Lectures on arithmetic Gevrey series

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Preface

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Chapter 1

Overview

1.1 Historical motivation

The subject of these lectures arose from attempts to generalise one of the most beautiful theorems in transcendental number theory, which at the end of the 19th century completely settled the question of understanding the arithmetic nature of the values of the exponential function at algebraic arguments.

Theorem 1.1 (Hermite–Lindemann–Weierstrass, 1885). *Let $n \geq 1$ be an integer, and let $\alpha_1, \ldots, \alpha_n$ be $\mathbf{Q}$-linearly independent algebraic numbers. Then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbf{Q}$.*

For $n = 1$, the theorem says that the exponential of a non-zero algebraic number is transcendental; in particular, the numbers e and π are transcendental (use $e^{\pi i} = -1$ for the latter). Thanks to the identity $e^{\alpha+\beta} = e^{\alpha}e^{\beta}$, the study of algebraic relations immediately reduces to that of linear relations, so it suffices to prove that the exponentials of distinct algebraic numbers are $\overline{\mathbf{Q}}$-linearly independent.

Because of the role that the functional equation of the exponential plays in its proof, it was unclear how to extend the Hermite–Lindemann–Weierstrass theorem to other special functions. Of particular interest where the *Bessel functions* of integral order $k \geq 0$, given by the formula

$$\begin{aligned} J_k(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} \left(\frac{z}{2}\right)^{2n+k} \\ &= \frac{1}{2\pi i} \int_{|x|=1} \exp\left(\frac{z}{2}\left(x - \frac{1}{x}\right)\right) \frac{dx}{x^{k+1}}. \end{aligned}$$

They appear in the solutions of the wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} & \text{if } x^2 + y^2 < 1, \\ u = 0 & \text{if } x^2 + y^2 = 1, \end{cases}$$

that governs the vibrations of a circular membrane. Here, $u(x, y, t)$ represents the height of the point of coordinates (x, y) at time t, and the boundary condition expresses the fact that the boundary of the membrane stays fixed. In polar coordinates, the radial part of a solution $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$ in separated variables is given by $R(r) = J_k(\lambda r)$ for $k = 0, 1, \ldots$ and a zero λ of the Bessel function. In deriving these solutions in his 1866 memoir [7], Bourget noticed that the sets of non-trivial zeros of Bessel functions of different order seem to be disjoint.

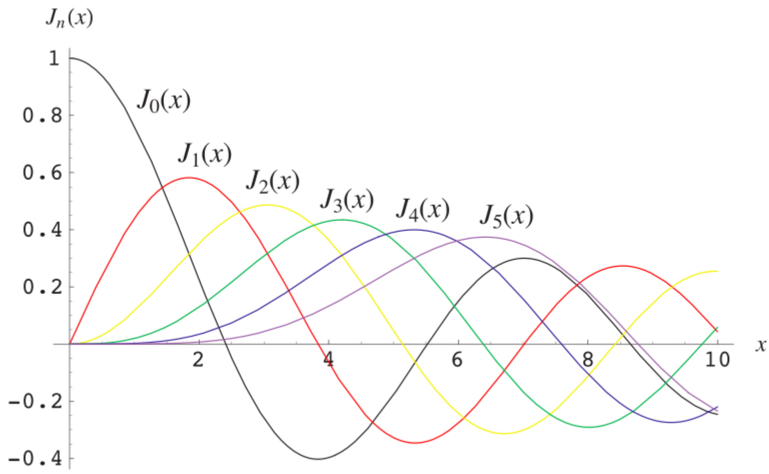

Figure 1.1. Real plots of Bessel functions of different orders

Conjecture 1.1 (Bourget's hypothesis, 1866). *Bessel functions of different orders do not have any non-zero common zero.*

This is elementary for Bessel functions of consecutive orders, combining the facts that $J_k(z)$ satisfies a differential equation of order 2, and that the derivative of $J_k(z)$ is equal to $\frac{k}{z}J_k(z) - J_{k+1}(z)$. Besides, there are recurrence relations

$$J_{k+m}(z) = J_k(z)R_{m,k}(z) - J_{k-1}(z)R_{m+1,k-1}(z)$$

for some explicit $R_{m,k} \in \mathbf{Q}[z^{-1}]$ called *Lommel polynomials*. From this, one derives that a non-trivial common zero would be algebraic. Therefore, a solution of Conjecture 1.1 would follow from the transcendence of non-zero zeros of the Bessel functions, or more generally from the statement that $J_k(\alpha)$ is transcendental for each non-zero algebraic number α. This was proved by Siegel in his 1929 monograph [16].

1.2 Definition of arithmetic Gevrey series

Definition 1.1. Let $s \in \mathbf{Q}$ be a rational number. An *arithmetic Gevrey series* of order s is a formal power series with algebraic coefficients

$$f(z) = \sum_{n=0}^{\infty} a_n (n!)^s z^n \in \overline{\mathbf{Q}}[[z]]$$

that satisfies the following conditions:

(1) f is a solution of a linear differential equation with polynomial coefficients; that is, there exists a non-zero differential operator $L \in \overline{\mathbf{Q}}[z, d/dz]$ satisfying

$$L \cdot f = 0;$$

(2) there exists a real number $C > 0$ such that

$$|\sigma(a_n)| \leq C^n \text{ and } d_n \leq C^n \text{ for all } n \geq 1 \text{ and all } \sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}),$$

where d_n denotes the smallest integer ≥ 1 such that $d_n a_0, \ldots, d_n a_n$ are algebraic integers ("the common denominator" of $a_0, \ldots, a_n$).

Remark 1.1. The condition that f satisfies a differential equation is equivalent to asking that the sequence $(a_n)_{n\geq 0}$ satisfies a linear recurrence with polynomial coefficients: there exist polynomials $P_0, \ldots, P_\mu \in \overline{\mathbf{Q}}[X]$ such that, for all $n \geq 1$,

$$P_0(n)a_n + P_1(n+1)a_{n+1} + \cdots + P_\mu(n+\mu)a_{n+\mu} = 0.$$

In particular, there exists a number field K such that $f(z)$ belongs to $K[[z]]$, so that in the second condition one only needs to check a finite number of Galois conjugates. In fact, every arithmetic Gevrey series of order s is an algebraic linear combination of arithmetic Gevrey series of order s with *rational* coefficients.

Most questions revolving around arithmetic Gevrey series can be reduced, by means of elementary manipulations, to questions about those of orders $s = -1$, $s = 0$, and $s = 1$. They are called, respectively, *E-functions*, *G-functions*, and Ǝ-functions. The first two names come from the case where $a_n = 1$ for all n, where one finds

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \textbf{E}\text{xponential function}, \qquad \sum_{n=0}^{\infty} z^n = \textbf{G}\text{eometric series}.$$

These were the notions introduced by Siegel. The name Ǝ-function is a tribute to Euler who studied the divergent series $\sum_{n=0}^{\infty}(-1)^n n!$ when he was in Saint Petersburg.

Remark 1.2. Actually, Siegel's 1929 definition of E-functions and G-functions was slightly different. Instead of condition (2), he required that, for every $\varepsilon > 0$ and large enough n, the quantities $|\sigma(a_n)|$ and d_n grow at most like $(n!)^\varepsilon$. We do not know of a single example where one can prove that a power series is an arithmetic Gevrey series in this larger sense but not in the stricter one. For solutions of a differential equations, the conditions $|\sigma(a_n)| \leq C^n$ and $|\sigma(a_n)| < (n!)^\varepsilon$ are in fact known to be equivalent, and the same is conjectured to be true for the denominators.

1.3 Examples

(1) Polynomials in $\overline{\mathbf{Q}}[z]$ are arithmetic Gevrey series of every order. Conversely, a formal power series which satisfies the conditions in Definition 1.1 for two different values of s is a polynomial with algebraic coefficients.

(2) *Exponential polynomials* $f(z) = \sum_{i=0}^{r} P_i(z)e^{\alpha_i z}$,with $P_i \in \overline{\mathbf{Q}}[X]$ and $\alpha_i \in \mathbf{Q}$ are examples of E-functions. The associated G-functions are rational functions. Indeed, for $e^{\alpha_i z} = \sum_{n=0}^{\infty} \alpha_i^n z^n/n!$ one finds $\sum_{n=0}^{\infty} \alpha_i^n z^n = 1/(1-\alpha_i z)$, and if f is an E-function with associated G-function g, then zf is an E-function with associated G-function $z^2 g'(z) + zg(z)$.

(3) All Bessel functions $J_k(z)$ are examples of E-functions. Indeed, they are solutions of the Bessel differential operator

$$L = z^2\left(\frac{d}{dz}\right)^2 + z\frac{d}{dz} + z^2 - k,$$

and rewriting their power series representation as

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+k}} \binom{2n+k}{n+k} \frac{z^{2n+k}}{(2n+k)!},$$

we have $|a_n| \leq 1$ for all $n \geq 0$ and $d_n = 2^n$.
For $k = 0$, the associated G-function is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \binom{2n}{n} z^{2n} = \frac{1}{\sqrt{1+z^2}},$$

which is in this case algebraic. It is an old theorem of Eisenstein [9] that algebraic functions which are regular at $z = 0$ are G-functions.

(4) A rich family of examples of arithmetic Gevrey series is given by *hypergeometric series*. For integers $p, q \geq 0$, and rational numbers $a_1, \dots, a_p \in \mathbf{Q}$ and $b_1, \dots, b_q \in \mathbf{Q} \setminus \mathbf{Z}_{\leq 0}$, they are given by

$$f\left(\begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \Big| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} z^n,$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ denotes the Pochhammer symbol. We claim that this is an arithmetic Gevrey series of order $p - q$. The only difficulty is to show that the denominators have the right growth, which follows from a lemma by Siegel [16] deducing that

$$\operatorname{den}\left(\frac{(a)_0}{(b)_0}, \frac{(a)_1}{(b)_1}, \dots, \frac{(a)_n}{(b)_n}\right) < C^n$$

from a weak form of the prime number theorem.

In his 1929 paper, Siegel asked whether every E-function can be written as a polynomial expression in E-functions of the form

$$f\left(\begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix} \middle| \lambda z^{p-q}\right)$$

for various values of $q > p \geq 0$, the parameters a_i's and b_j's, and $\lambda \in \overline{\mathbf{Q}}$. The answer turns out to be positive for E-functions which are a solution of a differential equation of order at most 2 (Gorelov, 2004), but negative starting from differential equations of order 3 (Fresán–Jossen, 2021). Fischler and Rivoal [13] raised the question whether every G-function is a polynomial expression in G-functions of the form

$$\mu(z) \cdot f\left(\begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_b \end{matrix} \middle| \lambda(z)\right),$$

where μ and λ are algebraic over $\overline{\mathbf{Q}}(z)$ and holomorphic at $z = 0$, and $\lambda(0) = 0$. The answer is expected to be negative already for G-functions that are solution of a differential equation of order 2.

1.4 Structure of the differential equations

Consider a differential operator of order μ, written as

$$L = P_\mu(z)\left(\frac{d}{dz}\right)^\mu + \cdots + P_0(z).$$

The *singularities* at finite distance of L are the zeros of the leading polynomial P_μ. To deal with the point at infinity, one considers the differential operator $\widetilde{L}$ obtained by formally replacing z with $1/z$ and d/dz with $-z^2 d/dz$ in L. One says that ∞ is a singularity of L if 0 is a pole of some of the rational coefficients of $\widetilde{L}$. If $\xi \in \mathbf{C}$ is not a singularity, then L admits a basis of holomorphic solutions in $\mathbf{C}[[z-\xi]]$. This might still hold for some singularities ξ, and one then says that they are *trivial*. A singularity $\xi \in \mathbf{C}$ is said to be *regular* if there exists a basis of solutions of the form[1]

$$\sum_{\text{finite}} (z-\xi)^{\alpha_i} \log^{k_i}(z-\xi) f_i(z-\varepsilon)$$

[1]The regularity of a singularity can be tested purely in terms of the differential operator by means of the so-called *Fuchs's criterium*: the singularity ξ is regular if and only if, for all i,

$$i - \operatorname{ord}_\xi(P_i) \leq \mu - \operatorname{ord}_\xi(P_\mu).$$

where α_i are complex numbers, called the *exponents*, k_i are non-negative integers, and f_i are complex power series. In case f_i is a $\mathbf{C}$-linear combination of arithmetic Gevrey series of order s, we call such an expression a *Nilson–Gevrey arithmetic function* of order s. We denote by $\mathrm{NGA}\{z-\xi\}_s$ the set of those.

Theorem 1.2 (André, Chudnovsky[2], Katz). *Let $L \in \overline{\mathbf{Q}}[z, d/dz]$ be a differential operator of minimal order annihilating a G-function. Then L has regular singularities, rational exponents, and a basis of solutions in $\mathrm{NGA}\{z-\xi\}_0$ for every $\xi \in \mathbf{C} \cup \{\infty\}$.*

Theorem 1.3 (André [3]). *Let $L \in \overline{\mathbf{Q}}[z, d/dz]$ be a differential operator of minimal order annihilating an E-function. Then:*

(1) *the only non-trivial singularities of L are 0 and ∞;*

(2) *0 is a regular singularity with rational exponents and there exists a basis of solutions in $\mathrm{NGA}\{z\}_{-1}$.*

(3) *∞ is in general an irregular singularity and there exists a basis of solutions of the form $u_i(1/z)\exp(\zeta_i z)$, with $u_i \in \mathrm{NGA}\{1/z\}_1$ and $\zeta_i \in \overline{\mathbf{Q}}$.*

1.5 Arithmetic of special values

By a *special value* of an arithmetic Gevrey series we will generally mean the value taken at an algebraic argument $\alpha \in \overline{\mathbf{Q}}$, but of course one needs to be careful with the various radiuses of convergence. E-functions have infinite radius of convergence; the set of their special values will be denoted by

$$\mathbf{E} = \{f(\alpha) \mid f \text{ an } E\text{-function }, \alpha \in \overline{\mathbf{Q}}\}.$$

Excluding the trivial case of polynomials, G-functions have finite non-zero radius of convergence. By a special value we will mean the evaluation at an argument inside the disc of convergence. One could consider the more general notion of evaluation at any algebraic argument of any analytic continuation of any G-function, but this doesn't change the set $\mathbf{G}$ of special values. An intriguing difference between the sets $\mathbf{E}$ and $\mathbf{G}$ comes from the results of Fischler and Rivoal [10, 11] that

$$\mathbf{G} = \{f(1) \mid f \text{ a } G\text{-function with coefficients in } \mathbf{Q}(i)\},$$

whether letting $\mathbf{E}_K$ denote the set of special values of E-functions with coefficients in a given number field K, multiplication induces an isomorphism $\mathbf{E}_K \otimes_K \overline{\mathbf{Q}} \to \mathbf{E}$.

Theorem 1.4 (Siegel–Shidlovsky–André–Beukers). *Let $n \geq 1$ be an integer, and let $f_1, \ldots, f_n$ be E-functions satisfying a linear system of differential equations*

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = A(z)\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \qquad A(z) \in M_{n\times n}(\overline{\mathbf{Q}}(z)).$$

Let $\alpha \in \overline{\mathbf{Q}}$ be a non-zero algebraic number at which none of the entries of $A(z)$ has a pole. Then every algebraic relation among the numbers $f_1(\alpha), \ldots, f_n(\alpha)$ arises by specialisation at $z = \alpha$ from an algebraic relation among the functions $f_1(z), \ldots, f_n(z)$.

Concretely, this means that for every polynomial $P \in \overline{\mathbf{Q}}[X_1, \ldots, X_n]$ satisfying

$$P(f_1(\alpha), \ldots, f_n(\alpha)) = 0,$$

there exists a polynomial $Q \in \overline{\mathbf{Q}}[z, X_1, \ldots, X_n]$ satisfying

$$Q(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n) \text{ and } Q(z, f_1(z), \ldots, f_n(z)) = 0.$$

In particular, there is an equality of transcendence degrees

$$\operatorname{trdeg}_{\overline{\mathbf{Q}}} \overline{\mathbf{Q}}(f_1(\alpha), \ldots, f_n(\alpha)) = \operatorname{trdeg}_{\mathbf{Q}(z)} \mathbf{Q}(z)(f_1, \ldots, f_n).$$

Such kind of results are in general totally false for G-functions (for those coming from geometry, this is related to the existence of Hodge loci).

For the functions $f_i(z) = e^{\alpha_i z}$, the Siegel–Shidlovsky theorem gives back the Hermite–Lindemann–Weierstrass theorem, on noting that all functional relations come from $\mathbf{Q}$-linear relations between the α_i's. In the case of the Bessel function $J_k(z)$ and its derivative, the system of linear differential equations takes the form

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{z^2-k}{z^2} & -\frac{1}{z} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

so the result applies to each non-zero algebraic number α. Since one can prove, for example using differential Galois theory, that the function $J_k(z)$ and its derivative are algebraically independent, it follows that $J_k(\alpha)$ is transcendental, and even algebraically independent with $J_k'(\alpha)$. One of the goals of these lectures is to explain, following André and Beukers, how one can prove such transcendence results purely from the structure of the differential equations satisfied by E-functions ("Transcendance sans transcendance"). The motivating example of the Hermite–Lindemann–Weierstrass theorem will be explained in the next chapter.

In the case of $\ni$-functions (again excluding the case of polynomials), the radius of convergence is 0. We will discuss two ways to give a meaning to, say, $\sum_{n=0}^{\infty} a_n n!$ for an $\ni$-function $\sum_{n=0}^{\infty} a_n n! z^n$ with rational coefficients:

(1) the series $\sum_{n=0}^{\infty} a_n n!$ converges in $\mathbf{Q}_p$ for almost all prime numbers p (we don't know much about these p-adic numbers, for example, $\sum_{n=0}^{\infty} n!$ is not known to be irrational for a single value of p);

(2) the associated G-function $g(z) = \sum_{n=0}^{\infty} a_n z^n$ has analytic continuation along almost all half rays $[0, \xi\infty]$, and one can consider the finite integral

$$\int_0^{\xi\infty} g(x)e^{-x}dx.$$

For example, for Euler's Э-function, one finds Gompertz's constant $\int_0^{\infty} \frac{e^{-x}}{1+x}dx$. An extension of the Siegel–Shidlovsky theorem to these values would imply that this number is transcendental.

1.6 Integral representations

Some classical G-functions, like Gauss's hypergeometric function admit integral representations that realise them as period functions of families of algebraic varieties:

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c-a)} {}_2F_1\left(\begin{matrix} ab \\ c \end{matrix} \middle| z\right) = \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}dt.$$

In this case, it is a family of curves $y^N = x^A(1-x)^B(1-tz)^C$ parameterized by t.

Theorem 1.5 (André, [2]). *Every period function $\int_\gamma \omega_z$ associated with a family $X \to \mathbb{P}^1 \setminus S$ of algebraic varieties over $\overline{\mathbf{Q}}$ belongs to*[2] $\mathrm{NGA}\{z\}_0$.

Conjecture 1.2 (Bombieri–Dwork). *Every G-function arises this way.*

The numerical aspect of this conjecture is that the set $\mathbf{G}$ of special values of G-functions should be equal to the ring of periods with π inverted. I will explain the "easy" inclusion. The fact that $1/\pi$ belongs to $\mathbf{G}$ follows from formulas of Ramanujan, e.g.

$$\frac{1}{\pi} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1)(1/2)_n^3}{n!^3},$$

where the right-hand side is a special value of a hypergeometric G-function.

What about E-functions? One way to find them in geometry is to consider integrals of the shape

$$\int_\sigma e^{-zf}\omega,$$

[2] one can be more precise about the coefficients in the complex linear combinations

where $f\colon X \to \mathbb{A}^1$ is a regular function on some algebraic variety X defined over $\overline{\mathbf{Q}}$, ω is an algebraic differential form on X and σ is a rapid decay cycle on $X(\mathbf{C})$, which roughly means that $\mathrm{Re}(f)$ goes to infinity along the boundary $\partial\sigma$.

Theorem 1.6 (Fresán–Jossen). *The exponential period function* $\int_\sigma e^{-zf}\omega$ *belongs to* $\mathrm{NGA}\{z\}_{-1}$*, and more precisely to*

$$\overline{\mathbf{Q}}(\mathit{periods}, \Gamma(1/d), \ldots, \Gamma((d-1)/d), \gamma)[z^{\frac{1}{d}\mathbf{Z}}, \log(z), E\text{-}\mathit{functions}],$$

where d *is the order of the finite part of the monodromy around* $z = 0$.

Corollary 1.1. *Every exponential period* $\int_\sigma e^{-f}\omega$ *is a polynomial expression in classical periods, special values of the gamma function, Euler's gamma constant, and special values of* E*-functions.*

Conjecture 1.3. *Every* E*-function arises this way.*

We will show that this is closely related to the Bombieri–Dwork conjecture.

1.7 Historical notes

Hermite (1873) proved that the exponentials of distinct rational numbers are $\mathbf{Q}$-linearly independent, which suffices to establish the transcendence of e. In 1882, Lindemann [14] generalised his method to show that e^α is transcendental for every non-zero algebraic number α, and stated without proof Theorem 1.1. A proof was given by Weierstrass (1885) and later simplified by Hilbert (1893).

Born tells that when he was hesitating between becoming a physicist or a mathematician, Hilbert suggested him to try to prove that some zeros of some Bessel functions are transcendental; after a few months he gave up and went into physics.

Weil tells in his memoir [**?**]

Chapter 2

The Hermite–Lindemann–Weierstrass theorem

The modern theory of E-functions started with the discovery by Bézivin and Robba [6] of the equivalence between the Hermite–Lindemann–Weierstrass theorem and a statement about exponential polynomials vanishing at 1.

2.1 Equivalent formulations

Recall that an *exponential polynomial* is a formal power series of the form

$$f(z) = \sum_{i=1}^{m} P_i(z) e^{\alpha_i z},$$

where P_i are polynomials with algebraic coefficients and α_i are algebraic numbers, which will always be assumed to be distinct. We say that f has *constant coefficients* if all P_i's are constant polynomials. If $\alpha_1, \dots, \alpha_m$ are distinct complex numbers, then the functions $e^{\alpha_i z}$ are $\mathbf{C}(z)$-linearly independent (see Lemma 2.1 below). In particular, an exponential polynomial is identically zero if and only if $P_i = 0$ for all i.

Proposition 2.1. *The following statements are equivalent:*

(1) *Let $\alpha_1, \dots, \alpha_n$ be distinct algebraic numbers. Then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbf{Q}}$.*

(2) *Let $\alpha_1, \dots, \alpha_m$ be algebraic numbers that are linearly independent over $\mathbf{Q}$. Then $e^{\alpha_1}, \dots, e^{\alpha_m}$ are algebraically independent over $\mathbf{Q}$.*

(3) *Let $f \in \mathbf{Q}[[z]]$ be an exponential polynomial with rational coefficients satisfying $f(1) = 0$. Then $f(z)/(z-1)$ is an exponential polynomial.*

(4) *Let $f \in \mathbf{Q}[[z]]$ be an exponential polynomial with constant coefficients. If $f(1) = 0$, then $f = 0$.*

Proof. We first prove that (1) implies (2). Let $\alpha_1, \dots, \alpha_m$ be distinct algebraic numbers, and let $P = \sum b_I X^I \in \mathbf{Q}[X_1, \dots, X_m]$ be a polynomial satisfying $P(e^{\alpha_1}, \dots, e^{\alpha_m}) = 0$. Using the functional equation $e^{z+w} = e^z e^w$, this can be written as $\sum b_I e^{\alpha_I} = 0$ for some algebraic numbers α_I, which are all distinct because of the $\mathbf{Q}$-linear independence of the α_i's. By (1), all b_I are zero, and hence $P = 0$.

Conversely, let $\sum_{i=1}^{n} \beta_i e^{\alpha_i} = 0$ be a $\overline{\mathbf{Q}}$-linear relation among the e^{α_i}'s, and let K be a Galois number field containing all β_i's. Then

$$P(X_1, \dots, X_n) = \prod_{\sigma \in \mathrm{Gal}(K/\mathbf{Q})} \big(\sigma(\beta_1) X_1 + \dots + \sigma(\beta_n) X_n\big)$$

is a polynomial with rational coefficients satisfying $P(e^{\alpha_1}, \ldots, e^{\alpha_n}) = 0$. By (2), P is the zero polynomial, and since all factors are Galois conjugates to each other, this implies that $\beta_1 = \cdots = \beta_n = 0$. Hence (1) and (2) are equivalent.

That (1) implies (3) is seen as follows. Let $f(z) = \sum_{i=0}^m P_i(z)e^{\alpha_i z} \in \mathbf{Q}[[z]]$ be an exponential polynomial. If the $\overline{\mathbf{Q}}$-linear combination $f(1) = \sum_{i=1}^m P_i(1)e^{\alpha_i}$ vanishes, then $P_i(1) = 0$ by (1), and hence there exist polynomials $Q_i(z)$ such that

$$f(z) = (z-1)\sum_{i=1}^m Q_i(z)e^{\alpha_i z}.$$

Let us now prove that (3) implies (4). Let $f = \sum_{i=1}^t \beta_i e^{\alpha_i z} \in \mathbf{Q}[[z]]$ be such that $f(1) = 0$. By (3), there exist polynomials $Q_i \in \overline{\mathbf{Q}}[z]$ such that

$$\sum_{i=1}^t \beta_i e^{\alpha_i z} = \sum_{j=1}^r (z-1)Q_i(z)e^{\gamma_i z},$$

but then all β_i vanish by the linear independence of the functions $e^{\alpha_i z}$, and hence $f = 0$.

Finally, that (4) implies (1) is seen as follows. Let $\alpha_1, \ldots, \alpha_m$ be distinct algebraic numbers and let $\sum_{i=1}^n \beta_i e^{\alpha_i} = 0$ be a $\overline{\mathbf{Q}}$-linear relation. Then

$$f(z) = \prod_{\sigma \in \mathrm{Gal}(K/\mathbf{Q})} \sum_{i=1}^n \sigma(\beta_i)e^{\sigma(\alpha_i)z} \in \mathbf{Q}[[z]]$$

is an exponential polynomial with constant rational coefficients satisfying $f(1) = 0$. By (4), this implies $f = 0$, and hence $\sum_{i=1}^n \beta_i e^{\alpha_i z} = 0$. Since the functions $e^{\alpha_i z}$ are linearly independent, we get $\beta_1 = \cdots = \beta_n = 0$. ∎

2.2 The modified Laplace transform

Definition 2.1. Let $f(z) = \sum_{n=0}^\infty \frac{f_n}{n!} z^n \in \mathbf{C}[[z]]$ be a formal power series. The *modified Laplace transform* of f is the formal power series

$$\hat{f}(y) = \sum_{n=0}^\infty f_n y^n \in \mathbf{C}[[z]].$$

Actually, the "true" Laplace transform is given by the integral

$$(\mathcal{L}f)(y) = \int_0^\infty f(z)e^{-yz}dz.$$

If f is of *exponential type* (that is, $|f(z)| = O(e^{|C|z})$ for some real number $C > 0$), then $\mathcal{L}f$ converges for $\mathrm{Re}(y) > C$ and

$$(\mathcal{L}f)(y) = \sum_{n=0}^\infty \frac{f_n}{n!}\int_0^\infty z^n e^{-yz}dz = \sum_{n=0}^\infty \frac{f_n}{y^{n+1}},$$

so that $\hat{f}(y) = \frac{1}{y}(\mathcal{L}f)(\frac{1}{y})$ has radius of convergence at least $1/C$. Note that this is in particular the case if f is an E-function.

We will use the following three properties of modified Laplace transform:

(1) f is of exponential type if and only if $\hat{f}$ has a non-zero radius of convergence.

(2) The modified Laplace transform of $zf(z)$ is $y^2\hat{f}'(y) + y\hat{f}(y)$.

(3) f is an exponential polynomial if and only if $\hat{f}$ is a rational function.

Property (2) follows from a direct computation. Indeed,

$$zf(z) = \sum_{n=0}^{\infty} \frac{f_n}{n!} z^{n+1} = \sum_{n=1}^{\infty} n f_{n-1} \frac{z^n}{n!}$$

has modified Laplace transform $\sum_{n=1}^{\infty} n f_{n-1} y^n$, and this formal power series is the same as $y^2\hat{f}'(y) + y\hat{f}(y)$.

Lemma 2.1. *Let $\alpha_1, \ldots, \alpha_n$ be distinct complex numbers. The functions $e^{\alpha_1 z}, \ldots, e^{\alpha_n z}$ are linearly independent over $\mathbf{C}(z)$.*

Proof. It suffices to prove that a non-trivial linear combination

$$f(z) = P_1(z)e^{\alpha_i z} + \cdots + P_n(z)e^{\alpha_n z}$$

where $P_i(z)$ lie in $\mathbf{C}[z]$, cannot be identically zero. This is obviously true for $n = 1$. If $n \geq 2$, then one of the exponents α_j is non-zero, and then the rational function $\hat{f}$ has a pole at $1/\alpha_j$, so it cannot be identically zero. ∎

2.3 Proof of Proposition 2.1

(3) *implies* (1). By contradiction, assume that there exists a non-trivial $\overline{\mathbf{Q}}$-linear combination

$$\sum_{i=1}^{m} \beta_i e^{\alpha_i z} = 0.$$

Let K be a Galois number field containing all α_i' and $\beta_i's$. Then

$$f(z) = \prod_{\sigma \in \mathrm{Gal}(K/\mathbf{Q})} \sum_{i=0}^{m} \sigma(\beta_i) e^{\sigma(\alpha_i) z}$$

is a formal power series with *rational* coefficients. Let us write it as

$$f(z) = \sum_{i=1}^{t} b_i e^{a_i z} = \sum_{n=0}^{\infty} \frac{f_n}{n!} z^n \in \mathbf{Q}[[z]],$$

where of course $f_n = \sum_{i=1}^t b_i a_i^n$, so that the modified Laplace transform of f is the rational function

$$\hat{f}(y) = \sum_{n=0}^{\infty} \Big(\sum_{i=1}^{t} b_i a_i^n \Big) z^n = \sum_{i=1}^{t} \frac{b_i}{1 - a_i y}.$$

Since f is not identically zero (Lemma 2.1) and $f(1) = 0$, we necessarily have $t \geq 2$ and at least one a_j is non-zero. Let us write $f(z) = (z-1)g(z)$ and take modified Laplace transform on both sides of the equality:

$$\hat{f}(y) = y^2 \hat{g}'(y) + (y-1)\hat{g}(y).$$

By (3), the formal power series $\hat{g}$ is a rational function. Since $\hat{f}$ has a pole at $1/a_j$, so does $\hat{g}$. But then $1/a_j$ is a pole of order ≥ 2 of $\hat{f}$, in contradiction with the explicit expression we found that only has simple poles. ∎

2.4 Proofs of the Hermite–Lindemann–Weierstrass theorem

In view of the equivalences from Proposition 2.1, I know of at least four different proofs of the Hermite–Lindemann–Weierstrass theorem.

- The proof by Bézivin and Robba [6], who rely on the Polya-Bertrandias rationality criterion for rationality of power series to show that if $u \in \mathbf{Q}[[z]]$ is a formal power series with non-zero radius of convergence such that

  $$z^2 u'(z) + (z-1)u(z)$$

  is a rational function, then u is itself a rational function.
- A proof by hand that Beukers found shortly after Bézivin and Robba announced their result [5].
- Two proofs by André [4] based on the structure theorems for differential equations of E-functions and G-functions.

  In what follows, I sketch some of those.

Proof by hand. We need to prove that the modified Laplace transform

$$\hat{g}(y) = \sum_{n=0}^{\infty} g_n y^n, \quad g_n = n! \sum_{k=0}^{n} \frac{f_n}{k!}, \quad f_n = \sum_{i=1}^{t} b_i a_i^n$$

is a rational function. If this is the case, then the poles of $\hat{g}(y)$ should be at the algebraic numbers $1/a_i$, so letting

$$(X - 1/a_1) \cdots (X - 1/a_t) = 1 - c_1 X - \cdots - c_t X^t,$$

we expect that there exists an integer $k \geq 0$ such that

$$(1 - c_1 y - \cdots - c_t y^t)^k \sum_{n=0}^{\infty} g_n y^n = \sum_{n=0}^{\infty} g_n(k) y^n$$

is a polynomial, that is, $g_n(k) = 0$ for all large enough n.

Note that the $c_i's$ are rational numbers, since the set of a_i is, by definition of f, stable under the action of the Galois group. Multiplying f by a suitable integer, we can arrange that $f_0, \ldots, f_{t-1}$ are integers. Let D be the common denominator of $c_1, \ldots, c_t$, and set $A = \max(1, |a_i|)$ and $C = 1 + |c_1| + \cdots + |c_t|$. Then using the recurrences

$$\begin{aligned} f_{n+t} &= c_1 f_{n+t-1} + \cdots + c_t u_n \\ g_n(k+1) &= g_n(k) - c_1 g_{n-1}(k) - \cdots - c_t g_{n-t}(k), \end{aligned}$$

one proves the following three properties for all $n \geq kt$:

(1) $|g_n(k)| \leq \lambda A^n C^k$ for some $\lambda > 0$,

(2) $D^n g_n(k)$ is an integer,

(3) $k!$ divides $D^n g_n(k)$.

If $n \geq kt$ and $g_n(k)$ is non-zero, then $k! \leq |D^n g_n(k)| \leq \lambda (AD)^n C^k$. But the lower bounds grows faster than the upper bound, so that if $k! > \lambda (AD)^n C^k$ and $n \geq kt$, then $g_n(k) = 0$. We then choose k_0 in such a way that $k! > \lambda (AD)^{10kt} C^k$ for all $k \geq 0$, and deduce $g_n(k) = 0$ for all (n, k) in the blue region in the picture. From the recurrence relation, we get $g_n(k) = 0$ for all (n, k) in the orange region, and hence $g_n(k) = 0$ for all $n \geq k_0 t$, as we wanted. ■

Proof by André using the differential equations of E-functions. To exploit the structure theorems for differential equations, we will first need:

Lemma 2.2. *Let $f(z) \in \mathbf{Q}[[z]]$ be an E-function with rational coefficients satisfying $f(1) = 0$. Then $f(z)/(z-1)$ is an E-function.*

Proof. Write $f(z) = \sum_{n=0}^{\infty} a_n z^n / n!$ and $g(z) = f(z)/(z-1)$, so that

$$g(z) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!} \quad \text{with} \quad b_n = n! \sum_{k=0}^{n} \frac{a_k}{k!}.$$

If f is annihilated by a differential operator L, then g is annihilated by the differential operator $L \cdot (z-1)$. Besides, a common denominator for $a_0, \ldots, a_n$ is also a common denominator for $b_0, \ldots, b_n$, so it only remains to check the growth of $|b_n|$. Here is where we use the vanishing of $f(1)$ in order to get $b_n = -n! \sum_{n=k+1}^{\infty} a_k / k!$, and hence

$$|b_n| \leq n! \sum_{k=n+1}^{\infty} \frac{|a_k|}{k!} \leq (n+1)! \sum_{k=n+1}^{\infty} \frac{C^k}{k!} = e^C C^{n+1}$$

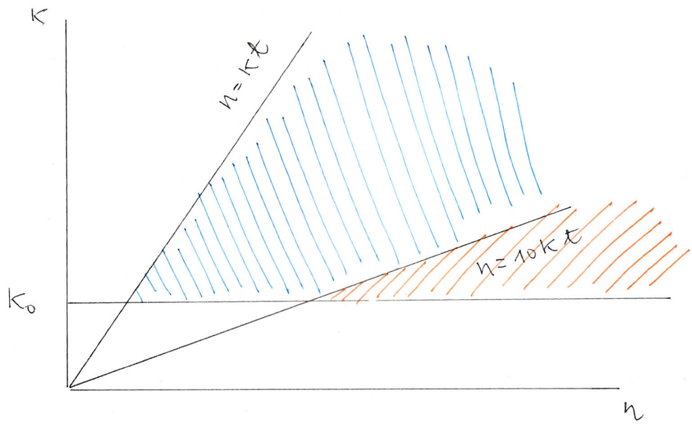

Figure 2.1. Proof by Beukers

if $|a_n| \leq C^n$ for all $n \geq 1$. Therefore, $|b_n| \leq D^n$ for some real number $D > 0$. ■

Remark 2.1. The result is still true for E-functions with algebraic coefficients. The proof is literally the same if one puts the stronger hypothesis that $\sum_{n=0}^{\infty} \frac{\sigma(a_n)}{n!} z^n$ vanishes at $z = 1$ for all $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$. With the weaker hypothesis, it is a bit harder since $f(1) = 0$ does not give a priori any information about the growth of $\sigma(a_n)$.

Let us recall from the overview (Theorem 1.3) that a differential operator of minimal order L annihilating a non-zero E-function f can only have non-trivial singularities at 0 and ∞. Assume $f(1) = 0$. Then $L \cdot (z-1)$ is a differential operator of minimal order annihilating the E-function $g(z) = f(z)/(z-1)$. Therefore, $L \cdot (z-1)$ has a basis of solutions in $\mathbf{Q}[[z-1]]$, which means that L has a basis of solutions in $(z-1)\mathbf{Q}[[z-1]]$. This means in particular that *all* solutions of L vanish at $z = 1$, which can only happen if 1 is a singularity of L.

But now let us return to our situation, where

$$f(z) = \sum_{i=1}^{t} b_i e^{a_i z}.$$

In that case, f is annihilated by a differential operator of minimal order with constant coefficients, namely $L = (\frac{d}{dz} - a_1) \cdots (\frac{d}{dz} - a_t)$. Since 1 is not a singularity of L, we get $f = 0$, and hence all b_i are zero. ■

Chapter 3

G-operators and the theorem of Chudnovsky

3.1 Galochkin's condition and the theorem of Chudnovsky

Let K be a number field. Given a linear system of differential equations

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} = A(z)\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} \qquad A(z) \in M_{\mu\times\mu}(K(z)), \tag{3.1}$$

we define a sequence $(A_n)_{n\geq 0}$ of matrices in $M_{\mu\times\mu}(K(z))$ by the recurrence relation

$$A_0 = I_\mu, \quad A_{n+1} = A_n A + \frac{d}{dz}A_n \quad \text{for } n \geq 0,$$

so that the following identity holds:

$$(\frac{d}{dz})^n \begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} = A_n(z)\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix}.$$

If $T(z) \in K[z]$ is a common denominator for the entries of $A(z)$, then all the entries of the matrix $T(z)^n A_n(z)$ lie in $K[z]$.

Definition 3.1. Let q_n denote the smallest integer ≥ 1 such that the condition

$$q_n \frac{T(z)^r A_r(z)}{r!} \in M_{\mu\times\mu}(\mathcal{O}_K[z])$$

holds for all $r = 1, \ldots, n$. We say that the linear system of differential equations (3.1) satisfies *Galochkin's condition* if there exists a real number $C > 0$ such that

$$q_n \leq C^n \text{ holds for all } n \geq 1.$$

Definition 3.2. A *G-operator* is a differential operator

$$L = \sum_{i=0}^{\mu} P_i(z)(\frac{d}{dz})^i \in K(z)[\frac{d}{dz}]$$

such that the associated linear system of differential equations

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & & & \\ -\frac{P_0}{P_\mu} & -\frac{P_1}{P_\mu} & \cdots & -\frac{P_{\mu-1}}{P_\mu} \end{pmatrix}\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix}$$

satisfies *Galochkin's condition*.

Example 3.1.1.

(1) The operator $L = \frac{d}{dz} - 1$ is not a G-operator. Indeed, it corresponds to the matrix $A(z) = 1$, so that all $A_r(z)$ are equal to 1 and $q_n = n!$, which does not grow geometrically.

(2) The operator $L_a = (1-z)\frac{d}{dz} - a$ is a G-operator for each $a \in \mathbf{Q}$. In this case, $A(z) = \frac{a}{1-z}$ and $A_r(z) = \frac{(a)_r}{(1-z)^r}$, where $(a)_r$ is the Pochhammer symbol. The result follows from Lemma A.1 according to which the common denominator of $(a)_1/1!, \ldots, (a)_n/n!$ grows geometrically.

It follows from the theorems of André and Katz stated below that a G-operator of order 1 is of the form

$$L = \frac{d}{dz} - \sum_{i=1}^{m} \frac{c_i}{z - \alpha_i}$$

for some integer $m \geq 0$, *rational* numbers $c_i \in \mathbf{Q}$ and algebraic numbers $\alpha_i \in \overline{\mathbf{Q}}$. In particular, if a is an algebraic but irrational number, L_a from Example 3.1.1 is not a G-operator. The classification of G-operators of order 2 seems to be out of reach[1].

Remark 3.1. The name of G-operator comes from the fact that, if L is a G-operator, then for each $\alpha \in \overline{\mathbf{Q}}$ which is not a singularity of L, there exists a basis of solutions

$$f_1(z-\alpha), \ldots, f_\mu(z-\alpha)$$

where $f_i \in \overline{\mathbf{Q}}[[z]]$ are G-functions. It is also true, "with monodromy", for singular points but harder to prove.

Remark 3.2. The condition of being a G-operator can be expressed purely in terms of L as follows. Let $L = P_\mu(z)(d/dz)^\mu + \cdots$ There exists a unique sequence $(L_r)_{r\geq 1}$ of elements of $K[z, \frac{d}{dz}]$ such that L_r is a left multiple of L and

$$L_r = \frac{1}{r!} P_\mu(z)^r \left(\frac{d}{dz}\right)^{\mu+r-1} + \sum_{j=0}^{\mu-1} Q_{r,j}(z) \left(\frac{d}{dz}\right)^j.$$

Then L satisfies Galochkin's condition if there exists a real number $C > 0$ such that, for all $n \geq 1$, the common denominator of the coefficients of $Q_{r,j}(z)$ for $j = 0, \ldots, \mu - 1$ and $m = 1, \ldots, n$ is bounded by C^n.

[1]The structure of those coming from inhomogeneous differential equations of order 1 is discussed in [12].

3.2 The theorem of Chudnovsky

Theorem 3.1 (Chudnovsky brothers). *Let $f_1, \ldots, f_\mu$ be $\overline{\mathbf{Q}}(z)$-linearly independent G-functions satisfying a system of linear differential equations*

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} = A(z)\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} \qquad A(z) \in M_{\mu\times\mu}(\overline{\mathbf{Q}}(z)).$$

Then $A(z)$ satisfies Galochkin's condition.

By definition, if L is a differential operator of minimal order μ annihilating a G-function f, then the G-functions $f, f', \ldots, f^{(\mu-1)}$ are $\overline{\mathbf{Q}}(z)$-linearly independent. Hence, an important corollary of Theorem 3.1 is the following.

Corollary 3.1. *A differential operator of minimal order annihilating a G-functions is a G-operator.*

Remark 3.3. In the ring $\overline{\mathbf{Q}}(z)[\frac{d}{dz}]$ there is Euclidean multiplication that makes every left ideal principal. In particular, given a power series $f \in \overline{\mathbf{Q}}[[z]]$ which is a solution of a non-zero differential operator, the annihilator ideal

$$\mathrm{Ann}(f) = \{L \in \overline{\mathbf{Q}}(z)[\frac{d}{dz}] \mid Lf = 0\}$$

is of the form $\overline{\mathbf{Q}}(z)[\frac{d}{dz}]\Theta$ for some differential operator Θ of minimal order among those annihilating f. In particular, if L is irreducible and $Lf = 0$, then L is necessarily of minimal order, but we can have operators of minimal order which are not irreducible, for example $L = (\frac{d}{dz} + 1)(\frac{d}{dz} - 1)$ for the series $f = e^z + e^{-z}$.

Another proof of the Hermite–Lindemann–Weierstrass theorem. Let $f \in \mathbf{Q}[[z]]$ be an exponential polynomial satisfying $f(1) = 0$, and write $f(z) = (z-1)g(z)$. Recall from Chapter 2 that we want to prove that g is an exponential polynomial, which amounts to proving that its Laplace transform $\hat{g}$ is a rational function. This G-function satisfies the inhomogeneous differential equation

$$\hat{f} = y^2 \frac{d}{dy}\hat{g} + (y-1)\hat{g},$$

which can be rewritten as the linear system of differential equations

$$\frac{d}{dy}\begin{pmatrix} 1 \\ \hat{g} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{\hat{f}}{y^2} & \frac{1-y}{y^2} \end{pmatrix}\begin{pmatrix} 1 \\ \hat{g} \end{pmatrix}.$$

Since $\hat{f}$ is a rational function, the matrix belongs to $M_{2\times 2}(\overline{\mathbf{Q}}(z))$. One then checks that this matrix does not satisfy Galochkin's condition, so by Theorem 3.1 the G-functions 1 and $\hat{g}$ are not $\overline{\mathbf{Q}}(z)$-linearly independent, which means that $\hat{g}$ is rational. ∎

3.3 Sketch of proof of Theorem 3.1

Chapter 4

Structure of G-operators

Let K be a number field with ring of integers $\mathcal{O}_K$, and let

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} = A(z)\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} \qquad A(z) \in M_{\mu\times\mu}(K(z)), \tag{4.1}$$

be a system of linear differential equations. Recall the sequence of matrices $(A_r)_{r\geq 1}$ from Chapter 3.

4.1 Generic radius of solvability

For each non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ above a prime number p, let

$$|\cdot|_\mathfrak{p} \colon K \longrightarrow \mathbf{R}_{\geq 0}$$

be the corresponding non-archimedean norm, normalised so that $|p|_\mathfrak{p} = 1/p$. Consider the field of rational functions $K(t)$ in an indeterminate t independent from z (a *generic* point) and endow it with the Gauss norm

$$\left|\frac{\sum a_i t^i}{\sum b_j t^j}\right|_G = \frac{\max |a_i|_\mathfrak{p}}{\max |b_j|_\mathfrak{p}},$$

which is well defined. Its valuation ring is given by

$$A = \{f \in K(t) \mid |f|_G \leq 1\} = \{P/Q \mid P, Q \in \mathcal{O}_K[t], Q \notin \mathfrak{p}[z]\},$$

and there is a reduction mod $\mathfrak{p}$ map

$$\begin{aligned} A &\longrightarrow (\mathcal{O}_K/\mathfrak{p})(t) \\ P/Q &\longmapsto \overline{P}/\overline{Q}. \end{aligned}$$

Let Ω be an algebraically closed field of $K(t)$, along with a complete norm extending the Gauss norm. Consider

$$\mathcal{U}(z) = \sum_{r=0}^{\infty} \frac{A_r(t)}{r!}(z-t)^r \in \mathrm{GL}_\mu(\Omega[[z-t]]),$$

a *fundamental matrix solution at the generic point*.

Definition 4.1. The *generic radius of solvability* at $\mathfrak{p}$ of the system (3.1) is

$$R_{\mathfrak{p}} = \sup\{R \mid \sum_{r=0}^{\infty} \frac{A_r(t)}{r!}(z-t)^r \text{ converges in } \Omega \text{ for } |z-t| < R\}.$$

Let us extend the Gauss norm to a matrix $B = (B_{ij})$ of rational functions by defining

$$|B|_G = \max |B_{ij}|_G.$$

The result is not a norm, but satisfies $|BC|_G \leq |B|_G |C|_G$.

Lemma 4.1. *The generic radius of solvability satisfies the inequality*

$$R_{\mathfrak{p}} \geq \frac{p^{-1/(p-1)}}{\max(1, |A|_G)}.$$

In particular, $R_{\mathfrak{p}}$ is positive, and $R_{\mathfrak{p}} \geq p^{-1/(p-1)}$ if $|A|_G = 1$.

Proof. The recurrence relation $A_{r+1} = A_r A + A_r'$ implies the inequality

$$|A_r|_G \leq \max(1, |A|_G)^r,$$

using that if $|f|_G \leq 1$, then $|f'|_G \leq 1$. Therefore,

$$\begin{aligned} \left|\frac{A_r(r)}{r!}\right|_G |z-t|^r &\leq \frac{\max(1, |A|_G)^r}{|r!|_{\mathfrak{p}}} |z-t|^r \\ &\leq \frac{\max(1, |A|_G)^r}{p^{-r/(p-1)}} |z-t|^r, \end{aligned}$$

where the second inequality comes from $v_p(r!) = \frac{r - S_r}{p-1} \leq \frac{r}{p-1}$ (see Appendix A). ■

Since all radii of solvability are non-zero, we can introduce the quantity

$$\rho(A) = \sum_{\mathfrak{p} \subset O_K} \log^+ \left(\frac{1}{R_{\mathfrak{p}}}\right) \in [0, +\infty],$$

where $\log^+(x) = \log \max(1, x)$.

Definition 4.2. The system of linear differential equations given by the matrix A satisfies *Bombieri's condition* if $\rho(A) < +\infty$.

Lemma 4.2. *If $\rho(A) < +\infty$, then the inequality $R_{\mathfrak{p}} > p^{-1/(p-1)}$ holds for every prime ideal $\mathfrak{p}$ above a set of prime numbers p with Dirichlet density equal to* 1.

Proof. Indeed, letting S denote the set of prime numbers p such that $R_{\mathfrak{p}} \leq p^{-1/(p-1)}$ for some prime ideal $\mathfrak{p}$ above p, we get the lower bound

$$\rho(A) \geq \sum_{p\in S} \log^{+}\left(\frac{1}{R_{\mathfrak{p}}}\right) \geq \sum_{p\in S} \frac{\log p}{p-1} \geq \sum_{p\in S} \frac{1}{p}.$$

If $\rho(A)$ converges, then so does $\sum_{p\in S} \frac{1}{p}$, which means that S has Dirichlet density equal to 0. ∎

The set $S = \{\mathfrak{p} \mid |A|_{G,\mathfrak{p}} > 1\}$ is finite. Outside this finite set, we can reduce the system of linear differential equations modulo $\mathfrak{p}$.

Proposition 4.1. *Assume* $|A|_{G,\mathfrak{p}} \leq 1$. *Then the system*

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} = A(z)\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} \qquad \overline{A}(z) \in M_{\mu\times\mu}(\mathbf{F}_q(z))$$

is nilpotent if and only if $R_{\mathfrak{p}} > p^{-1/(p-1)}$.

Proof. We prove that $R_{\mathfrak{p}} > p^{-1/(p-1)}$ implies that the system is nilpotent, the only part of the proposition that will be used later. For this, we first observe that $(\overline{A}_p)^n = \overline{A}_{pn}$, so it suffices to prove that $\overline{A}_{pn} = 0$ for some large enough n. Let $R_{\mathfrak{p}} > R > p^{-1/(p-1)}$. Then

$$\left|\frac{A_r}{r!}\right|_G R^r \longrightarrow 0 \text{ as } r \to +\infty.$$

Writing $R = \alpha p^{-1/(p-1)}$, we find

$$\frac{R^{p^j}}{|(p^j)!|} \geq (\alpha p^{-1/(p-1)})^{p^j} p^{\frac{p^j-1}{p-1}} = \alpha^{p^j} p^{-1/(p-1)} \longrightarrow +\infty \text{ as } j \to +\infty,$$

since $\alpha > 1$. From this it follows that $|A_{p^j}|_G \to 0$ as $j \to +\infty$. In particular, $|A_{p^j}|_G < 1$ for big enough j, and then $\overline{A}_{p\cdot p^{j-1}} = 0$. ∎

4.2 The theorem of André–Bombieri

Theorem 4.1. *The following inequalities hold:*

$$\rho(A) \leq \sigma(A) \leq \rho(A) + \mu - 1.$$

In particular, Bombieri's and Galochkin's conditions are equivalent.

4.3 The theorem of Katz

Theorem 4.2 (Katz). *Let K be a number field, and let $L \in K(z)[\frac{d}{dz}]$ be a differential operator.*

(1) *If there exists an infinite set of prime ideals $\mathfrak{p}$ such that $L_{\mathfrak{p}}$ is nilpotent, then L has regular singularities.*

(2) *If there exists a set of prime ideals with Dirichlet density 1 such that $L_{\mathfrak{p}}$ is nilpotent, then L has rational exponents.*

Corollary 4.1. *A differential operator $L \in K[z, \frac{d}{dz}]$ satisfying $\rho(L) < \infty$ has regular singularities and rational exponents.*

Proof. Write $L = \theta^\mu + B_{\mu-1}(z)\theta^{\mu-1} + \cdots + B_0(z)$ and consider the Laurent series expansions

$$B_j = \frac{b_{j,m}}{z^m} + \cdots + \frac{b_{j,1}}{z} + c_{j,0} + c_{j,1}z + \cdots \qquad b_{j,s}, c_{j,s} \in K.$$

The first goal is to show the vanishing $b_{j,1} = \cdots = b_{j,m} = 0$ for all j. ∎

4.4 Summary

Theorem 4.3. *A G-operator has regular singularities and rational exponents.*

Chapter 5

E-operators

5.1 Fourier–Laplace transform and E-operators

The *Fourier–Laplace transform* of a differential operator

$$L = \sum_{i=0}^{\mu} P_i(z)(\frac{d}{dz})^i \in K[z, \frac{d}{dz}]$$

is the differential operator

$$\mathrm{FT}(L) = \sum_{i=0}^{\mu} P_i(-d/dz)z^i \in K[z, \frac{d}{dz}]$$

obtained by replacing z by $-d/dz$ and d/dz by z in the expression of L. It is an automorphism of order 4, with inverse

$$\overline{\mathrm{FT}}(L) = \sum_{i=0}^{\mu} P_i(d/dz)(-z)^i$$

where we replace z by d/dz and d/dz by $-z$.

This definition is inspired by the properties of the Laplace transform

$$(\mathcal{L}f)(y) = \int_0^{\infty} f(z)e^{-yz}dz,$$

namely the following:

(1) $\frac{d}{dy}(\mathcal{L}f) = \mathcal{L}(-zf)$,

(2) $y\mathcal{L}f = \mathcal{L}(\frac{d}{dz}f) + f(0)$,

(3) $\frac{1}{y}\mathcal{L}f = \mathcal{L}(\int_0^z f)$,

(4) $\mathcal{L}f(y-a) = \mathcal{L}(e^{az}f)$,

(5) $\mathcal{L}f(\frac{y}{a}) = a\mathcal{L}(f(az))$.

If $f(z) = \sum_{n=0}^{\infty} a_n/n!z^n$ is an E-function, then $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is a G-function, and

$$(\mathcal{L}f)(y) = \frac{1}{y}g(\frac{1}{y}).$$

If g is annihilated by $\varphi \in K[z, d/dz]$ of order μ, then $\frac{1}{y}g(\frac{1}{y})$ is also annihilated by $\psi \in K[z, d/dz]$ of order μ by the change of variables $z \mapsto 1/z$. Let $\nu = \deg_z \psi$. Then

$$0 = \overline{\mathrm{FT}}\big((\frac{d}{dz})^{\nu}\psi\big)f = (-z)^{\nu}\overline{\mathrm{FT}}(\psi)f,$$

and hence f is annihilated by the differential operator $\Theta = \overline{\mathrm{FT}}(\psi)$. Conversely, if $Lf = 0$, then

$$(\frac{d}{dz})^{\mu}\mathrm{FT}(L)(\mathcal{L}f) = 0,$$

which means that $\mathrm{FT}(L)(\mathcal{L}f)$ is a polynomial of degree $< \mu$.

Definition 5.1. A differential operator $L \in \overline{\mathbf{Q}}[z, \frac{d}{dz}]$ is called an *E-operator* if its Fourier–Laplace transform $\mathrm{FT}(L)$ is a G-operator.

Equivalently, if L is a G-operator, then $\mathrm{FT}(L)$ is an E-operator.

By Chudnovsky's theorem, every E-function is annihilated by an E-operator. In fact, any differential operator of *minimal degree in z* annihilating an E-function is an E-operator.

If L is a differential operator of minimal order annihilating f, then there exists a differential operator L', an E-operator Φ, and a polynomial $Q(z)$ satisfying

$$L' \cdot L = Q(z) \cdot \Phi,$$

so f is a solution of Φ as well. But L is not necessarily an E-operator, as the following example shows.

Example 5.1.1. Consider the E-function $f(z) = (z-1)e^z$. Then $L = (z-1)\frac{d}{dz} - z$ is a differential operator of minimal order annihilating f. Its Fourier transform

$$\mathrm{FT}(L) = (-\frac{d}{dz} - 1)z + \frac{d}{dz} = (1-z)\frac{d}{dz} - (z+1)$$

is not a G-operator (for example, because ∞ is an irregular singularity, since in the usual notation $\deg P_0 - 0 = 1$ but $\deg P_1 - 1 = 0$, so Fuchs's criterion fails). In this case, we can take

$$\Phi = (\frac{d}{dz})^2 - 2\frac{d}{dz} + 1$$

as an E-operator. It satisfies $(\frac{d}{dz} - 1)L = (z-1)\Phi$. Note that L has a trivial singularity at $z = 1$, whereas Φ has only a singularity at ∞.

5.2 Structure of E-operators

Lemma 5.1. *If L has regular singularities at ∞, then the singularities of* $\mathrm{FT}(L)$ *are contained in* $\{0, \infty\}$.

Proof. Write the differential operator L as

$$L = (a_{\mu,r_\mu}z^{r_\mu} + \cdots)(\frac{d}{dz})^{\mu} + (a_{\mu-1,r_{\mu-1}}z^{r_{\mu-1}} + \cdots)(\frac{d}{dz})^{\mu-1} + \cdots$$

Fuchs's criterion at infinity gives $r_\mu > r_i$ for all $i = 0, \dots, \mu - 1$. Hence

$$\mathrm{FT}(L) = a_{\mu,r_\mu}(-\frac{d}{dz})^{-r_\mu} z^\mu + \cdots = (-1)^{r_\mu} a_{\mu,r_\mu}(\frac{d}{dz})^{r_\mu} + \cdots ,$$

so the only possible singularities are 0 and ∞. ∎

Therefore, an E-operator has singularities contained in $\{0, \infty\}$.

Theorem 5.1. *Let L be a differential operator of minimal order annihilating an E-function. Then L has a basis of holomorphic solutions at every point distinct from* 0 *and* ∞.

Proof. Indeed, $L'L = Q(z)\Phi$ for an E-operator Φ, so a non-holomorphic solution of L would give rise to a non-holomorphic solution of Φ, which is not possible. ∎

Appendix A

Factorials

The basic formula is that for the p-adic valuation of the factorial of an integer:

$$v_p(n!) = \sum_{j=1}^{\infty} \left[\frac{n}{p^j} \right] = \frac{n - S_n}{p - 1},$$

where S_n stands for the sum of the digits of n in base p. That is, if $n = a_0 + \cdots + a_\ell p^\ell$ with $a_i \in \{0, \ldots, p-1\}$ and $a_\ell \neq 0$, then $S_n = a_0 + \cdots + a_\ell$.

Lemma A.1. *Let $a \in \mathbf{Q}$ and $b \in \mathbf{Q} \setminus \mathbf{Z}_{\leq 0}$. There exists a real number $C > 0$ such that, for all $n \geq 1$, the following estimate holds:*

$$den\left(\frac{(a)_1}{(b)_1}, \ldots, \frac{(a)_n}{(b)_n}\right) \leq C^n.$$

Appendix B

A theorem of Kronecker (by Emmanuel Kowalski)

The theorem in question is:

Theorem B.1. *Let α be an algebraic number, and let $K = \mathbf{Q}(\alpha)$. If the set Q of non-zero prime ideals q of the ring of integers of K such that α mod q belongs to the prime field of the residue field of q has density* 1*, in the sense that*

$$\sum_{q \in Q} \frac{1}{|q|^{\sigma}} \sim -\log(\sigma - 1)$$

as $\sigma \to 1$, where $|q|$ is the norm of q, then $\alpha \in \mathbf{Q}$.

This result is discussed in the Bourbaki seminar of Chambert-Loir [8, Th. 2.2] as an elementary case of algebraicity theorems; the original reference of Kronecker (where the result is announced) is the paper [1].

In modern terms, this result is viewed as a simple consequence of the Chebotarev density theorem. For instance, one can argue that the set Q coincides (up to finitely many primes maybe) with the set of prime ideals q such that the Frobenius conjugacy class at q, in the Galois group G of a Galois closure of K, is conjugate to an element of the fixer H of α, hence we have

$$\sum_{q \in Q} \frac{1}{|q|^{\sigma}} \sim -\frac{|X|}{|G|} \log(\sigma - 1),$$

where $X \subset G$ is the set of elements of G conjugate to an element of H (or even the analogue with natural density). The assumption then implies that $H = G$ by one of Jordan's theorems, well-beloved of Serre [15]. But this is only possible if G is trivial, since G acts transitively on the conjugates of α.

We spell out here a proof which doesn't involve the full Chebotarev theorem.

Lemma B.1. *Let $E/\mathbf{Q}$ be a number field with ring of integers $\mathbf{Z}_E$.*

(1) *We have*

$$\sum_{q \subset \mathbf{Z}_E} \frac{1}{|q|^{\sigma}} = -\log(\sigma - 1) + O(1)$$

for $\sigma > 1$, where the sum ranges over non-zero prime ideals in $\mathbf{Z}_E$.

(2) *For p prime, let $\rho_E(p)$ denote the number of prime ideals of norm p in $\mathbf{Z}_E$. We have*

$$\sum_{p} \frac{\rho_E(p)}{p^{\sigma}} = -\log(\sigma - 1) + O(1)$$

for $\sigma > 1$.

Proof. (1) This is a consequence of the fact that the Dedekind zeta function of E has a simple pole at $s = 1$, together with its Euler product expansion.

(2) This follows from (1), since

$$\sum_{q \subset \mathbf{Z}_E} \frac{1}{|q|^\sigma} = \sum_p \frac{\rho_E(p)}{p^\sigma} + O\Big(\sum_p \frac{1}{p^{2\sigma}}\Big)$$

for $\sigma > 1$. ∎

Proof of Kronecker's Theorem. We may assume that α is an algebraic integer. Let $f \in \mathbf{Z}[X]$ be its minimal monic polynomial, d its degree, and let L be its splitting field; it is a Galois extension of $\mathbf{Q}$ containing K. For p prime, let $\rho(p)$ be the number of roots of f modulo p. We have $\rho(p) \geq 1$ for all but finitely many primes, by the assumption on α.

Let

$$F(s) = \sum_p \frac{\rho(p)}{p^s}$$

for $\mathrm{Re}(s) > 1$. Since $\rho(p)$ is (for all but finitely many primes) the number of prime ideals of norm p in $\mathbf{Z}_k$, we have

$$F(\sigma) = -\log(\sigma - 1) + O(1) \tag{B.1}$$

for $\sigma > 1$ by Lemma B.1, (2).

If we denote by T the set of primes p totally split in L, then noting that $\rho(p) = d$ for such primes, we see that the inequality

$$F(\sigma) \geq \sum_p \frac{1}{p^\sigma} + \sum_{p \in T} \frac{d-1}{p^\sigma} + O(1) \tag{B.2}$$

holds for $\sigma > 1$, where the bounded term $O(1)$ accounts again for the finitely many possible exceptional primes.

By Lemma B.1, (1), applied to $\mathbf{Q}$, we have

$$\sum_p \frac{1}{p^\sigma} = -\log(\sigma - 1) + O(1)$$

for $\sigma > 1$. But also by the second part of the lemma applied to L, we have

$$\sum_{p \in T} \frac{1}{p^\sigma} = \frac{1}{[L : \mathbf{Q}]} \sum_{q \subset \mathbf{Z}_L} \frac{1}{|q|^\sigma} + O(1)$$

for $\sigma > 1$, since $\rho_L(p) = [L : \mathbf{Q}]$ for $p \in T$. Combining this with (B.1) and (B.2), we derive

$$-\log(\sigma - 1) \geq -\Big(1 + \frac{d}{[L : \mathbf{Q}]}\Big) \log(\sigma - 1) + O(1)$$

for $\sigma > 1$, which is only possible if $d = 1$. ∎

Remark B.1. Instead of the behavior of the series as $\sigma \to 1$, one can also use either natural density or the behavior of the sums

$$\sum_{\substack{q \in Q \\ |q| \leq x}} \frac{1}{|q|}$$

as $x \to +\infty$.

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