutative square yields a commutative square with isomorphisms

$$\begin{tikzcd}
\mathrm{gr}^W_{kn+1}\,\mathrm{H}^{kn+1}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f_k}) \arrow[r, "\sim"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{gr}^W_{kn+1}\,\mathrm{H}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f_k})\\
\mathrm{gr}^W_{kn+1}\,\mathrm{H}^{kn+1}_{\mathrm{c}}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f_k}) \arrow[u] & \mathrm{gr}^W_{kn+1}\,\mathrm{H}^{kn+1}_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f_k}) \arrow[l, "\sim"'] \arrow[u]
\end{tikzcd}$$

of pure Hodge structures of weight $kn+1$ that are equivariant with respect to the action of $\mathfrak{S}_k \times \mu_{n+1}$. After taking the χ_n-isotypic components, the right vertical map is an isomorphism by Theorem 3.2, with source and target isomorphic to $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$. The same then holds for the remaining map.

Recall that the change of variables $(t,x) \mapsto (t,y)$ on $\mathbb{G}_{\mathrm{m}}^{kn+1}$ given by $x_{ji} = ty_{ji}$ transforms the Laurent polynomial $\widetilde{f_k}(t,x)$ into $tg^{\boxplus k}(y)$. On the new coordinates, $\mathfrak{S}_k$ permutes the indices j in y_{ji} and leaves t invariant, while $\zeta \in \mu_{n+1}$ acts through $t \mapsto \zeta t$ and $y_{ji} \mapsto \zeta^{-1} y_{ji}$. Applying Example A.27, we obtain isomorphisms

$$\begin{aligned}\mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f_k}) &\simeq \mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{A}^1 \times \mathscr{K}) \simeq \mathrm{H}_{\mathrm{c}}^{2}(\mathbb{A}^1) \otimes \mathrm{H}_{\mathrm{c}}^{kn-1}(\mathscr{K}) \\ &\simeq \mathrm{H}_{\mathrm{c}}^{kn-1}(\mathscr{K})(-1)\end{aligned}$$

and these isomorphisms are equivariant since μ_{n+1} acts trivially on $\mathrm{H}^2_{\mathrm{c}}(\mathbb{A}^1)$. Similarly, Example A.27 gives an equivariant isomorphism

$$\mathrm{H}^{kn+1}(\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f_k}) \simeq \mathrm{H}_{\mathscr{K}}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn}),$$

and the right-hand side is also isomorphic to $\mathrm{H}^{kn-1}(\mathscr{K})(-1)$ if $\mathscr{K}$ is smooth. □

Remark 3.10

The hypersurface $\mathscr{K}$ has at worst isolated singularities, its $\overline{\mathbf{Q}}$-singular points being those with coordinates $y_{ji} = \zeta_j$ for all $1 \le j \le k$ and $1 \le i \le n$, where $\zeta_j \in \mu_{n+1}$ are roots of unity satisfying $\sum_{j=1}^k \zeta_j = 0$. (This is reminiscent of the computation of the Swan conductor at infinity of the analogue of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ over finite fields in [21, Theorem 3.1].) For example, if $n+1$ is a prime number, then $\mathscr{K}$ is singular if and only if k is a multiple of $n+1$; this includes the case $n=1$, in which $\mathscr{K}$ is singular if and only if k is even.

3.2. *The case of characteristic p*

In this section, we prove an analogue of Theorem 3.8 for the étale and rigid cohomology of symmetric powers of Kloosterman sheaves over finite fields.

3.2.1. *Étale cohomology of symmetric powers of ℓ-adic Kloosterman sheaves*

Let p and ℓ be distinct prime numbers, let $\overline{\mathbf{F}}_p$ be an algebraic closure of the finite field $\mathbf{F}_p$, and let $\overline{\mathbf{Q}}_\ell$ be an algebraic closure of the field of ℓ-adic numbers $\mathbf{Q}_\ell$. All weights of ℓ-adic sheaves are understood to be considered with respect to a fixed isomorphism from $\overline{\mathbf{Q}}_\ell$ to $\mathbf{C}$. Let ζ be a primitive pth root of unity in $\overline{\mathbf{Q}}_\ell$, and let $\psi : \mathbf{F}_p \to \mathbf{Q}_\ell(\zeta)^\times$ be a nontrivial additive character. The Artin–Schreier sheaf $\mathscr{L}_\psi$ is the rank-1 lisse étale sheaf with coefficients in $\mathbf{Q}_\ell(\zeta)$ on the affine line $\mathbb{A}^1_{\mathbf{F}_p}$ on which geometric Frobenius acts as multiplication by $\psi(\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}(y))$ at each closed point y

defined over $\mathbf{F}_q$. Given a regular function $h\colon X \to \mathbb{A}^1$ on a smooth algebraic variety X over $\mathbf{F}_p$, we set $\mathscr{L}_{\psi(h)} = h^*\mathscr{L}_\psi$.

Recall the function $f\colon \mathbb{G}_{\mathrm{m}}^{n+1} \to \mathbb{A}^1$ from (2.1) and the projection to the first coordinate $\pi\colon \mathbb{G}_{\mathrm{m}}^{n+1} \to \mathbb{G}_{\mathrm{m},z}$, which we now view as defined over $\mathbf{F}_p$. For each $n \geq 1$, we define a complex of ℓ-adic sheaves with coefficients in $\mathbf{Q}_\ell(\zeta)$ on $\mathbb{G}_{\mathrm{m}}$ in a similar way as we did in (2.2); namely,

$$\mathrm{Kl}_{n+1} = \mathrm{R}\pi_*\mathscr{L}_{\psi(f)}[n].$$

Deligne proved in [8, Théorèmes 7.4, 7.8] that Kl_{n+1} is a lisse sheaf of rank $n+1$ concentrated in degree 0 that is pure of weight n; we shall call it the *ℓ-adic Kloosterman sheaf*. At each closed point $z \in \mathbb{G}_{\mathrm{m}}(\mathbf{F}_q)$, the action of geometric Frobenius has trace $(-1)^n \mathrm{Kl}_{n+1}(z;q)$, where

$$\mathrm{Kl}_{n+1}(z;q) = \sum_{x_1,\ldots,x_n \in \mathbf{F}_q^\times} \psi\bigl(\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}(x_1 + \cdots + x_n + z/x_1\cdots x_n)\bigr)$$

is the Kloosterman sum in many variables generalizing (1.1). Similarly, we define

$$\widetilde{\mathrm{Kl}}_{n+1} = [n+1]^*\,\mathrm{Kl}_{n+1} \simeq \mathrm{R}\widetilde{\pi}_*\mathscr{L}_{\psi(\widetilde{f})}[n],$$

where $\widetilde{f}(t,x) = f(t^{n+1},x)$ and $\widetilde{\pi}$ is the projection $(t,x) \mapsto t$. Note that $\widetilde{\mathrm{Kl}}_{n+1}$ is also concentrated in degree 0. If p divides $n+1$, say $n+1 = p^r m$ with $(p,m) = 1$, then the cover $[n+1]$ of $\mathbb{G}_{\mathrm{m}}$ factors as the composition of the étale cyclic cover $[m]$ and the purely inseparable cover $[p^r]$. The latter induces an equivalence of categories of étale covers (see, e.g., [18, Proposition 3.16]), and hence does not change the cohomology. In that case, the action of μ_{n+1} on $\widetilde{\mathrm{Kl}}_{n+1}$ factors through μ_m.

The proof of Proposition 2.10 adapts verbatim to this setting. Indeed, $\widetilde{\mathrm{Kl}}_{n+1}$ is the localized Fourier transform $j_0^*\,\mathrm{FT}_\psi$ of $\mathrm{R}g_*\mathbf{Q}_\ell(\zeta)_{\mathbb{G}_{\mathrm{m}}^{kn}}[n]$, which has thus nonconstant cohomology only in degree 0. From the fact that Fourier transformation exchanges tensor product and additive convolution (see, e.g., [37, Proposition 1.2.2.7]), we obtain isomorphisms $\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k} \simeq \mathrm{R}\widetilde{\pi}_{k*}\mathscr{L}_{\psi(\widetilde{f}_k)}$ and, arguing as in Corollary 2.11 and Proposition 2.13,

$$\mathrm{H}^1_{\text{ét},?}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \simeq \mathrm{H}^{kn+1}_{\text{ét},?}(\mathbb{G}^{kn+1}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathscr{L}_{\psi(\widetilde{f}_k)})^{\mathfrak{S}_k \times \mu_{n+1},\chi_n}$$

for $? = \varnothing, \mathrm{c}, \mathrm{mid}$. The fact that $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ has cohomology concentrated in degree 1 still holds for odd p or if $p = 2$ and n is odd, since in those cases the geometric monodromy group of Kl_{n+1} is either SL_{n+1} or Sp_{n+1} over $\mathbf{Q}_\ell(\zeta)$ by [33, Theorem 11.1]. However, if $p = 2$ and n is even, then the geometric monodromy group is either SO_{n+1} or G_2 and the symmetric powers of the standard representation may contain a copy of the trivial representation.

Besides, the natural morphism from étale cohomology with compact support to usual étale cohomology fits into a $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$-equivariant long exact sequence

$$\begin{aligned}
&(\mathrm{Sym}^k \mathrm{Kl}_{n+1})_{\overline{\eta}}^{I_0} \oplus (\mathrm{Sym}^k \mathrm{Kl}_{n+1})_{\overline{\eta}}^{I_\infty} \\
&\quad\longrightarrow \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \\
&\quad\longrightarrow \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \\
&\quad\longrightarrow \big((\mathrm{Sym}^k \mathrm{Kl}_{n+1})_{\overline{\eta}}\big)_{I_0}(-1) \oplus \big((\mathrm{Sym}^k \mathrm{Kl}_{n+1})_{\overline{\eta}}\big)_{I_\infty}(-1),
\end{aligned} \tag{3.11}$$

where I_0 and I_∞ stand for the inertia groups at zero and infinity acting on a geometric generic fiber of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ (see, e.g., [33, (2.0.7)]). Since this sheaf is pure of weight kn, it follows from Weil II [10, Lemme 1.8.1] that the leftmost term in the exact sequence is mixed of weights at most kn. By duality, the rightmost term is then mixed of weights at least $kn+2$, hence an isomorphism

$$\mathrm{gr}^W_{kn+1} \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \xrightarrow{\sim} \mathrm{gr}^W_{kn+1} \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}),$$

the image of which is the étale middle cohomology of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ (this is also proved in [64, Lemma 2.5.4]).

Let $\mathscr{K} \subset \mathbb{G}_\mathrm{m}^{kn}$ be the hypersurface over $\mathbf{F}_p$ defined by the same equation as in (3.9). From the Gysin long exact sequence for étale cohomology and the localization exact sequence for étale cohomology with compact support, we get as in the proof of Theorem 3.8 an isomorphism

$$\begin{aligned}
\mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) &\cong \mathrm{gr}^W_{kn+1} \mathrm{H}^{kn+1}_{\text{ét,c}}\big((\mathbb{A}^1 \times \mathbb{G}_\mathrm{m}^{kn})_{\overline{\mathbf{F}}_p}, \mathscr{L}_{\psi(\widetilde{f}_k)}\big)^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n} \\
&\cong \mathrm{gr}^W_{kn+1} \mathrm{H}^{kn+1}_{\text{ét}}\big((\mathbb{A}^1 \times \mathbb{G}_\mathrm{m}^{kn})_{\overline{\mathbf{F}}_p}, \mathscr{L}_{\psi(\widetilde{f}_k)}\big)^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}.
\end{aligned}$$

Finally, taking into account that $tg^{\boxplus k}$ vanishes at $\mathbb{A}^1 \times \mathscr{K}$, the localization exact sequence associated with the closed immersion $\mathbb{A}^1 \times \mathscr{K} \hookrightarrow \mathbb{A}^1 \times \mathbb{G}_\mathrm{m}^{kn}$ reads

$$\begin{aligned}
\cdots &\longrightarrow \mathrm{H}^{kn+1}_{\text{ét,c}}\big((\mathbb{A}^1 \times (\mathbb{G}_\mathrm{m}^{kn} \smallsetminus \mathscr{K}))_{\overline{\mathbf{F}}_p}, \mathscr{L}_{\psi(tg^{\boxplus k})}\big) \\
&\longrightarrow \mathrm{H}^{kn+1}_{\text{ét,c}}\big((\mathbb{A}^1 \times \mathbb{G}_\mathrm{m}^{kn})_{\overline{\mathbf{F}}_p}, \mathscr{L}_{\psi(tg^{\boxplus k})}\big) \\
&\longrightarrow \mathrm{H}^{kn+1}_{\text{ét,c}}\big((\mathbb{A}^1 \times \mathscr{K})_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell(\zeta)\big) \\
&\longrightarrow \mathrm{H}^{kn+2}_{\text{ét,c}}\big((\mathbb{A}^1 \times (\mathbb{G}_\mathrm{m}^{kn} \smallsetminus \mathscr{K}))_{\overline{\mathbf{F}}_p}, \mathscr{L}_{\psi(tg^{\boxplus k})}\big) \longrightarrow \cdots.
\end{aligned}$$

Since $g^{\boxplus k}$ is invertible on the complement $\mathbb{G}_\mathrm{m}^{kn} \smallsetminus \mathscr{K}$, the change of variables $(t, y_{ji}) \mapsto (s, y_{ji}) = (tg^{\boxplus k}, y_{ji})$ and the Künneth formula yield an isomorphism

$$\begin{aligned}
&\mathrm{H}^r_{\text{ét,c}}\big((\mathbb{A}^1\times(\mathbb{G}_{\mathrm{m}}^{kn}\smallsetminus\mathscr{K}))_{\overline{\mathbf{F}}_p},\mathscr{L}_{\psi(tg^{\boxplus k})}\big)\\
&\quad\cong \mathrm{H}^r_{\text{ét,c}}\big((\mathbb{A}^1\times(\mathbb{G}_{\mathrm{m}}^{kn}\smallsetminus\mathscr{K}))_{\overline{\mathbf{F}}_p},\mathscr{L}_{\psi(s)}\big)\\
&\quad\cong \bigoplus_{a+b=r}\mathrm{H}^a_{\text{ét,c}}(\mathbb{A}^1_{\overline{\mathbf{F}}_p},\mathscr{L}_\psi)\otimes \mathrm{H}^b_{\text{ét,c}}\big((\mathbb{G}_{\mathrm{m}}^{kn}\smallsetminus\mathscr{K})_{\overline{\mathbf{F}}_p},\mathbf{Q}_\ell(\zeta)\big),
\end{aligned}$$

which shows the vanishing of these groups in all degrees r since the Artin–Schreier sheaf $\mathscr{L}_\psi$ has trivial cohomology with compact support. Therefore,

$$\begin{aligned}
\mathrm{H}^{kn+1}_{\text{ét,c}}\big((\mathbb{A}^1\times\mathbb{G}_{\mathrm{m}}^{kn})_{\overline{\mathbf{F}}_p},\mathscr{L}_{\psi(tg^{\boxplus k})}\big)&\cong \mathrm{H}^{kn+1}_{\text{ét,c}}\big((\mathbb{A}^1\times\mathscr{K})_{\overline{\mathbf{F}}_p},\mathbf{Q}_\ell(\zeta)\big)\\
&\cong \mathrm{H}^{kn-1}_{\text{ét,c}}\big(\mathscr{K}_{\overline{\mathbf{F}}_p},\mathbf{Q}_\ell(\zeta)\big)(-1).
\end{aligned}$$

A similar argument involving the localization sequence for usual étale cohomology gives an isomorphism

$$\mathrm{H}^{kn+1}_{\text{ét}}\big((\mathbb{A}^1\times\mathbb{G}_{\mathrm{m}}^{kn})_{\overline{\mathbf{F}}_p},\mathscr{L}_{\psi(tg^{\boxplus k})}\big)\cong \mathrm{H}^{kn+1}_{\text{ét},\mathscr{K}_{\overline{\mathbf{F}}_p}}\big(\mathbb{G}_{\mathrm{m}}^{kn},\mathbf{Q}_\ell(\zeta)\big),$$

and the right-hand side is isomorphic to $\mathrm{H}^{kn-1}_{\text{ét}}(\mathscr{K}_{\overline{\mathbf{F}}_p},\mathbf{Q}_\ell(\zeta))(-1)$ if $\mathscr{K}$ is smooth. Passing to the χ_n-isotypic components and pulling everything together, we have thus proved the following result.

THEOREM 3.12
Assume that $kn\geq 2$. There is a canonical isomorphism of $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$*-modules*

$$\begin{aligned}
\mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p},\mathrm{Sym}^k\mathrm{Kl}_{n+1})&\cong \mathrm{gr}^W_{kn+1}\big[\mathrm{H}^{kn-1}_{\text{ét,c}}\big(\mathscr{K}_{\overline{\mathbf{F}}_p},\mathbf{Q}_\ell(\zeta)\big)(-1)\big]^{\mathfrak{S}_k\times\mu_{n+1},\chi_n}\\
&\cong \mathrm{gr}^W_{kn+1}\mathrm{H}^{kn+1}_{\text{ét},\mathscr{K}_{\overline{\mathbf{F}}_p}}\big(\mathbb{G}_{\mathrm{m}}^{kn},\mathbf{Q}_\ell(\zeta)\big)^{\mathfrak{S}_k\times\mu_{n+1},\chi_n}.
\end{aligned}$$

If $\mathscr{K}$ is smooth, then the last term is also isomorphic to

$$\mathrm{gr}^W_{kn+1}\big[\mathrm{H}^{kn-1}_{\text{ét}}\big(\mathscr{K}_{\overline{\mathbf{F}}_p},\mathbf{Q}_\ell(\zeta)\big)(-1)\big]^{\mathfrak{S}_k\times\mu_{n+1},\chi_n}.$$

3.2.2. Rigid cohomology of symmetric powers of Kloosterman F-isocrystals
Let $\overline{\mathbf{Q}}_p$ be an algebraic closure of $\mathbf{Q}_p$, and let $\varpi\in\overline{\mathbf{Q}}_p$ be an element satisfying $\varpi^{p-1}=-p$. As such elements are in bijection with primitive pth roots of unity in $\overline{\mathbf{Q}}_p$, this amounts to choosing a nontrivial $\overline{\mathbf{Q}}_p$-valued additive character on $\mathbf{F}_p$. The analogue of the Artin–Schreier sheaf is then Dwork's F-isocrystal $\mathscr{L}_\varpi$, which is the rank-1 connection $\mathrm{d}+\varpi\,\mathrm{d}z$ with Frobenius structure $\exp(\varpi(z^p-z))$ on the overconvergent structure sheaf of $\mathbb{A}^1$ over $K=\mathbf{Q}_p(\varpi)$. Given a regular function $h\colon X\to\mathbb{A}^1$ on a smooth algebraic variety X over $\mathbf{F}_p$, we set $\mathscr{L}_{\varpi h}=h^*\mathscr{L}_\varpi$. We refer the reader to Kedlaya's paper [35] for a summary of the properties of rigid

cohomology of overconvergent isocrystals that are used below, namely the fact that these groups carry a weight filtration for which the main results of Weil II hold.

The *Kloosterman F-isocrystal* Kl_{n+1} is the overconvergent F-isocrystal

$$\mathrm{Kl}_{n+1} = \mathrm{R}\pi_{\mathrm{rig}*}\mathscr{L}_{\varpi f}[n],$$

which has rank $n+1$ and is pure of weight n (see [6, Section 1, namely 1.5] for a detailed discussion). Similarly to the ℓ-adic case, writing $\widetilde{\mathrm{Kl}}_{n+1} = [n+1]^* \mathrm{Kl}_{n+1}$ as a localized Fourier transform $j_0^* \mathrm{FT}_\varpi$ yields isomorphisms

$$\mathrm{H}^1_{\mathrm{rig},?}(\mathbb{G}_{\mathrm{m}}/K, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) = \mathrm{H}^{kn+1}_{\mathrm{rig},?}(\mathbb{G}_{\mathrm{m}}^{kn+1}/K, \mathscr{L}_{\varpi \widetilde{f}_k})^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n} \tag{3.13}$$

for $? = \varnothing, \mathrm{c}, \mathrm{mid}$, and combining the analogue of the long exact sequence (3.11) relating rigid cohomology and rigid cohomology with compact support (see [35, (2.5.1)]) with the fact that the local contributions from 0 and ∞ are mixed of weights at most kn by [35, Proposition 5.1.4], we get

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m}}/K, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \cong \mathrm{gr}^W_{kn+1} \mathrm{H}^{kn+1}_{\mathrm{rig,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1}/K, \mathscr{L}_{\varpi \widetilde{f}_k})^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}. \tag{3.14}$$

Finally, the same argument as in the ℓ-adic case, considering the localization exact sequences for rigid cohomology and rigid cohomology with compact support (see [38, Proposition 8.2.18(ii)]) and the Gysin isomorphism (see [61, Theorem 4.1.1]), allows one to express the right-hand side of (3.14) in terms of the hypersurface $\mathscr{K}$, obtaining

$$\begin{aligned}\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m}}/K, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) &\cong \mathrm{gr}^W_{kn+1}\big[\mathrm{H}^{kn-1}_{\mathrm{rig,c}}(\mathscr{K}/K)(-1)\big]^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n} \\ &\cong \mathrm{gr}^W_{kn+1} \mathrm{H}^{kn+1}_{\mathrm{rig},\mathscr{K}/K}(\mathbb{G}_{\mathrm{m}}^{kn}/K)^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n},\end{aligned}$$

which is isomorphic to $\mathrm{gr}^W_{kn+1}[\mathrm{H}^{kn-1}_{\mathrm{rig}}(\mathscr{K}/K)(-1)]^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}$ if $\mathscr{K}$ is smooth.

4. Computation of the Hodge filtration

From now on, we focus on the case $n = 1$. In this section, we construct a basis of the de Rham cohomology of the symmetric power $\mathrm{Sym}^k \mathrm{Kl}_2$ that is well enough behaved with respect to the Hodge filtration for us to be able to prove Theorem 1.8.

4.1. Structure of Kl_2, $\widetilde{\mathrm{Kl}}_2$, *and their symmetric powers*

In preparation for the proof, we first make some of the statements of Section 2 more precise for $n = 1$. Among other things, this will allow us to compute the irregularity number (as defined, e.g., in [40, Chapitre III]) of $\mathrm{Sym}^k \mathrm{Kl}_2$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ at infinity.

Explicit bases of Kl_2 *and* $\widetilde{\mathrm{Kl}}_2$

Recall that Kl_2 is a rank-2 $\mathbf{C}[z,z^{-1}]$-module with connection that has a regular singularity at $z=0$ and an irregular singularity with slope $1/2$ at $z=\infty$. It is the localized Fourier transform $\mathrm{Kl}_2=\mathscr{F}(E^{1/y})$ of the rank-1 $\mathbf{C}[y,y^{-1}]$-module with connection $E^{1/y}$ generated by $\mathrm{e}^{1/y}$. Arguing as in the proof of Proposition 2.4, this interpretation shows that the generator $v_0=\mathrm{e}^{1/y}/y$ of $E^{1/y}$, viewed as an element of $j_0^+\,\mathrm{FT}\,E^{1/y}$, satisfies

$$(z\partial_z)^2v_0=zv_0.$$

We set $v_1=z\partial_z v_0$, so that $v_1=\mathrm{e}^{1/y}/y^2$ holds. Then $\{v_0,v_1\}$ is a $\mathbf{C}[z,z^{-1}]$-basis of Kl_2 and the matrix of the connection in this basis is given by

$$z\partial_z(v_0,v_1)=(v_0,v_1)\begin{pmatrix}0 & z\\ 1 & 0\end{pmatrix}. \tag{4.1}$$

Besides, by its definition as $\mathscr{H}^0\pi_+E^f$ with $f(z,x)=x+z/x$ and $\pi(z,x)=z$, the $\mathbf{C}[z,z^{-1}]$-module Kl_2 is equal to the cokernel of the relative de Rham complex

$$\mathbf{C}[x,x^{-1},z,z^{-1}]\xrightarrow{\mathrm{d}+\partial_x f\,\mathrm{d}x}\mathbf{C}[x,x^{-1},z,z^{-1}]\,\mathrm{d}x.$$

The elements v_0 and v_1 are respectively given by the classes of $\mathrm{d}x/x$ and $x\,\mathrm{d}x/x$ in this cokernel.

The basis $\{v_0,v_1\}$ lifts to a $\mathbf{C}[t,t^{-1}]$-basis $\{\widetilde{v}_0,\widetilde{v}_1\}$ of $\widetilde{\mathrm{Kl}}_2$ satisfying

$$\frac{1}{2}t\partial_t\widetilde{v}_0=\widetilde{v}_1,\qquad \frac{1}{2}t\partial_t\widetilde{v}_1=t^2\widetilde{v}_0. \tag{4.2}$$

Duality

Let $\{v_0^\vee,v_1^\vee\}$ denote the basis of $\mathrm{Kl}_2^\vee$ dual to $\{v_0,v_1\}$. The matrix of the dual connection in this basis is equal to $-\left(\begin{smallmatrix}0&1\\ z&0\end{smallmatrix}\right)$, and hence there is an isomorphism $\mathrm{Kl}_2\xrightarrow{\sim}\mathrm{Kl}_2^\vee$ given by $(v_0,v_1)\mapsto(v_1^\vee,-v_0^\vee)$. It induces a nondegenerate skew-symmetric pairing

$$\langle\,,\,\rangle\colon\ \mathrm{Kl}_2\otimes\mathrm{Kl}_2\longrightarrow\mathbf{C}[z,z^{-1}],\qquad \langle v_0,v_1\rangle=-\langle v_1,v_0\rangle=1,$$

which is compatible with the connection. There is a similar formula for $\widetilde{\mathrm{Kl}}_2$.

Structure at infinity

Recall the cover $[2]\colon\mathbb{G}_{\mathrm{m},t}\to\mathbb{G}_{\mathrm{m},z}$ induced by $t\mapsto z=t^2$. Since the irregular singularity of Kl_2 at $z=\infty$ has slope $1/2$, the formal stationary phase formula (see [49, Theorem 5.1]) gives the formal structure at infinity

$$\widehat{\mathrm{Kl}}_2=\mathbf{C}(\!(z^{-1})\!)\otimes_{\mathbf{C}[z,z^{-1}]}\mathrm{Kl}_2\simeq[2]_+(E^{2t}\otimes L_{-1}),$$

where L_{-1} is the rank-1 $\mathbf{C}((t^{-1}))$-module with connection

$$L_{-1} = \big(\mathbf{C}((t^{-1})), \mathrm{d} + \frac{1}{2}\mathrm{d}(t^{-1})/t^{-1}\big),$$

with monodromy -1. Let i be a square root of -1, and let

$$L_{\mathsf{i}} = \big(\mathbf{C}((z^{-1})), \mathrm{d} + \frac{1}{4}\mathrm{d}(z^{-1})/z^{-1}\big)$$

denote the rank-1 meromorphic connection with monodromy i around ∞, so that the equality $L_{-1} = [2]^+ L_{\mathsf{i}}$ holds. Then there is also an isomorphism

$$\widehat{\mathrm{Kl}}_2 \simeq \big([2]_+ E^{2t}\big) \otimes L_{\mathsf{i}}. \tag{4.3}$$

The pullback $\widetilde{\mathrm{Kl}}_2 = [2]^+ \mathrm{Kl}_2$ has slope 1 at $t = \infty$ and unipotent monodromy with one Jordan block at the origin. Its formal structure is given by

$$\widehat{\widetilde{\mathrm{Kl}}_2} = \mathbf{C}\big((t^{-1})\big) \otimes_{\mathbf{C}[t,t^{-1}]} \widetilde{\mathrm{Kl}}_2 = [2]^+\big([2]_+ E^{2t}\big) \otimes L_{-1} \simeq (E^{2t} \oplus E^{-2t}) \otimes L_{-1}.$$

The module $\widehat{\widetilde{\mathrm{Kl}}_2}$ carries the μ_2-action induced by $t \mapsto -t$ that exchanges both summands; the invariant submodule is identified with $[2]_+ E^{2t}$ as a $\mathbf{C}((z^{-1}))$-module.

$\widetilde{\mathrm{Kl}}_2$ *as a localized Fourier transform*

Recall the map $g\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{A}^1_\tau$ defined by $y \mapsto y + 1/y$. Its critical points are $y = \pm 1$ and its critical values are $\tau = \pm 2$. There is a decomposition $g_+ \mathscr{O}_{\mathbb{G}_{\mathrm{m}}} = \mathscr{O}_{\mathbb{A}^1} \oplus M_2$, where M_2 is an irreducible $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module with regular singularities at $\tau = \pm 2$ and $\tau = \infty$. In fact, M_2 is a rank-1 free $\mathbf{C}[\tau, (\tau \pm 2)^{-1}]$-module with connection, with monodromy $-\mathrm{Id}$ around each point $\tau = \pm 2$ and monodromy Id around ∞. Its Fourier transform $\mathrm{FT}(M_2)$ is then an irreducible $\mathbf{C}[t]\langle\partial_t\rangle$-module with a regular singularity at $t = 0$ and an irregular singularity of slope 1 at $t = \infty$. Equation (2.6) gives, in particular,

$$\widetilde{\mathrm{Kl}}_2 = j_0^+ \mathrm{FT}(g_+ \mathscr{O}_{\mathbb{G}_{\mathrm{m}}}) = j_0^+ \mathrm{FT}(M_2).$$

Therefore, $\widetilde{\mathrm{Kl}}_2$ is an irreducible $\mathbf{C}[t, t^{-1}]$-module with connection and $\mathrm{FT}(M_2)$ is equal to the intermediate extension $j_{0\dagger+}\widetilde{\mathrm{Kl}}_2$. It follows from the formulas (4.1) and (4.2) for the connection on Kl_2 and $\widetilde{\mathrm{Kl}}_2$ that the monodromy of the nearby cycles $\psi_z \mathrm{Kl}_2$ and $\psi_t \widetilde{\mathrm{Kl}}_2$ (2-dimensional vector spaces) is the unipotent automorphism with one Jordan block of size 2.

Bases of $\mathrm{Sym}^k \mathrm{Kl}_2$ *and* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$

Out of the bases $\{v_0, v_1\}$ of Kl_2 and $\{\widetilde{v}_0, \widetilde{v}_1\}$ of $\widetilde{\mathrm{Kl}}_2$, we obtain a $\mathbf{C}[z, z^{-1}]$-basis $\{u_a\}_{0 \le a \le k}$ of $\mathrm{Sym}^k \mathrm{Kl}_2$ and a $\mathbf{C}[t, t^{-1}]$-basis $\{\widetilde{u}_a\}_{0 \le a \le k}$ of $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ by considering the monomials

$$u_a = v_0^{k-a} v_1^a \qquad \text{and} \qquad \widetilde{u}_a = \widetilde{v}_0{}^{k-a} \widetilde{v}_1{}^a \quad (0 \le a \le k). \tag{4.4}$$

In the bases $\{u_a\}$ and $\{\widetilde{u}_a\}$, the connections of $\mathrm{Sym}^k \mathrm{Kl}_2$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ are given by

$$\begin{aligned} z\partial_z u_a &= (k-a)u_{a+1} + azu_{a-1}, \\ \frac{1}{2} t\partial_t \widetilde{u}_a &= (k-a)\widetilde{u}_{a+1} + at^2 \widetilde{u}_{a-1}, \end{aligned} \tag{4.5}$$

with the convention $u_b = \widetilde{u}_b = 0$ for $b < 0$ or $b > k$.

We gather in the following proposition the properties of $\mathrm{Sym}^k \mathrm{Kl}_2$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$.

PROPOSITION 4.6
The free $\mathbf{C}[z, z^{-1}]$*-module with connection* $\mathrm{Sym}^k \mathrm{Kl}_2$ *and the free* $\mathbf{C}[t, t^{-1}]$*-module with connection* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ *satisfy the following properties.*

(1) $\mathrm{rk}\, \mathrm{Sym}^k \mathrm{Kl}_2 = \mathrm{rk}\, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2 = k+1$ *and the monodromy of* $\mathrm{Sym}^k \mathrm{Kl}_2$ *around* $z = 0$ *(resp., of* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ *around* $t = 0$*) is unipotent with only one Jordan block of size* $k+1$.

(2) $\mathrm{Sym}^k \mathrm{Kl}_2$ *and* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ *are endowed with a* $(-1)^k$*-symmetric nondegenerate pairing.*

(3) *The formal structure of* $\mathrm{Sym}^k \mathrm{Kl}_2$ *at infinity is given by*

$$\mathrm{Sym}^k \widehat{\mathrm{Kl}}_2 \simeq \begin{cases} \bigoplus_{j=0}^{(k-1)/2} ([2]_+ E^{2(2j-k)t}) \otimes L_{\mathrm{i}}^{\otimes k} & \text{if } k \text{ is odd,} \\ L_{\mathrm{i}}^{\otimes k} \oplus \bigoplus_{j=0}^{k/2-1} ([2]_+ E^{2(2j-k)t}) \otimes L_{\mathrm{i}}^{\otimes k} & \text{if } k \text{ is even.} \end{cases}$$

In particular, the irregularity number of $\mathrm{Sym}^k \mathrm{Kl}_2$ *at infinity is equal to* $\mathrm{irr}_\infty(\mathrm{Sym}^k \mathrm{Kl}_2) = \lfloor (k+1)/2 \rfloor$ *and, for odd* k*,* $\mathrm{Sym}^k \widehat{\mathrm{Kl}}_2$ *is purely irregular at infinity.*

Proof
(1) This follows from the property that the kth symmetric power of the standard representation of $\mathfrak{sl}_2$ is an irreducible representation of dimension $k+1$.

(2) The $(-1)^k$-symmetric nondegenerate pairings of $\mathrm{Sym}^k \mathrm{Kl}_2$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ are induced by the skew-symmetric self-duality of Kl_2 and $\widetilde{\mathrm{Kl}}_2$, respectively.

(3) The formal structure of $\widehat{\mathrm{Kl}}_2$ at infinity given by (4.3) implies the following lemma.

LEMMA 4.7
There is an isomorphism $\mathrm{Sym}^k \widehat{\mathrm{Kl}}_2 \simeq \mathrm{Sym}^k([2]_+ E^{2t}) \otimes L_{\mathrm{i}}^{\otimes k}$*. Moreover,* $\mathrm{Sym}^k([2]_+ E^{2t})$ *is the invariant submodule of*

$$[2]^+ \mathrm{Sym}^k\bigl([2]_+ E^{2t}\bigr) \simeq \mathrm{Sym}^k\bigl([2]^+[2]_+ E^{2t}\bigr)$$

under $t \mapsto -t$. The latter $\mathbf{C}((t^{-1}))$-module with connection decomposes as

$$\bigoplus_{j=0}^{k} E^{2(2j-k)t}.$$

We know that $\mathrm{Sym}^k([2]_+ E^{2t})$ is the invariant part of $\bigoplus_{j=0}^{k} E^{2(2j-k)t}$ under $t \mapsto -t$. On the one hand, the invariant part of $E^{at} \oplus E^{-at}$ is $[2]_+ E^{at} = [2]_+ E^{-at}$ for $a \neq 0$. On the other hand, the invariant part of $(\mathbf{C}((t^{-1})), \mathrm{d})$ is $(\mathbf{C}((z^{-1})), \mathrm{d})$. Therefore, there is an isomorphism

$$\mathrm{Sym}^k\big([2]_+ E^{2t}\big) \simeq \begin{cases} \bigoplus_{j=0}^{(k-1)/2} [2]_+ E^{2(2j-k)t} & \text{if } k \text{ is odd,} \\ (\mathbf{C}((z^{-1})), \mathrm{d}) \oplus \bigoplus_{j=0}^{k/2-1} [2]_+ E^{2(2j-k)t} & \text{if } k \text{ is even,} \end{cases}$$

from which Proposition 4.6(3) follows. □

It follows from the proposition that the formal regular component $(\mathrm{Sym}^k \widehat{\mathrm{Kl}}_2)_{\mathrm{reg}}$ has rank 0 for odd k and rank 1 for even k, and in the latter case the formal monodromy has eigenvalue 1 if and only if $k \equiv 0 \bmod 4$. From the proof, we also get the irregularity number

$$\mathrm{irr}_\infty(\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) = \begin{cases} k+1 & \text{if } k \text{ is odd,} \\ k & \text{if } k \text{ is even.} \end{cases} \tag{4.8}$$

Similarly, the formalization of $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ at infinity is given by

$$\mathrm{Sym}^k \widehat{\widetilde{\mathrm{Kl}}_2} \simeq \bigoplus_{j=0}^{k} E^{2(2j-k)t} \otimes L_{-1}^{\otimes k}$$

both for odd and even values of k.

COROLLARY 4.9

Let $j_0 \colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1_z$ and $j_\infty \colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1_{1/z}$ denote the inclusions.

(1) *The natural morphism*

$$j_{\infty\dagger+} \mathrm{Sym}^k \mathrm{Kl}_2 \longrightarrow j_{\infty+} \mathrm{Sym}^k \mathrm{Kl}_2$$

is an isomorphism if $k \not\equiv 0 \bmod 4$. The same result holds for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ if $k \not\equiv 0 \bmod 2$.

(2) *Let N be a proper $\mathbf{C}[z]\langle\partial_z\rangle$-submodule of $j_{0+} \mathrm{Sym}^k \mathrm{Kl}_2$ satisfying the equality $j_0^+ N = \mathrm{Sym}^k \mathrm{Kl}_2$. Then $N = j_{0\dagger+} \mathrm{Sym}^k \mathrm{Kl}_2$ holds. The same result is true for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$.*

Proof

Statement (1) follows from Proposition 4.6(3). For (2), the question is local analytic around the point $z = 0$. Set $E = \psi_{z,1} \operatorname{Sym}^k \mathrm{Kl}_2$, and let N denote the nilpotent endomorphism on nearby cycles (see Section A.1). Giving N is equivalent to giving a subspace F of E stable by N, together with two morphisms $E \xrightarrow{\mathrm{can}} F \xrightarrow{\mathrm{var}} E$ commuting with N such that $\mathrm{var} \circ \mathrm{can} = \mathrm{N}$ and such that var is the natural inclusion. Hence, there is an inclusion $F \supset \operatorname{im} \mathrm{can} = \operatorname{im} \mathrm{N}$. Since N has only one Jordan block, its image $\operatorname{im} \mathrm{N}$ has codimension 1 in E and since $F \neq E$ by the properness assumption, this implies $F = \operatorname{im} \mathrm{N}$, as wanted. ☐

The inverse Fourier transform of $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$

Some results of Section 2.3 can be made more explicit for $n = 1$. Namely, applying the vanishing cycle functor at $t = 0$ to the exact sequence (2.17) with $n = 1$ we find the exact sequence

$$0 \longrightarrow \phi_{t,1}\, j_{0\dagger+}(\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2) \longrightarrow \phi_{t,1}\, j_{0+}(\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2) \longrightarrow \phi_{t,1} \widetilde{C}_{k,1} \longrightarrow 0.$$

By definition of the functors j_{0+} and $j_{0\dagger+}$, the middle term $\phi_{t,1}\, j_{0+}(\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ is canonically identified with the nearby cycle module $\psi_{t,1}\, j_{0+}(\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$, and the vanishing cycle module $\phi_{t,1}\, j_{0\dagger+}(\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ is then identified with the subspace $\operatorname{im} \widetilde{\mathrm{N}}$, where $\widetilde{\mathrm{N}}$ is the nilpotent endomorphism acting on $\psi_{t,1}\, j_{0+}(\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$. Since $\widetilde{\mathrm{N}}$ has only one Jordan block of size $k + 1$, it follows that $\phi_{t,1} \widetilde{C}_{k,1}$ is 1-dimensional, and that $\widetilde{\mathrm{N}}$ acting on $\phi_{t,1}\, j_{0\dagger+}(\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ has only one Jordan block of size k.

Let us consider the exact sequence (2.20) in the present setting. The origin $\tau = 0$ is a singular point for $\widetilde{M}$ and $\Pi(\widetilde{M})$ if and only if the formal regular component of $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$ at infinity is nonzero, and then $\dim \phi_\tau \widetilde{M} = \dim \phi_\tau \Pi(\widetilde{M})$ is equal to the rank of this formal regular component. Arguing as for Kl_2, this rank is equal to zero if k is odd and to one if k is even, and in the latter case the eigenvalue of the corresponding formal monodromy is $(-1)^k$.

Let us summarize the properties of $\widetilde{M}$ and $\Pi(\widetilde{M})$.

COROLLARY 4.10

Let $\widetilde{M}$ be the inverse Fourier transform of $j_{0\dagger+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$.

(1) *$\Pi(\widetilde{M})$ is a regular holonomic $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module, generically of rank $k+1$ with singularities at the points $\tau = 2(2j-k)$ for $j = 0, \dots, k$. The vanishing cycle space at each of these singularities has rank 1 with local monodromy equal to $(-1)^k \mathrm{Id}$. At $\tau = \infty$, the monodromy is unipotent, with only one Jordan block of size $k + 1$.*

(2) *$\widetilde{M}$ is a regular holonomic $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module, generically of rank k, with singularities at the points $\tau = 2(2j - k)$ for $j = 0, \dots, k$. The vanishing cycle*

space at each of these singularities has rank 1 *with local monodromy equal to* $(-1)^k \operatorname{Id}$. *At* $\tau = \infty$, *the monodromy is unipotent, with only one Jordan block of size* k.

4.2. *De Rham cohomology on* $\mathbb{G}_{\mathrm{m}}$

As we saw in Proposition 2.13, the de Rham cohomology of $\operatorname{Sym}^k \mathrm{Kl}_2$ is concentrated in degree 1. Thanks to the analogue of the Grothendieck–Ogg–Shafarevich formula for vector bundles with connection (see, e.g., [34, Theorem 2.9.9]), the dimension of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is equal to the irregularity number of $\operatorname{Sym}^k \mathrm{Kl}_2$ at infinity. From Proposition 4.6(3), we thus obtain

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) = \operatorname{irr}_\infty(\operatorname{Sym}^k \mathrm{Kl}_2) = \Big\lfloor \frac{k+1}{2} \Big\rfloor. \tag{4.11}$$

Similarly, using (4.8) we get

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2) = \operatorname{irr}_\infty(\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2) = \begin{cases} k+1 & \text{if } k \text{ is odd,} \\ k & \text{if } k \text{ is even.} \end{cases}$$

By self-duality (Proposition 4.6(2)) and Poincaré duality, there are isomorphisms

$$\begin{aligned} \mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) &\simeq \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)^\vee, \\ \mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2) &\simeq \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)^\vee. \end{aligned}$$

We consider the intermediate extension $\mathscr{D}_{\mathbb{P}^1}$-modules $j_{\dagger+} \operatorname{Sym}^k \mathrm{Kl}_2$ and $j_{\dagger+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$ with respect to the inclusion $j : \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{P}^1$, which according to Corollary 2.15 compute the middle de Rham cohomology. Recall from there that it is also concentrated in degree 1.

PROPOSITION 4.12
Let $\delta_{4\mathbf{Z}}$ *denote the characteristic function of multiples of* 4. *We have*

$$\begin{aligned} \dim \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) &= \Big\lfloor \frac{k-1}{2} \Big\rfloor - \delta_{4\mathbf{Z}}(k) = \begin{cases} \frac{k-1}{2} & \textit{if } k \textit{ is odd,} \\ 2\lfloor \frac{k-1}{4} \rfloor & \textit{if } k \textit{ is even,} \end{cases} \\ \dim \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2) &= \begin{cases} k & \textit{if } k \textit{ is odd,} \\ k-2 & \textit{if } k \textit{ is even.} \end{cases} \end{aligned} \tag{4.13}$$

Proof
We first consider the intermediate extension by $j_0 : \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1_z$. Corollary 4.9(2) and its proof imply that the cokernel of the injective morphism of $\mathbf{C}[z]\langle \partial_z \rangle$-modules

$$j_{0\dagger+}\operatorname{Sym}^k \mathrm{Kl}_2 \longrightarrow j_{0+}\operatorname{Sym}^k \mathrm{Kl}_2$$

is equal to $i_{0+}\mathbf{C}$, where $i_0 \colon \{0\} \hookrightarrow \mathbb{A}^1_z$ is the inclusion. Besides, for the intermediate extension by $j_\infty \colon \mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{A}^1_{1/z}$, we note that the natural morphism

$$j_{\infty\dagger+}\operatorname{Sym}^k \mathrm{Kl}_2 \longrightarrow j_{\infty+}\operatorname{Sym}^k \mathrm{Kl}_2$$

is an isomorphism if the formal completion of $\operatorname{Sym}^k \mathrm{Kl}_2$ at ∞ is purely irregular or has no monodromy invariants, that is, if $k \not\equiv 0 \bmod 4$ according to Proposition 4.6(3). Otherwise, since the formal regular component has rank 1 and monodromy equal to the identity, this morphism is injective with cokernel isomorphic to $i_{\infty+}\mathbf{C}$, where $i_\infty \colon \{\infty\} \hookrightarrow \mathbb{A}^1_{1/z}$ is the inclusion. Therefore, the equality

$$\dim \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2) = \begin{cases} \dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2) - 1 & \text{if } 4 \nmid k, \\ \dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2) - 2 & \text{if } 4 \mid k, \end{cases}$$

holds, and we conclude from (4.11). The proof for $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$ is similar. □

We can now give explicit bases of the de Rham cohomology of $\operatorname{Sym}^k \mathrm{Kl}_2$ and $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$.

PROPOSITION 4.14
The space $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2)$ *has a basis consisting of the classes*

$$z^j v_0^k \frac{\mathrm{d}z}{z}, \quad 0 \leq j < \Big\lfloor \frac{k+1}{2} \Big\rfloor,$$

and the space $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ *has a basis consisting of the classes*

$$t^j \widetilde{v}_0^{\,k} \frac{\mathrm{d}t}{t}, \quad 0 \leq j < 2\Big\lfloor \frac{k+1}{2} \Big\rfloor.$$

Proof
We will only consider the case of $\operatorname{Sym}^k \mathrm{Kl}_2$, that of $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$ being similar by replacing the grading below with the one for which $\deg t = 1$. The space $\mathrm{H}^r_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is identified with the cohomology of the two-term complex

$$G \xrightarrow{z\partial_z} G, \qquad G = \text{the } \mathbf{C}[z, z^{-1}]\langle z\partial_z\rangle\text{-module } \operatorname{Sym}^k \mathrm{Kl}_2.$$

Therefore, the map $z\partial_z$ is injective and the goal is to find a basis of its cokernel. Recall the $\mathbf{C}[z, z^{-1}]$-basis $\{u_0, \ldots, u_k\}$ of G from (4.4), and consider the $\mathbf{C}[z]$-submodule

$$G^+ = \bigoplus_{i=0}^k \mathbf{C}[z]u_i \subset G.$$

Formula (4.5) shows that G^+ is stable under the action of $z\partial_z$. In fact, the coherent sheaf on $\mathbb{A}^1$ associated with G^+ is Deligne's canonical extension of $\mathrm{Sym}^k \mathrm{Kl}_2$ to a logarithmic connection whose residue at zero has all eigenvalues equal to zero.

LEMMA 4.15
The inclusion $(G^+, z\partial_z) \to (G, z\partial_z)$ *is a quasi-isomorphism.*

Proof
This follows from the equality $G = \bigcup_{r\geq 0} z^{-r} G^+$ and the fact that $z\partial_z$ acts invertibly on $z^{-(r+1)}G^+/z^{-r}G^+$ (with eigenvalue $-(r+1)$ and one Jordan block) for all $r \geq 0$. $\square$

Let $\deg\colon G^+ \to (\mathbf{Z}_{\geq 0}, +)$ be the multiplicative *degree* map uniquely determined by the assignments

$$\deg z = 2, \qquad \deg u_i = i. \tag{4.16}$$

(This degree is the one induced from the Newton degree associated with the Laurent polynomial f_k that naturally appears in the computation of the tensor power $\mathrm{Kl}_2^{\otimes k}$; see Section 4.3.1 below). Then $z\partial_z$ is (inhomogeneous) of degree 1. Let $\mathrm{gr}\, G^+$ be the associated graded module. The induced graded $\mathbf{C}$-linear map $\overline{z\partial_z}\colon \mathrm{gr}\, G^+ \to \mathrm{gr}\, G^+[1]$ is $\mathbf{C}[z]$-linear and we shall regard it as a two-term complex $(\mathrm{gr}\, G^+, \overline{z\partial_z})$.

LEMMA 4.17
If k is odd, then $\mathrm{H}^0(\mathrm{gr}\, G^+, \overline{z\partial_z}) = 0$ *and the vector space* $\mathrm{H}^1(\mathrm{gr}\, G^+, \overline{z\partial_z})$ *is generated by the classes of $z^j u_0$, for $0 \leq j \leq (k-1)/2$.*

If k is even, then $\mathrm{H}^0(\mathrm{gr}\, G^+, \overline{z\partial_z})$ *and* $\mathrm{H}^1(\mathrm{gr}\, G^+, \overline{z\partial_z})$ *are the free rank-1 modules over the graded ring* $\mathbf{C}[z]$ *generated by* $\sum_{i=0}^{k/2}(-1)^i\binom{k/2}{i} z^i u_{k-2i}$ *and the class of u_0, respectively.*

Proof
We shall determine the structure of the endomorphism $\overline{z\partial_z}$ on the finitely generated module $\mathrm{gr}\, G^+$ over the principal ideal domain $\mathbf{C}[z]$. From formula (4.5) we see that $z\partial_z$ induces an isomorphism of $\mathbf{C}[z]$-modules

$$\overline{z\partial_z}\colon \bigoplus_{i=0}^{k-1} \mathbf{C}[z]u_i \longrightarrow \mathrm{gr}\, G^+/\mathbf{C}[z]u_0 \tag{4.18}$$

and that, with respect to the basis $\{u_i\}$, the operator $\overline{z\partial_z}$ has determinant $(k!!)^2(-z)^{(k+1)/2}$ if k is odd and zero otherwise. If k is odd, then the space $\mathrm{H}^1(\mathrm{gr}\, G^+, \overline{z\partial_z})$ has dimension $(k+1)/2$ and coincides with the image of $\mathbf{C}[z]u_0$

through the isomorphism (4.18). Therefore, $z^j u_0$, for $0 \leq j \leq (k-1)/2$, form a basis of $\mathrm{H}^1(\operatorname{gr} G^+, \overline{z\partial_z})$.

If k is even, then (4.18) gives an isomorphism $\mathbf{C}[z]u_0 \xrightarrow{\sim} \mathrm{H}^1(\operatorname{gr} G^+, \overline{z\partial_z})$. On the other hand, notice that the map $\overline{z\partial_z}$ splits as a direct sum $\overline{z\partial_z}' \oplus \overline{z\partial_z}''$, where

$$\overline{z\partial_z}': \bigoplus_{j=0}^{k/2} \mathbf{C}[z]u_{2j} \longrightarrow \bigoplus_{j=1}^{k/2} \mathbf{C}[z]u_{2j-1}, \qquad \overline{z\partial_z}'': \bigoplus_{j=1}^{k/2} \mathbf{C}[z]u_{2j-1} \longrightarrow \bigoplus_{j=0}^{k/2} \mathbf{C}[z]u_{2j}.$$

Moreover, $\overline{z\partial_z}'$ is surjective and $\overline{z\partial_z}''$ injective, and hence $\mathrm{H}^0(\operatorname{gr} G^+, \overline{z\partial_z})$ is contained in the submodule $\bigoplus_{j=0}^{k/2} \mathbf{C}[z]u_{2j}$. The statement then follows from an inspection of the coefficients a_i in the equation $\overline{z\partial_z}'(\sum_{i=0}^{k/2} a_i u_{k-2i}) = 0$. □

To finish the proof of Proposition 4.14, we use the spectral sequence

$$E_1^{p,q} = \mathrm{H}^p(\operatorname{gr}_{q-p} G^+ \xrightarrow{\overline{z\partial_z}} \operatorname{gr}_{q-p+1} G^+) \Longrightarrow \mathrm{H}^p(G^+, z\partial_z) \quad \big(p \in \{0,1\}, q \geq 0\big)$$

associated with the grading (4.16), which degenerates at the E_2-page. If k is odd, then all terms $E_1^{0,q}$ vanish by the first part of Lemma 4.17, and the spectral sequence yields an isomorphism of vector spaces $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) \simeq \mathrm{H}^1(\operatorname{gr} G^+, \overline{z\partial_z})$. The statement follows using Lemma 4.17 again. If k is even, then $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is isomorphic to the cokernel of the induced map

$$z\partial_z : \mathrm{H}^0(\operatorname{gr} G^+, \overline{z\partial_z}) \longrightarrow \mathrm{H}^1(\operatorname{gr} G^+, \overline{z\partial_z}). \tag{4.19}$$

For each $r \geq 0$, the equality

$$z\partial_z\Big(z^r \sum_{i=0}^{k/2} a_i z^i u_{k-2i}\Big) = \sum_{i=0}^{k/2} (r+i) a_i z^{r+i} u_{k-2i} \equiv c_r z^{r+k/2} u_0$$

holds in $\mathrm{H}^1(\operatorname{gr} G^+, \overline{z\partial_z})$ for some $c_r \in \mathbf{C}$. Therefore, the classes of the elements $z^j u_0$, for $0 \leq j \leq k/2 - 1$, are linearly independent in the cokernel of (4.19). Since there are as many as the dimension of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ by (4.11), they form a basis. □

4.3. *The Hodge filtration*

In this section, we prove Theorem 1.8. In order to do so, we first establish the analogue of this result for $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$, which is stated as follows.

PROPOSITION 4.20

The mixed Hodge structure $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ has weights at least $k+1$ and the following numerical data.

(1) *For odd k, it is mixed of weights $k+1$ and $2k+2$, with*

$$\dim \mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)^{p,q} = \begin{cases} 1 & p+q=k+1, p \in \{1,\dots,k\}, \\ 1 & p=q=k+1, \\ 0 & \textit{otherwise}. \end{cases}$$

(2) *For even k, it is mixed of weights $k+1$, $k+2$, and $2k+2$, with*

$$\begin{aligned} &\dim \mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)^{p,q} \\ &\quad = \begin{cases} 1 & p+q=k+1, p \in \{1,\dots,k\}, \textit{ and } p \neq k/2, k/2+1, \\ 1 & p=q=k/2+1, \\ 1 & p=q=k+1, \\ 0 & \textit{otherwise}. \end{cases} \end{aligned}$$

Furthermore, the mixed Hodge structure $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ is pure of weight $k+1$ and is equal to $W_{k+1}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$.

Proof
The weight properties of $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ and the purity of $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ were already obtained in the more general setting of Theorem 3.2. To compute the Hodge numbers, we take up the argument in its proof for $n=1$. Recall that the $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module $\widetilde{M}$ is irreducible, generically of rank k, and underlies a pure Hodge module of weight k (see Proposition 2.21 and Lemma 2.24). Let us describe the Hodge filtration on $\widetilde{M}^{\mathrm{H}}$. We start with the nearby cycles at infinity. Since the monodromy around infinity is maximally unipotent (see Corollary 4.10), the nonzero graded pieces of the weight filtration on $\psi_{1/\tau}\widetilde{M}^{\mathrm{H}} = \psi_{1/\tau,1}\widetilde{M}^{\mathrm{H}}$, which is the monodromy filtration associated with $\widetilde{\mathrm{N}}$ centered at $k-1$, are the $\widetilde{\mathrm{N}}^\ell \mathrm{P}_k$. They are hence of the form $\mathrm{gr}^W_{2j}\, \psi_{1/\tau}\widetilde{M}^{\mathrm{H}}$, for $0 \leq j \leq k-1$, and 1-dimensional. It follows that the mixed Hodge structure $\psi_{1/\tau}\widetilde{M}^{\mathrm{H}}$ is of Hodge–Tate type and that

$$\mathrm{gr}^p_F\, \psi_{1/\tau}\widetilde{M}^{\mathrm{H}} = \mathrm{gr}^W_{2p}\, \psi_{1/\tau}\widetilde{M}^{\mathrm{H}}, \quad p = 0, \dots, k-1,$$

has dimension 1. The compatibility property of [53, Section 3.2.1] between the Hodge filtration and the Kashiwara–Malgrange filtration of the filtered $\mathscr{D}$-modules underlying Hodge modules implies, in the case of smooth curves, the equality

$$\operatorname{rk} \mathrm{gr}^p_F\, \widetilde{M}^{\mathrm{H}} = \dim \mathrm{gr}^p_F\, \psi_{1/\tau}\widetilde{M}^{\mathrm{H}}.$$

Hence, $\mathrm{gr}^p_F\, \widetilde{M}^{\mathrm{H}}$ is generically a rank-1 bundle for $p = 0, \dots, k-1$.

Recall the equality $j_{0+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2 = \mathrm{FT}\, \Pi(\widetilde{M})$ from (2.19). From Proposition 2.21, we derive an exact sequence of mixed Hodge structures

$$0 \longrightarrow \mathrm{H}^1(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}^{\mathrm{H}}) \longrightarrow \mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) \longrightarrow \mathrm{H}^1(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}'^{\mathrm{H}}) \longrightarrow 0$$

by applying the functor of Notation A.29. Since $\widetilde{M}'^{\mathrm{H}}$ is pure of weight $2k+1$, Corollary A.31(3) says that $\mathrm{H}^1(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}'^{\mathrm{H}})$ is pure of weight $2k+2$. Besides, this space is 1-dimensional since $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, j_{0\dagger+}\,\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ has codimension 1 in $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ by the argument in the proof of Proposition 4.12. This yields the lines $p = q = k+1$ in (1) and (2).

If k is odd, then zero is not a singular point of $\widetilde{M}$, so Corollary A.31(2) applies. It follows that $\mathrm{H}^1(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}^{\mathrm{H}})$ is pure of weight $k+1$ and its Hodge numbers are the ranks of $\mathrm{gr}_F^{p-1} \widetilde{M}^{\mathrm{H}}$. Since $\mathrm{gr}_F^p \widetilde{M}^{\mathrm{H}}$ has rank 1 for $p = 0, \dots, k-1$ and is zero otherwise, this yields the rest of (1).

If k is even, then zero is a singular point of $\widetilde{M}$ and, according to Corollary A.31(1), there is an isomorphism of mixed Hodge structures

$$\mathrm{H}^1(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}^{\mathrm{H}}) \simeq \mathrm{coker}\big[\widetilde{\mathrm{N}} \colon \psi_{\tau,1}\widetilde{M}^{\mathrm{H}} \longrightarrow \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}(-1)\big].$$

Since $\widetilde{M}$ is an intermediate extension at $\tau = 0$ and $\dim \phi_{\tau,1}\widetilde{M} = 1$ by Corollary 4.10(2), the vanishing $\widetilde{\mathrm{N}}^2 = 0$ holds. Since $\widetilde{M}$ has generic rank k, the primitive parts of the Lefschetz decomposition of $\mathrm{gr}^W \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ are thus

- $\mathrm{P}_1 = \mathrm{gr}^W_k\, \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ of dimension 1,
- $\mathrm{P}_0 = \mathrm{gr}^W_{k-1}\, \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ of dimension $k-2$,

and we get the equality

$$\mathrm{gr}^W \mathrm{coker}\,\widetilde{\mathrm{N}} = \mathrm{P}_0(-1) \oplus \mathrm{P}_1(-1).$$

In particular, $\mathrm{gr}^W_{k+2}\mathrm{H}^1(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}^{\mathrm{H}}) = \mathrm{gr}^W_{k+2}\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ corresponds to the summand $\mathrm{P}_1(-1)$ and has dimension 1, so is of Hodge type $(k/2+1, k/2+1)$, yielding the corresponding line in (2). We conclude the proof by using the equality

$$\begin{aligned}\mathrm{rk}\,\mathrm{gr}_F^{p-1} \widetilde{M}^{\mathrm{H}} &= \dim \mathrm{gr}_F^{p-1} \psi_{\tau,1}\widetilde{M}^{\mathrm{H}} \\ &= \dim \mathrm{gr}_F^p\big(\mathrm{P}_1(-1)\big) + \dim \mathrm{gr}_F^p\big(\mathrm{P}_0(-1)\big) + \dim \mathrm{gr}_F^{p+1}\big(\mathrm{P}_1(-1)\big),\end{aligned}$$

which follows from the Hodge–Lefschetz decomposition on noting the equality $\mathrm{gr}_F^{p+1}\mathrm{P}_1 = \mathrm{gr}_F^p(\mathrm{N}\mathrm{P}_1)$. The leftmost term is 1-dimensional for $p = 1, \dots, k$ and zero otherwise. We already know that $\mathrm{gr}_F^p(\mathrm{P}_1(-1))$ is 1-dimensional for $p = k/2+1$ and zero otherwise. Hence $\mathrm{gr}_F^{p+1}(\mathrm{P}_1(-1))$ is 1-dimensional for $p = k/2$ and zero otherwise and we obtain $\dim \mathrm{gr}_F^p(\mathrm{P}_0(-1)) = 1$ for the remaining values of p, yielding the first line in (2). ◻

We can now show that the bases of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ and $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ given in Proposition 4.14 are adapted to the Hodge filtration if k is odd and that the first half of them are so if k is even. This information will suffice to prove Theorem 1.8 at the end of this section. (A full basis of $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ adapted to the Hodge filtration is constructed in [20, Corollary 3.28] by exploiting the explicit calculation of the intersection pairing on this space.) The proof will rely on the identification of the Hodge filtration on these spaces with their irregular Hodge filtration as de Rham fibers of exponential mixed Hodge structures (see Theorem A.24) and on toric techniques to compute the latter. In what follows, we still denote by $F^\bullet$ the irregular Hodge filtration.

PROPOSITION 4.21
With respect to the bases from Proposition 4.14:

(1) *the Hodge filtration on* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ *is given by*

$$F^p\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) = \Big\langle t^j \widetilde{v}_0^k \frac{\mathrm{d}t}{t} \;\Big|\; 0 \le j \le k+1-p \Big\rangle$$

if k is odd, or if k is even and $p > k/2$;

(2) *the Hodge filtration on* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *is given by*

$$F^p\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) = \Big\langle z^j v_0^k \frac{\mathrm{d}z}{z} \;\Big|\; 0 \le j \le \Big\lfloor \frac{k+1-p}{2} \Big\rfloor \Big\rangle$$

if k is odd, or if k is even and $p > k/2$.

Proof of the inclusion $\supset$

The inclusions

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \hookrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_2^{\otimes k}) \simeq \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k}),$$
$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) \hookrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \widetilde{\mathrm{Kl}}_2^{\otimes k}) \simeq \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{\widetilde{f}_k})$$

are strict with respect to the irregular Hodge filtration and map the basis elements $z^j v_0^k \mathrm{d}z/z$ of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ and $t^j \widetilde{v}_0^k \mathrm{d}t/t$ of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ to

$$w_j = z^j \frac{\mathrm{d}z}{z}\frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_k}{x_k} \in \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k}) \qquad \text{and}$$
$$\widetilde{w}_j = t^j \frac{\mathrm{d}t}{t}\frac{\mathrm{d}y_1}{y_1}\cdots\frac{\mathrm{d}y_k}{y_k} \in \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{\widetilde{f}_k}),$$

respectively. It is therefore enough to prove that

(i) w_j lies in $F^{k+1-2j}\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})$ for $j \ge 0$ if k is odd and $0 \le 2j \le k/2$ if k is even,

(ii) $\widetilde{w}_j$ lies in $F^{k+1-j}\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{\widetilde{f}_k})$ for $j \ge 0$ if k is odd and $0 \le j \le k/2$ if k is even.

4.3.1. Proof of (i) and (ii) in the case when k is odd

We start with (i). We identify the set of Laurent monomials in $z, x_i, \dots, x_k$ with the $\mathbf{Z}$-lattice $\mathbf{Z}^{k+1}$ in $\mathbf{R}^{k+1}$ by taking the exponents. Let $\{\alpha_i\}_{i=0}^k$ be the dual basis of the standard basis of $\mathbf{R}^{k+1}$. Regardless of the parity of k, the monomials appearing in $f_k = \sum_{j=1}^k x_j + z \sum_{j=1}^k 1/x_j$ all lie in the affine hyperplane $h = 1$ in $\mathbf{R}^{k+1}$ defined by the equation $h = 2\alpha_0 + \sum_{i=1}^k \alpha_i$. Thus the Newton polytope $\Delta \subset \mathbf{R}^{k+1}$ of f_k has only one facet that does not contain the origin; it lies on the hyperplane $h = 1$. The cone $\mathbf{R}_{\geq 0}\Delta$ is given by the 2^k inequalities

$$\alpha_0 + \sum_{i=1}^k \varepsilon_i \alpha_i \geq 0, \quad \varepsilon_i \in \{0, 1\}.$$

It is straightforward to check that f_k is nondegenerate with respect to Δ if and only if k is odd. In this case, the irregular Hodge filtration on $\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})$ arises from the Newton filtration on monomials $\mathbf{R}_{\geq 0}\Delta$ by [1, Theorem 1.4] and [63, Theorem 4.6]. In particular, if $m \in \mathbf{R}_{\geq 0}\Delta$ is a monomial with Newton degree $h(m)$ such that the top form $\omega = m \frac{\mathrm{d}z}{z} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_k}{x_k}$ represents a nontrivial class in $\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})$, then

$$\omega \in F^p \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k}) \qquad \text{if} \quad p \leq k + 1 - h(m).$$

In the case at hand, $z^j \in \mathbf{R}_{\geq 0}\Delta$ has degree $h(z^j) = 2j$, hence the assertion.

For (ii), we consider the function $\widetilde{h} = \alpha_0$ on the cone generated by the Newton polytope $\widetilde{\Delta}$ of the Laurent polynomial $\widetilde{f_k}$. If k is odd, then $\widetilde{f_k}$ is nondegenerate. Moreover, given $m \in \mathbf{R}_{\geq 0}\widetilde{\Delta}$ such that the class $\widetilde{\omega}$ of $m \frac{\mathrm{d}t}{t} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_k}{y_k}$ is nontrivial, $\widetilde{\omega}$ belongs to $F^p \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{\widetilde{f_k}})$ if $p \leq k + 1 - \widetilde{h}(m)$ holds. Since $\widetilde{h}(t^j) = j$, the result follows.

4.3.2. A toric compactification

Before proving (i) and (ii) for even k, we describe an explicit compactification of $(\mathbb{G}_{\mathrm{m}}^{k+1}, \widetilde{f_k})$ that will allow us to understand the Hodge filtration on the cohomology of $E^{\widetilde{f_k}}$. Since the construction is also used in Section 5.1 to study the étale realizations of the motive M_k, we take the base field to be $\mathbf{Q}$ before dealing with Hodge filtrations in the second half of this subsection. Let (U, f) be a pair consisting of a smooth quasiprojective variety U and a regular function $f : U \to \mathbb{A}^1$. After Mochizuki [43, Definition 2.6], we call a smooth compactification X of U *nondegenerate* along D if the boundary $D = X \smallsetminus U$ is a strict normal crossing divisor and f extends to a rational morphism

$$f : X \dashrightarrow \mathbb{P}^1$$

such that, locally for the analytic topology around each point of X, there is a coordinate system $\{\xi_1,\dots,\xi_a,\eta_1,\dots,\eta_b,\zeta_1,\dots,\zeta_c\}$ and a multi-index $e\in\mathbf{Z}^r_{>0}$ satisfying

$$D=(\xi\eta),\qquad f=1/\xi^e \text{ or } \zeta_1/\xi^e.$$

Recall the equality $\widetilde{f_k}=tg^{\boxplus k}=t\sum_{i=1}^k(y_i+1/y_i)$ from (2.9) and the isomorphism of Proposition 2.13 with $n=1$. We first compactify $(\mathbb{G}_\mathrm{m}^k,g^{\boxplus k})$. For this, let $M=\bigoplus_{i=1}^k\mathbf{Z}y_i$ be the lattice of Laurent monomials on $\mathbb{G}_\mathrm{m}^k$, and let $N=\bigoplus_{i=1}^k\mathbf{Z}e_i$ be the dual lattice with basis e_i dual to y_i. Consider the toric compactification X of $\mathbb{G}_\mathrm{m}^k$ attached to the simplicial fan F in $N_\mathbf{R}=\bigoplus_{i=1}^k\mathbf{R}e_i$ generated by the 3^k-1 rays

$$\mathbf{R}_{\geq 0}\cdot\sum_{i=1}^k\varepsilon_ie_i\quad\text{with } \varepsilon_i\in\{0,\pm 1\} \text{ and } (\varepsilon_1,\dots,\varepsilon_k)\neq 0. \tag{4.22}$$

There are $2^kk!$ simplicial cones of maximal dimension k in F, each of which provides an affine chart of X isomorphic to $\mathbb{A}^k$ on which the function $g^{\boxplus k}$ has the same structure. Explicitly as an example, consider the maximal cone of F generated by the k vectors

$$\sum_{i=1}^r e_i,\quad 1\leq r\leq k.$$

The affine ring associated with the dual cone in $M_\mathbf{R}$ is the polynomial ring $\mathbf{Q}[z_i]_{i=1}^k$ where

$$z_r=y_r/y_{r+1},\quad 1\leq r<k,\qquad\text{and}\qquad z_k=y_k.$$

On this chart $X_1=\operatorname{Spec}(\mathbf{Q}[z_i]_{i=1}^k)\cong\mathbb{A}^k$, the equality $g^{\boxplus k}=g_1/z_1\cdots z_k$ holds with

$$g_1=1+\sum_{r=2}^k z_1\cdots z_{r-1}+z_1\cdots z_k\sum_{r=1}^k z_r\cdots z_k\in\Gamma(X_1,\mathscr{O}_{X_1}). \tag{4.23}$$

The toric variety X provides an example of a nondegenerate compactification of $(\mathbb{G}_\mathrm{m}^k,g^{\boxplus k})$ as in a neighborhood of $X\smallsetminus\mathbb{G}_\mathrm{m}^k$, the closure of the zero locus of $g^{\boxplus k}$ and $X\smallsetminus\mathbb{G}_\mathrm{m}^k$ form a strict normal crossing divisor (see also the paragraph before Section 5.1.3).

Let us construct a nondegenerate compactification of $(\mathbb{G}_\mathrm{m}^{k+1},tg^{\boxplus k})\cong(\mathbb{G}_\mathrm{m}^{k+1},\widetilde{f_k})$ starting from $\mathbb{P}^1_t\times X$. For this, we order the 3^k-1 irreducible components $(S_i)_{1\leq i\leq 3^k-1}$ of $X\smallsetminus\mathbb{G}_\mathrm{m}^k$ corresponding to the rays (4.22) and consider the tower $\overline{X}\to\cdots\to\mathbb{P}^1_t\times X$ of 3^k-1 blowups along the intersection of the proper transform of $0\times X$ (on which $tg^{\boxplus k}$ has a simple zero) and the proper transform of $\mathbb{P}^1_t\times S_i$ (on which $tg^{\boxplus k}$ has a simple pole). Together with the function induced from the blowup maps, the resulting variety is a nondegenerate compactification of $(\mathbb{G}_{\mathrm{m}}^{k+1}, \widetilde{f_k})$ if k is odd. If k is even, then $\overline{X}$ is nondegenerate away from the $\binom{k}{k/2}$ points

$$(t, y_i) = (\infty, \varepsilon_i), \qquad \varepsilon_i \in \{\pm 1\} \qquad \sum_{i=1}^{k} \varepsilon_i = 0$$

(note that they are defined over $\mathbf{Q}$). Let x be such a point. For a suitable choice of (analytic or étale) local coordinates $z_1, \dots, z_k$ of X around x, the function $g^{\boxplus k}$ takes the form $z_1^2 + \cdots + z_k^2$, which means that x is an ordinary quadratic point of the zero locus of $g^{\boxplus k}$. We perform two blowups on $\overline{X}$: first at each x and then along the intersection of the exceptional divisor and the proper transform of $\infty \times X$. Let $\widetilde{X}$ be the resulting variety, and let E_1 and E_2 be the exceptional divisors from the first and second steps, respectively. A direct computation reveals that $\widetilde{X}$ is a nondegenerate compactification of the pair $(\mathbb{G}_{\mathrm{m}}^{k+1}, \widetilde{f_k})$ with $\operatorname{ord}_{E_1} \widetilde{f_k} = 1$ and $\operatorname{ord}_{E_2} \widetilde{f_k} = 0$.

4.3.3. Proof of (i) and (ii) in the case when k is even

We now start with (ii). Let $\widetilde{X}$ be the nondegenerate compactification of $(\mathbb{G}_{\mathrm{m}}^{k+1}, \widetilde{f_k})$ constructed above and $D = \widetilde{X} \smallsetminus \mathbb{G}_{\mathrm{m}}^{k+1}$. Since the indeterminacy locus of the rational map $\widetilde{f_k} \colon \widetilde{X} \dashrightarrow \mathbb{P}^1$ has codimension at least 2 in $\widetilde{X}$, one can define the pole divisor P of $\widetilde{f_k}$ as the closure of the pole divisor of a representative of $\widetilde{f_k}$, and similarly for the zero divisor. The exceptional divisors E_1 and E_2 are not contained in the support of P, and a direct computation shows that the form $\widetilde{\omega}_j$ lies in

$$\Gamma\big(\widetilde{X}, \Omega_{\widetilde{X}}^{k+1}(\log D)\big(jP - (k-j)E_1 - (k-2j)E_2\big)\big).$$

Accordingly, if the inequalities $0 \le j \le k/2$ hold (so that $\widetilde{\omega}_j$ is holomorphic generically on the divisors E_1 and E_2), then the form $\widetilde{\omega}_j$ lies in

$$\Gamma\big(\widetilde{X}, \Omega_{\widetilde{X}}^{k+1}(\log D)(jP)\big).$$

In this case, we claim that there is a natural map

$$\Gamma\big(\widetilde{X}, \Omega_{\widetilde{X}}^{k+1}(\log D)(jP)\big) \longrightarrow F^{k+1-j}\mathrm{H}_{\mathrm{dR}}^{k+1}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{\widetilde{f_k}}),$$

from which the statement will follow. Indeed, as described in [63, Section 4(b)] (especially in the paragraph containing diagram (26)), one can resolve the indeterminacies of $\widetilde{f_k}$ by taking a tower of blowups $\pi \colon \widetilde{X}' \to \widetilde{X}$ of $\widetilde{X}$ along the intersections of the zero divisor and the irreducible components of the pole divisor of the transforms of $\widetilde{f_k}$ such that $D' = \widetilde{X}' \smallsetminus \mathbb{G}_{\mathrm{m}}^{k+1}$ remains a strict normal crossing divisor and $\widetilde{f_k}$ extends to an everywhere defined morphism $\widetilde{f_k}' \colon \widetilde{X}' \to \mathbb{P}^1$. Let P' be the pole divisor of $\widetilde{f_k}'$. By [63, Proposition 4.4], the equality

$$\begin{aligned}&\mathrm{R}\Gamma\big(\widetilde{X}, \big(\Omega^\bullet_{\widetilde{X}}(\log D)\big((\bullet-p)P\big)_+, \mathrm{d}+\mathrm{d}\widetilde{f}_k\big)\big)\\&\qquad=\mathrm{R}\Gamma\big(\widetilde{X}', \big(\Omega^\bullet_{\widetilde{X}'}(\log D')\big((\bullet-p)P'\big)_+, \mathrm{d}+\mathrm{d}\widetilde{f}'_k\big)\big)\end{aligned}$$

holds, where we use the notation

$$\Omega^i_{\widetilde{X}}(\log D)\big((i-p)P\big)_+ = \begin{cases}\Omega^i_{\widetilde{X}}(\log D)((i-p)P) & i \geq p,\\ 0 & i < p,\end{cases}$$

and similarly for the other complex. (In [63, Proposition 4.4], the complex on the left-hand side is denoted by $F^p_{\mathrm{NP}}(\nabla)$ and that on the right-hand side by $F^p(\nabla)$, and Proposition 4.4 there implies that $F^p_{\mathrm{NP}}(\nabla)$ and $\mathrm{R}\pi_* F^p(\nabla)$ are quasi-isomorphic; see also the proof of Theorem 4.6 in the same work. Note that the assumption that the Laurent polynomial is nondegenerate is not needed for [63, Proposition 4.4].) On the other hand, by the E_1-degeneration of the irregular Hodge filtration proved in [15, Theorem 1.2.2], where $F^p(\nabla)$ is denoted by $F_0^{\mathrm{Yu},p}(\Omega_{\widetilde{X}'}(*D'), \nabla)$ instead, the equality

$$\mathbb{H}^{k+1}\big(\widetilde{X}', \big(\Omega^\bullet(\log D')\big((\bullet-p)P'\big)_+, \mathrm{d}+\mathrm{d}\widetilde{f}'_k\big)\big) = F^p\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}^{k+1}, E^{\widetilde{f}_k})$$

holds. This completes the claim, and hence the proof of (ii).

To prove (i), we observe that the equality

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mu_2} \tag{4.24}$$

is compatible with the Hodge filtration, so that we can check whether a form belongs to some step of the Hodge filtration by pulling it back by the double cover [2] given by $t \mapsto z = t^2$. Since the pullback $[2]^* z^j v_0^k \mathrm{d}z/z = 2t^{2j}\widetilde{v}_0^k \mathrm{d}t/t$ maps to $2\widetilde{w}_{2j}$, it lies in $F^{k+1-2j}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ for all $0 \leq 2j \leq k/2$ by part (ii). We thus get

$$z^j v_0^k \frac{\mathrm{d}z}{z} \in F^{k+1-2j}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \quad \text{if } 0 \leq 2j \leq k/2,$$

which ends the proof of the inclusion $\supset$ in Proposition 4.21. □

Proof of the equality in Proposition 4.21(1)
Since $t^j\widetilde{v}_0^k \mathrm{d}t/t$ form a basis of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ and the graded pieces of the Hodge filtration on this space are 1-dimensional by Proposition 4.20, the inclusion $\supset$ in Proposition 4.21(1) is necessarily an equality. □

Proof of the equality in Proposition 4.21(2) and of Theorem 1.8
By Proposition 4.20, the mixed Hodge structure $\mathrm{H}^1(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ has weights $k+1$, $k+2$ (for even k), and $2k+2$, and the graded piece $\mathrm{gr}^p_F \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$

is 1-dimensional for $p = 1, \dots, k+1$ (except for $p = k/2$ if k is even) and zero otherwise. Since the identification (4.24) is compatible with the weight and the Hodge filtrations, the possible weights of $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ are $k+1$, $k+2$ (for even k), and $2k+2$, with graded pieces of dimension at most 1 in the last two cases, and all Hodge numbers are zero or one, depending on whether μ_2 acts as multiplication by -1 or by $+1$ on $\mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$.

By Proposition 4.21(1), the class of $z^j v_0^k \mathrm{d}z/z$ in $\mathrm{gr}_F^{k+1-2j} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is nonzero for all j satisfying $0 \le 2j \le k$ if k is odd and $0 \le 2j \le k/2$ if k is even, since its pullback to $\mathrm{gr}_F^{k+1-2j} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ is nonzero.

If k is odd, then the nonvanishing of $\mathrm{gr}_F^{k+1-2j} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ for j satisfying $0 \le 2j \le k$ implies that this space is 1-dimensional, the class of $z^j v_0^k \mathrm{d}z/z$ being a basis. Since these classes form a basis of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$, all other graded pieces $\mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ vanish. This concludes the proof of both Theorem 1.8 and Proposition 4.21(2) for odd k.

If k is even, then the same argument shows that $\mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is 1-dimensional for $p = k+1, k-1, \dots, 2\lceil k/4 \rceil + 1$. Since, by Proposition 4.20(2), $\mathrm{gr}_F^{k+1} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ lies in weight $2k+2$ so does $\mathrm{gr}_F^{k+1} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ and the corresponding graded piece is 1-dimensional. This yields the Hodge numbers for the line $p = q = k+1$ in Theorem 1.8(2). On the other hand, since $k+1$ is odd, the space $\mathrm{gr}^W_{k+1} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is even-dimensional by Hodge symmetry. Since $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ has dimension $k/2$ by (4.11) and $\mathrm{gr}^W_{k+2} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ has dimension at most 1, we get the equality

$$\dim \mathrm{gr}^W_{k+2} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) = \begin{cases} 0 & k \not\equiv 0 \bmod 4, \\ 1 & k \equiv 0 \bmod 4. \end{cases}$$

If $k \not\equiv 0 \bmod 4$ (so that $2\lceil k/4 \rceil + 1 = k/2 + 2$ holds), then the spaces $\mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ lie in weight $k+1$ for $p = k-1, \dots, k/2+2$, and hence the space $\mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is 1-dimensional for $p = 2, 4, \dots, k/2 - 1$ by Hodge symmetry.

If $k \equiv 0 \bmod 4$ (so that we now have $2\lceil k/4 \rceil + 1 = k/2 + 1$), then the space $\mathrm{gr}^W_{k+2} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is 1-dimensional, and hence $\mathrm{gr}_F^{k/2+1} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ lies in weight $k+2$. This gives the line $p = q = k/2+1$ in Theorem 1.8(2). To get the remaining Hodge numbers, we argue as above: the spaces $\mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ lie in weight $k+1$ for $p = k-1, \dots, k/2+3$, and hence $\mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is 1-dimensional for $p = 2, 4, \dots, k/2 - 2$ by Hodge symmetry. This completes the proof of both Theorem 1.8(2) and Proposition 4.21(2) for even k. $\square$

5. L-Functions

In this section, we compute the L-function of the pure motive M_k over $\mathbf{Q}$ defined in (3.1). We first compare, in Theorems 5.8 and 5.17, the traces of Frobenius at unramified primes of its ℓ-adic realization with symmetric power moments of Kloosterman sums. These results largely overlap with Yun's (see [64, Theorem 1.1.6]). Up to semisimplification, the two approaches yield the same Galois representations as realizations of two different geometric models. In some sense, we have replaced the use of affine Grassmannians and homogeneous Fourier transformation in [64] with that of exponential mixed Hodge structures and the irregular Hodge filtration to obtain the easier geometric model $\mathscr{K}$ in terms of which the motive is defined (cf. [64, Section 4.1.6]). One advantage of this point of view is that it enables us to determine the structure of the Galois representations at ramified primes by means of the Picard–Lefschetz formula. In addition, we show in Proposition 5.23 that the Galois representations are crystalline at $p > k$ when k is odd (resp., $p > k/2$ when k is even) and we obtain lower bounds for the p-adic valuation of the eigenvalues of Frobenius in Corollary 5.27. We then compute the gamma factor in Corollary 5.30. Using the theorem of Patrikis and Taylor, we finally prove that the motives (3.1) are potentially automorphic in the last subsection. Theorems 1.2 and 1.3 from the introduction follow by pulling everything together.

5.1. Étale realizations

5.1.1. Cohomology of $\mathrm{Sym}^k \mathrm{Kl}_2$ *over finite fields*

Recall the ℓ-adic Kloosterman sheaf Kl_2 on $\mathbb{G}_{\mathrm{m}}$ over $\mathbf{F}_p$ from Section 3.2.1. In this subsection, we gather the main properties of the étale cohomology of its symmetric powers. All results below are due to Fu and Wan [21, Theorem 0.2] and Yun [64, Lemma 4.2.1, Corollaries 4.2.3, 4.3.5], who prove them by means of a thorough study of the structure of $\mathrm{Sym}^k \mathrm{Kl}_2$ at zero and infinity. Throughout, F_p denotes the geometric Frobenius in $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ and we consider the reciprocal characteristic polynomials

$$Z_k(p;T) = \det\big(1 - F_p T \mid \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)\big),$$

$$M_k(p;T) = \det\big(1 - F_p T \mid \mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)\big).$$

- If k is odd, then

$$\deg Z_k(p;T) = \begin{cases} \frac{k+1}{2} & p = 2, \\ \frac{k+1}{2} - \lfloor \frac{k}{2p} + \frac{1}{2} \rfloor & p \geq 3, \end{cases} \tag{5.1}$$

and there is a factorization

$$Z_k(p;T) = (1-T) M_k(p;T), \tag{5.2}$$

where the reciprocal roots of $M_k(p;T)$ are Weil numbers of weight $k+1$.

- If k is even and p is odd, then

$$\deg Z_k(p;T) = \frac{k}{2} - \left\lfloor \frac{k}{2p} \right\rfloor$$

and there is a factorization

$$Z_k(p;T) = (1-T)R_k(p;T)M_k(p;T) \tag{5.3}$$

such that the reciprocal roots of $M_k(p;T)$ are again Weil numbers of weight $k+1$. Above, the polynomial $R_k(p;T)$ is given by

$$R_k(p;T) = \big(1-(-1)^{(p-1)/2}p^{k/2}T\big)^{n_k(p)}(1-p^{k/2}T)^{m_k(p)-n_k(p)},$$
$$n_k(p) = \left\lfloor \frac{k}{4p} + \frac{1}{2} \right\rfloor, \qquad m_k(p) = \left\lfloor \frac{k}{2p} \right\rfloor + \delta_{4\mathbf{Z}}(k).$$

- There is also an explicit description for even k and $p=2$ in [64, Lemma 4.3.4, Corollary 4.3.5]; namely, $Z_k(2;T)$ has degree $\lfloor (k+2)/4 \rfloor$ and factors as

$$Z_k(2;T) = (1-T)(1-2^{k/2}T)^{a_k}(1+2^{k/2}T)^{b_k}M_k(2;T), \tag{5.4}$$

where $\deg M_k = 2\lfloor (k+2)/12 \rfloor - 2\delta_{12\mathbf{Z}}(k)$ and a_k and b_k are given by

$$a_k = \begin{cases} \lfloor \frac{k}{24} \rfloor + 1 & k \equiv 0, 8, 12, 16, 18, 20 \mod 24, \\ \lfloor \frac{k}{24} \rfloor & k \equiv 2, 4, 6, 10, 14, 22 \mod 24, \end{cases}$$

$$b_k = \begin{cases} \lfloor \frac{k}{24} \rfloor + 1 & k \equiv 6, 12, 14, 18, 20, 22 \mod 24, \\ \lfloor \frac{k}{24} \rfloor & k \equiv 0, 2, 4, 8, 10, 16 \mod 24. \end{cases}$$

In all three cases, the reciprocal roots α of the polynomial $M_k(p;T)$ are stable under the transformation $\alpha \mapsto p^{(k+1)/2}\alpha^{-1}$, which reflects the self-duality of the middle cohomology.

5.1.2. Galois representations of symmetric power moments

Recall from (3.9) the Laurent polynomial $g^{\boxplus k} = \sum_{i=1}^{k}(y_i + 1/y_i)$ on the torus $\mathbb{G}_{\mathrm{m}}^k$ and its zero locus $\mathscr{K} \subset \mathbb{G}_{\mathrm{m}}^k$. For each prime number ℓ, the ℓ-adic realization of the motive M_k is the $\mathbf{Q}_\ell$-vector space

$$V_{k,\ell} = \mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)^{\mathfrak{S}_k \times \mu_2, \chi}(-1) \tag{5.5}$$

equipped with the continuous representation

$$r_{k,\ell}\colon\ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}(V_{k,\ell}).$$

Writing $k = 2m+1$ for odd k and $k = 2m+2$ or $k = 2m+4$ with m an even integer for even k, the vector space $V_{k,\ell}$ is m-dimensional by (4.13).

The goal of the next two subsections is to compare the traces of Frobenius at unramified primes with symmetric power moments of Kloosterman sums. For this, we shall consider the toric compactification X of $\mathbb{G}_{\mathrm{m}}^k$ introduced in Section 4.3.2 and let $\overline{\mathscr{K}}$ be the closure of $\mathscr{K}$ in X. We also regard these varieties as defined over general rings (e.g., over $\mathbf{F}_p$, $\mathbf{Z}_p$, and so on). We claim that $\overline{\mathscr{K}}$ is smooth along the strict normal crossing divisor $D = X \smallsetminus \mathbb{G}_{\mathrm{m}}^k$ and that each irreducible component of D intersects $\overline{\mathscr{K}}$ in a smooth divisor in such a way that $\overline{\mathscr{K}} \smallsetminus \mathscr{K}$ forms a relative strict normal crossing divisor over $\mathbf{Z}$. Indeed, it is enough to check these properties on each of the $2^k k!$ affine charts of X corresponding to the cones of maximal dimension of the simplicial fan F. For example, on the chart $X_1 \cong \mathbb{A}^k = \mathrm{Spec}(\mathbf{Z}[z_i]_{i=1}^k)$, the function $g^{\boxplus k}$ is given by $g^{\boxplus k} = g_1/z_1 \cdots z_k$ and the equality $\overline{\mathscr{K}} \cap X_1 = (g_1)$ holds, with g_1 as in (4.23). One then checks the equalities

$$(g_1) \cap (z_1) = \varnothing, \qquad (g_1) \cap (z_r) = \big(1 + z_1(1 + z_2 + \cdots + z_2 \cdots z_{r-1})\big),$$
$$(\partial g_1/\partial z_1) \cap (z_r) = (1 + z_2 + \cdots + z_2 \cdots z_{r-1})$$

for $r = 2, \ldots, k$. From the first two, it follows that $\partial g_1/\partial z_1$ does not vanish on (z_r) for $r \geq 2$, hence the smoothness of $\overline{\mathscr{K}}$ along $(z_1 \cdots z_r)$. The smoothness of $(g_1) \cap (z_r)$ is also clear, and $\overline{\mathscr{K}} \cap (z_{r_1}) \cap \cdots \cap (z_{r_i})$ is smooth for any sequence of indices $2 \leq r_1 \leq \cdots \leq r_i \leq k$, which implies the strict normal crossing property. Besides, over $\mathbf{Q}$, the variety $\mathscr{K} \subset \mathbb{G}_{\mathrm{m}}^k$ is smooth when k is odd, while if k is even, its singular locus consists of $\binom{k}{k/2}$ ordinary quadratic points with coordinates $y_i \in \{\pm 1\}$ satisfying $\sum_{i=1}^k y_i = 0$.

5.1.3. The ℓ-adic case for odd symmetric powers

Let $k \geq 1$ be an odd integer, and let p be an odd prime number. The singular locus Σ of $\mathscr{K}$ over $\overline{\mathbf{F}}_p$ consists of $\lfloor k/2p + 1/2 \rfloor$ orbits of ordinary quadratic points under the action of $\mathfrak{S}_k \times \mu_2$. Indeed, the orbits are indexed by odd positive integers a such that $ap \leq k$, each of them being represented by the point with coordinates $y_i = 1$ (resp., -1) for $1 \leq i \leq (ap+k)/2$ (resp., $i > (ap+k)/2$). Locally around this point, writing $y_i = z_i + 1$ (resp., $y_i = z_i - 1$), the defining equation of $\mathscr{K}$ in $\mathbf{Z}_p[\![z_1, \ldots, z_k]\!]$ is given by

$$g^{\boxplus k}(z_1, \ldots, z_k) = 2ap + Q_{ap} + \text{higher order terms},$$

where Q_{ap} is the nondegenerate quadratic form

$$Q_{ap} = \sum_{i \leq (ap+k)/2} z_i^2 - \sum_{i > (ap+k)/2} z_i^2. \tag{5.6}$$

Write $k = 2m + 1$. After choosing a place of $\overline{\mathbf{Q}}$ above p, with each $x \in \Sigma$ is associated a *vanishing cycle class* δ_x in $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)(m)$ that is well defined up to sign. Letting $\langle , \rangle$ denote the pairing obtained from the intersection form and the identification $\mathrm{H}^{2k-2}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)(2m) \cong \mathbf{Q}_\ell$ given by the trace, these classes satisfy

$$\langle \delta_x, \delta_y \rangle = \begin{cases} (-1)^m 2 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

By the Picard–Lefschetz formula (see [11, Exposé XV, Théorème 3.4]), there is an exact sequence

$$0 \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell) \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) \xrightarrow{\gamma} \bigoplus_{x \in \Sigma} \mathbf{Q}_\ell(-m) \longrightarrow 0,$$

where the map γ is given by taking pairings with δ_x.

In what follows, we keep the notation ζ for a primitive pth root of unity in $\overline{\mathbf{Q}}_\ell$, denote by $-[\zeta]$ the scalar extension $- \otimes_{\mathbf{Q}_\ell} \mathbf{Q}_\ell(\zeta)$, and set

$$\Theta_p^+ = \{a \geq 1 \text{ odd integer} \mid ap \leq k \text{ with } v_p(a) \text{ odd}\},$$
$$\Theta_p^- = \{a \geq 1 \text{ odd integer} \mid ap \leq k \text{ with } v_p(a) \text{ even}\},$$

so that the following equality holds:

$$|\Theta_p^+| + |\Theta_p^-| = \Big\lfloor \frac{k}{2p} + \frac{1}{2} \Big\rfloor. \tag{5.7}$$

THEOREM 5.8

Let $k = 2m + 1$ be a positive odd integer, and let p and ℓ be distinct prime numbers. Let $V_{k,\ell}$ denote the ℓ-adic realization of the motive M_k, which is an m-dimensional $\mathbf{Q}_\ell$-representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Fix a place of $\overline{\mathbf{Q}}$ above p, and let I_p be the corresponding inertia subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

(1) *The representation $V_{k,\ell}$ is unramified at 2 and the $\mathrm{Gal}(\overline{\mathbf{F}}_2/\mathbf{F}_2)$-module $V_{k,\ell}[\zeta]$ is isomorphic to $\mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_2}, \mathrm{Sym}^k \mathrm{Kl}_2)$.*

(2) *If p is an odd prime, then $V_{k,\ell}$ is at most tamely ramified at p. More precisely, the restriction of $V_{k,\ell}$ to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ decomposes into an orthogonal sum $M \oplus E$, where*
   - $M[\zeta] = \mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)$,
   - *E is generated by vanishing cycles, one for each $a \in \Theta_p^+ \cup \Theta_p^-$, on which the Galois group acts through the character $\varepsilon_a \otimes \chi_{\mathrm{cyc}}^{-m-1}$, where $\varepsilon_a \colon \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \{\pm 1\}$ stands for the primitive character associated with the extension*

$$\mathbf{Q}_p\bigl(\sqrt{(-1)^{(1+ap)/2}2ap}\bigr) \quad \textit{of} \quad \mathbf{Q}_p.$$

In particular, decomposing $E = E^+ \oplus E^-$ according to whether a belongs to Θ_p^+ or Θ_p^-, the invariants under inertia are $V_{k,\ell}^{I_p} = M \oplus E^+$ and E^+ is a semisimple $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$-module with reciprocal characteristic polynomial of Frobenius

$$\det(1 - F_pT \mid E^+) = \prod_{a\in\Theta_p^+}\Bigl(1 - \Bigl(\frac{(-1)^{(1+ap)/2}2a'}{p}\Bigr)p^{m+1}T\Bigr),$$

where $a' = ap^{-v_p(a)}$ denotes the prime-to-p part of a and $(\cdot/p)$ denotes the Legendre symbol.

Proof
There is nothing to prove if $k = 1$, so we assume that $k \geq 3$. To shorten notation, we omit the coefficients $\mathbf{Q}_\ell$ from the cohomology and write $G = \mathfrak{S}_k \times \mu_2$, so that

$$V_{k,\ell} = \mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}})^{G,\chi}(-1).$$

Set $\overline{\mathscr{K}}^{(0)} = \overline{\mathscr{K}}$ and, for each $i \geq 1$, let $\overline{\mathscr{K}}^{(i)}$ denote the disjoint union of all i-fold intersections of distinct irreducible components of $\overline{\mathscr{K}} \smallsetminus \mathscr{K}$. The spectral sequence

$$E_1^{i,j} = \mathrm{H}^j_{\text{ét}}(\overline{\mathscr{K}}^{(i)}_{\overline{\mathbf{F}}}) \Longrightarrow \mathrm{H}^{i+j}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}}) \quad (i, j \geq 0)$$

computes the étale cohomology with compact support of $\mathscr{K}$ over $\mathbf{F} = \mathbf{Q}$ or $\mathbf{F}_p$.

For $\mathbf{F} = \mathbf{Q}$, the spectral sequence degenerates at E_2 since, $\overline{\mathscr{K}}^{(i)}_{\overline{\mathbf{Q}}}$ being a smooth proper variety for all $i \geq 0$, the source and the target of the differentials in the second page are pure of different weights. The graded piece of weight $k-1$ is thus isomorphic to $E_2^{0,k-1}$ and

$$\begin{aligned}\mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}) &= \ker\bigl\{\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}}) \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}^{(1)}_{\overline{\mathbf{Q}}})\bigr\}\\ &= \mathrm{im}\bigl\{\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}) \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})\bigr\},\end{aligned} \tag{5.9}$$

where the second map is the surjective edge map from the abutment $\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}})$ to $E_2^{0,k-1}$. On the other hand, by Theorem 3.12, there is an isomorphism

$$\mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k\,\mathrm{Kl}_2) \cong \mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_p})^{G,\chi}(-1)[\zeta].$$

This cohomology group is pure of weight $k+1$ and has dimension m if $p = 2$ and $m - \lfloor k/2p + 1/2\rfloor$ if $p \geq 3$ by (5.1). For each $i \geq 1$, the variety $\overline{\mathscr{K}}^{(i)}_{\overline{\mathbf{F}}_p}$ is smooth and proper. Hence, the étale cohomology $\mathrm{H}^{k-1-i}_{\text{ét}}(\overline{\mathscr{K}}^{(i)}_{\overline{\mathbf{F}}_p})$ is pure of weight $k-1-i$, and so is $E_\infty^{i,k-1-i}$ in the above spectral sequence for $\mathbf{F}=\mathbf{F}_p$. The only contribution of weight $k-1$ is thus given by

$$\mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_p}) = \mathrm{gr}^W_{k-1}\,\mathrm{im}\big\{\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_p}) \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p})\big\}. \tag{5.10}$$

Assume that $p=2$. The proper variety $\overline{\mathscr{K}}_{\overline{\mathbf{F}}_2}$ has a quadratic nonordinary isolated singularity, namely the point with coordinates $y_i=1$. When this is the case, the cohomology sheaf $\mathrm{R}^n\Phi$ of the vanishing cycle complex on $\overline{\mathscr{K}}_{\overline{\mathbf{F}}_2}$ is nonzero only in degree $n=k-1$ by [30, Corollaire 2.10], which implies that the cospecialization morphism $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_2}) \to \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})$ is injective. From the isomorphism (5.9) and the commutativity of the square

$$\begin{tikzcd}
\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_2}) \arrow[r] \arrow[d] & \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}) \arrow[d] \\
\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_2}) \arrow[r] & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})
\end{tikzcd}$$

we deduce an injection

$$\mathrm{im}\big\{\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_2}) \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_2})\big\}^{G,\chi} \longrightarrow \mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}})^{G,\chi} = V_{k,\ell}(1). \tag{5.11}$$

Since the χ-isotypic part of the left-hand side of (5.10) has dimension $m=\dim V_{k,\ell}$, it follows that (5.11) is indeed an isomorphism, hence an isomorphism

$$V_{k,\ell} = \mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_2})^{G,\chi}(-1)$$

of representations of $\mathrm{Gal}(\overline{\mathbf{F}}_2/\mathbf{F}_2)$. This concludes the proof of (1).

Now suppose that $p\geq 3$, and consider the G-equivariant commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_p}) \arrow[r] \arrow[d, "\beta"'] & \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}) \arrow[d, "\alpha"] & & \\
0 \arrow[r] & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p}) \arrow[r] \arrow[d] & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}}) \arrow[r, "\gamma"] \arrow[d] & \bigoplus_{x\in\Sigma}\mathbf{Q}_\ell(-m) \arrow[r] & 0 \\
 & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}^{(1)}_{\overline{\mathbf{F}}_p}) \arrow[r, "\sim"] & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}^{(1)}_{\overline{\mathbf{Q}}}) & &
\end{tikzcd}$$

in which the middle row is given by the Picard–Lefschetz formula. Let

$$\Delta = \bigoplus_{x\in\Sigma} \mathbf{Q}_\ell(-m)\delta_x$$

be the subspace of $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})$ generated by vanishing cycle classes, which is the orthogonal complement of the image of $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p})$. Let $\mathrm{R}\Psi$ be the complex of nearby cycles on $\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p}$. Since δ_x is a generator of the local cohomology $\mathrm{H}^{k-1}_{\{x\}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p}, \mathrm{R}\Psi(m))$ with support $\{x\}$ contained in $\mathscr{K}_{\overline{\mathbf{F}}_p}$, the subspace Δ lies in the image of α. The image of β and Δ being orthogonal as subspaces of $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})$, the equalities

$$\mathrm{im}(\beta)^{G,\chi} = \mathrm{gr}^W_{k-1}\,\mathrm{im}(\beta)^{G,\chi} \qquad \text{and} \qquad \mathrm{im}(\alpha)^{G,\chi} = \mathrm{im}(\beta)^{G,\chi} \oplus \Delta^{G,\chi}$$

hold, with $\dim \Delta^{G,\chi} = \lfloor k/2p + 1/2 \rfloor$ by a dimension count and (5.2). These are the factors M and E in part (2) of the theorem.

To compute the Galois action on E, recall the quadratic form Q_{ap} from (5.6), and consider the projective quadric $D = (2apw^2 + Q_{ap})$ in $\mathbb{P}^k_{\mathbf{Q}_p}$, as well as the hyperplane section $C = D \cap (w)$. In [11, Exposé XV, Proposition 2.2.3], the space $\mathbf{Q}_\ell(-m)\delta_x$ is described as $\mathrm{H}^{k-1}_{\text{ét,c}}((D \smallsetminus C)_{\overline{\mathbf{Q}}_p})$, which is equal to the primitive part of $\mathrm{H}^{k-1}_{\text{ét}}(D_{\overline{\mathbf{Q}}_p})$ by the localization sequence for étale cohomology with compact support. As a nondegenerate quadratic form over $\mathbf{Q}_p$, the defining equation of D has discriminant $d = (-1)^{(k-ap)/2}2ap$, and hence $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts on

$$\det \mathrm{H}^{k-1}_{\text{ét}}\big(D_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_\ell(m)\big) = \mathrm{H}^{k-1}_{\text{ét,prim}}(D_{\overline{\mathbf{Q}}_p})(m)$$

via the character ε_a corresponding to the extension $\mathbf{Q}_p(\sqrt{(-1)^{(k+1)/2}d})$ by [56, Section 5.2]. Noting the extra twist in the expression (5.5) of $V_{k,\ell}$ in terms of $\mathrm{H}^{k-1}_{\text{ét}}(\mathscr{K}_{\overline{\mathbf{Q}}})$, this proves the first statement about E. This extension is unramified if and only if $v_p(a)$ is odd, in which case it is equal to $\mathbf{Q}_p(\sqrt{(-1)^{(1+ap)/2}2a'})$ and the last assertion in (2) follows. □

Remark 5.12

In the case at hand, instead of invoking [56] one can directly see the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on the primitive cohomology of the quadric by regarding it as defined over $\mathbf{Z}$. Indeed, D has good reduction at all primes r with even $v_r(d)$ and, for example, by point counting over $\mathbf{F}_r$, Frobenius acts as multiplication by $(\frac{(-1)^{(k+1)/2}d}{r})r^m$. Chebotarev's density theorem then implies that $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on $\mathrm{H}^{k-1}_{\text{ét,prim}}(D_{\overline{\mathbf{Q}}})(m)$ through the character corresponding to the extension $\mathbf{Q}(\sqrt{(-1)^{(1+ap)/2}2ap})$. Note that D has good reduction at p if $v_p(a)$ is odd.

From Theorem 5.8 and Serre's recipe in [57], we immediately derive the local L-factors and the conductor of the system of Galois representations $\{V_{k,\ell}\}_\ell$ associated with the motive M_k. For each prime p, define $L_k(p;T)$ as the reciprocal of the polynomial with integer coefficients

$$\det(1-F_pT \mid V_{k,\ell}^{I_p}) = \begin{cases} M_k(2;T) & \text{if } p=2, \\ M_k(p;T)\prod_{a\in\Theta_p^+}(1-(\frac{(-1)^{(1+ap)/2}2a'}{p})p^{m+1}T) & \text{if } p\geq 3. \end{cases} \tag{5.13}$$

The L-function of M_k is the Euler product

$$L_k(s) = \prod_p L_k(p;p^{-s}),$$

which converges absolutely on the half-plane $\mathrm{Re}(s) > 1 + (k+1)/2$.

Recall from [57, (11), (29)] that the exponent of p in the global conductor of $\{V_{k,\ell}\}_\ell$ is given by the sum of the Swan conductor of the restriction of $V_{k,\ell}$ to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and the codimension of $V_{k,\ell}^{I_p}$. Since $V_{k,\ell}$ is at most tamely ramified at all primes $p\neq\ell$, the Swan conductor vanishes and we are left with $(k-1)/2-\dim V_{k,\ell}^{I_p}$, which in view of the formulas (5.1), (5.2), and (5.7) is equal to $|\Theta_p^-|$ if p is odd and to zero if $p=2$. Thus, the value of the conductor is

$$\mathfrak{N}_k = \prod_{p\text{ odd}} p^{|\Theta_p^-|} = 1_{\mathrm{s}}3_{\mathrm{s}}5_{\mathrm{s}}\cdots k_{\mathrm{s}}, \tag{5.14}$$

where n_{s} denotes the product of all primes p such that $v_p(n)$ is odd.

Remark 5.15
It is clear from Theorem 5.8 that both the L-factor $L_k(p;T)$ and the conductor $\mathfrak{N}_k$ remain unchanged if one replaces the input $\{V_{k,\ell}\}_\ell$ with its semisimplification $\{V_{k,\ell}^{\mathrm{ss}}\}_\ell$.

5.1.4. The ℓ-adic case for even symmetric powers
Let $k\geq 2$ be an even integer, and let p be an odd prime number. In this case, the singular locus of $\mathscr{K}_{\overline{\mathbf{F}}_p}$ consists of $1+\lfloor k/2p\rfloor$ orbits of ordinary quadratic points under the action of $\mathfrak{S}_k\times\mu_2$. They are indexed by nonnegative even integers b satisfying $bp\leq k$, with points with coordinates $y_i=1$ (resp., -1) for $1\leq i\leq (bp+k)/2$ (resp., $i>(bp+k)/2$) as representatives. Writing $y_i=z_i+1$ (resp., $y_i=z_i-1$), locally around each singularity the defining equation of $\mathscr{K}$ in $\mathbf{Z}_p[\![z_1,\ldots,z_k]\!]$ has the shape

$$2bp+Q_{bp}+\text{higher order terms},\quad Q_{bp}=\sum_{i\leq(bp+k)/2} z_i^2-\sum_{i>(bp+k)/2} z_i^2. \tag{5.16}$$

THEOREM 5.17
Let k be a positive even integer, either of the form $2m+2$ or $2m+4$ with even m, and let p and ℓ be distinct prime numbers with $p \geq 3$. Let $V_{k,\ell}$ be the ℓ-adic realization of the motive M_k, which is an m-dimensional $\mathbf{Q}_\ell$-representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Fix a place of $\overline{\mathbf{Q}}$ above p, and let $I_p \subset \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the corresponding inertia subgroup.

Then I_p acts unipotently on $V_{k,\ell}$. More precisely, the equality $(\sigma-1)^2 = 0$ holds for each $\sigma \in I_p$ acting on $V_{k,\ell}$ and there exists an isotropic subspace $U \subset V_{k,\ell}$ of dimension $\lfloor k/2p \rfloor$, the image of the logarithm of the monodromy operator, generated by vanishing cycles and such that the equality $V_{k,\ell}^{I_p} = U^\perp$ holds and such that the induced map $\sigma - 1\colon V_{k,\ell} \to V_{k,\ell}/U$ is zero. Moreover, there is an isomorphism of $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$-modules

$$V_{k,\ell}^{I_p}[\zeta] = \mathrm{H}^1_{\text{ét},\mathrm{c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)/E[\zeta],$$

where E is the trivial representation $\mathbf{Q}_\ell(0)$ if 4 does not divide k and an extension of $\mathbf{Q}_\ell(-k/2)$ by $\mathbf{Q}_\ell(0)$ otherwise.

Proof
There is nothing to prove for $k=2$, so we assume that $k \geq 4$. Again, we write $G = \mathfrak{S}_k \times \mu_2$ and we omit the coefficients $\mathbf{Q}_\ell$ from the étale cohomology groups. Let us first recall from Theorem 1.8 that, in characteristic 0, the mixed Hodge structure $\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K})^{G,\chi}/W_0$ is pure of weight $k-1$ and has dimension $(k-2)/2$ if $k \equiv 2 \bmod 4$, whereas if $k \equiv 0 \bmod 4$, it is mixed of weights $k-2$ and $k-1$ with graded pieces of dimension 1 and $(k-4)/2$, respectively.

Let S be the singular locus of $\mathscr{K}$ in characteristic 0, which consists of $\binom{k}{k/2}$ ordinary quadratic points, and let $\mathscr{K}'$ be the strict transform of $\mathscr{K}$ inside the blowup of $\mathbb{G}_{\mathrm{m}}^k$ at S. The preimage of S in $\mathscr{K}'$ is a disjoint union of quadrics that we denote by T. Consider the commutative diagram

$$\begin{array}{ccc} T & \subset & \mathscr{K}' \\ \downarrow & & \downarrow \\ S & \subset & \mathscr{K} \end{array}$$

and the corresponding commutative diagram with exact rows

$$\begin{tikzcd} \mathrm{H}^{k-2}(T) \arrow[r] & \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}' \smallsetminus T) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}') \arrow[r] \arrow[d] & \mathrm{H}^{k-1}(T) \\ \mathrm{H}^{k-2}(S) \arrow[r] & \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K} \smallsetminus S) \arrow[r] & \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}) \arrow[r] & \mathrm{H}^{k-1}(S) \end{tikzcd}$$

Since k is even and at least 4, the vanishing $\mathrm{H}^{k-2}(S)=\mathrm{H}^{k-1}(S)=\mathrm{H}^{k-1}(T)=0$ holds, and hence we get an isomorphism

$$\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}') \xrightarrow{\sim} \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}). \tag{5.18}$$

The above isomorphism remains true if one replaces $\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K})$ and $\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}')$ with $\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}})$ and $\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{F}}})$, respectively, for $\mathbf{F}=\mathbf{Q}$ or $\mathbf{F}_p$.

Recall the compactification X of $\mathbb{G}_{\mathrm{m}}^k$. Let $\overline{\mathscr{K}}'$ be the closure of $\mathscr{K}'$ in the blowup of X along S, and let $\overline{\mathscr{K}}'^{(i)}$ be the disjoint union of all i-fold intersections of distinct irreducible components of the boundary divisor $\overline{\mathscr{K}}' \smallsetminus \mathscr{K}'$, with the usual convention $\overline{\mathscr{K}}'^{(0)}=\overline{\mathscr{K}}'$. Consider the associated spectral sequence

$$(E_1^{i,j})_{\mathbf{F}}=\mathrm{H}^j_{\text{ét}}(\overline{\mathscr{K}}'^{(i)}_{\overline{\mathbf{F}}}) \Longrightarrow \mathrm{H}^{i+j}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{F}}}) \quad (i,j\geq 0).$$

In characteristic 0, since all $\overline{\mathscr{K}}'^{(i)}_{\overline{\mathbf{Q}}}$ are smooth and proper, the spectral sequence degenerates at E_2 and one gets

$$\begin{aligned}\mathrm{gr}^W_{k-1}\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{Q}}}) &= \ker\bigl\{\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}}) \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'^{(1)}_{\overline{\mathbf{Q}}})\bigr\}\\ &= \mathrm{im}\bigl\{\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{Q}}}) \xrightarrow{\alpha} \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})\bigr\}\end{aligned}$$

exactly as in the case where k is odd. The equality

$$V_{k,\ell}(1)=\mathrm{im}(\alpha)^{G,\chi} \tag{5.19}$$

then follows by taking χ-isotypic components. Moreover, the $E_2^{1,k-2}$-term reads

$$\mathrm{gr}^W_{k-2}\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{Q}}})=\frac{\ker\{\mathrm{H}^{k-2}_{\text{ét}}(\overline{\mathscr{K}}'^{(1)}_{\overline{\mathbf{Q}}}) \longrightarrow \mathrm{H}^{k-2}_{\text{ét}}(\overline{\mathscr{K}}'^{(2)}_{\overline{\mathbf{Q}}})\}}{\mathrm{im}\{\mathrm{H}^{k-2}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}}) \longrightarrow \mathrm{H}^{k-2}_{\text{ét}}(\overline{\mathscr{K}}'^{(1)}_{\overline{\mathbf{Q}}})\}}. \tag{5.20}$$

By (5.18), the right-hand side is isomorphic to $\mathrm{gr}^W_{k-2}\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}})$, and hence its χ-isotypic component has dimension $\delta_{4\mathbf{Z}}(k)$ as recalled at the beginning of the proof.

Since the singularities of $\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p}$ consist only of ordinary quadratic points supported on $\mathscr{K}'_{\overline{\mathbf{F}}_p}$, the Picard–Lefschetz formula and base change yield isomorphisms

$$\begin{aligned}&\mathrm{H}^n_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p}) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}}) \quad \text{for } n\leq k-2,\\ &\mathrm{H}^j_{\text{ét}}(\overline{\mathscr{K}}'^{(i)}_{\overline{\mathbf{F}}_p}) \xrightarrow{\sim} \mathrm{H}^j_{\text{ét}}(\overline{\mathscr{K}}'^{(i)}_{\overline{\mathbf{Q}}}) \quad \text{for } i\geq 1.\end{aligned}$$

In particular, the equality $(E_2^{i,j})_{\mathbf{F}_p}=(E_2^{i,j})_{\mathbf{Q}}$ holds for all $i+j=k-1$ with $i\geq 1$, hence the degeneration $(E_2^{1,k-2})^{G,\chi}_{\mathbf{F}_p}=(E_\infty^{1,k-2})^{G,\chi}_{\mathbf{F}_p}$. By (5.20), this space vanishes for $k \equiv 2 \bmod 4$ and is 1-dimensional of weight $k-2$ if 4 divides k. Consider again the G-equivariant commutative diagram of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-representations

$$\begin{tikzcd}
& \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{F}}_p}) \arrow[r] \arrow[d, "\beta"'] & \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{Q}}}) \arrow[d, "\alpha"] & \\
0 \arrow[r] & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p}) \arrow[r] & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}}) \arrow[r, "\gamma"] & \bigoplus_{x\in\Sigma}\mathbf{Q}_\ell(-k/2)
\end{tikzcd}$$

in which the second row is exact and the map γ is defined by pairing with vanishing cycle classes $\delta_x \in \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})((k-2)/2)$, one for each $x \in \Sigma$. Setting

$$C = \gamma\bigl(\mathrm{im}(\alpha)^{G,\chi}\bigr)$$

and regarding $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p})$ as a subspace of $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})$, we obtain a diagram

$$\begin{tikzcd}
& \mathrm{im}(\beta)^{G,\chi} \arrow[d, hook] & & & \\
0 \arrow[r] & \mathrm{im}(\alpha)^{G,\chi} \cap \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p}) \arrow[r] & \mathrm{im}(\alpha)^{G,\chi} \arrow[r] & C \arrow[r] \arrow[d, hook] & 0 \\
& & & \bigl(\bigoplus_{x\in\Sigma}\mathbf{Q}_\ell(-k/2)\bigr)^{G,\chi} &
\end{tikzcd} \tag{5.21}$$

in which the row is exact and the vertical arrows are injective. We now show that both of these inclusions are in fact equalities. Taking the identity $\mathrm{im}\,\beta = (E_\infty^{0,k-1})_{\mathbf{F}_p}$ into account, the spectral sequence yields an exact sequence

$$0 \longrightarrow (E_\infty^{1,k-2})^{G,\chi}_{\mathbf{F}_p} \longrightarrow \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)(1)/W'_0 \longrightarrow \mathrm{im}(\beta)^{G,\chi}/W'_0 \longrightarrow 0$$

of unramified $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-modules (here W' denotes the weight filtration on étale cohomology over finite fields given by the eigenvalues of Frobenius, in order to distinguish it from that for $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-representations). The calculation (5.3) of the action of Frobenius on $\mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)$ implies that the rightmost term $\mathrm{im}(\beta)^{G,\chi}/W'_0$ has weights $k-2$ and $k-1$, with graded pieces of dimensions

$$\dim \mathrm{gr}^{W'}_{k-2}\,\mathrm{im}(\beta)^{G,\chi} = \Bigl\lfloor \frac{k}{2p} \Bigr\rfloor \qquad \text{and}$$

$$\dim \mathrm{gr}^{W'}_{k-1}\,\mathrm{im}(\beta)^{G,\chi} = \frac{k-2}{2} - 2\Bigl\lfloor \frac{k}{2p} \Bigr\rfloor - \delta_{4\mathbf{Z}}(k).$$

In particular, $\mathrm{im}(\beta)^{G,\chi}$ has dimension at least

$$\frac{k-2}{2} - \delta_{4\mathbf{Z}}(k) - \Big\lfloor \frac{k}{2p} \Big\rfloor = \dim \mathrm{im}(\alpha)^{G,\chi} - \Big\lfloor \frac{k}{2p} \Big\rfloor,$$

and weights $k-2, k-1$, and possibly zero. On the one hand, the unramified representation $(\bigoplus_{x\in\Sigma} \mathbf{Q}_\ell(-k/2))^{G,\chi}$ has dimension $|G\backslash\Sigma| = \lfloor k/2p \rfloor$ and Frobenius acts on it as multiplication by $p^{k/2}$. On the other hand, taking (5.19) into account, the space $\mathrm{im}(\alpha)^{G,\chi}$ is equipped with a nondegenerate $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-equivariant Poincaré pairing with values in the unramified Tate twist $\mathbf{Q}_\ell(1-k)$ of (Frobenius) weight $2k-2$. Since $\overline{\mathscr{K}}'$ is a projective variety, the equality $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p}) = W'_{k-1}\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p})$ holds, and hence the orthogonal complement C' of the subspace $W'_{k-2}\,\mathrm{im}(\beta)^{G,\chi}$ of $\mathrm{im}(\alpha)^{G,\chi}$ contains $\mathrm{im}(\alpha)^{G,\chi} \cap \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p})$. From this we derive the inequalities

$$\Big\lfloor \frac{k}{2p} \Big\rfloor \le \dim W'_{k-2}\,\mathrm{im}(\beta)^{G,\chi} = \dim \mathrm{im}(\alpha)^{G,\chi}/C' \le \dim C \le \Big\lfloor \frac{k}{2p} \Big\rfloor.$$

It follows that the right vertical inclusion is an equality, and hence the left one as well since the dimension of the intersection $\mathrm{im}(\alpha)^{G,\chi} \cap \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p})$ is then equal to $\dim \mathrm{im}(\alpha)^{G,\chi} - \lfloor k/2p \rfloor \le \mathrm{im}(\beta)^{G,\chi}$. In addition, let $\Delta \subset \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})$ be the subspace generated by the vanishing cycle classes $\mathbf{Q}_\ell((2-k)/2)\delta_x$ for $x \in \Sigma$. Since $\dim \Delta^{G,\chi} \le \lfloor k/2p \rfloor$ and $\Delta \subset \Delta^\perp = \ker\gamma$, we get

$$\Delta^{G,\chi} = W'_{k-2}\,\mathrm{im}(\beta)^{G,\chi} \qquad \text{and} \qquad (\Delta^\perp)^{G,\chi} = \mathrm{im}(\beta)^{G,\chi}.$$

By the Picard–Lefschetz formula in [11, Exposé XV, Théorème 3.4], an element σ of the inertia group I_p acts on $v \in \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})$ as

$$\sigma(v) = v - (-1)^{k/2} t_\ell(\sigma) \sum_{x\in\Sigma} \langle v, \delta_x \rangle \delta_x,$$

where $\langle v, \delta_x \rangle \in \mathrm{H}^{2k-2}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})((k-2)/2) \cong \mathbf{Q}_\ell(-k/2)$ and $t_\ell\colon I_p \to \varprojlim \mu_{\ell^n}(\overline{\mathbf{Q}}_\ell)$ is the fundamental tame character. From this we derive the vanishing $(\sigma-1)^2 = 0$ for each $\sigma \in I_p$ acting on $V_{k,\ell}$ and the equality $V_{k,\ell}^{I_p} = \mathrm{im}(\beta)^{G,\chi}(-1)$. Observe that we have proved that $V_{k,\ell}$ satisfies the *weight-monodromy conjecture*, that is, that the associated Weil–Deligne representation is pure of weight $k+1$, in the terminology of Corollary 5.39 below. (Conversely, that corollary can be used to show that the vertical arrows in (5.21) are equalities.) This completes the proof in the case $k \equiv 2 \bmod 4$.

Finally, we look at the action of Frobenius on the vanishing cycles Δ. Recall that each $V_x = \mathbf{Q}_\ell((2-k)/2)\delta_x$ corresponds to the singularity defined by the equation (5.16) for an even positive integer b. Consider the quadric $C = (Q_{bp}) \subset \mathbb{P}^{k-1}_{\mathbf{F}_p}$, whose primitive cohomology $\mathrm{H}^{k-2}_{\text{ét,prim}}(C_{\overline{\mathbf{F}}_p})$ coincides with V_x by [11, Exposé XV, Proposition 2.2.3]. The quadratic form Q_{bp} has discriminant $d = (-1)^{(k-bp)/2}$, and therefore F_p acts on the primitive cohomology as multiplication by

$$\Big(\frac{(-1)^{k/2}d}{p}\Big)p^{(k-2)/2} = (-1)^{bp(p-1)/4}p^{(k-2)/2}.$$

For $p \equiv 1 \bmod 4$, the sign is always positive, whereas, for $p \equiv 3 \bmod 4$, there are $\lfloor k/4p + 1/2 \rfloor$ values of b such that the sign is negative. Comparing with the eigenvalues of F_p in (5.3), one concludes that in the case $k \equiv 0 \bmod 4$, the kernel of the map $\beta\colon \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{F}}_p})^{G,\chi}/W_0 \to \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p})^{G,\chi}$ is a factor $\mathbf{Q}_\ell((2-k)/2)$. This completes the proof. $\square$

Similarly to the case of odd symmetric powers, the above theorem gives the local L-factors and the conductor away from $p = 2$. Indeed, defining

$$L_k(p;T) = \det\big(1 - F_pT \mid V_{k,\ell}^{I_p}\big)^{-1}$$

for a prime number p, Theorem 5.17 and (5.3) imply the equalities

$$L_k(p;T)^{-1} = \begin{cases} (1-p^{k/2}T)^{\lfloor k/2p \rfloor}M_k(p;T) & \text{if } p \equiv 1 \bmod 4, \\ (1+p^{k/2}T)^{\lfloor \frac{k}{4p}+\frac{1}{2} \rfloor}(1-p^{k/2}T)^{\lfloor \frac{k}{2p} \rfloor - \lfloor \frac{k}{4p}+\frac{1}{2} \rfloor}M_k(p;T) & \text{if } p \equiv 3 \bmod 4. \end{cases} \tag{5.22}$$

The L-function of M_k is the Euler product

$$L_k(s) = \prod_p L_k(p;p^{-s}),$$

which again converges absolutely for $\mathrm{Re}(s) > 1 + (k+1)/2$.

As for the conductor, Serre's recipe yields in this case that the exponent of an odd prime p is given by $\lfloor k/2p \rfloor$. The conductor is thus equal to

$$2^{r_k}\prod_{p \text{ odd}} p^{\lfloor k/2p \rfloor} = 2^{r_k}\cdot 2_{\mathrm{u}}4_{\mathrm{u}}6_{\mathrm{u}}\cdots k_{\mathrm{u}},$$

where $r_k = \mathrm{Sw}(V_{k,\ell}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_2/\mathbf{Q}_2)}) + \mathrm{codim}\,V_{k,\ell}^{I_2}$ and n_{u} stands for the odd part of the radical (i.e., the product of all odd primes dividing n). Broadhurst and Roberts conjecture that $r_k = \lfloor k/6 \rfloor$.

5.1.5. *The p-adic case*

We keep the setting of Section 3.2.2, and let $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{crys}}$, and $\mathbf{B}_{\mathrm{st}}$ denote Fontaine's p-adic de Rham, crystalline, and semistable period rings over $\mathbf{Q}_p$. Recall from (3.13) and (3.14) that, for any prime p, there is an isomorphism of Frobenius modules

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m}}/K, \mathrm{Sym}^k \mathrm{Kl}_2) = \mathrm{gr}^W_{k+1} \mathrm{H}^1_{\mathrm{rig,c}}(\mathbb{G}_{\mathrm{m}}/K, \mathrm{Sym}^k \mathrm{Kl}_2).$$

By [46, Theorem B], this K-vector space has dimension $\lfloor (k-1)/2 \rfloor - \delta_{4\mathbf{Z}}(k)$ for all $p > k$ if k is odd and for all $p > k/2$ if k is even.

PROPOSITION 5.23
Fix an integer $k \geq 1$, a prime number p, and a place of $\overline{\mathbf{Q}}$ above p. The p-adic representation $V_{k,p}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is de Rham. If p is odd, then $V_{k,p}$ is semistable over $\mathbf{Q}_p(\sqrt{-p})$ and there is an inclusion of Frobenius modules

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m}}/K, \mathrm{Sym}^k \mathrm{Kl}_2) \longrightarrow (V_{k,p} \otimes \mathbf{B}_{\mathrm{st}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p(\sqrt{-p}))} \otimes K. \tag{5.24}$$

Under the extra assumption that $p > k$ if k is odd (resp., $p > k/2$ if k is even), the representation $V_{k,p}$ is crystalline and there is an isomorphism of Frobenius modules

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m}}/K, \mathrm{Sym}^k \mathrm{Kl}_2) \cong (V_{k,p} \otimes \mathbf{B}_{\mathrm{crys}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)} \otimes K.$$

Proof
The p-adic representations $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(-1)$ and $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(-1)$ arising from the smooth proper varieties $\overline{\mathscr{K}}$ and $\overline{\mathscr{K}}'$ are de Rham (see, e.g., [3, Sections 3.3(i), 3.4]), and any subquotient of a de Rham representation is still de Rham. Hence, the first assertion follows from (5.9) for odd k, and from (5.19) along with (5.18) for even k.

For the remaining statements, we assume that p is odd. We first treat the case of even k. As in the proof of Theorem 5.17, consider the resolution of singularities $\mathscr{K}'$ of $\mathscr{K}$ and its compactification $\overline{\mathscr{K}}'$ induced from the blowup of the ambient torus and the explicit toric compactification X over $\mathbf{Z}_p$. Recall from Section 3.2.2 that the localization sequence for rigid cohomology with compact support yields

$$\big(\mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}_{\mathbf{F}_p}/\mathbf{Q}_p)^{\mathfrak{S}_k \times \mu_2, \chi}\big)(-1)[\varpi] \cong \mathrm{gr}^W_{k+1} \mathrm{H}^1_{\mathrm{rig,c}}(\mathbb{G}_{\mathrm{m}}/K, \mathrm{Sym}^k \mathrm{Kl}_2).$$

Besides, arguing as in (5.18), there are isomorphisms

$$\mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}'_{\mathbf{F}_p}/\mathbf{Q}_p) \xrightarrow{\sim} \mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}_{\mathbf{F}_p}/\mathbf{Q}_p),$$
$$\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) \xrightarrow{\sim} \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$$

of Frobenius modules and $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-modules, respectively.

For rigid cohomology, consider the spectral sequence

$$E_1^{i,j} = \mathrm{H}^j_{\mathrm{rig}}(\overline{\mathscr{K}}'^{(i)}_{\mathbf{F}_p}/\mathbf{Q}_p) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{rig,c}}(\mathscr{K}'_{\mathbf{F}_p}/\mathbf{Q}_p) \quad (i, j \geq 0),$$

as we did in the ℓ-adic setting (see [38, Propositions 8.2.17, 8.2.18(ii)]), and let

$$\alpha\colon \mathrm{H}^{k-1}_{\mathrm{rig}}(\overline{\mathscr{K}}'_{\mathbf{F}_p}/\mathbf{Q}_p) \longrightarrow \mathrm{H}^{k-1}_{\mathrm{rig}}(\overline{\mathscr{K}}'^{(1)}_{\mathbf{F}_p}/\mathbf{Q}_p)$$

denote the differential from $E_1^{0,k-1}$ to $E_1^{1,k-1}$. Since the varieties $\overline{\mathscr{K}}'^{(i)}$ are smooth and proper for all $i \geq 1$, the only contribution of weight $k-1$ to the abutment of the spectral sequence comes from the kernel of α, hence an isomorphism

$$\mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}'_{\mathbf{F}_p}/\mathbf{Q}_p) \xrightarrow{\sim} \mathrm{gr}^W_{k-1} \ker\alpha.$$

Let L be the ramified quadratic extension of $\mathbf{Q}_p$ contained in K, which is given by $L = \mathbf{Q}_p(\sqrt{-p})$ thanks to the equality $\sqrt{-p} = \varpi^{(p-1)/2}$. Since the singularities of $\overline{\mathscr{K}}'$ consist only of ordinary quadratic points supported on $\mathscr{K}'_{\mathbf{F}_p}$, the p-adic Picard–Lefschetz formula in [42, Theorem 1.1] yields a commutative diagram of L-modules

$$\begin{tikzcd}
\mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}'_{\mathbf{F}_p}/\mathbf{Q}_p)_L \arrow[r] & \mathrm{H}^{k-1}_{\mathrm{rig}}(\overline{\mathscr{K}}'_{\mathbf{F}_p}/\mathbf{Q}_p)_L \arrow[r, "\alpha_L"] \arrow[d, hook, "\beta"] & \mathrm{H}^{k-1}_{\mathrm{rig}}(\overline{\mathscr{K}}'^{(1)}_{\mathbf{F}_p}/\mathbf{Q}_p)_L \arrow[d, equal] \\
& \mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'_{\mathbf{Q}_p})_L \arrow[r] & \mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'^{(1)}_{\mathbf{Q}_p})_L
\end{tikzcd}$$

in which the map β is injective. Hence, β induces an inclusion

$$\ker\alpha_L \longrightarrow \ker\{\mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'_{\mathbf{Q}_p})_L \longrightarrow \mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'^{(1)}_{\mathbf{Q}_p})_L\}. \tag{5.25}$$

Besides, over the ring of integers $\mathbf{Z}_p[\sqrt{-p}]$ of L, with uniformizer $\sqrt{-p}$, each ordinary quadratic point of $\overline{\mathscr{K}}'$ is formally defined by an equation

$$Q - u\cdot(\sqrt{-p})^2,$$

where u is some unit and Q equals Q_{ap} from (5.6) for odd k and Q_{bp} from (5.16) for even k. In both cases, the equation $Q = 0$ defines a smooth quadric in $\mathbb{P}^{k-1}$ over $\mathbf{Z}_p[\sqrt{-p}]$, so that the assumptions of [42, (2.3)] are satisfied. Let $\overline{\mathscr{K}}''$ be the blowup of $\overline{\mathscr{K}}' \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\sqrt{-p}]$ along the ordinary quadratic points. Then $\overline{\mathscr{K}}''$ is semistable over $\mathbf{Z}_p[\sqrt{-p}]$ by [42, (2.3)], and hence any subquotient of $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}''_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ is a semistable $\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$-representation, for example, by [3, Section 3.3(iii)]. In particular, on noting the equalities $\mathscr{K}'_L = \mathscr{K}''_L$ and $\overline{\mathscr{K}}'^{(i)}_L = \overline{\mathscr{K}}''^{(i)}_L$, the representation $V_{k,p}(1)$ is semistable and the expression (5.19) yields

$$\begin{aligned}
&\bigl(\mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}''_{\overline{\mathbf{Q}}_p},\mathbf{Q}_p)\otimes\mathbf{B}_{\mathrm{st}}\bigr)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)}\otimes L\\
&\quad=\bigl(\mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{Q}}_p},\mathbf{Q}_p)\otimes\mathbf{B}_{\mathrm{dR}}\bigr)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)}\\
&\quad=\ker\bigl\{\mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'_{\mathbf{Q}_p})_L\longrightarrow\mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'^{(1)}_{\mathbf{Q}_p})_L\bigr\}
\end{aligned}\tag{5.26}$$

by the p-adic Hodge comparison theorem. In addition, since both the Frobenius structure on the L-vector space $\mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'_{\mathbf{Q}_p})_L=\mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}''_{L})$ and the map β are constructed by means of logarithmic de Rham-Witt complexes (see the proof of [42, Theorem 2.13]), β is compatible with the Frobenius action. The inclusion (5.24) of Frobenius modules follows by extending scalars to K and taking χ-isotypic components in (5.25) and (5.26).

If k is even and $p>k/2$, then $\overline{\mathscr{K}}'$ is smooth. Moreover, β is induced by the isomorphism

$$\mathrm{H}^{k-1}_{\mathrm{rig}}(\overline{\mathscr{K}}'_{\mathbf{F}_p}/\mathbf{Q}_p)\longrightarrow\mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'_{\mathbf{Q}_p})$$

and $\ker\alpha$ is pure of weight $k-1$. By [3, Section 3.3(iii)] and a similar argument as in the above case, we obtain the identity

$$(V_{k,p}\otimes\mathbf{B}_{\mathrm{crys}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}[\varpi]=\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m},\mathbf{F}_p}/K,\mathrm{Sym}^k\,\mathrm{Kl}_2),$$

thus finishing the proof for even k.

In case k is odd, there is no need to perform the first resolution of singularities, so we simply take $\mathscr{K}'=\mathscr{K}$ in what precedes and do the same proof as for even k. □

COROLLARY 5.27
Let $k\geq 1$ be an integer, and let p be an odd prime number. The Newton polygon of the Frobenius module $\mathrm{H}^1_{\mathrm{rig,c}}(\mathbb{G}_{\mathrm{m},\mathbf{F}_p}/K,\mathrm{Sym}^k\,\mathrm{Kl}_2)$ lies above the Hodge polygon of $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}},\mathrm{Sym}^k\,\mathrm{Kl}_2)$. In case $p>k$ if k is odd or $2p>k$ if k is even, the endpoints of both polygons coincide.

Proof
Considered as a representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p(\sqrt{-p}))$, the p-adic étale realization $V_{k,p}$ of M_k is semistable, and hence the associated filtered (φ,N)-module

$$\bigl(V_{k,p}\otimes\mathbf{B}_{\mathrm{st}}\bigr)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p(\sqrt{-p}))}$$

is (weakly) admissible. This means precisely that its Newton polygon lies above its Hodge polygon, both having the same endpoints. Notice that these polygons are additive with respect to the Minkowski sum (i.e., the sum of the convex sets above the polygons in the plane $\mathbf{R}^2$) for filtered Frobenius modules. Since the Frobenius module $\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m},\mathbf{F}_p}/K, \mathrm{Sym}^k \mathrm{Kl}_2)$ injects (after extending scalars to K) into the Frobenius module associated with $V_{k,p}$ by Proposition 5.23, the Newton polygon of the former lies above that of the latter, which in turns lies above the Hodge polygon of $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ by admissibility. Moreover, under the condition $p > k$ (resp., $p > k/2$) if k is odd (resp., even), the two Frobenius modules are equal by Proposition 5.23, and hence both polygons have the same endpoints. We conclude the proof by putting the trivial factor back, which is 1-dimensional with Frobenius and Hodge slopes 0 if $k \not\equiv 0 \bmod 4$ and 2-dimensional with Frobenius and Hodge slopes 0 and $k/2$ otherwise. $\square$

Remark 5.28
Writing $Z_k(p;T) = \sum c_n T^n$, this corollary implies in particular the inequality

$$v_p(c_n) \geq n(n-1) \tag{5.29}$$

for all $p \geq 3$. This sharpens a theorem of Haessig [25, Theorem 1.1], who obtained the lower bound $(1 - 1/(p-1))n(n-1)$ for all $p \geq 5$ using p-adic analysis à la Dwork. While our article was being refereed, he managed to prove (5.29) for all $p \geq 2$ by strengthening his previous arguments in [26]. As we explain in Remark 5.41 below, it is also possible to obtain this lower bound in all cases except for $p = 2$ and even k from the potential automorphy of the motive M_k. Note that, when k is even, the Hodge polygon of $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ lies strictly above the polygon with vertices $n(n-1)$, as Figure 1 shows.

5.2. *The gamma factor*

We first recall Serre's recipe in [57, Section 3] describing the conjectural shape of the gamma factor at infinity in the complete L-function of a pure motive over $\mathbf{Q}$. Let V be a finite-dimensional vector space over $\mathbf{C}$ together with an $\mathbf{R}$-Hodge decomposition of weight w, that is, the data of a grading $V = \bigoplus_{p \in \mathbf{Z}} V^p$ and a $\mathbf{C}$-linear involution σ of V such that $\sigma(V^p) = V^{w-p}$ holds. Given an $\mathbf{R}$-Hodge decomposition, we set $h(p) = \dim_{\mathbf{C}} V^p$ and

$$h(w/2)^{\pm} = \dim_{\mathbf{C}} \{ v \in V^{w/2} \mid \sigma(v) = \pm(-1)^{w/2} v \}$$

if w is even and $h(w/2)^{\pm} = 0$ otherwise. Setting

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1),$$

the *gamma factor* $\Gamma_V(s)$ of V is defined as

$$\Gamma_V(s) = \Gamma_{\mathbf{R}}(s - w/2)^{h(w/2)^+} \Gamma_{\mathbf{R}}(s - w/2 + 1)^{h(w/2)^-} \prod_{p < w/2} \Gamma_{\mathbf{C}}(s - p)^{h(p)}.$$

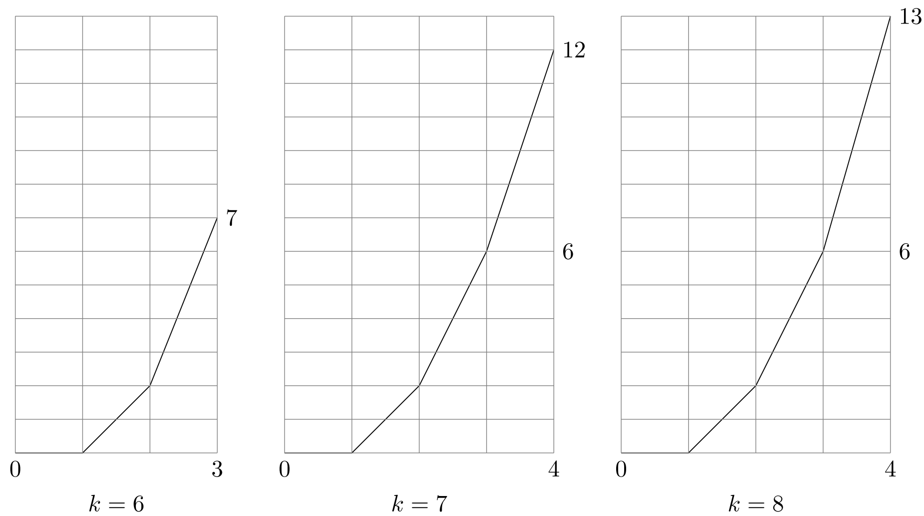

Figure 1. The Hodge polygons of $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$.

COROLLARY 5.30
For each integer $k \geq 1$, the gamma factor of the motive M_k is equal to

$$L_k(\infty, s) = \pi^{-ms/2} \prod_{j=1}^{m} \Gamma\Big(\frac{s-j}{2}\Big), \qquad m = \Big\lfloor \frac{k-1}{2} \Big\rfloor - \delta_{4\mathbf{Z}}(k).$$

Proof
In our geometric setting, the grading is given by

$$V^p = \mathrm{gr}^p_F \, \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$$

and the $\mathbf{R}$-structure comes from the maps σ induced by complex conjugation $\mathscr{K}(\mathbf{C}) \to \mathscr{K}(\mathbf{C})$ on the singular cohomology $\mathrm{H}^{k-1}(\mathscr{K}(\mathbf{C}))$ and the singular cohomology with compact support $\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}(\mathbf{C}))$ (see [57, Section 3.3(b)]). These form an $\mathbf{R}$-Hodge decomposition of weight $w = k+1$.

Observe that the middle degree factor $V^{w/2}$ is nontrivial if and only if $k = 4r+3$ for some integer $r \geq 0$, in which case the weight is $w = 4r+4$ and $V^{w/2}$ has dimension 1. Assuming this, let $\varepsilon \in \{\pm 1\}$ denote the sign of the action of σ on $V^{w/2}$. Since V has dimension $2r+1$ and σ interchanges V^p and V^{w-p}, the equality $\det \sigma = (-1)^r \varepsilon$ holds in $\det \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. Therefore, it suffices to compute $\det \sigma$. Thanks to the orthogonal pairing (3.4), the above determinant is, up to a twist, the de Rham realization of the rank-1 Artin motive associated with a quadratic field extension of $\mathbf{Q}$ and one only needs to decide whether this field is real or imaginary.

To do so, we look at the ℓ-adic representation $r_{k,\ell}\colon \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \operatorname{GL}(V_{k,\ell})$. For each odd prime p, the determinant of Frobenius was computed in [23, Theorem 0.1]:

$$\begin{aligned}&\det\bigl(F_p \bigm| \mathrm{H}^1_{\text{ét},\mathrm{mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \operatorname{Sym}^k \mathrm{Kl}_2)\bigr)\\ &\qquad = p^{(k+1)\dim \mathrm{H}^1_{\mathrm{mid}}/2}\Bigl(\frac{2}{p}\Bigr)^{\lfloor k/2p+1/2\rfloor} \prod_{\substack{0\le j\le (k-1)/2\\ p\nmid 2j+1}} \Bigl(\frac{(-1)^j(2j+1)}{p}\Bigr).\end{aligned}$$

From this we immediately derive that, for all primes $p > k$, the equality

$$\det\bigl(r_{k,\ell}(\mathrm{Frob}_p)\bigr) = \Bigl(\frac{(-3)\cdot 5\cdots(-1)^{(k-1)/2}k}{p}\Bigr)p^{(k^2-1)/4} = \Bigl(\frac{p}{k!!}\Bigr)p^{(k^2-1)/4}$$

holds, with $k!! = 3\cdot 5\cdots k$. Chebotarev's density theorem then yields the formula $\det r_{k,\ell} = (\cdot/k!!)\,\chi_{\mathrm{cyc}}^{(1-k^2)/4}$. It follows that the quadratic number field which this character gives rise to through class field theory is equal to $\mathbf{Q}(\sqrt{\pm k!!})$, with the sign adjusted by the condition that the radicand is congruent to 1 modulo 4 (otherwise, 2 would be a ramified prime). Noting that $k = 4r+3$, this sign is given by $(-1)^{r+1}$ and the power of the cyclotomic character appearing in $\det r_{k,\ell}$ is even. Putting everything together, one derives $\varepsilon = -1 = -(-1)^{w/2}$, and hence the missing information $h(w/2)^+ = 0$ and $h(w/2)^- = 1$ to compute the gamma factor. $\square$

5.3. Potential automorphy, meromorphic continuation, and functional equation

In this final subsection, we pull everything together to prove Theorems 1.2 and 1.3 from the introduction. We first compute the ε-factors of the Galois representations $V_{k,\ell}$ and recall the particular case of the theorem of Patrikis and Taylor that will imply potential automorphy.

5.3.1. Weil–Deligne representations and ε-factors

For each integer $k \ge 1$, consider the system $\{V_{k,\ell}\}_\ell$ of ℓ-adic realizations of the motive M_k. We investigate its global ε-factor $\varepsilon_k(s)$ by means of the information obtained in Theorems 5.8 and 5.17. We refer the reader to [60] for an accessible introduction to Weil–Deligne representations.

As inputs for defining the local ε-factor of $\{V_{k,\ell}\}$ at each place p of $\mathbf{Q}$, we fix the additive character ψ and the Haar measure $\mathrm{d}x$ on $\mathbf{Q}_p$ as follows. If $p < \infty$, then ψ is the composition

$$\mathbf{Q}_p \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p = \mathbf{Z}[1/p]/\mathbf{Z} \longrightarrow \mathbf{C}^\times,$$

where the first map is the quotient and the second map sends α to $\exp(2\pi\mathrm{i}\alpha)$. The Haar measure $\mathrm{d}x$ is normalized so that $\int_{\mathbf{Z}_p}\mathrm{d}x = 1$ holds; note that it is self-dual with respect to ψ. For $p=\infty$, we set $\psi(\alpha)=\exp(-2\pi\mathrm{i}\alpha)$ for $\alpha\in\mathbf{R}$, and we take as $\mathrm{d}x$ the usual Lebesgue measure. Letting $\mathbf{A}_{\mathbf{Q}}$ denote the adele ring of $\mathbf{Q}$, these local characters and Haar measures are compatible in the sense that the product of the ψ's induces a character of $\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$ and the compact quotient $\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$ has volume 1 with respect to the induced measure (see [7, Section 3.10]).

For each $p<\infty$, let $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be the Weil group of $\mathbf{Q}_p$, that is, the subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ consisting of those elements whose image in $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ is an integral power of Frobenius together with the topology making I_p with its usual topology into an open subgroup, and let $F_p\in W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be a lifting of the geometric Frobenius. Local class field theory provides an isomorphism between $\mathbf{Q}_p^\times$ and the maximal abelian quotient $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)^{\mathrm{ab}}$; following the convention of [7, Section 2.3], we normalize it so that p is mapped to F_p. For $s\in\mathbf{C}$, let

$$\omega_s\colon W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)\longrightarrow\mathbf{C}^\times$$

be the homomorphism defined by the composition of the quotient map to $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)^{\mathrm{ab}}\cong\mathbf{Q}_p^\times$ with the map from $\mathbf{Q}_p^\times$ to $\mathbf{C}^\times$ sending α to $\|\alpha\|^s$, with the normalization $\|p\|=1/p$.

With a continuous representation ρ of $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on a discrete topological vector space V over a field of characteristic 0, to which we shall refer as a *Weil representation*, is associated a local ε-factor $\varepsilon_0(\rho,s)=\varepsilon_0(\rho\cdot\omega_s,0)$, depending on ψ and $\mathrm{d}x$, in [7, Théorème 4.1]. By (5.5.2) there, the equality

$$\varepsilon_0(\rho,s)=\omega_s(p^{a(\rho)})\cdot\varepsilon_0(\rho,0)=p^{-a(\rho)s}\cdot\varepsilon_0(\rho,0) \tag{5.31}$$

holds, where $a(\rho)$ denotes the conductor of ρ and we regard ω_s as a map

$$\mathbf{Q}_p^\times\cong W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)^{\mathrm{ab}}\longrightarrow\mathbf{C}^\times.$$

A *Weil–Deligne representation* (ρ,N) on V consists of a Weil representation ρ on V as above and a nilpotent endomorphism N, called the *logarithm of the unipotent part of the local monodromy*, such that the equality $\rho(w)\mathrm{N}\rho(w)^{-1}=p^{-v(w)}\mathrm{N}$ holds for all $w\in W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, where $v(w)$ denotes the power of F_p to which w is mapped in $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. There is a canonical way to attach a Weil–Deligne representation to an ℓ-adic representation r of $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$: by Grothendieck's quasiunipotency theorem, there exists a unique nilpotent endomorphism N satisfying $r(\sigma)=\exp(t_\ell(\sigma)\mathrm{N})$ for all σ in a finite index subgroup of I_p, and one sets

$$\rho(\sigma F_p^n)=r(\sigma F_p^n)\exp\bigl(-t_\ell(\sigma)\mathrm{N}\bigr) \tag{5.32}$$

for all $\sigma\in I_p$ and all $n\in\mathbf{Z}$ (see [7, Section 8.4]). Setting

$$\varepsilon_1\bigl((\rho,\mathrm{N}),s\bigr)=\det\bigl(-p^{-s}F_p\bigm| V^{\rho(I_p)}/\ker(\mathrm{N})^{\rho(I_p)}\bigr),$$

the local ε-factor of the Weil–Deligne representation (ρ, N) defined in [9, above Remarque 5.2.1] is equal to the product

$$\varepsilon\big((\rho, \mathrm{N}), s\big) = \varepsilon_0(\rho, s) \cdot \varepsilon_1\big((\rho, \mathrm{N}), s\big). \tag{5.33}$$

Notice the equality $V^{r(I_p)} = \ker(\mathrm{N})^{\rho(I_p)}$ from [7, Section 8.12].

Let ℓ be a prime number distinct from p. For $s \in \mathbf{Z}$, we also regard ω_s as a homomorphism to $\mathbf{Q}_\ell^\times$. We consider the Weil–Deligne representation (ρ, N) on $V_{k,\ell}$ corresponding to the ℓ-adic representation $V_{k,\ell}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and denote its ε-factor by

$$\varepsilon_k(p, s) = \varepsilon((\rho, \mathrm{N}), s).$$

Suppose that k is odd. For $2 < p < \infty$, the representation $V_{k,\ell}$ of the inertia group I_p is tame and factors through characters of subgroups of index at most 2 by Theorem 5.8. The associated Weil–Deligne representation (ρ, N) has thus $\mathrm{N} = 0$ and ρ equals the restriction of $V_{k,\ell}$ to $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, so that the equality $\varepsilon_k(p, s) = \varepsilon_0(\rho, s)$ holds in this case. By definition (see [7, (4.5.4)]), the conductor of ρ is given by

$$a(\rho) = \dim V_{k,\ell} - \dim V_{k,\ell}^{\rho(I_p)} = |\Theta_p^-|$$

and from the *formulaire* in [7, (4.5.4)], we find

$$\begin{aligned}
1 &= \varepsilon_0(\rho, 0) \cdot \varepsilon_0(\rho^\vee \cdot \omega_1, 0) \cdot \det(\rho)(-1) \quad \text{by [7, (5.4), (5.7.1)]} \\
&= \varepsilon_0(\rho, 0) \cdot \varepsilon_0(\rho \cdot \omega_{k+2}, 0) \cdot \det(\rho)(-1) \quad \text{since } V_{k,\ell}^\vee = V_{k,\ell}(k+1) \\
&= \varepsilon_0(\rho, 0)^2 \cdot (p^{|\Theta_p^-|})^{-(k+2)} \cdot \det(\rho)(-1) \quad \text{by (5.31).}
\end{aligned} \tag{5.34}$$

Recall from the proof of Corollary 5.30 that $\det(\rho)$ is the nontrivial character associated with the quadratic extension $\mathbf{Q}_p(\sqrt{\pm k!!})$ with positive sign if $k \equiv 1, 7 \bmod 8$ and negative sign otherwise. Therefore, $\det(\rho)(-1)$ is given by the Hilbert symbol $(-1, \pm k!!)$ and there exists a fourth root of unity $w_p \in \mu_4(\mathbf{C})$ with $w_p^2 = (-1, \pm k!!)$ such that

$$\varepsilon_k(p, s) = w_p \cdot (p^{|\Theta_p^-|})^{(k+2)/2-s}.$$

Moreover, if $V_{k,\ell}$ is unramified, then the equality $\varepsilon_k(p, s) = 1$ holds (recall that this includes the case $p = 2$). According to [9, Section 5.3], at $p = \infty$ the associated ε-factor $\varepsilon_k(\infty, s)$ is given, in the notation of Section 5.2 above, by $\mathsf{i} = \sqrt{-1}$ raised to the power

$$\sum_{p<q} (q - p + 1)h(p) + h\big((k+1)/2\big)^- = \frac{k^2 - 1}{8}.$$

Now the product formula for Hilbert symbols implies that $\varepsilon_k(\infty, s) \cdot \prod_{p<\infty} w_p$ belongs to $\{\pm 1\}$, and putting everything together we get

$$\varepsilon_k(s) = \prod_{p \leq \infty} \varepsilon_k(p, s) = \pm \mathfrak{N}_k^{(k+2)/2-s},$$

where $\mathfrak{N}_k$ is the integer defined in (5.14).

Remark 5.35
It is obvious that in this case the ε-factors remain unchanged if one replaces the input $\{V_{k,\ell}\}_\ell$ with its semisimplification $\{V_{k,\ell}^{\mathrm{ss}}\}_\ell$.

We now suppose that k is even and keep notation from Theorem 5.17. For each $2 < p < \infty$, there exist a basis $\{e_i\}$ of U and elements $\{e_i'\}$ inducing a basis of $V_{k,\ell}/U^\perp$ satisfying $\ker(\mathrm{N}) = U^\perp$ and $\mathrm{N}(e_i') = -(-1)^{k/2} e_i$. In this case, the inertia group acts trivially on the Weil–Deligne representation ρ given by (5.32), hence the equalities $a(\rho) = 0$ and $\varepsilon_0(\rho, s) = 1$. From the identity $V_{k,\ell}^\vee = V_{k,\ell}(k+1)$ as representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, we derive $\det(\rho(F_p), V_{k,\ell})^2 = p^{m(k+1)}$, and since the duality pairing is symplectic the determinant has positive sign, so that $\det(\rho(F_p), V_{k,\ell}) = p^{m(k+1)/2}$ (note that $m = \dim V_{k,\ell}$ is even). Using the definition in (5.33) and Theorem 5.17, we obtain

$$\varepsilon_k(p, s) = (-1)^{v_p} \cdot p^{\lfloor k/2p \rfloor((k+2)/2-s)}, \qquad v_p = \begin{cases} \lfloor \frac{k}{2p} \rfloor & \text{if } p \equiv 1 \bmod 4, \\ \lfloor \frac{k}{4p} \rfloor & \text{if } p \equiv 3 \bmod 4. \end{cases} \tag{5.36}$$

Besides, the computation of Hodge numbers yields

$$\varepsilon_k(\infty, s) = \left.\begin{cases} 1 & \text{if } k \equiv 2 \bmod 4, \\ (-1)^{(k-4)/4} & \text{if } k \equiv 0 \bmod 4 \end{cases}\right\} = (-1)^{\delta_{8\mathbf{Z}}(k)},$$

from which we get the value of the global epsilon factor away from $p = 2$:

$$\prod_{2<p\leq\infty} \varepsilon_k(p, s) = (-1)^{v'} \mathfrak{N}_k'^{\,(k+2)/2-s},$$

$$v' = \sum_{p\equiv 1(4)} \left\lfloor \frac{k}{2p} \right\rfloor + \sum_{p \equiv 3(4)} \left\lfloor \frac{k}{4p} \right\rfloor + \delta_{8\mathbf{Z}}(k).$$

Remark 5.37
The factor $\varepsilon_k(s)$ is of the form AB^{-s} since all its local factors are. Suppose that there is a functional equation

$$\widehat{L}_k(s) = \varepsilon_k(s) \cdot \widehat{L}_k(k+2-s), \qquad \widehat{L}_k(s) = L_k(\infty, s) \cdot \prod_{p<\infty} L_k(p, p^{-s}).$$

By applying it twice, we get $A^2 = B^{k+2}$. Suppose that k is even and $p = 2$, and let $a = a(\rho)$ be the conductor of the associated Weil–Deligne representation. The same computation as in (5.34) gives $|\varepsilon_0(\rho,0)| = 2^{a(k+2)/2}$. On the other hand, suppose that the quotient $V^{\rho(I_2)}/\ker(\mathrm{N})^{\rho(I_2)}$ has dimension r and that $\det(F_2)$ acts as δ. Plugging this into (5.33), we obtain

$$\varepsilon_k(2,s) = w''|\delta|2^{a(k+2)/2}2^{-(a+r)s}$$

for some $|w''| = 1$. Under the assumption that the functional equation holds with (5.36) we get $|\delta| = 2^{r(k+2)/2}$, and hence the equality

$$\varepsilon_k(s) = w \cdot (2^{a+r}\mathfrak{N}'_k)^{(k+2)/2-s}, \qquad w = (-1)^{v'}w''.$$

Based on the numerical data, the equality $a + r = \lfloor k/6 \rfloor$, which is also the exponent of 3 in $\mathfrak{N}'_k$, is conjectured in [5]. It is further conjectured that $w'' = (-1)^{v''}$ with $v'' = \lfloor k/8 \rfloor$. One possible structure fitting these data would be that $V_{k,\ell}$ is tamely ramified at 2, that the inertia group acts trivially on the associated Weil–Deligne representation, and that the reciprocal characteristic polynomial of Frobenius is

$$\det(1 - F_2 T \mid V_{k,\ell}^{I_2}) = (1 - 2^{k/2}T)^{\lfloor k/8 \rfloor}(1 + 2^{k/2}T)^{b_k} M_k(2;T)$$

of degree $(k-2)/2 - \delta_{4\mathbf{Z}}(k) - \lfloor k/6 \rfloor$, where b_k and $M_k(2;T)$ are defined in (5.4).

5.3.2. The theorem of Patrikis and Taylor

Let $m \geq 1$ be an integer, and let S be a finite set of prime numbers. We consider a *weakly compatible* system of continuous semisimple representations

$$r_\ell \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_m(\overline{\mathbf{Q}}_\ell)$$

with ℓ running over all prime numbers. The notion of being "weakly compatible" is borrowed from [2, Section 5.1] and means that the following three conditions hold:

- if $p \notin S$, then for all $\ell \neq p$ the representation r_ℓ is unramified at p and the characteristic polynomial of $r_\ell(\mathrm{Frob}_p)$ lies in $\mathbf{Q}[T]$ and is independent of ℓ;
- each representation r_ℓ is de Rham and in fact crystalline if $\ell \notin S$;
- the Hodge–Tate weights of r_ℓ are independent of ℓ.

THEOREM 5.38 (Patrikis and Taylor [44, Theorem A])

Let $\mathscr{R} = \{r_\ell\}$ be a weakly compatible system that satisfies the following properties:

- *(Purity) There exists an integer w such that, for each prime $p \notin S$, the roots of the common characteristic polynomial of $r_\ell(\mathrm{Frob}_p)$ are Weil numbers of weight w.*
- *(Regularity) The representation r_ℓ has m distinct Hodge–Tate weights.*

- *(Odd essential self-duality) Either each r_ℓ factors through a map to* $\mathrm{GO}_m(\overline{\mathbf{Q}}_\ell)$ *with even similitude character or each r_ℓ factors through a map to* $\mathrm{GSp}_m(\overline{\mathbf{Q}}_\ell)$ *with odd similitude character. Moreover, these characters form a weakly compatible system.*

Then there exists a finite, Galois, totally real number field over which all of the r_ℓ become automorphic. In particular, the partial L-function

$$L^S(\mathscr{R}, s) = \prod_{p \notin S} \det\bigl(1 - r_\ell(\mathrm{Frob}_p)\, p^{-s}\bigr)^{-1}$$

admits a meromorphic continuation to the complex plane.

Let p and ℓ be distinct prime numbers, and let (ρ, N) be the Weil–Deligne representation on $V \simeq \overline{\mathbf{Q}}_\ell^m$ associated with r_ℓ. There is a unique increasing *monodromy filtration* $V^{\leq \bullet} \subset V$ attached to the nilpotent endomorphism N such that, for each integer a, there is an inclusion $\mathrm{N}V^{\leq a} \subset V^{\leq a-2}$ and such that, for each $a \geq 0$, the map $V^{\leq a}/V^{\leq a-1} \to V^{\leq -a}/V^{\leq -a-1}$ induced by N^a is an isomorphism. Recall that (ρ, N) is called *pure* of weight w if the eigenvalues of Frobenius F_p acting on $V^{\leq a}/V^{\leq a-1}$ are p-Weil numbers of weight $w + a$ for all a. Building on vast work on constructions of Galois representations attached to automorphic representations, which is partly summarized in [2, Theorem 2.1.1], Theorem 5.38 implies the following.

COROLLARY 5.39 ([44, Corollary 2.2(ii)])
Let $\mathscr{R} = \{r_\ell\}$ be a weakly compatible system that is pure of weight w, regular, and odd essential self-dual. Then, for any distinct primes p and ℓ, the Weil–Deligne representation $\mathrm{WD}_p(\mathscr{R})$ *of* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *associated with r_ℓ is pure of weight w. Moreover, with the notation of [2, Section 5.1], the completed L-function*

$$\Lambda(\mathscr{R}, s) = L_\infty(\mathscr{R}, s) \cdot \prod_{p \in S} L\bigl(\mathrm{WD}_p(\mathscr{R}), s\bigr) \cdot L^S(\mathscr{R}, s)$$

satisfies the functional equation $\Lambda(\mathscr{R}, s) = \varepsilon(\mathscr{R}, s)\Lambda(\mathscr{R}^\vee, 1 - s)$.

5.3.3. Proofs of Theorems 1.2 and 1.3
For each integer $k \geq 1$, the ℓ-adic representations

$$r_{k,\ell} \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}(V_{k,\ell} \otimes \overline{\mathbf{Q}}_\ell) \simeq \mathrm{GL}_m(\overline{\mathbf{Q}}_\ell)$$

are pure of weight $k + 1$. After choosing an embedding $\mathbf{Q}_p \hookrightarrow \mathbf{C}$, we get a filtered isomorphism

$$\begin{aligned}
&(V_{k,p} \otimes \mathbf{B}_{\mathrm{dR}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)} \otimes \mathbf{C} \\
&\quad = \big(\mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)^{\mathfrak{S}_k \times \mu_2, \chi} \otimes \mathbf{B}_{\mathrm{dR}}\big)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}(-1) \otimes \mathbf{C} \\
&\quad = \mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_{\mathrm{dR,c}}(\mathscr{K}_{\mathbf{Q}_p})^{\mathfrak{S}_k \times \mu_2, \chi}(-1) \otimes \mathbf{C} \\
&\quad \cong \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)
\end{aligned}$$

by the p-adic comparison theorem. By definition, the Hodge–Tate weights of $V_{k,p}$ are those integers a such that the graded piece $\mathrm{gr}^a_F \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ is nonzero, counted with multiplicity its dimension. As the Hodge numbers are either zero or one by Theorem 1.8, the system $\{r_{k,\ell}\}$ is regular. Besides, the existence of the $(-1)^{k+1}$-symmetric perfect pairing (3.4) implies that the $r_{k,\ell}$ factor through $\mathrm{GO}_m(\overline{\mathbf{Q}}_\ell)$ (resp., $\mathrm{GSp}_m(\overline{\mathbf{Q}}_\ell)$) if k is odd (resp., even) with similitude character $\chi_{\mathrm{cyc}}^{-k-1}$. Choose a basis of $\mathbf{Q}_\ell(-k-1)$ and regard the perfect pairing $V_{k,\ell} \times V_{k,\ell} \to \mathbf{Q}_\ell(-k-1)$ as a compatible nondegenerate bilinear form on the module $V_{k,\ell}$ over the group ring $\mathbf{Q}_\ell[\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})]$ with the involution $g \mapsto \chi_{\mathrm{cyc}}^{-k-1}(g)g^{-1}$. By [58, Theorem 4.2.1], the semisimplification $r^{\mathrm{ss}}_{k,\ell}$ also factors through $\mathrm{GO}_m(\overline{\mathbf{Q}}_\ell)$ (resp., $\mathrm{GSp}_m(\overline{\mathbf{Q}}_\ell)$) with similitude character $\chi_{\mathrm{cyc}}^{-k-1}$. Moreover, $r^{\mathrm{ss}}_{k,\ell}$ is de Rham at all primes ℓ and crystalline if $\ell > k$ by Proposition 5.23. Therefore, the $r^{\mathrm{ss}}_{k,\ell}$ form a weakly compatible system satisfying the assumptions of the theorem of Patrikis and Taylor, and the partial L-function of $\{r^{\mathrm{ss}}_{k,\ell}\}$ has meromorphic continuation and satisfies the expected automorphic functional equation.

We now show that the L-function and the ε-factor of $\{r^{\mathrm{ss}}_{k,\ell}\}$ coincide with those of $\{r_{k,\ell}\}$. For odd k, this was the content of Remarks 5.15 and 5.35. For even k, we rely on the following lemma, which is certainly well known to experts. (We thank one of the referees for suggesting the statement and sketching a proof, which we include for lack of an appropriate reference.)

LEMMA 5.40

Let p and ℓ be distinct prime numbers, and let $r\colon \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{GL}(V)$ be an ℓ-adic representation. Suppose that there exists a sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_c = V$$

of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-stable subspaces such that the Weil–Deligne representation associated with the induced representation $\bar{r}$ on $\overline{V} = \bigoplus_{i=1}^c V_i/V_{i-1}$ is pure. Then the Weil–Deligne representation associated with r is pure as well, and r and $\bar{r}$ have the same L- and ε-factors.

Proof

Let (ρ, N) and $(\bar{\rho}, \overline{\mathrm{N}})$ be the Weil–Deligne representations associated with r and $\bar{r}$.

By its defining properties (see [7, Théorème 4.1(1)]), the factor $\varepsilon_0(\rho, s)$ in (5.33) depends only on the semisimplification of r. Set $\varphi = \rho(F_p)$. It follows from the relation $\mathrm{N}\varphi = p\varphi\mathrm{N}$ that if α is an eigenvalue of φ acting on $V^{\leq a}/V^{\leq a-1}$ with multiplicity μ, then αp^{-b} is an eigenvalue of φ acting on $V^{\leq a-2b}/V^{\leq a-2b-1}$ with multiplicity at least μ for all $0 \leq b \leq a$. Besides, by the uniqueness of the monodromy of ℓ-adic representations in Grothendieck's quasiunipotency theorem, N restricts to the logarithm on each V_i, and hence induces the monodromy $\overline{\mathrm{N}}$ on $\overline{V}$. Since $(\bar{\rho}, \overline{\mathrm{N}})$ is pure and $\det(1-\varphi T) = \det(1-\bar{\varphi}T)$, we conclude that the monodromy filtrations on (ρ, N) and $(\bar{\rho}, \overline{\mathrm{N}})$ have the same dimension on each graded piece and that (ρ, N) is pure as well. In addition, the two Weil representations $\rho|_{\ker(\mathrm{N})}$ and $\bar{\rho}|_{\ker(\overline{\mathrm{N}})}$ have the same semisimplification. Finally, observe that taking I_p-invariants is exact on Weil representations since I_p acts through finite quotients, hence the identities

$$\det\bigl(1-\varphi T \mid \ker(\mathrm{N})^{\rho(I_p)}\bigr) = \det\bigl(1-\bar{\varphi}T \mid \ker(\overline{\mathrm{N}})^{\bar{\rho}(I_p)}\bigr),$$
$$\det\bigl(\varphi T \mid V^{\rho(I_p)}/\ker(\mathrm{N})^{\rho(I_p)}\bigr) = \det\bigl(\bar{\varphi}T \mid \overline{V}^{\bar{\rho}(I_p)}/\ker(\overline{\mathrm{N}})^{\bar{\rho}(I_p)}\bigr),$$

from which the equality of the L- and the ε-factors follows. □

The discussion of Section 5.3.1 above then implies that this functional equation is, up to sign, precisely the one from Theorems 1.2 and 1.3. To conclude, we need to show that the sign is always positive for odd k; for this we use Saito's result in [55] that the sign of the functional equation of the L-function of an orthogonal motive of even weight is always positive.

Remark 5.41

As explained in [44, Corollary 2.2(ii)], another consequence of the potential automorphy of the weakly compatible system $\{r_{k,\ell}\}_\ell$ is that it is indeed *strictly* compatible (see [44, p. 214] for this notion). Given a prime p, this allows one to transfer some properties of the ℓ-adic representations $r_{k,\ell}$ for $\ell \neq p$, to the p-adic representation $r_{k,p}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, since their associated Weil–Deligne representations $\rho_{k,\ell}$ and $\rho_{k,p}$ are isomorphic up to semisimplification. (See, e.g., [60, Section 1] for the construction of the Weil–Deligne representation associated with a p-adic de Rham representation.) In particular, the representation $r_{k,p}$ is semistable over $L = \mathbf{Q}_p$ if k is even and $L = \mathbf{Q}_p(\sqrt{-p})$ if k is odd since $\rho_{k,\ell}(I_L) = \{1\}$, where $I_L \subset \mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$ denotes the inertia group (see Section 5.3.1). Moreover, if k is odd, then $\mathrm{N} = 0$, and hence $r_{k,p}$ is indeed crystalline over L. This strengthens the statement of Proposition 5.23, where semistability was only proved over $\mathbf{Q}_p(\sqrt{-p})$. Besides, we have shown in (5.13) and (5.22) that the polynomial $M_k(p;T)$ is a factor of $\det(1-F_pT \mid V_{k,\ell}^{I_p})$ for all primes p if k is odd and for all primes $p \neq 2$ if k is even. From the (weak) admissibility of $r_{k,p}$ and the equalities

$$\begin{aligned}\det(1 - F_p T \mid V_{k,\ell}^{I_p}) &= \det\bigl(1 - F_p T \mid \ker(\mathrm{N})^{\rho(I_p)}\bigr)\\ &= \det\bigl(1 - \varphi T \mid (V_{k,p} \otimes \mathbf{B}_{\mathrm{st}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)}\bigr),\end{aligned}$$

it follows that the p-adic Newton polygon of $M_k(p;T)$ lies above the Hodge polygon of $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. We then recover Corollary 5.27 by adding back the trivial factor, including also the case of $p = 2$ and odd k, which is not treated in [44, p. 214].

Appendix. Exponential mixed Hodge structures and irregular Hodge filtrations

In this appendix, we prove some of the theorems used in the main text concerning mixed Hodge structures obtained from exponential mixed Hodge structures. We start by recalling the necessary material on $\mathscr{D}$-modules, mixed Hodge modules, and exponential mixed Hodge modules. Proposition A.13 then provides us with a condition for getting a mixed Hodge structure from the irregular Hodge filtration, and a criterion for this condition to be satisfied is proved in Theorem A.24. Finally, Theorem A.30 gives a way to compute the corresponding Hodge and weight filtrations. As it is customary, we adopt the convention that filtrations with lower (resp., upper) indices are increasing (resp., decreasing), and we pass from increasing to decreasing filtrations by setting $F^p_{\mathrm{irr}} = F^{\mathrm{irr}}_{-p}$ for any $p \in \mathbf{Q}$, and similarly for the ordinary Hodge filtration. In the theory of mixed Hodge modules one usually considers increasing filtrations, while the Hodge filtration of mixed Hodge structures is usually decreasing. We will make use of both conventions without further explanation.

A.1. Notation and results from the theory of $\mathscr{D}$-modules

We refer the reader, for example, to [27] and [31] for this section, although the notation therein may be somewhat different.

Given an algebraic morphism $h\colon X \to Y$ between smooth complex algebraic varieties, we denote by h_+ (resp., h^+) the derived pushforward (resp., pullback) in the sense of $\mathscr{D}$-modules; that is, for a $\mathscr{D}_X$-module or a bounded complex of $\mathscr{D}_X$-modules (resp., $\mathscr{D}_Y$-modules) M, we set

$$h_+M = \mathrm{R}h_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} M) \quad (\text{resp., } h^+M = \mathscr{D}_{X\longrightarrow Y} \overset{\mathrm{L}}{\otimes}_{h^{-1}\mathscr{D}_Y} h^{-1}M).$$

We denote by $\mathrm{DR}\,M$ the analytic de Rham complex of M, with M^{an} sitting in degree 0, and by ${}^{\mathrm{p}}\mathrm{DR}M$ the shifted complex $\mathrm{DR}\,M[\dim X]$, which is a perverse sheaf. We also denote by $h_\dagger$ the adjoint by duality of h_+ (i.e., the functor $h_\dagger = \boldsymbol{D}_Y h_+ \boldsymbol{D}_X$, where $\boldsymbol{D}$ denotes the duality functor in the category of $\mathscr{D}$-modules), so that there is a natural morphism $h_\dagger M \to h_+M$. In particular, given a holonomic $\mathscr{D}_X$-module M with regular singularities at infinity, the morphism $h_\dagger M \to h_+M$ induces the natural morphism $\mathrm{R}h_!{}^{\mathrm{p}}\mathrm{DR}M \to \mathrm{R}h_*{}^{\mathrm{p}}\mathrm{DR}M$ upon application of the shifted de Rham functor and taking the isomorphisms ${}^{\mathrm{p}}\mathrm{DR}h_{+}M \simeq \mathrm{R}h_{*}{}^{\mathrm{p}}\mathrm{DR}M$ and ${}^{\mathrm{p}}\mathrm{DR}h_{\dagger}M \simeq \mathrm{R}h_{!}{}^{\mathrm{p}}\mathrm{DR}M$ into account.

Given a $\mathscr{D}_X$-module M on a complex manifold (resp., smooth algebraic variety) X, we denote by $\mathrm{H}^k_{\mathrm{dR}}(X, M)$ the hypercohomology in degree k of the analytic (resp., algebraic) de Rham complex of M. Note that here we do not shift the de Rham complex as it is usually done in the theory of $\mathscr{D}$-modules. When dealing with an affine variety X, we will identify algebraic $\mathscr{D}_X$-modules with their global sections.

Let $j : U \hookrightarrow X$ be the open embedding of the complement of a divisor D on X. Given a holonomic $\mathscr{D}_U$-module M, the extension $j_{+}M$ is the holonomic $\mathscr{D}_X$-module on which any local equation of D acts in an invertible way. We denote by $j_{\dagger+}M$ the intermediate extension, defined as the maximal $\mathscr{D}_X$-submodule of $j_{+}M$ that has no quotient supported on D. For a holonomic $\mathscr{D}_U$-module M with regular singularities at infinity, the inclusion $j_{\dagger+}M \to j_{+}M$ corresponds to the natural morphism $j_{!*}{}^{\mathrm{p}}\mathrm{DR}M \to \mathrm{R}j_{*}{}^{\mathrm{p}}\mathrm{DR}M$ via the shifted de Rham functor.

If $h\colon X \to Y$ is smooth, then h^{+} sends holonomic $\mathscr{D}_Y$-modules to holonomic $\mathscr{D}_X$-modules. This functor corresponds to the usual pullback of vector bundles with connection. If $j : U \hookrightarrow X$ is an open embedding, then j^{+} is the usual restriction functor from holonomic $\mathscr{D}_X$-modules to holonomic $\mathscr{D}_U$-modules.

Instrumental for the theory of mixed Hodge modules (see [54]) is the notion of *nearby cycle* and *vanishing cycle functors* ψ_f and ϕ_f along a function $f : X \to \mathbb{A}^1$ on the category of perverse sheaves and holonomic $\mathscr{D}$-modules on X. If M is a holonomic $\mathscr{D}_X$-module, then $\psi_f M$ and $\phi_f M$ are holonomic $\mathscr{D}_X$-modules supported on $f^{-1}(0)$, which are defined in terms of the *Kashiwara–Malgrange filtration* of M. Both $\psi_f M$ and $\phi_f M$ are equipped with an automorphism T and decompose with respect to its eigenvalues. We denote by $\psi_{f,\lambda}M$ and $\phi_{f,\lambda}M$ the generalized eigencomponents corresponding to an eigenvalue $\lambda \in \mathbf{C}^*$. If $\lambda \neq 1$, there is an isomorphism $\psi_{f,\lambda}M \simeq \phi_{f,\lambda}M$ compatible with T, whereas for $\lambda = 1$ there is a quiver

$$\begin{tikzcd} \psi_{f,1}M \arrow[r, bend left, "\mathrm{can}"] & \phi_{f,1}M \arrow[l, bend left, "\mathrm{var}"] \end{tikzcd}$$

with the property that the maps $\exp(2\pi\mathrm{i}\,\mathrm{var}\circ\mathrm{can})$ and $\exp(2\pi\mathrm{i}\,\mathrm{can}\circ\mathrm{var})$ coincide with the unipotent automorphism T on $\psi_{f,1}M$ and $\phi_{f,1}M$, respectively. The nilpotent endomorphisms $\mathrm{var}\circ\mathrm{can}$ and $\mathrm{can}\circ\mathrm{var}$ on $\psi_{f,1}M$ and $\phi_{f,1}M$ are denoted by N. Up to a suitable shift, the nearby and vanishing cycles of a regular holonomic $\mathscr{D}_X$-module M correspond to the nearby and vanishing cycles of the perverse sheaf ${}^{\mathrm{p}}\mathrm{DR}M$.

For a meromorphic function $\varphi \in \Gamma(X, \mathscr{O}_X(*P))$ on X with pole divisor P, set

$$E^{\varphi} = \bigl(\mathscr{O}_X(*P), \mathrm{d} + \mathrm{d}\varphi\bigr),$$

and denote by e^φ the generator $1 \in \mathscr{O}_X(*P)$ of E^φ. Consider the product $\mathbb{A}^1_t \times \mathbb{A}^1_\tau$ of two affine lines with coordinates t and τ, and denote by p_t and p_τ the projections to the first and the second factors, respectively. The *Fourier transform* of a $\mathscr{D}$-module (or a bounded complex of $\mathscr{D}$-modules) M on the affine line $\mathbb{A}^1_\tau$ is the complex

$$\mathrm{FT}_\tau\, M = p_{t+}(p_\tau^+ M \otimes E^{t\tau}).$$

If M is a holonomic $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module, then $\mathrm{FT}_\tau\, M$ has cohomology concentrated in degree 0 (i.e., is a holonomic $\mathbf{C}[t]\langle\partial_t\rangle$-module). This yields a functor

$$\mathrm{FT}_\tau \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{\mathbb{A}^1_\tau}) \longrightarrow \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{\mathbb{A}^1_t}).$$

If M has regular singularities everywhere, including at infinity, then the only singularity of $\mathrm{FT}_\tau\, M$ on $\mathbb{A}^1_t$ is the origin, which is also regular.

Let $s\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ denote the sum map. The *additive $*$-convolution* $M_1 \star_* M_2$ of M_1 and M_2 is the object $s_+(M_1 \boxtimes M_2)$ of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{\mathbb{A}^1})$. This operation is associative and corresponds to the derived tensor product through Fourier transformation:

$$\mathrm{FT}(M_1 \star_* M_2) \simeq \mathrm{FT}\, M_1 \overset{\mathrm{L}}{\otimes} \mathrm{FT}\, M_2 = \delta^+(\mathrm{FT}\, M_1 \boxtimes \mathrm{FT}\, M_2),$$

where $\delta\colon \mathbb{A}^1 \hookrightarrow \mathbb{A}^1 \times \mathbb{A}^1$ is the diagonal embedding. For holonomic $\mathscr{D}$-modules with regular singularities M_1 and M_2, the cohomologies in nonzero (i.e., negative) degrees of the complex $\mathrm{FT}\, M_1 \otimes^{\mathrm{L}} \mathrm{FT}\, M_2$ are supported at the origin of $\mathbb{A}^1$, so the corresponding cohomologies of $M_1 \star_* M_2$ are constant $\mathscr{D}_{\mathbb{A}^1}$-modules. We refer the reader to [14, Section 1.1] for details.

A.2. Notation and results from the theory of mixed Hodge modules

Let X be a smooth complex projective variety, and let $M^{\mathrm{H}} = (M, F_\bullet M, \mathscr{F}_{\mathbf{Q}}, \alpha)$ be the data of a regular holonomic left $\mathscr{D}_X$-module M, an increasing good filtration $F_\bullet M$ on M, a $\mathbf{Q}$-perverse sheaf $\mathscr{F}_{\mathbf{Q}}$ on X, and an isomorphism $\alpha\colon {}^{\mathrm{p}}\mathrm{DR} M \xrightarrow{\sim} \mathscr{F}_{\mathbf{C}}$. As it is customary, we shall omit α from the notation. Let Sp denote the Spencer complex of a right $\mathscr{D}_X$-module. We say that M^{H} is a *pure polarizable Hodge module* of weight w if the associated right filtered $\mathscr{D}_X$-module

$$(\omega_X \otimes_{\mathscr{O}_X} M, \omega_X \otimes_{\mathscr{O}_X} F_{\bullet - \dim X} M),$$

together with $\mathscr{F}_{\mathbf{Q}}$ and the isomorphism $\mathrm{Sp}(\omega_X \otimes M) = {}^{\mathrm{p}}\mathrm{DR} M \simeq \mathscr{F}_{\mathbf{C}}$ forms a pure polarizable Hodge module of weight w in the sense of Saito [53, Sections 5.1.6, 5.2.10] (see also the introduction there). There is a similar definition for left *mixed Hodge modules*, with the left-to-right correspondence $W_\bullet M \leftrightarrow \omega_X \otimes W_\bullet M$ between weight filtrations. We refer the reader to [54] for the properties of the category $\mathsf{MHM}(X)$ of *algebraic mixed Hodge modules*, which are always assumed to be graded-polarizable. The derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$ is endowed with a six-functor formalism, and we denote the functors with a lower left index H in order to keep the category in mind. For example, the pushforward functor ${}_{\mathrm{H}}f_*$ by a morphism f for left mixed Hodge modules is obtained by composing the similar functor for mixed Hodge modules with the side-changing functor. There is a similar definition for other functors. In particular, the additive $*$-convolution of holonomic $\mathscr{D}_{\mathbb{A}^1}$-modules lifts to $\mathsf{MHM}(\mathbb{A}^1)$ (see formula (A.5) below). Be aware that some of these functors may not correspond to the corresponding functors on the underlying $\mathscr{D}$-modules: for example, ${}_{\mathrm{H}}\otimes$ does not correspond to $\otimes^{\mathrm{L}}$ as used above.

Assume that $(M, F_\bullet M)$ underlies a pure Hodge module M^{H} of weight w that is smooth (i.e., M is an $\mathscr{O}_X$-locally free module of finite rank with integrable connection ∇), and consider the associated decreasing filtration $F^pM = F_{-p}M$. Then the triple $(M, \nabla, F^\bullet M)$ is a polarizable variation of pure Hodge structures of weight $w - \dim X$.

Let U be a smooth complex quasiprojective variety of dimension d. We let ${}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}$ denote the pure Hodge module of weight d whose underlying perverse sheaf is ${}^{\mathrm{p}}\mathbf{Q}_U = \mathbf{Q}_U[d]$ and whose underlying filtered $\mathscr{D}_U$-module is $(\mathscr{O}_U, F_\bullet\mathscr{O}_U)$ with $\mathrm{gr}_p^F\,\mathscr{O}_U = 0$ except for $p = 0$ (it is denoted by $\mathbf{Q}_U^{\mathrm{H}}$ in [54]).

The duality functor ${}_{\mathrm{H}}\boldsymbol{D}$ is a contravariant anti-t-exact involution on the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$, and hence preserves $\mathsf{MHM}(X)$. There is a natural isomorphism ${}_{\mathrm{H}}\boldsymbol{D}\,{}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}} \simeq {}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}(-d)$.

Given a morphism $f : U \to \mathbb{A}^1_\theta$, there are functors

$$ {}_{\mathrm{H}}f_*, {}_{\mathrm{H}}f_! \colon \mathsf{D}^{\mathrm{b}}\big(\mathsf{MHM}(U)\big) \longrightarrow \mathsf{D}^{\mathrm{b}}\big(\mathsf{MHM}(\mathbb{A}^1)\big). $$

For the open immersion $j : U \smallsetminus D \hookrightarrow U$ of the complement of a divisor D, the localization functor ${}_{\mathrm{H}}j_*{}_{\mathrm{H}}j^*$ is simply denoted by $[*D]$ (and the corresponding functor for $\mathscr{D}$-modules by $(*D)$), while the dual localization functor ${}_{\mathrm{H}}j_!{}_{\mathrm{H}}j^*$ is denoted by $[!D]$ (and by $(!D)$, respectively). If $i : D \hookrightarrow U$ denotes the closed immersion, then there are distinguished dual triangles in $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(U))$ (see [54, (4.4.1)]):

$$ \begin{aligned} {}_{\mathrm{H}}i_*{}_{\mathrm{H}}i^!M^{\mathrm{H}} \longrightarrow M^{\mathrm{H}} \longrightarrow M^{\mathrm{H}}[*D] \xrightarrow{+1}, \\ M^{\mathrm{H}}[!D] \longrightarrow M^{\mathrm{H}} \longrightarrow {}_{\mathrm{H}}i_*{}_{\mathrm{H}}i^*M^{\mathrm{H}} \xrightarrow{+1}, \end{aligned} \tag{A.1} $$

which, for $M^{\mathrm{H}} \in \mathsf{MHM}(U)$, reduce to the dual exact sequences

$$ \begin{gathered} 0 \longrightarrow \mathscr{H}^0{}_{\mathrm{H}}i_*{}_{\mathrm{H}}i^!M^{\mathrm{H}} \longrightarrow M^{\mathrm{H}} \longrightarrow M^{\mathrm{H}}[*D] \longrightarrow \mathscr{H}^1{}_{\mathrm{H}}i_*{}_{\mathrm{H}}i^!M^{\mathrm{H}} \longrightarrow 0, \\ 0 \longrightarrow \mathscr{H}^{-1}{}_{\mathrm{H}}i_*{}_{\mathrm{H}}i^*M^{\mathrm{H}} \longrightarrow M^{\mathrm{H}}[!D] \longrightarrow M^{\mathrm{H}} \longrightarrow \mathscr{H}^0{}_{\mathrm{H}}i_*{}_{\mathrm{H}}i^*M^{\mathrm{H}} \longrightarrow 0. \end{gathered} $$

We also make use of the nearby and vanishing cycle functors $\psi_f = \psi_{f,1} \oplus \psi_{f,\neq 1}$ and $\phi_{f,1}$ on $\mathsf{MHM}(U)$, with the associated nilpotent operator N. If f is smooth in the neighborhood of $f^{-1}(0)$, then we regard them as taking values in $\mathsf{MHM}(f^{-1}(0))$.

Example A.2
Set $U = \mathbb{A}^1$ and $f = \mathrm{Id}$, and let M^{H} be a mixed Hodge module on $\mathbb{A}^1$. Since we are considering left modules, the convention for filtrations is that $\dim F^p \psi_f M$ equals $\mathrm{rk}\, F^p M$ and, if 0 is not a singular point of M, then $W_\ell \psi_f M = \psi_f W_{\ell+1} M$.

If M^{H} is pure, then $M^{\mathrm{H}} = M_1^{\mathrm{H}} \oplus M_2^{\mathrm{H}}$ is the direct sum of a Hodge module M_1^{H} whose underlying $\mathscr{D}_{\mathbb{A}^1}$-module M_1 has no section supported at zero and a Hodge module M_2^{H} supported at zero. Then the following hold:

$$\mathscr{H}^0{}_{\mathrm{H}}i_{*\mathrm{H}}i^! M_1^{\mathrm{H}} = 0, \qquad \mathscr{H}^1{}_{\mathrm{H}}i_{*\mathrm{H}}i^! M_2^{\mathrm{H}} = 0, \qquad \mathscr{H}^0{}_{\mathrm{H}}i_{*\mathrm{H}}i^! M_2^{\mathrm{H}} \simeq {}_{\mathrm{H}}i_{*\mathrm{H}}i^! M_2^{\mathrm{H}}.$$

Moreover, $\mathscr{H}^1{}_{\mathrm{H}}i_{*\mathrm{H}}i^! M_1^{\mathrm{H}}$ is isomorphic to the mixed Hodge structure

$$\operatorname{coker} \mathrm{N}_\tau \colon \psi_{\tau,1} M_1^{\mathrm{H}} \longrightarrow \psi_{\tau,1} M_1^{\mathrm{H}}(-1),$$

where N_τ denotes the nilpotent part of the monodromy operator for its eigenvalue one on the nearby cycles of M^{H} at the origin. Indeed, considering the diagram

$$\begin{tikzcd}
\phi_{\tau,1} M_1^{\mathrm{H}} \arrow[r] \arrow[d, "\mathrm{var}"'] & \phi_{\tau,1}(M_1^{\mathrm{H}}[*0]) \arrow[d, "\mathrm{var}", "\wr"'] \\
\psi_{\tau,1} M_1^{\mathrm{H}}(-1) \arrow[r, "\sim"] & \psi_{\tau,1}(M_1^{\mathrm{H}}[*0])(-1)
\end{tikzcd}$$

where the horizontal arrows are functorially obtained from $M_1^{\mathrm{H}} \to M_1^{\mathrm{H}}[*0]$, the mixed Hodge structure $\mathscr{H}^1{}_{\mathrm{H}}i_{*\mathrm{H}}i^! M_1^{\mathrm{H}}$ is identified with the cokernel of the upper horizontal arrow, and hence of the left vertical one. Since $\mathrm{can}\colon \psi_{\tau,1} M_1^{\mathrm{H}} \to \phi_{\tau,1} M_1^{\mathrm{H}}$ is an epimorphism and $\mathrm{N}_\tau = \mathrm{var} \circ \mathrm{can}$, the conclusion follows.

Example A.3
Given a reduced divisor $D \subset U$, define ${}^{\mathrm{p}}\mathbf{Q}_D^{\mathrm{H}} = {}_{\mathrm{H}}a_D^* \mathbf{Q}_{\operatorname{Spec}\mathbf{C}}^{\mathrm{H}}[\dim D]$, where a_D denotes the structure morphism. Then there is an isomorphism

$${}_{\mathrm{H}}i^{*\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}} = \mathscr{H}^0{}_{\mathrm{H}}i^{*\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}} \simeq {}^{\mathrm{p}}\mathbf{Q}_D^{\mathrm{H}}$$

and an exact sequence

$$0 \longrightarrow {}_{\mathrm{H}}i_*{}^{\mathrm{p}}\mathbf{Q}_D^{\mathrm{H}} \longrightarrow {}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}[!D] \longrightarrow {}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}} \longrightarrow 0.$$

If, moreover, D is smooth, then there is also an isomorphism

$$ {}_{\mathrm{H}}i^{!\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}[1] = \mathscr{H}^1{}_{\mathrm{H}}i^{!\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}} \simeq {}^{\mathrm{p}}\mathbf{Q}_D^{\mathrm{H}}(-1), $$

and the above exact sequence can be completed to a diagram

$$
\begin{tikzcd}
0 \arrow[r] & {}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & {}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}[*D] \arrow[r] & {}_{\mathrm{H}}i_*{}^{\mathrm{p}}\mathbf{Q}_D^{\mathrm{H}}(-1) \arrow[r] & 0, \\
0 & {}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}} \arrow[l] & {}^{\mathrm{p}}\mathbf{Q}_U^{\mathrm{H}}[!D] \arrow[l] \arrow[u] & {}_{\mathrm{H}}i_*{}^{\mathrm{p}}\mathbf{Q}_D^{\mathrm{H}} \arrow[l] & 0 \arrow[l]
\end{tikzcd}
\tag{A.4}
$$

in which the lower row is the dual exact sequence and the square commutes.

A.3. A review of exponential mixed Hodge structures

Let $\mathbb{A}^1_\theta$ be the affine line with coordinate θ. In [36, Section 4], Kontsevich and Soibelman define the category EMHS of *exponential mixed Hodge structures* as the full subcategory of $\mathsf{MHM}(\mathbb{A}^1_\theta)$ consisting of those objects N^{H} whose underlying perverse sheaf has vanishing global cohomology. The assignment

$$ N^{\mathrm{H}} \longmapsto \Pi_\theta(N^{\mathrm{H}}) = N^{\mathrm{H}} \star {}_{\mathrm{H}}j_!\mathscr{O}^{\mathrm{H}}_{\mathbb{G}_{\mathrm{m}}}, $$

where $\star$ stands for additive $*$-convolution defined as

$$ N_1^{\mathrm{H}} \star N_2^{\mathrm{H}} = {}_{\mathrm{H}}s_*(N_1^{\mathrm{H}} \boxtimes N_2^{\mathrm{H}}) \tag{A.5} $$

and $j : \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1$ for the inclusion, yields an exact functor

$$ \Pi_\theta : \ \mathsf{MHM}(\mathbb{A}^1_\theta) \longrightarrow \mathsf{MHM}(\mathbb{A}^1_\theta), $$

which is a projector onto EMHS that is left adjoint to its inclusion as a subcategory. In particular, there is a natural morphism $N^{\mathrm{H}} \to \Pi_\theta(N^{\mathrm{H}})$ in $\mathsf{MHM}(\mathbb{A}^1_\theta)$. Its kernel and cokernel are constant mixed Hodge modules on $\mathbb{A}^1_\theta$. More precisely, the $\mathscr{D}_{\mathbb{A}^1_\theta}$-module underlying the kernel is the maximal constant submodule of N. We will also write Π for Π_θ when the coordinate is clear.

Each object of EMHS is endowed with a weight filtration, defined from that in the category $\mathsf{MHM}(\mathbb{A}^1_\theta)$ by the formula $W_n^{\mathrm{EMHS}}\Pi(N^{\mathrm{H}}) = \Pi(W_nN^{\mathrm{H}})$ for any mixed Hodge module N^{H} on $\mathbb{A}^1_\theta$.

Remark A.6

This weight filtration has the following properties.

- If $N = \Pi(N)$ (and hence $N^{\mathrm{H}} = \Pi(N^{\mathrm{H}})$), then there is a functorial morphism

$$ W_nN^{\mathrm{H}} \longrightarrow W_n^{\mathrm{EMHS}}N^{\mathrm{H}}. $$

- If N^{H} is mixed of weights at most n (resp., at least n) in $\mathsf{MHM}(\mathbb{A}^1_\theta)$, then $\Pi(N^{\mathrm{H}})$ is mixed of weights at most n (resp., at least n) in EMHS; therefore, if N^{H} is pure of weight n, then $\Pi(N^{\mathrm{H}})$ (which in general is mixed of weights at least n as an object of $\mathsf{MHM}(\mathbb{A}^1_\theta)$) is a pure object of EMHS of weight n.
- For the sake of simplicity, when there is no risk of confusion, we write $W_\bullet\Pi(N^{\mathrm{H}})$ instead of $W_\bullet^{\mathrm{EMHS}}\Pi(N^{\mathrm{H}})$ for the weight filtration of an exponential mixed Hodge structure $\Pi(N^{\mathrm{H}})$.

The category EMHS is endowed with a tensor structure induced by the additive convolution $\star$ on $\mathsf{MHM}(\mathbb{A}^1)$ such that the equality

$$\Pi(N_1^{\mathrm{H}}) \star \Pi(N_2^{\mathrm{H}}) = \Pi(N_1^{\mathrm{H}} \star N_2^{\mathrm{H}})$$

holds. Moreover, the weight filtration is strictly compatible with the tensor product (see [36, Proposition 4.3]). Note that $N_1^{\mathrm{H}} \star N_2^{\mathrm{H}}$ could be a complex, but its cohomologies in nonzero degrees are constant (this is checked on the underlying $\mathscr{D}$-modules), and hence are annihilated by Π; on the other hand, the cohomology of $\Pi(N_1^{\mathrm{H}}) \star \Pi(N_2^{\mathrm{H}})$ is concentrated in degree 0.

Recall that the functor

$$\mathsf{Mod}_{\mathrm{holreg}}(\mathscr{D}_{\mathbb{A}^1_\theta}) \longrightarrow \mathsf{Vect}_{\mathbf{C}}, \qquad N \longmapsto \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta) \tag{A.7}$$

is exact and that $\mathrm{H}^j_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta)$ vanishes for $j \neq 1$. Moreover, $N \to \Pi(N)$ induces an isomorphism

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{A}^1_\theta, \Pi(N) \otimes E^\theta\big).$$

Let N^{H} be a mixed Hodge module on $\mathbb{A}^1_\theta$, and let $\Pi(N^{\mathrm{H}})$ be the associated exponential mixed Hodge structure. Let $a_{\mathbb{A}^1_\theta}$ denote the structure morphism of $\mathbb{A}^1_\theta$. The *de Rham fiber functor* on EMHS is given by

$$\Pi(N^{\mathrm{H}}) \longmapsto \mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{A}^1_\theta, \Pi(N) \otimes E^\theta\big) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta) = \mathrm{H}^0 a_{\mathbb{A}^1_\theta,+}(N \otimes E^\theta). \tag{A.8}$$

An object of EMHS is zero if and only if its de Rham fiber is zero. It follows from the exactness of (A.7) that the natural morphism

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, W_\bullet N \otimes E^\theta) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta) \tag{A.9}$$

is injective. The filtration $W_\bullet \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta)$ is defined as its image. Then any morphism $N_1^{\mathrm{H}} \to N_2^{\mathrm{H}}$ induces a strictly filtered morphism

$$\big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N_1 \otimes E^\theta), W_\bullet\big) \longrightarrow \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N_2 \otimes E^\theta), W_\bullet\big).$$

On the other hand, letting $j_\infty\colon \mathbb{A}^1_\theta \hookrightarrow \mathbb{P}^1_\theta$ denote the inclusion, the (decreasing) *irregular Hodge filtration* on $j_{\infty+}(N \otimes E^\theta)$ is defined in [50, Section 6.b] out of the Hodge filtration $F^\bullet N$ of N^{H}. It is indexed by $\mathbf{Q}$ and denoted there by $F^\bullet_{\mathrm{Del}}$, whereas here we denote it by $F^\bullet_{\mathrm{irr}}$, to emphasize the compatibility with other possible definitions (see [15], [52]).

PROPOSITION A.10
The natural morphism

$$\mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{P}^1_\theta, F^\bullet_{\mathrm{irr}} j_{\infty+}(N \otimes E^\theta)\big) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{P}^1_\theta, j_{\infty+}(N \otimes E^\theta)\big) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta)$$

is injective and, defining $F^\bullet_{\mathrm{irr}}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta)$ as its image, the functor from $\mathsf{MHM}(\mathbb{A}^1_\theta)$ to bifiltered vector spaces

$$N^{\mathrm{H}} \longmapsto \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet\big) \tag{A.11}$$

factors through Π. Any morphism in $\mathsf{MHM}(\mathbb{A}^1_\theta)$ (or in EMHS) gives rise by means of (A.11) to a strictly bifiltered morphism.

Proof
Injectivity is proved in [50, Theorem 6.1] for polarized Hodge modules. The case of mixed Hodge modules is deduced from it in [15, Theorem 3.3.1] (in a more general setting). Since the kernel and cokernel of $N^{\mathrm{H}} \to \Pi(N^{\mathrm{H}})$ are constant mixed Hodge modules and $\mathrm{H}^j_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta)$ vanishes in all degrees j for constant N, it is clear that (A.11) factors through Π. The last statement is obtained by successively applying Theorems 0.2(2), 0.3(4), and 0.3(2) of [51]. $\square$

A.4. Mixed Hodge structures as exponential mixed Hodge structures
Let MHS be the category of mixed Hodge structures, endowed with its natural tensor product. We consider the open immersion $j_0\colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1_\theta$ and the closed immersion $i_0\colon \{0\} \hookrightarrow \mathbb{A}^1_\theta$. Letting ${}_{\mathrm{H}}i_{0!}$ denote the pushforward functor $\mathsf{MHS} \to \mathsf{MHM}(\mathbb{A}^1_\theta)$, there is an isomorphism $\phi_{\theta,1} \circ {}_{\mathrm{H}}i_{0!} \simeq \mathrm{Id}_{\mathsf{MHS}}$. Let us now consider the composed functor $\Pi \circ {}_{\mathrm{H}}i_{0!} : \mathsf{MHS} \to \mathsf{EMHS}$. Then $\phi_{\theta,1} \circ (\Pi \circ {}_{\mathrm{H}}i_{0!}) \simeq \mathrm{Id}_{\mathsf{MHS}}$, since $\phi_{\theta,1} \circ \Pi \simeq \phi_{\theta,1}$ (because $\phi_{\theta,1}$ of a constant object in $\mathsf{MHM}(\mathbb{A}^1_\theta)$ is zero). From standard properties of mixed Hodge modules one checks the following lemma.

LEMMA A.12
The functor $\Pi \circ {}_{\mathrm{H}}i_{0!}\colon$ $\mathsf{MHS} \to \mathsf{EMHS}$ is compatible with tensor products and makes MHS into a full tensor subcategory of EMHS. Moreover, if V^{H} is a mixed Hodge structure with weight filtration $W_\bullet V^{\mathrm{H}}$ and associated exponential mixed Hodge structure $\Pi_\theta({}_{\mathrm{H}}i_{0!}V^{\mathrm{H}})$, then the equality $W_\bullet\Pi_\theta({}_{\mathrm{H}}i_{0!}V^{\mathrm{H}}) = \Pi_\theta({}_{\mathrm{H}}i_{0!}W_\bullet V^{\mathrm{H}})$ holds (see Remark A.6 for the notation).

PROPOSITION A.13
Let N^{H} *be an object of* $\mathsf{MHM}(\mathbb{A}^1_\theta)$ *such that* $\Pi(N^{\mathrm{H}})$ *belongs to* MHS. *Then the bifiltered vector space* $(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet)$ *is naturally isomorphic to that associated with the mixed Hodge structure* $\phi_{\theta,1} N^{\mathrm{H}}$.

Proof
Let V^{H} be a mixed Hodge structure. The vector space

$$\big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, i_{0+}V \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet\big)$$

endowed with its two filtrations is easily identified with $(V, F^\bullet, W_\bullet)$. Taking the isomorphism

$$\big(\mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{A}^1_\theta, \Pi(i_{0+}V) \otimes E^\theta\big), F^\bullet_{\mathrm{irr}}, W_\bullet\big) \simeq \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, i_{0+}V \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet\big)$$

into account, we deduce an isomorphism

$$\big(\mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{A}^1_\theta, \Pi(i_{0+}V) \otimes E^\theta\big), F^\bullet_{\mathrm{irr}}, W_\bullet\big) \simeq (V, F^\bullet, W_\bullet). \tag{A.14}$$

Now, if $\Pi(N^{\mathrm{H}})$ belongs to MHS, then $\Pi(N^{\mathrm{H}}) = \Pi({}_{\mathrm{H}}i_{0!}V^{\mathrm{H}})$ for some mixed Hodge structure V^{H}, which satisfies $V^{\mathrm{H}} \simeq \phi_{\theta,1}\Pi(N^{\mathrm{H}}) \simeq \phi_{\theta,1}N^{\mathrm{H}}$. Then (A.14) gives

$$\begin{aligned}(\phi_{\theta,1}N, F^\bullet, W_\bullet) &\simeq \big(\mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{A}^1_\theta, \Pi(i_{0+}\phi_{\theta,1}N) \otimes E^\theta\big), F^\bullet_{\mathrm{irr}}, W_\bullet\big)\\ &\simeq \big(\mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{A}^1_\theta, \Pi(N) \otimes E^\theta\big), F^\bullet_{\mathrm{irr}}, W_\bullet\big)\\ &\simeq \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet\big),\end{aligned}$$

as we wanted to show. □

A.5. The exponential mixed Hodge structures associated with a function
Let U be a smooth complex quasiprojective variety of dimension d, together with a regular function $f : U \to \mathbb{A}^1_\theta$. For each object N^{H}_U of $\mathsf{MHM}(U)$ and each integer r, we define the mixed Hodge modules

$$\begin{aligned}(N^{\mathrm{H}})^r_* &= \mathscr{H}^{r-d}\,{}_{\mathrm{H}}f_* N^{\mathrm{H}}_U, \qquad (N^{\mathrm{H}})^r_! = \mathscr{H}^{r-d}\,{}_{\mathrm{H}}f_! N^{\mathrm{H}}_U,\\ (N^{\mathrm{H}})^r_{\mathrm{mid}} &= \mathrm{im}\big[(N^{\mathrm{H}})^r_! \longrightarrow (N^{\mathrm{H}})^r_*\big]\end{aligned}$$

on $\mathbb{A}^1_\theta$, and we denote by N^r_*, $N^r_!$, N^r_{mid} the respective underlying $\mathscr{D}_{\mathbb{A}^1_\theta}$-modules. Upon application of the projector Π_θ, they define objects of EMHS:

$$\begin{aligned}\mathrm{H}^r(U, N^{\mathrm{H}}, f) &= \Pi_\theta\big((N^{\mathrm{H}})^r_*\big), \qquad \mathrm{H}^r_{\mathrm{c}}(U, N^{\mathrm{H}}, f) = \Pi_\theta\big((N^{\mathrm{H}})^r_!\big),\\ \mathrm{H}^r_{\mathrm{mid}}(U, N^{\mathrm{H}}, f) &= \Pi_\theta\big((N^{\mathrm{H}})^r_{\mathrm{mid}}\big) = \mathrm{im}\big[\mathrm{H}^r_{\mathrm{c}}(U, N^{\mathrm{H}}, f) \longrightarrow \mathrm{H}^r(U, N^{\mathrm{H}}, f)\big].\end{aligned}$$

Example A.15
Let a_U denote the structure morphism of U. For f the zero function, the above are the mixed Hodge structures

$$\mathrm{H}^r(U, N^{\mathrm{H}}, 0) = \mathrm{H}^{r-d}({}_{\mathrm{H}}a_{U*}, N^{\mathrm{H}}), \qquad \mathrm{H}^r_{\mathrm{c}}(U, N^{\mathrm{H}}, 0) = \mathrm{H}^{r-d}_{\mathrm{c}}({}_{\mathrm{H}}a_{U*}, N^{\mathrm{H}}).$$

There are filtered isomorphisms between the filtered de Rham fibers

$$\big(\mathrm{H}^r_{\mathrm{dR}}(U, N_U \otimes E^f), F^\bullet_{\mathrm{irr}}\big) \simeq \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N^r_* \otimes E^\theta), F^\bullet_{\mathrm{irr}}\big), \tag{A.16}$$

$$\big(\mathrm{H}^r_{\mathrm{dR,c}}(U, N_U \otimes E^f), F^\bullet_{\mathrm{irr}}\big) \simeq \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N^r_! \otimes E^\theta), F^\bullet_{\mathrm{irr}}\big). \tag{A.17}$$

This follows from [52, Theorem 1.3(4)] applied to a compactification of the morphism f and by taking the pushforward by the structure morphism. For the second isomorphism, we use the equality $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{A}^1_\theta, N^r_! \otimes E^\theta) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N^r_! \otimes E^\theta)$, which follows from the fact that $N^r_! \otimes E^\theta$ has a purely irregular singularity at ∞.

The natural morphism $(\mathrm{H}^r_{\mathrm{dR,c}}(U, N_U \otimes E^f), F^\bullet_{\mathrm{irr}}) \to (\mathrm{H}^r_{\mathrm{dR}}(U, N_U \otimes E^f), F^\bullet_{\mathrm{irr}})$ is strict and its image $\mathrm{H}^r_{\mathrm{dR,mid}}(U, N_U \otimes E^f)$ is endowed with the quotient (equivalently, sub-) filtration $F^\bullet_{\mathrm{irr}}$.

On the other hand, the right-hand sides of (A.16) and (A.17) underlie bifiltered de Rham fibers $(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_\theta, N^r_* \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet)$ and $(\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{A}^1_\theta, N^r_! \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet)$, the weight filtration $W_\bullet$ being defined as the image of (A.9). Thanks to the last statement of Proposition A.10, the bifiltered de Rham fiber $(\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{A}^1_\theta, N^r_! \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet)$ is unambiguously defined.

Definition A.18 ([36, Section 5])
Associated with a function $f : U \to \mathbb{A}^1_\theta$ and an integer r as above are the exponential mixed Hodge structures

$$\mathrm{H}^r(U, f) = \Pi_\theta\big(({}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U)^r_*\big), \qquad \mathrm{H}^r_{\mathrm{c}}(U, f) = \Pi_\theta\big(({}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U)^r_!\big),$$
$$\mathrm{H}^r_{\mathrm{mid}}(U, f) = \Pi_\theta\big(({}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U)^r_{\mathrm{mid}}\big),$$

with corresponding bifiltered de Rham fibers

$$\big(\mathrm{H}^r_{\mathrm{dR}}(U, E^f), F^\bullet_{\mathrm{irr}}, W_\bullet\big), \qquad \big(\mathrm{H}^r_{\mathrm{dR,c}}(U, E^f), F^\bullet_{\mathrm{irr}}, W_\bullet\big),$$
$$\big(\mathrm{H}^r_{\mathrm{dR,mid}}(U, E^f), F^\bullet_{\mathrm{irr}}, W_\bullet\big).$$

PROPOSITION A.19
The exponential mixed Hodge structure $\mathrm{H}^d_{\mathrm{c}}(U, f)$ is mixed of weights at most d, $\mathrm{H}^d(U, f)$ is mixed of weights at least d, and $\mathrm{H}^d_{\mathrm{mid}}(U, f)$ is pure of weight d. Moreover, the following properties are equivalent:

(1) *the natural morphism* $\mathrm{gr}^W_d \mathrm{H}^d_{\mathrm{c}}(U,f) \to \mathrm{gr}^W_d \mathrm{H}^d(U,f)$ *is an isomorphism, and*

(2) *the equality* $\mathrm{H}^d_{\mathrm{mid}}(U,f) = W_d\mathrm{H}^d(U,f)$ *holds.*

Proof
It follows from the standard behavior of weights with respect to pushforward functors (here applied to mixed Hodge modules; see [54, (4.5.2)]) that the object $\mathscr{H}^0{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U$ of $\mathsf{MHM}(\mathbb{A}^1_\theta)$ has weights at most d, and hence that the object $\mathrm{H}^d_{\mathrm{c}}(U,f)$ of EMHS has weights at most d by Remark A.6. The argument for $\mathrm{H}^d(U,f)$ and $\mathrm{H}^d_{\mathrm{mid}}(U,f)$ is the same.

For the last assertion, let us denote by $\mathscr{H}^0{}_{\mathrm{H}}f_{!*}{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U$ the image in $\mathsf{MHM}(\mathbb{A}^1_\theta)$ of the natural morphism $\mathscr{H}^0{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U \to \mathscr{H}^0{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U$. Then the exact sequence

$$0 \longrightarrow \mathscr{H}^0{}_{\mathrm{H}}f_{!*}{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U \longrightarrow \mathscr{H}^0{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U \longrightarrow N^{\mathrm{H}} \longrightarrow 0$$

in $\mathsf{MHM}(\mathbb{A}^1_\theta)$, which defines the object N^{H}, dualizes (with Tate twist $-d$) to

$$0 \longrightarrow N^{\mathrm{H}}_1 \longrightarrow \mathscr{H}^0{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U \longrightarrow \mathscr{H}^0{}_{\mathrm{H}}f_{!*}{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U \longrightarrow 0,$$

with $N^{\mathrm{H}}_1 = {}_{\mathrm{H}}\boldsymbol{D}N^{\mathrm{H}}(-d)$ and the composition of the above morphisms

$$\mathscr{H}^0{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U \longrightarrow \mathscr{H}^0{}_{\mathrm{H}}f_{!*}{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U \longrightarrow \mathscr{H}^0{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U$$

is the natural morphism. By Remark A.6, applying Π and gr^W_d gives exact sequences

$$\begin{aligned}
&0 \longrightarrow \mathrm{H}^d_{\mathrm{mid}}(U,f) \longrightarrow \mathrm{gr}^W_d \mathrm{H}^d(U,f) \longrightarrow \mathrm{gr}^W_d \Pi(N^{\mathrm{H}}) \longrightarrow 0,\\
&0 \longrightarrow \mathrm{gr}^W_d \Pi(N^{\mathrm{H}}_1) \longrightarrow \mathrm{gr}^W_d \mathrm{H}^d_{\mathrm{c}}(U,f) \longrightarrow \mathrm{H}^d_{\mathrm{mid}}(U,f) \longrightarrow 0.
\end{aligned}$$

Property (2) is thus equivalent to the vanishing $\mathrm{gr}^W_d \Pi(N^{\mathrm{H}}) = 0$, which means that $\mathrm{gr}^W_d N^{\mathrm{H}}$ is constant. By duality, this is equivalent to $\mathrm{gr}^W_d N^{\mathrm{H}}_1$ being constant, and hence to the vanishing $\mathrm{gr}^W_d \Pi(N^{\mathrm{H}}_1) = 0$, which is precisely property (1). ☐

Example A.20
Let D be a divisor on which the function f vanishes. By applying ${}_{\mathrm{H}}f_!$ and the exact functor Π_θ to the lower exact sequence in (A.4), we obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{r-1}_{\mathrm{c}}(D) \longrightarrow \mathrm{H}^r_{\mathrm{c}}(U \smallsetminus D, f) \longrightarrow \mathrm{H}^r_{\mathrm{c}}(U,f) \longrightarrow \mathrm{H}^r_{\mathrm{c}}(D) \longrightarrow \cdots \tag{A.21}$$

on noting the equality $\mathrm{H}^j_{\mathrm{c}}(D,0) = \mathrm{H}^j_{\mathrm{c}}(D)$ from Example A.15. If, moreover, D is smooth, then by also applying ${}_{\mathrm{H}}f_*$ to the upper exact sequence in (A.4), we obtain a diagram of exponential mixed Hodge structures

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^{r-2}(D)(-1) \arrow[r] & \mathrm{H}^r(U,f) \arrow[r] & \mathrm{H}^r(U\smallsetminus D,f) \arrow[r] & \mathrm{H}^{r-1}(D)(-1) \arrow[r] & \cdots \\
\cdots & \mathrm{H}^r_{\mathrm{c}}(D) \arrow[l] & \mathrm{H}^r_{\mathrm{c}}(U,f) \arrow[l] \arrow[u] \arrow[r, phantom, "\circlearrowleft"] & \mathrm{H}^r_{\mathrm{c}}(U\smallsetminus D,f) \arrow[l] \arrow[u] & \mathrm{H}^{r-1}_{\mathrm{c}}(D) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

where the rows are long exact sequences and the square commutes.

Example A.22
In this example, we construct isomorphisms

$$\mathrm{H}^k(\mathbb{G}_{\mathrm{m}}^k, x_1+\cdots+x_k)\simeq \mathbf{Q}(-k) \qquad \text{and} \qquad \mathrm{H}^k_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}}^k, x_1+\cdots+x_k)\simeq \mathbf{Q}(0)$$

for each integer $k\geq 1$. For this, we first observe that $\mathrm{H}^r(\mathbb{A}^1_x,x)=\mathrm{H}^r_{\mathrm{c}}(\mathbb{A}^1_x,x)=0$ for all r since the pushforward of ${}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1_x}$ by the identity map is the constant Hodge module concentrated in degree 0, and hence is killed by the projector Π. The long exact sequences from Example A.20 then yield

$$\mathrm{H}^1(\mathbb{G}_{\mathrm{m}},x)\simeq \mathrm{H}^0\big(\{0\}\big)(-1)=\mathbf{Q}(-1) \qquad \text{and} \qquad \mathrm{H}^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}},x)\simeq \mathrm{H}^0_{\mathrm{c}}\big(\{0\}\big)=\mathbf{Q}(0).$$

For $k\geq 2$, let $f\colon U=\mathbb{G}_{\mathrm{m}}^k\to\mathbb{A}^1$ be the sum map $x_1+\cdots+x_k$. Then ${}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U$ is the k-fold convolution product of ${}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1}[*0]$ with itself. After applying Π to the cohomology modules, we obtain the vanishing $\mathrm{H}^r(U,f)=0$ (i.e., $\mathscr{H}^{r-k}{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U$ is constant) for $r\neq k$ and an isomorphism $\mathrm{H}^k(U,f)\simeq \mathrm{H}^1(\mathbb{G}_{\mathrm{m}},x)^{\otimes k}$ in EMHS. Since MHS is a full tensor subcategory of EMHS by Lemma A.12 and $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}},x)=\mathbf{Q}(-1)$ lies in MHS by the above computation, so does $\mathrm{H}^k(U,f)$ and there is an isomorphism of mixed Hodge structures

$$\mathrm{H}^k(\mathbb{G}_{\mathrm{m}}^k, x_1+\cdots+x_k)\simeq \mathbf{Q}(-k). \tag{A.23}$$

Finally, with the above assumptions, the isomorphism

$$\mathscr{H}^j{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U\simeq ({}_{\mathrm{H}}\boldsymbol{D}\mathscr{H}^{-j}{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U)(-k)$$

implies the vanishing $\mathrm{H}^r_{\mathrm{c}}(U,f)=0$ for all $r\neq k$. From (A.23) we know that the successive quotients of the weight filtration on $\mathscr{H}^0{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U$ as an object of $\mathsf{MHM}(\mathbb{A}^1)$ are all constant except for one that is isomorphic to ${}_{\mathrm{H}}i_{0!}\mathbf{Q}(-k)$. Dually, the successive quotients of the weight filtration on $\mathscr{H}^j{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_U$ as an object of $\mathsf{MHM}(\mathbb{A}^1)$ are all constant except for one that is isomorphic to ${}_{\mathrm{H}}i_{0!}\mathbf{Q}(0)$. There is thus an isomorphism

$$\mathrm{H}^k_{\mathrm{c}}(\mathbb{G}_{\mathrm{m}}^k, x_1+\cdots+x_k)\simeq \mathbf{Q}(0).$$

A.6. A criterion for an object of EMHS *to belong to* MHS

We now give a criterion ensuring that, for certain $f : U \to \mathbb{A}^1$ and N^{H} as above, the objects $\mathrm{H}^r(U, N^{\mathrm{H}}, f)$ and $\mathrm{H}^r_{\mathrm{c}}(U, N^{\mathrm{H}}, f)$ of EMHS belong to the subcategory MHS, that is, are usual mixed Hodge structures. This will allow us to apply Proposition A.13 to identify their Hodge and irregular Hodge filtrations.

THEOREM A.24 (see also [63, Theorem 3.3], [19, Lemma 6.5.3])
Assume that U and f are of the form $U = \mathbb{A}^1_t \times V$ and $f = tg$ for some smooth quasiprojective variety V and some regular function $g\colon V \to \mathbb{A}^1$, and let M^{H}_V be an object of $\mathsf{MHM}(V)$.

(1) *If $N^{\mathrm{H}}_U = {}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes M^{\mathrm{H}}_V$, then the exponential mixed Hodge structures*

$$\mathrm{H}^r(U, N^{\mathrm{H}}, f) \qquad \textit{and} \qquad \mathrm{H}^r_{\mathrm{c}}(U, N^{\mathrm{H}}, f)$$

belong to MHS *for all r, and their bifiltered de Rham fibers*

$$(\mathrm{H}^r_{\mathrm{dR}}(U, N_U \otimes E^f), F^\bullet_{\mathrm{irr}}, W_\bullet) \qquad \textit{and} \qquad (\mathrm{H}^r_{\mathrm{dR,c}}(U, N_U \otimes E^f), F^\bullet_{\mathrm{irr}}, W_\bullet)$$

underlie the corresponding mixed Hodge structures.

(2) *If $N^{\mathrm{H}}_U = {}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1_t}[*0] \boxtimes M^{\mathrm{H}}_V$, then the exponential mixed Hodge structure*

$$\mathrm{H}^r(U, N^{\mathrm{H}}, f)$$

belongs to MHS *for all r and its bifiltered de Rham fiber*

$$(\mathrm{H}^r_{\mathrm{dR}}(U, N_U \otimes E^f), F^\bullet_{\mathrm{irr}}, W_\bullet)$$

underlies the corresponding mixed Hodge structure.

(3) *If $N^{\mathrm{H}}_U = {}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1_t}[!0] \boxtimes M^{\mathrm{H}}_V$, then the exponential mixed Hodge structure*

$$\mathrm{H}^r_{\mathrm{c}}(U, N^{\mathrm{H}}, f)$$

belongs to MHS *for all r and its bifiltered de Rham fiber*

$$(\mathrm{H}^r_{\mathrm{dR,c}}(U, N_U \otimes E^f), F^\bullet_{\mathrm{irr}}, W_\bullet)$$

underlies the corresponding mixed Hodge structure.

The last statements in (1)–(3) follow from Proposition A.13. We are thus reduced to proving the first statements.

Proof of (1)

We start with $\mathrm{H}^r(U, N^{\mathrm{H}}, f)$. Consider the divisor $D = \mathbb{A}^1_t \times g^{-1}(0)$ on U. The object

$\mathscr{N}_{U*}^{\mathrm{H}} = [N_U^{\mathrm{H}} \to N_U^{\mathrm{H}}[*D]]$ of $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(U))$ is supported on the zero locus of f. For each r, there exists a mixed Hodge structure V^{H} such that $\mathscr{H}^{r-d}{}_{\mathrm{H}} f_* \mathscr{N}_{U*}^{\mathrm{H}} = {}_{\mathrm{H}} i_{0!} V^{\mathrm{H}}$ holds, and hence the object $\Pi_\theta(\mathscr{H}^{r-d}{}_{\mathrm{H}} f_* \mathscr{N}_{U*}^{\mathrm{H}})$ of EMHS belongs to MHS for all r. Thanks to Proposition A.13, it suffices to prove that the de Rham fibers $\mathrm{H}^r_{\mathrm{dR}}(U, N_U(*D) \otimes E^f)$ of $\mathrm{H}^r(U, N_U^{\mathrm{H}}[*D], f) = \Pi_\theta(N_U^{\mathrm{H}}[*D])^r_*$ vanish in all degrees r. By considering the pushforward by the map $(t, g)\colon \mathbb{A}^1_t \times V \to \mathbb{A}^1_t \times \mathbb{A}^1_\tau$, we can reduce the proof to the case $V = \mathbb{A}^1_\tau$ and $g = \tau$. We then simply write $M = M_{\mathbb{A}^1_\tau}$, and we are reduced to proving

$$a_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau, +}\big((\mathscr{O}_{\mathbb{A}^1_t} \boxtimes M(*0)) \otimes E^{t\tau}\big) = 0. \tag{A.25}$$

Let $p_t\colon \mathbb{A}^1_t \times \mathbb{A}^1_\tau \to \mathbb{A}^1_t$ denote the projection. The equality $a_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau} = a_{\mathbb{A}^1_t} \circ p_t$ holds. We note that the complex $p_{t+}((\mathscr{O}_{\mathbb{A}^1_t} \boxtimes M(*0)) \otimes E^{t\tau})$ is nothing but the Fourier transform $\mathrm{FT}_\tau(M(*0))$, and in particular is concentrated in degree 0. Then, identifying a $\mathscr{O}_{\mathbb{A}^1_t}$-module with connection with a $\mathbf{C}[t]$-module with connection, $a_{\mathbb{A}^1_t,+} \mathrm{FT}_\tau(M(*0))$ is represented by the complex

$$\big[\mathrm{FT}_\tau\big(M(*0)\big) \xrightarrow{\partial_t} \underset{\bullet}{\mathrm{FT}_\tau\big(M(*0)\big)}\big] \simeq \big[M(*0) \xrightarrow{\tau} \underset{\bullet}{M(*0)}\big],$$

where $\bullet$ indicates the term in degree 0. Since τ acts invertibly on $M(*0)$, the left-hand side is thus quasi-isomorphic to zero.

For $\mathrm{H}^r_{\mathrm{c}}(U, N^{\mathrm{H}}, f)$, we argue similarly, considering $\mathscr{N}_{U!}^{\mathrm{H}} = [N_U^{\mathrm{H}}[!D] \to N_U^{\mathrm{H}}]$ instead and noting that $\Pi_\theta(\mathscr{H}^{r-d}{}_{\mathrm{H}} f_! \mathscr{N}_{U!}^{\mathrm{H}})$ belongs to MHS for all r. It is then enough to prove the vanishing $\mathrm{H}^r_{\mathrm{dR,c}}(U, N_U(!D) \otimes E^f) = 0$ in all degrees r, which reduces to

$$a_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau, \dagger}\big((\mathscr{O}_{\mathbb{A}^1_t} \boxtimes M(!0)) \otimes E^{t\tau}\big) = 0 \tag{A.26}$$

by taking the proper pushforward by (t, g). It is known that the complex

$$p_{t\dagger}\big((\mathscr{O}_{\mathbb{A}^1_t} \boxtimes M(!0)) \otimes E^{t\tau}\big)$$

is also isomorphic to $\mathrm{FT}_\tau(M(!0))$, and in particular is concentrated in degree 0 (see, e.g., [40, Appendice 2, Proposition 1.7]). From the isomorphism

$$\mathrm{FT}_\tau\big(M(!0)\big) \simeq \iota^+ \boldsymbol{D}\, \mathrm{FT}_\tau\big((\boldsymbol{D} M)(*0)\big),$$

where ι is the involution $t \mapsto -t$, we thus get the vanishing

$$a_{\mathbb{A}^1_\tau, \dagger} \iota^+ \boldsymbol{D}\, \mathrm{FT}_\tau\big((\boldsymbol{D} M)(*0)\big) \simeq \boldsymbol{D} a_{\mathbb{A}^1_\tau,+} \mathrm{FT}_\tau\big((\boldsymbol{D} M)(*0)\big) \simeq 0$$

by the first part of the proof applied to $\boldsymbol{D} M$. $\square$

Proof of (2)
As in (1), we reduce to the case where $V = \mathbb{A}^1_\tau$ and $g = \tau$, so that $f = t\tau$, and we simply denote the pushforward ${}_{\mathrm{H}}g_* M^{\mathrm{H}}_V$ by $M^{\mathrm{H}} \in \mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(\mathbb{A}^1_\tau))$. We extend the functor Π_τ to an endofunctor of $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(\mathbb{A}^1_\tau))$ that commutes with taking cohomology, and we consider the morphisms

$$\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*0] \boxtimes M^{\mathrm{H}} \longrightarrow \mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*0] \boxtimes \Pi_\tau(M^{\mathrm{H}}) \longleftarrow \mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes \Pi_\tau(M^{\mathrm{H}})$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau))$ and their pushforwards

$${}_{\mathrm{H}}f_*\big(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*0] \boxtimes M^{\mathrm{H}}\big) \longrightarrow {}_{\mathrm{H}}f_*\big(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*0] \boxtimes \Pi_\tau(M^{\mathrm{H}})\big) \longleftarrow {}_{\mathrm{H}}f_*\big(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes \Pi_\tau(M^{\mathrm{H}})\big)$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(\mathbb{A}^1_\theta))$. We will prove that, after applying the projector Π_θ, they induce cohomology isomorphisms in EMHS. Since the projections of the cohomologies of the rightmost term belong to MHS according to (1) (note that (A.25) holds for a complex M if it holds for its cohomology modules), then so will the projections of the cohomologies of the leftmost term.

For the left morphism, we note that the cohomologies of the simple complex associated with the double complex $M^{\mathrm{H}} \to \Pi_\tau M^{\mathrm{H}}$ are constant mixed Hodge modules. Indeed, by [59, Theorem 4.20], it is enough to check that the underlying $\mathscr{D}$-modules are constant, and this follows from the long exact sequence in cohomology, upon noting that an extension of constant $\mathscr{D}_{\mathbb{A}^1_\tau}$-modules is constant. We are thus reduced to proving that, for any constant mixed Hodge module M^{H} on $\mathbb{A}^1_\tau$ and any j, the mixed Hodge module $\mathscr{H}^j\,{}_{\mathrm{H}}f_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*0] \boxtimes M^{\mathrm{H}})$ on $\mathbb{A}^1_\theta$ is constant. Again, it is enough to prove that the underlying $\mathscr{D}$-module is constant, which amounts to proving that the de Rham fiber (A.8) of its projection to EMHS is zero. This fiber is isomorphic to

$$\mathscr{H}^j a_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau,+}\big((\mathscr{O}_{\mathbb{A}^1_t}(*0) \boxtimes M) \otimes E^{t\tau}\big).$$

By first projecting to $\mathbb{A}^1_t$, we find

$$p_{t+}\big((\mathscr{O}_{\mathbb{A}^1_t}(*0) \boxtimes M) \otimes E^{t\tau}\big) \simeq \mathscr{O}_{\mathbb{A}^1_t}(*0) \otimes \mathrm{FT}_\tau(M) = 0,$$

because the Fourier transform of a constant $\mathscr{D}$-module is supported at zero.

For the right morphism, a similar argument reduces the proof to showing that the cohomology of the double complex that it defines has constant cohomology or, equivalently, that the cohomology of the double complex

$$\begin{aligned}&\big\{a_{\mathbb{A}^1_\theta}\big[f_+\big(\mathscr{O}_{\mathbb{A}^1_t} \boxtimes \Pi_\tau(M)\big) \otimes E^\theta\big] \longrightarrow a_{\mathbb{A}^1_\theta}\big[f_+\big(\mathscr{O}_{\mathbb{A}^1_t}(*0) \boxtimes \Pi_\tau(M)\big) \otimes E^\theta\big]\big\}\\ &\quad\simeq \big\{a_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau,+}\big((\mathscr{O}_{\mathbb{A}^1_t} \boxtimes \Pi_\tau(M)) \otimes E^{t\tau}\big)\\ &\qquad\longrightarrow a_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau,+}\big((\mathscr{O}_{\mathbb{A}^1_t}(*0) \boxtimes \Pi_\tau(M)) \otimes E^{t\tau}\big)\big\}\end{aligned}$$

$$\simeq \{a_{\mathbb{A}^1_t,+}\,\mathrm{FT}_\tau\big(\Pi_\tau(M)\big) \longrightarrow a_{\mathbb{A}^1_t,+}\big(\mathscr{O}_{\mathbb{A}^1_t}(*0)\otimes \mathrm{FT}_\tau\big(\Pi_\tau(M)\big)\big)\}$$

is zero. This follows from the equality $\mathrm{FT}_\tau(\Pi_\tau(M)) = \mathscr{O}_{\mathbb{A}^1_t}(*0)\otimes \mathrm{FT}_\tau(\Pi_\tau(M))$, which is readily checked from the definition of Π_τ (see [34, Proposition 12.3.5]). ☐

Proof of (3)
The dual mixed Hodge module ${}_{\mathrm{H}}\boldsymbol{D}(N_U^{\mathrm{H}})$ is of the form considered in (2), and there is an isomorphism ${}_{\mathrm{H}}f_!N_U^{\mathrm{H}} \simeq {}_{\mathrm{H}}\boldsymbol{D}\,{}_{\mathrm{H}}f_{*\mathrm{H}}\boldsymbol{D}(N_U^{\mathrm{H}})$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(\mathbb{A}^1_\theta))$. Therefore, each cohomology (in $\mathsf{MHM}(\mathbb{A}^1_\theta)$) of ${}_{\mathrm{H}}f_!N_U^{\mathrm{H}}$ is dual to some cohomology of ${}_{\mathrm{H}}f_{*\mathrm{H}}\boldsymbol{D}(N_U^{\mathrm{H}})$. We can then conclude by using the fact that the projection $\Pi_\theta(M^{\mathrm{H}})$ of a mixed Hodge module $M^{\mathrm{H}}\in\mathsf{MHM}(\mathbb{A}^1_\theta)$ belongs to MHS if and only if $\Pi_\theta({}_{\mathrm{H}}\boldsymbol{D}M^{\mathrm{H}})$ does. Indeed, the former property is equivalent to $\mathrm{FT}_\theta(M)(*0)$ being a constant flat bundle with connection and, letting ι denote the involution $\theta\mapsto-\theta$, there is an isomorphism

$$(\mathrm{FT}_\theta(\boldsymbol{D}M))(*0)\simeq \mathrm{FT}_\theta(\iota^+M)(*0)^\vee. \qquad ☐$$

Example A.27
Let us apply Theorem A.24 to $M_V^{\mathrm{H}} = {}^{\mathrm{p}}\mathbf{Q}_V^{\mathrm{H}}$. Setting $\mathscr{K}=g^{-1}(0)$, the divisor D in its proof is given by $D=\mathbb{A}^1_t\times\mathscr{K}$. The vanishing

$$\mathrm{H}^r_{\mathrm{c}}\big(\mathbb{A}^1_t\times(V\smallsetminus\mathscr{K}),tg\big)=\mathrm{H}^r\big(\mathbb{A}^1_t\times(V\smallsetminus\mathscr{K}),tg\big)=0$$

holds for all r by (A.25) and (A.26). Then, according to the exact sequence (A.21), the mixed Hodge structure $\mathrm{H}^r_{\mathrm{c}}(\mathbb{A}^1_t\times V,tg)$ provided by Theorem A.24(1) is isomorphic to $\mathrm{H}^r_{\mathrm{c}}(\mathbb{A}^1_t\times\mathscr{K})$. Furthermore, by the Künneth formula, there is an isomorphism

$$\mathrm{H}^r_{\mathrm{c}}(\mathbb{A}^1\times\mathscr{K})=\mathrm{H}^2_{\mathrm{c}}(\mathbb{A}^1)\otimes\mathrm{H}^{r-2}_{\mathrm{c}}(\mathscr{K})=\mathrm{H}^{r-2}_{\mathrm{c}}(\mathscr{K})(-1),$$

so that, finally, we obtain an isomorphism of mixed Hodge structures

$$\mathrm{H}^r_{\mathrm{c}}(\mathbb{A}^1_t\times V,tg)\simeq\mathrm{H}^{r-2}_{\mathrm{c}}(\mathscr{K})(-1).$$

On the other hand, let $i_{\mathscr{K}}$ and $j_{\mathscr{K}}$ be the complementary closed and open immersions attached to the divisor $\mathbb{A}^1\times\mathscr{K}$ on $\mathbb{A}^1\times V$. Applying ${}_{\mathrm{H}}(tg)_*$ to the triangle

$${}_{\mathrm{H}}i_{\mathscr{K},*\mathrm{H}}i^!_{\mathscr{K}}\,{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1\times V}\longrightarrow {}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1\times V}\longrightarrow {}_{\mathrm{H}}j_{\mathscr{K},*\mathrm{H}}j^*_{\mathscr{K}}\,{}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1\times V}\xrightarrow{+1}$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(\mathbb{A}^1\times V))$ (see [54, (4.4.1)]) and noting that $(tg)_*\circ i_{\mathscr{K},*}$ is the zero map, the vanishing of $\mathrm{H}^r(\mathbb{A}^1_t\times(V\smallsetminus\mathscr{K}),tg)$ for all r yields an isomorphism of mixed Hodge structures

$$\mathrm{H}^r(\mathbb{A}^1_t\times V,tg)\simeq\mathrm{H}^r_{\mathbb{A}^1_t\times\mathscr{K}}(\mathbb{A}^1_t\times V)\simeq\mathrm{H}^r_{\mathscr{K}}(V).$$

If $\mathscr{K}$ is smooth, then the rightmost term is also isomorphic to $\mathrm{H}^{r-2}(\mathscr{K})(-1)$.

A.7. Computation of the Hodge and the weight filtrations

Let M^{H} be a mixed Hodge module on the affine line $\mathbb{A}^1_\tau$, and let $N^{\mathrm{H}} = {}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes M^{\mathrm{H}}$. According to Theorem A.24(1), the exponential mixed Hodge structures

$$\mathrm{H}^r(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, N^{\mathrm{H}}, t\tau) \qquad \text{and} \qquad \mathrm{H}^r_{\mathrm{c}}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, N^{\mathrm{H}}, t\tau)$$

are usual mixed Hodge structures. As already noticed in the proof of Theorem A.24, writing $a_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau} = a_{\mathbb{A}^1_t} \circ p_t$ we get the following result.

LEMMA A.28

The de Rham fibers of $\mathrm{H}^{j+1}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, N^{\mathrm{H}}, t\tau)$ *and* $\mathrm{H}^{j+1}_{\mathrm{c}}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, N^{\mathrm{H}}, t\tau)$ *are, respectively,* $\mathrm{H}^j_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\, M)$ *for* $j = 0, 1$ *and* $\mathrm{H}^j_{\mathrm{dR,c}}(\mathbb{A}^1_t, \mathrm{FT}\, M)$ *for* $j = 1, 2$, *and zero otherwise.*

Notation A.29

We denote by $\mathrm{H}^j(\mathbb{A}^1_t, \mathrm{FT}\, M^{\mathrm{H}})$ and $\mathrm{H}^j_{\mathrm{c}}(\mathbb{A}^1_t, \mathrm{FT}\, M^{\mathrm{H}})$ the mixed Hodge structures $\mathrm{H}^{j+1}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, N^{\mathrm{H}}, t\tau)$ and $\mathrm{H}^{j+1}_{\mathrm{c}}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, N^{\mathrm{H}}, t\tau)$, respectively. Their associated bifiltered de Rham fibers are, respectively, $(\mathrm{H}^j_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\, M), F^\bullet_{\mathrm{irr}}, W_\bullet)$ and $(\mathrm{H}^j_{\mathrm{dR,c}}(\mathbb{A}^1_t, \mathrm{FT}\, M), F^\bullet_{\mathrm{irr}}, W_\bullet)$.

A priori, there might be a source of ambiguity in the notation for the irregular Hodge filtration, since $\mathrm{FT}\, M$ also underlies an irregular mixed Hodge module on $\mathbb{P}^1_t$ in the sense of [51], by means of which $\mathrm{H}^j_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\, M)$ acquires an irregular Hodge filtration. However, due to the known E_1-degeneration results for the irregular Hodge filtration, both filtrations on $\mathrm{H}^j_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\, M)$ agree. We shall not use this property.

THEOREM A.30

Let $i_0 \colon \{0\} \to \mathbb{A}^1_t$ *be the inclusion. For each mixed Hodge module* M^{H} *on the affine line, the mixed Hodge structures* $\mathrm{H}^j(\mathbb{A}^1_t, \mathrm{FT}\, M^{\mathrm{H}})$ *and* $\mathrm{H}^j({}_{\mathrm{H}}i_0^! M^{\mathrm{H}})$ *are isomorphic.*

Proof

Let us first consider the case where M^{H} is supported at the point 0; that is, M^{H} is isomorphic to ${}_{\mathrm{H}}i_{0,*}V^{\mathrm{H}}$ for some mixed Hodge structure V^{H}. Set $N^{\mathrm{H}} = {}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes M^{\mathrm{H}}$, and let $f : \mathbb{A}^1_t \times \mathbb{A}^1_\tau \to \mathbb{A}^1_\theta$ be the function $(t, \tau) \mapsto t\tau = \theta$. Then N^{H} is supported on $f = 0$ and ${}_{\mathrm{H}}f_* N^{\mathrm{H}}[-1] \simeq \mathscr{H}^{-1}{}_{\mathrm{H}}f_* N^{\mathrm{H}}$ is supported at $\theta = 0$ and is isomorphic to ${}_{\mathrm{H}}i_{0,*}V^{\mathrm{H}}$ (here i_0 denotes the inclusion of 0 in $\mathbb{A}^1_\theta$). Therefore, $\mathrm{H}^1(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, N^{\mathrm{H}}, f)$ is isomorphic to V^{H} and all other cohomologies vanish.

Now, for a general object M^{H} of $\mathsf{MHM}(\mathbb{A}^1_\tau)$, we consider the exact sequence (A.1) for the divisor $D = \{0\}$. According to (A.25), the vanishing

$$\mathrm{H}^j_{\mathrm{dR}}\big(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, \big(\mathscr{O}_{\mathbb{A}^1_t} \boxtimes M(*0)\big) \otimes E^{t\tau}\big) = 0$$

holds for all j. Setting $\mathrm{H}^j({}_{\mathrm{H}}i_0^! M^{\mathrm{H}}) = V_j^{\mathrm{H}}$, for $j = 0, 1$, and $N_j^{\mathrm{H}} = {}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes {}_{\mathrm{H}}i_{0,*} V_j^{\mathrm{H}}$, we thus get an isomorphism of mixed Hodge structures

$$\mathrm{H}^{j+1}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, {}^{\mathrm{p}}\mathbf{Q}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes M^{\mathrm{H}}, t\tau) \simeq \mathrm{H}^1(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, N_j^{\mathrm{H}}, t\tau)$$

and the right-hand side underlies V_j^{H} by the first part of the proof. □

COROLLARY A.31

Assume that M^{H} is a pure object of $\mathsf{MHM}(\mathbb{A}^1_\tau)$ of weight w whose underlying $\mathscr{D}_{\mathbb{A}^1_\tau}$-module M has no nonzero section supported at the origin (in particular, M is an intermediate extension at the origin). Then $\mathrm{H}^{j+1}_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau\, M) = 0$ for $j \neq 0$ and

(1) $\mathrm{H}^1(\mathbb{A}^1_t, \mathrm{FT}_\tau\, M^{\mathrm{H}})$ *is isomorphic to the mixed Hodge structure*

$$\operatorname{coker}\big[\mathrm{N}\colon \psi_{\tau,1} M^{\mathrm{H}} \longrightarrow \psi_{\tau,1} M^{\mathrm{H}}(-1)\big];$$

(2) *if 0 is not a singular point of M, then $\mathrm{H}^1(\mathbb{A}^1_t, \mathrm{FT}_\tau\, M^{\mathrm{H}})$ is a pure Hodge structure of weight $w+1$ and the equality $\dim \mathrm{gr}^p_{F_{\mathrm{irr}}} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau\, M) = \mathrm{rk}\, \mathrm{gr}_F^{p-1} M$ holds.*

If, moreover, M has no nonzero constant submodule, then there is an exact sequence

$$0 \longrightarrow M^{\mathrm{H}} \longrightarrow \Pi_\tau(M^{\mathrm{H}}) \longrightarrow M'^{\mathrm{H}} \longrightarrow 0$$

in $\mathsf{MHM}(\mathbb{A}^1)$, where M'^{H} is a constant mixed Hodge module on $\mathbb{A}^1_\tau$ of weights at least $w+1$ with

(3) $\dim \mathrm{gr}^p_{F_{\mathrm{irr}}} \mathrm{gr}^W_\ell \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau\, M') = \mathrm{rk}\, \mathrm{gr}_F^{p-1} \mathrm{gr}^W_{\ell-1} M'$, *for all $\ell, p \in \mathbf{Z}$.*

Proof

The assumptions on M imply that the semisimple $\mathscr{D}_{\mathbb{A}^1_t}$-module $\mathrm{FT}_\tau\, M$ has no constant submodules, and hence its de Rham cohomology in degree 0 vanishes. Statement (1) follows from Theorem A.30 and Example A.2. If 0 is not a singular point of M, then $\psi_{\tau,1} M^{\mathrm{H}} = \psi_\tau M^{\mathrm{H}}$ and the operator N is identically zero, so (1) identifies $\mathrm{H}^1(\mathbb{A}^1_t, \mathrm{FT}_\tau\, M^{\mathrm{H}})$ with the mixed Hodge structure $\psi_\tau M^{\mathrm{H}}(-1)$, which is pure of weight $w+1$ and satisfies

$$\dim \mathrm{gr}^p_{F_{\mathrm{irr}}} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau\, M) = \dim \mathrm{gr}^p_F\, \psi_\tau M_1^{\mathrm{H}}(-1) = \mathrm{rk}\, \mathrm{gr}_F^{p-1} M$$

by the formulas recalled at the beginning of Example A.2. This proves (2). The statement about weights in the last point follows from the same argument we used in the proof of Proposition 2.21: were the inclusion $M^{\mathrm{H}} \subset W_w \Pi_\tau(M^{\mathrm{H}})$ not an equality, $\Pi_\tau(M)$ would have a nonzero constant submodule, which contradicts the vanishing of its cohomology. Finally, (3) is obtained by applying Example A.2 once again. □

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Fresán
Centre de Mathématiques Laurent Schwartz, École Polytechnique, Palaiseau, France;
javier.fresan@polytechnique.edu; http://javier.fresan.perso.math.cnrs.fr

Sabbah
Centre de Mathématiques Laurent Schwartz, CNRS, École Polytechnique, Institut Polytechnique de Paris, Palaiseau, France;
Claude.Sabbah@polytechnique.edu; http://www.math.polytechnique.fr/perso/sabbah

Yu
Department of Mathematics, National Taiwan University, Taipei, Taiwan;
jdyu@ntu.edu.tw; http://homepage.ntu.edu.tw/~jdyu/