ON THE GROSS-DELIGNE CONJECTURE FOR VARIATIONS OF HODGE-DE RHAM STRUCTURES

MASANORI ASAKURA AND JAVIER FRESÁN

Abstract. We prove an alternating variant of the Gross-Deligne conjecture for periods of variations of Hodge–de Rham structures endowed with a multiplication by a number field.

0. Introduction

A CM elliptic curve is an elliptic curve defined over $\overline{\mathbb{Q}}$ endowed with multiplication by an imaginary quadratic field K. The multiplication gives rise to an action of K on the Betti and de Rham cohomology groups $H^1(E,\mathbb{Q})$, $H^1_{\mathrm{dR}}(E/\overline{\mathbb{Q}})$. Since $\dim_K H^1_B(E,\mathbb{Q}) = 1$ one has the eigendecompositions $(H^1_B(E,\overline{\mathbb{Q}})^\chi, H^1_{\mathrm{dR}}(E/\overline{\mathbb{Q}})^\chi)$ which are one-dimensional over $\overline{\mathbb{Q}}$ for each embedding $\chi : K \to \overline{\mathbb{Q}}$. The difference of the two $\overline{\mathbb{Q}}$-structures via the comparison isomorphism is called the *period* of the χ-part. Roughly speaking, the Lerch-Chowla-Selberg formula says that the period of a CM elliptic curve is a product of values of the gamma function at certain rational numbers. The Gross-Deligne conjecture is a generalization of the Lerch-Chowla-Selberg formula for the cohomology groups of smooth projective varieties equipped with multiplication by an abelian number field [Gro78]. See §2 below for the precise statement. The Gross-Deligne conjecture is still wide open, although some recent results give new evidence supporting it.

Maillot and Rössler proved a variant of the conjecture for the complex absolute values of periods [MR04]. More recently, the second author proved the conjecture for the alternating product of the cohomology groups of a variety acted upon by an automorphism of finite order [Fre13].

The purpose of this paper is to prove the period conjecture of Gross-Deligne for the alternating product of the cohomology of *fibrations* equipped with relative multiplication. Theorem 5.1 is the main result in this paper. The proof is divided into two parts, one is describing the period as a product of gamma values, and the other is computing the Hodge indices. The former goes in the same way as in [Fre13], and it relies heavily on the seminal paper by Saito-Terasoma [ST97]. The latter discussion, namely computing the Hodge indices goes in a different way from [Fre13]. We apply Saito's Jacobi sum formula [Sai94] to obtain an explicit description of the Hodge-Tate representation on p-adic cohomology. Then the Hodge-Tate conjecture proved by Faltings *et al.* gives the relationship between the Hodge-Tate representation and the (usual) Hodge decomposition of de Rham cohomology.

This paper is organized as follows. §1 is a section of preliminaries for the notion of Hodge–de Rham structures and the periods. In §2 we give a precise statement of the period conjecture. The techinical key to the formulation is the lemma of Deligne-Koblitz-Ogus (Lemma 2.1). In §3, we review the period formula of Saito–Terasoma. In §4, we apply Saito's Jacobi sum formula to obtain the Hodge-Tate representation and then we compute the Hodge indices. The main result and its applications for the cohomology of fibrations are given in §5. In the proof of the main theorem (Theorem 5.1), we use algebraic correspondences on weight spectral sequences of non-complete smooth varieties. Since the authors were unable to find a suitable reference for it, an exposition is given in §6 for the convenience of the reader.

0.1. **Notation and conventions.** Throughout the paper, $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. By a variety over a field k we mean an integral separated k-scheme of finite type. For a scheme X of finite type over $\mathbb{C}$ (resp. $\overline{\mathbb{Q}}$), we denote by X^{an} the analytic site associated to X (resp. $X \times_{\overline{\mathbb{Q}}} \mathbb{C}$). The structure sheaf on the analytic site is denoted by $\mathscr{O}^{an}$. The Zariski site is simply denoted by X or by X^{zar} whenever we want to emphasize it.

We denote by $\mu_d(F)$ the set of d-th roots of unity contained in a field F.

For a free R-module M of rank n with R a commutative ring, we denote the determinant of M over R by $\det_R M = \bigwedge_R^n M$. By convention, $\det_R\{0\} = R$.

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1. Preliminaries

1.1. **Periods of rank one Betti–de Rham structures.** Let $K \subseteq \mathbb{C}$ be a subfield. A *Betti–de Rham structure* over K is a triple $H = (H_{dR}, H_B, \iota)$ consisting of

- a finite-dimensional K-vector space H_{dR},
- a finite-dimensional $\overline{\mathbb{Q}}$-vector space H_B,
- an isomorphism of complex vector spaces $\iota \colon H_{\mathrm{dR}} \otimes_K \mathbb{C} \to H_B \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

Assume further that $K \subseteq \overline{\mathbb{Q}}$. To a rank one Betti–de Rham structure H, it is attached a *period* as follows: let $e_{\mathrm{dR}} \in H_{\mathrm{dR}}$ and $e_B \in H_B$ be bases over K and $\overline{\mathbb{Q}}$ respectively. Then there exists a complex number $\mathrm{per}(H)$ satisfying

$$\iota(e_{\mathrm{dR}}) = \mathrm{per}(H) e_B.$$

This is well-defined up to multiplication by $\overline{\mathbb{Q}}^\times$, thus yielding a class $\mathrm{per}(H) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$.

1.2. **Hodge–de Rham structures.** Let $K \subseteq \mathbb{C}$ be a subfield. A *Hodge–de Rham structure* over K is a triple $H = ((H_{\mathrm{dR}}, F^\bullet), H_B, \iota)$ consisting of

- a finite dimensional K-vector space H_{dR} together with a decreasing filtration $F^\bullet H_{dR}$,
- a finite dimensional $\mathbb{Q}$-vector space H_B,
- an isomorphism of complex vector spaces $\iota \colon H_{\mathrm{dR}} \otimes_K \mathbb{C} \to H_B \otimes_{\mathbb{Q}} \mathbb{C}$

such that $F^\bullet$ induces a pure Hodge structure on H_B via ι.

De Rham and Betti cohomologies of smooth proper varieties X over F, together with the comparison isomorphism, define a Hodge–de Rham structure $H^n(X)$. We will say that a Hodge–de Rham structure has *geometric origin* if... There is a forgetful functor from the category of Hodge–de Rham structures over F to that of Hodge structures. If $F = \mathbb{C}$, the functor is the identity, in other words, a Hodge–de Rham structure over $\mathbb{C}$ is nothing other than a Hodge structure.

Let F be a commutative $\mathbb{Q}$-algebra. An *F-multiplication* on a Hodge-de Rham structure H is a pair of F-actions on H_{dR} and H_B which are compatible with ι and $F^\bullet$. In other words, an F-action amounts to a ring morphism $F \to \mathrm{End}_{\mathrm{HdR}}(H)$. We say that H *has maximal multiplication* if F is a number field and $\dim_F H_B = 1$.

Suppose that $K \subseteq \overline{\mathbb{Q}}$. For each embedding $\chi\colon F \to \overline{\mathbb{Q}}$, the χ-components

$$H^\chi_{\mathrm{dR}} = (\overline{\mathbb{Q}} \otimes_K H_{\mathrm{dR}})^\chi, \qquad H^\chi_B = (\overline{\mathbb{Q}} \otimes_K H_B)^\chi$$

are defined as the subspaces $H^\chi_{dR} \subset \overline{\mathbb{Q}} \otimes_K H_{\mathrm{dR}}$ (resp. $H^\chi_B \subset \overline{\mathbb{Q}} \otimes_K H_B$) on which $1 \otimes g$ acts by multiplication by $\chi(g)$ for all $g \in F$. Whenever the F-multiplication is maximal, the triple $(H^\chi_B, H^\chi_{\mathrm{dR}}, \iota)$ forms a rank one Betti–de Rham structure in the sense of §1.1. Thus the period of the χ-component is defined and will be denoted by $\mathrm{per}(H^\chi) \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$.

Example 1.1. *For each integer r, the Tate Hodge–de Rham structure (over $\mathbb{Q}$) is defined as $\mathbb{Q}(r) = H^1(\mathbb{G}_m, \mathbb{Q})^{\otimes -r} \simeq H^2(\mathbb{P}^1, \mathbb{Q})^{\otimes -r}$. Its period is $\mathrm{per}(\mathbb{Q}(r)) = (2\pi i)^{-r}$.*

1.3. **Connections.** Let $K \subseteq \mathbb{C}$ be a subfield, and U a smooth quasi-projective variety over K. A *connection over U* is a triple $\mathscr{M} = ((M_{dR}, \nabla), M_B, \iota)$ consisting of

- a locally-free $\mathscr{O}_U$-module of finite rank M_{dR} on U, together with an integrable connection $\nabla\colon M_{\mathrm{dR}} \to M_{\mathrm{dR}} \otimes_{\mathscr{O}_U} \Omega^1_U$,
- a local system of $\overline{\mathbb{Q}}$-vector spaces M_B on the analytic complex manifold $U^{\mathrm{an}}_{\mathbb{C}}$,
- an isomorphism $\iota\colon M_B \otimes_{\overline{\mathbb{Q}}} \mathscr{O}^{\mathrm{an}}_U \to M_{\mathrm{dR}} \otimes_{\mathscr{O}_U} \mathscr{O}^{\mathrm{an}}_U$

such that the analytic connection ∇^{an} annhilates $\iota(M_B \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$.

We say that the connection $(M_{\mathrm{dR}}, \nabla)$ has *regular singularities* if the following holds. Let $j : U \hookrightarrow X$ be a smooth compactification such that $D = X \setminus U$ is a simple normal crossings divisor. Then there exists a locally free subsheaf $\overline{M}_{\mathrm{dR}} \subset j_* M_{\mathrm{dR}}$ on the Zariski site such that $j^*\overline{M}_{\mathrm{dR}} = M_{\mathrm{dR}}$ and

$$\nabla(\overline{M}_{\mathrm{dR}}) \subset \overline{M}_{\mathrm{dR}} \otimes \Omega^1_X(\log D).$$

Given a connection $\mathscr{M}$ over U and a closed point x, we define the fiber $\mathscr{M}_x = (M_{B,x}, M_{\mathrm{dR},x}, \iota)$.

1.4. **Variations of Hodge–de Rham structure.** Let U be a smooth quasi-projective variety over $K \subseteq \mathbb{C}$. We say that the data $\mathscr{M} = ((M_{\mathrm{dR}}, \nabla, F^\bullet), M_B, \iota)$ forms a *variation of Hodge–de Rham structures* on U if

- M_B is a smooth sheaf of finite-dimensional $\mathbb{Q}$-vector spaces on U^{an},
- M_{dR} is a locally free $\mathscr{O}^{\mathrm{zar}}_U$-module, and $F^\bullet$ is a descending filtration,
- $\iota\colon \mathbb{C} \otimes_{\mathbb{Q}} M_B \to \mathscr{O}^{an}_U \otimes_{\mathscr{O}^{\mathrm{zar}}} M_{\mathrm{dR}}$ is a homomorphism,
- $(M_{\mathrm{dR}}, \nabla)$ is an integrable connection on U^{zar} with regular singularities,
- $(M_B, \mathscr{O}^{an}_U \otimes_{\mathscr{O}^{\mathrm{zar}}} M_{\mathrm{dR}}, \iota, F^\bullet, \nabla)$ is a variation of Hodge structures on U^{an}.

If $K = \mathbb{C}$, a smooth sheaf of Hodge–de Rham structure is nothing other than a variation of Hodge structures. A polarization form is a pair of homomorphisms $Q_B : M_B \otimes M_B \to \mathbb{Q}$ and $Q_{\mathrm{dR}} : M_{\mathrm{dR}} \otimes M_{\mathrm{dR}} \to \mathscr{O}^{\mathrm{zar}}_U$ which induce the polarization on the variation of Hodge structures $(M_B, \mathscr{O}^{an}_U \otimes_{\mathscr{O}^{\mathrm{zar}}} M_{\mathrm{dR}}, \iota, F^\bullet, \nabla)$.

Let K be a number field. A multiplication on $\mathscr{M}$ by a number field K is a ring morphism $\rho : K \to \mathrm{End}(\mathscr{M})$. We say that $\mathscr{M}$ has maximal multiplication by K if the fibers of the local system M_B have dimension one over K.

1.5. **Local period.**

Theorem 1.2. *Let $\mathscr{M} = ((M_{\mathrm{dR}}, \nabla), M_B, \iota)$ be a rank one connection over $\overline{\mathbb{Q}}$. Then the period* $\mathrm{per}(\mathscr{M}_x)$ *does not depend on* x.

Proposition 1.3. *Let $\mathscr{M} = ((M_{\mathrm{dR}}, \nabla, F^\bullet), M_B, \iota)$ be a polarizable variation of Hodge–de Rham structures endowed with maximal multiplication by a number field F. Assume $K \subseteq \overline{\mathbb{Q}}$. Let $\mathscr{M}_x = (M_{B,x}, M_{\mathrm{dR},x}, \iota, F^\bullet)$ denote the restriction at a point $x \in U(\overline{\mathbb{Q}})$. Then for any $\chi\colon F \hookrightarrow \overline{\mathbb{Q}}$, the period* $\mathrm{per}(\mathscr{M}_x^\chi)$ *does not depend on* x.

Proof. We first show that the monodromy representation $\pi_1(U^{an}, x) \to \mathrm{GL}(M_{B,x})$ factors through a finite quotient. Let $\chi : K \hookrightarrow \overline{\mathbb{Q}}$ and $M_B^\chi \subset \overline{\mathbb{Q}} \otimes M_{B,t}$ the χ-part. Then $\pi_1(U^{an})$ acts on $M_B^\chi \cong \overline{\mathbb{Q}}$. We want to show that for any $T \in \pi_1(U^{an}, x)$ the eigenvalue λ_χ of T on M_B^χ is a root of unity or, equivalently, that the residues of (M_{dR}, ∇) are rational numbers. Let α_χ be such a residue. Since M_{dR} is defined over $\overline{\mathbb{Q}}$, we have $\alpha \in \mathbb{Q}$. On the other hand, In view of the previous argument, there exists a finite étale cover $h\colon U' \to U$ such that h^*M_B is a constant sheaf. Then $(M_B^\chi, M_{\mathrm{dR}}^\chi, \iota, \nabla)$ is a connection with regular singularities of rank one and M_B^χ is a constant sheaf. Now the assertion follows from Lemma 1.4. ☐

Lemma 1.4. *Let $\mathscr{M} = ((M_{\mathrm{dR}}, \nabla), M_B, \iota)$ be a connection of rank one with regular singularities. Suppose that M_B is a constant sheaf. Let $e_B \in \Gamma(U^{an}, M_B)$ and $e_{\mathrm{dR}} \in \Gamma(U^{\mathrm{zar}}, M_{\mathrm{dR}})$ be bases and $f^{an} \in \mathscr{O}(U^{an})^\times$ an analytic function satisfying*

$$\iota(e_B) = f^{an} \otimes e_{\mathrm{dR}} \in \Gamma(U^{an}, \mathscr{O}_U^{an} \otimes_{\mathscr{O}_U} M_{\mathrm{dR}}).$$

Then there are $c \in \mathbb{C}^\times$ and $f \in \mathscr{O}(U^{\mathrm{zar}})^\times$ such that $f^{an} = cf$. In particular, if $F \subseteq \overline{\mathbb{Q}}$, then the period $\mathrm{per}(\mathscr{M}_x)$ *does not depend on the choice of a point $x \in U(\overline{\mathbb{Q}})$.*

Proof. Write $U_{\mathbb{C}} = U \times_F \mathbb{C}$. Since M has regular singularities, $(M_B, M_{\mathrm{dR}}, \iota, \nabla)$ is a trivial connection on $U_{\mathbb{C}}$ by the Riemann-Hilbert correspondence (cf. [Mal87] (1,1) and (7.2.1)). Therefore $\mathbb{C} \otimes_F M_{\mathrm{dR}}$ is a free $\mathscr{O}_{U_{\mathbb{C}}}^{\mathrm{zar}}$-module of rank one and $\iota(e_B)$ is a free basis of $\Gamma(U_{\mathbb{C}}^{\mathrm{zar}}, \mathbb{C} \otimes_F M_{\mathrm{dR}})$. Therefore $f^{an} \in \mathscr{O}(U_{\mathbb{C}}^{\mathrm{zar}})^\times = \mathbb{C}^\times \cdot \mathscr{O}(U^{\mathrm{zar}})^\times$. ☐

2. The Gross-Deligne conjecture

In this section, we give the precise statement of the Gross-Deligne conjecture [Gro78], as well as a self-contained proof of a lemma by Koblitz-Ogus which plays a crucial role in the formulation.

2.1. **The gamma distribution.**

Let $\langle \alpha \rangle$ be symbols indexed by $\mathbb{Q}/\mathbb{Z}$ and consider the free $\mathbb{Q}$-module S generated by them

$$S = \bigoplus_{\alpha \in \mathbb{Q}/\mathbb{Z}} \mathbb{Q}\ \langle \alpha \rangle.$$

Let $T \subset S$ denote the sub-$\mathbb{Q}$-module of S generated by the symbols

$$\langle 0 \rangle, \qquad \langle \alpha \rangle + \langle -\alpha \rangle - 2\langle 1/2 \rangle, \qquad \sum_{i=0}^{m-1} \langle \alpha + i/m \rangle - (m-1)\langle 1/2 \rangle - \langle m\alpha \rangle,$$

where $\alpha \in \mathbb{Q}/\mathbb{Z}$ is non-zero and $m \geq 1$.

Let $\{-\} : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q} \cap [0, 1)$ denote the inverse map of the natural bijection $\mathbb{Q} \cap [0, 1) \to \mathbb{Q}/\mathbb{Z}$. It induces the $\mathbb{Q}$-linear map $\{-\} : S/T \to \mathbb{Q}$ such that $\{\langle \alpha \rangle\} = \{\alpha\}$, which we write by the same notation.

The group $\hat{\mathbb{Z}}^\times$ acts on $\mathbb{Q}/\mathbb{Z}$ in a natural way and it extends to the action on S/T. Then the Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ acts on S/T via the cyclotomic character $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \xrightarrow{\sim} \hat{\mathbb{Z}}^\times$.

Let

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$$

be the gamma function. Define a $\mathbb{Q}$-linear map $\widetilde{\mathbf{\Gamma}} : S \longrightarrow \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ by $\widetilde{\mathbf{\Gamma}}(\langle\alpha\rangle) := \Gamma(-\tilde{\alpha})$ where $\tilde{\alpha} \in \mathbb{Q}$ is a lifting of $\alpha \in \mathbb{Q}/\mathbb{Z}$ such that $\tilde{\alpha} > 0$. One sees that this does not depend on the choice of the lifting by the formula $\Gamma(s+1) = s\Gamma(s)$. The classical identites

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \quad \prod_{r=0}^{m-1}\Gamma(s+r/m) = (2\pi)^{\frac{1}{2}(m-1)}m^{\frac{1}{2}-ms}\Gamma(ms)$$

yield that $\widetilde{\mathbf{\Gamma}}$ kills T so that we have a $\mathbb{Q}$-linear map

$$\mathbf{\Gamma} : S/T \longrightarrow \mathbb{C}^\times/\overline{\mathbb{Q}}^\times.$$

2.2. **Lemma of Koblitz-Ogus.** Let F be an abelian extension of $\mathbb{Q}$. Put $G_F := \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/F)$. We denote by $(S/T)^{G_F}$ the fixed part by G_F. We then consider the $\mathbb{Q}$-linear map

$$\begin{aligned}\theta_F \colon (S/T)^{G_F} &\longrightarrow \mathbb{Q}[\mathrm{Gal}(F/\mathbb{Q})]\\ \xi &\longmapsto \sum_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})}\{\sigma\xi\}\,\sigma^{-1}.\end{aligned}$$

This is a $\mathrm{Gal}(F/\mathbb{Q})$-equivariant map.

Lemma 2.1 (Koblitz-Ogus)**.** *The map θ_F is injective and its image is generated by*

$$T_F := \sum_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})}\sigma, \quad \tau - \overline{\tau} \quad (\tau \in \mathrm{Gal}(F/\mathbb{Q})) \tag{1}$$

where $\overline{\tau}$ denotes the composition with complex conjugation In other words, θ_F induces a bijection

$$(S/T)^{G_F} \simeq \mathbb{Q}T_F \oplus \mathbb{Q}[\mathrm{Gal}(F/\mathbb{Q})]^-$$

where $\mathbb{Q}[\mathrm{Gal}(F/\mathbb{Q})]^-$ denotes the (-1)-eigenspace for the action of complex conjugation

We shall give a self-contained proof of Lemma 2.1 below.

Remark 2.2.

2.3. **The Gross-Deligne conjecture.**

Conjecture 2.3 (Gross-Deligne)**.** *Let $H = ((H_{\mathrm{dR}}, F^\bullet), H_B, \iota)$ be a Hodge–de Rham structure over $\overline{\mathbb{Q}}$ of geometric origin. Suppose that H has maximal multiplication by an abelian number field F, and fix an embedding $i \colon F \hookrightarrow \mathbb{C}$. For each embedding $\chi \colon F \hookrightarrow \mathbb{C}$, let $(p_\chi, q_\chi) \in \mathbb{Z}^2$ denote the unique couple of integers such that $H^\chi_{\mathrm{dR}} \subset H^{p_\chi,q_\chi}$. Let $\wp_i \in (S/T)^{G_F}$ be the unique element satisfying $\theta_K(\wp_i) = \sum_\sigma p_{i\sigma}\sigma^{-1}$ by Lemma 2.1. Then the equality*

$$\mathrm{per}(H^{i\sigma}) = \mathbf{\Gamma}(\wp_i^\sigma)$$

holds in $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ for all $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$.

Note that $\sum_\sigma p_{i\sigma}\sigma^{-1} \in \mathbb{Q}T_F \oplus \mathbb{Q}[\mathrm{Gal}(F/\mathbb{Q})]^-$ since $p_\chi + p_{\bar{\chi}}$ is a constant function of χ by the Hodge symmetry.

Remark 2.4. *The Gross-Deligne conjecture does not depend on the choice of the embedding* $i\colon F \hookrightarrow \mathbb{C}$. *Indeed, if* i *is replaced by* $i' = i\sigma_0$, *with* $\sigma_0 \in \mathrm{Gal}(F/\mathbb{Q})$, *then*

$$\theta_K(\wp_{i'}) = \sum_\sigma p_{i'\sigma}\sigma^{-1} = \sigma_0 \sum_\sigma p_{i\sigma}\sigma^{-1} = \sigma_0(\theta_K(\wp_i)) = \theta_K(\wp_i^{\sigma_0}).$$

By Lemma 2.1, this implies that $\wp_{i'} = \wp_i^{\sigma_0}$ *and hence* $\mathrm{per}(H^{i\sigma}) = \mathbf{\Gamma}(\wp_i^\sigma)$ *for all* $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ *if and only if* $\mathrm{per}(H^{i'\sigma}) = \mathbf{\Gamma}(\wp_{i'}^\sigma)$ *for all* $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$.

For later use, we formulate the period conjecture for F not necessarily an abelian field.

Conjecture 2.5 (A variant of the Gross-Deligne conjecture)**.** *Let the notation be as before, but* F *an arbitrary number field. Put* $F_0 = F \cap \mathbb{Q}(\mu_\infty)$. *Assume that* (p_χ, q_χ) *depends only on* $\chi_0 := \chi|_{F_0}$. *Then* $\mathrm{per}(H^\chi)$ *depends only on* χ_0 *and, letting* $\wp_i \in (S/T)^{G_{F_0}}$ *denote the unique element satisfying* $\theta_{F_0}(\wp_i) = \sum_\sigma p_{i\sigma}\sigma^{-1}$, *the equality*

$$\mathrm{per}(H^{i\sigma}) = \mathbf{\Gamma}(\wp_i^\sigma)$$

holds in $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ *for all* $\sigma \in \mathrm{Gal}(F_0/\mathbb{Q})$.

2.4. **Proof of Lemma 2.1.** Suppose $F \subseteq \mathbb{Q}(\mu_m)$. Write $G_m := \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}(\mu_m))$ and $G_K := \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/F)$. Then $G = G_1 \supset G_F \supseteq G_m$. There is a commutative diagram

$$\begin{array}{ccc}
(S/T)^{G_m} & \xrightarrow{\theta_m} & \mathbb{Q}[G/G_m] \\
\uparrow & & \uparrow \\
(S/T)^{G_F} & \xrightarrow{\theta_K} & \mathbb{Q}[G/G_F]
\end{array}$$

where $\theta_m = \theta_{\mathbb{Q}(\mu_m)}$, the left vertical arrow is the natural inclusion and the right vertical arrow is given by

$$\sigma \longmapsto \sum_{\tau \in G_K/G_m} \tau\tilde{\sigma}, \quad (\tilde{\sigma} \in G \text{ are liftings}).$$

Since the G_F/G_m-fixed part of the arrow θ_m coincides with θ_K, it is enough to prove Lemma 2.1 in the case where $F = \mathbb{Q}(\mu_m)$ for $m \geq 3$.

2.4.1. *Surjectivity of* θ_m. For $k \in \hat{\mathbb{Z}}^\times$, let $\sigma_k \in G$ denote the automorphism given by $\sigma_k(\zeta) = \zeta^k$, $\zeta \in \mu_\infty$. We want to show

$$\theta_m : \mathbb{C} \otimes_\mathbb{Q} (S/T)^{G_m} \longrightarrow \mathbb{C}T_m \oplus \mathbb{C}[G/G_m]^-, \quad T_m := \sum_{k \in (\mathbb{Z}/m\mathbb{Z})^\times} \sigma_k \tag{2}$$

is surjective. The surjectivity onto the component $\mathbb{C}T_m$ is immediate as

$$\theta_m \left\langle \frac{1}{2} \right\rangle = \sum_k \frac{1}{2}\sigma_k^{-1} = \frac{\varphi(m)}{2} T_m. \tag{3}$$

Note

$$\mathbb{C}[G/G_m]^- = \bigoplus_{\chi:\text{odd}} \mathbb{C}[G/G_m]^\chi,$$

where $\chi : G/G_m \cong (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ is a character and "odd" means $\chi(\sigma_{-1}) = -1$. We want to show that $\mathbb{C}[G]^\chi$ belongs to the image of (2). Note that each $\mathbb{C}[G]^\chi$ is one-dimensional.

We associate to χ a Dirichlet character $\chi : \mathbb{Z} \to \mathbb{C}$ in the customary way. Namely letting f_χ be the conductor of χ, define $\chi(k) = \chi(\sigma_k)$ if $(k, f_\chi) = 1$ and $= 0$ if $(k, f_\chi) \neq 1$. Put

$$e_\chi := \sum_{i=1}^{m-1} \chi^{-1}(i)\langle \frac{i}{m} \rangle = \sum_{i=1}^{m-1} \overline{\chi}(i)\langle \frac{i}{m} \rangle.$$

Since

$$\sigma_k(e_\chi) = \sum_{i=1}^{m-1} \chi^{-1}(i)\langle \frac{ki}{m} \rangle = \sum_{i=1}^{m-1} \chi^{-1}(ik^{-1})\langle \frac{i}{m} \rangle = \chi(k)e_\chi.$$

one has $\theta_m(e_\chi) \in \mathbb{C}[G]^\chi$. Thus if one can show that $\theta_m(e_\chi) \neq 0$, then this finishes the proof. Since

$$\theta(e_\chi) = \sum_{i=1}^{m-1} \sum_{k \in (\mathbb{Z}/m\mathbb{Z})^\times} \chi^{-1}(i) \left\{ \frac{ki}{m} \right\} \sigma_k^{-1}$$

it is enough to show nonvanishing of

$$\sum_{i=1}^{m-1} \chi^{-1}(i) \left\{ \frac{i}{m} \right\} = \frac{1}{m} \sum_{i=0}^{m-1} i\overline{\chi}(i) = \frac{1}{m} \sum_{j=0}^{m/f_\chi - 1} \sum_{i=0}^{f_\chi - 1} (i + jf_\chi)\overline{\chi}(i) = \frac{1}{f_\chi} \sum_{i=0}^{f_\chi - 1} i\overline{\chi}(i).$$

However, as is well-known, the last term is the special value of Dirichlet L-function for the odd character χ

$$L(1, \chi) = \pi\sqrt{-1}\frac{\tau(\chi)}{f_\chi} \sum_{i=0}^{f_\chi - 1} i\overline{\chi}(i)$$

and hence it is nonzero.

2.4.2. *Injectivity of θ_m.* Put

$$S_m := \bigoplus_{\alpha \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}, \text{ or } \alpha = 1/2} \mathbb{Q}\langle \alpha \rangle \subset S^{G_m}, \quad \overline{S}_m := \mathrm{Image}(S_m) \subset (S/T)^{G_m}.$$

The proof in §2.4.1 shows that $\overline{S}_m$ is onto $\mathbb{Q}T_m \oplus \mathbb{Q}[G/G_m]^-$. Therefore it is enough to show

$$\overline{S}_m = (S/T)^{G_m} \tag{4}$$

and

$$\dim_{\mathbb{Q}} \overline{S}_m \leq \frac{1}{2}\varphi(m) + 1 \tag{5}$$

where we define

$$\varphi(m) := \begin{cases} \sharp(\mathbb{Z}/m\mathbb{Z})^\times & m > 2 \\ 0 & m = 1, 2 \end{cases}$$

We first show (4). For $m|n$, let $I_{n,m} := \ker[(\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/m\mathbb{Z})^\times]$ and define a $\mathbb{Q}$-linear map

$$\mathrm{Tr}_{n,m} := \sum_{k \in I_{n,m}} \sigma_k : S^{G_n} \longrightarrow S^{G_m}.$$

Then $(S/T)^{G_m}$ is generated by the set

$$\left\{ \mathrm{Tr}_{n,m} \left\langle \frac{i}{n} \right\rangle \right\}_{n,i}$$

where n, i run over the integers such that $m|n$ and $(n,i)=1$. Hence it is enough to show that they belongs to $\overline{S}_m$. Since $\mathrm{Tr}_{n',n} \circ \mathrm{Tr}_{n,m} = \mathrm{Tr}_{n',m}$, one can reduce it to the case $n = pm$ with p a prime number.

Case ($p \not| m$). Let a be an integer such that $1 \le a \le p-1$ and $am \equiv -1 \bmod p$. Then $I_{mp,m}$ is the set of the elements

$$1 + km, \quad (0 \le k \le p-1,\ k \ne a).$$

Hence

$$\begin{aligned} \mathrm{Tr}_{pm,m} \left\langle \frac{i}{pm} \right\rangle &= \sum_{k \ne a} \left\langle \frac{i(1+km)}{pm} \right\rangle \\ &= \sum_{k=0}^{p-1} \left\langle \frac{i}{pm} + \frac{ik}{p} \right\rangle - \left\langle \frac{i(1+am)}{pm} \right\rangle \\ &\equiv \left\langle \frac{i}{m} \right\rangle + \frac{p-1}{2} \left\langle \frac{1}{2} \right\rangle - \left\langle \frac{i(1+am)}{pm} \right\rangle \mod T \end{aligned}$$

and the last term belongs to S_m as $p|(1+am)$.

Case ($p|m$). In this case $I_{mp,m}$ is the set of the elements

$$1 + km, \quad (0 \le k \le p-1).$$

Hence

$$\begin{aligned} \mathrm{Tr}_{pm,m} \left\langle \frac{i}{pm} \right\rangle &= \sum_{k=0}^{p-1} \left\langle \frac{i}{pm} + \frac{ik}{p} \right\rangle \\ &\equiv \left\langle \frac{i}{m} \right\rangle + \frac{p-1}{2} \left\langle \frac{1}{2} \right\rangle \mod T \end{aligned}$$

and the last term belongs to S_m.

Next we show (5). Let $\widetilde{S}_m$ be defined by

$$0 \longrightarrow \mathbb{Q}\langle \frac{1}{2} \rangle \longrightarrow \overline{S}_m \longrightarrow \widetilde{S}_m \longrightarrow 0$$

Here the injectivity of the left arrow follows from the fact that $\theta_m\langle \frac{1}{2} \rangle \ne 0$, cf. (3). Then (5) is equivalent to the following

$$\dim_{\mathbb{Q}} \widetilde{S}_m \le \frac{1}{2}\varphi(m). \tag{6}$$

Let $Q_m \subset S_m$ be the sub $\mathbb{Q}$-module generated by

$$\langle 0 \rangle, \quad \left\langle \frac{1}{2} \right\rangle, \quad \sum_{r=0}^{d-1} \left\langle \frac{i}{m} + \frac{r}{d} \right\rangle - \left\langle \frac{di}{m} \right\rangle, \quad (d|m,\ 1 \le i \le m-1).$$

Put $M := S_m/Q_m$ on which $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ acts. Then one easily sees that the natural map

$$M^- \longrightarrow \widetilde{S}_m$$

is surjective where M^- denotes the part on which the complex conjugation $\sigma_{-1} \in \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ acts by multiplication by -1. Hence it is enough to show

$$\dim_{\mathbb{Q}} M^- \leq \frac{1}{2}\varphi(m). \tag{7}$$

Define sub $\mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})]$-modules

$$M^{\leq d} := \mathrm{Image}\left[\bigoplus_{i=0}^{d-1} \mathbb{Q}\langle \frac{i}{d}\rangle \longrightarrow M\right] \supset M^{<d} := \sum_{e|d, e\neq d} M^e$$

for $d|m$. Obviously $M^{\leq e} \subset M^{\leq d}$ if $e|d$, and $M^{\leq 2} = 0$, $M = M^{\leq m}$. The quotient $M^{\leq d}/M^{<d}$ is a $\mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})]$-module generated by $\langle \frac{1}{d}\rangle$. Moreover it is annihilated by elements

$$\mathrm{Tr}_{d,e} := \sum_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q}(\mu_e))} \sigma, \quad (e|d,\ e\neq d).$$

Hence there is the surjection

$$L^d := \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})]/(\mathrm{Tr}_{d,e})_{e|d,e\neq d} \longrightarrow M^{\leq d}/M^{<d}, \quad \alpha \longmapsto \alpha\langle \frac{1}{d}\rangle$$

of $\mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})]$-module. Hence to show (7), it is enough to show

$$\sum_{d|m, d\neq 1} \dim_{\mathbb{Q}}(L^d)^- = \sum_{d|m} \dim_{\mathbb{Q}}(L^d)^- \leq \frac{1}{2}\varphi(m) \tag{8}$$

(note $(L^1)^- = 0$). Let $d = \prod_i p_i^{r_i}$ be the prime decomposition. Then there is a canonical isomorphism

$$L^d \cong \bigotimes_i L^{p_i^{r_i}}$$

of $\mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})]$-algebra where the tensor product is taken over $\mathbb{Q}$ and $\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$ acts on the right hand side diagonally. One can easily sees

$$L^{p^r} \cong \begin{cases} \mathbb{Q}[x]/\left(\frac{x^{p^{n-1}(p-1)}-1}{x^{p^{n-2}(p-1)}-1}\right) & n\geq 2 \\ \mathbb{Q}[x]/\left(\frac{x^{p-1}-1}{x-1}\right) & n = 1 \end{cases} \qquad (p\geq 3),$$

$$L^1 = L^2 = \mathbb{Q}, \quad L^4 \cong \mathbb{Q}[\sigma_{-1}]/(\sigma_{-1}+1),$$

$$L^{2^n} \cong \mathbb{Q}[\sigma_{-1}, y]/(\sigma_{-1}^2 - 1, y^{2^{n-3}}+1), \quad n\geq 3,$$

where σ_{-1} is the complex conjugation, x corresponds to a cyclic generator of $\mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$ and y corresponds to a cyclic generator of $\ker[\mathrm{Gal}(\mathbb{Q}(\mu_{2^n})/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\mu_4)/\mathbb{Q})] \cong$

$\mathbb{Z}/2^{n-2}\mathbb{Z}$. We thus have, if $m = \prod_i p_i^{n_i}$,

$$\begin{aligned}\prod_{d|m} L^d &= \prod_{d|m}\left(\bigotimes_i L^{p_i^{r_i}}\right)\\ &= \bigotimes_i \left(\prod_{k=0}^{n_i} L^{p_i^k}\right)\\ &= \bigotimes_i \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_{p_i^{n_i}})/\mathbb{Q})]\\ &= \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})].\end{aligned}$$

Hence

$$\sum_{d|m}\dim(L^d)^- = \dim\left(\mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})]^-\right) = \frac{1}{2}\varphi(m)$$

as desired. This completes the proof of (5).

3. The Saito-Terasoma theorem

Let U be a smooth quasi-projective variety, and X a smooth projective variety containing U as the complement of a simple normal crossings divisor $D = X\setminus U$, with everything defined over $\overline{\mathbb{Q}}$.

Let $\mathscr{M} = ((M_{dR},\nabla), M_B, \iota)$ be a connection with regular singularities. We define

$$\begin{aligned}\det \mathit{\Gamma}(U, M_B) &= \bigotimes_i \det H^i(U, M_B)^{\otimes(-1)^i},\\ \det \mathit{\Gamma}(U, M_{\mathrm{dR}}) &= \bigotimes_i \det \mathbb{H}^i(U, DR(M_{\mathrm{dR}},\nabla))^{\otimes(-1)^i}\end{aligned}$$

where the tensor product is taken over $\overline{\mathbb{Q}}$. There is the comparison isomorphism (also denoted by ι) and then we have a rank one Betti–de Rham structure

$$\det R\mathit{\Gamma}(U,\mathscr{M}) := (\det \mathit{\Gamma}(U, M_B), \det \mathit{\Gamma}(U, M_{\mathrm{dR}}), \iota)$$

We denote the period simply by $\mathrm{per} R\mathit{\Gamma}(U,\mathscr{M})$.

The period of the unit object $\mathrm{per}R\mathit{\Gamma}(U,\mathbb{Q})$ is defined in the same way. However, since the Hodge–de Rham structure $\det R\mathit{\Gamma}(U,\mathbb{Q})$ is a Tate twist $\mathbb{Q}(-r_U)$, we simply have $\mathrm{per}R\mathit{\Gamma}(U,\mathbb{Q}) = (2\pi i)^{-r_U}$ (Example 1.1). The value of r_U is computed in [Fre13, Prop. 2.2 and Lemma 3.2]:

$$r_U = \sum_{p=0}^{n}(-1)^p p\chi(X, \Omega_X^p(\log D)).$$

The *gamma factor* $\Gamma(\nabla : M_{\mathrm{dR}})$ is defined in the following way. Since (M_{dR},∇) has regular singulaities, one has the log connection

$$\nabla : \overline{M}_{\mathrm{dR}} \longrightarrow \Omega^1_X(\log D)\otimes \overline{M}_{\mathrm{dR}},$$

and it gives rise to the residue map $\mathrm{Res}_{D_i}(\nabla)$ on $\mathscr{O}_{D_i}\otimes\overline{M}_{\mathrm{dR}}$ along an irreducible component D_i of D. Let $\alpha_{D_i,j}$ $(j = 1,\ldots,\mathrm{rank}\mathscr{M})$ be the eigenvalues of $\mathrm{Res}_{D_i}(\nabla)$. Put $D_i^\circ = D_i \setminus \bigcup_{k\neq i} D_k$.

Then the gamma factor is defined as

$$\Gamma(\nabla : M_{\mathrm{dR}}) := \prod_{i,j} \Gamma(\alpha_{D_i,j})^{\chi^{\mathrm{top}}(D_i^\circ)} \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$$

where χ^{top} denotes the topological Euler characteristic.

Saito introduced the relative canonical cycle $c_{X,U}$ which is a zero-cycle of degree $\chi^{\mathrm{top}}(U)$ on U modulo some equivalence relation which is finer than the rational equivalence. We refer to [Sai93] §1 or [Fre13] 1.4 for the detailed construction. Let $c_{X,U}^* \det \mathscr{M}$ denote the Betti–de Rham structure consisting of restrictions $c_{X,U}^* \det_{\overline{\mathbb{Q}}} M_B$ and $c_{X,U}^* \det_{\overline{\mathbb{Q}}} M_{\mathrm{dR}}$.

The main theorem of [ST97] is stated as follows.

Theorem 3.1 (Saito–Terasoma)**.** *The following equality holds in* $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$:

$$\mathrm{per} R\Gamma(U,\mathscr{M}) = \mathrm{per} R\Gamma(U,\mathbb{Q})^{\mathrm{rank}\mathscr{M}} \cdot \Gamma(-\nabla : M_{\mathrm{dR}}) \cdot \mathrm{per}(c_{X,U}^* \det \mathscr{M}). \tag{9}$$

3.1. We shall apply the theorem of Saito-Terasoma to the following case.

Theorem 3.2. *Let* $\mathscr{M} = (M_B, M_{\mathrm{dR}}, \iota, F^\bullet, \nabla)$ *be a polarizable variation of Hodge–de Rham structure on* U *endowed with multiplication by a number field* K*. Put*

$$H(\mathscr{M}) := \det_K R\Gamma(U,\mathscr{M}) \otimes_K \det_K(\mathscr{M}_x)^{\otimes_K - \chi^{\mathrm{top}}(U)} \otimes_{\mathbb{Q}} \det R\Gamma(U,\mathbb{Q})^{\otimes - \mathrm{rank}\mathscr{M}}$$

a Hodge–de Rham structure with maximal multiplication by K *for* $x \in U(\overline{\mathbb{Q}})$*. Then for* $\chi : K \hookrightarrow \overline{\mathbb{Q}}$ *one has*

$$\mathrm{per} H(\mathscr{M})^\chi = \Gamma(-\nabla : M_{\mathrm{dR}}^\chi) = \prod_{i,j} \Gamma(-\alpha_{D_i,j}^\chi)^{\chi^{\mathrm{top}}(D_i^\circ)}$$

where $\alpha_{D_i,j}^\chi$ $(j = 1, \ldots, \mathrm{rank}\mathscr{M}^\chi)$ *are eigenvalues of* $\mathrm{Res}_{D_i}(\nabla)$ *on the* χ*-part* M_{dR}^χ.

Proof. Note

$$H(\mathscr{M})^\chi = \det_{\overline{\mathbb{Q}}} R\Gamma(U,\mathscr{M}^\chi) \otimes_{\overline{\mathbb{Q}}} \det_{\overline{\mathbb{Q}}}(\mathscr{M}_x^\chi)^{\otimes - \chi^{\mathrm{top}}(U)} \otimes_{\mathbb{Q}} \det R\Gamma(U,\mathbb{Q})^{\otimes - \mathrm{rank}\mathscr{M}}.$$

Apply Theorem 3.1 to $\mathscr{M}^\chi$. It is enough to check

$$\mathrm{per}(c_{X,U}^* \det \mathscr{M}^\chi) = \mathrm{per}(\det_{\overline{\mathbb{Q}}} \mathscr{M}_x^\chi)^{\chi^{\mathrm{top}}(U)}.$$

Since a variation $\det \mathscr{M} = (\det_{\mathbb{Q}} M_B, \det_{\overline{\mathbb{Q}}} M_{\mathrm{dR}}, \iota, F^\bullet, \nabla)$ has maximal multiplication by K and satisfies the assumption in Proposition 1.3, the period $\mathrm{per}(\det_{\overline{\mathbb{Q}}} \mathscr{M}_x^\chi)$ does not depend on $x \in U(\overline{\mathbb{Q}})$. Then the above follows from the fact that the degree of $c_{X,U}$ is $\chi^{\mathrm{top}}(U)$. □

4. Riemann-Roch

Let X be a smooth projective complex variety and D a simple normal crossings divisor on X. Let us denote by U the complement of D in X and by $j\colon U \hookrightarrow X$ the inclusion. Let (H,∇) be a vector bundle together with an integrable connection $\nabla\colon H \to H \otimes_{\mathcal{O}_U} \Omega_U^1$ with regular singularities. Then the cohomology groups

$$H_{dR}^j(U,H) = \mathbb{H}^j(X, H \otimes_{\mathcal{O}_U} \Omega_U^\bullet)$$

are equipped with a mixed Hodge structure. According to [Sai90, Prop. 3.11], the Hodge filtration can be described as follows. Recall that Deligne's canonical extension is a locally free $\mathcal{O}_X$-module $\overline{H}$ together with a logarithmic integrable connection $\overline{\nabla}\colon \overline{H} \to \overline{H} \otimes_{\mathcal{O}_X} \Omega_X^1(\log D)$ such that $(\overline{H},\overline{\nabla})_{|U} \simeq (H,\nabla)$ and that the eigenvalues λ of the residues of $\overline{\nabla}$ along the irreducible components of D satisfy $\mathrm{Re}(\lambda) \in [0,1)$. Define $F^p\overline{H} = \overline{H} \cap j_* F^p$. It follows from the nilpotent orbit theorem that the sheaves $F^p\overline{H}$ are locally free.

Lemma 4.1. *One has*

$$F^p H^j_{dR}(U, H) = \mathbb{H}^i(X, F^{p-\bullet}\overline{H} \otimes \Omega^\bullet_X(\log D)),$$

hence

$$\mathrm{Gr}^p_F H^j(U, \mathscr{M}) = \mathbb{H}^j(X, \mathrm{Gr}^{p-\bullet}_F \mathscr{M} \otimes \Omega^\bullet_X(\log D)).$$

4.1.

Theorem 4.2. *Let $\mathscr{M}$ be a variation of Hodge-de Rham structures with multiplication by a number field F (not necessarily maximal). Then:*

$$p_\chi(\det_F R\Gamma(U, \mathscr{M})) = \chi(U) p_\chi(\det_F \mathscr{M}_t) + \mathrm{rk}(\mathscr{M}^\chi) r_U + \sum_{i,j} \chi(D_i^\circ)\{\alpha^\chi_{D_i,j}\}$$

Proof.

$$\begin{aligned} p_\chi(\det_F R\Gamma(U, \mathscr{M})) &= \sum_{j\geq 0} (-1)^j p_\chi(\det_F H^j(U, \mathscr{M})) \\ &= \sum_{j\geq 0} (-1)^j \sum_{p\geq 0} p \dim \mathrm{Gr}^p_F H^j(U, \mathscr{M}^\chi) \\ &= \sum_{j\geq 0} (-1)^j \sum_{p\geq 0} p \dim \mathbb{H}^j(U, \mathrm{Gr}^{p-\bullet}_F \mathscr{M}^\chi \otimes \Omega^\bullet_X(\log D)) \\ &= \sum_{j\geq 0} (-1)^j \sum_{p\geq 0} p \sum_{q\geq 0} (-1)^q h^j(U, \mathrm{Gr}^{p-q}_F \mathscr{M}^\chi \otimes \Omega^q_X(\log D)) \\ &= \sum_{i,j,k} (-1)^{i+j}(p+j) h^i(\mathrm{Gr}^p_F \mathscr{M}^\chi \otimes \Omega^j_X(\log D)) \end{aligned}$$

We now compute each of the terms in the last sum:

Proposition 4.3. *Let t be any closed point in U. Then:*

$$\sum_{q\geq 0} (-1)^q \sum_{p\geq 0} p\chi(\mathrm{Gr}^p_F \mathscr{M}^\chi \otimes \Omega^q_X(\log D)) = \chi(U) \cdot p_\chi(\det_F \mathscr{M}_t) \tag{10}$$

Proof. By the Hirzebruch-Riemann-Roch theorem,

$$\begin{aligned} \sum_{q\geq 0} (-1)^q \chi(\mathrm{Gr}^p_F \mathscr{M}^\chi \otimes \Omega^q_X(\log D)) &= \int_X \mathrm{ch}(\mathrm{Gr}^p_F \mathscr{M}^\chi) \sum_{q\geq 0} (-1)^q \mathrm{ch}(\Lambda^q \Omega^1_X(\log D)) \mathrm{td}(TX) \\ &= (-1)^n \int_X \mathrm{ch}(\mathrm{Gr}^p_F \mathscr{M}^\chi) c_n(\Omega^1_X(\log D)) \\ &= \mathrm{rk}(\mathrm{Gr}^p_F \mathscr{M}^\chi) \cdot \chi(U). \end{aligned}$$

Above the first equality follows from the identity

$$\sum_{q\geq 0} (-1)^q \mathrm{ch}(\Lambda^q \mathscr{F}) = (-1)^r c_r(\mathscr{F}) + \text{higher order}$$

for a rank r vector bundle $\mathscr{F}$, and the second uses $\chi(U) = (-1)^n \int_X c_n(\Omega^1_X(\log D))$. Hence the left hand side of (10) is equal to

$$\sum_{p\geq 0} p\mathrm{rk}(\mathrm{Gr}^F_p \mathscr{M}^\chi)\chi(U) = \chi(U)p_\chi(\det{}_F \mathscr{M}_t)$$

for any closed point t of U. □

Proposition 4.4. *The following equality holds:*

$$\sum_{p\geq 0}(-1)^p p\chi(U, \mathscr{M}^\chi \otimes \Omega^p_X(\log D)) = \mathrm{rk}(\mathscr{M}^\chi)r_U + \sum_{i,j}\chi(D_i^\circ)\{\alpha^\chi_{D_i,j}\}$$

Proof. By the Hirzebruch-Riemann-Roch theorem,

$$\begin{aligned}\chi(U, \mathscr{M}^\chi \otimes \Omega^p_X(\log D)) &- \mathrm{rk}(\mathscr{M}^\chi)\cdot\chi(U, \Omega^p_X(\log D)) \\ &= \int_X [\mathrm{ch}(\mathscr{M}^\chi) - \mathrm{rk}(\mathscr{M}^\chi)]\mathrm{ch}(\Omega^p_X(\log D))\mathrm{Td}(TX),\end{aligned}$$

so we are reduced to compute

$$\begin{aligned}\int_X [\mathrm{ch}(\mathscr{M}^\chi) - \mathrm{rk}(\mathscr{M}^\chi)]\cdot[\sum_{p\geq 0}(-1)^p p\mathrm{ch}(\Lambda^p\Omega^1_X(\log D)))]\cdot\mathrm{Td}(TX) \\ = (-1)^n\int_X c_1(\mathscr{M}^\chi)\cdot c_{n-1}(\Omega^1_X(\log D)) \\ = \sum_{i\in I}\sum_{j=1}^{\mathrm{rk}(\mathscr{M}^\chi)}\chi(D_i^\circ)\{\alpha^\chi_{D_i,j}\}.\end{aligned}$$

The first equality above follows from the identity

$$\sum_{q\geq 0}(-1)^q q\mathrm{ch}(\Lambda^q\mathscr{F}) = (-1)^r c_{r-1}(\mathscr{F}) + \text{higher order}$$

and the second from the combination of the identity [Fre13, Lemma 3.4]

$$\chi(D_i^\circ) = (-1)^{n-1}\deg c_{n-1}(\Omega^1_X(\log D))_{|D_i}$$

and the fact that the first Chern class of a vector bundle equipped with a connection with regular singularities can be expressed in terms of the residues [EV87, Appendix B]

$$c_1(\mathscr{M}^\chi) = -\sum_i \mathrm{Tr}(\mathrm{Res}_{D_i}\nabla)[D_i] = -\sum_i\sum_j\{\alpha^\chi_{D_i,j}\}$$

This completes the proof. □

□

5. Main Theorem and its applications

Theorem 5.1 (Main Theorem). *Let $\mathscr{M} = (M_B, M_{\mathrm{dR}}, \iota, F^\bullet, \nabla)$ be a polarizable variation of Hodge–de Rham structure on a smooth variety Y endowed with multiplication by a number field F. Then*

(11) $\det_F R\Gamma(Y, \mathscr{M}) \otimes (\det_F \mathscr{M}_t)^{\otimes -\chi(Y)}$

satisfies the period conjecture of Gross-Deligne 2.5.

Proof. By Lemma 5.2 below (11) satisfies the assumption in Conjecture 2.5. Then the assertion is immediate from Theorems 3.1 and 4.2. □

Lemma 5.2. *Let $F_0 = F \cap \mathbb{Q}(\mu_\infty)$. Then, for $\chi\colon F \hookrightarrow \overline{\mathbb{Q}}$*

$$p_\chi(\det_F R\Gamma(Y, \mathscr{M}) \otimes (\det_F \mathscr{M}_t)^{\otimes -\chi(Y)}) = \mathrm{rk}(\mathscr{M}) r_Y + \sum_{i,j} \chi(D_i^\circ)\{\alpha^\chi_{D_i,j}\} \tag{12}$$

depends only on $\chi|_{F_0}$ (the equality follows from Theorem 4.2). More precisely, let $\tau : \mathbb{C} \to \mathbb{C}$ be any homomorphism of $\mathbb{Q}$-algebra and let $\chi' = \tau\chi$. Let $s \in \hat{\mathbb{Z}}^\times$ be the associated element to $\tau|\mathbb{Q}(\mu_\infty)$ via the cyclotomic character $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\times$. Then the collection $\{s\alpha^\chi_{D_i,j} \in \mathbb{Q}/\mathbb{Z}\}_j$ coincides with $\{\alpha^{\chi'}_{D_i,j} \in \mathbb{Q}/\mathbb{Z}\}_j$.

Proof. It is enough to see that $\sum_j \{\alpha^\chi_{D_i,j}\}$ depends only on $\chi|_{F_0}$. Recall that $\alpha^\chi_{D_i,j}$ are defined to be the eigenvalues of $\mathrm{Res}_{D_i}(\nabla)$ on the χ-part $\mathscr{M}^\chi_{\mathrm{dR}}$. Then, as is well-known, $\exp(2\pi\sqrt{-1}\alpha^\chi_{D_i,j})$ are the eigenvalues of the local monodromy around D_j. More precisely, let $\rho : \pi_1(Y, t) \to \mathrm{GL}(\mathscr{M}^\chi_t)$ be the monodromy representation. Let ρ_{D_i} be the restriction on $\mathbb{Z}(1)_{D_i} = 2\pi i\mathbb{Z} \subset \pi_1(Y)$ the subgroup generated by the local monodromy T_{D_i} around D_i. By the monodromy theorem its semisimplification $\rho^{ss}_{D_i}$ is a product of characters $\rho^{ss}_{D_i,j} \in \mathrm{Hom}(\mathbb{Z}(1), \mu_\infty)$ of finite orders. Under the canonical identification $\mathrm{Hom}(\mathbb{Z}(1), \mu_\infty) \cong \mathbb{Q}/\mathbb{Z}$, the collection $\{\rho^{ss}_{D_i,j}\}_j$ is associated to $\{\alpha^\chi_{D_i,j}\}_j$.

Letting $\chi' : F \hookrightarrow \mathbb{C}$ be another embedding, we compare two characteristic polynomials $\det(1 - zT_{D_i} : \mathscr{M}^\chi_t)$ and $\det(1 - zT_{D_i} : \mathscr{M}^{\chi'}_t)$. Let $\tau : \mathbb{C} \to \mathbb{C}$ satisfies $\chi' = \tau\chi$. Since

$$\mathscr{M}^\chi_t = \ker(\chi(\sigma) \otimes 1 - 1 \otimes \sigma : \mathbb{C} \otimes \mathscr{M}_t \to \mathbb{C} \otimes \mathscr{M}_t), \quad \exists \sigma \in F$$

by definition, one has $(\tau \otimes 1)(\mathscr{M}^\chi_t) \subset \mathscr{M}^{\chi'}_t$. This implies

$$\det(1 - zT_{D_i} : \mathscr{M}^\chi_t)^\tau = \det(1 - zT_{D_i} : \mathscr{M}^{\chi'}_t).$$

Hence the collection $\{\tau \exp(2\pi\sqrt{-1}\alpha^\chi_{D_i,j}) = \exp(2\pi\sqrt{-1}s\alpha^\chi_{D_i,j})\}_j$ coincides with the collection $\{\exp(2\pi\sqrt{-1}\alpha^{\chi'}_{D_i,j})\}_j$, namely $\{s\alpha^\chi_{D_i,j}\}_j = \{\alpha^{\chi'}_{D_i,j}\}_j$. □

5.1. Multiplication by automorphism of finite order. The following is an immediate corollary of Theorem 5.1.

Theorem 5.3. *Let $f : X \to Y$ be a smooth projective morphism over $\overline{\mathbb{Q}}$. Let $S \subset \prod_i \mathrm{End}(R^i f_* \mathbb{Q})$ be a finite dimensional semisimple commutative $\mathbb{Q}$-algebra. Let $e : S \to F$ be a projector onto a number field F. Then the period conjecture holds for*

$$\det_F eR\Gamma(X, \mathbb{Q}) \otimes (\det_F eR\Gamma(X_t, \mathbb{Q}))^{\otimes -\chi(Y)}$$

where $X_t = f^{-1}(t)$ denotes the general fiber.

Of particular interest is the case that S is generated by an auomorphims of finite order. Let W be a smooth projective variety over $\overline{\mathbb{Q}}$. Let $\sigma : W \to W$ be an automorphism of order $n \geq 2$. Let $f : W \to V = W/\langle\sigma\rangle$ be the quotient variety by σ. Let $Y \subset V$ be a Zariski open set in the smooth locus such that $X := f^{-1}(Y) \to Y$ is finite etale. Let $S \subset \prod_i \mathrm{End}(R^i f_* \mathbb{Q})$ be the $\mathbb{Q}$-algbera generated by σ. Let $e : S \to \mathbb{Q}(\mu_n)$ be a projector such that $e(\sigma)$ is a primitive n-th root of unity. Theorem 5.3 yields the period conjecture for $\det_F eR\Gamma(X, \mathbb{Q}) \otimes (\det_F eR\Gamma(X_t, \mathbb{Q}))^{\otimes -\chi(Y)}$. Since X_t is 0-dimensional, one can remove the term "$\det_F eR\Gamma(X_t, \mathbb{Q})$". On the other hand, one finds that $\det_F eR\Gamma(X, \mathbb{Q}) = \det_F eR\Gamma(W, \mathbb{Q})$ ([Fre13] 3.1). We thus have the following theorem.

Corollary 5.4 ([Fre13] Theorem A)**.** *Let $\sigma : W \to W$ be an automorphism of order $n \geq 2$. Then the period conjecture of Gross-Deligne for $\det_{\mathbb{Q}(\mu_n)} eR\Gamma(W, \mathbb{Q})$ holds.*

5.2. Removing $\det_F \mathscr{M}_t$. It is not obvious to remove the term $\det_F \mathscr{M}_t$ from the main theorem. Here we give sufficient conditions.

Theorem 5.5. *Assume that either of the following conditions holds.*

- **(a)** *There is a point t_0 such that $\det_F \mathscr{M}_{t_0}$ satisfies the period conjecture.*
- **(b)** *There is a non-constant map $h : C \to Y$ with C a smooth curve and a point $P \in \overline{C} \setminus C$ with $\overline{C}$ a compactification, such that the period conjecture holds for the nearby cohomology $\det_F \psi_P(h^* \mathscr{M})$ of Hodge–de Rham structure.*

Then the period conjecture holds for $\det_F \mathscr{M}_t$ and hence for $\det_F R\Gamma(Y, \mathscr{M})$.

Proof. The key is the fact that a variation of Hodge–de Rham structures with maximal multiplication has constant periods (Proposition 1.3). In particular the periods of $\det_F \mathscr{M}_t$ do not depend on $t \in Y(\overline{\mathbb{Q}})$, and they also agree with the periods of the nearby cohomology $\det_F \psi_P(h^* \mathscr{M})$. □

Corollary 5.6. *Let Y be a smooth variety and $f : X \to Y$ a family of abelian varieties endowed with mulptiplication by F. Assume that either of the following holds.*

1. *there is a point $t \in Y(\overline{\mathbb{Q}})$ such that $H^1(f^{-1}(t), \mathbb{Q})$ has a maximal multiplication by a number field $F' \supset F$.*
2. *f has a totally degeneration at a boundary point.*

Then the period conjecture holds for $\det_F R\Gamma(Y, R^i f_* \mathbb{Q})$ for any i.

Proof. In case of (1), the period conjecture for $H^1(f^{-1}(t), \mathbb{Q})$ is known by Shimura, Deligne and Anderson. In case of (2), the nearby cohomology $\det_F \psi_P(h^* \mathscr{M})$ turns out to be of mixed Tate type as a Hodge–de Rham structure. Now the assertion is straightforward from Theorem 5.5. □

5.3. Hypergeometric Fibrations. Even when one is able to remove the term $\det_F \mathscr{M}_t$ from the main theorem, it seems still very difficult to separate the contributions coming from different degrees and weights, which would yield the conjecture for all $\det_F \mathrm{Gr}_m^W H^j(Y, \mathscr{M})$. We end this section by showing a particular example, i.e. the period conjecture for the weight graded piece of cohomology of hypergeometric fibrations. Recall the definition from [AO15].

Definition 5.7. *Let F be a number field. A hypergeometric fibration with relative multiplication by F is a smooth projective morphism $f \colon X \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$, together with a relative multiplication on $R^1 f_* \mathbb{Q}$, such that the following two conditions hold:*

- *(i)* $\dim_F R^1 f_* \mathbb{Q} = 2$.
- *(ii) The local monodromy T_1 of $R^1 f_* \mathbb{Q}$ around $t = 1 \in \mathbb{P}^1$ is unipotent and*

$$2 \cdot \mathrm{rk}(T_1 - 1) = \dim_{\mathbb{Q}} H^1(X_t, \mathbb{Q}).$$

For $\chi : F \hookrightarrow \mathbb{C}$, we denote by $\alpha_{0,i}^{\chi}$ $(i = 1, 2)$ the eigenvalues of $\mathrm{Res}(\nabla)$ on $R^1 f_* \mathbb{C}^{\chi}$ at $t = 0$, and $\alpha_{\infty,i}^{\chi}$ at $t = \infty$.

Let g be a non-constant rational function on $\mathbb{P}^1$ such that there is no g_0 such that $g = g_0^n$, $n \geq 2$. Consider the cyclic covering $\pi : C \to \mathbb{P}^1$ of degree $l \geq 1$ obtained from the field extension $\overline{\mathbb{Q}}(t, g^{1/l}) / \overline{\mathbb{Q}}(t)$. Put a smooth motivic sheaf

$$\mathscr{M} := \pi_* \mathbb{Q} \otimes R^1 f_* \mathbb{Q}$$

on $Y = \mathbb{P}^1 \backslash \{0, 1, \infty, \mathrm{Supp}\ \mathrm{div}(g)\}$ on which $F[\mu_l]$ acts in a natural way. A pair of $k \in \mathbb{Z}/l\mathbb{Z}$ and $\chi : F \hookrightarrow \mathbb{C}$ determines a morphism of $\mathbb{Q}$-algebras $\varepsilon_k \otimes \chi : F[\mu_l] \to \mathbb{C}$ such that $(\varepsilon_k \otimes \chi)(\zeta_l) = \zeta_l^k$ and $(\varepsilon_k \otimes \chi)|_F = \chi$, and all morphisms are given in such a way.

The following is a vast generalization of [AO15], Theorem 4.1.

Theorem 5.8. *Let $e\colon F[\mu_l] \to E$ be a projector onto a number field E. Fix a complex embedding of E and let $\varepsilon_k \otimes \chi : F[\mu_l] \to \mathbb{C}$ factor through e. Put $r_0 := \mathrm{ord}_{t=0}(g)$ and $r_\infty := \mathrm{ord}_{t=\infty}(g)$. Assume $r_\infty k/l + \alpha^\chi_{\infty,i} \notin \mathbb{Z}$ for $i = 1, 2$ and that either of the following the conditions holds.*

(1) $r_0 k/l + \alpha^\chi_{0,i} \notin \mathbb{Z}$ for $i = 1, 2$,

(2) $\alpha^\chi_{0,1}$ or $\alpha^\chi_{0,2} \in \mathbb{Z}$.

Then the period conjecture for $\det_E W_2 H^1(Y, e\mathscr{M})$ holds.

Remark 5.9. *Note that $r_0 k/l + \alpha^\chi_{0,i}$ and $r_\infty k/l + \alpha^\chi_{\infty,i}$ are the eigenvalues of $\mathrm{Res}(\nabla)$ on $\mathscr{M}$. It follows from Lemma 5.2 that the conditions in Theorem 5.8 depend only on e but not on the choice of (ε_k, χ) factoring through e.*

Proof. Let $j : Y \hookrightarrow \mathbb{P}^1$ denote the inclusion. Then there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\mathbb{P}^1, j_*\mathscr{M}) & \longrightarrow & H^1(Y, \mathscr{M}) & \longrightarrow & H^0(\mathbb{P}^1, R^1 j_*\mathscr{M}) & \longrightarrow & 0. \\ & & \| & & & & & & \\ & & W_2 H^1(Y, \mathscr{M}) & & & & & & \end{array}$$

Since $H^0(Y, \mathscr{M}) = 0$ and $\mathscr{M}$ satisfies condition **(b)** in Theorem 5.5, one has the period conjecture for $\det_E H^1(Y, e\mathscr{M})$. Therefore it is enough to show that the period conjecture for the determinant of

$$(R^1 j_*\mathscr{M})_\alpha = \mathrm{Coker}[T_\alpha - 1 : \psi_\alpha \mathscr{M} \to \psi_\alpha \mathscr{M} \otimes \mathbb{Q}(-1)]$$

holds for each $\alpha \in \{0, 1, \infty, \mathrm{Supp}\ \mathrm{div}(f)\}$ where T_α is the local monodromy at $t = \alpha$, and $\psi_\alpha \mathscr{M} = \psi_\alpha \pi_* \mathbb{Q} \otimes \psi_\alpha R^1 f_* \mathbb{Q}$ is the nearby cohomology equipped with the limiting mixed Hodge–de Rham structure at $t = \alpha$.

In case $\alpha = \infty$, since $r_\infty k/l + \alpha^\chi_{\infty,i} \notin \mathbb{Z}$ this implies $(R^1 j_*\mathscr{M})_\infty = 0$.

In case $\alpha = 1$ the assertion follows from the fact that $\psi_1 \mathscr{M}_t$ is a mixed Tate Hodge–de Rham structure.

In case $\alpha \in \mathrm{Supp}\ \mathrm{div}(f) \setminus \{0, 1, \infty\}$, one finds

$$(R^1 j_*\mathscr{M})_\alpha = \mathrm{Coker}[T_\alpha - 1 : H^0(\pi^{-1}(t)) \to H^0(\pi^{-1}(t))] \otimes_{\mathbb{Q}} (R^1 f_* \mathbb{Q})_\alpha$$

and hence

$$\det_E e(R^1 j_*\mathscr{M})_\alpha = \det_E e[\mathrm{Coker}(T_\alpha - 1) \otimes_{\mathbb{Q}} \det_F (R^1 f_* \mathbb{Q})_\alpha].$$

Since $\det_F (R^1 f_* \mathbb{Q})_t$ is of Tate type, the period conjecture also holds.

There remains the case $\alpha = 0$. If condition (1) is satisfied, then $(R^1 j_*\mathscr{M})_0 = 0$ so there is nothing to prove. Suppose that condition (2) is satisfied. Let

$$\psi_0 R^1 f_* \mathbb{Q} = \psi_0^{\mathrm{unip}} \oplus \psi_0^{\neq \mathrm{unip}}$$

be the decomposition into the unipotent part and the non-unipotent part with respect to the local monodromy T_0 at $t = 0$. If both $\alpha^\chi_{0,i} \in \mathbb{Z}$, then T_0 is unipotent and non-trivial because $H^1(X_t)^\chi$ is an irreducible $F[\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})]$-module ([AO15] Lemma 3.8). This implies that $\psi_0^{\neq \mathrm{unip}} = 0$ and ψ_0^{unip} is of mixed Tate type. Hence the period conjecture for $\det_F e(R^1 j_* \mathscr{M})_0$ follows.

Suppose that $\alpha_{0,1}^{\chi} \in \mathbb{Z}$ and $\alpha_{0,2}^{\chi} \notin \mathbb{Z}$. In this case $\psi_0^{\neq \mathrm{unip}}$ and ψ_0^{unip} are one-dimensional over F. We first claim that both of $\psi_0^{\neq \mathrm{unip}}$ and ψ_0^{unip} (with multiplication by F) satisfy the period conjecture. Indeed, it is enough to see it only for ψ_0^{unip} because $\det_F R^1 f_* \mathbb{Q} = \psi_0^{\mathrm{unip}} \otimes_F \psi_0^{\neq \mathrm{unip}}$ is of Tate type. Note that both of $\alpha_{\infty,i}^{\chi}$ cannot be integers (if $\exists \alpha_{\infty,i}^{\chi} \in \mathbb{Z}$, then $H^1(X_t)^{\chi}$ is a reducible $F[\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})]$-module, which does not happen by [AO15] Lemma 3.8). This implies $\dim_F \mathrm{Coker}(T_\infty - 1) = 0$. Since $\dim_F \mathrm{Coker}(T_1 - 1) = 1$ and $\dim_F H^1(\mathbb{P}^1 \setminus \{0,1,\infty\}, R^1 f_* \mathbb{Q}) = 2$, one has

$$H^1(\mathbb{P}^1 \setminus \{0,1,\infty\}, R^1 f_* \mathbb{Q}) \cong \mathrm{Coker}(T_0 - 1) \oplus \mathrm{Coker}(T_1 - 1) = \psi_0^{\mathrm{unip}} \oplus \mathrm{Coker}(T_1 - 1).$$

Since $R^1 f_* \mathbb{Q}$ satisfies **(b)** in Theorem 5.5, $\det_F H^1(\mathbb{P}^1 \setminus \{0,1,\infty\}, R^1 f_* \mathbb{Q})$ satisfies the period conjecture. Since $\mathrm{Coker}(T_1 - 1)$ is of Tate type, the period conjecture for ψ_0^{unip} follows.

We now have the period conjecture for ψ_0^{unip} and $\psi_0^{\neq \mathrm{unip}}$, and hence for $e(\psi_0 \pi_* \mathbb{Q} \otimes \psi_0^{\mathrm{unip}})$ and $e(\psi_0 \pi_* \mathbb{Q} \otimes \psi_0^{\neq \mathrm{unip}})$ with multiplication by E. Since $e(\psi_0 \pi_* \mathbb{Q} \otimes \psi_0^{\mathrm{unip}})$ is one-dimensional over E,

$$e\mathrm{Coker}[T_0 - 1 : \psi_0 \pi_* \mathbb{Q} \otimes \psi_0^{\mathrm{unip}} \to \psi_0 \pi_* \mathbb{Q} \otimes \psi_0^{\mathrm{unip}} \otimes \mathbb{Q}(-1)] = \psi_0 \pi_* \mathbb{Q} \otimes \psi_0^{\mathrm{unip}} \otimes \mathbb{Q}(-1) \text{ or } 0,$$

and the same holds for $\psi_0^{\neq \mathrm{unip}}$. Hence

$$eR^1 j_* \mathscr{M} = \psi_0 \pi_* \mathbb{Q} \otimes \psi_0^{\mathrm{unip}} \otimes \mathbb{Q}(-1) \text{ or } \psi_0 \pi_* \mathbb{Q} \otimes \psi_0^{\neq \mathrm{unip}} \otimes \mathbb{Q}(-1) \text{ or } 0 \cong E \text{ or } 0$$

and this satisfies the period conjecture. This completes the proof in case $\alpha = 0$. □

6. Hodge-Tate decomposition

6.1. Infinite type of algebraic Hecke characters. Let F and E be number fields. We denote by $\mathbb{Z}[\mathrm{Hom}(F, \overline{E})]$ the free abelian group generated by the embeddings of F into a fixed algebraic closure $\overline{E}$ of E. If $\mathfrak{f}$ is an integral ideal of F, we denote by $I_{F,\mathfrak{f}}$ the multiplicative group of fractional ideals of F prime to $\mathfrak{f}$.

Definition 6.1. *An algebraic Hecke character with conductor dividing* $\mathfrak{f}$ *of infinite type* $T = \sum_\sigma n_\sigma \sigma \in \mathbb{Z}[\mathrm{Hom}(F, \overline{E})]$ *is a homomorphism* $\Phi : I_{F,\mathfrak{f}} \to E^\times$ *which satisfies*

$$\Phi((\alpha)) = \alpha^T = \prod \sigma(\alpha)^{n_\sigma} \in E$$

if $\alpha \in F^\times$ *is totally positive and* $\alpha \equiv 1$ *mod* $\mathfrak{f}$.

We refer the reader to [Sch88, Ch. 0].

Let $\mathbb{A}_F^\times = (\mathbb{R} \otimes_{\mathbb{Q}} F)^\times \times (\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} F)^\times$ be the group of ideles of F. Using the approximation theorem ([Neu99] Ch II, (3.4)), one can show that the algebraic Hecke character Φ induces a unique continuous homomorphism

$$\widetilde{\Phi} : \mathbb{A}_F^\times \to E^\times$$

satisfying the following conditions:

(i) $\widetilde{\Phi}(\mathbf{s}) = \prod_{\mathfrak{l}} \Phi(\mathfrak{l})^{\mathrm{ord}_{\mathfrak{l}}(\mathbf{s})}$ for $\mathbf{s}$ a finite idele prime to $\mathfrak{f}$,

(ii) $\widetilde{\Phi}(\alpha) = \alpha^T$ for $\alpha \in F^\times$.

The infinite type T can be viewed as an homomorphism $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ of algebraic groups in a canonical way. Hence it induces a homomorphism $T_{\mathbb{A}} : \mathbb{A}_F^\times \to \mathbb{A}_E^\times$. For a prime $\mathfrak{p}$ of E, we write $T_{\mathfrak{p}} := \mathrm{pr}_{\mathfrak{p}} \circ T_{\mathbb{A}} : \mathbb{A}_F^\times \to E_{\mathfrak{p}}^\times$ where $\mathrm{pr}_{\mathfrak{p}}$ is the projection onto the component at $\mathfrak{p}$. We then have a well-defined continuous character

$$\Phi_{\mathbb{A},\mathfrak{p}} : \mathbb{A}_F^\times/F^\times \longrightarrow E_{\mathfrak{p}}^\times, \quad \Phi_{\mathbb{A},\mathfrak{p}}(\mathbf{s}) := \widetilde{\Phi}(\mathbf{s})T_{\mathfrak{p}}(\mathbf{s})^{-1}.$$

The following is an easy exersice (and well-known).

Lemma 6.2. $\Phi_{\mathbb{A},\mathfrak{p}}$ *factors through a finite quotient if and only if* $T = 0$.

Let $\mathfrak{l}$ be a prime of F and $F_{\mathfrak{l}}$ the completion. Suppose that both of $\mathfrak{l}$ and $\mathfrak{p}$ are prime to $\mathfrak{f}$. Then the composition

$$\rho_{\mathfrak{l}} : O_{F_{\mathfrak{l}}}^\times \hookrightarrow F_{\mathfrak{l}}^\times \longrightarrow \mathbb{A}_F^\times/F^\times \longrightarrow E_{\mathfrak{p}}^\times$$

is given by $\rho_{\mathfrak{l}}(\alpha) = \prod' \sigma(\alpha)^{-n_\sigma}$ where $\prod'$ means that σ runs over the embeddings such that $\mathfrak{l}|\sigma^{-1}(\mathfrak{p}O_{\overline{E}})$.

Lemma 6.3. *Suppose that* $F = E$ *is a finite Galois extension of* $\mathbb{Q}$ *and* $\mathfrak{l} = \mathfrak{p}$. *Let* $V(\rho_{\mathfrak{p}})$ *be the one-dimensional Galois representation of* $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ *over* $F_{\mathfrak{p}}$ *defined by* $\rho_{\mathfrak{p}}$. *Let* $\mathbb{C}_p$ *be the completion of* $\overline{F}_{\mathfrak{p}}$ *and* $\mathbb{C}_p(r) := \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$ *denotes the Tate twist. Then there is an isomorphism*

$$\mathbb{C}_p \otimes_{F_{\mathfrak{p}}} V(\rho_{\mathfrak{p}}) \cong \mathbb{C}_p(n_{\mathrm{id}}) \tag{13}$$

of $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$*-modules where* $g \in \mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ *acts on the left hand side by* $g \otimes g$

Proof. See [Ser98] Chapter III, A.6, Exercise 2. □

6.2. Smooth sheaves of mixed realization. Let F be a number field embedded into $\overline{\mathbb{Q}}$. Let X be a smooth projective smooth variety over F and D a simple normal crossings divisor on X with irreducible components D_i. Let $U = X \setminus D$ denote the complement of D in X. We assume that X and all D_i are geometrically irreducible over F.

We say that a collection $\mathscr{M} = (M_B, M_{\mathrm{dR}}, F^\bullet, W_\bullet, \nabla, M_{\mathbb{Q}_l}, \iota_{\mathbb{C}}, \iota_l)$ with l primes is a *smooth sheaf of mixed realization* on U if

- M_B is a smooth sheaf of $\mathbb{Z}$-modules of finite rank on U^{an} equipped with the weight filtration $W_\bullet$ on $M_{B,\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M_B$,
- $(M_{\mathrm{dR}}, W_\bullet, F^\bullet)$ is a locally free sheaf of $\mathscr{O}_U$-module of finite rank with the weight filtration $W_\bullet$ and the Hodge filtrartion $F^\bullet$,
- $\iota_{\mathbb{C}} : (\mathscr{O}_U^{an} \otimes_{\mathbb{Z}} M_B, W_\bullet) \cong (\mathscr{O}_U^{an} \otimes_{\mathscr{O}_U} M_{\mathrm{dR}}, W_\bullet)$ a comparison isomorphism of filtered modules,
- $\nabla : M_{\mathrm{dR}} \to \Omega^1_{U/k} \otimes M_{\mathrm{dR}}$ is an integrable connection,
- $(M_B, M_{\mathrm{dR}}, F^\bullet, W_\bullet, \nabla, \iota_{\mathbb{C}})$ is an admissible VMHS,
- $(M_{\mathbb{Q}_l}, W_\bullet)$ is a filtered finite dimensional smooth $\mathbb{Q}_l$-sheaf on $U^{\text{ét}}$ which is defined on a model of U of finite type over $\mathbb{Z}$,
- $\iota_l : (\mathbb{Q}_l \otimes_{\mathbb{Z}} M_B, W_\bullet) \cong (M_{\mathbb{Q}_l}, W_\bullet)$ comparison isomorphisms of filtered modules on U^{an},

If $U = \mathrm{Spec} F$ we simply say $\mathscr{M}$ a *mixed realization over* F. The standard operations $\otimes$, $\mathcal{H}om$ are defined in the customary way. For $f : U \to V$ a morphism of smooth varieties, the pull-back f^* is defined in a canonical way. Moreover if the sheaf $R^i f_* M_B$ is a smooth sheaf, then $R^if_*\mathscr{M}$ carries a mixed realization in such a way that the underlying admissible VMHS is given by the theory of Hodge modules due to M. Saito. There is the functor

$$\mathrm{Gr}^W_\bullet : (\text{Mixed realizations over } F) \longrightarrow (\text{Hodge–de Rham structures over } F)$$

by taking a weight piece. Letting $\pi : \mathbb{G}_m \times_{\mathbb{Z}} U \to U$ the *Tate twist* is defined as $\mathbb{Q}(r) := (R^1\pi_*\mathbb{Q})^{\otimes -r}$. If there is a finite extension F'/F such that each graded piece $\mathrm{Gr}^W_w\mathscr{M}|_{U\times_F F'}$ is isomorphic to a product of copies of Tate twists, then we say it a sheaf of *mixed Tate realization.*

Suppose that $\mathscr{M}$ has multiplication by a number field K (we do not fix an embedding $K \hookrightarrow \overline{\mathbb{Q}}$). Put

$$m = \mathrm{rank}_K\mathscr{M}, \quad \det_{\mathbb{Q}} R\Gamma(U, \mathbb{Q}) = \mathbb{Q}(-r_U).$$

For a point $i_x : \mathrm{Spec}F \to U$, we write $\mathscr{M}_x = i_x^*(\mathscr{M})$ the pull-back. This is a mixed realization over F and the determinant $\det_K(\mathscr{M}_x)$ does not depend on x.

We shall discuss a mixed realization

$$H = H(\mathscr{M}) := \det_K R\Gamma(U, \mathscr{M}) \otimes_K \overbrace{\det_K(\mathscr{M}_x) \otimes_K \cdots \otimes_K \det_K(\mathscr{M}_x)}^{-\chi^{\mathrm{top}}(U)} \otimes_{\mathbb{Q}} \mathbb{Q}(r_U m), \tag{14}$$

especially the Hodge decomposition on $H_{\mathrm{dR}}(\mathscr{M})$.

Let $\chi : K \hookrightarrow \overline{\mathbb{Q}}$ be an embedding. Let M_B^χ be the χ-component of $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} M_B$, i.e. the subspace on which $1 \otimes g$ acts by multiplication by $\chi(g) \otimes 1$ for all $g \in K$. Let ρ be the monodromy representation of $\pi_1^B(U, x)$ on $M_{B,x}$ and ρ_{D_i} the restriction on $\mathbb{Z}(1)_{D_i} = 2\pi i\mathbb{Z} \subset \pi_1^B(U)^{ab} \cong H_1(U, \mathbb{Z})$ the subgroup generated by the local monodromy around D_i. By the monodromy theorem, its semisimplification $\rho_{D_i}^{ss}$ factors through a finite quotient. Hence one has a decomposition

$$\rho_{D_i}^{ss}|M_B^\chi \cong \bigoplus_{j=1}^m \rho_{D_i,j}^\chi$$

where $\rho_{D_i,j}^\chi : \mathbb{Z}(1) \to \mathbb{Q}/\mathbb{Z}(1) \cong \mu_\infty \subset \overline{\mathbb{Q}}^\times$ is a character of finite order. Let $\alpha_{D_i,j}^\chi \in \mathbb{Q}/\mathbb{Z}$ be the corresponding element to $\rho_{D_i,j}^\chi$ under the canonical isomorphism $\mathrm{Hom}(\mathbb{Z}(1), \mathbb{Q}/\mathbb{Z}(1)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$. Fix an integer $d \geq 1$ such that $\alpha_{D_i,j}^\chi \in \frac{1}{d}\mathbb{Z}/\mathbb{Z}$ for all i, j (e.g. the least common multiple of orders of $\alpha_{D_i,j}^\chi$'s).

Lemma 6.4. *Let $\chi' = \sigma\chi$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then, by changing the numbering 'j' if necessary,*

$$\exp(2\pi i\alpha_{D_i,j}^{\chi'}) = \exp(2\pi i\alpha_{D_i,j}^\chi)^\sigma.$$

In particular if $\chi = \chi'$ on $K_0 = K \cap \mathbb{Q}(\mu_\infty)$ then $\alpha_{D_i,j}^\chi = \alpha_{D_i,j}^{\chi'}$.

Proof. This follows from the fact that the isomorphism $\sigma\otimes 1 : \overline{\mathbb{Q}}\otimes M_B \to \overline{\mathbb{Q}}\otimes M_B$ is compatible with the action of $\pi_1(U, x)$ and it satisfies $(\sigma \otimes 1)(M_B^\chi) \subset M_B^{\chi'}$. □

6.3. Saito's Jacobi sum formula [Sai94]**.** Let p be a prime number such that $(p,d)=1$. Fix an embedding $\tau:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$. Let $\mathfrak{p}$ be a prime of K associated to the embedding $\tau\circ\chi$, and $K_\mathfrak{p}$ the completion. Hence $\tau\circ\chi$ is naturally extended to an embedding $K_\mathfrak{p}\hookrightarrow\overline{\mathbb{Q}}_p$. Let $M^\chi_{K_\mathfrak{p}}\subset K_\mathfrak{p}\otimes_{\mathbb{Q}_p}M_{\mathbb{Q}_p}$ be the part on which $1\otimes g$ acts by multiplication by $\chi(g)\otimes 1$. Then

$$H(\mathscr{M})^\chi_{K_\mathfrak{p}}\cong \det{}_{K_\mathfrak{p}}R\Gamma(U,M^\chi_{K_\mathfrak{p}})\otimes_{K_\mathfrak{p}}\det{}_{K_\mathfrak{p}}(M^\chi_{K_\mathfrak{p},x})^{\otimes-\chi^{\mathrm{top}}(U)}\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(r_Um) \tag{15}$$

is a one-dimensional $\mathbb{Q}_p$-sheaf on $\mathrm{Spec}(F)$ where the tensor product "$\otimes-\chi^{\mathrm{top}}(U)$" in the middle term is taken over $K_\mathfrak{p}$.

In the paper [Sai94], Saito introduced a sheaf $J_{D,M^\chi_{K_\mathfrak{p}}}$ of Jacobi Hecke sum character which is determined by the ramification data of $M^\chi_{K_\mathfrak{p}}$ and showed that there is an isomorphism

$$H(\mathscr{M})^\chi_{K_\mathfrak{p}}\cong J_{D,M^\chi_{K_\mathfrak{p}}} \tag{16}$$

of sheaves on $U=U^{\text{ét}}$. Let us briefly recall the definition of $J_{D,M^\chi_{K_\mathfrak{p}}}$.

Let $\mu_d=\mu_d(\overline{\mathbb{Q}})\subset\overline{\mathbb{Q}}^\times$ be the group of d-th roots of unity in $\overline{\mathbb{Q}}$. Let $\phi_{ij}:\hat{\mathbb{Z}}(1)_{\overline{\mathbb{Q}}}\to\mu_d\subset\overline{\mathbb{Q}}^\times$ be the homomorphism associated to $\alpha^\chi_{D_i,j}\in\frac{1}{d}\mathbb{Z}/\mathbb{Z}$. Put $T_{ij}:=\mathrm{Spec}F(\mu_d)$ and $n_{ij}:=\chi^{\mathrm{top}}(D_i^\circ)$ where $D_i^\circ=D_i\setminus\bigcup_{k\neq i}(D_i\cap D_k)$. Then the datum $\{(T_{ij})_{i,j},(\tau\circ\phi_{ij})_{i,j},(n_{ij})_{i,j}\}$ defines the Jacobi datum on $\mathrm{Spec}F$ with coefficient in $K_\mathfrak{p}$ in the sense of [Sai94] p.416, Definition (see, loc.cit. p.424, Theorem 1). It defines a one-dimensional $K_\mathfrak{p}$-sheaf on $\mathrm{Spec}F$, which we denotre by $J_{D,M^\chi_{K_\mathfrak{p}}}$ (loc.cit. p.417, Proposition 2). Moreover, by the Hasse-Davenport theorem, one can show that it is isomorphic to the sheaf arising from an algebraic Hecke character (loc.cit. p.418, Corollary). More precisely let $I_{\mathbb{Q}(\mu_d),d}$ be the group of fractional ideals of $\mathbb{Q}(\mu_d)$ prime to d. Let $\mathfrak{p}_0$ be a prime of $\mathbb{Q}(\mu_d)$ corresponding to $\mathbb{Q}(\mu_d)\subset\overline{\mathbb{Q}}\xrightarrow{\tau}\overline{\mathbb{Q}}_p$, so that we have an embedding $\mathbb{Q}(\mu_d)_{\mathfrak{p}_0}\hookrightarrow\overline{\mathbb{Q}}_p$. Then there is an algebraic Hecke character

$$\Phi\colon I_{\mathbb{Q}(\mu_d),d^2}\longrightarrow\mathbb{Q}(\mu_d)^\times \tag{17}$$

with conductor dividing d^2 such that

$$K_\mathfrak{p}(\mu_d)\otimes_{K_\mathfrak{p}}u^*J_{D,\mathscr{M}^\chi_{K_\mathfrak{p}}}\cong K_\mathfrak{p}(\mu_d)\otimes_{\mathbb{Q}(\mu_d)_{\mathfrak{p}_0}}v^*J_{\Phi,\mathbb{Q}(\mu_d)} \tag{18}$$

where $u:\mathrm{Spec}F(\mu_d)\to\mathrm{Spec}F$, $v:\mathrm{Spec}F(\mu_d)\to\mathrm{Spec}\mathbb{Q}(\mu_d)$ and $J_{\Phi,\mathbb{Q}(\mu_d)}$ is the one-dimensional $\mathbb{Q}(\mu_d)_\mathfrak{p}$-sheaf on $\mathrm{Spec}\mathbb{Q}(\mu_d)$ associated to the Hecke character

$$\Phi_{\mathbb{A},\mathfrak{p}}:\mathbb{A}^\times_{\mathbb{Q}(\mu_d)}/\mathbb{Q}(\mu_d)^\times\to\mathbb{Q}(\mu_d)_\mathfrak{p}$$

at $\mathfrak{p}$ arising from Φ.

Let us give an explicit description of Φ (17). Let $\mathfrak{l}$ be a prime of $\mathbb{Q}(\mu_d)$ which does not divide d. Let $\kappa(\mathfrak{l})$ be the residue field of $\mathfrak{l}$ and $\mu_d(\kappa(\mathfrak{l}))$ denotes the group of d-th roots of unity in $\kappa(\mathfrak{l})$. Since the reduction $\mu_d=\mu_d(\overline{\mathbb{Q}})\to\mu_d(\kappa(\mathfrak{l}))$ modulo $\mathfrak{l}$ is bijective, there is a unique homomorphism $\overline{\phi}_{ij}:\kappa(\mathfrak{l})^\times\to\mu_d$ such that

$$\overline{\phi}_{ij}(x)\equiv x^{(l^s-1)\alpha^\chi_{D_i,j}}\mod\mathfrak{l},\quad l^s:=\sharp\kappa(\mathfrak{l}).$$

Fix a nontrivial additive character $\psi:\kappa(\mathfrak{l})\to\overline{\mathbb{Q}}$. Let $\tau(\overline{\phi}_{ij}):=-\sum_{x\in\kappa(\mathfrak{l})^\times}\overline{\phi}_{ij}^{-1}(x)\psi(x)$ be the Gauss sum. The Jacobi sum is defined to be

$$J(\mathfrak{l}):=\prod_{i,j}\tau(\overline{\phi}_{ij})^{n_{ij}}=\prod_{i,j}\tau(\overline{\phi}_{ij})^{\chi^{\mathrm{top}}(D_i^\circ)}\in\mathbb{Q}(\mu_d)$$

and this is independent of the choice of ψ. Then the algebraic Hecke character (17) is given by

$$\Phi(\mathbf{s}) = \prod_{\mathfrak{l}} J(\mathfrak{l})^{-\mathrm{ord}_{\mathfrak{l}}(\mathbf{s})}, \quad \mathbf{s} \in I_{\mathbb{Q}(\mu_d),d^2}.$$

By the theorem of Weil [Wei52], the infinite type of Φ is

$$T = \sum_{t\in(\mathbb{Z}/d\mathbb{Z})^\times} \left(\sum_{i,j} -\chi^{\mathrm{top}}(D_i^\circ)\{t\alpha^\chi_{D_i,j}\} \right) \sigma_t^{-1} \in \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})] \tag{19}$$

where σ_t is defined by $\zeta^{\sigma_t} = \zeta^t$.

Proposition 6.5 (Hodge-Tate representation of $H(\mathscr{M})_{\mathbb{Q}_p}$)**.** *Let $\wp$ be the prime of F associated to $F \subset \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $F_\wp$ the completion. Let $F'_\wp \subset \overline{\mathbb{Q}}_p$ be the composition of fields $F_\wp$ and $K_\mathfrak{p}$. Then there is an isomorphism*

$$\mathbb{C}_p \otimes_{K_\mathfrak{p}} H(\mathscr{M})^\chi_{K_\mathfrak{p}} \cong \mathbb{C}_p(-p_\chi^{\mathrm{HT}}), \quad p_\chi^{\mathrm{HT}} := \sum_{i,j} \chi^{\mathrm{top}}(D_i^\circ)\{\alpha^\chi_{D_i,j}\}$$

of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F'_\wp)$-modules where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ and $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F'_\wp)$ acts on the left hand side by $g \otimes g$.

Proof. We may replace F by any finite extension. Hence we can assume $\mathbb{Q}(\mu_d) \subset F$. By (15) and (18) we can reduce the assertion to the case of $J_{\Phi,\mathbb{Q}(\mu_d)}$, and then this follows from Lemma 6.3 and (19). □

6.4. Hodge-Tate conjecture for cohomology with coefficients in a mixed realization. Let H be a mixed realization over F. Let $\wp$ be a prime of F above p and $F_\wp$ the completion. Let $S \subset \mathrm{End}(H)$ be a subring. We say that the *S-equivariant Hodge-Tate conjecture for H over $F_\wp$* holds if there is a $\mathbb{C}_p$-linear isomorphism

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H_{\mathbb{Q}_p} \cong \bigoplus_i \mathbb{C}_p(-i) \otimes_{F_\wp} \mathrm{Gr}^i_F H_{\mathrm{dR},F_\wp}$$

which is compatible with the actions of S and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\wp)$. Here $H_{\mathrm{dR},F_\wp} = F_\wp \otimes_F H_{\mathrm{dR}}$ and $F^\bullet H_{\mathrm{dR}}$ denotes the Hodge filtration. The action of $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\wp)$ is given by $g \otimes g$ on the left hand side, and $g \otimes 1$ on the right hand side. The action of $r \in S$ is given by $1 \otimes r$ on both sides.

Proposition 6.6. *Let the notation be as in §6.2. Let $\wp$ be a primeof F above p prime to d. Assume that the K-equivariant Hodge-Tate conjecture for $H(\mathscr{M})$ over $F_\wp$ holds. Then*

$$p_\chi = p_\chi^{\mathrm{HT}} = \sum_{i,j} \chi^{\mathrm{top}}(D_i^\circ)\{\alpha^\chi_{D_i,j}\}$$

for each $\chi : K \hookrightarrow \overline{\mathbb{Q}}$. Here p_χ is the unique integer satisfying that the χ-component $H(\mathscr{M})^\chi_{\mathrm{dR}}$ belongs to the Hodge (p_χ, q_χ)-component. In particular, the period conjecture of Gross-Deligne holds for $H(\mathscr{M})$.

Proof. Let $\tau : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ be an embedding extending $F \hookrightarrow F_\wp \subset \overline{\mathbb{Q}}_p$. Let $\mathfrak{p}$ be the prime of K associated to $\tau \circ \chi$ and $K_\mathfrak{p}$ the completion. Then the K-equivariant Hodge-Tate conjecture for $H(\mathscr{M})$ over $F_\wp$ immediately implies an isomorphism

$$\mathbb{C}_p \otimes_{K_\mathfrak{p}} H(\mathscr{M})^\chi_{K_\mathfrak{p}} \cong \mathbb{C}_p(-p_\chi) \otimes_\tau H(\mathscr{M})^\chi_{\mathrm{dR}} \tag{20}$$

of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F'_\wp)$-modules where $F'_\wp = F_\wp K_\mathfrak{p}$. Thus the assertion follows from Proposition 6.5 □

The following is a variant of Proposition 6.6, and the proof goes in the same way.

Proposition 6.7. *Let $\mathscr{M}_i$ be smooth sheaves of mixed realizations equipped with multiplication by a number field K. Assume that the K-equivariant Hodge-Tate conjecture for*

$$H := \bigotimes_i H(\mathscr{M}_i)^{\otimes n_i}$$

over $F_\wp$ holds where the tensor product is taken over K. Then the period conjecture of Gross-Deligne holds for H.

7. Appendix : Note on S-equivariant Hodge-Tate conjecture for graded weight pieces

Let $f : X \to Y$ be a smooth proper morphism of quasi-projective smooth varieties over a field. Let $\Gamma \in \mathrm{CH}^r(X \times_Y X)$ be an algebraic cycle of codimension r. It induces an algebraic correspondence

$$\Gamma_* : H^\bullet(X, \mathbb{Q}) \longrightarrow H^{\bullet+2r-2d}(X, \mathbb{Q}(r-d)), \quad d := \dim X - \dim Y$$

on any Borel-Moore cohomology groups which is the composition of the maps

$$H^\bullet(X,\mathbb{Q}) \xrightarrow{p_1^*} H^\bullet(X\times_Y X,\mathbb{Q}) \xrightarrow{\cup[\Gamma]} H^{\bullet+2r}(X\times_Y X,\mathbb{Q}(r)) \xrightarrow{p_{2*}} H^{\bullet+2r-2d}(X,\mathbb{Q}(r-d))$$

where the middle arrow denotes the cup-product with the cycle class $[\Gamma] \in H^{2r}(X \times_Y, X, \mathbb{Q}(r))$.

Suppose that the base field is a p-adic local field k. Write $X_{\overline{k}} := X \times_k \overline{k}$. Then the algebraic correspondence induces maps

$$\Gamma_* : \mathrm{Gr}^W_w H^\bullet_{\mathrm{dR}}(X/k) \longrightarrow \mathrm{Gr}^W_{w+2r-2d} H^{\bullet+2r-2d}_{\mathrm{dR}}(X/k)$$

$$\Gamma_* : \mathrm{Gr}^W_w H^\bullet_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_p) \longrightarrow (\mathrm{Gr}^W_{w+2r-2d} H^{\bullet+2r-2d}_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_p)) \otimes \mathbb{Q}_p(r-d)$$

on the graded weight pieces. In the proof of Theorem **??**, we used the fact that there is a Galois equivariant isomorphism

$$\mathbb{C}_p \otimes \mathrm{Gr}^W_w H^\bullet_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_p) \cong \bigoplus_i \mathbb{C}_p(-i) \otimes \mathrm{Gr}^i_F \mathrm{Gr}^W_w H^\bullet_{\mathrm{dR}}(X/k) \tag{21}$$

which is compatible with the algebraic correspondence Γ_*. In this section we give an explicit construction of it. To do this it is enough to construct the Hodge-Tate isomorphism (21) for each smooth varieties X over k which satisfies the following compatibilities.

(a) Compatibility with the pull-back of morphisms,
(b) Compatibility with the Gysin maps f_* for proper morphisms f,
(c) Compatibility with the cup-product $\cup[Z]$ for $Z \in \mathrm{CH}^r(X)$.

If X is a smooth projective variety, then the the Hodge-Tate isomorphim is given by Faltings et al and it is compatible with the pull-back (cf. [Ill94] 3.1.2). It is also compatible with the Gysin map f_* for $f : X_1 \to X_2$ a morphism of projective smooth varieties, since it is the dual map of the pull-back f^*. For non-complete X, let $\overline{X} \supset X$ be a smooth compactification such that the complement $D = \cup_i D_i = \overline{X} \setminus X$ is a simple normal crossing divisor. There is the exact sequence

$$\cdots \longrightarrow H^j_D(\overline{X}) \cong H_{2\dim X-j}(D) \xrightarrow{i_1} H^j(\overline{X}) \xrightarrow{i_2} H^j(X) \longrightarrow \cdots$$

Note that i_1 is the Gysin map. Hence

$$\mathrm{Gr}^W_w H^j(X) = \begin{cases} \mathrm{coker}[\mathrm{Gr}^W_j H_{2\dim X-j}(D) \to H^j(\overline{X})] & w = j \\ \ker[\mathrm{Gr}^W_{j+1} H_{2\dim X-j-1}(D) \to H^{j+1}(\overline{X})] & w = j+1 \\ \mathrm{Gr}^W_w H_{2\dim X-j-1}(D) & w > j+1 \end{cases}$$

Since $\mathrm{Gr}^W_w H_\bullet(D)$ is described by the cohomology groups $H_{2\dim X-j}(D^{[q]})$ where $D^{[q]}$ denotes the sum of q-fold intersections of components of D, one obtains the Hodge-Tate isomorphism for the graded pieces $\mathrm{Gr}^W_w H_\bullet(D)$ and hence for $\mathrm{Gr}^W_w H_\bullet(X)$.

Next we see the compatibility (a). Let $f : X_1 \to X_2$ be a morphism of smooth varieties. Let $\overline{X}_i \supset X_i$ be smooth compctifications such that the complement $D_i = \overline{X}_i \setminus X_i$ is a simple normal crossing divisor and f extends to a map $\bar{f} : \overline{X}_1 \to \overline{X}_2$. To see the comaptibility (a), it is enough to see the comaptibility of the Hodge-Tate isomorphism for $\overline{f}^*$ on $H^\bullet(\overline{X}_i)$ and on $\mathrm{Gr}^W_w H_\bullet(D_i)$. The former is immediate. To see the latter, we look at the map $\bar{f}^* : \mathrm{Gr}^W_w H^\bullet_{D_2}(\overline{X}_2) \to \mathrm{Gr}^W_w H^\bullet_{D_1}(\overline{X}_1)$ in an explicit way. Let $D_i = \cup_k D_{i,k}$ be the irreducible decomposition, and let $\bar{f}^{-1}(D_{2,k}) = \sum_l e_{kl} D_{1,l}$ be the scheme-theoretic inverse image. The pull-back $\bar{f}^*$ on $\mathrm{Gr}^W_w H^\bullet_{D_i}(\overline{X}_i)$ is induced from the map $\bar{f}^{-1} R^q j_{2*}\mathbb{Q} \to R^q j_{1*}\mathbb{Q}$ where $j_i : X_i \hookrightarrow \overline{X}_i$. One has the composition

$$\bar{f}^{-1}\mathbb{Q}_{D_{2,k_1}\cap\cdots\cap D_{2,k_q}} \to \mathbb{Q}_{D_2^{[q]}} \cong \bar{f}^{-1} R^q j_{2*}\mathbb{Q} \to R^q j_{1*}\mathbb{Q} \cong \mathbb{Q}_{D_1^{[q]}} \to \mathbb{Q}_{D_{1,l_1}\cap\cdots\cap D_{2,l_q}} \tag{22}$$

and this is described in the following way. If $\det(e_{k_s,l_t})_{1\le s,t\le q} = 0$ then (22) is zero. If $\det(e_{k_s,l_t})_{1\le s,t\le q} \neq 0$, then $\bar{f}(D_{2,k_1} \cap \cdots \cap D_{2,k_q}) \subset D_{1,l_1} \cap \cdots \cap D_{2,l_q}$ and (22) is equal to $\det(e_{k_s,l_t}) \cdot (\bar{f}|_{D_{2,k_1}\cap\cdots\cap D_{2,k_q}})^*$. Now the desired compatibility is immediate.

Let us see the compatibility (b). Let $f : X_1 \to X_2$ and $\bar{f}$ be as before and assume that f is proper. Then $\bar{f}^{-1}(D_2) = D_1$ (as f is proper). The compatibility for f_* on $\mathrm{Gr}^W_w H^\bullet(X_i)$ is reduced to the comatibility for f_* on $H^\bullet(\overline{X}_i)$ and on $\mathrm{Gr}^W_w H_\bullet(D_i)$. The former is clear because it is the dual map of the pull-back. To see the latter it is enough to see the compatibility of the Hodge-Tate isomorphism for $\bar{f}^* : \mathrm{Gr}^W_w H^\bullet(D_2) \to \mathrm{Gr}^W_w H^\bullet(D_1)$. The graded piece $\mathrm{Gr}^W_w H^\bullet(D_i)$ is described by the cohomology of the components of a proper hypercovering of D_i ([Del74] 8.3.5), and the Hodge-Tate isomorphism is induced from them. Now the assertion follows from the functoriality of proper hypercoverings (loc.cit. 6.2.8).

We finally see the compatibility (c). We now have isomorphims (21) which satisfy the compatibilities (a) and (b). Moreover it is also compatible with the Künneth decomposition. To see (c), we may assume that Z is an irreducible subvariety in X. Let $Z' \to Z$ be a desingularization and let $i : Z' \to X$ be the morphism into X. Then letting $i_* : \mathbb{Q} \cong H^0(Z', \mathbb{Q}) \to H^{2r}(X, \mathbb{Q}(r))$ be the Gysin map, one has $[Z] = i_*(1)$ and the cup-product $\cup[Z]$

is given as the composition

$$H^i(X) \cong H^0(Z') \otimes H^i(X) \xrightarrow{i_* \otimes 1} H^{2r}(X) \otimes H^i(X) \xrightarrow{\Delta^*} H^{2r+i}(X)$$

where $\Delta : X \to X \times_k X$ denotes the diagonal embedding. Thus the compatibility (c) follows.

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Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: `asakura@math.sci.hokudai.ac.jp`

ETH Zürich, D-MATH, Rämistrasse 101, CH-8092 Zürich, Switzerland
E-mail address: `javier.fresan@math.ethz.ch`