# Exponential motives 

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#### Abstract

Following ideas of Katz, Kontsevich, and Nori, we construct a neutral $\mathbb{Q}$-linear tannakian category of exponential motives over a subfield $k$ of the complex numbers. This category is endowed with Betti and de Rham realisation functors, as well as a comparison isomorphism between them. When $k$ is algebraic, the coefficients of this comparison isomorphism are called exponential periods and form a class of complex numbers including the special values of the gamma and the Bessel functions, the Euler-Mascheroni constant, and other interesting numbers that are not expected to be periods of usual motives. In particular, we attach to exponential motives a Galois group that conjecturally governs all algebraic relations among their periods.


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## CHAPTER 1

## Introduction

What motives are to algebraic varieties, exponential motives are to varieties with a potential, that is, to pairs $(X, f)$ consisting of an algebraic variety $X$ over some field $k$ and a regular function $f$ on $X$. These objects have attracted considerable attention in recent years, especially in connection with mirror symmetry, where one seeks to associate with a Fano variety $Y$ a Landau-Ginzburg mirror $(X, f)$ so that certain invariants of $Y$ such as the Hodge numbers are reflected by the geometry of $f$, namely its critical locus. Our motivation is somewhat different: exponential motives provide a framework to deal with many interesting numbers that are not expected to be periods in the usual sense of algebraic geometry. Following ideas of Katz, Kontsevich, and Nori, we shall construct a $\mathbb{Q}$-linear neutral tannakian category of exponential motives over a subfield $k$ of the complex numbers and compute a few examples of Galois groups. Classical results and conjectures of transcendence theory may then be interpreted - in the spirit of Grothendieck's period conjecture as instances of the statement that the transcendence degree of the field generated by the periods of an exponential motive agrees with the dimension of its Galois group.

### 1.1. Exponential periods

1.1.1. - To get in tune, let us introduce two cohomology theories for varieties with a potential. The first one, rapid decay cohomology, appears implicitly in the classical study of the asymptotic behaviour of the solutions of differential equations with irregular singularities. To our knowledge, it was first considered in a 1976 letter from Deligne to Malgrange [25, p. 17]. We learnt the definition below from Kontsevich, see [59, §4.3] and [58, Def. 4.1].

Let $n \geqslant 0$ be an integer. For each real number $r$, we let $S_{r} \subseteq \mathbb{C}$ denote the closed half-plane $\{\operatorname{Re}(z) \geqslant r\}$. Given a complex algebraic variety $X$ and a regular function $f: X \rightarrow \mathbb{C}$, the rapid decay homology in degree $n$ of the pair $(X, f)$ is defined as the limit

$$
\begin{equation*}
H_{n}^{\mathrm{rd}}(X, f)=\lim _{r \rightarrow+\infty} H_{n}\left(X(\mathbb{C}), f^{-1}\left(S_{r}\right) ; \mathbb{Q}\right) \tag{1.1.1.1}
\end{equation*}
$$

On the right-hand side stands the singular homology with rational coefficients of the topological space $X(\mathbb{C})$ relative to the closed subspace $f^{-1}\left(S_{r}\right)$, and the limit is taken in the category of vector spaces with respect to the transition maps induced by the inclusions $f^{-1}\left(S_{t}\right) \subseteq f^{-1}\left(S_{r}\right)$ for all $t \geqslant r$. For big enough $r$, all these maps are in fact isomorphisms and the fibre $f^{-1}(r)$
is homotopically equivalent to $f^{-1}\left(S_{r}\right)$, so one may as well think of rapid decay homology as the homology of $X(\mathbb{C})$ relative to a general fibre of the function $f$. The reason for the name will become apparent soon. The rapid decay cohomology in degree $n$ of the pair $(X, f)$ is then defined as the linear dual $H_{\mathrm{rd}}^{n}(X, f)$ of the rapid decay homology space, that is:

$$
\begin{equation*}
H_{\mathrm{rd}}^{n}(X, f)=\operatorname{Hom}_{\mathbb{Q}}\left(H_{n}^{\mathrm{rd}}(X, f), \mathbb{Q}\right)=\operatorname{colim}_{r \rightarrow+\infty} H^{n}\left(X(\mathbb{C}), f^{-1}\left(S_{r}\right) ; \mathbb{Q}\right) \tag{1.1.1.2}
\end{equation*}
$$

This cohomology theory for varieties with a potential enjoys many of the usual properties: the vector space $H_{\mathrm{rd}}^{n}(X, f)$ has finite dimension, depends functorially on $(X, f)$, satisfies a Künneth formula, fits into a Mayer-Vietoris long exact sequence, etc. Whenever $f$ is constant, the subspace $f^{-1}\left(S_{r}\right)$ is empty for big enough $r$, so $H_{\mathrm{rd}}^{n}(X, f)$ agrees with the usual Betti cohomology of $X$.

As for usual cohomology, rapid decay cohomology admits a purely algebraic counterpart. Let $X$ be a smooth variety over a field $k$ of characteristic zero, $f: X \rightarrow \mathbb{A}^{1}$ a regular function, and

$$
\mathcal{E}^{f}=\left(\mathcal{O}_{X}, d_{f}\right)
$$

the trivial rank one vector bundle on $X$ together with the integrable connection $d_{f}: \mathcal{O}_{X} \rightarrow \Omega_{X}^{1}$ uniquely determined by $d_{f}(1)=-d f$. If $k$ is a subfield of the complex numbers, the local system of analytic horizontal sections of $\mathcal{E}^{f}$ is the trivial local system on $X(\mathbb{C})$ generated by the exponential of $f$, which justifies the notation and our choice of the minus sign. However, the connection $\mathcal{E}^{f}$ is non-trivial as long as $f$ is non-constant, a reflection of the fact that it has then irregular singularities at infinity. Let $D R\left(\mathcal{E}^{f}\right)$ be the de Rham complex of $\mathcal{E}^{f}$, namely the complex of Zariski sheaves

$$
D R\left(\mathcal{E}^{f}\right): \quad \mathcal{O}_{X} \xrightarrow{d_{f}} \Omega_{X}^{1} \xrightarrow{d_{f}} \cdots \xrightarrow{d_{f}} \Omega_{X}^{\operatorname{dim} X},
$$

where the differential $d_{f}: \Omega_{X}^{p} \rightarrow \Omega_{X}^{p+1}$ is given by $d_{f}(\omega)=d \omega-d f \wedge \omega$ on local sections $\omega$. By definition, the de Rham cohomology of the pair $(X, f)$ is the cohomology of this complex:

$$
\begin{equation*}
H_{\mathrm{dR}}^{n}(X, f)=H^{n}\left(X, D R\left(\mathcal{E}^{f}\right)\right) \tag{1.1.1.3}
\end{equation*}
$$

As we shall see in Section 7.1, using standard techniques in homological algebra, this definition generalises to singular varieties and yields another cohomology theory for varieties with a potential. If $f$ is constant, then $d_{f}$ is nothing but the exterior derivative, and hence $H_{\mathrm{dR}}^{n}(X, f)$ agrees with the usual de Rham cohomology of $X$.
1.1.2. - Let $(X, f)$ be a smooth variety with a potential defined over a subfield $k$ of $\mathbb{C}$. By a series of works starting from the aforementioned letter and continuing with Dimca-Saito [28], Sabbah [68], Hien-Roucairol [45], and Hien [44], there is a canonical comparison isomorphism

$$
H_{\mathrm{dR}}^{n}(X, f) \otimes_{k} \mathbb{C} \xrightarrow{\sim} H_{\mathrm{rd}}^{n}(X, f) \otimes_{\mathbb{Q}} \mathbb{C}
$$

which we shall most conveniently regard as a perfect pairing

$$
\begin{equation*}
H_{\mathrm{dR}}^{n}(X, f) \otimes H_{n}^{\mathrm{rd}}(X, f) \rightarrow \mathbb{C} \tag{1.1.2.1}
\end{equation*}
$$

between de Rham cohomology and rapid decay homology. Intuitively, elements of $H_{n}^{\mathrm{rd}}(X, f)$ are homology classes of topological cycles $\gamma$ on $X(\mathbb{C})$ which are not necessarily compact, but are only unbounded in the directions where $\operatorname{Re}(f)$ tends to infinity. More precisely, we view them as classes of compatible systems $\gamma=\left(\gamma_{r}\right)_{r \in \mathbb{R}}$ of compact cycles $\gamma_{r}$ in $X(\mathbb{C})$ whose boundary $\partial \gamma_{r}$ is contained
in $f^{-1}\left(S_{r}\right)$. Besides, when $X$ is affine, de Rham cohomology can be computed using global sections of the complex $D R\left(\mathcal{E}^{f}\right)$, so that elements of $H_{\mathrm{dR}}^{n}(X, f)$ are represented by $n$-forms $\omega$ on $X$. In this case, the pairing (1.1.2.1) sends $[\omega] \otimes[\gamma]$ to the integral

$$
\int_{\gamma} e^{-f} \omega=\lim _{r \rightarrow+\infty} \int_{\gamma_{r}} e^{-f} \omega
$$

which is absolutely convergent since $e^{-f}$ decays rapidly in the directions where $\gamma$ is unbounded. The value of the integral is independent of the choice of representatives by Stokes' theorem: for example, two cohomologous forms will differ by $d_{f}(\eta)$ for some $\eta \in \Omega_{X}^{n-1}(X)$, and we have

$$
\int_{\gamma} e^{-f} d_{f}(\eta)=\int_{\gamma} d\left(e^{-f} \eta\right)=\lim _{r \rightarrow+\infty} \int_{\partial \gamma_{r}} e^{-f} \eta=0
$$

because $\eta$ is algebraic and $e^{-f}$ goes to zero faster than any polynomial along the boundary of $\gamma_{r}$.
If the base field $k$ is further assumed to lie inside $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, one calls exponential periods the complex numbers arising as values of the pairing (1.1.2.1). Note that, when $f$ is a non-zero constant function, although we are dealing with usual de Rham and singular cohomology of $X$, the comparison isomorphism is twisted by $e^{-f}$. For this reason, exponentials of algebraic numbers are exponential periods associated with zero-dimensional varieties, and will play in what follows a similar role to algebraic numbers in the classical theory of periods.
1.1.3. - We now present two more elaborated examples of exponential periods that appear, under various guises, in the work of Bloch-Esnault [11, §5], [12, p. 360-361], Kontsevich-Zagier [59, §4.3], Deligne [25, p. 115-128], Hien-Roucairol [45, p. 529-530], and Bertrand [10, §6].

Example 1.1.4. - Let $X=\operatorname{Spec} k[x]$ be the affine line and $f=a_{n} x^{n}+\ldots+a_{0}$ a polynomial of degree $n \geqslant 2$. The global de Rham complex of the connection $\mathcal{E}^{f}$ reads:

$$
\begin{aligned}
k[x] & \xrightarrow{d_{f}} k[x] d x \\
g & \longmapsto\left(g^{\prime}-f^{\prime} g\right) d x .
\end{aligned}
$$

Since $d_{f}$ is injective, the only non-trivial cohomology group is $H_{\mathrm{dR}}^{1}(X, f)=\operatorname{coker}\left(d_{f}\right)$. A basis is given by the differentials $d x, x d x, \ldots, x^{n-2} d x$. Indeed, these classes are linearly independent because the image of $d_{f}$ consists of elements of the form $h d x$ with $h$ of degree at least $n-1$. That they generate the whole cohomology can be seen by induction on noting that, for each $m \geqslant 0$, there is a polynomial $h_{m}$ of degree at most $n+m-2$ with

$$
x^{n+m-1} d x-h_{m} d x=d_{f}\left(\frac{1}{n a_{n}} x^{m}\right) .
$$

Let us now turn to rapid decay homology. The asymptotics of $\operatorname{Re}(f)$ being governed by the leading term of the polynomial, we may assume without loss of generality that $f=a_{n} x^{n}$ and write $a_{n}=u e^{i \alpha}$ with $u>0$ and $\alpha \in[0,2 \pi)$. Given a real number $r>0$, the subspace $f^{-1}\left(S_{r}\right) \subseteq \mathbb{C}$ consists of the $n$ disjoint regions

$$
\coprod_{j=0}^{n-1}\left\{s e^{i \theta} \left\lvert\, \frac{-\alpha+\left(2 j-\frac{1}{2}\right) \pi}{n}<\theta<\frac{-\alpha+\left(2 j+\frac{1}{2}\right) \pi}{n}\right., \quad s \geqslant\left(\frac{r}{u \cos (\alpha+n \theta)}\right)^{\frac{1}{n}}\right\},
$$

which are concentrated around the half-lines

$$
\sigma_{j}=\left\{s e^{i \theta} \left\lvert\, \theta=\frac{-\alpha+2 \pi j}{n}\right., s \geqslant 0\right\}, \quad j=0, \ldots, n-1
$$

We orient each $\sigma_{i}$ from zero to infinity. A basis of $H_{1}^{\mathrm{rd}}(X, f)$ is then given by the cycles

$$
\gamma_{i}=\sigma_{i}-\sigma_{0}, \quad i=1, \ldots, n-1
$$

Figure 1.1.1 illustrates the case of a polynomial of degree $n=5$ whose leading term $a_{n}$ is a positive real number: the subspace $f^{-1}\left(S_{r}\right)$ is drawn in blue and the half-lines $\sigma_{j}$ in green.


Figure 1.1.1. A basis of the rapid decay homology of a polynomial of degree 5 with positive leading term

With respect to these bases, the matrix of the period pairing (1.1.2.1) is

$$
P=\left(\int_{\gamma_{i}} x^{j-1} e^{-f(x)} d x\right)_{i, j=1, \ldots, n-1}
$$

Assuming that the base field $k$ is algebraic, the entries of $P$ are exponential periods. Let us see a few examples of familiar numbers which appear this way:
(i) Given a quadratic polynomial $f=a x^{2}+b x+c$, the cohomology is one-dimensional. In this case, the cycle $-\gamma_{1}$ is the "rotated" real line $e^{-\frac{i \arg (a)}{2}} \mathbb{R}$, with its usual orientation, and one gets:

$$
\begin{equation*}
\int_{e^{-\frac{i \arg (a)}{2}} \mathbb{R}} e^{-a x^{2}-b x-c} d x=e^{\frac{b^{2}}{4 a}-c} \sqrt{\frac{\pi}{a}} \tag{1.1.4.1}
\end{equation*}
$$

A particular case, for $f=x^{2}$, is the Gaussian integral

$$
\begin{equation*}
\int_{\mathbb{R}} e^{-x^{2}} d x=\sqrt{\pi} \tag{1.1.4.2}
\end{equation*}
$$

which is not expected to be a period in the usual sense since, granted a theory of weights for periods, it would hint at the existence of a one-dimensional pure Hodge structure of weight one. We will prove in Section 12.2 that, assuming the analogue of the Grothendieck period conjecture for exponential motives, $\sqrt{\pi}$ is not a period of a usual motive.
(ii) More generally, consider the polynomial $f=x^{n}$ for $n \geqslant 2$. Set $\xi=e^{\frac{2 \pi i}{n}}$ and let $\Gamma$ be the classical gamma function. Then the entries of $P$ are the exponential periods

$$
\int_{\gamma_{i}} x^{j-1} e^{-x^{n}} d x=\frac{\xi^{i j}-1}{n} \int_{0}^{+\infty} x^{\frac{j}{n}-1} e^{-x} d x=\frac{\xi^{i j}-1}{n} \Gamma\left(\frac{j}{n}\right)
$$

To get the special value of the gamma value alone, i.e. without the cyclotomic factor, it suffices to observe that the relation $\sum_{i=1}^{n-1} \xi^{i j}=-1$ yields

$$
\begin{equation*}
\Gamma\left(\frac{j}{n}\right)=\int_{-\gamma_{1}-\ldots-\gamma_{n-1}} x^{j-1} e^{-x^{n}} d x \tag{1.1.4.3}
\end{equation*}
$$

Again, one does not expect single gamma values to be periods in the usual sense. However, we can obtain periods by taking suitable monomials in them.

Using geometric techniques inspired from the stationary phase formula-which will carry over to exponential motives-, Bloch and Esnault computed the determinant of the period matrix $P$ in [11, Prop. 5.4]:

$$
\begin{equation*}
\operatorname{det} P \sim_{k^{\times}} \sqrt{(-1)^{\frac{(n-1)(n-2)}{2}} s} \cdot \pi^{\frac{n-1}{2}} \cdot \exp \left(-\sum_{f^{\prime}(\alpha)=0} f(\alpha)\right) \tag{1.1.4.4}
\end{equation*}
$$

where $s=1$ if $n$ is odd and $s=n a_{n} / 2$ if $n$ is even. The symbol $\sim_{k \times}$ means that the left and the right-hand side agree up to multiplication by an element of $k^{\times}$. Note the particular case (1.1.4.1).

Example 1.1.5. - Consider the torus $X=\operatorname{Spec} k\left[x, x^{-1}\right]$, together with the Laurent polynomial

$$
f=-\frac{\lambda}{2}\left(x-\frac{1}{x}\right)
$$

for some $\lambda \in k^{\times}$, which we view for the moment as a fixed parameter. Arguing as before, one sees that coker $\left(d_{f}\right)$ is generated by $x^{p} d x$, for $p \in \mathbb{Z}$, modulo the relations

$$
x^{p} d x+\frac{2 p}{\lambda} x^{p-1} d x+x^{p-2} d x=0
$$

It follows that the de Rham cohomology $H_{\mathrm{dR}}^{1}(X, f)$ is two-dimensional, a basis being given by the classes of the differentials $x^{-p-1} d x$ and $x^{-p} d x$ for any choice of an integer $p$.

On the rapid decay side, the subspace $f^{-1}\left(S_{r}\right) \subseteq \mathbb{C}^{\times}$consists of two disjoint regions which are roughly a half-plane where $\operatorname{Re}(-\bar{\lambda} x)$ is large and the inversion with respect to the unit circle of the half-plane where $\operatorname{Re}(\lambda x)$ is large (see Figure 1.1.2 below). By passing to the limit $r \rightarrow+\infty$ in the long exact sequence of relative homology
$\cdots \rightarrow H_{1}\left(f^{-1}\left(S_{r}\right), \mathbb{Q}\right) \rightarrow H_{1}\left(\mathbb{C}^{\times}, \mathbb{Q}\right) \rightarrow H_{1}\left(\mathbb{C}^{\times}, f^{-1}\left(S_{r}\right) ; \mathbb{Q}\right) \rightarrow H_{0}\left(f^{-1}\left(S_{r}\right), \mathbb{Q}\right) \rightarrow H_{0}\left(\mathbb{C}^{\times}, \mathbb{Q}\right) \rightarrow \cdots$ one sees that $H_{1}^{\mathrm{rd}}(X, f)$ is two-dimensional and contains $H_{1}\left(\mathbb{C}^{\times}, \mathbb{Q}\right)$. Therefore, a loop $\gamma_{1}$ winding once counterclockwise around 0 defines a class in rapid decay homology. To complete it to a basis, we consider any path joining the two connected components of $f^{-1}\left(S_{r}\right)$, for example the cycle $\gamma_{2}$ in $\mathbb{C}^{\times}$consisting of the straight line from 0 (not included) to $\lambda$, the positive arc from $\lambda$ to $-\bar{\lambda}$ and the half-line from $-\bar{\lambda}$ towards $-\bar{\lambda} \infty$, as shown in Figure 1.1.2. Alternatively, we note that, on the
vertical axis $\{x=i t \mid t \in \mathbb{R}\}$, the real part of $f$ is given by $\operatorname{Re}(f)=\operatorname{Im}(\lambda)\left(t+\frac{1}{t}\right)$, so, as long as $\lambda$ is not real, we can take the path $\gamma_{2}: \mathbb{R}_{>0} \rightarrow \mathbb{C}^{\times}$defined by

$$
\gamma_{2}(t)= \begin{cases}i t & \text { if } \operatorname{Im}(\lambda)>0 \\ -i t & \text { if } \operatorname{Im}(\lambda)<0\end{cases}
$$



Figure 1.1.2. The subspaces $f^{-1}\left(S_{r}\right)$ and a basis of the rapid decay homology $H_{1}^{\text {rd }}(X, f)$ when $\lambda=1+i$ (left) and $\lambda=1$ (right)

Recall that, given an integer $n$, the Bessel function of the first kind of order $n$ is defined by

$$
J_{n}(z)=\frac{1}{2 \pi i} \int_{\gamma_{1}} e^{\frac{z}{2}\left(x-\frac{1}{x}\right)} \frac{d x}{x^{n+1}}, \quad z \in \mathbb{C},
$$

and the Bessel function of the third kind of order $n$ is defined by

$$
H_{n}(z)=\frac{1}{\pi i} \int_{\gamma_{2}} e^{\frac{z}{2}\left(x-\frac{1}{x}\right)} \frac{d x}{x^{n+1}}, \quad z \in \mathbb{C}^{\times}
$$

We adopt the conventions from $[88,6.21]$. The function $J_{n}(z)$ is entire whereas $H_{n}(z)$ is holomorphic on $\mathbb{C} \backslash i \mathbb{R}$ if the cycle $\gamma_{2}$ is given by the first description. The functions $J_{n}(z)$ and $H_{n}(z)$ are two linearly independent solutions of the second order linear differential equation

$$
\begin{equation*}
\frac{d^{2} u}{d z^{2}}+\frac{1}{z} \frac{d u}{d z}+\left(1-\frac{n^{2}}{z^{2}}\right) u=0 \tag{1.1.5.1}
\end{equation*}
$$

for an unknown function $u$ in one variable $z$. Observe that (1.1.5.1) has a regular singular point at $z=0$ and an irregular singularity at infinity.

The matrix of the period pairing (1.1.2.1) with respect to the basis $x^{-n-1} d x$ and $x^{-n} d x$ of de Rham cohomology and $\gamma_{1}, \gamma_{2}$ of rapid decay homology reads

$$
P=\left(\begin{array}{cc}
2 \pi i J_{n}(\lambda) & 2 \pi i J_{n-1}(\lambda)  \tag{1.1.5.2}\\
\pi i H_{n}(\lambda) & \pi i H_{n-1}(\lambda)
\end{array}\right) .
$$

### 1.2. Exponential motives

1.2.1 (An abelian category after Nori). - According to the philosophy of motives, the existence of two cohomology theories for varieties with potential, as well as a comparison isomorphism between them, suggests looking for a universal cohomology with values in a tannakian category, from which any other reasonable cohomology theory would be obtained by composition with realisation functors. Such a category of exponential motives over a fixed subfield $k$ of $\mathbb{C}$ indeed exists, and we shall construct it using Nori's formalism [65].

Extending slightly the definition of rapid decay cohomology, we associate with a $k$-variety $X$, a closed subvariety $Y \subseteq X$, a regular function $f$ on $X$, and two integers $n$ and $i$ the vector space

$$
\begin{align*}
\rho([X, Y, f, n, i]) & =H_{\mathrm{rd}}^{n}(X, Y, f)(i) \\
& =\underset{r \rightarrow+\infty}{\operatorname{colim}} H^{n}\left(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right) ; \mathbb{Q}\right)(i), \tag{1.2.1.1}
\end{align*}
$$

where the twist $(i)$ means tensoring $-i$ times with the one-dimensional vector space $H^{1}\left(\mathbb{G}_{m}, \mathbb{Q}\right)$. Note that we do not require any compatibility between the function and the subvariety.

Let us preliminarily write $\mathrm{Q}^{\exp }(k)$ for the category with objects the tuples $[X, Y, f, n, i]$ as above, and morphisms the maps of $k$-varieties compatible with the subvarieties and the functions in the obvious way. Then the assignment (1.2.1.1) defines a functor

$$
\begin{equation*}
\rho: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}} . \tag{1.2.1.2}
\end{equation*}
$$

The basic idea is to look at the endomorphism algebra of $\rho$, that is,

$$
\begin{equation*}
\operatorname{End}(\rho)=\left\{\left(e_{q}\right) \in \prod_{q \in Q^{\exp }(k)} \operatorname{End}(\rho(q)) \mid e_{q} \circ \rho(f)=\rho(f) \circ e_{p} \text { for all } f: p \rightarrow q\right\} \tag{1.2.1.3}
\end{equation*}
$$

Filtering $\mathrm{Q}^{\exp }(k)$ by subcategories with a finite number of objects and morphisms, one sees that $\operatorname{End}(\rho)$ has a canonical structure of pro-algebra over $\mathbb{Q}$. Bearing this in mind, we tentatively define the category of exponential motives as

$$
\mathbf{M}^{\exp }(k)=\left\{\begin{array}{c}
\text { finite-dimensional } \mathbb{Q} \text {-vector spaces endowed }  \tag{1.2.1.4}\\
\text { with a continuous } \operatorname{End}(\rho) \text {-action }
\end{array}\right\} .
$$

The category $\mathbf{M}^{\exp }(k)$ is abelian, $\mathbb{Q}$-linear, and the functor $\rho$ lifts canonically to a functor $\tilde{\rho}: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{M}^{\exp }(k)$. The images of the objects of $Q^{\exp }(k)$ will be denoted by

$$
H^{n}(X, Y, f)(i)=\tilde{\rho}([X, Y, f, n, i])
$$

When $Y$ is empty or $i=0$, we will usually drop them from the notation. In general, an exponential motive is a subquotient of a finite direct sum of objects of the form $H^{n}(X, Y, f)(i)$.

So far, there are no morphisms between objects of $\mathrm{Q}^{\exp }(k)$ with different $n$ or $i$. Yet, given a closed subvariety $Z$ of $Y$, there is a canonical morphism of vector spaces

$$
\begin{equation*}
\rho\left(\left[Y, Z,\left.f\right|_{Y}, n-1, i\right]\right) \rightarrow \rho([X, Y, f, n, i]) \tag{1.2.1.5}
\end{equation*}
$$

which is induced, after passing to the limit, by the connecting morphism in the long exact sequence for the closed immersions $Z \cup f^{-1}\left(S_{r}\right) \subseteq Y \cup f^{-1}\left(S_{r}\right) \subseteq X$. We would like to lift this morphism to
the category $\mathbf{M}^{\exp }(k)$. To achieve this, we simply add to $\mathrm{Q}^{\exp }(k)$ an artificial morphism

$$
[X, Y, f, n, i] \rightarrow\left[Y, Z,\left.f\right|_{Y}, n-1, i\right]
$$

and declare its image under $\rho$ to be (1.2.1.5). As we do not specify any composition law for the new morphisms, $\mathrm{Q}^{\exp }(k)$ ceases to be a category, and is now only a quiver (or a diagram in Nori's terminology). By that, we understand a collection of objects, morphisms with source and target, and specified identity morphisms (see Section 4.1 for a reminder).

The definitions (1.2.1.3) and (1.2.1.4) are still meaningful, and now the morphisms (1.2.1.5) obviously lift to $\mathbf{M}^{\exp }(k)$. After introducing a second class of extra morphisms to $\mathrm{Q}^{\exp }(k)$, which relate objects having different twists, we arrive at our final definition of the quiver $\mathrm{Q}^{\exp }(k)$ and the category $\mathbf{M}^{\exp }(k)$. We will call Betti realisation the forgetful functor

$$
\begin{equation*}
R_{B}: \mathbf{M}^{\exp }(k) \longrightarrow \mathbf{V e c}_{\mathbb{Q}} \tag{1.2.1.6}
\end{equation*}
$$

Adapted to our context, Nori's main theorem about the categories associated with quiver representations $[65,48]$ says that $\mathbf{M}^{\exp }(k)$ is universal for all cohomology theories which are comparable to rapid decay cohomology. More precisely, one has the following result:

Theorem 1.2.2 (Nori). - Let $F$ be a field of characteristic zero and $\mathbf{A}$ an abelian, F-linear category together with an exact, F-linear, faithful functor $\mathbf{A} \rightarrow \operatorname{Vec}_{F}$. Let $h: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{A}$ be a functor, and suppose that natural isomorphisms of vector spaces

$$
h([X, Y, f, n, i]) \simeq \rho([X, Y, f, n, i]) \otimes_{\mathbb{Q}} F
$$

are given for each object $[X, Y, f, n, i]$. Then there exists a unique functor, up to isomorphism, $R_{\mathbf{A}}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{A}$ such that $h$ is the composite of $R_{\mathbf{A}}$ and the canonical lift $\widetilde{\rho}: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{M}^{\exp }(k)$.

This universal property will be used to construct other realisation functors. Important examples are the period and the perverse realisations, which we now discuss.
1.2.3 (The period realisation). - A period structure over $k$ is a triple ( $V, W, \alpha$ ) consisting of a $\mathbb{Q}$-vector space $V$, a $k$-vector space $W$, and an isomorphism $\alpha: V \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow W \otimes_{k} \mathbb{C}$ of complex vector spaces. Together with the obvious morphisms, period structures form an abelian $\mathbb{Q}$-linear category $\mathbf{P S}(k)$. There is a forgetful functor $\mathbf{P S}(k) \rightarrow \operatorname{Vec}_{\mathbb{Q}}$ sending $(V, W, \alpha)$ to $V$.

Extending the definition of de Rham cohomology and the comparison isomorphism from 1.1.1 and 1.1.2 to the relative setting and singular varieties, one obtains a functor $\mathrm{Q}^{\exp }(k) \rightarrow \mathbf{P S}(k)$, whose composition with the forgetful functor is nothing else but $\rho$. Therefore, Nori's Theorem 1.2.2 yields an exact and faithful functor

$$
R_{\mathbf{P}}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{P S}(k)
$$

which we call the period realisation. Composing with the functor $\mathbf{P S}(k) \rightarrow \mathbf{V e c}_{k}$ sending ( $V, W, \alpha$ ) to $W$, we obtain the de Rham realisation

$$
R_{\mathrm{dR}}: \mathbf{M}^{\exp }(k) \longrightarrow \mathbf{V e c}_{k}
$$

1.2.4 (The perverse realisation). - We now turn to another realisation functor which takes values in a subcategory of perverse sheaves with rational coefficients on the complex affine line. Recall that, given two objects $A$ and $B$ of the derived category of constructible sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{A}^{1}(\mathbb{C})$, one defines their additive convolution by

$$
A * B=\operatorname{Rsum}_{*}\left(\operatorname{pr}_{1}^{*} A \otimes \operatorname{pr}_{2}^{*} B\right)
$$

where sum: $\mathbb{A}^{2} \rightarrow \mathbb{A}^{1}$ is the summation map, and $\operatorname{pr}_{i}: \mathbb{A}^{2} \rightarrow \mathbb{A}^{1}$ the projections onto the two factors. Even if we start with two perverse sheaves, their additive convolution fails to be perverse in general. To remedy this, we will restrict to the full subcategory $\mathbf{P e r v}_{0}$ of $\mathbb{Q}$-perverse sheaves on $\mathbb{A}^{1}(\mathbb{C})$ consisting of those objects $C$ without global cohomology, i.e. such that $R \pi_{*} C=0$ for $\pi$ the structure morphism of $\mathbb{A}^{1}$. A typical object of this category is $E(0)=j!j^{*} \underline{\mathbb{Q}}[1]$, where $j: \mathbb{G}_{m} \hookrightarrow \mathbb{A}^{1}$ stands for the natural inclusion. Indeed, we shall see that all the objects of $\mathbf{P e r v}_{0}$ are of the form $F[1]$ for some constructible sheaf of $\mathbb{Q}$-vector spaces $F$ satisfying $H^{*}\left(\mathbb{A}^{1}(\mathbb{C}), F\right)=0$. This enables us to define the "nearby fibre at infinity" $\Psi_{\infty}: \mathbf{P e r v}_{0} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ as

$$
\Psi_{\infty}(F[1])=\lim _{r \rightarrow+\infty} F\left(S_{r}\right)
$$

Besides, the inclusion of $\operatorname{Perv}_{0}$ into Perv admits a left adjoint $\Pi$ : Perv $\rightarrow \mathbf{P e r v}_{0}$ which is given by additive convolution with the object $E(0)$, that is, $\Pi(C)=C * E(0)$.

For a variety $X$ and a closed subvariety $Y \subseteq X$, let $\beta: X \backslash Y \hookrightarrow X$ be the inclusion of the complement and $\mathbb{Q}_{[X, Y]}=\beta_{!} \beta^{*} \underline{\mathbb{Q}}$ the sheaf computing the relative cohomology of the pair $(X(\mathbb{C}), Y(\mathbb{C}))$. We define a functor $\mathrm{Q}^{\exp }(k) \rightarrow \operatorname{Perv}_{0}$ by assigning to $[X, Y, f, n, i]$ the perverse sheaf

$$
\Pi\left({ }^{p} \mathcal{H}^{n}\left(R f_{*} \underline{\mathbb{Q}}_{[X, Y]}\right)\right)(i),
$$

where ${ }^{p} \mathcal{H}^{n}$ stands for the cohomology with respect to the $t$-structure defining Perv inside the derived category of constructible sheaves. As we shall prove in 3.2 , the composition of this functor with $\Psi_{\infty}$ gives back the rapid decay cohomology. Invoking the universal property again, this yields the perverse realisation

$$
R_{\text {Perv }}: \mathbf{M}^{\exp }(\mathbf{k}) \longrightarrow \text { Perv }_{\mathbf{0}}
$$

1.2 .5 (The tensor structure). - Given two pairs $\left(X_{1}, f_{1}\right)$ and $\left(X_{2}, f_{2}\right)$ of varieties with potential, the cartesian product $X_{1} \times X_{2}$ is equipped with the Thom-Sebastiani sum

$$
\begin{equation*}
\left(f_{1} \boxplus f_{2}\right)\left(x_{1}, x_{2}\right)=f_{1}\left(x_{1}\right)+f_{2}\left(x_{2}\right) \tag{1.2.5.1}
\end{equation*}
$$

There is a cup-product in rapid decay cohomology

$$
H_{\mathrm{rd}}^{n_{1}}\left(X_{1}, Y_{1}, f_{1}\right) \otimes H_{\mathrm{rd}}^{n_{2}}\left(X_{2}, Y_{2}, f_{2}\right) \longrightarrow H_{\mathrm{rd}}^{n_{1}+n_{2}}\left(X_{1} \times X_{2},\left(Y_{1} \times X_{2}\right) \cup\left(X_{1} \times Y_{2}\right), f_{1} \boxplus f_{2}\right)
$$

which induces an isomorphism of $\mathbb{Q}$-vector spaces (Künneth formula):

$$
\bigoplus_{a+b=n} H_{\mathrm{rd}}^{a}\left(X_{1}, Y_{1}, f_{1}\right) \otimes H_{\mathrm{rd}}^{b}\left(X_{2}, Y_{2}, f_{2}\right) \simeq H_{\mathrm{rd}}^{n}\left(X_{1} \times X_{2},\left(Y_{1} \times X_{2}\right) \cup\left(X_{1} \times Y_{2}\right), f_{1} \boxplus f_{2}\right)
$$

The technical heart of this work is the following theorem:

THEOREM 1.2.6 (cf. Theorem 4.4.1). - There exists a unique monoidal structure on $\mathbf{M}^{\exp }(k)$ which is compatible with the Betti realisation $R_{B}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ and with cup-products. With respect to this monoidal structure, $\mathbf{M}^{\exp }(k)$ is a neutral tannakian category with $R_{B}$ as fibre functor.

The difficulty of constructing the tensor product stems from the fact that the obvious rule

$$
\left[X_{1}, Y_{1}, f_{1}, n_{1}, i_{1}\right] \otimes\left[X_{2}, Y_{2}, f_{2}, n_{2}, i_{2}\right]=\left[X_{1} \times X_{2},\left(Y_{1} \times X_{2}\right) \cup\left(X_{1} \times Y_{2}\right), f_{1} \boxplus f_{2}, n_{1}+n_{2}, i_{1}+i_{2}\right]
$$ is not compatible with the Künneth formula unless the rapid decay cohomology of the triples $\left(X_{i}, Y_{i}, f_{i}\right)$ is concentrated in a single degree. As for usual Nori motives, the problem is solved by showing that every object admits a "cellular filtration". More precisely, the key ingredient is the following statement, which—thanks to the perverse realisation-follows from Beilinson's most general form of the basic lemma.

Theorem 1.2.7 (Exponential basic lemma, cf. Corollary 3.3.3). - Let $X$ be an affine variety of dimension $d$, together with a regular function $f: X \rightarrow \mathbb{A}$, and let $Y \subsetneq X$ be a closed subvariety, There exists a closed subvariety $Z \subseteq X$ of dimension at most $d-1$ and containing $Y$ such that $H^{i}(X, Z, f)=0$ for all $i \neq d$.

Once we have the tensor product at our disposal, many relations between exponential periods can be proved to be of motivic origin. For instance, the value (1.1.4.2) of the Gaussian integral reflects the isomorphism of motives

$$
H^{1}\left(\mathbb{A}^{1}, x^{2}\right)^{\otimes 2}=H^{1}\left(\left\{x^{2}+y^{2}=1\right\}\right)
$$

which will be established in Section 12.2.
1.2.8 (Relation with usual Nori motives). - Nori's category of (non-effective cohomological) mixed motives over $k$ is related to our construction as follows. Let $\mathrm{Q}(k)$ be the full subquiver of $\mathrm{Q}^{\exp }(k)$ consisting of those tuples $[X, Y, f, n, i]$ with $f=0$. The restriction of the representation $\rho$ to this subquiver is nothing other than the usual Betti cohomology of the pair $(X(\mathbb{C}), Y(\mathbb{C}))$. Nori's category of mixed motives $\mathbf{M}(k)$ is the category of finite-dimensional $\mathbb{Q}$-vector spaces equipped with a continuous $\operatorname{End}\left(\left.\rho\right|_{\mathrm{Q}(k)}\right)$-action. From the inclusion $\mathrm{Q}(k) \rightarrow \mathrm{Q}^{\exp }(k)$, one obtains a restriction homomorphism $\operatorname{End}(\rho) \rightarrow \operatorname{End}\left(\left.\rho\right|_{Q(k)}\right)$, and hence a canonical functor $\iota: \mathbf{M}(k) \rightarrow \mathbf{M}^{\exp }(k)$ which, by the general formalism, is faithful and exact.

THEOREM 1.2.9 (cf. Theorem 5.1.1). - The functor $\iota: \mathbf{M}(k) \longrightarrow \mathbf{M}^{\exp }(k)$ is full.

This enables us to identify the category of classical Nori motives with a full subcategory of the category of exponential motives. However, the image of $\mathbf{M}(k)$ in $\mathbf{M}^{\exp }(k)$ is not stable under extension. In Chapter 12, we shall construct an extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$ whose period matrix is given by

$$
\left(\begin{array}{cc}
1 & \gamma \\
0 & 2 \pi i
\end{array}\right)
$$

where $\gamma=\lim _{n \rightarrow \infty}\left(\sum_{k=1}^{n} \frac{1}{k}-\log (n)\right)$ denotes the Euler-Mascheroni constant.

### 1.3. The motivic exponential Galois group

By the fundamental theorem of tannakian categories, $\mathbf{M}^{\exp }(k)$ is equivalent to the category of representations of an affine group scheme $G^{\exp }(k)$ over $\mathbb{Q}$, which will be called the motivic exponential Galois group. A formal consequence of the construction of the tannakian category $\mathbf{M}^{\exp }(k)$ and the realisations functors will be the following:

Proposition 1.3.1 (cf. Proposition 8.3.1). - The scheme of tensor isomorphisms

$$
\underline{\operatorname{Isom}}^{\otimes}\left(R_{\mathrm{dR}}, R_{B}\right)
$$

is a torsor under the motivic exponential Galois group.

Given an exponential motive $M$, one can look at the smallest tannakian subcategory $\langle M\rangle^{\otimes}$ of $\mathbf{M}^{\exp }(k)$ containing $M$. Invoking again the general formalism, $\langle M\rangle^{\otimes}$ is equivalent to $\boldsymbol{\operatorname { R e p }}\left(G_{M}\right)$ for a linear algebraic group $G_{M} \subseteq \mathrm{GL}\left(R_{B}(M)\right)$ which we call the Galois group of $M$. It follows from Proposition 1.3.1 that, when $k$ is a number field, the dimension of $G_{M}$ is an upper bound for the transcendence degree of the field generated by the periods of $M$. Indeed, one conjectures:

Conjecture 1.3.2 (Exponential period conjecture, cf. Conjecture 8.2.6). - Given an exponential motive $M$ over a number field, one has

$$
\operatorname{trdeg} \overline{\mathbb{Q}}(\text { periods of } M)=\operatorname{dim} G_{M}
$$

A number of classical results and conjectures in transcendence theory may be seen as instances of this equality. For example, we will show in Section 12.1 that the Lindemann-Weierstrass theorem (given $\mathbb{Q}$-linearly independent algebraic numbers $\alpha_{1}, \ldots, \alpha_{n}$, their exponentials $e^{\alpha_{1}}, \ldots, e^{\alpha_{n}}$ are algebraically independent) is the exponential period conjecture for the motive

$$
M=\bigoplus_{i=1}^{n} H^{0}\left(\operatorname{Spec} k,-\alpha_{i}\right)
$$

where $k$ denotes the number field generated by $\alpha_{1}, \ldots, \alpha_{n}$.
1.3.3 (Gamma motives and the abelianisation of the exponential Galois group). - For each integer $n \geqslant 2$, consider the following exponential motive over $\mathbb{Q}$ :

$$
\begin{equation*}
M_{n}=H^{1}\left(\mathbb{A}^{1}, x^{n}\right) \tag{1.3.3.1}
\end{equation*}
$$

By Example 1.1.4, all the values of the gamma function at rational numbers of denominator $n$ are periods of $M_{n}$, so it makes sense to call (1.3.3.1) a gamma motive. Avatars of the $M_{n}$ already appeared in the work of Anderson [1] under the name of ulterior motives. The rationale behind his
choice of terminology was that, while $M_{n}$ are "not themselves motives, motives may be constructed from them via the operations of linear algebra" (loc.cit. p.154). As a striking illustration, he showed that, for all $m \geqslant 2$, the tensor product $M_{n}^{\otimes m}$ contains a submotive isomorphic to the primitive motive of Fermat hypersurface $X=\left\{x_{1}^{n}+\ldots+x_{m}^{n}=0\right\} \subseteq \mathbb{P}^{m-1}$. We shall recover this fact in a very natural way in Chapter 13, cf. Proposition 13.3.3.

Conjecture 1.3.4 (Lang). - Let $n \geqslant 3$ be an integer. The transcendence degree of the field generated over $\overline{\mathbb{Q}}$ by the gamma values $\Gamma\left(\frac{a}{n}\right)$, for $a=1, \ldots, n-1$, is equal to $1+\varphi(n) / 2$.

At the time of writing, the conjecture is only known for $n=3,4,6$, as a corollary of Chudnovsky's theorem that the transcendence degree of the field of periods of an elliptic curve over $\overline{\mathbb{Q}}$ is at least 2 and the Chowla-Selberg formula [19]. As observed by André [2, 24.6], this conjecture follows from Grothendieck's period conjecture, although in a rather indirect way which requires to know that periods of abelian varieties with complex multiplication by a cyclotomic field can be expressed in terms of gamma values. We shall prove that the Galois group of the motive $M_{n}$ fits into an exact sequence

$$
0 \rightarrow \mu_{n} \rightarrow G_{M_{n}} \longrightarrow \mathcal{S}^{\mathbb{Q}\left(\mu_{n}\right)} \rightarrow 0
$$

where $\mathcal{S}^{\mathbb{Q}\left(\mu_{n}\right)}$ stands for the Serre torus of the cyclotomic field $\mathbb{Q}\left(\mu_{n}\right)$. This implies that $G_{M_{n}}$ has dimension $1+\varphi(n) / 2$ and enables us to see Lang's conjecture as an instance of Conjecture 1.3.2.

### 1.4. Outline

Briefly, the text is organised as follows. We refer the reader to the introductions of each chapter for a more precise description.

Chapter 2 contains some preliminaries about perverse sheaves that will be used in the sequel. The main result is that the category $\mathbf{P e r v}_{0}$ is tannakian with respect to the monoidal structure given by additive convolution and the nearby fibre at infinity functor. We then discuss another fibre functor, given by the total vanishing cycles. A careful study of the local monodromies of the additive convolution allows one to see the compatibility with the tensor structures as a reformulation of the Thom-Sebastiani theorem.

In Chapter 3, we study the basic properties of rapid decay cohomology. Besides the elementary definition, we give two alternative descriptions. The first one, as the nearby fibre at infinity of a perverse sheaf, is used to obtain the exponential basic lemma. The second one, in terms of the oriented real blow-up, will play a pivotal role in the proof of the comparison isomorphism.

Chapter 4 is the technical core of this work. After some preliminaries about Nori's formalism, we define $\mathbf{M}^{\exp }(k)$ as an abelian category. We then move to the construction of the tensor product using the exponential basic lemma from the previous chapter. In the last sections, we show that the Gysin long exact sequence is motivic and we complete the proof that $\mathbf{M}^{\exp }(k)$ is tannakian.

In Chapter 5, we prove that classical Nori motives form a full subcategory of exponential motives. We then explain how the categories $\mathbf{M}^{\exp }(k)$ behave with respect to base field extension. We end the chapter with a brief discussion of the conjectural relation with a Voevodsky-like category of exponential motives and the Grothendieck ring of varieties with exponentials.

Chapter 6 presents the construction of the perverse realisation. The main result of this chapter is that an exponential motives comes from a classical motive if and only if its perverse realisation is trivial. We also give a few examples of situations where the knowledge of the fundamental group of the perverse realisation allows one to compute the whole motivic fundamental group.

Chapter 7 is devoted to the comparison isomorphism between rapid decay and de Rham cohomology. Revisiting work of Hien and Roucairol, we prove a Poincaré lemma for the moderate growth twisted de Rham complex and use it to construct the period pairing.

Chapter 8 exploits the results of the previous chapter to obtain the period realisation functor. We then discuss a number of related topics, especially the notion of motivic exponential period and the coaction of the motivic Galois group.

In Chapter 9, we introduce a realisation functor with values in the category of $\mathscr{D}$-modules over the affine line and we explain how to use Fourier transform to obtain the de Rham realisation out of it. This gives a new interpretation of the fundamental group of the object of $\mathbf{P e r v}_{0}$ underlying an exponential motives as a differential Galois group.

Chapter 10 contains a brief discussion about how to associate an $\ell$-adic perverse sheaf with an exponential motive. Over a number field, one can then reduce modulo a prime ideal this $\ell$-adic sheaf and compute traces of Frobenius.

Chapter 11 deals with exponential Hodge theory. We upgrade the perverse realisation to a Hodge realisation with values in a subcategory of mixed Hodge modules on the affine line. We then prove that the weight filtration is motivic and discuss briefly the irregular Hodge filtration.

In Chapter 12, we present a collection of examples of exponential motives and compute their periods and Galois groups. These include exponentials of algebraic numbers, the motive $\mathbb{Q}(1 / 2)$, special values of the Bessel functions, and the Euler-Mascheroni constant.

Finally, in Chapter 13 we examine the gamma motives $M_{n}$. We compute their Galois groups and show that their dimensions are in accordance with Lang's conjecture. From this we obtain a conjectural description of the abelianisation of the exponential motivic Galois group.

The text is supplemented by an appendix where we gather a few results from the theory of tannakian categories that are often used in the main text.
1.4.1 (Notation and conventions). - Throughout, $k$ denotes a subfield of $\mathbb{C}$. By a variety over $k$ we mean a quasi-projective separated scheme of finite type over $k$. We shall call normal crossing divisor what is usually called a simple or strict normal crossing divisor, i.e. the irreducible components are smooth. Although this assumption is not indispensable for all constructions, there will be no lost in making it. Given a variety $X$, a closed subvariety $Y \subseteq X$, and a constructible sheaf $F$ on $X$, we set $F_{[X, Y]}=\beta_{!} \beta^{*} F$ where $\beta: X \backslash Y \hookrightarrow X$ is the inclusion of the complement.
1.4.2 (Acknowledgments). - This work secretly started when Emmanuel Kowalski and Henryk Iwaniec asked the first author to present the main results from Katz's book [56] at the ITS informal analytic number theory seminar. We are grateful to Piotr Achinger, Yves André, Joseph Ayoub, Daniel Bertrand, Spencer Bloch, Jean-Benoît Bost, Francis Brown, Pierre Colmez, Johan Commelin, Clément Dupont, Hélène Esnault, Martin Gallauer, Marco Hien, Annette Huber, Florian Ivorra, Daniel Juteau, Bruno Kahn, Maxim Kontsevich, Marco Maculan, Yuri Manin, Sophie Morel, Simon Pepin-Lehalleur, Richard Pink, Claude Sabbah, Will Sawin, Lenny Taelman, JeanBaptiste Teyssier, and Jeng-Daw Yu for frutiful discussions. We would like to thank the MPIM Bonn where part of the work was done. During the preparation of this work the first author was supported by the SNSF grant 200020-162928.

## CHAPTER 2

## The category Perv 0

In this chapter, we study the category $\operatorname{Perv}_{0}$ of perverse sheaves with vanishing cohomology on the complex affine line, which was originally introduced by Katz [55, 12.6] and further discussed by Kontsevich and Soibelman [58, 4.2]. The main result is that Perv $_{0}$ has the structure of a neutral tannakian category, with additive convolution as a tensor product and the nearby fibre at infinity as a fibre functor. This category plays a pivotal role in the description of rapid decay cohomology and the proof of the exponential lemma in Chapter 3. Later on, it will also be indispensable for the construction of the Hodge realisation functor.

### 2.1. Preliminaries on perverse sheaves

In this section, we collect a few basic definitions and facts about perverse sheaves that will be used in the sequel. Our standard references are [8], [20], [27], [53], [79]. We convene that "sheaf" means "sheaf of finite-dimensional $\mathbb{Q}$-vector spaces" unless otherwise indicated. We will try to systematically stick to the following naming convention:

$$
\begin{array}{ll}
A, B, C, \ldots & \text { complexes of sheaves }(e . g . \text { perverse sheaves), } \\
F, G, \ldots & \text { sheaves or complexes concentrated in degree zero. }
\end{array}
$$

2.1.1 (Constructible sheaves and the six functors formalism). - Given an algebraic variety $X$ over a subfield $k$ of $\mathbb{C}$, we denote by $\operatorname{Sh}(X)$ the abelian category of sheaves on the topological space $X(\mathbb{C})$, by $\mathrm{D}(X)$ the derived category of $\operatorname{Sh}(X)$, and by $\mathrm{D}^{b}(X)$ the bounded derived category. A sheaf $F$ in $\operatorname{Sh}(X)$ is said to be constructible if there exist closed subvarieties

$$
\emptyset=X_{-1} \subseteq X_{0} \subseteq X_{1} \subseteq \cdots \subseteq X_{r}=X
$$

such that, for each $p=0, \ldots, r$, the restriction of $F$ to $X_{p}(\mathbb{C}) \backslash X_{p-1}(\mathbb{C})$ is a local system of finite rank. If two out of three terms in a short exact sequence of sheaves on $X$ are constructible, then so is the third one. Constructible sheaves thus form an abelian subcategory of $\operatorname{Sh}(X)$, which is moreover stable under tensor products and internal Hom.

Definition 2.1.2. - The bounded derived category of constructible sheaves $\mathrm{D}_{c}^{b}(X)$ is the full subcategory of $\mathrm{D}^{b}(X)$ consisting of those complexes $A$ whose cohomology sheaves $\mathcal{H}^{q}(A)$ are constructible for all integers $q$. Sometimes we will also call constructible sheaf an object $A$ of $\mathrm{D}_{c}^{b}(X)$ such that $\mathcal{H}^{q}(A)=0$ unless $q=0$.

The terminology is not completely abusive. Writing $\mathrm{D}^{b}(\operatorname{Constr}(X))$ for the bounded derived category of the abelian category of constructible sheaves on $X$, the obvious functor

$$
\mathrm{D}^{b}(\operatorname{Constr}(X)) \longrightarrow \mathrm{D}_{c}^{b}(X)
$$

is an equivalence of categories by a theorem of Nori $[66$, Theorem $3(\mathrm{~b})]$.
2.1.3. - Associated with each morphism $f: X \rightarrow Y$ of algebraic varieties, there are functors

$$
\begin{aligned}
& f^{*}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X) \\
& \text { inverse image } \\
& f_{*}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y) \\
& \text { direct image } \\
& f_{!}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y) \\
& \text { direct image with compact support. }
\end{aligned}
$$

The inverse image functor $f^{*}$ is exact, whereas the two direct image functors $f_{*}$ and $f_{!}$are only left exact. Taking their derived functors yields $f^{*}: \mathrm{D}(Y) \rightarrow \mathrm{D}(X)$ and $R f_{*}, R f_{!}: \mathrm{D}(X) \rightarrow \mathrm{D}(Y)$. The functors $f^{*}$ and $R f_{*}$ are adjoint to each other, so there is a natural adjunction isomorphism

$$
\operatorname{Hom}_{\mathrm{D}(Y)}\left(A, R f_{*} B\right)=\operatorname{Hom}_{\mathrm{D}(X)}\left(f^{*} A, B\right)
$$

for all objects $A$ of $\mathrm{D}(Y)$ and $B$ of $\mathrm{D}(X)$. It is a non-trivial result that the functor $R f_{!}$admits a right adjoint $f^{!}: \mathrm{D}(Y) \rightarrow \mathrm{D}(X)$, so there is a natural adjunction isomorphism

$$
\operatorname{Hom}_{\mathrm{D}(Y)}\left(R f_{!} B, A\right)=\operatorname{Hom}_{\mathrm{D}(X)}\left(B, f^{!} A\right)
$$

for all objects $A$ of $\mathrm{D}(Y)$ and $B$ of $\mathrm{D}(X)$. This adjoint $f^{!}$only exists on the derived categories; the functor $f_{!}$between the abelian categories of sheaves has in general no right adjoint. The situation is summarised in the following diagram:

where functors on top are right adjoint to functors below.
The functor sheaf of homomorphisms that associates with sheaves $F$ and $G$ on $X$ the sheaf $\mathcal{H o m}(F, G)$ on $X$ can be derived as a left exact functor in $G$, giving rise to the functor

$$
R \mathcal{H o m}: \mathrm{D}(X)^{\mathrm{op}} \times \mathrm{D}(X) \longrightarrow \mathrm{D}(X) .
$$

Since we will only consider sheaves of vector spaces, the functor associating with sheaves $F$ and $G$ on $X$ the tensor product sheaf $F \otimes G$ is exact in both variables and there is no need to derive it. Given objects $A, B, C$ of $\mathrm{D}(X)$, the usual adjunction formula holds: there is a canonical isomorphism

$$
R \mathcal{H o m}(A \otimes B, C)=R \mathcal{H o m}(A, R \mathcal{H o m}(B, C))
$$

in the derived category $\mathrm{D}(X)$ which is natural in the three arguments. The functors

$$
R f_{*}, f^{*}, R f_{!}, f^{!}, \otimes, R \mathcal{H} \text { om }
$$

are usually referred to as the six operations.

Theorem 2.1.4 (Verdier's constructibility theorem). - The six operations preserve the derived categories of constructible sheaves.

To our knowledge, Verdier never stated this theorem explicitly. The stability under the three operations $f^{*}, \otimes$, and $R \mathcal{H} o m$ is straightforward. As explained in $[8,2.1 .13$ and 2.2.1], the statement that $R f_{*}, R f_{!}$, and $f^{!}$preserve constructibility follows formally from the fact that every stratification of an algebraic variety can be refined into a Whitney stratification, which is proven by Verdier in [86, Théorème 2.2]. One can also prove it by induction on the dimension of supports, using the fact that, for every morphism of complex algebraic varieties $f: X \rightarrow Y$, there exists a non-empty Zariski open subset $U \subseteq Y$ such that $f^{-1}(U) \rightarrow U$ is a fibre bundle for the complex topology. This statement is the content of [86, Corollaire 5.1], and can also be proved using resolution of singularities and Ehresmann's fibration theorem. A quite different approach is taken by Nori in [66, Theorem 4], where he shows that $R f_{*}$ can be computed using a resolution by constructible sheaves. Therefore, in order to show that $R f_{*}$ preserves constructibility it suffices to show that $f_{*}$ does so, which is not difficult. A proof of Verdier's constructibility theorem in a more general context is given in [79, Chapter 4].
2.1.5. - Let $f: X \rightarrow Y$ be a morphism of algebraic varieties. In special cases, depending on the quality of $f$, direct and inverse image functors between derived categories of constructible sheaves satisfy useful relations, that we collect here pour mémoire:
(1) If $f$ is proper, then $R f_{*}=R f_{!}$.
(2) If $f$ is a smooth morphism of relative dimension $d$, then $f^{!}=f^{*}[2 d]$.
(3) If $f$ is a closed immersion, then $f_{*}$ is exact.
(4) If $f$ is an open immersion, then $f$ ! is exact and $f^{!}=f^{*}$.
2.1.6 (Base change theorems). - Consider a cartesian square of complex algebraic varieties

which simply means that $X^{\prime}$ is the fibre product of $X$ and $Y^{\prime}$ over $Y$. For every sheaf $F$ on $X$, or more generally for every object $A$ of $\mathrm{D}(X)$, there is a canonical and natural morphism

$$
\begin{equation*}
g_{Y}^{*} R f_{*} A \longrightarrow R f_{*}^{\prime} g_{X}^{*} A \tag{2.1.6.2}
\end{equation*}
$$

on the derived category of $Y^{\prime}$ called base change morphism. In general, (2.1.6.2) is not an isomorphism. There are, however, two important geometric situations in which it is. The proper base change theorem states that, if $f$ is a proper morphism, then (2.1.6.2) is an isomorphism for all objects $A$ of $\mathrm{D}(X)$. In particular, there is an isomorphism

$$
\begin{equation*}
g_{Y}^{*} R f_{!} A \xrightarrow{\cong} R f_{!}^{\prime} g_{X}^{*} A \tag{2.1.6.3}
\end{equation*}
$$

without any condition on $f$. The smooth base change theorem states that, if $g_{Y}$ is smooth, then (2.1.6.2) is an isomorphism for all objects $A$ of $D_{c}^{b}(X)$. This can be deduced from point (2) of 2.1.5 and the base change theorem for the exceptional inverse image, which gives a canonical and natural isomorphism

$$
\begin{equation*}
g_{Y}^{!} R f_{*} A \xrightarrow{\cong} R f_{*}^{\prime} g_{X}^{!} A \tag{2.1.6.4}
\end{equation*}
$$

without any condition on $g_{Y}$. Proofs can be found in [53, Proposition 2.5.11] for proper base change, and [53, Proposition 3.1.9] for smooth base change.
2.1.7. - Let $X$ be a variety over $k$ and $\pi: X \rightarrow \operatorname{Spec}(k)$ the structure morphism. The dualising complex of $X$ (often dualising sheaf, although it is not really a sheaf) is the object

$$
\omega_{X}=\pi^{!} \mathbb{Q}
$$

of the category $\mathrm{D}_{c}^{b}(X)$. More generally, the relative dualising complex for a morphism $f: X \rightarrow Y$ is defined as $\omega_{X / Y}=f^{!} \mathbb{Q}_{Y}$. One then defines the Verdier dual of an object $A$ of $D_{c}^{b}(X)$ as

$$
\mathbb{D}(A)=R \mathcal{H o m}\left(A, \omega_{X}\right)
$$

which is again an object of the derived category of constructible sheaves on $X$.

Theorem 2.1.8 (Local Verdier duality). - Given a morphism $f: X \rightarrow Y$ of algebraic varieties and objects $A$ of $\mathrm{D}_{c}^{b}(X)$ and $B$ of $D_{c}^{b}(Y)$, there is a natural isomorphism

$$
\begin{equation*}
R \mathcal{H o m}\left(R f_{!} A, B\right) \cong R f_{*} R \mathcal{H o m}\left(A, f^{!} B\right) \tag{2.1.8.1}
\end{equation*}
$$

in the category $\mathrm{D}_{c}^{b}(X)$. In particular, for all objects $A$ of $\mathrm{D}_{c}^{b}(X)$, there are natural isomorphisms

$$
\mathbb{D}\left(R f_{!} A\right) \cong R f_{*} \mathbb{D}(A) \quad \text { and } \quad \mathbb{D}(\mathbb{D}(A)) \cong A
$$

References are [53, Proposition 3.1.10] or [27, Theorem 3.2.3]. Taking global sections on both sides of (2.1.8.1) yields the global form of Verdier's duality theorem.
2.1.9. - The dualising complex $\omega_{X}$ on $X$ has the following explicit description. For any open set $U \subseteq X(\mathbb{C})$, let $\dot{U}=U \cup\{\cdot\}$ be the one-point compactification of $U$, and let

$$
C_{*}(\dot{U},\{\cdot\})=\left[\cdots \longrightarrow C_{2}(\dot{U},\{\cdot\}) \longrightarrow C_{1}(\dot{U},\{\cdot\}) \longrightarrow C_{0}(\dot{U},\{\cdot\})\right]
$$

be the singular chain complex with rational coefficients of the pair $(U,\{\cdot\})$. We view this as a complex concentrated in degrees $\leqslant 0$. For any inclusion of open sets $V \subseteq U$, there is a canonical map $\dot{U} \rightarrow \dot{V}$ contracting the complement $U \backslash V$ to the point $\cdot \in V$. This map yields a morphism
of complexes $C_{*}(\dot{U},\{\cdot\}) \rightarrow C_{*}(\dot{V},\{\cdot\})$. The dualising complex is the complex of sheaves associated with the presheaves $U \longmapsto C_{*}(\dot{U},\{\cdot\})$. In particular, $\mathcal{H}^{p}\left(\omega_{X}\right)$ is the sheaf associated with the presheaf given by reduced singular homology $U \longmapsto \widetilde{H}_{p}(\dot{U})$. The easiest example where this recipe for computing the dualising complex yields a concrete description is the case where $X$ is smooth of dimension $d$. In that case, $X(\mathbb{C})$ is locally homeomorphic to an open ball of real dimension $2 d$, so every point of $X(\mathbb{C})$ has a fundamental system of open neighbourhoods $U$ for which $\dot{U}$ is homeomorphic to a sphere of dimension $2 d$. Hence, the dualising complex is isomorphic to

$$
\omega_{X} \cong \mathbb{Q}_{X}[2 d] .
$$

A useful consequence is that the Verdier dual of a local system $L$ on $X$ is given by $\mathbb{D}(L)=L^{\vee}[2 d]$, where $L^{\vee}=\mathcal{H o m}\left(L, \mathbb{Q}_{X}\right)$ denotes the dual local system.

To see what Verdier's local duality theorem 2.1.8 has to do with more classical duality theorems, consider a smooth variety $X$, and take for $f$ the structure morphism. Choose for $A$ the constant sheaf $\mathbb{Q}_{X}$ on $X$, and for $B$ the sheaf $\mathbb{Q}$ on the point. The complex $R f_{!} A$ computes the cohomology with compact support $H_{c}^{p}(X(\mathbb{C}), \mathbb{Q})$ of $X$, whereas $R \mathcal{H} o m\left(R f_{!} A, B\right)$ computes its linear dual $H_{c}^{p}(X(\mathbb{C}), \mathbb{Q})^{\vee}$. The sheaf $R \mathcal{H o m}\left(A, f^{!} B\right)$ is the dualising sheaf $\omega_{X}=\mathbb{Q}_{X}[2 d]$, and hence $R f_{*} R \mathcal{H}$ om $\left(A, f^{!} B\right)$ computes the homology of $X$. Bookkeeping the shifting, the canonical isomorphism in Verdier's duality theorem boils down to the classical Poincaré duality pairing

$$
H_{c}^{p}(X(\mathbb{C}), \mathbb{Q}) \otimes H^{2 d-p}(X(\mathbb{C}), \mathbb{Q}) \longrightarrow \mathbb{Q}
$$

between cohomology and cohomology with compact support.

Theorem 2.1.10 (Artin's vanishing theorem). - Let $X$ be an affine variety over $k$, and let $F$ be a constructible sheaf on $X$. Then $H^{q}(X, F)=0$ for all $q>\operatorname{dim} X$.

The original reference is Artin's Exposé XIV in SGA 4 [5]. An analytic proof, relying on the Riemann-Hilbert correspondence, is given by Esnault in [33].
2.1.11 (Perverse sheaves). - Beilinson, Bernstein, Deligne, and Gabber [8] defined an abelian subcategory $\operatorname{Perv}(X)$ of $\mathrm{D}_{c}^{b}(X)$ whose objects are called perverse sheaves. Recall that the support of a sheaf $F$ on $X$ is the closed subset

$$
\operatorname{supp} F=\overline{\left\{x \in X(\mathbb{C}) \mid F_{x} \neq 0\right\}}
$$

Definition 2.1.12. - An object $A$ of $\mathrm{D}_{c}^{b}(X)$ is called semiperverse if the inequality

$$
\operatorname{dim}\left(\operatorname{supp} \mathcal{H}^{-q}(A)\right) \leqslant q
$$

holds for all integers $q$. A perverse sheaf is an object $A$ of $D_{c}^{b}(X)$ such that both $A$ and its Verdier dual $\mathbb{D}(A)$ are semiperverse.

Example 2.1.13. - Let $X$ be a smooth variety of dimension $d$. For each local system $L$ on $X$, the complex $L[d]$ is a perverse sheaf. Indeed, its only non-trivial cohomology sheaf is $\mathcal{H}^{-d}(A)=L$,
which has full support. This shows that $A$ is semiperverse. Thanks to the shift by the dimension, its Verdier dual is of the same shape, namely $\mathbb{D}(A)=L^{\vee}[d]$, so it is also semiperverse.
2.1.14. - Let ${ }^{p} \mathrm{D}_{c}^{\leqslant 0}(X)$ be the full subcategory of $\mathrm{D}_{c}^{b}(X)$ consisting of semiperverse complexes, and ${ }^{p} \mathrm{D}_{c}^{\geqslant 0}(X)$ the full subcategory of complexes $A$ such that $\mathbb{D}(A)$ is semiperverse. One of the main results of $[8]$ is that the pair

$$
\left({ }^{p} \mathrm{D}_{c}^{\leqslant 0}(X),{ }^{p} \mathrm{D}_{c}^{\geqslant 0}(X)\right)
$$

forms a $t$-structure on the triangulated category $\mathrm{D}_{c}^{b}(X)$. Perverse sheaves are precisely the objects of the heart ${ }^{p} \mathrm{D}_{c}^{\leqslant 0}(X) \cap^{p} \mathrm{D}_{c}^{\geqslant 0}(X)$ and thus form an abelian category. This allows one to define cohomology functors

$$
{ }^{p} \mathcal{H}^{n}: \mathrm{D}_{c}^{b}(X) \longrightarrow \operatorname{Perv}(X)
$$

Theorem 2.1.15 (Artin's vanishing theorem for perverse sheaves). - Let $X$ be an affine variety and $A$ a perverse sheaf on $X$. Then $H^{q}(X, A)=0$ for all $q>0$, and $H_{c}^{q}(X, A)=0$ for all $q<0$.

Theorem 2.1.16 (Artin). - Let $f: X \rightarrow Y$ be an affine morphism. Then $R f_{*}$ is $t$-right exact and $R f_{!}$is $t$-left exact for the perverse $t$-structure.

Example 2.1.17. - Let $X$ be a smooth variety of dimension $d$ and let $\beta: U \rightarrow X$ be the inclusion of the complement of a divisor $D$. Then $\beta_{!} \mathbb{Q}_{U}[d]$ is a perverse sheaf on $X$. Indeed, the morphism $\beta$ is affine and $R \beta_{!}=\beta_{!}$by 2.1.5, so Artin's vanishing theorem implies that $\beta_{!}$sends perverse sheaves on $U$ to perverse sheaves on $X$.
2.1.18 (Perverse sheaves on the affine line). - Since we will be mainly dealing with perverse sheaves on the affine line, we now specialize to this setting. A perverse sheaf on the complex affine line is a bounded complex $A$ of sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{A}^{1}(\mathbb{C})$ with constructible homology sheaves $\mathcal{H}^{n}(A)$ such that the following three conditions hold:
(a) $\mathcal{H}^{n}(A)=0$ for $n \notin\{-1,0\}$,
(b) $\mathcal{H}^{-1}(A)$ has no non-zero global sections with finite support,
(c) $\mathcal{H}^{0}(A)$ is a skyscraper sheaf.
2.1.19 (Nearby and vanishing cycles). - Let $A$ be an object of the derived category of constructible sheaves on the complex affine line. Let $S$ be the set of singularities of $A$. For every point $z \in \mathbb{C}$, we denote by $\Phi_{z}(A)$ the complex of vanishing cycles of $A$ at $z$. It is a complex of vector spaces given as follows: Let $\alpha:\{z\} \rightarrow \mathbb{C}$ be the inclusion, let $\beta: D_{0} \rightarrow \mathbb{C}$ be the inclusion of a small punctured disk around $z$, not containing any of the singularities of $A$, and let $e: U \rightarrow D_{0}$ be a universal covering. We define the following complexes of vector spaces (sheaves on a point)

$$
\begin{align*}
\Psi_{z}(A) & =\alpha^{*} \beta_{*} e_{*} e^{*} \beta^{*} A[-1]  \tag{2.1.19.1}\\
\Phi_{z}(A) & =\operatorname{cone}\left(\alpha^{*} A \longrightarrow \alpha^{*} \beta_{*} e_{*} e^{*} \beta^{*} A\right)[-1] \tag{2.1.19.2}
\end{align*}
$$

where the map in (2.1.19.2) is given by adjunction. We call $\Psi_{z}(A)$ the complex of nearby cycles and $\Phi_{z}(A)$ the complex of vanishing cycles of $A$ at $z$. If $z \notin S$, the complex of vanishing cycles is nullhomotopic. Notice that the definition of nearby and vanishing cycles depends on the choice of a universal covering $U \rightarrow D_{0}$. A different choice $U^{\prime} \rightarrow D_{0}$ yields different functors $\Psi_{z}^{\prime}$ and $\Phi_{z}^{\prime}$. Any isomorphism of covers $\gamma: U \rightarrow U^{\prime}$ induces isomorphisms $\gamma^{*}: \Psi_{z}^{\prime} \rightarrow \Psi_{z}$ and $\gamma^{*}: \Phi_{z}^{\prime} \rightarrow \Phi_{z}$. In particular, the deck transformation $U \rightarrow U$ coming from the action of the standard generator of $\pi_{1}\left(D_{0}\right)$ induces an automorphism of vector spaces

$$
\gamma_{z}: \Psi_{z}(A) \longrightarrow \Psi_{z}(A)
$$

called the local monodromy operator.
The following lemma is a special case of the general fact that, whenever $A$ is a perverse sheaf, the nearby and vanishing cycles are perverse sheaves as well.

Lemma 2.1.20. - Let $A$ be a perverse sheaf on $\mathbb{C}$. The complexes $\Psi_{z}(A)$ and $\Phi_{z}(A)$ are homologically concentrated in degree 0.

Proof. Let $z \in S$. Without loss of generality, we may restrict $A$ to a small disk $D$ around $z$ not containing any other singularity of $A$. This means that the sheaves $\mathcal{H}^{n}(A)$ on $D$ are constructible with respect to the stratification $\{z\} \subseteq D$. The complex $A$ fits into the exact truncation triangle $\mathcal{H}^{-1}(A)[1] \rightarrow A \rightarrow \mathcal{H}^{0}(A)[0]$, and $\Psi_{z}(A)$ and $\Phi_{z}(A)$ are triangulated functors, so it is enough to prove the lemma in the case where $A$ is a skyscraper sheaf sitting in degree 0 , and in the case where $A$ is a constructible sheaf with no non-zero sections with finite support sitting in degree -1 . For a skyscraper sheaf, $\Phi_{z}(A)$ is zero and $\Psi_{z}(A)$ is the stalk at $z$ sitting in degree 0 . In the case of a constructible sheaf, $\Phi_{z}(A)$ is the vector space of global sections of the local system $e^{*} \beta^{*} A[-1]$ on the universal cover of $D \backslash\{z\}$, viewed as a complex of sheaves on $\{z\}$ concentrated in degree 0 . Finally, the kernel of the adjunction map $\alpha^{*} A[-1] \longrightarrow \alpha^{*} \beta_{*} e_{*} e^{*} \beta^{*} A[-1]$ is the vector space of sections of $A[-1]$ supported on $\{z\}$, but this space is zero because $A$ is perverse. Therefore, the adjunction map is injective and its cone $\Psi_{z}(A)$ is homologically concentrated in degree 0 .
2.1.21. - Let $F$ be a constructible sheaf on $\mathbb{A}^{1}$ and let $z \in \mathbb{C}$. The nearby cycles $\Psi_{z}(F)$ and the vanishing cycles $\Phi_{z}(F)$ can be constructed in a less intrinsic, but more effectively conveying way. The fibre of $F$ at $z$ is defined as the colimit $F_{z}=\operatorname{colim} F\left(U_{n}\right)$, where $\left(U_{n}\right)_{n=1}^{\infty}$ is a fundamental system of open neighbourhoods of $z$. The nearby fibre can be defined as the colimit

$$
\Psi_{z}(F)=\operatorname{colim} F\left(V_{n}\right),
$$

where $\left(V_{n}\right)_{n=1}^{\infty}$ is the filter of open sets $V_{n}=\left\{u \in U_{n} \mid \operatorname{Re}(u)>\operatorname{Re}(z)\right\}$, or any other filter equivalent to it. The restriction maps $F\left(U_{n}\right) \rightarrow F\left(V_{n}\right)$ induce a map $F_{z} \rightarrow \Psi_{z}(F)$, which is called cospecialisation. The two-term complex $\Phi_{z}(F)=\left[F_{z} \rightarrow \Psi_{z}(F)\right]$ is the complex of vanishing cycles.


Figure 2.1.1. Filters $\left(U_{n}\right)_{n=1}^{\infty}$ and $\left(V_{n}\right)_{n=1}^{\infty}$

### 2.2. Computing the cohomology of constructible sheaves on the affine line

In this section, we describe the cohomology of constructible sheaves on the affine line using cochains. This is reminiscent of the cochain description of group cohomology, and will be helpful for concrete computations, in particular when we want to handle specific cohomology classes. We will come back to this description in Section 2.6, where cochains are used to compute the additive convolution of perverse sheaves. Throughout, all vector spaces are understood to be finite-dimensional vector spaces over $\mathbb{Q}$.
2.2.1. - We first interpret constructible sheaves on the complex plane $\mathbb{C}$ in terms of group representations. Let $S \subseteq \mathbb{C}$ be a finite set, $X=\mathbb{C} \backslash S$ its complement, and denote by

$$
S \xrightarrow{\alpha} \mathbb{C} \stackrel{\beta}{\leftarrow} X
$$

the inclusions. A constructible sheaf $F$ on $\mathbb{C}$ with singularities in $S$ is uniquely described by the following data:
(1) A local system $F_{X}$ on $X$.
(2) A sheaf $F_{S}$ on the discrete set $S$, and a morphism of sheaves $F_{S} \rightarrow \alpha^{*} \beta_{*} F_{X}$ on $S$.

Fix a base point $x \in X$, set $G=\pi_{1}(X, x)$ and denote by $V$ the fibre of $F$ at $x$. The local system $F_{X}$ corresponds to a representation $\rho: G \rightarrow \mathrm{GL}(V)$. The sheaf $F_{S}$ is given by a collection of vector spaces $\left(V_{s}\right)_{s \in S}$. For every path $p:[0,1] \rightarrow \mathbb{C}$ with $p(0)=s, p(1)=x$ and $p(t) \in X$ for all $t>0$, the gluing data (2) determines a linear map $\rho_{s}(p): V_{s} \rightarrow V$ called cospecialisation. If now $\alpha$ and $\beta$ denote the inclusions

$$
\{0\} \xrightarrow{\alpha}[0,1] \stackrel{\beta}{\longleftrightarrow}(0,1],
$$

then $\rho_{s}(p)$ is the linear map $V_{s} \rightarrow \alpha^{*} \beta_{*}\left(\left.p\right|_{(0,1]}\right)^{*} F_{X}$ composed with the canonical isomorphism

$$
\alpha^{*} \beta_{*}\left(\left.p\right|_{(0,1]}\right)^{*} F_{X} \cong \Gamma\left(\left(\left.\tau\right|_{(0,1]}\right)^{*} F_{X}\right) \cong V .
$$

The linear map $\rho_{s}(p)$ only depends on the class of $p$ up to homotopies in $X$ leaving $p(0)=s$ and $p(1)=x$ fixed. This makes sense despite the fact that $s$ is not in $X$. Write

$$
P_{s}=\{\text { paths from } s \text { to } x \text { in } X\} / \simeq_{\text {homotopy }}
$$

for the set of these classes. The fundamental group $G$ acts transitively on $P_{s}$ by concatenation of paths, and for $g \in G$ and $p \in P_{s}$ the relation $\rho_{s}(g p)=\rho(g) \rho_{s}(p)$ holds. Once a base point $x$ is chosen, we may thus describe constructible sheaves on $\mathbb{C}$ with singularities in $S$ by the following data:
(1) A vector space $V$, and a linear representation $\rho: G \rightarrow \mathrm{GL}(V)$.
(2) For every $s \in S$, a vector space $V_{s}$ and, for every path $p \in P_{s}$, a cospecialisation map $\rho_{s}(p): V_{s} \rightarrow V$ such that $\rho_{s}(g p)=\rho(g) \rho_{s}(p)$ holds for all $p \in P_{s}$ and all $g \in G$.

For fixed $S$ and $x$, the tuples $\left(V, \rho,\left(V_{s}, \rho_{s}\right)_{s \in S}\right)$ form an abelian category in the evident way, which is equivalent to the category of constructible sheaves on $\mathbb{C}$ with singularities contained in $S$. We can now forget about the geometric origin of $G$ and the $P_{s}$, and are lead to the following definition.

Definition 2.2.2. - Let $G$ be a group, and let $P_{S}=\left(P_{s}\right)_{s \in S}$ be a finite, possibly empty collection of non-empty $G$-sets. A representation of $\left(G, P_{S}\right)$ consists of a vector space $V$ and vector spaces $\left(V_{s}\right)_{s \in S}$, a group homomorphism $\rho: G \rightarrow \operatorname{GL}(V)$, and maps $\rho_{s}: P_{s} \rightarrow \operatorname{Hom}\left(V_{s}, V\right)$ satisfying the equality $\rho_{s}(g p)=\rho(g) \rho_{s}(p)$ for all $g \in G$ and $p \in P_{s}$. Morphisms of representations and their composition are defined in the evident way. We denote the resulting category by $\boldsymbol{\operatorname { R e p }}\left(G, P_{S}\right)$.
2.2.3. - The category $\operatorname{Rep}\left(G, P_{S}\right)$ is abelian, and it is indeed the category of sheaves on an appropriate site. Given a representation $V$ of $\left(G, P_{S}\right)$, we call invariants the subspace

$$
V^{\left(G, P_{S}\right)} \subseteq V \oplus \bigoplus_{s \in S} V_{s}
$$

of tuples $\left(v,\left(v_{s}\right)_{s \in S}\right)$ satisfying $g v=v$ for all $g \in G$ and $p v_{s}=v$ for all $p \in P_{s}$. Here, as we shall do from now on if no confusion seems possible, we suppressed $\rho$ and $\rho_{s}$ from the notation. We can regard the invariants also as a homomorphism set

$$
V^{\left(G, P_{S}\right)}=\operatorname{Hom}_{\left(G, P_{S}\right)}(\mathbb{Q}, V)
$$

where $\mathbb{Q}$ stands for the constant representation, in which all involved vector spaces are $\mathbb{Q}$ and all maps $\rho(g)$ and $\rho_{s}(p)$ are the identities. Associating with a representation its space of invariants defines a left exact functor from $\operatorname{Rep}\left(G, P_{S}\right)$ to the category of vector spaces. We can thus define cohomology groups

$$
H^{n}\left(G, P_{S}, V\right)
$$

using the right derived functor of the invariants functor. As for ordinary group cohomology, there is an explicit, functorial chain complex which computes this cohomology. Namely, define

$$
\begin{aligned}
C^{0}\left(G, P_{S}, V\right) & =V \oplus \bigoplus_{s \in S} V_{s} \\
C^{n}\left(G, P_{S}, V\right) & =\operatorname{Maps}\left(G^{n}, V\right) \oplus \bigoplus_{s \in S} \operatorname{Maps}\left(G^{n-1} \times P_{s}, V\right), \quad(n \geqslant 1)
\end{aligned}
$$

and call elements of $C^{n}\left(G, P_{S}, V\right)$ cochains. Alternatively, we will also think of cochains as $V$ valued functions on the disjoint union of $G^{n}$ and the various $G^{n-1} \times P_{s}$. This can make notation shorter. Define differentials

$$
\begin{equation*}
C^{0}\left(G, P_{S}, V\right) \xrightarrow{d^{0}} C^{1}\left(G, P_{S}, V\right) \xrightarrow{d^{1}} C^{2}\left(G, P_{S}, V\right) \xrightarrow{d^{2}} \cdots \tag{2.2.3.1}
\end{equation*}
$$

as follows. We set

$$
d^{0}\left(v,\left(v_{s}\right)_{s \in S}\right)(g)=v-g v \quad \text { and } \quad d^{0}\left(v,\left(v_{s}\right)_{s \in S}\right)(p)=v-p v_{s}
$$

and, for $n \geqslant 1$ and $c \in C^{n}\left(G, P_{S}, V\right)$, we define $d^{n} c$ by the usual formula

$$
\begin{aligned}
& \left(d^{n} c\right)\left(g_{1}, \ldots, g_{n}, y\right)=g_{1} c\left(g_{2}, \ldots, g_{n}, y\right)+ \\
& \quad+\sum_{i=1}^{n-1}(-1)^{i} c\left(g_{1}, \ldots, g_{i} g_{i+1}, \ldots, g_{n}, y\right)+(-1)^{n} c\left(g_{1}, \ldots, g_{n} y\right)+(-1)^{n+1} c\left(g_{1}, \ldots, g_{n}\right)
\end{aligned}
$$

where $y$ is either an element of $G$ or an element of $P_{s}$ for some $s \in S$. The verification that the spaces $C^{n}\left(G, P_{S}, V\right)$ and the differentials $d^{n}$ form a complex is straightforward. The chain complex $C^{*}\left(G, P_{S}, V\right)$ depends functorially on the representation $V$ in the evident way. The kernel of $d^{0}$ is the space of invariants, and if $S$ is empty, we get back the standard cochain complex computing group cohomology.

Lemma 2.2.4. - The chain complex (2.2.3.1) computes the right derived functor of the invariants functor $\operatorname{Hom}_{\left(G, P_{S}\right)}(\mathbb{Q},-)$.

Proof. We can compute $R \operatorname{Hom}_{\left(G, P_{S}\right)}\left(\mathbb{Q},\left(V,\left(V_{s}\right)_{s \in S}\right)\right)$ either by choosing an injective resolution of $V$ or by choosing a projective resolution of the constant representation $\mathbb{Q}$. Let us construct a projective resolution as follows. Set

$$
L_{0}=\left(\mathbb{Q}[G] \oplus \bigoplus_{s \in S} \mathbb{Q}\left[P_{s}\right], \quad(\mathbb{Q})_{s \in S}\right)
$$

and let $G$ act on $\mathbb{Q}[G] \oplus \bigoplus_{s \in S} \mathbb{Q}\left[P_{s}\right]$ by left multiplication, and for $p \in P_{s}$ define $\rho_{s}(p)(1)=1 \cdot p$. This makes $L_{0}$ into a $\left(G, P_{S}\right)$-representation. For $n \geqslant 1$ set

$$
L_{n}=\left(\mathbb{Q}\left[G^{n+1}\right] \oplus \bigoplus_{s \in S} \mathbb{Q}\left[G^{n} \times P_{s}\right], \quad(0)_{s \in S}\right)
$$

and endow $L_{n}$ with a $\left(G, P_{S}\right)$-action by letting $G$ act via multiplication on the left. Differentials are given by

$$
d_{n}\left(g_{0}, \ldots, g_{n-1}, y\right)=\sum_{i=0}^{n}(-1)^{i}\left(g_{0}, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{n-1}, y\right)+(-1)^{n}\left(g_{0}, \ldots, g_{n-1}\right)
$$

for $n \geqslant 1$ and $d_{0}: L_{0} \rightarrow \mathbb{Q}$ by $d_{0}(g)=d_{0}(p)=1$ and $d_{0, s}(1)=1$. A straightforward computation shows that $\cdots \rightarrow L_{2} \rightarrow L_{1} \rightarrow L_{0} \rightarrow \mathbb{Q} \rightarrow 0$ is an exact complex of $\left(G, P_{S}\right)$-representations, and that there is a natural isomorphism of chain complexes

$$
\operatorname{Hom}_{\left(G, P_{S}\right)}\left(L_{*}, V\right) \cong C^{*}\left(G, P_{S}, V\right)
$$

for every representation $V$ of $\left(G, P_{S}, V\right)$. In particular, the functor $\operatorname{Hom}_{\left(G, P_{S}\right)}\left(L_{n},-\right)$ is exact, so $L_{*}$ is a projective resolution of $\mathbb{Q}$.
2.2.5. - We keep the notation from paragraph 2.2.3, and have a closer look at the first cohomology group $H^{1}\left(G, P_{S}, V\right)$. The space of cocycles $Z^{1}\left(G, P_{S}, V\right)=\operatorname{ker}\left(d^{1}\right)$ is the space of tuples $\left(c,\left(c_{s}\right)_{s \in S}\right)$ consisting of maps $c: G \rightarrow V$ and $c_{s}: P_{s} \rightarrow V$ satisfying the cocycle relations

$$
c(g h)=c(g)+g c(h) \quad \text { and } \quad c_{s}(g p)=c(g)+g c_{s}(p)
$$

for all $g, h \in G$ and $p_{s} \in P_{s}$, and the space of coboundaries $B^{1}\left(G, P_{S}, V\right)=\operatorname{im}\left(d^{0}\right)$ is the space of those tuples of the form

$$
c(g)=v-g v \quad \text { and } \quad c_{s}(p)=v-p v_{s}
$$

for some $v \in V$ and $v_{s} \in V_{s}$. For general $\left(G, P_{S}\right)$ and $V$ nothing more can be said.
2.2.6. - A particular case is interesting to us: pick for every $s \in S$ an element $p_{s} \in P_{s}$, and suppose that $G$ acts transitively on the sets $P_{s}$, and that the stabilisers $G_{s}=\left\{g \in G \mid g p_{s}=p_{s}\right\}$ generate $G$. In that case, the whole cocycle $c$ is determined by the values $c_{s}\left(p_{s}\right)$, and in particular, $H^{1}(G, P, V)$ is finite-dimensional. Indeed, if $c$ is a cocycle satisfying $c\left(p_{s}\right)=0$ for all $s \in S$, then

$$
c\left(g p_{s}\right)=c(g)+g c\left(p_{s}\right)=c(g)
$$

for all $g \in G$. In particular we find $c(g)=0$ for all $g \in G_{s}$. Since $c: G \rightarrow V$ is an ordinary cocycle and the stabilisers $G_{s}$ generate $G$, we find $c(g)=0$ for all $g \in G$. But then, since $G$ acts transitively on $P_{s}$, we find $c(p)=0$ for all $p \in P_{s}$ as well, so $c=0$. A particular case of this is the situation where $G$ is the free group on generators $\left\{g_{s} \mid s \in S\right\}$, and $P_{s}=G /\left\langle g_{s}\right\rangle$ is the quotient of $G$ by the equivalence relation $g g_{s} \sim g$, and $p_{s}$ is the class of the unit element. In that case, the injective map

$$
\begin{equation*}
Z^{1}\left(G, P_{S}, V\right) \rightarrow \bigoplus_{s \in S} V \tag{2.2.6.1}
\end{equation*}
$$

sending $c$ to $c\left(p_{s}\right)_{s \in S}$ is also surjective, and the complex $C^{0}\left(G, P_{S}, V\right) \rightarrow Z^{1}\left(G, P_{S}, V\right)$ takes the following shape:

$$
\begin{align*}
V \oplus \bigoplus_{s \in S} V_{s} & \xrightarrow{d} \bigoplus_{s \in S} V  \tag{2.2.6.2}\\
v,\left(v_{s}\right)_{s \in S} & \longmapsto\left(v-p_{s} v_{s}\right)_{s \in S}
\end{align*}
$$

This is of course precisely the situation at which we arrived in 2.2.1, where $G$ was the fundamental group of $X=\mathbb{C} \backslash S$ based at $x \in X$, and $P_{s}$ the $G$-set of homotopy classes of paths from $s \in S$ to $X$. The complex (2.2.6.2) computes thus the cohomology $H^{*}\left(\mathbb{A}^{1}, F\right)$, where $F$ is the constructible sheaf corresponding to the representation $V$.
2.2.7. - Let us now come back to the geometric situation described in 2.2.1, where $G$ is the fundamental group of the complement of a finite set $S \subseteq \mathbb{C}$, and $P$ the $G$-set of paths from $S$ to the base point $x \in \mathbb{C} \backslash S$. We can use the cochain complex (2.2.3.1), or the more economic variant (2.2.6.2) to compute the cohomology of constructible sheaves with singularities in $S$. Let us now explain how this computation works in families.

We consider the following setup: Let $\left(\gamma_{j}\right)_{j \in J}$ be a finite collection of paths $\gamma_{j}:[0,1] \rightarrow \mathbb{C}$ which are disjoint at all times $t \in[0,1]$, and never meet the base point $x \in \mathbb{C}$. We also denote by $\gamma_{j}$ the set $\left\{\left(\gamma_{j}(t), t\right) \mid t \in[0,1]\right\}$. The sets $\gamma_{j}$ are the strands of a braid in $\mathbb{C} \times[0,1]$ from initial points $S^{(0)}=\left\{\gamma_{j}(0) \mid j \in J\right\}$ to endpoints $S^{(1)}=\left\{\gamma_{j}(1) \mid j \in J\right\}$, as illustrated in Figure 2.2.2.

For each $t \in[0,1]$, let $G^{(t)}$ be the fundamental group of $\mathbb{C} \backslash S^{(t)}$ based at $x$. For each $j \in J$, the group $G^{(t)}$ acts on the set $P_{j}^{(t)}$ of paths from $\gamma_{j}(t)$ to $x$. By continuously deforming loops and


Figure 2.2.2. A general braid
paths as $t$ moves from 0 to 1 , we obtain a group isomorphism and bijections

$$
\beta: G^{(0)} \rightarrow G^{(1)} \quad \text { and } \quad \beta: P_{j}^{(0)} \rightarrow P_{j}^{(1)}
$$

compatible with group actions.
Let $F$ be a sheaf on $\mathbb{C} \times[0,1]$ which is constructible with respect to the stratification given by the strands $\gamma_{j}$, so $F$ is locally constant on each strand, and on the complement of the strands. For each time $t \in[0,1]$, the restriction

$$
\begin{aligned}
F^{(t)}=\iota_{t}^{*} F \quad \quad \iota_{t}: \mathbb{C} & \rightarrow \mathbb{C} \times[0,1] \\
z & \longmapsto(z, t)
\end{aligned}
$$

is a constructible sheaf with singularities contained in $S^{(t)}=\left\{\gamma_{j}(t) \mid j \in J\right\}$. We write $V^{(t)}$ for the fibre of $F^{(t)}$ at $x$, and $V_{s}^{(t)}$ for the fibre of $F^{(t)}$ at the singular point $s=\gamma_{j}(t) \in S^{(t)}$. By constructibility of $F$, there are given parallel transport isomorphisms

$$
\tau: V^{(0)} \rightarrow V^{(1)} \quad \text { and } \quad \tau: V_{\gamma_{j}(0)}^{(0)} \rightarrow V_{\gamma_{j}(1)}^{(1)}
$$

for all $j \in J$.
Let $\pi: \mathbb{C} \times[0,1] \rightarrow[0,1]$ be the projection. The sheaf $R^{n} \pi_{*} F$ on $[0,1]$ is a local system on $[0,1]$, whose fibre at $t \in[0,1]$ is

$$
\begin{equation*}
\left(R^{n} \pi_{*} F\right)_{t} \cong H^{n}\left(\mathbb{C}, F^{(t)}\right) \cong H^{n}\left(\left(G^{(t)}, P^{(t)}\right), V^{(t)}\right) \tag{2.2.7.1}
\end{equation*}
$$

by smooth base change. The following lemma expresses parallel transport for the locally constant sheaf $R^{n} \pi_{*} F$ in terms of cocycles.

Lemma 2.2.8. - The parallel transport morphism $\left(R^{n} \pi_{*} F\right)_{0} \rightarrow\left(R^{n} \pi_{*} F\right)_{1}$ is, via the isomorphism (2.2.7.1), induced by the morphism of chain complexes

$$
\begin{gathered}
C^{0}\left(\left(G^{(0)}, P^{(0)}\right), V^{(0)}\right) \stackrel{d}{\longrightarrow} C^{1}\left(\left(G^{(0)}, P^{(0)}\right), V^{(0)}\right) \longrightarrow \cdots \\
\quad \varphi^{0} \downarrow \\
C^{0}\left(\left(G^{(1)}, P^{(1)}\right), V^{(1)}\right) \stackrel{d}{\longrightarrow} C^{1}\left(\left(G^{(1)}, P^{(1)}\right), V^{(1)}\right) \longrightarrow
\end{gathered}
$$

given in degree zero by parallel transport morphisms

$$
\varphi^{0}: V^{(0)} \oplus \bigoplus_{j \in J} V_{j}^{(0)} \rightarrow V^{(1)} \oplus \bigoplus_{j \in J} V_{j}^{(1)} \quad \varphi^{0}\left(v,\left(v_{s}\right)_{s \in S^{(0)}}\right)=\left(\tau v, \tau v_{s}\right)_{s \in S^{(1)}}
$$

and in degrees $n>0$ by parallel transport and deformation of paths

$$
\begin{aligned}
\varphi^{n}(c)\left(g_{1}, \ldots, g_{n}\right) & =\tau c\left(\beta^{-1} g_{1}, \ldots, \beta^{-1} g_{n}\right) \\
\varphi^{n}(c)\left(g_{1}, \ldots, g_{n-1}, p\right) & =\tau c\left(\beta^{-1} g_{1}, \ldots, \beta^{-1} g_{n-1}, \beta_{j}^{-1} p\right)
\end{aligned}
$$

for all $c \in C^{n}\left(\left(G^{(0)}, P^{(0)}\right), V^{(0)}\right)$, and all $g_{1}, \ldots, g_{n} \in G^{(1)}$ and $p \in P_{j}^{(1)}$.
Proof. The parallel transport morphism $\left(R^{n} \pi_{*} F\right)_{0} \rightarrow\left(R^{n} \pi_{*} F\right)_{1}$ is obtained in sheaf cohomology terms as the composite of base change morphisms

$$
\begin{equation*}
H^{n}\left(\mathbb{C}, F^{(0)}\right) \stackrel{\cong}{\longleftarrow} H^{n}(\mathbb{C} \times[0,1], F) \xrightarrow{\cong} H^{n}\left(\mathbb{C}, F^{(1)}\right) \tag{2.2.8.1}
\end{equation*}
$$

induced by inclusions $\iota_{t}: \mathbb{C} \rightarrow \mathbb{C} \times[0,1]$ for $t=0,1$. Let us denote by $G$ be the fundamental group of the complement of the strands $\gamma_{j}$ in $\mathbb{C} \times[0,1]$, with base point the contractible set $\{x\} \times[0,1]$, so elements of $G$ are paths starting and ending in $\{x\} \times[0,1]$ and not meeting the strands $\gamma_{j}$, modulo appropriate homotopies. For $j \in J$, let $P_{j}$ be the $G$-set of homotopy classes of paths from $\gamma_{j}$ to $\{x\} \times[0,1]$, not meeting strands $\gamma_{i}$ for $i \neq j$, modulo appropriate homotopies. The inclusion $\iota_{t}: \mathbb{C} \rightarrow \mathbb{C} \times[0,1]$ induces a group isomorphism and bijections

$$
G^{(t)} \rightarrow G \quad \text { and } \quad P_{j}^{(t)} \rightarrow P_{j}
$$

compatible with group actions. The category of sheaves on $\mathbb{C} \times[0,1]$ which are constructible with respect to the stratification given by the strands $\gamma_{j}$ is equivalent to the category of $(G, P)$ representations, via the equivalence of categories sending the sheaf $F$ to the ( $G, P$ ) representation on the vector spaces

$$
V=H^{0}(x \times[0,1], F) \quad \text { and } \quad V_{j}=H^{0}\left(\gamma_{j}, F\right)
$$

Moreover, the this equivalence is compatible with the equivalence between $\left(G^{(t)}, P^{(t)}\right)$ representations and constructible sheaves on $\mathbb{C}$ with singularities in $S^{(t)}$. In particular, the chain complex $C^{*}((G, P), V)$ computes the cohomology $H^{*}(\mathbb{C} \times[0,1], F)$, and the specialisation maps

$$
H^{n}(\mathbb{C} \times[0,1], F) \rightarrow H^{n}(\mathbb{C} \times[0,1], F)_{t} \cong H^{n}\left(\mathbb{C}, F^{(t)}\right)
$$

are obtained from the morphism of chain complexes

$$
\begin{gathered}
C^{0}((G, P), V) \xrightarrow{d} C^{1}((G, P), V) \longrightarrow \cdots \\
\quad \varphi_{t}^{0} \downarrow \\
C^{0}\left(\left(G^{(t)}, P^{(t)}\right), V^{(t)}\right) \stackrel{d}{\longrightarrow} C^{1}\left(\left(G^{(t)}, P^{(t)}\right), V^{(t)}\right) \longrightarrow
\end{gathered}
$$

given by specialisation maps $V \rightarrow V^{(t)}$ and $V_{j} \rightarrow V_{j}^{(t)}$ in degree $n=0$, and by composition with isomorphism $G^{(t)} \rightarrow G$ and $P^{(t)} \rightarrow P$ and specialisation $V \rightarrow V^{(t)}$ in degree $n \geqslant 1$. Together with the observation the group isomorphism $\beta$ and the bijections $\beta_{j}$ all are composites

$$
\beta: G^{(0)} \xrightarrow{\cong} G \cong G^{(1)} \quad \text { and } \quad \beta_{j}: P_{j}^{(0)} \xrightarrow{\cong} P_{j} \cong P_{j}^{(1)}
$$

and that the parallel transport maps $\tau$ and $\tau_{j}$ are composites
this proves the lemma.
2.2.9. - We shall also need to understand how to express cospecialisation in terms of cocycle cohomology. We consider the same setup as in 2.2.7, except that now we allow the strands $\gamma_{j}$ to meet at time $t=0$, and allow the sheaf $F$ to be constructible with respect to the finer stratification given by the strands $\gamma_{j}$, their startpoints $\gamma_{j}(0)$ and the $t=0$ plane. The sheaf $R^{n} \pi_{*} F$ is then a constructible sheaf on $[0,1]$, with respect to the stratification $\{0\} \subseteq[0,1]$. Instead of parallel transport, we now have a cospecialisation map

$$
H^{n}\left(\mathbb{C}, F^{(0)}\right) \cong H^{n}(\mathbb{C} \times[0,1], F)_{0} \rightarrow H^{n}(\mathbb{C} \times[0,1], F)_{1} \cong H^{n}\left(\mathbb{C}, F^{(1)}\right)
$$

which we want to understand in terms of cocycles. The recipe is very similar to the one given by Lemma 2.2.8. By continuous deformation of paths and loops, we obtain a group homomorphism and maps

$$
\beta: G^{(0)} \rightarrow G^{(1)} \quad \text { and } \quad \beta_{j}: P_{j}^{(0)} \rightarrow P_{j}^{(1)}
$$

compatible with group actions. From the sheaf $F$ we obtain vector spaces $V^{(t)}$ and $V_{s}^{(t)}$ for $s \in S^{(t)}$ which constitute a representation of $\left(G^{(t)}, P^{(t)}\right)$. The cospecialisation map translates to a morphism in the derived category of vector spaces, given explicitly by the roof between chain complexes

which we shall describe presently. Elements of $C_{\text {tot }}^{n}$ are pairs $\left(c^{0}, c^{1}\right)$ consisting of a cocycle $c^{0} \in$ $C^{n}\left(\left(G^{(0)}, P^{(0)}\right), V^{(0)}\right)$ and a cocycle $c^{1} \in C^{n}\left(\left(G^{(0)}, P^{(0)}\right), V^{(0)}\right)$ such that the equalities

$$
\begin{aligned}
c^{1}\left(\beta g_{1}, \ldots, \beta g_{n}\right) & =\tau c^{0}\left(g_{1}, \ldots, g_{n}\right) \\
c^{1}\left(\beta g_{1}, \ldots, \beta g_{n-1}, \beta p\right) & =\tau c^{0}\left(g_{1}, \ldots, g_{n-1}, p\right)
\end{aligned}
$$

hold for all $g_{1}, \ldots, g_{n} \in G^{(0)}$ and $p \in P_{j}^{(0)}$. The differential in $C_{\text {tot }}^{n}$ is given by $d\left(c^{0}, c^{1}\right)=\left(d c^{0}, d c^{1}\right)$, and the morphisms in (2.2.9.1) are given by projections $\left(c^{0}, c^{1}\right) \longmapsto c^{0}$ and $\left(c^{0}, c^{1}\right) \longmapsto c^{1}$. One can check that the leftward morphism in (2.2.9.1) is indeed a quasi-isomorphism. The complex $C_{\text {tot }}^{n}$ computes the cohomology $H^{*}(\mathbb{C} \times[0,1], F)$, and the morphism

$$
H^{n}\left(\left(G^{(0)}, P^{(0)}\right), V^{(0)}\right) \rightarrow H^{n}\left(\left(G^{(1)}, P^{(1)}\right), V^{(1)}\right)
$$

induced by the roof (2.2.9.1) identifies with the cospecialisation map.
2.2.10. - For concrete calculations, it is useful to have a standard ordered set of generators of the free group $G=\pi_{1}(\mathbb{C} \backslash S, x)$ at our disposal. Here it is. Let $S \subseteq \mathbb{C}$ be finite, and let $x \in \mathbb{R}$ be a real number, larger than the real part of any $s \in S$, serving as a base point. Let us enumerate the set $S$ in western reading order (left-to-right, top-to-bottom), so that we have $S=\left\{s_{1}, s_{2}, \ldots, s_{n}\right\}$ with

$$
\operatorname{Im}\left(s_{i}\right) \geqslant \operatorname{Im}\left(s_{i+1}\right) \quad \text { and } \quad \operatorname{Im}\left(s_{i}\right)=\operatorname{Im}\left(s_{i+1}\right) \Longrightarrow \operatorname{Re}\left(s_{i}\right)<\operatorname{Re}\left(s_{i+1}\right)
$$

We declare standard paths $p_{i}$ from $s_{i} \in S$ to $x$ to be paths in $\overline{\mathbb{C}} \backslash S$ such that

$$
\operatorname{Re}\left(p_{i}\left(t_{1}\right)\right)=\operatorname{Re}\left(p_{j}\left(t_{2}\right)\right) \Longrightarrow \operatorname{Im}\left(p_{i}\left(t_{1}\right)\right)>\operatorname{Im}\left(p_{j}\left(t_{2}\right)\right)
$$

holds for all $1 \leqslant i<j \leqslant n$ and $t_{1}, t_{2} \in[0,1)$. In other words, for $i<j$ it is required that the path $p_{i}$ lies strictly above the path $p_{j}$ in the complex plane. Up to homotopy, the paths $p_{i}$ are uniquely determined by this requirement. We declare standard loops around $s_{i}$ to be the loops composed in the accustomed way by the paths $p_{i}$ and their inverses, and a small, positively oriented simple loop around $s_{i}$.


Figure 2.2.3. Standard paths and and standard loops

With this convention for standard paths, a constructible sheaf on $\mathbb{A}^{1}$ can be described by the following data:
(1) A finite set $S=\left\{s_{1}, \ldots, s_{n}\right\} \subseteq \mathbb{C}$, ordered in western reading order.
(2) Vector spaces $V$ and $V_{1}, V_{2}, \ldots, V_{n}$
(3) Automorphisms $g_{i} \in \mathrm{GL}(V)$ and homomorphisms $p_{i}: V_{i} \rightarrow V$ satisfying $g_{i} p_{i}=p_{i}$ for all $i=1,2, \ldots, n$.

Given the ordered set $S$, this data can be described effectively by a finite list of matrices with rational coefficients. If for some $i$ the map $p_{i}: V_{i} \rightarrow V$ is an isomorphism, then $g_{i}$ is the identity, and $s_{i} \in \mathbb{C}$ is not a singular point of the constructible sheaf. In that case, deleting $s_{i}$ from $S$ leads to a shorter description of the same sheaf. On the other hand, we may add an additional point $s_{i}$ to $S$, and set $V_{i}=V$ and $p_{i}=g_{i}=\mathrm{id}$.
2.2.11. - We are particularly interested in constructible sheaves $F$ on $\mathbb{C}$ satisfying $H^{*}(\mathbb{A}, F)=0$, that is, $R \pi_{*} F=0$ for the map $\pi$ from $\mathbb{C}$ to a point. Let again $S \subseteq \mathbb{C}$ be a finite set containing the singularities of $F$, and regard $F$ as a representation $V$ of $\left(G, P_{S}\right)$ as in 2.2.1. The cohomology
$H^{n}\left(\mathbb{A}^{1}, F\right) \cong H^{n}\left(G, P_{S}, V\right)$ is zero for $n \geqslant 2$. Therefore, $R \pi_{*} F=0$ holds if and only if the differential $d: C^{0}\left(G, P_{S}, V\right) \rightarrow Z^{1}\left(G, P_{S}, V\right)$ is an isomorphism. Explicitly, this means that for all $s \in S$ the $\operatorname{map} p_{s}: V_{s} \rightarrow V$ is injective, and that

$$
\bigcap_{s \in S} p_{s} V_{s}=\{0\} \quad \text { and } \quad \sum_{s \in S} \operatorname{dim}\left(V / p_{s} V_{s}\right)=\operatorname{dim}(V)
$$

holds. It follows from this description that, given constructible sheaves $F_{1}$ and $F_{2}$ on $\mathbb{C}$ such that $R \pi_{*} F_{1}=R \pi_{*} F_{2}=0$, a morphism $\varphi: F_{1} \rightarrow F_{2}$ which induces an isomorphism between the fibres over $x$ is an isomorphism. More generally, the functor

$$
\left\{\begin{array}{l}
\text { Constructible sheaves } F  \tag{2.2.11.1}\\
\text { on } \mathbb{C} \text { with singularities } \\
\text { in } S \text { and } R \pi_{*} F=0
\end{array}\right\} \rightarrow \operatorname{Vec}_{\mathbb{Q}}
$$

sending $F$ to its fibre $V=F_{x}$ is exact and faithful.

Lemma 2.2.12. - Let $F$ and $G$ be constructible sheaves on $\mathbb{C}$. Suppose that $F$ has no non-zero global sections, and that $G$ has no non-zero global sections with finite support. Then $F \otimes G$ has no non-zero global sections.

Proof. Choose a sufficiently large finite set $S \subseteq \mathbb{C}$ containing the singularities of both $F$ and $G$. In the notation of 2.2.1, the sheaves $F$ and $G$ correspond to representations $V$ and $W$ of $\left(G, P_{S}\right)$. Fix elements $p_{s} \in P_{s}$, that is, paths from $s \in S$ to the base point $x$ avoiding $S$ along the way. We get complexes

$$
V \oplus \bigoplus_{s \in S} V_{s} \xrightarrow{d_{V}} \bigoplus_{s \in S} V \quad \text { and } \quad W \oplus \bigoplus_{s \in S} W_{s} \xrightarrow{d_{W}} \bigoplus_{s \in S} W
$$

with $d_{V}\left(v,\left(v_{s}\right)_{s \in S}\right)=\left(v-p_{s} v_{s}\right)_{s \in S}$ and $d_{W}\left(w,\left(w_{s}\right)_{s \in S}\right)=\left(w-p_{s} w_{s}\right)_{s \in S}$. The representation of $\left(G, P_{S}\right)$ given by the vector spaces $V \otimes W$ and $\left(V_{s} \otimes W_{s}\right)_{s \in S} S$ with the diagonal actions

$$
g(v \otimes w)=g v \otimes g w \quad \text { and } \quad p_{s}\left(v_{s} \otimes w_{s}\right)=p_{s} v_{s} \otimes p_{s} w_{s}
$$

corresponds to the sheaf $F \otimes G$. That $F$ and $G$ have no non-zero sections with finite support means that the maps $p_{s}: V_{s} \rightarrow V$ and $p_{s}: W_{s} \rightarrow W$ are injective, and that $F$ has no non-zero sections means that moreover the intersection of the $p_{s} V_{s}$ in $V$ is zero. It follows that $p_{s}: V_{s} \otimes W_{s} \rightarrow V \otimes W$ is injective for every $s \in S$, and hence $F \otimes G$ has no non-zero sections with finite support. We also have

$$
\bigcap_{s \in S} p_{s}\left(V_{s} \otimes W_{s}\right) \subseteq \bigcap_{s \in S}\left(p_{s} V_{s} \otimes W\right)=\left(\bigcap_{s \in S} p_{s} V_{s}\right) \otimes W=\{0\} \otimes W=\{0\}
$$

so $F \otimes G$ has no non-zero global sections at all.

Lemma 2.2.13. - Let $F$ and $G$ be constructible sheaves on $\mathbb{C}$ with disjoint sets of singularities. The Euler characteristics of $F, G$, and $F \otimes G$ are related by

$$
\chi(F \otimes G)+\operatorname{rk}(F \otimes G)=\operatorname{rk}(G) \chi(F)+\operatorname{rk}(F) \chi(G)
$$

where $\operatorname{rk}(F)$ and $\operatorname{rk}(G)$ are the dimensions of the local systems underlying $F$ and $G$.

Proof. Let $S$ and $T$ be the sets of singularities of $F$ and $G$ respectively. Fix a base point $x \in \mathbb{C} \backslash(S \cup T)$ and choose a path from each element of $S \cup T$ to $x$. The cohomology of $F$ and $G$ is then computed by the complexes

$$
V \oplus \bigoplus_{s \in S} V_{s} \xrightarrow{d_{V}} \bigoplus_{s \in S} V \quad \text { and } \quad W \oplus \bigoplus_{t \in T} W_{s} \xrightarrow{d_{W}} \bigoplus_{t \in T} W
$$

and the Euler characteristics of $F$ and $G$ are the Euler characteristics of these complexes. Explicitly, these are

$$
\chi(F)=(1-\# S) n+\sum_{s \in S} n_{s} \quad \text { and } \quad \chi(G)=(1-\# T) m+\sum_{t \in T} m_{t}
$$

where we set $n=\operatorname{dim} V=\operatorname{rk}(F)$ and $n_{s}=\operatorname{dim} V_{s}$, and similarly $m=\operatorname{dim} W=\operatorname{rk}(G)$ and $m_{t}=\operatorname{dim} W_{t}$. The constructible sheaf $F \otimes G$ has singularities in $S \cup T$, and its cohomology is computed by the complex

$$
(V \otimes W) \oplus \bigoplus_{s \in S}\left(V_{s} \otimes W\right) \oplus \bigoplus_{t \in T}\left(V \otimes W_{t}\right) \xrightarrow{d_{V \otimes W}} \bigoplus_{s \in S}(V \otimes W) \oplus \bigoplus_{t \in T}(V \otimes W),
$$

whose Euler characteristic is that of $F \otimes G$. An elementary computation shows the equality

$$
\begin{aligned}
\chi(F \otimes G) & =(-\# S-\# T+1) n m+m \sum_{s \in S} n_{s}+n \sum_{t \in T} m_{t} \\
& =m \chi(F)+n \chi(G)-n m
\end{aligned}
$$

which is what we wanted to prove.

### 2.3. The category $\operatorname{Perv}_{0}$

In this section, we introduce the category $\operatorname{Perv}_{0}$ and derive some of its basic properties. Throughout, we let $\pi: \mathbb{A}_{\mathbb{C}}^{1} \rightarrow \operatorname{Spec}(\mathbb{C})$ denote the structure morphism and

$$
\operatorname{Perv}=\operatorname{Perv}\left(\mathbb{A}^{1}(\mathbb{C}), \mathbb{Q}\right)
$$

the abelian category of perverse sheaves with rational coefficients on the complex affine line. Recall that its objects are bounded complexes $C$ of sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{A}^{1}(\mathbb{C})$ with constructible cohomology and satisfying the conditions from 2.1.18.

Definition 2.3.1. - The category Perv $_{0}$ is the full subcategory of Perv consisting of those objects $A$ with no global cohomology, that is, $R \pi_{*} A=H^{*}\left(\mathbb{A}^{1}(\mathbb{C}), A\right)=0$.
2.3.2. - Here are some premonitions of what is to become of the category Perv ${ }_{0}$. As we shall show in Proposition 2.3.7, it is an abelian category. It will turn out in Proposition 2.4.3 that the inclusion $\operatorname{Perv}_{0} \rightarrow \mathbf{P e r v}$ has a left adjoint $\Pi$ : Perv $\rightarrow \mathbf{P e r v}_{0}$. Once we understand the basic
structure of objects of $\operatorname{Perv}_{0}$, we will be able to define functors nearby fibre at infinity and total vanishing cycles

$$
\Psi_{\infty}: \operatorname{Perv}_{0} \rightarrow \mathbf{V e c}_{\mathbb{Q}} \quad \text { and } \quad \Phi: \operatorname{Perv}_{0} \rightarrow \mathbf{V e c}_{\mathbb{Q}}
$$

which are exact and faithful. As a consequence, $\operatorname{Perv}_{0}$ is artinian and noetherian, and we can associate a dimension with every object $A$ of $\operatorname{Perv}_{0}$ by declaring that it is the dimension of the vector space $\Psi_{\infty}(A)$. In Section 2.4 we will introduce a tannakian structure on $\operatorname{Perv}_{0}$, for which we will verify later in Section 2.8 that $\Psi_{\infty}$ as well as $\Phi$ are fibre functors. In Section 3.2 we will relate objects of $\operatorname{Perv}_{0}$ with rapid decay cohomology (1.1.1.2) by establishing a canonical and natural isomorphism

$$
H_{\mathrm{rd}}^{n}(X, f) \cong \Psi_{\infty}\left(\Pi\left({ }^{p} R^{n} f_{*} \underline{\mathbb{Q}}_{X}\right)\right)
$$

where $\underline{\mathbb{Q}}_{X}$ is the constant sheaf with value $\mathbb{Q}$ on $X$. This isomorphism can be seen as an enrichment of the vector space $H_{\mathrm{rd}}^{n}(X, f)$ with an additional structure, namely that of an object of $\mathbf{P e r v}_{0}$.

Lemma 2.3.3 ([57], proof of Theorem 2.29). - An object $C$ of the derived category of constructible sheaves on $\mathbb{A}^{1}(\mathbb{C})$ belongs to $\mathbf{P e r v}_{0}$ if and only if it is of the form $C=F[1]$ for some constructible sheaf $F$ satisfying $R \pi_{*} F=0$.

Proof. If $F$ is a constructible sheaf on $\mathbb{A}^{1}$, then $\mathcal{H}^{n}(F[1])=0$ for $n \neq-1$ and $\mathcal{H}^{-1}(F[1])=F$, so to ensure that $F[1]$ is perverse one only needs to check that the condition $R \pi_{*} F=0$ implies that $F$ has no non-zero global sections with finite support. This is clear since $F$ has no non-zero global sections at all. Conversely, let $C$ be a perverse sheaf on $\mathbb{A}^{1}$. Invoking the exact triangle

$$
\mathcal{H}^{-1}(C)[1] \rightarrow C \rightarrow \mathcal{H}^{0}(C)[0]
$$

it suffices to prove that both $\mathcal{H}^{0}(C)$ and $R \pi_{*} \mathcal{H}^{-1}(C)$ vanish under the assumption $R \pi_{*} C=0$. This will follow from the spectral sequence

$$
E_{2}^{p, q}=H^{p}\left(\mathbb{A}^{1}, \mathcal{H}^{q}(C)\right) \Longrightarrow H^{p+q}\left(\mathbb{A}^{1}, C\right)
$$

Combining the facts that $\mathcal{H}^{n}(C)=0$ for $n \notin\{-1,0\}$ and $\mathcal{H}^{0}(C)$ is a skyscraper sheaf with Artin's vanishing theorem 2.1.10, the spectral sequence degenerates at $E_{2}$ and we have:

$$
\begin{aligned}
H^{-1}\left(\mathbb{A}^{1}, C\right) & =H^{0}\left(\mathbb{A}^{1}, \mathcal{H}^{-1}(C)\right) \\
H^{0}\left(\mathbb{A}^{1}, C\right) & =H^{1}\left(\mathbb{A}^{1}, \mathcal{H}^{-1}(C)\right) \oplus H^{0}\left(\mathbb{A}^{1}, \mathcal{H}^{0}(C)\right) .
\end{aligned}
$$

Therefore, the condition $R \pi_{*} C=0$ implies $H^{0}\left(\mathbb{A}^{1}, \mathcal{H}^{0}(C)\right)=0$ and $R \pi_{*} \mathcal{H}^{-1}(C)=0$. Since $\mathcal{H}^{0}(C)$ is a skyscraper sheaf, we necessarily have $\mathcal{H}^{0}(C)=0$.

Example 2.3.4. - Let $s \in \mathbb{C}$ be a point, and denote by $j(s): \mathbb{C} \backslash\{s\} \hookrightarrow \mathbb{C}$ the inclusion. The constructible sheaf $j(s)!j(s)^{*} \mathbb{Q}$ has trivial cohomology. Therefore,

$$
\begin{equation*}
E(s)=j(s)!j(s)^{*} \underline{\mathbb{Q}}[1] \tag{2.3.4.1}
\end{equation*}
$$

defines an object of the category $\operatorname{Perv}_{0}$. More generally, for every local system $L$ on $\mathbb{C} \backslash\{s\}$, the object $j(s)$ ! $L[1]$ belongs to $\mathbf{P e r v}_{0}$. Conversely, if a constructible sheaf $F$ has trivial cohomology
and only one singular fibre, located at the point $s \in \mathbb{C}$, then $F$ is of the form $j(s)!L$ for the local system $L=j(s)^{*} F$ on $\mathbb{C} \backslash\{s\}$.

Definition 2.3.5. - We call nearby fibre at infinity the functor

$$
\begin{aligned}
\Psi_{\infty}: \operatorname{Perv}_{0} & \longrightarrow \operatorname{Vec}_{\mathbb{Q}} \\
F[1] & \longmapsto \operatorname{colim}_{r \rightarrow+\infty} F\left(S_{r}\right)
\end{aligned}
$$

defined in the evident way on morphisms. Recall that $S_{r}$ is the closed half-plane $\{z \in \mathbb{C} \mid \operatorname{Re}(z) \geqslant r\}$.

Remark 2.3.6. - The nearby fibre at infinity functor is related to the nearby cycles functor from 2.1.19 as follows: writing $j: \mathbb{G}_{m} \rightarrow \mathbb{A}^{1}$ for the inclusion and $i: \mathbb{G}_{m} \rightarrow \mathbb{G}_{m}$ for the inversion $i(z)=z^{-1}$, there is a natural isomorphism

$$
\Psi_{\infty}(F[1])=\Psi_{0}\left(j!i^{*} j^{*} F[1]\right),
$$

where nearby cycles at $z=0$ on the right-hand side are computed as in 2.1.21, using the filter of contractible sets $i\left(S_{r}\right)_{r>0}$.

Proposition 2.3.7. - The category $\operatorname{Perv}_{0}$ is a $\mathbb{Q}$-linear abelian category and the functor nearby fibre at infinity $\Psi_{\infty}: \operatorname{Perv}_{0} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ is faithful and exact.

Proof. The category $\operatorname{Perv}_{0}$ is a full additive subcategory of the abelian $\mathbb{Q}$-linear category of rational perverse sheaves on $\mathbb{A}^{1}(\mathbb{C})$, so $\operatorname{Perv}_{0}$ is itself a $\mathbb{Q}$-linear category. If $f: F \rightarrow G$ is a morphism between constructible sheaves on $\mathbb{C}$ satisfying $R \pi_{*} F=R \pi_{*} G=0$, then one has $R \pi_{*}(\operatorname{ker} f)=0$ and $R \pi_{*}(\operatorname{coker} f)=0$, as one can read off the long exact sequences associated with the exact triangles

$$
[0 \rightarrow G] \rightarrow[F \rightarrow G] \rightarrow[F \rightarrow 0] \quad \text { and } \quad[\operatorname{ker} f \rightarrow 0] \rightarrow[F \rightarrow G] \rightarrow[0 \rightarrow \operatorname{coker} f],
$$

noting that $[0 \rightarrow$ coker $f]$ is quasi-isomorphic to $[F / \operatorname{ker} f \rightarrow G]$. Thus, kernels and cokernels of a morphism in $\mathbf{P e r v}_{0}$ are its kernel and cokernel in Perv, and if any two objects in an exact sequence in Perv belong to Perv 0 , then so does the third, again because $R \pi_{*}$ is a triangulated functor.

The functor $\Psi_{\infty}$ is exact: indeed, pick any exact sequence $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ of constructible sheaves on $\mathbb{A}^{1}(\mathbb{C})$. For every sufficiently big $r$, the restrictions of these sheaves to $S_{r}$ are local systems, and hence constant sheaves since $S_{r}$ is simply connected. Thus, for every sufficiently $\operatorname{big} r$, the sequence $0 \rightarrow F\left(S_{r}\right) \rightarrow G\left(S_{r}\right) \rightarrow H\left(S_{r}\right) \rightarrow 0$ is exact. Finally, we prove that $\Psi_{\infty}$ is faithful. Let $f: F \rightarrow G$ be a morphism of constructible sheaves with vanishing global cohomology such that the induced map $F\left(S_{r}\right) \rightarrow G\left(S_{r}\right)$ is the zero map for some $r$. We need to show that $f=0$, that is, $f_{z}: F_{z} \rightarrow G_{z}$ for any $z \in \mathbb{C}$. The choice of a path starting at $z$, ending in $S_{r}$ and avoiding the singularities of $F$ and $G$ induces functorial cospecialisation maps $F_{z} \rightarrow F\left(S_{r}\right)$ and $G_{z} \rightarrow G\left(S_{r}\right)$. By 2.2.11, these maps are injective by the assumption on the vanishing of cohomology, and therefore $f_{z}=0$.
2.3.8. - Let $A=F[1]$ be an object of $\mathbf{P e r v}_{0}$. We can describe the constructible sheaf $F$ on $\mathbb{A}^{1}$ in terms of representations as follows. The sheaf $F$ corresponds to data
(1) A finite set $S=\left\{s_{1}, \ldots, s_{n}\right\}$ of complex numbers.
(2) A vector space $V$ and subspaces $V_{1}, V_{2}, \ldots, V_{n}$ of $V$, such that the linear map

$$
V \oplus \bigoplus_{i=1}^{n} V_{i} \xrightarrow{d} \bigoplus_{i=1}^{n} V
$$

given by $d\left(v,\left(v_{i}\right)_{i=1}^{n}\right)=\left(v-v_{i}\right)_{i=1}^{n}$ is an isomorphism.
(3) Automorphisms $g_{1}, \ldots, g_{n}$ of $V$ satisfying $V_{i} \subseteq V^{\left\langle g_{i}\right\rangle}$.

Here, we think of $V$ as the fibre of $F$ near infinity and of $V_{i}$ as the fibre of $F$ at the point $s_{i}$. The inclusion $V_{i} \rightarrow V$ is the cospecialisation map along the standard path $p_{i}$ from $s_{i}$ to $+\infty$, and the automorphisms $g_{i}$ are the corresponding monodromy operators.

Lemma 2.3.9. - Let $V$ be a finite-dimensional vector space, and let $V_{1}, \ldots, V_{n}$ be subspaces of $V$ such that the linear map

$$
\begin{equation*}
V \oplus \bigoplus_{i=1}^{n} V_{i} \rightarrow \bigoplus_{i=1}^{n} V \quad\left(v,\left(v_{i}\right)_{i=1}^{n}\right) \longmapsto\left(v-v_{i}\right)_{i=1}^{n} \tag{2.3.9.1}
\end{equation*}
$$

is an isomorphism. For $i \in\{1,2, \ldots, n\}$, define $V_{i}^{\prime}=\bigcap_{j \neq i} V_{j}$. For every subset $I \subseteq\{1,2, \ldots, n\}$, the equality

$$
\begin{equation*}
\bigoplus_{i \notin I} V_{i}^{\prime}=\bigcap_{i \in I} V_{i} \tag{2.3.9.2}
\end{equation*}
$$

holds. In particular, $V$ is the direct sum of its subspaces $V_{i}^{\prime}$.
Proof. Write $d$ for the dimension of $V$, write $d_{i}$ for the dimension of $V_{i}$ and $d_{i}^{\prime}$ for the dimension of $V_{i}^{\prime}$. For every subset $I \subseteq\{1,2, \ldots, n\}$, define

$$
V_{I}=\bigcap_{i \in I} V_{i}
$$

and $d_{I}=\operatorname{dim} V_{I}$. For every $i$, the subspace of $V$ generated by the $V_{j}^{\prime}$ with $j \neq i$ is contained in $V_{i}$, hence the inclusion

$$
V_{i}^{\prime} \cap \sum_{j \neq i} V_{j}^{\prime} \subseteq V_{i}^{\prime} \cap V_{i}=\bigcap_{i=1}^{n} V_{i}=\{0\}
$$

where the last equality follows from the injectivity of the map (2.3.9.1). It follows that the canonical map

$$
\begin{equation*}
\bigoplus_{i=1}^{n} V_{i}^{\prime} \rightarrow V \tag{2.3.9.3}
\end{equation*}
$$

is injective. The codimension of an intersection of subspaces is at most the sum of the codimension of the subspaces, so the inequality

$$
d-d_{i}^{\prime} \leqslant \sum_{j \neq i}\left(d-d_{j}\right)
$$

holds. Summing over all $i$ and combining the equality $d+\sum_{i=1}^{n} d_{i}=n d$, which holds since (2.3.9.1) is an isomorphism, we find $d=\sum_{i=1}^{n} d_{i}^{\prime}$. Hence, (2.3.9.3) is indeed an isomorphism. This shows
the desired equality (2.3.9.2) in the essential case $I=\varnothing$. The general case is shown similarly by counting dimensions.
2.3.10. - Let $A=F[1]$ be an object of Perv 0 . With the help of Lemma 2.3.9, we can describe the constructible sheaf $F$ on $\mathbb{A}^{1}$ in the following equivalent way: the sheaf $F$ corresponds to data
(1) a finite set $S=\left\{s_{1}, \ldots, s_{n}\right\}$ of complex numbers,
(2) a vector space $V$ together with an endomorphism $g \in V$ and a decomposition

$$
V=\bigoplus_{i=1}^{n} V_{i}^{\prime}
$$

such that the linear maps $V_{i}^{\prime} \xrightarrow{\text { incl. }} V \xrightarrow{g} V \xrightarrow{\text { proj. }} V_{i}^{\prime}$ are invertible for all $i=1,2, \ldots, n$. This presentation of objects of $\operatorname{Perv}_{0}$ is related to the presentation given in 2.3.8 as follows: the subspaces $V_{i} \subseteq V$ are given by

$$
V_{i}=\bigoplus_{j \neq i} V_{j}^{\prime}
$$

and the monodromy operators are given by $g_{i}=g \pi_{i}^{\prime}+\pi_{i}$, where $\pi_{i}^{\prime}: V \rightarrow V$ is the projection onto the factor $V_{i}^{\prime}$, and $\pi_{i}=1-\pi_{i}^{\prime}$ is the projection onto the complement $V_{i}$ of $V_{i}^{\prime}$.

EXAMPLE 2.3.11 (The perverse realisation of a polynomial). - Let $n \geqslant 2$ be an integer and let $f \in \mathbb{C}[x]$ be a polynomial of degree $n$. We shall view $f$ as a finite morphism $f: \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$. The constructible sheaf $f_{*} \mathbb{Q}$ contains the constant sheaf $\mathbb{Q}$ as the image of the adjunction map $\mathbb{Q} \rightarrow f_{*} \mathbb{Q}=f_{*} f^{*} \mathbb{Q}$, and we are interested in the quotient $F=f_{*} \mathbb{Q} / \mathbb{Q}$. Since $f$ is a finite morphism, the higher direct images $R^{q} f_{*} \mathbb{Q}$ vanish for all $q \geqslant 1$, and it then follows from the Leray spectral sequence that the cohomology $H^{*}\left(\mathbb{A}^{1}, F\right)$ vanishes. Hence, $F[1]$ belongs to $\mathbf{P e r v}_{0}$.

Let us describe the singularities and the local system underlying $F$. First, it is a general fact that the singularities of a constructible sheaf with no punctual sections are the points at which the fibre has dimension strictly smaller than the generic rank. The singularities of $F$, which are the same as those of $f_{*} \mathbb{Q}$, are thus given by the set

$$
S=\left\{f(a) \mid a \in \mathbb{C}, f^{\prime}(a)=0\right\}=\left\{s \in \mathbb{C} \mid \# f^{-1}(s)<n\right\}
$$

of critical values of $f$. Fix a base point $x \in X=\mathbb{C} \backslash S$ with large real part. The restriction of $f$ to $f^{-1}(X)$ is a connected covering $f^{-1}(X) \rightarrow X$, corresponding to a transitive action

$$
\rho: \pi_{1}(X, x) \rightarrow \operatorname{Perm}\left(f^{-1}(x)\right)
$$

The local system underlying $f_{*} \mathbb{Q}$ corresponds to the permutation representation $\operatorname{Map}\left(f^{-1}(x), \mathbb{Q}\right)$ of $\pi_{1}(X, x)$, which contains a one-dimensional trivial subrepresentation given by constant functions. The quotient $V=\operatorname{Map}\left(f^{-1}(x), \mathbb{Q}\right) / \mathbb{Q}$ is the $(n-1)$-dimensional representation associated with the underlying local system of $F$.

For every element $s \in S$, let $g_{s}:[0,1] \rightarrow X$ be a standard loop around $s$ based at $x$, as described in 2.2.10. The cycle type of the permutation $\rho\left(g_{s}\right)$ can be read from the factorisation of
the polynomial $f(x)-s$. Indeed, if

$$
f(x)-s=\prod_{t \in f^{-1}(s)}(x-t)^{e_{t}}
$$

then the cycle type of $\rho\left(g_{s}\right)$ is the partition $n=\sum e_{t}$. The Riemann-Hurwitz Formula for the ramified covering $f: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1}$ gives the equality

$$
\begin{equation*}
-2=-2 n+\sum_{s \in S} \sum_{t \in f^{-1}(s)}\left(e_{t}-1\right)+\left(e_{\infty}-1\right)=-n-1+\sum_{s \in S}\left(n-\# f^{-1}(s)\right) \tag{2.3.11.1}
\end{equation*}
$$

where the ramification index at infinity is $e_{\infty}=n$ since $f^{-1}(\infty)=\{\infty\}$. We can express the dimensions of the special fibres of $F$ and the vanishing cycles in terms of these invariants, namely:

$$
\operatorname{dim} V_{s}=\# f^{-1}(s)-1, \quad \operatorname{dim} \Phi_{s}(F)=n-\# f^{-1}(s)
$$

Then the equality (2.3.11.1) becomes ${ }^{1}$

$$
\sum_{s \in S} \operatorname{dim} \Phi_{s}(F)=n-1
$$

which is the content of the vanishing $H^{1}\left(\mathbb{A}^{1}, F\right)=0$ according to the discussion in 2.2.11.
2.3.12 (Simple objects of $\mathbf{P e r v}_{0}$ ). - We end this section with a description of the simple objects of the category $\operatorname{Perv}_{0}$ and of certain extension groups.

Lemma 2.3.13. - Let $F[1]$ be a simple object of $\operatorname{Perv}_{0}$. Let $S \subseteq \mathbb{C}$ be the set of singular points of $F$ and denote by $j: \mathbb{C} \backslash S \rightarrow \mathbb{C}$ the inclusion. Then either $S$ consists of a single point and $F=j!j^{*} \mathbb{Q}$, or the local system $j^{*} F$ on $\mathbb{C} \backslash S$ is simple and $F=j_{*} j^{*} F$.

Proof. Suppose first that there exists some $s \in S$ such that $F$ has a non-zero section over $\mathbb{C} \backslash\{s\}$, or in other words, that there exists a non-zero morphism $j(s)^{*} \mathbb{Q} \rightarrow j(s)^{*} F$. In that case, we obtain a non-zero morphism

$$
j(s)!j(s)^{*} \mathbb{Q}[1] \rightarrow F[1]
$$

in the category $\operatorname{Perv}_{0}$, which must be an isomorphism since $F$ is simple. Let us now suppose that $H^{0}\left(\mathbb{C} \backslash\{s\}, j(s)^{*} F\right)=0$ for all $s \in S$. For every $s \in S$, the adjunction morphism $F \rightarrow j(s)_{*} j(s)^{*} F$ is injective, and in the short exact sequence

$$
0 \rightarrow F \rightarrow j(s)_{*} j(s)^{*} F \rightarrow G \rightarrow 0
$$

the sheaf $G$ is a skyscraper sheaf supported at $s$. In the associated long exact sequence

$$
0 \rightarrow H^{0}(\mathbb{C}, F) \rightarrow H^{0}\left(\mathbb{C}, j(s)_{*} j(s)^{*} F\right) \rightarrow H^{0}(\mathbb{C}, G) \rightarrow H^{1}(\mathbb{C}, F) \rightarrow \cdots
$$

the map $H^{0}\left(\mathbb{C}, j^{*}(s) j(s)_{*} F\right) \rightarrow H^{0}(\mathbb{C}, G)$ is an isomorphism and all other terms vanish, because $F[1]$ belongs to $\mathbf{P e r v}_{0}$. Since $j(s)^{*} F$ has no non-zero sections also $H^{0}\left(\mathbb{C}, j(s)_{*} j(s)^{*} F\right)$ and hence $H^{0}(\mathbb{C}, G)$ is zero, so $G=0$ because it is a skyscraper sheaf. It follows that the adjunction morphism $F \rightarrow j(s)_{*} j(s)^{*} F$ is an isomorphism for all $s \in S$. But then, also the adjunction morphism $F \rightarrow j_{*} j^{*} F$ is an isomorphism because locally around any $s \in S$ it is. Finally, if $j^{*} F$ was not

[^0]simple, say $j^{*} F=F_{1} \oplus F_{2}$, then we could write $F$ as $j_{*} F_{1} \oplus j_{*} F_{2}$. If a direct sum of constructible sheaves has trivial cohomology, then both summands have trivial cohomology, and therefore both $j_{*} F_{1}[1]$ and $j_{*} F_{2}[1]$ are objects of $\mathbf{P e r v}_{0}$, which conflicts our hypothesis that $F$ was simple.

Remark 2.3.14. - For every finite set $S \subseteq \mathbb{C}$ containing at least two elements, there exist local systems on $\mathbb{C} \backslash S$ which do not come from objects in Pervo. For example, let $L$ be a local system of rank $r>0$ on $\mathbb{C} \backslash S$ with the property that, for each $s \in S$, the local monodromy operator around $s$, acting on the fibre of $L$ near $s$, has no non-zero fixed points. Then $j_{*} L=j_{!} L$ has non-trivial cohomology; in fact, $H^{1}(\mathbb{C}, j!L)$ is a vector space of dimension $(\# S-1) r$.

Example 2.3.15. - Let us go back to the perverse sheaf associated with a polynomial $f \in \mathbb{C}[x]$ of degree $n \geqslant 2$ as in Example 2.3.11. The local system underlying $f_{*} \mathbb{Q}$ is obtained by linearising the permutation representation

$$
\rho: \pi_{1}(\mathbb{C} \backslash S, x) \rightarrow \operatorname{Perm}\left(f^{-1}(x)\right) \simeq \mathfrak{S}_{n}
$$

corresponding to the covering of $\mathbb{C} \backslash S$ defined by $f$. The image $G \subseteq \mathfrak{S}_{n}$ of $\rho$ is a transitive subgroup containing an $n$-cycle, which we can assume to be $(1,2, \ldots, n)$. The standard $n$-dimensional permutation representation of $G$ splits canonically as $\mathbb{Q} \oplus V$, where $G$ acts trivially on $\mathbb{Q}$, and the local system underlying $F=f_{*} \mathbb{Q} / \mathbb{Q}$ corresponds to the representation $V$ of $G$, or of $\pi_{1}(\mathbb{C} \backslash S, x)$. If this representation is simple, then the object $A=F[1]$ of $\mathbf{P e r v}_{0}$ is simple and vice versa. Thus, in order to decide whether $A$ is simple, we only need to compute the finite group $G$ as a subgroup of $\mathfrak{S}_{n}$ and understand its action on $V$. For instance, $A$ is simple in the following two cases:
(1) if $n$ is a prime number
(2) if $f$ has $n-1$ distinct critical values

Indeed, if the degree $n$ of $f$ is prime, then $V$ contains no non-trivial subspace invariant under the $n$-cycle. If $f$ has $n-1$ distinct critical values, then $G \subseteq \mathfrak{S}_{n}$ is a transitive subgroup generated by $n-1$ transpositions, and hence is equal to $\mathfrak{S}_{n}$.
2.3.16 (Extensions by $E(0)$ ). - We can also compute extension groups in Pervo. For the time being, we restrict our attention to $\operatorname{Ext}^{n}(E(0), A)$, where $E(0)=j!j^{*} \mathbb{Q}[1]$ is as in Example 2.3.4 with $j=j(0)$ the inclusion $\mathbb{A}^{1} \backslash\{0\} \rightarrow \mathbb{A}^{1}$. We will see later that $\operatorname{Perv}_{0}$ is a tannakian category with $E(0)$ as neutral object for the tensor product. This allows then for the computation of general extension groups. Let $A=F[1]$ be an arbitrary object of $\mathbf{P e r v}_{0}$. We have

$$
\operatorname{Hom}(E(0), A)=\operatorname{Hom}\left(j!j^{*} \mathbb{Q}, F\right)=\operatorname{Hom}\left(j^{*} \underline{\mathbb{Q}}, j^{*} F\right),
$$

hence the equality

$$
\operatorname{Ext}^{n}(E(0), A)=H^{n}\left(\mathbb{A}^{1} \backslash\{0\}, j^{*} F\right)
$$

for all $n \geqslant 0$. This shows that $\operatorname{Ext}^{n}(E(0), A)=0$ for $n \geqslant 2$. Let $S \subseteq \mathbb{C}$ be a finite set containing 0 and all singularities of $F$, pick a base point $x \in \mathbb{C} \backslash S$. For every $s \in S$, choose a path $p_{s}$ from $s$ to $x$ (in $\mathbb{C} \backslash S$ except for the starting point), and denote by $g_{s} \in \pi_{1}(\mathbb{C} \backslash S, x)$ the corresponding generator. The sheaf $F$ corresponds to the following data:
(1) a vector space $V$, the fibre of $F$ at $x$;
(2) vector spaces $\left(V_{s}\right)_{s \in S}$, the fibres of $F$ at $s \in S$;
(3) a linear action of $G=\pi_{1}(\mathbb{C} \backslash S, x)$ on $V$, the global monodromy;
(4) for every $s \in S$ a map $p_{s}: V_{s} \rightarrow V$, the cospecialisation along the chosen path.

Since $A=F[1]$ belongs to $\operatorname{Perv}_{0}$, the map

$$
V \oplus \bigoplus_{s \in S} V_{s} \xrightarrow{d} \bigoplus_{s \in S} V \quad d\left(v,\left(v_{s}\right)_{s \in S}\right)=\left(v-p_{s} v_{s}\right)_{s \in S}
$$

is an isomorphism. Let us write $S^{\times}=S \backslash\{0\}$, and $g_{0} \in G$ for the class of the simple loop around 0 determined by the path $p_{0}$. The two term complex

$$
V \oplus \bigoplus_{s \in S^{\times}} V_{s} \xrightarrow{d} V \oplus \bigoplus_{s \in S^{\times}} V \quad d\left(v,\left(v_{s}\right)_{s \in S^{\times}}\right)=\left(v-g_{0} v,\left(v-p_{s} v_{s}\right)_{s \in S^{\times}}\right)
$$

computes the cohomology $H^{*}\left(\mathbb{A}^{1} \backslash\{0\}, j^{*} F\right)$. This makes it easy to compute the dimension of $\operatorname{Ext}^{1}(E(0), A)$. Set $h^{0}=\operatorname{dim} \operatorname{Hom}(E(0), A)$ and $h^{1}=\operatorname{dim} \operatorname{Ext}^{1}(E(0), A)$. We find

$$
\begin{aligned}
h^{1} & =\operatorname{dim} V+\sum_{s \in S^{\times}} \operatorname{dim} V-\left(\operatorname{dim} V+\sum_{s \in S^{\times}} \operatorname{dim} V_{s}\right)+h^{0} \\
& =\sum_{s \in S}\left(\operatorname{dim} V-\operatorname{dim} V_{s}\right)-\left(\operatorname{dim} V-\operatorname{dim} V_{0}\right)+h^{0} \\
& =\operatorname{dim} V-\left(\operatorname{dim} V-\operatorname{dim} V_{0}\right)+h^{0} \\
& =\operatorname{dim} V_{0}+h^{0}
\end{aligned}
$$

In particular, if $A=F[1]$ is simple and different from $E(0)$, then we have $h^{0}=0$ and hence $h^{1}=\operatorname{dim} V_{0}$.

Example 2.3.17. - In the case $A=E(0)$ we find $V_{0}=0$ and $h^{0}=1$, so $\operatorname{Ext}^{1}(E(0), E(0))$ is one-dimensional. A nontrivial extension of $E(0)$ by $E(0)$ is the perverse sheaf with only one singularity $S=\{0\}$, corresponding to the vector space $V=\mathbb{Q}^{2}$ and monodromy operator $g_{0}\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$.

In the case $A=E(s)$, where $s$ is some nonzero complex number, we find $V_{0}=1$ and $h^{0}=0$, so $\operatorname{Ext}^{1}(E(0), E(s))$ is one-dimensional. A nontrivial extension of $E(0)$ by $E(s)$ is the perverse sheaf with two singularities $S=\{0, s\}$, corresponding to the vector space $V=\mathbb{Q} \oplus \mathbb{Q}$, its two subspaces $V_{0}=\mathbb{Q} \oplus 0$ and $V_{s}=0 \oplus \mathbb{Q}$, together with monodromy operators

$$
g_{0}=\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right) \quad g_{s}=\left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right)
$$

for the chosen loops around 0 and $s$. The subobject $E(s)$ corresponds to the subspace $\mathbb{Q} \oplus 0$ of $V$.

Example 2.3.18. - Let $N>0$ be an integer, and let $A$ be the object of Perv $_{0}$ whose only singularity is zero,

### 2.4. Additive convolution

In this section, we introduce the additive convolution of perverse sheaves on the affine line, and prove that $\operatorname{Perv}_{0}$ is stable under additive convolution. We verify that $\mathbf{P e r v}_{0}$ is a $\mathbb{Q}$-linear tannakian category with respect to the tensor product given by additive convolution, with the exception of the existence of a fibre functor. The discussion of fibre functors will be postponed to Section 2.8. It will turn out that the nearby fibre at infinity $\Psi_{\infty}$ as well as the total vanishing cycles functor $\Phi$ are $\mathbb{Q}$-linear fibre functors.

Definition 2.4.1. - Let $A$ and $B$ be objects of $D_{c}^{b}\left(\mathbb{A}^{1}\right)$, the bounded derived category of constructible sheaves on $\mathbb{A}^{1}$. We define the additive convolution of $A$ and $B$ as

$$
A * B=R \operatorname{sum}_{*}\left(\operatorname{pr}_{1}^{*} A \otimes \operatorname{pr}_{2}^{*} B\right)
$$

where sum: $\mathbb{A}^{2} \rightarrow \mathbb{A}^{1}$ is the summation map, and $\operatorname{pr}_{1}, \operatorname{pr}_{2}: \mathbb{A}^{2} \rightarrow \mathbb{A}^{1}$ are the projection maps. We define the functor $\Pi: D_{c}^{b}\left(\mathbb{A}^{1}\right) \rightarrow D_{c}^{b}\left(\mathbb{A}^{1}\right)$ as

$$
\begin{equation*}
\Pi(A)=A * j!j^{*} \underline{\mathbb{Q}}[1], \tag{2.4.1.1}
\end{equation*}
$$

where $j: \mathbb{A}^{1} \backslash\{0\} \rightarrow \mathbb{A}^{1}$ is the inclusion.

Lemma 2.4.2. - Let $A$ and $B$ be objects of $D_{c}^{b}\left(\mathbb{A}^{1}\right)$. For every $z \in \mathbb{C}$, there is a natural isomorphism

$$
(A * B)_{z} \xrightarrow{\sim} R \pi_{*}\left(A \otimes \tau_{z}^{*} B\right)
$$

in the derived category of vector spaces, where $\tau_{z}: \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ is the reflection map $\tau_{z}(x)=z-x$.
Proof. We first suppose that $A$ and $B$ are constructible sheaves $F$ and $G$ concentrated in degree zero. Let $S$ and $T$ be finite sets containing the singular points of $F$ and $G$ respectively, and set $Y=S \times \mathbb{C} \cup \mathbb{C} \times T \rightarrow \mathbb{C}$. The following holds:
(1) The summation map sum: $\mathbb{C}^{2} \rightarrow \mathbb{C}$ is a fibre bundle.
(2) The morphism sum: $Y \rightarrow \mathbb{C}$ is proper.
(3) Outside $Y$, the sheaf $\operatorname{pr}_{1}^{*} F \otimes \operatorname{pr}_{2}^{*} G$ is a local system.

It was observed by Nori [65, Lemma 2.7] or [66, Proposition 1.3A] that (1), (2) and (3) imply via a combination of proper base change and the Künneth formula that the base change morphisms

$$
\begin{equation*}
\left(R \operatorname{sum}_{*}\left(\operatorname{pr}_{1}^{*} F \otimes \operatorname{pr}_{2}^{*} G\right)\right)_{z} \rightarrow R \pi_{*}\left(\left.\left(\operatorname{pr}_{1}^{*} F \otimes \operatorname{pr}_{2}^{*} G\right)\right|_{\operatorname{sum}^{-1}(z)}\right) \tag{2.4.2.1}
\end{equation*}
$$

are isomorphisms in the derived category of vector spaces. The base change morphism (2.4.2.1) is a natural morphism for arbitrary sheaves or complexes of sheaves, and hence it follows from a dévissage argument that (2.4.2.1) is an isomorphism also when $F$ and $G$ are replaced by objects $A$ and $B$ of the bounded derived category of constructible sheaves. The fibre

$$
\operatorname{sum}^{-1}(z)=\left\{\left(x, \tau_{z}(x)\right) \mid x \in \mathbb{C}\right\}
$$

is an affine line, and with respect to the coordinate $x$ the restriction of $\operatorname{pr}_{1}^{*} A \otimes \operatorname{pr}_{2}^{*} B$ to this line is the sheaf $A \otimes \tau_{z}^{*} B$. Hence we obtain a natural isomorphism as claimed.

Proposition 2.4.3. - Let $A$ and $B$ be objects of $D_{c}^{b}\left(\mathbb{A}^{1}\right)$.
(1) There is a natural isomorphism $R \pi_{*}(A * B) \cong R \pi_{*}(A) \otimes R \pi_{*}(B)$ in the derived category of vector spaces.
(2) If $A$ is perverse and $B$ is an object of $\mathbf{P e r v}_{0}$, then $A * B$ is an object of $\mathbf{P e r v}_{0}$. In particular, the endofunctor $A \longmapsto A * B$ on the derived category of constructible sheaves is exact for the perverse $t$-structure.
(3) The functor $\Pi$ from (2.4.1.1) sends Perv to $\operatorname{Perv}_{0}$ and is left adjoint to the inclusion Perv $_{0} \rightarrow$ Perv.
(4) If $A$ is an object of $\mathbf{P e r v}_{0}$, then the canonical morphism $A \rightarrow \Pi(A)$ is an isomorphism.

Proof. Denote by $\pi^{2}: \mathbb{C}^{2} \rightarrow \operatorname{Spec}(\mathbb{C})$ the structure morphism and set $A \boxtimes B=\operatorname{pr}_{1}^{*} A \otimes \operatorname{pr}_{2}^{*} B$. The composite isomorphism

$$
R \pi_{*}(A * B) \cong R \pi_{*}^{2}\left(\operatorname{pr}_{1}^{*} A \otimes \operatorname{pr}_{2}^{*} B\right) \cong R \pi_{*}^{2}\left(\operatorname{pr}_{1}^{*} A\right) \otimes R \pi_{*}^{2}\left(\operatorname{pr}_{2}^{*} B\right) \cong R \pi_{*}(A) \otimes R \pi_{*}(B)
$$

yields (1). The second isomorphism is explained by the fact that a tensor product of flasque sheaves is flasque, and that for arbitrary sheaves $F$ and $G$ we have $\pi_{*}^{2}(F \boxtimes G) \cong \pi_{*} F \otimes \pi_{*} G$ (the presheaf tensor product is already a sheaf).

Now suppose that the objects $A$ and $B$ are perverse, and that $R \pi_{*} B=0$ holds. By (1) we have $R \pi_{*}(A * B)=0$, so it will be enough to convince ourselves that $A * B$ is a constructible sheaf placed in degree -1 . According to Lemma 2.3.3, the complex $A \boxtimes B$ of constructible sheaves on $\mathbb{C}^{2}$ sits in degrees -2 and -1 , and hence $A * B$ is supported in cohomological degrees $-2,-1$, and 0 . We must show that $\mathcal{H}^{-2}(A * B)=\mathcal{H}^{0}(A * B)=0$. For any $z \in \mathbb{C}$, there is according to Lemma 2.4.2 a canonical isomorphism of rational vector spaces

$$
\mathcal{H}^{q}(A * B)_{z} \cong R^{q} \pi_{*}\left(A \otimes \tau_{z}^{*} B\right)
$$

The complex of sheaves $C=A \otimes \tau_{z}^{*} B$ is cohomologically supported in degrees -2 and -1 and

$$
\mathcal{H}^{-2}(C)=\mathcal{H}^{-1}(A) \otimes \mathcal{H}^{-1}\left(\tau_{z}^{*} B\right), \quad \mathcal{H}^{-1}(C)=\mathcal{H}^{0}(A) \otimes \mathcal{H}^{-1}\left(\tau_{z}^{*} B\right)
$$

Using the spectral sequence $R^{p} \pi_{*}\left(\mathbb{A}^{1}, \mathcal{H}^{q}(C)\right) \Rightarrow R^{p+q} \pi_{*} C$, we compute

$$
\begin{aligned}
\mathcal{H}^{-2}(A * B)_{z} & =R^{-2} \pi_{*} C=H^{0}\left(\mathcal{H}^{-1}(A) \otimes \mathcal{H}^{-1}(B)\right) \\
\mathcal{H}^{0}(A * B)_{z} & =\pi_{*} C=H^{1}\left(\mathcal{H}^{0}(A) \otimes \mathcal{H}^{-1}(B)\right)
\end{aligned}
$$

Since $A$ is perverse and $B$ belongs to $\operatorname{Perv}_{0}$, the sheaf $\mathcal{H}^{-1}(A)$ has no non-zero sections with finite support and $\mathcal{H}^{-1}(B)$ has no global sections, Lemma 2.2.12 implies that $\mathcal{H}^{-2}(A * B)_{z}=0$. Since $A$ is perverse, $\mathcal{H}^{0}(A)$ is a skyscraper sheaf, and hence so is $\mathcal{H}^{0}(A) \otimes \mathcal{H}^{-1}(B)$, thus $\mathcal{H}^{0}(A * B)_{z}=0$.

We have already seen in Example 2.3.4 that $j!j^{*} \underline{\mathbb{Q}}[1]$ has trivial cohomology, so it is an object of $\mathbf{P e r v}_{0}$. We need to find a natural isomorphism

$$
\begin{equation*}
\operatorname{Hom}(\Pi(A), B) \cong \operatorname{Hom}(A, B) \tag{2.4.3.1}
\end{equation*}
$$

for all perverse sheaves $A$ and $B$ with $R \pi_{*} B=0$. Let $i:\{0\} \rightarrow \mathbb{C}$ be the inclusion. The canonical exact sequence $0 \rightarrow j!j^{*} \underline{\mathbb{Q}} \rightarrow \underline{\mathbb{Q}} \rightarrow i_{*} i^{*} \underline{\mathbb{Q}} \rightarrow 0$ induces for any object $A$ in $D_{c}^{b}\left(\mathbb{A}^{1}\right)$ the following
exact triangle:

$$
R \operatorname{sum}_{*}\left(A \boxtimes j_{!} j^{*} \underline{\mathbb{Q}}\right) \rightarrow R \operatorname{sum}_{*}(A \boxtimes \underline{\mathbb{Q}}) \rightarrow R \operatorname{sum}_{*}\left(A \boxtimes i_{*} i^{*} \underline{\mathbb{Q}}\right) \rightarrow R \operatorname{sum}_{*}\left(A \boxtimes j!j^{*} \underline{\mathbb{Q}}\right)[1] .
$$

The complex $R \operatorname{sum}_{*}\left(A \boxtimes i_{*} i^{*} \underline{\mathbb{Q}}\right)$ is just $A$, and $R \operatorname{sum}_{*}(A \boxtimes \underline{\mathbb{Q}})$ is the complex of constant sheaves $\pi^{*} R \pi_{*} A$, so we may rewrite the triangle as follows:

$$
\begin{equation*}
\Pi(A)[-1] \rightarrow \pi^{*} R \pi_{*} A \rightarrow A \rightarrow \Pi(A) \tag{2.4.3.2}
\end{equation*}
$$

The triangle is functorial in $A$, and hence produces a natural map $A \rightarrow \Pi(A)$ which is an isomorphism if $R \pi_{*} A=0$. The adjunction (2.4.3.1) sends a morphism $\Pi(A) \rightarrow B$ to the composite $A \rightarrow \Pi(A) \rightarrow B$, and in the opposite direction a morphism $A \rightarrow B$ to the induced morphism $\Pi(A) \rightarrow \Pi(B)$ composed with the isomorphism $B \cong \Pi(B)$. This shows (3) and (4).

Corollary 2.4.4. - An object $A$ of $D_{c}^{b}\left(\mathbb{A}^{1}\right)$ satisfies $\Pi(A)=0$ if and only if $A$ is constant.

Proof. It follows from (2.4.3.2).

Example 2.4.5. - Let $L$ be a local system on $\mathbb{A}^{1} \backslash S$ for some finite set $S$, and consider the perverse sheaf $j_{!} L[1]$ on $\mathbb{A}^{1}$, where $j: \mathbb{A}^{1} \backslash S \rightarrow \mathbb{A}^{1}$ is the inclusion. We have seen that $\Pi(j!L[1])$ is of the form $F[1]$ for some constructible sheaf $F$, and we want to understand $F$. The fibre of $F$ at a point $z \in \mathbb{C}$ is the cohomology group

$$
F_{z}=H^{1}\left(\mathbb{C}, j(z)!j(z)^{*} j!L\right)
$$

where $j(z)$ is the inclusion of $\mathbb{C} \backslash\{z\}$ into $\mathbb{C}$. The sheaf $j(z)!j(z)^{*} j_{!} L$ is given by the local system $L$ outside $S \cup\{z\}$ and has trivial fibres at each point of $S \cup\{z\}$. We see that $F$ is given by a local system of $\operatorname{rank} \# S \cdot \operatorname{rank}(L)$ on $\mathbb{C} \backslash S$, and that its fibre at $s \in S$ is a vector space of dimension $(\# S-1) \cdot \operatorname{rank}(L)$. Later we will see how to effectively calculate the monodromy of the local system given by $F$ on $\mathbb{C} \backslash S$.
2.4.6 (Additive convolution with support). - There is a variant of additive convolution defined using direct image with compact support, namely

$$
A *!B=R \operatorname{sum}_{!}(A \boxtimes B)
$$

Verdier duality exchanges the two convolutions, in the sense that

$$
\mathbb{D}(A * B)=\mathbb{D}(A) *!\mathbb{D}(B)
$$

In general, the object $A *_{!} B$ of $D_{c}^{b}\left(\mathbb{A}^{1}\right)$ is not a perverse sheaf, even if $A$ and $B$ belong to Perv $\mathbf{P}_{0}$ One has however the following, which was already proved in [58, Lemma 4.1]:

Lemma 2.4.7. - Let $A$ be a perverse sheaf and let $B$ be an object in $\mathbf{P e r v}_{0}$. Then the forget supports map $A *!B \rightarrow A * B$ induces an isomorphism $\Pi(A * B) \cong A * B$.

Proof. Let $\lambda: \mathbb{A}^{2} \rightarrow \mathbb{A}^{1} \times \mathbb{P}^{1}$ be the open immersion sending $(x, y)$ to $(x+y,[1: x-y])$ and let $\kappa: \mathbb{A}^{1} \rightarrow \mathbb{A}^{1} \times \mathbb{P}^{1}$ be the complementary closed immersion. The composition of $\lambda$ with the projection $p$ to the first coordinate is the summation map sum: $\mathbb{A}^{2} \rightarrow \mathbb{A}^{1}$ and $\lambda$ is indeed a relative compactification. Therefore, $A *!B=R p_{*} \lambda_{!}(A \boxtimes B)$. Let $L=R \lambda_{*}(A \boxtimes B)$ and consider the exact triangle $\lambda_{!} \lambda^{*} L \longrightarrow L \longrightarrow \kappa_{*} \kappa^{*} L$. Applying $R p_{*}$ to it, we find

$$
R p_{*} \kappa_{*} \kappa^{*} L[-1] \rightarrow R p_{*} \lambda_{!} \lambda^{*} L \longrightarrow R p_{*} L \rightarrow R p_{*} \kappa_{*} \kappa^{*} L
$$

In view of Corollary 2.4.4, it suffices to show that $R p_{*} \kappa_{*} \kappa^{*} L$ is a constant sheaf on $\mathbb{A}^{1}$. Indeed, $\kappa^{*} L$ is already a constant sheaf on $\mathbb{A}^{1}$ because the singularities of $A \boxtimes B$ are horizontal and vertical lines in $\mathbb{A}^{2}$ which do not meet the line at infinity.

Lemma 2.4.8. - For every object $A$ of $D_{c}^{b}\left(\mathbb{A}^{1}\right)$, the canonical morphism $A \rightarrow \Pi(A)$ induces an isomorphism $\Pi(\mathbb{D}(\Pi(A))) \rightarrow \Pi(\mathbb{D}(A))$.

Proof. The perverse sheaf $\mathbb{D}(E(0))=R \operatorname{Hom}(E(0), \mathbb{Q}[2])$ is an extension of the skyscraper sheaf $\delta_{0}$ with fibre $\mathbb{Q}$ at 0 by the constant sheaf $\mathbb{Q}$ on $\mathbb{A}^{1}$, and hence $\Pi(\mathbb{D}(E(0)))=\Pi\left(\delta_{0}\right)=E(0)$. Using this and Lemma 2.4.7, we obtain a string of natural isomorphisms

$$
\begin{aligned}
\Pi(\mathbb{D}(\Pi(A))) & =\Pi(\mathbb{D}(A * E(0)))=\Pi(\mathbb{D}(A) * \mathbb{D}(E(0))=\Pi(\mathbb{D}(A) * \mathbb{D}(E(0)) \\
& =\mathbb{D}(A) * \mathbb{D}(E(0)) * E(0)=\mathbb{D}(A) * \Pi(\mathbb{D}(E(0)))=\mathbb{D}(A) * E(0)=\Pi(\mathbb{D}(A))
\end{aligned}
$$

whose composite is indeed the morphism obtained by applying $\Pi \circ \mathbb{D}$ to $A \rightarrow \Pi(A)$.

Proposition 2.4.9. - Let $[-1]: \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ be the involution sending $x$ to $-x$. Given an object A of $\mathbf{P e r v}_{0}$, define

$$
\begin{equation*}
A^{\vee}=\Pi\left(\mathbb{D}\left([-1]^{*} A\right)\right)=\mathbb{D}\left([-1]^{*} A\right) * E(0) \tag{2.4.9.1}
\end{equation*}
$$

There is a canonical bijection $\operatorname{Hom}(A * B, C) \cong \operatorname{Hom}\left(A, C * B^{\vee}\right)$, natural in $A, B$, and $C$. In particular, $A^{\vee}$ is a dual of $A$.

Proof. The statement follows from the conjunction of
(a) $\operatorname{Hom}\left(E(0), X^{\vee} * Y\right)=\operatorname{Hom}(X, Y)$,
(b) $(X * Y)^{\vee}=X^{\vee} * Y^{\vee}$.

Indeed, taking these properties for granted, one has:

$$
\operatorname{Hom}(X * C, Y)=\operatorname{Hom}\left(E(0),(X * C)^{\vee} * Y\right)=\operatorname{Hom}\left(E(0), X^{\vee} * C^{\vee} * Y\right)=\operatorname{Hom}\left(X, Y * C^{\vee}\right)
$$

To prove (a), recall that the inclusion of Perv 0 into Perv has a right adjoint functor $\Pi=$ $-* E(0)$, and notice that $E(0)=\Pi\left(\delta_{0}\right)$ for $\delta_{0}$ the skyscraper sheaf with fibre $\mathbb{Q}$ at 0 . Let $\iota:\{x+y=$
$0\} \hookrightarrow \mathbb{A}^{2}$ denote the inclusion of the antidiagonal. Then:

$$
\begin{align*}
\operatorname{Hom}\left(E(0), M^{\vee} * N\right) & =\operatorname{Hom}\left(\delta_{0}, M^{\vee} * N\right) & & \text { (adjunction) } \\
& =\operatorname{Hom}\left(\delta_{0}, \Pi\left(\mathbb{D}\left([-1]^{*} M\right)\right) * N\right) & & \text { (definition of } \left.M^{\vee}\right) \\
& =\operatorname{Hom}\left(\delta_{0}, \mathbb{D}\left([-1]^{*} M\right) * N\right) & & \text { (Prop. 2.4.3 (2)) }  \tag{2}\\
& =\operatorname{Hom}\left(\delta_{0}, R \operatorname{sum}_{*}\left(\mathbb{D}\left([-1]^{*} M\right) \boxtimes N\right)\right) & & \text { (definition of } *) \\
& =\operatorname{Hom}\left(\operatorname{sum}^{*} \delta_{0}, \mathbb{D}\left([-1]^{*} M \boxtimes N\right)\right. & & \text { (adjunction) } \\
& =\operatorname{Hom}\left(\iota \mathbb{Q}, \mathbb{D}\left([-1]^{*} M \boxtimes N\right)\right. & & \text { (inspection) } \\
& =\operatorname{Hom}\left(\mathbb{Q}, \iota^{!}\left(\mathbb{D}\left([-1]^{*} M \boxtimes N\right)\right)\right. & & \text { (adjunction) } \\
& =\operatorname{Hom}\left(\mathbb{Q}, \Delta^{!}(\mathbb{D}(M) \boxtimes N)\right) & & (\Delta=\iota \circ([-1], i d)) .
\end{align*}
$$

To conclude, we use: Let $\Delta: \mathbb{A}^{1} \hookrightarrow \mathbb{A}^{2}$ be the diagonal embedding. Then, for each pair of objects $A$ and $B$ of $D_{c}^{b}\left(\mathbb{A}^{1}\right)$, the following holds:

$$
\operatorname{Hom}_{D_{c}^{b}\left(\mathbb{A}^{1}\right)}(A, B)=\operatorname{Hom}_{D_{c}^{b}\left(\mathbb{A}^{1}\right)}\left(\mathbb{Q}, \Delta^{!}(\mathbb{D}(A) \boxtimes B)\right)
$$

Using the basic properties of $\mathbb{D}$, we find:

$$
\begin{aligned}
\operatorname{Hom}(A, B) & =\operatorname{Hom}(A, \mathbb{D}(\mathbb{D}(B)))=\operatorname{Hom}(A, R \operatorname{Hom}(\mathbb{D}(B), \underline{\mathbb{Q}}[2])=\operatorname{Hom}(A \otimes \mathbb{D}(B), \underline{\mathbb{Q}}[2]) \\
& =\operatorname{Hom}\left(\underline{\mathbb{Q}}, R \operatorname{Hom}\left(A \otimes \mathbb{D}(B), \omega_{X}\right)\right)=\operatorname{Hom}(\underline{\mathbb{Q}}, \mathbb{D}(A \otimes \mathbb{D}(B))) .
\end{aligned}
$$

Therefore, we are reduced to show that $\Delta^{!}(\mathbb{D}(A) \boxtimes B)=\mathbb{D}(A \otimes \mathbb{D}(B))$, which follows from the relation $A \otimes B=\Delta^{*}(A \boxtimes B)$ and Verdier duality.

We now turn to property (b).

$$
\begin{array}{rlrl}
(A * B)^{\vee} & =\Pi\left(\mathbb{D}[-1]^{*} R \operatorname{sum}_{*}(A \boxtimes B)\right) & \\
& =\Pi\left(\mathbb{D} R \operatorname{sum}_{*}\left([-1]^{*} A \boxtimes[-1]^{*} B\right)\right) & \\
& =\Pi\left(R \operatorname{sum}_{1}\left(\mathbb{D}\left([-1]^{*} A\right) \boxtimes \mathbb{D}\left([-1]^{*} B\right)\right)\right) & & \text { (Verdier duality) } \\
& =\Pi\left(A^{\vee} * B^{\vee}\right) & \\
& =A^{\vee} * B^{\vee} & \text { (Lemma 2.4.7) } \tag{Lemma2.4.7}
\end{array}
$$

We are done.
2.4.10. - We have already shown in Proposition 2.3.7 that $\mathbf{P e r v}_{0}$ is a $\mathbb{Q}$-linear abelian category. Moreover, by Proposition 2.4.3, the category $\mathbf{P e r v}_{0}$ is stable under additive convolution. The functor

$$
*: \operatorname{Perv}_{0} \times \operatorname{Perv}_{0} \rightarrow \operatorname{Perv}_{0}
$$

is additive in both variables. It is even exact in both variables: Given an exact sequence $0 \rightarrow A \rightarrow$ $A^{\prime} \rightarrow A^{\prime \prime} \rightarrow 0$ and an object $B$ in $\operatorname{Perv}_{0}$, we get an exact sequence

$$
0 \rightarrow A \boxtimes B \rightarrow A^{\prime} \boxtimes B \rightarrow A^{\prime \prime} \boxtimes B \rightarrow 0
$$

of perverse sheaves on $\mathbb{C}^{2}$. Applying $R$ sum $_{*}$ yields a long exact sequence of perverse sheaves on $\mathbb{C}$, of which only the part

$$
0 \rightarrow A * B \rightarrow A^{\prime} * B \rightarrow A^{\prime \prime} * B \rightarrow 0
$$

is non-zero. As already mentioned, we regard additive convolution as a tensor product on $\operatorname{Perv}_{0}$. Our next task is to choose unit, associativity and commutativity constraints. The unit object is $E(0)=j!\mathbb{Q}[1]$, and as the unit constraint

$$
\begin{equation*}
A * E(0) \cong A \cong E(0) * A \tag{2.4.10.1}
\end{equation*}
$$

we choose the canonical isomorphism of Proposition 2.4 .3 part (4) when the unit $E(0)$ stands on the right, and the analogous isomorphism when $E(0)$ stands on the left. The associativity constraint (we choose it)

$$
\begin{equation*}
(A * B) * C=\operatorname{Rsum}_{*}^{3}(A \boxtimes B \boxtimes C) \cong A *(B * C) \tag{2.4.10.2}
\end{equation*}
$$

is given by the associativity constraint for complexes of sheaves on $\mathbb{C}^{3}$ and associativity of the sum of complex numbers. The commutativity constraint

$$
\begin{equation*}
A * B=R \operatorname{sum}_{*}(A \boxtimes B) \cong R \operatorname{sum}_{*}(B \boxtimes A)=B * A \tag{2.4.10.3}
\end{equation*}
$$

is given by the commutativity constraint for complexes of sheaves on $\mathbb{C}^{2}$ and commutativity of the sum of complex numbers. Be careful and don't make the same mistakes as the authors: the commutativity constraint for the tensor product of complexes of sheaves is given by the Koszul rule, and objects $A, B$ of $\operatorname{Perv}_{0}$ are concentrated in degree -1 . The present discussion and Proposition 2.4.9 are summarised in the following theorem.

Theorem 2.4.11. - Additive convolution defines a tensor product on the $\mathbb{Q}$-linear abelian category $\operatorname{Perv}_{0}$ with respect to which it is a symmetric monoidal closed category. Constraints are given by (2.4.10.1) for units, (2.4.10.2) for associativity, and (2.4.10.3) for commutativity.

In different terminology (we explain conventions in Section A.1), Theorem 2.4.11 states that $\operatorname{Perv}_{0}$, equipped with additive convolution is a tensor category which is ACU and rigid, and $\operatorname{End}(E(0))=\mathbb{Q}$ holds. What remains to show in order to prove that $\operatorname{Perv}_{0}$ is a tannakian category, is to find a fibre functor. The nearby fibre at infinity $\Psi_{\infty}: \operatorname{Perv}_{0} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ is an obvious candidate, but to show that it is indeed a fibre functor is more difficult than it seems, and will keep us busy for a while.

### 2.5. A braid group action

In the previous section, we introduced and studied additive convolution using the six-functors formalism. Given objects $A$ and $B$ of $\mathbf{P e r v}_{0}$, we would now also like to describe the monodromy representation and the singular fibres of the convolution $A * B$ in terms of those of $A$ and $B$. This is feasible, but not straightforward, and will only be achieved in Theorem 2.6.2. At the heart of the description is the action of a braid group on fundamental groups, which is what we aim to describe in the present section.
2.5.1. - We write $\overline{\mathbb{C}}$ for the compactification of $\mathbb{C}$ by a circle at infinity, so $\overline{\mathbb{C}}=\mathbb{C} \sqcup S^{1}$, where a system of open neighbourhoods of $z \in S^{1}=\{z \in \mathbb{C}| | z \mid=1\}$ is given by the sets

$$
\left\{w \in \mathbb{C}||w|>R,|\arg (w)-\arg (z)|<\varepsilon\} \sqcup\left\{z^{\prime} \in S^{1}| | \arg \left(z^{\prime}\right)-\arg (z) \mid<\varepsilon\right\}\right.
$$

for large $R$ and small $\varepsilon$, as illustrated in Figure 2.5.4. The space $\overline{\mathbb{C}}$ is called the oriented real blow-up of $\mathbb{P}^{1}(\mathbb{C})$ at infinity. For a non-zero complex number $z$, we write $z \infty$ for the element of the boundary $S^{1}$ of $\overline{\mathbb{C}}$ with $\operatorname{argument} \arg (z)$.


Figure 2.5.4. A neighbourhood of $z \infty$
2.5.2. - Let $S$ and $T$ be finite, not necessarily disjoint sets of points in the complex plane $\mathbb{C}$, and define $S+T=\{s+t \mid s \in S, t \in T\}$. A point $u \in \mathbb{C}$ does not belong to $S+T$ if and only if the sets $S$ and $u-T=\{u-t \mid t \in T\}$ are disjoint. For each $u \in \mathbb{C} \backslash(S+T)$, let us denote by $G(u)$ the fundamental group of the space $\overline{\mathbb{C}} \backslash(S \cup(u-T))$ relative to the base point $1 \infty$. It is the same as the fundamental group of $\mathbb{C} \backslash(S \cup(u-T))$ with respect to a large real number as base point. The groups $G(u)$ form a local system on $\mathbb{C} \backslash(S+T)$, and we may consider its monodromy. Concretely, pick a base point $u_{0} \in \mathbb{C} \backslash(S+T)$ and define a group homomorphism

$$
\begin{equation*}
\beta: \pi_{1}\left(\mathbb{C} \backslash(S+T), u_{0}\right) \rightarrow \operatorname{Aut}\left(G\left(u_{0}\right)\right) \tag{2.5.2.1}
\end{equation*}
$$

as follows: given a loop $\gamma:[0,1] \rightarrow \mathbb{C} \backslash(S+T)$ based at $u_{0}$ and a loop $g:[0,1] \rightarrow \overline{\mathbb{C}} \backslash\left(S \cup\left(u_{0}-T\right)\right)$ based at $1 \infty$, we define $\beta(\gamma)(g)$ to be the homotopy class of any loop $g^{\prime}$ in $\overline{\mathbb{C}} \backslash\left(S \cup\left(u_{0}-T\right)\right)$ such that $g^{\prime} \times 1$ is homotopic to $\tau(g \times 0) \tau^{-1}$ in the space

$$
(\overline{\mathbb{C}} \times[0,1]) \backslash\{(z, t) \mid z \in S \cup(\gamma(t)-T)\}
$$

where $\tau$ is the path $t \longmapsto(1 \infty, t)$.
2.5.3. - In a similar fashion, the group $\pi_{1}\left(\mathbb{C} \backslash(S+T), u_{0}\right)$ acts on homotopy classes of paths. Pick a point $x \in S \cup\left(u_{0}-T\right)$, and let us denote by $P_{x}\left(u_{0}\right)$ the set of homotopy classes of paths in $\overline{\mathbb{C}} \backslash\left(S \cup\left(u_{0}-T\right)\right)$ which start tangentially at $x$ and end in $1 \infty$. The group $G\left(u_{0}\right)$ acts on the set
$P_{x}\left(u_{0}\right)$ by composition of paths. Similarly to the action $\beta$ given in (2.5.2.1), there is a canonical action

$$
\begin{equation*}
\beta_{x}: \pi_{1}\left(\mathbb{C} \backslash(S+T), u_{0}\right) \rightarrow \operatorname{Aut}\left(P_{x}\left(u_{0}\right)\right) \tag{2.5.3.1}
\end{equation*}
$$

of $\pi_{1}\left(\mathbb{C} \backslash(S+T), u_{0}\right)$ on the $G\left(u_{0}\right)$-set $P_{x}\left(u_{0}\right)$. It is defined by $\beta_{x}(\gamma)(p)=p^{\prime}$, where $p^{\prime}$ is a path from $x$ to $1 \infty$ in $\overline{\mathbb{C}} \backslash\left(S \cup\left(u_{0}-T\right)\right)$ such that $p^{\prime} \times 1$ is homotopic to $\tau^{-1}(p \times 0) \tau_{x}$, where $\tau_{x}$ is the path $t \longmapsto(x, t)$ if $x \in S$, and $t \longmapsto\left(\gamma(t)-u_{0}-x, t\right)$ if $x \in u_{0}-T$. Figure 2.5.5 illustrates this for $x \in u_{0}-T$.


Figure 2.5.5. The homotopy $\left(p^{\prime} \times 1\right) \simeq \tau^{-1}(p \times 0) \tau_{x}$
The two actions $\beta$ and $\beta_{x}$ are compatible in the sense that the equality

$$
\beta_{x}(\gamma)(g p)=\beta(\gamma)(g) \beta_{x}(\gamma)(p)
$$

holds for all $\gamma \in \pi_{1}\left(\mathbb{C} \backslash(S+T), u_{0}\right)$, all $g \in G\left(u_{0}\right)$ and all $p \in P_{x}\left(u_{0}\right)$. Since $G\left(u_{0}\right)$ acts transitively on the set $P_{x}\left(u_{0}\right)$ the map $\beta_{x}(\gamma)$ is described by its value on a single element.

Definition 2.5.4. - We call braid-actions the actions

$$
\beta: \pi_{1}\left(\mathbb{C} \backslash(S+T), u_{0}\right) \rightarrow \operatorname{Aut}\left(G\left(u_{0}\right)\right) \quad \text { and } \quad \beta_{x}: \pi_{1}\left(\mathbb{C} \backslash(S+T), u_{0}\right) \rightarrow \operatorname{Aut}\left(P_{x}\left(u_{0}\right)\right)
$$

defined in (2.5.2.1) and (2.5.3.1) respectively.
2.5.5. - We can give a visually more appealing description of the action (2.5.2.1) by regarding braids as isotopy classes. Start with a loop $g$ based at $1 \infty$ in $\overline{\mathbb{C}}$ avoiding the points $S \cup\left(u_{0}-T\right)$. Then, as $t$ moves from 0 to 1 , the set $\gamma(t)-T$ moves and never touches $S$, and we can deform the ambient space along with this motion, leaving the circle at infinity fixed. In particular the loop $g$ deforms together with the ambient space, at all times avoiding points in $S \cup(\gamma(t)-T)$. As $t$ reaches 1 , we obtain a new loop in $\overline{\mathbb{C}}$ avoiding the points $S \cup\left(u_{0}-T\right)$, which we declare to be $\beta(\gamma)(g)$. This works exactly in the same way also for paths. As a concrete example, take $S=\{0,1\}, T=\{0, i\}$ and $u_{0}=2$, so $S+T=\{0,1, i, 1+i\}$ and $S \cup\left(u_{0}-T\right)$ consists of the four elements $\{0,1,2,2-i\}$. Now pick a loop $\gamma$ based at $u_{0}$ avoiding $S+T$ and for each $x \in S \cup\left(u_{0}-T\right)$ a simple loop around
$x$ based at $1 \infty$ avoiding $S \cup\left(u_{0}-T\right)$, named as as in Figure 2.5.6. We will systematise this way of choosing and naming loops later.


Figure 2.5.6. The loops $\gamma$ (left) and $g_{1}, g_{2}, h_{1}, h_{2}$ (right)
The fundamental group $G\left(u_{0}\right)$ is the free group generated by the $g_{1}, g_{2}, h_{1}, h_{2}$. On the right hand picture, we now move the elements $\{2,2-i\}$ of $u_{0}-T$ along $\gamma(t)-T$, and deform the four loops accordingly. In the left part Figure 2.5 .7 we have drawn $g_{2}$ and the trajectory of $\gamma(t)-T$ as $t$ moves from 0 to 1 . Deforming $g_{2}$ results in a new path $\beta(\gamma)\left(g_{2}\right)$, which is drawn on the right in Figure 2.5.7. This path is $\beta(\gamma)\left(g_{2}\right)=g_{2}^{-1} h_{1}^{-1} g_{2} h_{1} g_{2}$.


Figure 2.5.7. The loop $g_{2}$ (left) and $\beta(\gamma)\left(g_{2}\right)$ (right)
We determine the effect of $\gamma$ on $g_{1}, h_{1}, h_{2}$ in a similar way, and find $\beta(\gamma)$ to be the following automorphism of $G\left(u_{0}\right)$.

$$
\begin{align*}
& g_{1} \longmapsto g_{1} \\
& g_{2} \longmapsto \\
& g_{2}^{-1} h_{1}^{-1} g_{2} h_{1} g_{2}  \tag{2.5.5.1}\\
& h_{1} \longmapsto \\
& h_{2}^{-1} h_{1} g_{2} \\
& h_{2} \longmapsto h_{2}
\end{align*}
$$

Notice that we have $\beta(\gamma)\left(g_{i}\right)=a_{i} g_{i} a_{i}^{-1}$ and similarly $\beta(\gamma)\left(h_{i}\right)=b_{i} g_{i} b_{i}^{-1}$ for some $a_{i}, b_{i}$ depending on $\gamma$. The elements $a_{1}$ and $b_{2}$ are trivial, because only $1 \in S$ and $2-i \in 2-T$ are entangled by the action of $\gamma$. The description works as well for the braid action on paths. Let us call $p_{1}, p_{2}, q_{1}, q_{2}$
the paths corresponding to the chosen simple loops, as indicated on the left hand part of Figure 2.5.8. The paths $\beta(\gamma)\left(p_{1}\right), \ldots, \beta(\gamma)\left(q_{2}\right)$ are the ones on the right.


Figure 2.5.8. The paths $p_{0}, p_{1}, q_{1}, q_{2}$ (left) and their image under $\beta(\gamma)$ (right)

The action of $\gamma$ on path spaces $P_{x}\left(u_{0}\right)$ is uniquely determined by (2.5.3.1) and the following values

$$
\begin{align*}
p_{0} & \longmapsto p_{1} \\
p_{1} & \longmapsto
\end{align*} g_{2}^{-1} h_{1}^{-1} p_{2} .
$$

Notice that we have indeed $\beta(\gamma)\left(p_{i}\right)=a_{i} p_{i}$ and $\beta(\gamma)\left(q_{i}\right)=b_{i} p_{i}$ for the same $a_{i}$ and $b_{i}$ we found in (2.5.5.1). This is so because the simple loops $g_{i}$ and $h_{i}$ are obtained from the paths $p_{i}$ and $q_{i}$ and small loops around the corresponding starting points. This shows that if we understand the braid action on loops, then we understand it also on paths, and vice versa.
2.5.6. - We devote the rest of this section to a systematic combinatorial description of the braid actions, which will be helpful for explicit computations. The idea is to produce in a more or less systematic way the group elements $a_{i}$ and $b_{i}$ which appeared in (2.5.5.1) and (2.5.5.2). This in turn has to do much with the choice of generators of the of the involved fundamental groups. We now come back to the standard paths and loops introduced in Paragraph 2.2.10.

Consider two finite sets of points $S=\left\{s_{1}, \ldots, s_{n}\right\}$ and $T=\left\{t_{1}, \ldots, t_{m}\right\}$ in the complex plane, listed in western reading order. Choose a large real number $r \gg 0$, greater than the norm of all $s \in S$ and all $t \in T$, and set $\tau_{2 r}(z)=2 r-z$. We write

$$
G=\pi_{1}(\mathbb{C} \backslash S, r) \cong \pi_{1}(\overline{\mathbb{C}} \backslash S, 1 \infty) \quad \text { and } \quad H=\pi_{1}(\mathbb{C} \backslash T, r) \cong \pi_{1}(\overline{\mathbb{C}} \backslash T, 1 \infty)
$$

and choose standard paths $p_{i}$ from $s_{i} \in S$ to $r$ giving rise to generators $g_{i} \in G$, as well as standard paths $q_{i}$ from $t_{i} \in T$ to $r$ giving rise to generators $h_{j} \in H$. Let $\omega$ be the path $t \longmapsto r\left(2-e^{-\pi i t}\right)$ from $r$ to $3 r$ in $\overline{\mathbb{C}}$, and extend it along the real half-line $[3 r, \infty]$. Using the Seifert-van-Kampen theorem and the path $\tau_{2 r} \circ \omega$, we identify

$$
\begin{equation*}
G * H \cong \pi_{1}\left(\overline{\mathbb{C}} \backslash\left(S \cup \tau_{2 r}(T)\right), 1 \infty\right) \tag{2.5.6.1}
\end{equation*}
$$

Explicitly, the isomorphism (2.5.6.1) sends $g_{i} \in G * H$ to the concatenation $\omega \cdot g_{i}$, and $h_{j} \in G * H$ to the concatenation $\omega \cdot\left(\tau_{2 r} \circ h_{j}\right)$. Similarly, the paths $p_{i}$ and $q_{j}$ yield a paths $\omega \cdot p_{i}$ and $\omega \cdot\left(\tau_{2 r} \circ q_{j}\right)$. The situation is illustrated in Figure 2.5.9.


Figure 2.5.9. The set $S \cup(2 r-T)$ and the path $\omega$

Definition 2.5.7. - Let $S=\left\{s_{1}, \ldots, s_{n}\right\}$ and $T=\left\{t_{1}, \ldots, t_{m}\right\}$ be finite sets of complex numbers, ordered in western reading order, and let $u \in S+T$. We call entanglement list of $u$ the list of pairs of integers

$$
\mathrm{TL}(u)=\left\{\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right), \ldots,\left(i_{d}, j_{d}\right)\right\}
$$

which contains all pairs of integers $(i, j)$ such that $s_{i}+t_{j}=u$, and which is ordered so that $i_{1}<i_{2}<\cdots<i_{d}$, and hence $j_{1}>j_{2}>\cdots>j_{d}$, holds.

Proposition 2.5.8. - Pick $u \in S+T$, and let $\gamma_{u} \in \pi_{1}(\mathbb{C} \backslash(S+T), 2 r)$ be the standard simple loop around $u$. For every $1 \leqslant i \leqslant n$, define an element $a_{i} \in G * H$ by

$$
a_{i}= \begin{cases}\left(g_{i_{1}}^{-1} h_{j_{1}}^{-1} g_{i_{1}} h_{j_{1}}\right)\left(g_{i_{2}}^{-1} h_{j_{2}}^{-1} g_{i_{2}} h_{j_{2}}\right) \cdots\left(g_{i_{l}}^{-1} h_{j_{l}}^{-1} g_{i_{l}} h_{j_{l}}\right) & \text { if } i_{l}<i \\ \left(g_{i_{1}}^{-1} h_{j_{1}}^{-1} g_{i_{1}} h_{j_{1}}\right)\left(g_{i_{2}}^{-1} h_{j_{2}}^{-1} g_{i_{2}} h_{j_{2}}\right) \cdots\left(g_{i_{l}}^{-1} h_{j_{l}}^{-1}\right) & \text { if } i_{l}=i\end{cases}
$$

where $i_{l}$ is the largest integer $\leqslant i$ appearing in the entanglement list $\mathrm{TL}(u)$ of $u$ and, for every $1 \leqslant j \leqslant m$, define an element $b_{j} \in G * H$ by

$$
b_{j}= \begin{cases}\left(g_{i_{1}}^{-1} h_{j_{1}}^{-1} g_{i_{1}} h_{j_{1}}\right)\left(g_{i_{2}}^{-1} h_{j_{2}}^{-1} g_{i_{2}} h_{j_{2}}\right) \cdots\left(g_{i_{l}}^{-1} h_{j_{l}}^{-1} g_{i_{l}} h_{j_{l}}\right) & \text { if } j_{l}>j \\ \left(g_{i_{1}}^{-1} h_{j_{1}}^{-1} g_{i_{1}} h_{j_{1}}\right)\left(g_{i_{2}}^{-1} h_{j_{2}}^{-1} g_{i_{2}} h_{j_{2}}\right) \cdots\left(g_{i_{l}}^{-1}\right) & \text { if } j_{l}=j\end{cases}
$$

where $j_{l}$ is the smallest integer $\geqslant j$ appearing in the entanglement list of $u$. Via the isomorphism (2.5.6.1), the braid action of the standard path $\gamma_{u}$ on $G * H$ is given by

$$
\beta\left(\gamma_{u}\right)\left(g_{i}\right)=a_{i} g_{i} a_{i}^{-1} \quad \text { and } \quad \beta\left(\gamma_{u}\right)\left(p_{i}\right)=a_{i} p_{i}
$$

$$
\beta\left(\gamma_{u}\right)\left(h_{j}\right)=b_{j} h_{j} b_{j}^{-1} \quad \text { and } \quad \beta\left(\gamma_{u}\right)\left(q_{j}\right)=b_{j} q_{j}
$$

Proof. We can deform the sets $S$ and $T$ without changing the presentation of the fundamental groups and the braid action of $\gamma_{u}$, as long as we ensure that orderings of $S$ and $T$ as well as the entanglement list $\mathrm{TL}(u)$ remain the same. We can arrange that $u=0$ and that $S$ and $T$ consist of purely imaginary numbers. Thus, the imaginary parts of $s_{1}, s_{2}, \ldots, s_{n}$ as well as those of $t_{1}, t_{2} \ldots, t_{m}$ are strictly decreasing, and $(i, j)$ is in the entanglement list of $u=0$ if and only if $\operatorname{im}\left(2 r+t_{j}\right)=\operatorname{im}\left(s_{i}\right)$. Figure 2.5.10 shows this configuration of points in the complex plane, together with the trajectory of $\gamma_{u}-T$. The formula of the proposition can now be shown by examining how


Figure 2.5.10. Entanglement list $\operatorname{TL}(u)=\{(2,6),(4,3),(8,1)\}$, with the path $p_{5}$ shown. As $T$ moves, $p_{5}$ will pick up a factor $g_{2}^{-1} h_{6}^{-1} g_{2} h_{6}$ from the entanglement $(2,6)$ and a factor $g_{4}^{-1} h_{3}^{-1} g_{4} h_{3}$ from the entanglement $(4,3)$. The entanglement $(8,1)$ has no influence.
in this configuration the paths $p_{i}$ and $q_{j}$ deform. This can be done explicitly with a drawing. There are essentially three cases to consider: a pair $(i, j)$ in the entanglement list affects those paths $p$ that start below the horizontal line through $s_{i}$ and $2 r-t_{j}$, and it affects the path $p_{i}$ and the path $q_{j}$. These three cases are illustrated in Figure 2.5.11 As the point $2 r-t_{j}$ moves around $s_{i}$ along


Figure 2.5.11. Paths before deformation
$\gamma_{u}-t_{j}$, the paths $p, p_{i}$ and $q_{j}$ deform continuously to the paths displayed in Figure 2.5.12, which represent $\beta\left(\gamma_{u}\right)(p), \beta\left(\gamma_{u}\right)\left(p_{i}\right)$ and $\beta\left(\gamma_{u}\right)\left(q_{j}\right)$. The path $p$ has changed by a factor $g_{i}^{-1} h_{j}^{-1} g_{i} h_{j}$, and there will be one such factor for every entangled pair $p$ crosses in forward order. After deformation, the path $p_{i}$ ends in $g_{i}^{-1} h_{j}^{-1} p_{i}$, and finally, after deformation, $q_{j}$ ends in $g_{i}^{-1} q_{j}$.


Figure 2.5.12. Paths after deformation
2.5.9. - It will be also useful to describe the braid action of $\gamma_{u}^{-1}$. It is given as follows: In the setup of Proposition 2.5.8, define $\widetilde{a}_{i} \in G$ by

$$
\tilde{a}_{i}= \begin{cases}\left(h_{j_{1}} g_{i_{1}} h_{j_{1}}^{-1} g_{i_{1}}^{-1}\right)\left(h_{j_{2}} g_{i_{2}} h_{j_{2}}^{-1} g_{i_{2}}^{-1}\right) \cdots\left(h_{j_{l}} g_{i_{l}} h_{j_{l}}^{-1} g_{i_{l}}^{-1}\right) & \text { if } i_{l}<i \\ \left(h_{j_{1}} g_{i_{1}} h_{j_{1}}^{-1} g_{i_{1}}^{-1}\right)\left(h_{j_{2}} g_{i_{2}} h_{j_{2}}^{-1} g_{i_{2}}^{-1}\right) \cdots\left(h_{j_{l}}\right) & \text { if } i_{l}=i\end{cases}
$$

where $i_{l}$ is the largest integer $\leqslant i$ appearing in the entanglement list of $u$ and, for every $1 \leqslant j \leqslant m$, define $\widetilde{b}_{j} \in G$ by

$$
\widetilde{b}_{j}= \begin{cases}\left(h_{j_{1}} g_{i_{1}} h_{j_{1}}^{-1} g_{i_{1}}^{-1}\right)\left(h_{j_{2}} g_{i_{2}} h_{j_{2}}^{-1} g_{i_{2}}^{-1}\right) \cdots\left(h_{j_{l}} g_{i_{l}} h_{j_{l}}^{-1} g_{i_{l}}^{-1}\right) & \text { if } j_{l}>j \\ \left(h_{j_{1}} g_{i_{1}} h_{j_{1}}^{-1} g_{i_{1}}^{-1}\right)\left(h_{j_{2}} g_{i_{2}} h_{j_{2}}^{-1} g_{i_{2}}^{-1}\right) \cdots\left(h_{j_{l}} g_{i_{l}}\right) & \text { if } j_{l}=j\end{cases}
$$

where $j_{l}$ is the smallest integer $\geqslant j$ appearing in the entanglement list of $u$. The braid action of $\gamma_{u}^{-1} \in \pi_{1}(\mathbb{C} \backslash(S+T), 2 r)$ is then given by

$$
\begin{array}{llll}
\beta\left(\gamma_{u}^{-1}\right)\left(g_{i}\right)=\widetilde{a}_{i} g_{i} \widetilde{a}_{i}^{-1} & \text { and } & \beta\left(\gamma_{u}^{-1}\right)\left(p_{i}\right)=\widetilde{a}_{i} p_{i} \\
\beta\left(\gamma_{u}^{-1}\right)\left(h_{j}\right)=\widetilde{b}_{j} h_{j} \widetilde{b}_{j}^{-1} & \text { and } & \beta\left(\gamma_{u}^{-1}\right)\left(q_{j}\right)=\widetilde{b}_{j} q_{j}
\end{array}
$$

This can be shown with the same geometric arguments as Proposition 2.5.8.

Example 2.5.10. - Here is a numerical example illustrating Proposition 2.5.8. We consider the set $S=\{4 i,-4,4,-4 i\}=\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}$, already appropriately ordered, and $T=S=\left\{t_{1}, t_{2}, t_{3}, t_{4}\right\}$. In a later example we will come back to this set and regard it as the set of critical values of the polynomial $f(x)=x^{5}-5 x$, but for the moment we do not care about that. The fundamental groups $G$ and $H$ are free groups

$$
G=\left\langle g_{1}, g_{2}, g_{3}, g_{4}\right\rangle \quad \text { and } \quad H=\left\langle h_{1}, h_{2}, h_{3}, h_{4}\right\rangle
$$

and the set $S+T$ is $S+T=\{8 i, 4 i-4,4 i+4,-8,0,8,-4 i-4,-4 i+4,-8 i\}$. The fundamental group $\pi_{1}(\overline{\mathbb{C}} \backslash S+T, 1 \infty)$ is thus free on 9 generators, one of which is the standard loop $\gamma_{u}$ around the point $u=-4-4 i$. The element $u \in S+T$ can be written in two different ways as a sum of an element of $S$ and an element of $T$, namely $u=s_{2}+t_{4}=s_{4}+t_{2}$. Its entanglement list is thus

$$
\mathrm{TL}(-4-4 i)=\{(2,4),(4,2)\}
$$



The elements $a_{1}, \ldots, a_{4}, b_{1}, \ldots, b_{4}$ of $G * H$ described in Proposition 2.5.8 are

$$
\begin{array}{ll}
a_{1}=1 & b_{1}=\left(g_{2}^{-1} h_{4}^{-1} g_{2} h_{4}\right)\left(g_{4}^{-1} h_{2}^{-1} g_{4} h_{2}\right) \\
a_{2}=g_{2}^{-1} h_{4}^{-1} & b_{2}=\left(g_{2}^{-1} h_{4}^{-1} g_{2} h_{4}\right) g_{4}^{-1} \\
a_{3}=\left(g_{2}^{-1} h_{4}^{-1} g_{2} h_{4}\right) & b_{3}=\left(g_{2}^{-1} h_{4}^{-1} g_{2} h_{4}\right) \\
a_{4}=\left(g_{2}^{-1} h_{4}^{-1} g_{2} h_{4}\right) g_{4}^{-1} h_{2}^{-1} & b_{4}=g_{2}^{-1} .
\end{array}
$$

The action of $\gamma_{-4-4 i}$ on $G * H$ can be written down using these strings, as stated in Proposition 2.5.8. The element $8 \in S+T$ can be written on only one way as a sum of an element of $S$ and an element of $T$, namely $8=4+4=s_{3}+t_{3}$. The entanglement list for $u=8$ has hence just one entry, namely $\operatorname{TL}(8)=\{(3,3)\}$. The elements $a_{1}, \ldots, a_{4}, b_{1}, \ldots, b_{4}$ for $u=8$ are accordingly simpler. The entanglement list of $u=0$ is $\operatorname{TL}(0)=\{(1,4),(2,3),(3,2),(4,1)\}$, and the $a_{i}$ and $b_{j}$ are accordingly more complicated.
2.5.11. - In 2.2.9 we have explained how to express cospecialisation for the cohomology of constructibe sheaves in families in terms of cocycle cohomology. We will describe now in terms of the standard loops $g_{i}$ and $h_{j}$, and paths $p_{i}$ and $q_{j}$ the involved cospecialisation map for the fundamental group $\pi_{1}\left(\overline{\mathbb{C}} \backslash\left(S \cup \tau_{z}(T)\right), 1 \infty\right)$ and the corresponding path spaces, as $z$ runs along the standard path from $u \in S+T$ to $2 r$. Pick an element $u \in S+T$. After moving $S$ and $T$ to a configuration as in the proof of Proposition 2.5.8, we can consider the paths from $S+(u-T)$ to $1 \infty$ as illustrated in on the left-hand side in Figure 2.5.13. This path, starting at an element


Figure 2.5.13. Cospecialisation of paths. On the right, $p_{8}^{\prime}=h_{6} h_{3} p_{8}$ is shown.
of $S \cup(u-T)$, runs upwards while passing unentagled elements of $u-T$ to the left and elements of $S$ to the right. If this path starts at an element $s_{i} \in S$, then we can deform it to a path $p_{i}^{\prime}$ in $\overline{\mathbb{C}} \backslash\left(S \cup \tau_{2 r} T\right)$ starting at $s_{i}$. The path $p_{i}^{\prime}$ is related to the standard path $p_{i}$ by

$$
\begin{equation*}
p_{i}^{\prime}=h_{j_{1}} h_{j_{2}} \cdots h_{j_{l}} p_{i} \tag{2.5.11.1}
\end{equation*}
$$

where $h_{j_{1}}, h_{j_{2}}, \ldots$ are the usual standard paths, and $l$ ist the largest integer such that $j_{l}$ appears in the entanglement list $\operatorname{TL}(u)=\left\{\left(i_{1}, j_{1}\right)\left(i_{2}, j_{2}\right), \ldots,\right\}$ of $u$, and such that $i_{l}<i$ holds. In case of a path starting at an element $u-t_{j}$ of $u-T$, we can deform it to a path $q_{j}^{\prime}$ starting at $2 r-t_{j}$. In terms of standard paths, the path $q_{j}^{\prime}$ is given by

$$
\begin{equation*}
q_{j}^{\prime}=h_{j_{1}} h_{j_{2}} \cdots h_{j_{l}} q_{j} \tag{2.5.11.2}
\end{equation*}
$$

where $l$ is the largest integer such that $j_{l}$ appears in $\mathrm{TL}(u)$ and such that $j_{l}>j$ holds. From the formulas (2.5.11.1) and (2.5.11.2), one can derive formulas for the cospecialisation of loops associated with paths.

### 2.6. Computation of the global monodromy of a convolution

In this section, we give an explicit description of the additive convolution of perverse sheaves in terms of group representations. We are interested in the particular case of convolution of perverse sheaves which belong to $\operatorname{Perv}_{0}$, since additive convolution is the tensor product in the tannakian category Perv ${ }_{0}$.
2.6.1. - For us, the most convenient presentation of objects of $\operatorname{Perv}_{0}$ is the one given in 2.3.10, which we briefly recall. An object $A$ of $\operatorname{Perv}_{0}$ is thus described by the following data: A finite set of singularities $\left\{s_{1}, s_{2}, \ldots, s_{n}\right\} \subseteq \mathbb{C}$ listed in western reading order, a vector space $V=\Psi_{\infty}(A)$, an endomorphism $g \in \operatorname{End}(V)$ and a vanishing cycles decomposition

$$
V=\bigoplus_{i=1}^{n} V_{i}^{\prime}
$$

The complement $V_{i} \subseteq V$ of $V_{i}^{\prime}$ in this decomposition is to be interpreted as the fibre of $A$ at the singularity $s_{i}$, identified with a subspace of $V$ via cospecialisation along the standard path $p_{i}$, and $V_{i}^{\prime}=\Phi_{s_{i}}(A)$ is the corresponding space of vanishing cycles. Writing $\pi_{i}^{\prime}$ and $\pi_{i}=\mathrm{id}-\pi_{i}^{\prime}$ for the projections onto the subspaces $V_{i}^{\prime}$ and $V_{i}$, the linear map $g_{i}: V \rightarrow V$ defined by $g_{i} v=g \pi_{i}^{\prime}(v)+\pi_{i}(v)$ is the monodromy action on $V$ of the standard loop around $s_{i}$.

Theorem 2.6.2. - Let $A$ and $B$ be objects of $\operatorname{Perv}_{0}$ and set $V=\Psi_{\infty}(A)$ and $W=\Psi_{\infty}(B)$. Let $S=\left\{s_{1}, \ldots, s_{n}\right\}$ be the set of singularities of $A$ and let $T=\left\{t_{1}, \ldots, t_{m}\right\}$ be the set of singularities of $T$ in western reading order. With respect to standard paths, let

$$
V=\bigoplus_{i=1}^{n} V_{i}^{\prime}, \quad g \in \operatorname{End}(V) \quad \text { and } \quad W=\bigoplus_{j=1}^{m} W_{j}^{\prime}, \quad h \in \operatorname{End}(W)
$$

be the vanishing cycles decompositions and endomorphisms encoding the global monodromy of $A$ and $B$. Write $\pi_{i}^{\prime}: V \rightarrow V_{i}^{\prime} \subseteq V$ and $\pi_{j}^{\prime}: W \rightarrow W_{j}^{\prime} \subseteq W$ for the projections, and set $g_{a}^{c}=\pi_{c}^{\prime} \circ g \circ \pi_{a}^{\prime}$ and $h_{b}^{d}=\pi_{d}^{\prime} \circ h \circ \pi_{b}^{\prime}$. The additive convolution $A * B$ has the following description in these terms:
(1) The set of singularities is $S+T=\{s+t \mid s \in S, t \in T\}=\left\{u_{1}, \ldots, u_{l}\right\}$.
(2) The nearby fibre at infinity is $\Psi_{\infty}(A * B)=V \otimes W$.
(3) The vanishing cycles decomposition is given by

$$
V \otimes W=\bigoplus_{k=1}^{l}\left(\bigoplus_{(i, j) \in \mathrm{TL}\left(u_{k}\right)}\left(V_{i}^{\prime} \otimes W_{j}^{\prime}\right)\right)
$$

(4) The endomorphism $e \in \operatorname{End}(V \otimes W)$ describing the global monodromy of the local system underlying $A * B$ is uniquely determined by $e_{a b}^{c d}=\left(\pi_{c}^{\prime} \otimes \pi_{d}^{\prime}\right) \circ e \circ\left(\pi_{a}^{\prime} \otimes \pi_{b}^{\prime}\right)$ given as follows

$$
\begin{array}{rll}
(i) & e_{a b}^{c d}=-g_{a}^{c} \otimes h_{b}^{d} & \text { if } a<c \text { and } b<d, \\
(i i) & e_{a b}^{c d}=g_{a}^{c} \otimes \mathrm{id} & \text { if } a<c \text { and } b=d, \\
(i i i) & e_{a b}^{c d}=0 & \text { if } a<c \text { and } b>d, \\
(i v) & e_{a b}^{c d}=\mathrm{id} \otimes h_{b}^{d} & \text { if } a=c \text { and } b<d, \\
(v) & e_{a b}^{c d}=g_{a}^{a} \otimes h_{b}^{b} & \text { if } a=c \text { and } b=d, \\
(v i) & e_{a b}^{c d}=g_{a}^{a} \otimes h_{b}^{d} & \text { if } a=c \text { and } b>d, \\
(v i i) & e_{a b}^{c d}=0 & \text { if } a>c \text { and } b<d, \\
(v i i i) & e_{a b}^{c d}=g_{a}^{c} \otimes h_{b}^{b} & \text { if } a>c \text { and } b=d, \\
(i x) & e_{a b}^{c d}=g_{a}^{c} \otimes h_{b}^{d} & \text { if } a>c \text { and } b>d
\end{array}
$$

2.6.3. - Here is an overview on the proof of Theorem 2.6.2. For a complex number $u$, let $\tau_{u}: \mathbb{C} \rightarrow \mathbb{C}$ be the reflection given by $\tau_{u}(z)=u-z$. According to Lemma 2.4.2, the fibre of $(A * B)[-1]$ at the point $u \in \mathbb{C}$ is the cohomology group

$$
H^{1}\left(\mathbb{A}^{1}, A[-1] \otimes \tau_{u}^{*} B[-1]\right)
$$

If $u \notin S+T$, then the singularities of $A$ and $\tau_{u}^{*} B$ are disjoint, and assertion (1) follows from this. Next, let $r$ be a real number, larger than the absolute value of any $s \in S$ and $t \in T$. Then $2 r$ is larger than the real part of any element of $S+T$, and hence $\Psi_{\infty}(A * B)$ is canonically isomorphic to the cohomology group $H^{1}\left(\mathbb{A}^{1}, A[-1] \otimes \tau_{2 r}^{*} B[-1]\right)$ which we will compute in terms of cocycles. Set

$$
G_{2 r}=\pi_{1}(\overline{\mathbb{C}} \backslash(S \cup(2 r-T)), 1 \infty)
$$

and write $P_{2 r}$ for the set of homotopy classes of paths from elements of $S \cup(2 r-T)$ to $1 \infty$. The constructible sheaf $A[-1] \otimes \tau_{2 r}^{*} B[-1]$ corresponds to a representation $\left(G_{2 r}, P_{2 r}\right)$ whose underlying vector space we identify with $V \otimes W$, and there is a canonical and natural isomorphism

$$
\begin{equation*}
(A * B)[-1]_{2 r} \cong H^{1}\left(\mathbb{A}^{1}, A[-1] \otimes \tau_{2 r}^{*} B[-1]\right) \cong H^{1}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right) \tag{2.6.3.1}
\end{equation*}
$$

The fundamental group $\pi_{1}(\overline{\mathbb{C}} \backslash(S+T), 2 r)$ acts on the left-hand side by monodromy-this is the action we want to understand. The fundamental group $\pi_{1}(\overline{\mathbb{C}} \backslash(S+T), 2 r)$ also acts on $G_{2 r}$ and $P_{2 r}$ via the braid action $\beta$ introduced in Definition 2.5.4, and hence it acts by precomposition on cochains, and hence on cohomology. Proposition 2.6 .5 below states that these two actions are
compatible via (2.6.3.1). After explaining this in some more detail, the next step is to construct a natural isomorphism of vector spaces

$$
\alpha_{A, B}: \Psi_{\infty}(A) \otimes \Psi_{\infty}(B) \rightarrow \Psi_{\infty}(A * B)
$$

which justifies statement (2) of the theorem. To do so, we will produce a linear map

$$
V \otimes W \rightarrow Z^{1}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right)
$$

associating with every $x \in V \otimes W$ a cocycle $c_{x}$. The isomorphism $\alpha_{A, B}$ is natural in $A$ and $B$, and it will turn out later that it is compatible with constraints for tensor products, and hence that $\Psi_{\infty}$ defines a fibre functor on the category $\operatorname{Perv}_{0}$. The isomorphim $\alpha_{A, B}$ is however not a canonical isomorphism, since it depends on our choice of standard paths. Statements (3) and most importantly (4) are then somewhat unspectacular computations, for which we use the explicit braid group actions computed in the previous section.
2.6.4. - The sheaf $A[-1] \otimes \tau_{2 r}^{*} B[-1]$ is constructible. Its fibre at a point $u \in \mathbb{C}$ is

$$
\left(A[-1] \otimes \tau_{2 r}^{*} B[-1]\right)_{u}=A[-1]_{u} \otimes B[-1]_{2 r-u}
$$

and hence its singularities are contained in the set $S \cup(2 r-T)$. In particular, the fibre of $A[-1] \otimes$ $\tau_{2 r}^{*} B[-1]$ at the point $r \in \mathbb{C}$ is $A[-1]_{r} \otimes B[-1]_{r} \cong V \otimes W$. We identify this fibre with the nearby fibre at infinity

$$
\Psi_{\infty}\left(\left(A \otimes \tau_{2 r}^{*} B\right)[-1]\right) \cong \Psi_{\infty}(A) \otimes \Psi_{\infty}(B)=V \otimes W
$$

using the path $\omega: t \longmapsto r\left(2-e^{-\pi i t}\right)$. Note that this is the same path we already used for the identification (2.5.6.1) (see Figure 2.5.9), which was

$$
G * H \cong G_{2 r}
$$

with $G=\pi_{1}(\overline{\mathbb{C}} \backslash S, 1 \infty)$ and $H=\pi_{1}(\overline{\mathbb{C}} \backslash T, 1 \infty)$. The monodromy action of $\pi_{1}(\mathbb{C} \backslash(S \cup(2 r-T)), 2 r)$ on the fibre near infinity $\Psi_{\infty}\left(\left(A \otimes \tau_{2 r}^{*} B\right)[-1]\right)$ corresponds to the action

$$
G * H \rightarrow \mathrm{GL}(V \otimes W)
$$

where $g \in G$ acts as $g \otimes \mathrm{id}_{W}$ via the monodromy representation $G \rightarrow \mathrm{GL}(V)$ from $A$, and $h \in H$ acts as id ${ }_{V} \otimes h$ via the monodromy representation $H \rightarrow \mathrm{GL}(W)$ from $B$. The fibre of $A[-1] \otimes \tau_{2 r}^{*} B[-1]$ at the singular point $s_{i} \in S$ is $V_{i} \otimes W$, via cospecialisation along the path $p_{i} \in P_{2 r}$ obtained by concatenating the standard path from $s_{i}$ to $r$ and $\omega$. Similarly, the fibre at the singular point $2 r-t_{j} \in 2 r-T$ is $V \otimes W_{j}$, via cospecialisation along the path $q_{j} \in P_{2 r}$ obtained by composing the standard path from $t_{j}$ to $r$ with $\tau_{2 r}$ and concatenating with $\omega$. These paths are illustrated in Figure 2.5.9. We have now described the constructible sheaf $A[-1] \otimes \tau_{2 r}^{*} B[-1]$ as a representation of $\left(G_{2 r}, P_{2 r}\right)$. We denote this representation simply by $V \otimes W$.

Proposition 2.6.5. - The monodromy action of $\pi_{1}(\overline{\mathbb{C}} \backslash(S+T), 2 r)$ on the fibre $(A * B)[1]_{2 r}$ corresponds, via the isomorphisms (2.6.3.1), to the action

$$
\pi_{1}(\overline{\mathbb{C}} \backslash(S+T), 2 r) \rightarrow \mathrm{GL}\left(H^{1}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right)\right)
$$

sending $\gamma \in \pi_{1}(\overline{\mathbb{C}} \backslash(S+T), 2 r)$ to the linear map defined by $c \longmapsto c \circ \beta\left(\gamma^{-1}\right)$ on cocycles, where $\beta$ is the braid action of $\pi_{1}(\overline{\mathbb{C}} \backslash(S+T), 2 r)$ on $G_{2 r}$ and $P_{2 r}$.

Proof. This is a straightforward application of Lemma 2.2.8. Fix an path $\gamma:[0,1] \rightarrow \mathbb{C} \backslash(S+T)$ representing an element of $\pi_{1}(\overline{\mathbb{C}} \backslash(S+T), 2 r)$. We consider the map $\mu: \overline{\mathbb{C}} \times[0,1] \rightarrow \overline{\mathbb{C}}^{2}$ defined by

$$
\mu(z, t)= \begin{cases}(z, \gamma(t)-z) & \text { for } z \in \mathbb{C} \\ \mu(z, t)=(z,-z) & \text { for } z \in \partial \overline{\mathbb{C}}\end{cases}
$$

and the sheaf $F=\mu^{*}(A[-1] \boxtimes B[-1])$ on $\overline{\mathbb{C}} \times[0,1]$. The fibre of $F$ at $(z, t)$ is $A[-1]_{z} \otimes B[-1]_{\gamma(t)-z}$, and the fibre of $F$ at $(1 \infty, t)$ is $A[-1]_{\infty} \otimes B[-1]_{-\infty}$ for all $t \in[0,1]$. The sheaf $F$ is constructible with respect to the stratification given by the braid with strands $\gamma_{i}(t)=s_{i}$ for $s_{i} \in S$ and $\gamma_{j}(t)=\gamma(t)-t_{j}$ for $t_{j} \in T$. The parallel transport isomorphism (2.2.8.1) specialises to the monodromy action, and the braid action

$$
\beta(\gamma)^{-1}:\left(G_{2 r}, P_{2 r}\right) \rightarrow\left(G_{2 r}, P_{2 r}\right)
$$

is in the setup of Lemma 2.2 .8 the isomorphism denoted $\beta:\left(G^{(0)}, P^{(0)}\right) \rightarrow\left(G^{(1)}, P^{(1)}\right)$, obtained by continuous deformation of loops and paths.
2.6.6. - The next step towards the proof of Theorem 2.6 .2 consists in identifying the nearby fibre at infinity of $A * B$ with $V \otimes W=\Psi_{\infty}(A) \otimes \Psi_{\infty}(B)$. To do so, we shall construct a natural isomorphism

$$
V \otimes W \rightarrow H^{1}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right) \cong \Psi_{\infty}(A * B)
$$

by defining for every element $x \in V \otimes W$ a cocycle $c_{x} \in Z^{1}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right)$. As we discussed in 2.2.6, a 1-cocycle $c \in Z^{1}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right)$ is uniquely determined by the values it takes on the particular paths $p_{1}, \ldots, p_{n}, q_{1}, \ldots, q_{m}$ from elements of $S \cup(2 r-T)$ to $1 \infty$, and any choice of values $c\left(p_{i}\right) \in V \otimes W$ and $c\left(q_{j}\right) \in V \otimes W$ determine a cocycle. More precisely, the two-term complex

$$
\begin{equation*}
(V \otimes W) \oplus \bigoplus_{i=1}^{n}\left(V_{i} \otimes W\right) \oplus \bigoplus_{j=1}^{m}\left(V \otimes W_{j}\right) \quad \xrightarrow{d} \quad \bigoplus_{i=1}^{n}(V \otimes W) \oplus \bigoplus_{j=1}^{m}(V \otimes W) \tag{2.6.6.1}
\end{equation*}
$$

computes the cohomology $H^{*}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right)$, the class of a cocycle $c \in Z^{1}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right)$ corresponding to the element $\left(c\left(p_{i}\right)_{i=1}^{n}, c\left(q_{j}\right)_{j=1}^{m}\right)$ in degree 1 of (2.6.6.1). The differential $d$ is injective, hence the equality

$$
\begin{equation*}
\operatorname{dim}(V \otimes W)=n m=\operatorname{dim} H^{1}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right) \tag{2.6.6.2}
\end{equation*}
$$

LEMMA 2.6.7. - Let $c:\left(G_{2 r}, P_{2 r}\right) \rightarrow V \otimes W$ be a cocycle. There exists a unique element $x \in V \otimes W$ such that $c$ is cohomologous to the cocycle $c_{x}$ determined by

$$
c_{x}\left(p_{1}\right)=\cdots=c_{x}\left(p_{n}\right)=x \quad \text { and } \quad c_{x}\left(q_{1}\right)=\cdots=c_{x}\left(q_{m}\right)=0
$$

This element $x \in V \otimes W$ is given by

$$
x=\kappa(c)=\sum_{i=1}^{n}\left(\pi_{i}^{\prime} \otimes \mathrm{id}\right)\left(c\left(p_{i}\right)\right)-\sum_{j=1}^{m}\left(\mathrm{id} \otimes \pi_{j}^{\prime}\right)\left(c\left(q_{j}\right)\right)
$$

Proof. Taking the equality of dimensions (2.6.6.2) into account, the first statement will follow from the fact that the map $x \longmapsto\left[c_{x}\right]$ from $V \otimes W$ to $H^{1}\left(\left(G_{2 r}, P_{2 r}\right), V \otimes W\right)$ is injective. For this, we will prove that the map $[c] \longmapsto \kappa(c)$ is a well-defined retraction. Indeed, the identities $\pi_{1}^{\prime}+\cdots+\pi_{n}^{\prime}=\mathrm{id}_{V}$ in $\operatorname{End}(V)$ and $\pi_{1}^{\prime}+\cdots \pi_{m}^{\prime}=\mathrm{id}_{W}$ in $\operatorname{End}(W)$ show that $\kappa\left(c_{x}\right)=x-0=x$ holds for all $x \in V \otimes W$, and it suffices thus to show that $\kappa(b)=0$ holds for every coboundary $b:(G, P) \rightarrow V \otimes W$. By inspection of the complex (2.6.6.1), one sees that coboundaries are linearly spanned by three kinds of elements. First, for every $y \in V \otimes W$ the cocycle $b_{y}$ determined by

$$
b_{y}\left(p_{1}\right)=\cdots=b_{y}\left(p_{n}\right)=b_{y}\left(q_{1}\right)=\cdots=b_{y}\left(q_{m}\right)=y
$$

is a coboundary, and the same identities show that $\kappa\left(b_{y}\right)=y-y=0$ holds. Secondly, for every $y \in V_{i} \otimes W$, the cocycle $b_{y}^{i}$ determined by

$$
b_{y}^{i}\left(p_{i}\right)=y \quad \text { and } \quad b_{y}^{i}\left(p_{l}\right)=0 \text { for } l \neq i \quad \text { and } \quad b_{y}\left(q_{1}\right)=\cdots=b_{y}\left(q_{m}\right)=0
$$

is a coboundary. The equality $\kappa\left(b_{y}^{i}\right)=0$ holds since $V_{i}$ is the kernel of the projector $\pi_{i}^{\prime} \in \operatorname{End}(V)$. Lastly, for every $y \in V \otimes W_{j}$, the cocycle $b_{y}^{j}$ determined by

$$
b_{y}\left(p_{1}\right)=\cdots=b_{y}\left(p_{n}\right)=0 \quad \text { and } \quad b_{y}^{i}\left(q_{j}\right)=y \quad \text { and } \quad b_{y}^{j}\left(q_{l}\right)=0 \text { for } l \neq j
$$

is a coboundary, and $\kappa\left(b_{y}^{j}\right)=0$ holds since $W_{j}$ is the kernel of the projector $\pi_{j}^{\prime} \in \operatorname{End}(W)$. Hence, we have indeed $\kappa(b)=0$ for every coboundary $b$.

Definition 2.6.8. - For any objects $A$ and $B$ of $\mathbf{P e r v}_{0}$, we denote by

$$
\alpha_{A, B}: \Psi_{\infty}(A) \otimes \Psi_{\infty}(B) \stackrel{\cong}{\cong} \Psi_{\infty}(A * B)
$$

the isomorphism which sends an element $x \in V \otimes W=\Psi_{\infty}(A) \otimes \Psi_{\infty}(B)$ to the class of the cocycle $c_{x}$ given in Lemma 2.6.7.
2.6.9. - The construction of $\alpha_{A, B}$ is slightly asymmetric in $A$ and $B$. On two occasions we treated $A$ and $B$ differently. First, in the choice of the parametrisation $z \longmapsto(z, 2 r-z)$ of the affine line $\left\{(x, y) \in \mathbb{A}^{2} \mid x+y=2 r\right\}=\operatorname{sum}^{-1}(2 r)$, and secondly in the definition of the cocycle $c_{x}$. Interchanging the role of $A$ and $B$ in the construction of $c_{x}$ leads to the isomorphism $-\alpha_{A, B}$.

In the course of the proof of Theorem 2.6.2, we will need to make the braid action on the cocycles $c_{x} \circ$ explicit. This is the content of the next lemma.

Lemma 2.6.10. - Pick $u \in S+T$, with entanglement list $T L(u)=\left\{\left(i_{1}, j_{1}\right), \ldots,\left(i_{d}, j_{d}\right)\right\}$, and pick $x \in V \otimes W$. The values of the cocycle $c_{x} \circ \beta\left(\gamma_{u}^{-1}\right)$ at the paths $p_{1}, \ldots, p_{n}, q_{1}, \ldots, q_{m}$ are given as follows:

$$
\begin{aligned}
& c_{x}\left(\beta\left(\gamma_{u}^{-1}\right)\left(p_{i}\right)\right)=-\sum_{i_{l}<i}\left(g_{i_{l}}-1\right)\left(h_{j_{l}}-1\right) x+ \begin{cases}x & \text { if } i \text { does not occur in } \mathrm{TL}(u) \\
h_{j_{l}} x & \text { if } i=i_{l} \text { occurs in } \mathrm{TL}(u)\end{cases} \\
& c_{x}\left(\beta\left(\gamma_{u}^{-1}\right)\left(q_{j}\right)\right)=-\sum_{j_{l}>j}\left(g_{i_{l}}-1\right)\left(h_{j_{l}}-1\right) x+ \begin{cases}0 & \text { if } j \text { does not occur in } \mathrm{TL}(u) \\
h_{j_{l}}\left(1-g_{i_{l}}\right) x & \text { if } j=j_{l} \text { occurs in } \mathrm{TL}(u)\end{cases}
\end{aligned}
$$

Proof. Before proving the two formulas, we make the following useful observation: The group $G$ acts on $V$, and the group $H$ acts on $W$, and hence the free product $G * H$ acts on $V \otimes W$. Commutators $g h g^{-1} h^{-1}$ act trivially on $V \otimes W$. Thus, if $c \in Z^{1}\left(G_{2 r}, P_{2 r}, V \otimes W\right)$ is a 1-cocycle, then the relation

$$
c\left(g h g^{-1} h^{-1} y\right)=c(g)+g c(h)+g h c\left(g^{-1}\right)+h c\left(h^{-1}\right)+c(y)
$$

holds. Suppose now that $c=c_{x}$ for some element $x \in V \otimes W$. Then we find

$$
x=c\left(p_{i}\right)=c\left(g_{i} p_{i}\right)=c\left(g_{i}\right)+g_{i} c\left(p_{i}\right)=c\left(g_{i}\right)+g_{i} x
$$

and hence $c\left(g_{i}\right)=\left(1-g_{i}\right) x$, and similarly $c\left(g_{i}^{-1}\right)=\left(1-g_{i}^{-1}\right) x$ and $c\left(h_{j}\right)=c\left(h_{j}^{-1}\right)=0$. This yields in particular the relation

$$
c\left(g_{i} h_{j} g_{i}^{-1} h_{j}^{-1} y\right)=c\left(g_{i}\right)+g_{i} h_{j} c\left(g^{-1}\right)+c(y)=-\left(g_{i}-1\right)\left(h_{j}-1\right) x+c(y)
$$

The two formulas in the lemma follow from this relation and the formulas given in 2.5.9 for the inverse braid action.

Proof of Theorem 2.6.2. Statement (1) is clear from the discussion in 2.6.3, and we have defined an isomorphism $\Psi_{\infty}(A * B) \cong V \otimes W$ in Definition 2.6.8, settling statement (2). We proceed to proving (4). Fix an element $u \in S+T$, and denote by $\gamma_{u}$ the standard loop around $u$. As a monodromy operator, $\gamma_{u}$ acts on $V \otimes W$ by sending an element $x \in V \otimes W$ to the unique element $y=\gamma_{u} x \in V \otimes W$ such that the cocycle $c_{x} \circ \beta\left(\gamma_{u}^{-1}\right)$ is cohomologous to $c_{y}$. This element is $y=\kappa\left(c_{x} \circ \beta\left(\gamma_{u}^{-1}\right)\right)$, where $\kappa$ is the map given in Lemma 2.6.7. Set $x=v \otimes w$. We calculate $\gamma_{u}(v \otimes w)$ using the formulas from Lemma 2.6.7 and Lemma 2.6.10. That is,

$$
\begin{aligned}
& \gamma_{u}(v \otimes w)=\sum_{i=1}^{n}\left(\pi_{i}^{\prime} \otimes 1\right)\left(c_{x}\left(\beta\left(\gamma_{u}^{-1}\right)\left(p_{i}\right)\right)\right)-\sum_{j=1}^{m}\left(1 \otimes \pi_{j}^{\prime}\right)\left(c_{x}\left(\beta\left(\gamma_{u}^{-1}\left(q_{j}\right)\right)\right)\right. \\
& \quad=\sum_{i=1}^{n}\left[-\sum_{i_{l}<i}\left(\pi_{i}^{\prime} \otimes 1\right)\left(v \otimes w-g_{i_{l}} v \otimes w-v \otimes h_{j_{l}} w+g_{i_{l}} v \otimes h_{j_{l}} w\right)+\left\{\begin{array}{c}
\pi_{i}^{\prime} v \otimes w \\
\text { or } \\
\pi_{i}^{\prime} v \otimes h_{j} w
\end{array}\right]\right. \\
& -\sum_{j=1}^{m}\left[-\sum_{j_{l}>j}\left(1 \otimes \pi_{i}^{\prime}\right)\left(v \otimes w-g_{i_{l}} v \otimes w-v \otimes h_{j_{l}} w+g_{i_{l}} v \otimes h_{j_{l}} w\right)+\left\{\begin{array}{c}
0 \\
\text { or } \\
v \otimes \pi_{j}^{\prime} h_{j} w-g_{i} v \otimes \pi_{j}^{\prime} h_{j} w
\end{array}\right]\right.
\end{aligned}
$$

The case distinctions in the first sum are as follows: add $\pi_{i}^{\prime} v \otimes w$ if $(i, j)$ does not occur in $\mathrm{TL}(u)$, and add $\pi_{i}^{\prime} v \otimes h_{j} w$ if $(i, j)$ occurs in $\mathrm{TL}(u)$. Similarly for the second sum, add 0 if $(i, j)$ does not occur in the entanglement list of $u$, and add the given summand if $(i, j)$ occurs.

Now assume that we have $v \in V_{a}^{\prime}$ and $w \in W_{b}^{\prime}$ for some $(a, b) \in \mathrm{TL}(u)$, so we have $\pi_{i}^{\prime} v=\delta_{i a} v$ and $\pi_{j}^{\prime} w=\delta_{j b} w$. The space $V_{a}^{\prime}$ is contained in the invariants of $g_{i}$ for all $i \neq a$ and similarly $W_{b}^{\prime}$ is contained in the invariants of $h_{j}$ for all $j \neq b$. We find:

$$
\sum_{i_{l}<i}\left(\pi_{i}^{\prime} \otimes 1\right)(\underbrace{v \otimes w-g_{i_{l}} v \otimes w-v \otimes h_{j_{l}} w+g_{i_{l}} v \otimes h_{j_{l}} w}_{=0 \text { if } i_{l} \neq a})= \begin{cases}0 & \text { if } a \geqslant i \\ -\pi_{i}^{\prime} g_{a} v \otimes w+\pi_{i}^{\prime} g_{a} v \otimes h_{b} w & \text { if } a<i\end{cases}
$$

for all $1 \leqslant i \leqslant n$, and similarly

$$
\sum_{j_{l}>j}\left(1 \otimes \pi_{j}^{\prime}\right)(\underbrace{v \otimes w-g_{i_{l}} v \otimes w-v \otimes h_{j_{l}} w+g_{i_{l}} v \otimes h_{j_{l}} w}_{=0 \text { if } j_{l} \neq b})= \begin{cases}0 & \text { if } b \leqslant j \\ -v \otimes \pi_{j}^{\prime} h_{b} w+g_{a} v \otimes \pi_{j}^{\prime} h_{b} w & \text { if } b>j\end{cases}
$$

for all $i \leqslant j \leqslant m$. We substitute this in the above expression for $\gamma_{u}(v \otimes w)$ and observe that among the summands where we have case distinctions, only the terms with $i_{l}=a$ and $j_{l}=b$ contribute. The expression for $\gamma_{u}(v \otimes w)$ becomes thus after some elementary simplifications

$$
\begin{equation*}
\gamma_{u}(v \otimes w)=v \otimes h_{b} w+\sum_{i=a+1}^{n} \pi_{i}^{\prime} g_{a} v \otimes\left(w-h_{b} w\right)-\sum_{j=1}^{b}\left(v-g_{a} v\right) \otimes \pi_{j}^{\prime} h_{b} w \tag{2.6.10.1}
\end{equation*}
$$

Interestingly enough, the entanglement list is not involved anymore in this formula. In order to isolate the piece $e_{a b}^{c d}$ of the endomorphism $e$, we have to apply the projector $\pi_{c}^{\prime} \otimes \pi_{d}^{\prime}$ to this expression, since we already assume $(v \otimes w)=\left(\pi_{a}^{\prime} \otimes \pi_{b}^{\prime}\right)(v \otimes w)$. We distinguish nine cases, as in the statement of the theorem:
(i) If $a<c$ and $b<d$, only the term $i=c$ from in the expression (2.6.10.1) contributes, and we get $e_{a b}^{c d}(v \otimes w)=-\pi_{c}^{\prime} g_{a} \pi_{a}^{\prime} v \otimes \pi_{d}^{\prime} h_{b} \pi_{b}^{\prime} w=-g_{a}^{c} v \otimes h_{b}^{d} w$.
(ii) If $a<c$ and $b=d$, the two terms $i=c$ and $j=b=d$ in (2.6.10.1) contribute, and we get $e_{a b}^{c d}(v \otimes w)=\pi_{c}^{\prime} g_{a} v \otimes \pi_{d}^{\prime}\left(w-h_{b} w\right)-\pi_{c}^{\prime}\left(v-g_{a} v\right) \otimes \pi_{d}^{\prime} \pi_{b}^{\prime} h_{b} w=\pi_{c}^{\prime} g_{a} \pi_{a}^{\prime} v \otimes w=g_{a}^{c} v \otimes w$.
(iii) If $a<c$ and $b>d$, the two terms $i=c$ and $j=d$ in (2.6.10.1) contribute, and we get $e_{a b}^{c d}(v \otimes w)=\pi_{c}^{\prime} g_{a} v \otimes \pi_{d}^{\prime}\left(w-h_{b} w\right)-\pi_{c}^{\prime}\left(v-g_{a} v\right) \otimes \pi_{d}^{\prime} h_{b} w=0$.
(iv) If $a=c$ and $b<d$, only the first term in the expression (2.6.10.1) contributes, and we get $e_{a b}^{c d}(v \otimes w)=v \otimes \pi_{d}^{\prime} h_{b} \pi_{b}^{\prime} w=v \otimes h_{b}^{d} w$.
(v) If $a=c$ and $b=d$, the first term and the term $j=b=d$ in (2.6.10.1) contribute, so $e_{a b}^{c d}(v \otimes w)=v \otimes \pi_{b}^{\prime} h_{b} w-\pi_{a}^{\prime}\left(v-g_{a} v\right) \otimes \pi_{b}^{\prime} h_{b} w=\pi_{a}^{\prime} g_{a} \pi_{a}^{\prime} v \otimes \pi_{b}^{\prime} h_{b} \pi_{b}^{\prime} w=g_{a}^{a} v \otimes h_{b}^{b} w$.
(vi) If $a=c$ and $b>d$, the first term and the term $j=d$ in the expression (2.6.10.1) contribute, so $e_{a b}^{c d}(v \otimes w)=v \otimes \pi_{d}^{\prime} h_{b} w-\pi_{a}^{\prime}\left(v-g_{a} v\right) \otimes \pi_{d}^{\prime} h_{b} w=\pi_{a}^{\prime} g_{a} \pi_{a}^{\prime} v \otimes \pi_{d}^{\prime} h_{b} \pi_{b}^{\prime} w=g_{a}^{a} v \otimes h_{b}^{d} w$.
(vii) If $a>c$ and $b<d$, none of the terms in (2.6.10.1) contributes, so $e_{a b}^{c d}(v \otimes w)=0$.
(viii) If $a>c$ and $b=d$, only the term $j=b=d$ in the expression (2.6.10.1) contributes, and we find $e_{a b}^{c d}(v \otimes w)=-\pi_{c}^{\prime}\left(v-g_{a} v\right) \otimes \pi_{d}^{\prime} h_{b} w=\pi_{c}^{\prime} g_{a} v \otimes \pi_{b}^{\prime} h_{b} \pi_{b}^{\prime} w=g_{a}^{c} v \otimes h_{b}^{b} w$.
(ix) If $a>c$ and $b>d$, only the term $j=d$ in the expression (2.6.10.1) contributes, and we find $e_{a b}^{c d}(v \otimes w)=-\pi_{c}^{\prime}\left(v-g_{a} v\right) \otimes \pi_{d}^{\prime} h_{b} w=\pi_{c}^{\prime} g_{a} v \otimes \pi_{d}^{\prime} h_{b} w=g_{a}^{c} v \otimes h_{b}^{d} w$.

It remains to show statement (3) of the theorem. First, let us check (for safety reasons only) that if $(a, b)$ does not occur in $\mathrm{TL}(u)$, then $v \otimes w \in V_{a}^{\prime} \otimes W_{b}^{\prime}$ is fixed under $\gamma_{u}$. There are cases to discuss: The index $a$ may occur in $\operatorname{TL}(u)$, but if so, then in a pair $(a, d)$ with $b \neq d$. Similarly, $b$ may or may not occur in $\mathrm{TL}(u)$, but if so, then in a pair $(c, b)$ with $a \neq c$. The expression for
$\gamma_{u}(v \otimes w)$ simplifies to

$$
\begin{aligned}
\gamma_{u}(v \otimes w)= & \begin{array}{ll}
v \otimes w & \text { if } a \operatorname{not} \operatorname{in} \operatorname{TL}(u) \\
\sum_{i=a+1}^{n}\left(\pi_{i}^{\prime} g_{a} v \otimes w-\pi_{i}^{\prime} g_{a} v \otimes h_{d} w\right)+v \otimes h_{d} w & \text { if }(a, d) \in \operatorname{TL}(u)
\end{array} \\
& + \begin{cases}0 & \text { if } b \operatorname{not} \text { in } \operatorname{TL}(u) \\
\sum_{j=1}^{b-1}\left(-v \otimes \pi_{j}^{\prime} h_{b} w+g_{a} v \otimes \pi_{j}^{\prime} h_{b} w\right)+v \otimes \pi_{b}^{\prime} h_{b} w-g_{c} v \otimes \pi_{b}^{\prime} h_{b} w & \text { if }(c, b) \in \operatorname{TL}(u)\end{cases}
\end{aligned}
$$

But as $d \neq b$, the vector $w \in W_{b}^{\prime}$ is invariant under $h_{d}$, and as $c \neq a$, the vector $v \in V_{a}^{\prime}$ is invariant under $g_{c}$. Hence we find $\gamma_{u}(v \otimes w)=v \otimes w$ in all cases, as we wanted. This shows that the space

$$
\begin{equation*}
\bigoplus_{(a, b) \notin \mathrm{TL}(u)} V_{a}^{\prime} \otimes W_{b}^{\prime} \quad \subseteq V \otimes W \tag{2.6.10.2}
\end{equation*}
$$

is contained in the invariants of $\gamma_{u}$ acting on $V \otimes W$, as implied by statement (3). If the monodromy operators $g_{a}^{c}$ and $h_{b}^{d}$ are sufficiently generic, then the subspace (2.6.10.2) is exactly the space of invariants, and hence statement (3) follows by a dimension count.

We now give a direct argument for statement (3). We have to show that the subspace of $V \otimes W$ defined by (2.6.10.2) is contained in the image of the cospecialisation map

$$
\begin{equation*}
H^{1}\left(\mathbb{A}^{1}, A[-1] \otimes \tau_{u} B[-1]\right) \rightarrow H^{1}\left(\mathbb{A}^{1}, A[-1] \otimes \tau_{2 r} B[-1]\right) \cong V \otimes W \tag{2.6.10.3}
\end{equation*}
$$

along the standard path $\gamma_{u}$ from $u \in S+T$ to $2 r$. To check this, pick an element

$$
x=\sum_{(a, b) \notin \mathrm{TL}(u)} v_{a} \otimes w_{b}
$$

and define a cocycle $c:\left(G_{2 r}, P_{2 r}\right) \rightarrow V \otimes W$ by setting

$$
c\left(p_{i}\right)=\left(\pi_{i}^{\prime} \otimes \mathrm{id}\right)(x) \quad \text { and } \quad c\left(q_{j}\right)= \begin{cases}\left(\pi_{i}^{\prime} \otimes \mathrm{id}\right)(x) & \text { if }(i, j) \in \mathrm{TL}(u) \\ 0 & \text { otherwise }\end{cases}
$$

for $i=1,2, \ldots, n$ and $j=1,2, \ldots, m$. The class of $c$ in $H^{1}\left(\mathbb{A}^{1}, A[-1] \otimes \tau_{2 r} B[-1]\right)$ corresponds via the isomorphism given in Definition 2.6.8 to the element $x$ of $V \otimes W$ as the computation

$$
\kappa(c)=\sum_{i=1}^{n}\left(\pi_{i}^{\prime} \otimes \mathrm{id}\right)\left(c\left(p_{i}\right)\right)-\sum_{j=1}^{m}\left(\mathrm{id} \otimes \pi_{j}^{\prime}\right)\left(c\left(q_{j}\right)\right)=\sum_{i=1}^{n}\left(\pi_{i}^{\prime} \otimes \mathrm{id}\right)(x)-\sum_{(i, j) \in \mathrm{TL}(u)}\left(\pi_{i}^{\prime} \otimes \pi_{j}^{\prime}\right)(x)=x-0
$$

confirms. In other words, $c$ is cohomologous to the standard cocycle $c_{x}$. It remains to check that the cocycle $c$ is in the image of the cospecialisation map (2.6.10.3). As in 2.5.11 define paths

$$
p_{i}^{\prime}=h_{j_{1}} h_{j_{2}} \cdots h_{j_{l}} p_{i} \quad \text { and } \quad q_{j}^{\prime}=h_{j_{1}} h_{j_{2}} \cdots h_{j_{k}} q_{j}
$$

where $l$ is the largest integer such that $j_{l}$ occurs in $\mathrm{TL}(u)$ and $i_{l}<i$, and where $k$ is the largest integer such that $j_{l}$ occurs in $\mathrm{TL}(u)$ and $j_{l}>j$. According to the discussions in 2.2.9 and 2.5.11, if we manage to show that for all $(i, j) \in \mathrm{TL}(u)$ the equality

$$
c\left(p_{i}^{\prime}\right)=c\left(q_{j}^{\prime}\right)
$$

holds, then the class of $c$ is in the image of the cospecialisation map. Indeed, for a pair $(i, j) \in \mathrm{TL}(u)$ we have $p_{i}^{\prime}=h_{j_{1}} h_{j_{2}} \cdots h_{j_{l}} p_{i}$ and $q_{j}^{\prime}=h_{j_{1}} h_{j_{2}} \cdots h_{j_{l}} q_{j}$ for the same integer $l$. Therefore, setting $h=h_{j_{1}} h_{j_{2}} \cdots h_{j_{l}}$, we find

$$
c\left(p_{i}^{\prime}\right)=c\left(h p_{i}\right)=c(h)+h c\left(p_{i}\right)=c(h)+h c\left(q_{j}\right)=c\left(h q_{j}\right)=c\left(q_{j}^{\prime}\right)
$$

as we wanted to show. This finishes the proof of the theorem.
2.6.11. - Theorem 2.6.2 allows us to explicitly compute convolution powers of an object $A$ of $\mathbf{P e r v}_{0}$, and thus it allows us in principle to determine the type and multiplicity of subobjects of any convolution power $A * A * \cdots * A$. This is of course crucial in determining the tannakian fundamental group of $A$ in the tannakian category $\mathbf{P e r v}_{0}$, but to do so we also need to understand the dual $A^{\vee}=\Pi \mathbb{D}\left([-1]^{*} A\right)$ of $A$. There are now two ways to compute the dual explicitly, in the spirit of Theorem 2.6.2. One way is similar to what was used in the proof of 2.6.2, that is, understanding how $\Pi, \mathbb{D}$ and $[-1]^{]}$ast act on standard loops and paths and then expressing the global monodromy of $A^{\vee}$ in terms of an action on cocycles. A second possibility is to guess the form of $A^{\vee}$, and to verify that the guess is correct by writing down the evaluation pairing $A^{\vee} * A \rightarrow E(0)$. We choose the latter.

Theorem 2.6.12. - Let $A$ be an object of $\operatorname{Perv}_{0}$ and set $V=\Psi_{\infty}(A)$. Let $S=\left\{s_{1}, \ldots, s_{n}\right\}$ be the set of singularities of $A$ in western reading order. With respect to standard paths, let

$$
V=\bigoplus_{i=1}^{n} V_{i}^{\prime}, \quad g \in \operatorname{End}(V)
$$

be the vanishing cycles decomposition and endomorphism encoding the global monodromy of $A$. Write $\pi_{i}^{\prime}: V \rightarrow V_{i}^{\prime} \subseteq V$, and set $g_{a}^{c}=\pi_{c}^{\prime} \circ g \circ \pi_{a}^{\prime}$. The dual object $A^{\vee}=\Pi \mathbb{D}\left([-1]^{*} A\right)$ has the following description in these terms:
(1) The set of singularities is $-S=T=\left\{t_{1}, \ldots, t_{n}\right\}$ with $t_{i}=-s_{n+1-i}$.
(2) The nearby fibre at infinity is $\Psi_{\infty}\left(A^{\vee}\right)=W=\operatorname{Hom}(V, \mathbb{Q})$, the vector space dual to $V$.
(3) The vanishing cycles decomposition is given by

$$
W=\bigoplus_{i=1}^{n} W_{i}^{\prime}, \quad W_{i}^{\prime}=\operatorname{Hom}\left(V_{n+1-i}^{\prime}, \mathbb{Q}\right)
$$

(4) The endomorphism $h \in \operatorname{End}(W)$ describing the global monodromy of the local system underlying $A^{\vee}$ is uniquely determined by $h_{a}^{c}=\pi_{c}^{\prime} \circ h \circ \pi_{a}^{\prime}$ given as follows

$$
h_{a}^{c}=
$$

The evaluation map $A * A^{\vee} \rightarrow E(0)$ and coevaluation map $E(0) \rightarrow A * A^{\vee}$ correspond to the evaluation map $V \otimes W \rightarrow \mathbb{Q}$, respectively coevaluation map $\mathbb{Q} \rightarrow V \otimes W$.

Proof. Let us write $B$ for the object of $\operatorname{Perv}_{0}$ described in the statement of the theorem. We may calculate $A * B$ using Theorem 2.6.2, and all we have to do to prove that $B$ is dual to $A$ is to verify that the evaluation map $\varepsilon: V \otimes W \rightarrow \mathbb{Q}$ comes from a morphism in $\mathbf{P e r v}_{0}$, meaning that $\varepsilon$ is compatible with the vanishing cycle decomposition and with monodromy.

Example 2.6.13. - We end this section with a concrete example in which we compute an additive convolution, illustrating Theorem 2.6.2. Consider the polynomial

$$
f(x)=x^{5}-5 x \in \mathbb{C}[x]
$$

of degree 5 , and regard $f$ as a potential on the variety $X=\mathbb{A}^{1}$. As we have seen in Example 2.3.11, the object

$$
A=\left(f_{*} \underline{\mathbb{Q}} / \underline{\mathbb{Q}}\right)[1]=\Pi\left(f_{*} \underline{\mathbb{Q}}\right)
$$

belongs to $\operatorname{Perv}_{0}$. Its singularities are the set $S$ of critical values of $f$. We aim to describe the monodromy representation and vanishing cycles decomposition of $A$ and compute the convolution $A * A$ in these terms. We order $S$ in western reading order, that is:

$$
S=\{4 i,-4,4,-4 i\}=\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}
$$

Standard loops $g_{1}, g_{2}, g_{3}, g_{4}$ around these points, generating $G=\pi_{1}(\overline{\mathbb{C}} \backslash S, 1 \infty)$ are illustrated in Figure 2.6.14. Notice that $g_{2}$ stays above $g_{3}$, as per convention of standard paths laid out in 2.5.6. The monodromy representation associated with $f_{*} \underline{\mathbb{Q}}$ is the permutation representation associated


Figure 2.6.14. Standard paths $g_{i}$ around $s_{i}$
with the topological covering of degree 5

$$
f: \overline{\mathbb{C}} \backslash\{1, i,-1,-i\} \rightarrow \overline{\mathbb{C}} \backslash S
$$

The fibre at a point $z \in \mathbb{C}$ of this covering is the set of roots of the polynomial $f(x)-z$. The fibre at $1 \infty$ is the set $-\mu_{5} \infty$, where $\mu_{5}$ stands for fifth roots of unity. If we enumerate fifth roots of unity in the standard way by $\exp (2 \pi i k / 5)$ for $k=1,2,3,4,5$, the permutation representation $\rho: G \rightarrow \mathfrak{S}_{5}$ is given by $\rho\left(g_{1}\right)=(34), \rho\left(g_{2}\right)=(35), \rho\left(g_{3}\right)=(23)$ and $\rho\left(g_{4}\right)=(12)$. This can be seen by tracing the roots of $f(x)-p_{i}(t)$ as $t$ runs from 0 to 1 .


Figure 2.6.15. Trajectories of the roots of $f(x)-p_{i}(t)$ as $t$ runs from 0 to 1

That $\rho\left(g_{i}\right)$ is a transposition stems from the fact that at the critical value $s_{i}=p_{i}(0)$, the polynomial $f(x)-s_{i}$ has one double root, and three simple roots. Let $e_{1}, e_{2}, \ldots, e_{5}$ denote the canonical basis of $\mathbb{Q}\left[-\mu_{5} \infty\right]=\mathbb{Q}^{5}$. The vector $e_{1}+e_{2}+\cdots+e_{5}$ generates the global invariants of the representation of $G$ on $\mathbb{Q}^{5}$. With respect to the basis

$$
v_{1}=e_{4}, \quad v_{2}=e_{5}, \quad v_{3}=e_{1}+e_{2}, \quad v_{4}=e_{1}
$$

of $V=\mathbb{Q}^{5} /\left\langle e_{1}+\cdots+e_{5}\right\rangle$, the monodromy representation $G \rightarrow \mathrm{GL}_{4}(\mathbb{Q})$ associated with the constructible sheaf $A[-1]=f_{*} \underline{\mathbb{Q}} / \underline{\mathbb{Q}}$ is given by

$$
g_{1} \longmapsto\left(\begin{array}{cccc}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), \quad g_{2} \longmapsto\left(\begin{array}{rrrr}
1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), \quad g_{3} \longmapsto\left(\begin{array}{ccccc}
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1
\end{array}\right), \quad g_{4} \longmapsto\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1
\end{array}\right) .
$$

The vanishing cycles decomposition of $V=\Psi_{\infty}(A)$ is given by $V_{i}^{\prime}=\left\langle v_{i}\right\rangle$, and the endomorphism $g \in \operatorname{End}(V)$ is given by the matrix

$$
g=\left(\begin{array}{cccc}
-1 & -1 & -1 & 0 \\
-1 & -1 & -1 & 0 \\
-1 & -1 & -1 & 1 \\
0 & 0 & 1 & -1
\end{array}\right)=\left(\begin{array}{cccc}
g_{1}^{1} & g_{2}^{1} & g_{3}^{1} & g_{4}^{1} \\
g_{1}^{2} & g_{2}^{2} & g_{3}^{2} & g_{4}^{2} \\
g_{1}^{3} & g_{2}^{3} & g_{3}^{3} & g_{4}^{3} \\
g_{1}^{4} & g_{2}^{4} & g_{3}^{4} & g_{4}^{4}
\end{array}\right)
$$

made from the relevant columns of $g_{i}$. We now want to compute the convolution $A * A$ with the help of Theorem 2.6.2. The set of singularities of $A * A$ is the set

$$
S+S=\{8 i,-4+4 i, 4+4 i,-8,0,8,-4-4 i, 4-4 i,-8 i\}
$$

which we already considered in Example 2.5.10. The endomorphism $e \in \operatorname{End}(V \otimes V)$ is, with respect to the lexicographically ordered basis $v_{1} \otimes v_{1}, v_{1} \otimes v_{2}, \ldots, v_{4} \otimes v_{4}$ is

$$
e=\left(\begin{array}{ccclll}
e_{11}^{11} & e_{12}^{11} & e_{13}^{11} & \cdots & e_{44}^{11} \\
e_{11}^{12} & e_{12}^{12} & e_{13}^{12} & \cdots & e_{44}^{12} \\
e_{11}^{13} & e_{12}^{13} & e_{13}^{13} & \cdots & e_{44}^{13} \\
\vdots & \vdots & \vdots & & \vdots \\
e_{11}^{44} & e_{12}^{11} & e_{13}^{11} & \cdots & e_{44}^{44}
\end{array}\right)=\left(\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & -1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 \\
-1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \\
-1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1
\end{array}\right) .
$$

As $\mathrm{TL}(0)=\{(1,4),(2,3),(3,2),(4,1)\}$, the vanishing cycles at 0 correspond to the subspace of $V \otimes V$ generated by the vectors $v_{1} \otimes v_{4}, v_{2} \otimes v_{3}, v_{3} \otimes v_{2}$ and $v_{4} \otimes v_{1}$. The matrix $e_{0}$ of the monodromy of $A * A$ around the standard loop around $0 \in S+S$ can thus be read from the matrix $e$ as follows: Replace all columns $\left(e_{a b}^{* *}\right)$ of $e$ with $(a, b) \notin \mathrm{TL}(0)$ with the corresponding columns of
the identity matrix - et voilà
stands the monodromy operator acting on $V \otimes V=\Psi_{\infty}(A * A)$. Substracting the identity from $e$ and deleting all columns $\left(e_{a b}^{* *}\right)$ with $(a, b) \notin \mathrm{TL}(0)$ yields the matrix $\left.\left(e_{0}-\mathrm{id}\right)\right|_{\Phi_{0}(A * A)}$ displayed above. Its kernel corresponds to the largest trivial subobject of $A * A$, that is to say, the image of the canonical map $\mathcal{H o m}(E(0), A * A) \rightarrow A * A$ given by evaluation. This matrix has rank three, in fact, the vector

$$
\left(v_{1} \otimes v_{4}+v_{2} \otimes v_{3}\right)-\left(v_{3} \otimes v_{2}+v_{4} \otimes v_{1}\right)
$$

is annihilated by it. In tannakian terms, once introduced, this means that the tannakian fundamental group $G_{A} \subseteq \mathrm{GL}_{4}$ of $A$ fixed a nondegenerate alternating bilinear form. This is no surprise: the dual of the exponential motive $M$ of the form $M=H^{1}\left(\mathbb{A}^{1}, f\right)$ is given by $M^{\vee}=H^{1}\left(\mathbb{A}^{1},-f\right)$. The polynomial $f(x)=x^{5}-5 x$ is odd, thus $M$ is isomorphic to its dual via the automorphism $x \longmapsto-x$ of $\mathbb{A}^{1}$, and the perverse sheaf $A$ we considered here is the perverse realisation of $M$.

### 2.7. Monodromic vector spaces

In this section, we introduce a local variant of the category $\mathbf{P e r v}_{0}$, which we call the category of monodromic vector spaces. In essence, a monodromic vector space is just a vector space with an automorphism, but viewed as a perverse sheaf with vanishing cohomology on a small disk. The tensor product of monodromic vector spaces is given by additive convolution. We will show that this category is a tannakian category, equivalent to the category of vector spaces with an automorphism. There are two isomorphic neutral fibre functors for it, the nearby fibre at infinity functor $\Psi_{\infty}$, and the vanishing cycles functor $\Phi_{0}$. Among all things to check, the main issue is to show that $\Psi_{\infty}$ and $\Phi_{0}$ are compatible with associativity and commutativity constraints.

Definition 2.7.1. - We call category of monodromic vector spaces and denote by Vec ${ }^{\mu}$ the full tannakian subcategory of $\operatorname{Perv}_{0}$ consisting of those objects whose only singularity is $0 \in \mathbb{C}$.
2.7.2. - There are several interesting enrichments of the category Vec ${ }^{\mu}$, which we will examine in detail in due time. For instance, one may consider monodromic $\ell$-adic Galois representations, which are defined as lisse, tamely ramified $\ell$-adic sheaves on a formal punctured neighbourhood of $0 \in \mathbb{A}^{1}$. That these can be extended to lisse $\ell$-adic sheaves on $\mathbb{G}_{m}$ was shown by Katz in [54]. We
can also enrich local systems to variations of Hodge structures on a punctured disk, so to obtain the category of monodromic Hodge structures. These structures made their first appearance in [76]. In light of the existence of these enrichments, we try to separate as much as possible topological arguments as in the previous section from arguments that can be formulated in terms of the six functors formalism.
2.7.3. - Consider a perverse sheaf $A$ on a disk $D=\{z \in \mathbb{C}| | z \mid<\varepsilon\}$, and suppose that 0 is the only singularity of $A$. There is a unique way of extending $A$ to a perverse sheaf on $\mathbb{C}$ whose only singularity is $0 \in \mathbb{C}$. In this section, the distinction between such a sheaf $A$ on $D$ and its extension to the whole complex plane is irrelevant. In particular, we shall think of monodromic vector spaces as being perverse sheaves $A$ defined on some disk of unspecified but ideally small size, locally constant outside 0 and with trivial fibre at 0 .

As an abelian category, $\mathbf{V e c}^{\mu}$ is equivalent to the category of local systems on a punctured disk, or alternatively, to the category $\operatorname{Rep}(\mathbb{Z})$ of vector spaces with an automorphism. Equivalences of categories inverse to each other are the functors

$$
\Phi_{0}: \operatorname{Vec}^{\mu} \rightarrow \boldsymbol{\operatorname { R e p }}(\mathbb{Z}) \quad \text { and } \quad(-)_{!}[1]: \boldsymbol{\operatorname { R e p }}(\mathbb{Z}) \rightarrow \mathbf{V e c}^{\mu}
$$

sending an object of $\mathbf{V e c}^{\mu}$ to its vanishing cycles near 0 , respectively sending a representation of $\mathbb{Z}$ corresponding to a local system $L$ on $\mathbb{C} \backslash\{0\}$ to the perverse sheaf $j!L[1]$, where $j: \mathbb{C} \backslash\{0\} \rightarrow \mathbb{C}$ is the inclusion, see example 2.3.4. Notice that, when restricted to $\mathrm{Vec}^{\mu}$, the functor of vanishing cycles $\Phi_{0}$ agrees with the functor of nearby cycles $\Psi_{0}$, and the functor $\Psi_{\infty}$ is obtained as the composite of $\Phi_{0}$ with the functor $\operatorname{Rep}(\mathbb{Z}) \rightarrow \operatorname{Rep}(\mathbb{Z})$ sending a vector space with automorphism ( $V, \rho$ ) to ( $V, \rho^{-1}$ ).

Theorem 2.7.4. - The functor of vanishing cycles $\Phi_{0}: \operatorname{Vec}^{\mu} \rightarrow \boldsymbol{\operatorname { R e p }}(\mathbb{Z})$ and its inverse $(-)![1]: \operatorname{Rep}(\mathbb{Z}) \rightarrow \mathbf{V e c}^{\mu}$ are equivalences of tannakian categories.
2.7.5. - Let us spell out in detail what the statement of Theorem 2.7.4 is, and at the same time outline its proof. First of all, for any two objects $A$ and $B$ of $\mathbf{V e c}^{\mu}$, there is a canonical isomorphism of vector spaces

$$
\begin{equation*}
\alpha_{A, B}: \Phi_{0}(A) \otimes \Phi_{0}(B) \xrightarrow{\cong} \Phi_{0}(A * B) \tag{2.7.5.1}
\end{equation*}
$$

which is functorial in $A$ and $B$. We have already constructed such an isomorphism in 2.6.8. Next, we have to check that $\alpha_{A, B}$ is compatible with the monodromy automorphisms on both sides of (2.7.5.1), which is an easy consequence of Theorem 2.6.2. Finally, we have to check that the construction of $\alpha_{A, B}$ is compatible with associativity and commutativity constraints, which turns out to be a non-trivial issue. For any two objects $A$ and $B$ of $\mathbf{V e c}^{\mu}$, the diagram

has to commute, where x are commutativity constraints, and for any three objects $A, B$ and $C$ of $\mathbf{V e c}^{\mu}$, the diagram

$$
\begin{align*}
& \Phi_{0}(A) \otimes \Phi_{0}(B) \otimes \Phi_{0}(C) \xrightarrow{\alpha_{A, B} \otimes \mathrm{id}} \Phi_{0}(A * B) \otimes \Phi_{0}(C) \\
& \begin{array}{c}
\text { id } \otimes \alpha_{B, C} \\
\Phi_{0}(A) \otimes \Phi_{0}(B * C) \xrightarrow{\alpha_{A, B * C}} \Phi_{0}(A * B * C)
\end{array} \downarrow^{\downarrow}{ }^{\alpha_{A * B, C}} \tag{2.7.5.3}
\end{align*}
$$

has to commute. In the upper left and lower right corner of (2.7.5.3), the associativity constraints for the usual tensor product of vector spaces, respectively for the additive convolution are hidden. The devil made these diagrams! In order to deal with them, we shall study for $n=1,2,3$ the categories $\mathbf{C}(n)$ of sheaves on $\mathbb{A}^{n}$ which are zero on the coordinate planes and local systems outside the coordinate planes, and describe explicitly the following functors between them:

$$
\mathbf{C}(3) \xrightarrow[R(\operatorname{sum} \times \mathrm{id})_{*}]{R\left(\mathrm{id} \times \mathrm{sum}_{*}\right.} \mathbf{C}(2) \xrightarrow{R \mathrm{sum}_{*}} \mathbf{C}(1) .
$$

Proposition 2.7.6. - For any two objects $A$ and $B$ of $\mathbf{V e c}^{\mu}$, the isomorphism $\alpha_{A, B}$ is compatible with monodromy operators.

Proof. Set $A=j!L[1]$ and $B=j_{!} M[1]$ for local systems $L$ and $M$ on $\mathbb{C} \backslash\{0\}$. The only singularity of $A * B$ is 0 , and hence its fibre at 0 vanishes and $A * B$ is uniquely determined by a local system on $\mathbb{C} \backslash\{0\}$. We have to show that this local system is $L \otimes M$. This is a simple instance of Theorem 2.6.2 where only the case ( $v$ ) occurs. For convenience, we give a direct proof.

Fix $r>0$, and let $p$ and $q$ be standard paths in $\overline{\mathbb{C}}$ from 0 to $1 \infty$ and from $2 r$ to $1 \infty$ respectively, and let $g$ and $h$ be the corresponding positively oriented generators of $G=\pi_{1}(\overline{\mathbb{C}} \backslash\{0,2 r\}, 1 \infty)$. Let $\gamma$ be the positively oriented generator of $\pi_{1}(\mathbb{C} \backslash\{0\}, 2 r)$. The braid action on path spaces is given


Figure 2.7.16. Loops $g$ and $h$ (left) and $\gamma^{-1}$ (right)
by $\beta\left(\gamma^{-1}\right)(p)=h p$ and $\beta\left(\gamma^{-1}\right)(q)=h g q$. Let us denote by $V$ and $W$ the fibres of $L$ and $M$ at $r$. The fibre at $r$ of $L \otimes \tau_{2 r}^{*} M$ is then $V \otimes W$. We can identify the fibre

$$
(A * B)[-1]_{2 r} \cong H^{1}\left(\mathbb{A}^{1}, \otimes \tau_{2 r}^{*} j_{!} M\right)
$$

with $V \otimes W$ via the map sending $v \otimes w$ to the class of the unique cocycle $c_{v \otimes w}=c$ satisfying $c(p)=v \otimes w$ and $c(q)=0$. This identification is the one we used to produce the isomorphism $\alpha_{A, B}$ in Definition 2.6.8. According to Proposition 2.6.5, the monodromy action of $\gamma$ on $H^{1}\left(\mathbb{A}^{1}, j!L \otimes \tau_{2 r}^{*} j_{!} M\right)$ sends the class of the cocycle $c$ to the cocycle $c^{\prime}=c \circ \beta\left(\gamma^{-1}\right)$, whose values at $p$ and $q$ are

$$
c^{\prime}(p)=c\left(\beta\left(\gamma^{-1}\right)(p)\right)=c(h p)=c(h)+h c(p)
$$

and

$$
c^{\prime}(q)=c\left(\beta\left(\gamma^{-1}\right)(q)\right)=c(h g q)=c(h)+h c(g)+h g c(q) .
$$

We have $v \otimes w=c(p)=c(g p)=c(g)+g c(p)$, and hence $c(g)=v \otimes w-g v \otimes w$ and by the same reasoning $c(h)=0$, and hence

$$
c^{\prime}(p)=v \otimes h w \quad \text { and } \quad c^{\prime}(q)=v \otimes h w-g v \otimes h w
$$

Adding to $c^{\prime}$ the coboundary $b$ defined by $b(p)=b(q)=-v \otimes h w+g v \otimes h w$ we find that the cocycle $c^{\prime \prime}=c^{\prime}+b$ with values

$$
c^{\prime \prime}(p)=g v \otimes h w \quad \text { and } \quad c^{\prime \prime}(q)=0
$$

represents the same cohomology class as $c^{\prime}$. Hence the equality $\left[c_{v \otimes w} \circ \beta\left(\gamma^{-1}\right)\right]=\left[c_{g v \otimes h w}\right]$ in cohomology, which is what we wanted to show.

REmark 2.7.7. - The reader with a fondness for stars and shrieks might wonder at this point whether there is a six operations proof of Proposition 2.7.6. As it turns out, such a proof cannot exist. The reason for that is that the statement of the proposition is false in other contexts with six functor formalism, for example in the framework of mixed Hodge modules.
2.7.8. - For each integer $n \geqslant 1$, let $\mathbf{C}(n)$ be the category of sheaves on $\mathbb{A}^{n}$ which are constructible with respect to the stratification given by coordinate planes. We give a combinatorial description of $\mathbf{C}(n)$. We equip the set $\{0,1\}^{n}$ with its natural partial order, and let $I(n)$ be the category whose objects are the elements of $\{0,1\}^{n}$, and whose morphisms from $\alpha$ to $\beta$ are

$$
\operatorname{Mor}(\beta, \alpha)= \begin{cases}\left(\beta \cdot \mathbb{Z}^{n}\right) p_{\beta \alpha} & \text { if } \beta \leqslant \alpha \\ \varnothing & \text { otherwise }\end{cases}
$$

where $p_{\beta \alpha}$ is just a symbol for recovering the source and the target of a morphism. The identity of $\alpha$ is $0 p_{\alpha \alpha}$, and the composition law is given by

$$
v p_{\beta \alpha} \circ u p_{\gamma \beta}=\gamma \cdot(u+v) p_{\gamma \alpha}
$$

for $\gamma \leqslant \beta \leqslant \alpha$. To give a functor from $I(n)$ to the category of vector spaces is to give for every $\alpha \in\{0,1\}^{n}$ a vector space $V_{\alpha}$, maps $V_{\beta} \rightarrow V_{\alpha}$ for $\beta \leqslant \alpha$ corresponding to the morphisms $0 p_{\beta \alpha}$, and for each non-zero coordinate of $\alpha$ an automorphism of $V_{\alpha}$. These automorphisms are required to commute with each other and be compatible with the maps $V_{\beta} \rightarrow V_{\alpha}$ in the appropriate way. The category $\mathbf{C}(n)$ is canonically equivalent to the category of functors from $I(n)$ to finite-dimensional vector spaces. With a sheaf $F$, we associate the functor $I(n) \rightarrow \operatorname{Vec}_{\mathbb{Q}}$ given by the collection of vector spaces $V_{\alpha}$, where $V_{\alpha}$ is the fibre of $F$ at $\alpha \in\{0,1\}^{n} \subseteq \mathbb{C}^{n}$, the cospecialisation maps $V_{\beta} \rightarrow V_{\alpha}$
for straight paths from $\beta$ to $\alpha$, and the monodromy operators on $V_{\alpha}$. In particular for $n=1,2,3$, an object of $\mathbf{C}(n)$ is a commutative diagram of vector spaces of the following shape:


These vector spaces $V_{\alpha}$ come equipped with commuting automorphisms, one for every non-zero coordinate of $\alpha$. For example, on $V_{110}$ we are given commuting automorphisms $g_{\bullet 10}$ and $g_{1 \bullet 0}$. For the given map $p_{1 \bullet 0}: V_{100} \rightarrow V_{110}$, the relations

$$
p_{1 \bullet 0} g_{\bullet 00}=g_{\bullet 10} p_{1 \bullet 0} \quad \text { and } \quad p_{1 \bullet 0}=g_{1 \bullet 0} p_{1 \bullet 0}
$$

are satisfied. There is a canonical action of the symmetric group $\mathfrak{S}_{n}$ on $I(n)$, and hence on $\mathbf{C}(n)$. In terms of diagrams (2.7.8.1) this action permutes indices; in terms of sheaves this same action is given by push-forward along the corresponding permutation action on $\mathbb{A}^{n}$.
2.7.9. - We shall now describe the functor $R \operatorname{sum}_{*}: D_{c}^{b}\left(\mathbb{A}^{2}\right) \rightarrow D_{c}^{b}\left(\mathbb{A}^{1}\right)$ on objects of $\mathbf{C}(2)$ in the combinatorial terms introduced in 2.7.8. Let $F$ be an object of $\mathbf{C}(2)$, corresponding to a commutative diagram of vector spaces

and automorphisms $g_{\bullet 0} \in \mathrm{GL}\left(V_{10}\right), g_{0 \bullet} \in \mathrm{GL}\left(V_{01}\right)$, and $g_{\bullet 1}, g_{1 \bullet} \in \mathrm{GL}\left(V_{11}\right)$ satisfying appropriate compatibilities. Set $g_{\bullet \bullet}=g_{1} g_{\bullet 1}$ and $p_{\bullet \bullet}=p_{\bullet} p_{0 \bullet}$. We associate with $F$ the two-term complex

$$
S(F)=\left[S^{0}(F) \rightarrow S^{1}(F)\right]
$$

in $\mathbf{C}(1)$, in degrees 0 and 1 , where $S^{0}(F)$ and $S^{1}(F)$ correspond to the columns in the diagram

with cospecialisation map $p_{\bullet}\left(v_{00}\right)=\left(p_{\bullet 0}\left(v_{00}\right), p_{0 \bullet}\left(v_{00}\right), p_{\bullet \bullet}\left(v_{00}\right)\right)$ together with the monodromy automorphisms $\left(g_{\bullet 0}, g_{0 \bullet}, g_{\bullet \bullet}\right)$ of $V_{10} \oplus V_{01} \oplus V_{11}$ and $\left(g_{\bullet \bullet}, g_{\bullet \bullet}\right)$ of $V_{11} \oplus V_{11}$. The differential $d$ is given by

$$
d\left(v_{10}, v_{01}, v_{11}\right)=\left(v_{11}-p_{1} \bullet\left(v_{10}\right), v_{11}-p_{\bullet 1}\left(v_{01}\right)\right)
$$

which makes the whole diagram commute, and is compatible with the automorphisms. As we have already noticed, the transposition $\sigma \in \mathfrak{S}_{2}$ acts on $\mathbf{C}(2)$ by permuting indices or coordinates. This action induces a natural isomorphism $S(F) \cong S\left(\sigma_{*} F\right)$ given by

with $s_{0}\left(v_{10}, v_{01}, v_{11}\right)=\left(v_{01}, v_{10}, v_{11}\right)$ and $s_{1}\left(v, v^{\prime}\right)=\left(v^{\prime}, v\right)$.

LEMMA 2.7.10. - There is a canonical and natural isomorphism $R_{\operatorname{sum}_{*}}(F) \cong S(F)$ in the derived category of constructible sheaves on $\mathbb{A}^{1}$.

Proof. There is an obvious natural isomorphism $\operatorname{sum}_{*} F \cong H^{0}(S(F))$. In order to prove the lemma, we need to show that for any object $F$ of $\mathbf{C}(2)$ given in terms of data (2.7.9.1), there is a natural isomorphism

$$
\left(R \operatorname{sum}_{*}(F)\right)_{1} \cong\left[V_{10} \oplus V_{01} \oplus V_{11} \xrightarrow{d} V_{11} \oplus V_{11}\right]
$$

in the derived category of vector spaces. The fibre $\left(R \operatorname{sum}_{*}(F)\right)_{1}$ is the complex computing the cohomology of $F$ restricted to the affine line $\mathbb{A}^{1} \simeq \operatorname{sum}^{-1}(1)$, and hence Lemma 2.2.4 and the particular presentation of the cochain complex (2.2.6.2) provide the canonical isomorphism.
2.7.11. - We can now reinterpret the isomorphism $\alpha_{A, B}$ from Proposition ?? and reprove Proposition 2.7.6 using Lemma 2.7.10. Given objects $A$ and $B$ of $\mathbf{V e c}^{\mu}$, the constructible sheaves $A[-1], B[-1]$, and $A[-1] \boxtimes B[-1]$ correspond to objects

in $\mathbf{C}(1)$ and $\mathbf{C}(2)$ respectively, with $V=\Phi_{0}(A)$ and $W=\Phi_{0}(B)$. On $V$ and $W$ we are given monodromy automorphisms $g_{V}$ and $g_{W}$, and on $V \otimes W$ we are given the two commuting automorphisms
$g_{\bullet 1}=g_{V} \otimes \mathrm{id}_{W}$ and $g_{\bullet \bullet}=\mathrm{id}_{V} \otimes g_{W}$. According to Lemma 2.7.10, $A * B$ is the complex

in $\mathbf{C}(1)$, in degrees -2 and -1 , where $d$ is the diagonal map, and the monodromy automorphism on $V \otimes W$ is $g_{\bullet \bullet}=g_{V} \otimes g_{W}$. The isomorphism $\alpha_{A, B}$ is in these terms the map sending the element $v \otimes w$ of $V \otimes W=\Phi_{0}(A) \otimes \Phi_{0}(B)$ to the class of $(v \otimes w, 0 \otimes 0)$ in $\Phi_{0}(A * B)=$ coker $d$.

Proposition 2.7.12. - The diagram 2.7.5.2 commutes.
Proof. Pick objects $A$ and $B$ of $\mathbf{V e c}^{\mu}$ and set $V=\Phi_{0}(A)$ and $W=\Phi_{0}(B)$. Recall from 2.4.10 that the commutativity constraint in $\mathbf{P e r v}_{0}$ is the isomorphism $\mathbf{x}: A * B \rightarrow B * A$ given by the following composition of natural isomorphisms:

$$
\begin{aligned}
A * B & =R \operatorname{sum}_{*}\left(\operatorname{pr}_{1}^{*} A \otimes \operatorname{pr}_{2}^{*} B\right) & & \text { definition } \\
& =R \operatorname{sum}_{*}\left(\operatorname{pr}_{2}^{*} B \otimes \operatorname{pr}_{1}^{*} A\right) & & a \otimes b \longmapsto-b \otimes a \\
& =R \operatorname{sum}_{*} \sigma_{*}\left(\operatorname{pr}_{1}^{*} B \otimes \operatorname{pr}_{2}^{*} A\right) & & \\
& =B * A & & \text { sum } \circ \sigma=\operatorname{sum}
\end{aligned}
$$

The catch here is the minus sign, coming from the commutativity constraint in $D_{c}^{b}\left(\mathbb{A}^{2}\right)$ and the fact that $A$ and $B$ sit in degree -1 . Rephrased in combinatorial terms, this string of isomorphisms corresponds to the following isomorphism of complexes in $\mathbf{C}(1)$ :

$$
\begin{aligned}
A * B & =[V \otimes W \xrightarrow{d}(V \otimes W) \oplus(V \otimes W)] & & \text { definition } \\
& =[W \otimes V \xrightarrow{d}(W \otimes V) \oplus(W \otimes V)] & & v \otimes w \longmapsto-w \otimes v \\
& =[W \otimes V \xrightarrow{d}(W \otimes V) \oplus(W \otimes V)] & & \left(s_{0}, s_{1}\right) \text { from (2.7.9.2) } \\
& =B * A & &
\end{aligned}
$$

Here, each of the two term complexes is in fact a complex of the shape (2.7.11.1), we omitted the zeroes. The diagram (2.7.5.2) takes the following shape:

where the right-hand vertical map is induced by $\left(v \otimes w, v^{\prime} \otimes w^{\prime}\right) \longmapsto\left(-w^{\prime} \otimes v^{\prime},-w \otimes v\right)$. The map $\alpha_{A, B}$ sends $v \otimes w$ to the class of $(v \otimes w, 0 \otimes 0)$ and the map $\alpha_{B, A}$ sends $w \otimes v$ to the class of $(w \otimes v, 0 \otimes 0)$. Commutativity of the diagram now follows from the equality

$$
(0 \otimes 0,-w \otimes v)=(w \otimes v, 0 \otimes 0)
$$

in $\operatorname{coker}(W \otimes V \rightarrow(W \otimes V) \oplus(W \otimes V))$.
2.7.13. - Our next task is to describe the functors $R(\mathrm{id} \times \operatorname{sum})_{*}: D_{c}^{b}\left(\mathbb{A}^{3}\right) \rightarrow D_{c}^{b}\left(\mathbb{A}^{2}\right)$ and also $R\left(\mathrm{sum}^{3}\right)_{*}: D_{c}^{b}\left(\mathbb{A}^{3}\right) \rightarrow D_{c}^{b}(\mathbb{A})$ on objects of $\mathbf{C}(3)$ in similar terms. Fix an object $F$ of $\mathbf{C}(3)$ given by a diagram as on the far right of (2.7.8.1). We associate with it two-term complexes $S_{1,2}(F)$ and $S_{2,1}(F)$ in $\mathbf{C}(2)$, and a three-term complex $S_{3}(F)$ in $\mathbf{C}(1)$. Analogously to Lemma 2.7.10, there will be natural isomorphisms

$$
\begin{align*}
S_{1,2}(F) & \cong R(\mathrm{id} \times \mathrm{sum})_{*}(F)  \tag{2.7.13.1}\\
S_{2,1}(F) & \cong R(\operatorname{sum} \times \mathrm{id})_{*}(F)  \tag{2.7.13.2}\\
S_{3}(F) & \cong R \operatorname{sum}_{*}^{3}(F) \tag{2.7.13.3}
\end{align*}
$$

in $D_{c}^{b}\left(\mathbb{A}^{2}\right)$ and $D_{c}^{b}\left(\mathbb{A}^{1}\right)$ respectively, where sum ${ }^{3}: \mathbb{A}^{3} \rightarrow \mathbb{A}^{1}$ is the summation of all three coordinates. We define $S_{1,2}(F)$ to be the two-term complex

in $\mathbf{C}(2)$, where the differential from left to right is given by

$$
d\left(v_{110}, v_{101}, v_{111}\right)=\left(v_{111}-p_{11 \bullet}\left(v_{110}\right), v_{111}-p_{1 \bullet 1}\left(v_{101}\right)\right)
$$

on $V_{110} \oplus V_{101} \oplus V_{111}$. The two monodromy operators on the source of $d$ are $\left(g_{\bullet 10}, g_{\bullet 01}, g_{\bullet 11}\right)$ and $\left(g_{1 \bullet 0}, g_{10 \bullet}, g_{1 \bullet \bullet}\right)$, and the two monodromy operators on the target of $d$ are $\left(g_{\bullet} 11, g_{\bullet 11}\right)$ and $\left(g_{1 \bullet \bullet}, g_{1 \bullet \bullet}\right)$, where $g_{1 \bullet \bullet}=g_{11 \bullet} \circ g_{1 \bullet 1}$. There is only one sensible way of defining the rest of the data in the complex defining $S_{1,2}(F)$ in order to have a natural isomorphism (2.7.13.1). Later, we will use the complex $S_{1,2}(F)$ only in the special case where all spaces $V_{\alpha}$, except possibly $V_{111}$ are zero, so in good conscience we leave the remaining details to the reader. The definition of $S_{1,2}(F)$ is similar. Applying the construction $S(-)$ from 2.7 .9 to the complex $\left[S_{1,2}^{0}(F) \rightarrow S_{1,2}^{1}(F)\right.$ ] yields a double complex in $\mathbf{C}(1)$ whose associated total complex is the three-term complex in $\mathbf{C}(1)$ which we call $S_{3}(F)$. Explicitly, this complex is

with differentials

$$
d^{0}\left(\begin{array}{c}
v_{100} \\
v_{010}, v_{001}, v_{011} \\
v_{110}, v_{101}, v_{111}
\end{array}\right)=\left(\begin{array}{r}
v_{110}-p_{\bullet \bullet 0}\left(v_{000}\right), v_{101}-p_{\bullet 0 \bullet}\left(v_{000}\right), v_{111}-p_{\bullet \bullet \bullet}\left(v_{000}\right) \\
v_{110}-p_{\bullet 10}\left(v_{010}\right), v_{101}-p_{\bullet 01}\left(v_{001}\right), v_{111}-p_{\bullet 11}\left(v_{011}\right) \\
v_{011}-p_{01 \bullet}\left(v_{010}\right), v_{011}-p_{0 \bullet 1}\left(v_{001}\right) \\
v_{111}-p_{11 \bullet}\left(v_{110}\right), v_{111}-p_{1 \bullet 1}\left(v_{101}\right)
\end{array}\right)
$$

and

$$
\begin{aligned}
d^{1}\left(\begin{array}{c}
v_{110}, v_{101}, v_{111} \\
v_{110}^{\prime}, v_{101}^{\prime}, v_{111}^{\prime} \\
v_{011}^{\prime \prime}, v_{011}^{\prime \prime \prime} \\
v_{111}^{\prime \prime}, v_{111}^{\prime \prime \prime}
\end{array}\right)= & \binom{v_{111}-p_{11}\left(v_{110}\right), v_{111}-p_{1 \bullet 1}\left(v_{101}\right)}{v_{111}^{\prime}-p_{11}\left(v_{110}^{\prime}\right), v_{111}^{\prime}-p_{1 \bullet}\left(v_{101}^{\prime}\right)} \\
& -\binom{v_{11}^{\prime \prime}, v_{111}^{\prime \prime \prime}}{v_{111}^{\prime \prime}-p_{\bullet 11}\left(v_{011}^{\prime \prime}\right), v_{111}^{\prime \prime \prime}-p_{\bullet 11}\left(v_{011}^{\prime \prime \prime}\right)} .
\end{aligned}
$$

We could as well have defined $S^{3}(F)$ as the total complex associated with the double complex that one obtains by applying $S(-)$ to the complex $\left[S_{2,1}^{0}(F) \rightarrow S_{2,1}^{1}(F)\right]$ in $\mathbf{C}(2)$, the outcome is the same.

Proposition 2.7.14. - The diagram (2.7.5.3) commutes.

Proof. Let $A, B$, and $C$ be objects of $\operatorname{Perv}_{0}$ and set $U=\Phi_{0}(A), V=\Phi_{0}(B)$, and $W=\Phi_{0}(C)$. The constructible sheaf $(A \boxtimes B \boxtimes C)[-3]$ on $\mathbb{A}^{3}$ is the object of $\mathbf{C}(3)$ corresponding to the diagram

with monodromy operators $g_{\bullet 11}=g_{U} \otimes \mathrm{id}_{V} \otimes \mathrm{id}_{W}, g_{1 \bullet 1}=\mathrm{id}_{U} \otimes g_{V} \otimes \mathrm{id}_{W}$, and $g_{11} \bullet=\mathrm{id}_{U} \otimes \mathrm{id}_{V} \otimes g_{W}$. The convolution $A * B$ is given by the two term complex

in $\mathbf{C}(1)$ placed in degrees -2 and -1 , with differential the diagonal map. The triple convolution $A * B * C=R \operatorname{sum}_{*}^{3}(A \boxtimes B \boxtimes C)$ is the three term complex $S_{3}(A \boxtimes B \boxtimes C)$

in $\mathbf{C}(1)$, placed in degrees $-3,-2$ and -1 , where $d_{U, V, W}^{0}$ is the diagonal map and

$$
d_{U, V, W}^{1}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=\left(x_{3}-x_{1}, x_{4}-x_{1}, x_{3}-x_{2}, x_{4}-x_{2}\right)
$$

In these terms, the diagram (2.7.5.3) whose commutativity we are about to check takes the following shape.


The map $\alpha_{A, B} \otimes \mathrm{id}$ is $u \otimes v \otimes w \longmapsto[u \otimes v, 0 \otimes 0] \otimes w$, and id $\otimes \alpha_{B, C}$ is given by the analogous formula $u \otimes v \otimes w \longmapsto u \otimes[v \otimes w, 0 \otimes 0]$. The isomorphisms $\alpha_{A * B, C}$ and $\alpha_{A, B * C}$ are

$$
\begin{aligned}
\alpha_{A * B, C}: & {\left[u \otimes v, u^{\prime} \otimes v^{\prime}\right] \otimes w } & \longmapsto\left[u \otimes v \otimes w, u^{\prime} \otimes v^{\prime} \otimes w, 0,0\right] \\
\alpha_{A, B * C}: & u \otimes\left[v \otimes w, v^{\prime} \otimes w^{\prime}\right] & \longmapsto\left[u \otimes v \otimes w, 0, u \otimes v^{\prime} \otimes w^{\prime}, 0\right]
\end{aligned}
$$

and hence both compositions in the square send $u \otimes v \otimes w$ to the class $[u \otimes v \otimes w, 0,0,0]$.

### 2.8. The nearby fibre at infinity and vanishing cycles as fibre functors

In this section, we prove that the nearby fibre at infinity functor $\Psi_{\infty}: \operatorname{Perv}_{0} \rightarrow \mathbf{V e c}$ is a fibre functor on the tannakian category $\operatorname{Perv}_{0}$. We already know from Proposition 2.3.7 that $\Psi_{\infty}$ is faithful and exact, and it remains to show that $\Psi_{\infty}$ is compatible with tensor products. Besides the fibre functor $\Psi_{\infty}$, there is another interesting and useful fibre functor

$$
\Phi: \operatorname{Perv}_{0} \rightarrow \operatorname{Vec}_{\mathbb{Q}}
$$

which sends an object of $\operatorname{Perv}_{0}$ to the sum over $z \in \mathbb{C}$ of its vanishing cycles at $z$. We obtain from $\Phi$ not just a vector space, but a $\mathbb{C}$-graded vector space, where each graded piece is equipped with an automorphism given by the local monodromy operator as was recalled in 2.1.19. That the functor $\Phi$ is compatible with the additive convolution, even if we enrich it to a functor

$$
\Phi: \operatorname{Perv}_{0} \rightarrow\{\mathbb{C} \text {-graded representations of } \mathbb{Z}\}
$$

is essentially the statement of the Thom-Sebastiani theorem for functions in one variable, except that we deal here with a global version of it.

## Theorem 2.8.1. - The nearby fibre at infinity $\Psi_{\infty}: \mathbf{P e r v}_{0} \rightarrow \mathbf{V e c}$ is a fibre functor.

Proof. Given an object $F[1]$ of $\operatorname{Perv}_{0}$, let us call monodromic nearby fibre at infinity the monodromic vector space

$$
\Psi_{\infty}^{\mu}(F[1])=j!u^{*} F[1]
$$

where $j: \mathbb{C} \backslash\{0\} \rightarrow \mathbb{C}$ is the inclusion and $u: \mathbb{C} \backslash\{0\} \rightarrow \mathbb{C}$ is the function $u\left(c e^{i \theta}\right)=\max \left(r, c^{-1}\right) e^{-i \theta}$ for some real $r$, larger than the absolute value of each singularity of $F$. The functor $\Psi_{\infty}$ factors as

$$
\operatorname{Perv}_{0} \xrightarrow{\Psi_{\infty}^{\mu}} \mathbf{V e c}^{\mu} \xrightarrow{\Phi_{0}} \mathbf{V e c}
$$

and by Theorem 2.7.4, the vanishing fibre functor $\Phi_{0}$ is a fibre functor on the tannakian category of monodromic vector spaces $\mathbf{V e c}^{\mu}$. It suffices thus to show that the monodromic nearby fibre at infinity functor $\Psi_{\infty}^{\mu}$ is compatible with tensor products. This is a six-operations exercise.


We have to show that for a sheaf $F \boxtimes G$ on $\mathbb{C} \times \mathbb{C}$ in the upper left corner of the diagram, we have a natural isomorphism of sheaves

$$
R \operatorname{sum}_{*}(j \times j)_{!}(u \times u)^{*}(F \boxtimes G) \cong j_{!} u^{*} R \operatorname{sum}_{*}(F \boxtimes G)
$$

on the copy of $\mathbb{C}$ sitting on the lower left corner in the diagram.

Definition 2.8.2. - We call total vanishing cycles functor the functor

$$
\begin{aligned}
\Phi: \operatorname{Perv}_{0} & \longrightarrow\{\mathbb{C} \text {-graded representations of } \mathbb{Z}\} \\
A & \longmapsto \bigoplus_{z \in \mathbb{C}} \Phi_{z}(A)
\end{aligned}
$$

where the $\mathbb{Z}$-action on the vanishing cycles $\Phi_{z}(A)$ is induced by the local monodromy on the fibre of $A$ near $z$, as explained in 2.1.19.

Theorem 2.8.3 (Thom-Sebastiani). - The total vanishing cycles functor is exact, faithful and monoidal: for all objects $A$ and $B$ of $\mathbf{P e r v}_{0}$, there exist functorial isomorphisms

$$
\begin{align*}
\Phi(A * B) & \cong \Phi(A) \otimes \Phi(B)  \tag{2.8.3.1}\\
\Phi\left(A^{\vee}\right) & \cong \Phi(A)^{\vee} \tag{2.8.3.2}
\end{align*}
$$

in the category of $\mathbb{C}$-graded representations of $\mathbb{Z}$, which are compatible with associativity, commutativity, and unit constraints.

Corollary 2.8.4. - The composite of $\Phi$ with the forgetful functor to vector spaces is a fibre functor on the tannakian category $\mathbf{P e r v}_{0}$.
2.8.5. - Before we start with the proof, let us summarise what we have to show and how we will show it. First of all, we need to check that the functor $\Phi$ is faithful and exact. This is not difficult, and part of Proposition 2.8 .6 where we show that $\Psi_{\infty}$ and $\Phi$ are isomorphic as additive functors $\operatorname{Perv}_{0} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$. The essential part of Theorem 2.8.3 is of course the existence of the isomorphisms (2.8.3.1) and (2.8.3.2). We produce them in two steps. First, we interpret vanishing
cycles as objects in the tannakian category $\mathbf{V e c}_{\mathbb{Q}}^{\mu}$ of local systems on a small punctured disk with additive convolution as tensor product. The latter is equivalent to the full tannakian subcategory of $\operatorname{Perv}_{0}$ consisting of those objects whose only singularity is at $0 \in \mathbb{C}$. If vanishing cycles are interpreted this way, it is quite straightforward to check that the total vanishing cycles functor is monoidal. The second step consists in the use of Theorem 2.7.4, which states that the category $\mathbf{V e c}_{\mathbb{Q}}^{\mu}$ is equivalent as a tannakian category to the category of representations of $\mathbb{Z}$ for the usual tensor product.

PROPOSITION 2.8.6. - There exists an isomorphism of additive functors between the nearby fibre at infinity $\Psi_{\infty}: \mathbf{P e r v}_{0} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ and the total vanishing cycles functor $\Phi: \mathbf{P e r v}_{0} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$. In particular, $\Phi$ is faithful and exact.

Proof. By virtue of Lemma 2.1.20, the vanishing cycles functor $\Phi_{z}:$ Perv $\rightarrow \mathbf{V e c}_{\mathbb{Q}}$ is exact for every $z \in \mathbb{C}$, and the inclusion $\operatorname{Perv}_{0} \rightarrow$ Perv is exact, and hence $\Phi$ is exact. Fix a finite set of singularities $S \subseteq \mathbb{C}$, and denote by $\operatorname{Perv}_{0}(S)$ the full subcategory of $\operatorname{Perv}_{0}$ consisting of those objects whose singularities are contained in $S$. We can regard $\operatorname{Perv}_{0}(S)$ as the category of representations of $\left(G, P_{S}\right)$, where $G$ is the fundamental group of $\overline{\mathbb{C}} \backslash S$ based at $1 \infty$, and $P_{s}$ is the $G$-set of paths from $s$ to $1 \infty$. Let $F[1]$ be an object of $\operatorname{Perv}_{0}(S)$ corresponding to a representation $V=\left(V,\left(V_{s}\right)_{s \in S}, \rho,\left(\rho_{s}\right)_{s \in S}\right)$. By choosing for each $s \in S$ a path $p_{s}^{0} \in P_{s}$ from $s$ to $1 \infty$ we can identify the nearby cycles of $F$ at $s$ with the fibres at $1 \infty$, that is, with the vector space $V$. In particular, vanishing cycles are in these terms identified with

$$
\Phi_{s}(F) \cong \operatorname{coker}\left(p_{s}^{0}: V_{s} \rightarrow V\right)
$$

functorially for morphisms in $\operatorname{Perv}_{0}(S)$. Since the cohomology $H^{*}\left(\mathbb{A}^{1}, F\right)$ vanishes, the diagonal morphism

$$
\Psi_{\infty}(F[1])=V \xrightarrow{\text { diag }} \bigoplus_{s \in S} \operatorname{coker}\left(p_{s}^{0}: V_{s} \rightarrow V\right)=\Phi(F[1])
$$

is an isomorphism, functorial for $F[1]$ in $\operatorname{Perv}_{0}(S)$. We have thus far constructed an isomorphism of functors

$$
\begin{equation*}
\left.\left.\Psi_{\infty}\right|_{\operatorname{Perv}_{0}(S)} \rightarrow \Phi\right|_{\operatorname{Perv}_{0}(S)} \tag{2.8.6.1}
\end{equation*}
$$

from $\operatorname{Perv}_{0}(S)$ to $\operatorname{Vec}_{\mathbb{Q}}$ depending on the choice of a path $p_{s}^{0} \in P_{s}$ for each $s \in S$. Consider now a finite set $S^{\prime} \subseteq \mathbb{C}$ containing $S$, and the corresponding fundamental group $G^{\prime}$ and $G^{\prime}$-sets of paths $P_{s}^{\prime}$ for $s \in S^{\prime}$. The canonical group homomorphism $G^{\prime} \rightarrow G$ is surjective, and also the maps $P_{s}^{\prime} \rightarrow P_{s}$ are surjective for each $s \in S$. This implies that the isomorphism of functors (2.8.6.1) can be extended to the larger subcategory $\operatorname{Perv}_{0}\left(S^{\prime}\right)$. The choice of an element

$$
p_{s}^{0} \in \lim _{s \in S} P_{s}
$$

for each $s \in \mathbb{C}$, where the limit runs over all finite subsets $S \subseteq \mathbb{C}$ containing $s$, yields an isomorphism of additive functors $\Psi_{\infty} \simeq \Phi$ as claimed.

We define monodromic vanishing cycles at $z \in \mathbb{C}$ as the functor

$$
\begin{aligned}
\Phi_{z}^{\mu}: \operatorname{Perv}_{0} & \rightarrow \mathbf{V e c}_{\mathbb{Q}}^{\mu} \\
F[1] & \longmapsto \Pi_{0}\left(\delta_{z}^{*} F[1]\right)
\end{aligned}
$$

where $\delta_{z}$ is the translation map sending a small $D$ disk around 0 to a small disk around $z$ containing no singularities of $F$ except possibly $z$, and $\Pi_{0}$ is the functor sending a perverse sheaf $A$ on $D$ to

$$
\Pi_{0}(A)=\operatorname{coker}\left(\pi^{*} \pi_{*} A \rightarrow A\right)
$$

where $\pi$ is the map from $D$ to a point. The total vanishing fibres functor as introduced in Definition 2.8.2 is thus the composite of the functor

$$
\begin{aligned}
\operatorname{Perv}_{0} & \xrightarrow{\Phi^{\mu}}\{\mathbb{C} \text {-graded monodromic vector spaces }\} \\
F[1] & \longmapsto \bigoplus_{z \in \mathbb{C}} \Phi_{z}^{\mu}(F[1])
\end{aligned}
$$

and the functor $\Phi_{0}$.

Proposition 2.8.7. - The total monodromic vanishing cycles functor is exact, faithful and monoidal: for all objects $F[1]$ and $G[1]$ of $\mathbf{P e r v}_{0}$, there exist functorial isomorphisms

$$
\begin{align*}
\Phi^{\mu}(F[1] * G[1]) & \cong \Phi^{\mu}(F[1]) \otimes \Phi^{\mu}(G[1])  \tag{2.8.7.1}\\
\Phi^{\mu}\left(F[1]^{\vee}\right) & \cong \Phi^{\mu}(G[1])^{\vee} \tag{2.8.7.2}
\end{align*}
$$

in the category of $\mathbb{C}$-graded monodromic vector spaces which are compatible with associativity, commutativity and unit constraints.

Proof. Constructing a functorial isomorphism of graded monodromic vector spaces (2.8.3.1) for objects $F[1]$ and $G[1]$ of $\operatorname{Perv}_{0}$ amounts to constructing, for any fixed $t \in S_{F}+S_{G}$, an isomorphism of monodromic vector spaces

$$
\begin{equation*}
\Phi_{t}^{\mu}(F[1] * G[1]) \xrightarrow{\cong} \bigoplus_{r+s=t} \Phi_{r}^{\mu}(F[1]) \otimes \Phi_{s}^{\mu}(G[1]) \tag{2.8.7.3}
\end{equation*}
$$

where the direct sum ranges over all pairs or singularities $(r, s) \in S_{F} \times S_{G}$ with $r+s=t$. Fix a pair $(r, s) \in S_{F} \times S_{G}$ with $r+s=t$ and disks $\delta_{r}: D_{r} \rightarrow \mathbb{C}, \delta_{s}: D_{s} \rightarrow \mathbb{C}$, and $\delta_{t}: D_{t} \rightarrow \mathbb{C}$ centered at $r, s$, and $t$. By choosing $D_{r}$ and $D_{s}$ small enough, we may suppose that the box $D_{r, s}=D_{r} \times D_{s}$ is contained in $U=\operatorname{sum}^{-1}\left(D_{t}\right)$, as pictured in Figure 2.8.17.

The tensor products on the right-hand side of (2.8.7.3) are tensor products of monodromic vector spaces, so in fact additive convolutions, namely

$$
\begin{aligned}
\Phi_{r}^{\mu}(F[1]) \otimes \Phi_{s}^{\mu}(G[1]) & =R \operatorname{sum}_{*}\left(\Pi_{0} \delta_{r}^{*} F[1] \boxtimes \Pi_{0} \delta_{s} G[1]\right) \\
& =R^{1} \operatorname{sum}_{*}\left(\delta_{r, s}^{*}(F \boxtimes G)\right)[1] \\
& =\delta_{t}^{*} R^{1} \operatorname{sum}_{*}\left(\left.(F \boxtimes G)\right|_{D_{r, s}}\right)[1]
\end{aligned}
$$

On the other hand, the monodromic vanishing cycles of $F[1] * G[1]$ near $t$ are by definition

$$
\begin{aligned}
\Phi_{t}^{\mu}(F[1] * G[1]) & =\Pi_{0} \delta_{t}^{*} R \operatorname{sum}_{*}(F \boxtimes G) \\
& =\delta_{t}^{*} R^{1} \operatorname{sum}_{*}\left(\left.(F \boxtimes G)\right|_{U}\right)[1]
\end{aligned}
$$



Figure 2.8.17. Boxes and disks
so there is a natural morphism of monodromic vector spaces

$$
p_{r, s}: \Phi_{t}^{\mu}(F[1] * G[1]) \rightarrow \Phi_{r}^{\mu}(F[1]) \otimes \Phi_{s}^{\mu}(G[1])
$$

induced by the inclusion of $D_{r, s}$ into $U$. By collecting these morphisms for all $(r, s) \in S_{F} \times S_{G}$ with $r+s=t$ we obtain a morphism as displayed in (2.8.7.3), and hence a morphism

$$
\begin{equation*}
\Phi^{\mu}(F[1] * G[1]) \longrightarrow \Phi^{\mu}(F[1]) \otimes \Phi^{\mu}(G[1]) \tag{2.8.7.4}
\end{equation*}
$$

of $\mathbb{C}$-graded monodromic vector spaces. This morphism is functorial in $A$ and $B$, and compatible with commutativity, associativity and unit constraints. Duals in $\mathbf{P e r v}_{0}$ are given by

$$
F[1]^{\vee}=\Pi\left([-1]^{*} \mathbb{D}(F[1])\right)
$$

as we saw in Proposition 2.4.9, and duals in $\operatorname{Vec}_{\mathbb{Q}}^{\mu}$ are similarly defined using $\Pi_{0}$ instead of $\Pi$. We find a functorial isomorphism

$$
\begin{equation*}
\Phi^{\mu}\left(F[1]^{\vee}\right) \longrightarrow \Phi^{\mu}(F[1])^{\vee} \tag{2.8.7.5}
\end{equation*}
$$

and it remains to show that (2.8.7.4) is an isomorphism. This follows by general tannakian nonsense from (2.8.7.5) and compatibility of (2.8.7.4) with the constraints. Indeed, an inverse to (2.8.7.5) is the composite morphism

$$
\begin{aligned}
\Phi^{\mu}(F[1]) \otimes \Phi^{\mu}(G[1]) & \rightarrow \Phi^{\mu}\left(F[1] * G[1] * G[1]^{\vee}\right) \otimes \Phi^{\mu}(G[1]) \\
& \rightarrow \Phi^{\mu}(F[1] * G[1]) \otimes \Phi^{\mu}(G[1])^{\vee} \otimes \Phi^{\mu}(G[1]) \\
& \rightarrow \Phi^{\mu}(F[1] * G[1])
\end{aligned}
$$

where at first we use coevaluation in $\operatorname{Perv}_{0}$, then we use (2.8.7.4) and (2.8.7.5), and at last we use evaluation in $\mathbf{V e c}_{\mathbb{Q}}^{\mu}$.

Proof of Theorem 2.8.3. All the work is done, we just have to summarise.

### 2.9. The structure of the fundamental group of $\operatorname{Perv}_{0}$

In this section, we explore structural properties of the tannakian fundamental group of the category $\mathbf{P e r v}_{0}$, such as connected components, characters, abelianisation, etc. This will of course always have to be done through categorical properties of $\mathbf{P e r v}_{0}$.
2.9.1 (Connected components). - We first show that the group of connected components of the fundamental group of $\mathbf{P e r v}_{0}$ is isomorphic to $\widehat{\mathbb{Z}}$, viewed as a constant group scheme over $\mathbb{Q}$.

Lemma 2.9.2. - Denote by $j: \mathbb{A}^{1} \backslash\{0\} \rightarrow \mathbb{A}^{1}$ the inclusion. If the tannakian fundamental group of an object $A=F[1]$ of $\mathbf{P e r v}_{0}$ is finite, then $j^{*} F$ is a local system with finite monodromy on $\mathbb{A}^{1} \backslash\{0\}$ and $F=j!j^{*} F$ holds. Conversely, for any local system $V$ on $\mathbb{A}^{1} \backslash\{0\}$ with finite monodromy, $j_{!} V[1]$ is an object of $\mathbf{P e r v}_{0}$ with finite tannakian fundamental group.

Proof. Recall that a tannakian category has a finite fundamental group if and only if it is generated as an abelian linear category by finitely many objects, as explained in A.2.1 from the appendix. Thus, if the tannakian fundamental group of $A$ is finite, there exists a finite set $S \subseteq \mathbb{C}$ containing the singularities of $A$ and of all tensor constructions of $A$. But, if $s \in \mathbb{C}$ is a singularity of $A$, then $2 s$ is a singularity of the tensor square of $A$, and hence $2 S \subseteq S$. This forces $S=\{0\}$ as required, and in particular $j^{*} F$ is a local system.

For any object $A=F[1]$ of $\operatorname{Perv}_{0}$ whose only singularity is 0 , the equality $A=j!j^{*} F[1]$ holds, and for any local system $V$ on $\mathbb{A}^{1} \backslash\{0\}$ the perverse sheaf $j!V[1]$ belongs to $\operatorname{Perv}_{0}$. To complete the proof, it remains to show that the tannakian fundamental group of $A$ is the same as the monodromy group of the local system $j^{*} F$. Indeed, the functor

$$
j_{!}(-)[1]:\left\{\text { Local systems on } \mathbb{A}^{1} \backslash\{0\}\right\} \rightarrow \mathbf{P e r v}_{0}
$$

is fully faithful and compatible with tensor products and duals for the usual tensor structure on local systems by Theorem 2.7.4.

Theorem 2.9.3. - Let $G$ be the tannakian fundamental group of the category $\mathbf{P e r v}_{0}$, and denote by $G^{0} \subseteq G$ the connected component of the unity. There is a canonical short exact sequence

$$
1 \rightarrow G^{0} \rightarrow G \rightarrow \widehat{\mathbb{Z}} \rightarrow 0
$$

where $\widehat{\mathbb{Z}}=\lim \mathbb{Z} / n \mathbb{Z}$ is the profinite completion of $\mathbb{Z}$, viewed as a constant group scheme over $\mathbb{Q}$.

Proof. This follows from Lemma 2.9.2 and the general tannakian formalism. Indeed, it suffices to observe that in any tannakian category $\mathbf{T}$ with fundamental group $G$, the full subcategory $\mathbf{T}_{0}$ consisting of those objects that have finite fundamental groups is a tannakian subcategory, corresponding to representations of the group of connected components $G / G^{0}$. Lemma 2.9.2 states that objects of $\operatorname{Perv}_{0}$ with finite tannakian fundamental group are local systems on $\mathbb{A}^{1} \backslash\{0\}$ with finite monodromy, or equivalently, $\mathbb{Q}$-linear representations of $\mathbb{Z}$ with finite image. The tannakian fundamental group of the category of $\mathbb{Q}$-linear representations of $\mathbb{Z}$ with finite image is $\widehat{\mathbb{Z}}$.
2.9.4 (The torus of singularities). - In studying the fundamental group of an object in a tannakian category, it is often useful to look for tori contained in it. In the case of Perv $\mathbf{v}_{0}$, there is a remarkable one: from Theorem 2.8.3 we obtain a morphism of affine group schemes

$$
\Phi^{*}: \pi_{1}(\mathbb{C} \mathbf{R e p}(\mathbb{Z})) \longrightarrow \pi_{1}\left(\mathbf{P e r v}_{0}, \Phi\right)
$$

which will turn out to be a closed immersion. We want to understand its image. If we forget the $\mathbb{Z}$-action on the right-hand side, the corresponding fundamental group becomes a protorus with character group the additive group $\mathbb{C}$, viewed as the union of its finitely generated subgroups. We obtain thus a split subtorus of $\pi_{1}\left(\mathbf{P e r v}_{0}, \Phi\right)$. We call it torus of singularities.
2.9.5. - The tannakian category of $\mathbb{C}$-graded, $\mathbb{Q}$-linear representations of $\mathbb{Z}$ is easy to understand. It comes with a forgetful functor to the category of vector spaces which we take as a fibre functor. We can of course also just forget the $\mathbb{C}$-grading and keep the $\mathbb{Z}$-action, or vice versa. The fundamental group of the category of $\mathbb{C}$-graded representations of $\mathbb{Z}$ is indeed a product

$$
\pi_{1}(\mathbb{C} \operatorname{Rep}(\mathbb{Z}))=T \times G
$$

where $T$ is the fundamental group of the tannakian category of $\mathbb{C}$-graded vector spaces, and $G$ is the fundamental group of the tannakian category of vector spaces with an automorphism. The groups $T$ and $G$ can be described explicitly as follows, though this description is not particularly useful. The group $T$ is the protorus defined by

$$
T=\lim _{\Gamma \subseteq \mathbb{C}} \operatorname{Hom}\left(\Gamma, \mathbb{G}_{m}\right),
$$

where the limit runs over all finitely generated subgroups $\Gamma$ of $\mathbb{C}$ ordered by inclusion, and transition maps $\operatorname{Hom}\left(\Gamma, \mathbb{G}_{m}\right) \rightarrow \operatorname{Hom}\left(\Gamma^{\prime}, \mathbb{G}_{m}\right)$ are given by restriction for $\Gamma^{\prime} \subseteq \Gamma$. Alternatively, we can define $T=\operatorname{Spec} A$, where $A$ is the Hopf algebra over $\mathbb{Q}$ generated by a set of variables $\left\{X_{z} \mid z \in \mathbb{C}\right\}$, modulo the relations $X_{w} X_{z}=X_{w+z}$. These relations imply in particular $X_{0}=1$ and $X_{z}^{n}=X_{n z}$. The comultiplication is defined by $X_{z} \longmapsto X_{z} \otimes X_{z}$ and the antipode by $X_{z} \longmapsto X_{-z}$. The group $G$ is the proalgebraic completion of $\mathbb{Z}$. It can be given as a limit

$$
G=\lim _{\left(\varphi, G_{\varphi}\right)} G_{\varphi}
$$

running over all pairs $\left(\varphi, G_{\varphi}\right)$ consisting of an algebraic group $G_{\varphi}$ over $\mathbb{Q}$ and a group homomorphism $\varphi: \mathbb{Z} \rightarrow G_{\varphi}(\mathbb{Q})$ with Zariski dense image. Transition maps are the evident ones. Notice that, while $T$ is connected, the group of connected components of $G$ is canonically isomorphic to the profinite completion of $\mathbb{Z}$, seen as a constant affine group scheme over $\mathbb{Q}$.

Proposition 2.9.6. - The morphism $\Phi^{*}$ is a closed immersion. Moreover, $\Phi^{*}$ induces an isomorphism on the groups of connected components.

Proof. By Lemma A.3.2, the morphism $\Phi^{*}$ is a closed immersion if and only if every $\mathbb{C}$-graded representation of $\mathbb{Z}$ is a subquotient of $\Phi(A)$ for some $A \in \mathbf{P e r v}_{0}$. Let $V$ be a $\mathbb{C}$-graded representation of $\mathbb{Z}$, and let us prove that there exists an object $A$ of $\operatorname{Perv}_{0}$ such that $\Phi(A)$ is isomorphic
to $V$ as $\mathbb{C}$-graded representation of $\mathbb{Z}$. We assume without loss of generality that $V$ is indecomposable, and hence pure of some degree $z \in \mathbb{C}$ for the $\mathbb{C}$-grading. In other words, we are given a complex number $z$, a vector space $V$ and an automorphism $\mu: V \rightarrow V$, and it suffices to construct an object $A$ of $\operatorname{Perv}_{0}$ whose space of vanishing cycles at $z$ is isomorphic to $V$, as a vector space with automorphism. There is indeed a canonical choice for $A$, namely

$$
A=\left(j_{z}\right)!L[1]
$$

where $j_{z}: \mathbb{C} \backslash\{z\} \rightarrow \mathbb{C}$ is the inclusion, and $L$ is the local system on $\mathbb{C} \backslash\{z\}$ with general fibre $V$ and monodromy $\mu$ around $z$. To say that $\Phi^{*}$ induces an isomorphism on groups of connected components is to say that the functor $\Phi$ restricts to an equivalence $\Phi^{f}$ between the respective tannakian subcategories of objects with finite fundamental group, as shown in the diagram


This is obvious from the characterisation of objects with finite fundamental group in $\operatorname{Perv}_{0}$ and the fact that an object of ${ }_{C} \operatorname{Rep}(\mathbb{Z})^{f}$ has a finite fundamental group if and only if it is pure of degree $0 \in \mathbb{C}$ and $1 \in \mathbb{Z}$ acts through an automorphism of finite order. An inverse to the functor $\Phi^{f}$ is given by regarding a representation of $\mathbb{Z}$ as a local system $V$ on $\mathbb{A}^{1} \backslash\{0\}$ and associating with it the object $j!~ V[1]$ of $\operatorname{Perv}_{0}$.
2.9.7. - Let $A$ be an object of $\mathbf{P e r v}_{0}$ with fundamental group $G$. By Proposition 2.9.6, we can identify the tannakian fundamental group of $\Phi(A)$ with a closed subgroup $H$ of $G$. It is not true in general that this inclusion induces an isomorphism on the groups of connected components, but the homomorphism of finite groups

$$
H / H^{\circ} \longrightarrow G / G^{\circ}
$$

is surjective. According to Lemma A.3.2, this amounts to say that the functor $\Phi^{\prime}:\langle A\rangle^{f} \rightarrow\langle\Phi(A)\rangle^{f}$ induced by $\Phi$ is fully faithful and that its essential image is stable under taking subquotients. Indeed, $\Phi^{\prime}$ is fully faithful because it is also the restriction of the equivalence of categories $\Phi^{f}$. Being the category of representations of a finite group, $\langle\Phi A\rangle^{f}$ is semisimple. Each subobject of an object $\Phi^{\prime}(B)$ in the essential image of $\Phi^{\prime}$ is therefore the kernel of a projector which lifts to a projector of $B$. Hence, the essential image of $\Phi^{\prime}$ is stable under taking subobjects, and similarly for quotients.

Example 2.9.8. - Let $A$ be an object of Perv $_{0}$ whose set of singularities $S$ has as many elements as the dimension of $A$. Denote by $G$ the fundamental group of $A$ and by $H \subseteq G$ the fundamental group of $\Phi(A)$. The $\mathbb{C}$-graded representation $\Phi(A)$ of $\mathbb{Z}$ can be written as

$$
\Phi(A)=\bigoplus_{s \in S} \mathbb{Q}\left(s, \lambda_{s}\right),
$$

where $\mathbb{Q}(s, \lambda)$ denotes the one-dimensional object of ${ }_{C} \boldsymbol{\operatorname { R e p }}(\mathbb{Z})^{f}$ with degree $s$ and where $1 \in \mathbb{Z}$ acts as multiplication by $\lambda \in \mathbb{Q}^{\times}$. Every object of the category $\langle\Phi(A)\rangle$ is a finite sum of one-dimensional objects of the form $\mathbb{Q}(t, \alpha)$ with $t=\sum_{s} a_{s} s$ and $\alpha=\prod_{s} \lambda_{s}^{a_{s}}$ for some integers $\left(a_{s}\right)_{s \in S}$. Objects with finite fundamental group are sums of those $\mathbb{Q}(t, \alpha)$ where $t=0$ and $\alpha \in\{1,-1\}$. Thus, the group $H / H^{\circ}$ of connected components of $\Phi(A)$ is equal to $\mathbb{Z} / 2 \mathbb{Z}$ if there exist integers $\left(a_{s}\right)_{s \in S}$ with the property

$$
\sum_{s \in S} a_{s} s=0 \quad \text { and } \quad \prod_{s \in S} \lambda_{s}^{a_{s}}=-1
$$

and $H$ is connected otherwise. In the former case, $G / G^{\circ}$ is either trivial or $\mathbb{Z} / 2 \mathbb{Z}$, and in the latter case also $G$ is connected.

Let us examine the particular case where $A=\left(f_{*} \mathbb{Q} / \mathbb{Q}\right)[1]$ for some polynomial $f \in \mathbb{C}[x]$ of degree $n$ with $n-1$ distinct critical values $S$. Since the fibre $f^{-1}(s)$ at each $s \in S$ consists of exactly $n-1$ points, the local monodromy around $s$ of the ramified cover defined by $f$ is a transposition according to Example 2.3.11, and hence $\lambda_{s}=-1$ for all $s$. From the general discussion, we thus deduce that the tannakian fundamental group of the object $A \otimes \mathbb{Q}(-1 / 2)$ of $\operatorname{Perv}_{0}$ is connected. Therefore, the fundamental group $G$ of $A$ is either connected or has two connected components.

Definition 2.9.9. - Let $M$ be an object of a neutral tannakian category with fibre functor $\omega$, and let $G$ be the tannakian fundamental group of $M$ acting on the vector space $V=\omega(M)$. We say that $M$ is Lie-irreducible or Lie-simple if the corresponding Lie algebra representation of $\operatorname{Lie}(G)$ on $V$ is irreducible.
2.9.10. - Let $G$ be an algebraic group with Lie algebra $\mathfrak{g}$. If a representation of $G$ is Lieirreducible, then it is irreducible, but there may exist irreducible representations of $G$ which are not Lie-irreducible (e.g. any irreducible representation of dimension $\geqslant 2$ of a finite group). For connected groups, the notions of irreducibility and Lie-irreducibility coincide. This follows from the fact that the faithful exact functor

$$
\{\text { Representations of } G\} \rightarrow\{\text { Representations of } \mathfrak{g}\}
$$

is full if $G$ is connected. Indeed, if $G$ is connected, then the equality $V^{G}=V^{\mathfrak{g}}$ holds for any representation $V$ of $G$ [84, Cor. 24.3.3], and hence the equality

$$
\operatorname{Hom}_{G}(V, W)=\left(V^{\vee} \otimes W\right)^{G}=\left(V^{\vee} \otimes W\right)^{\mathfrak{g}}=\operatorname{Hom}_{\mathfrak{g}}(V, W)
$$

holds for any two $G$-representations $V$ and $W$. A representation of a group $G$ is thus Lie-irreducible if and only if its restriction to the connected component of the unity is irreducible. We have already classified the simple objects of $\operatorname{Perv}_{0}$ in Lemma 2.3.13, and now want to understand which among them are Lie-irreducible. This requires some understanding of the group of connected components of the tannakian fundamental group of $\mathbf{P e r v}_{0}$.

Theorem 2.9.12. - Let $G$ be the tannakian fundamental group of the category $\mathbf{P e r v}_{0}$ and let $V$ be a finite-dimensional representation of $G$. Then $H^{n}(G, V)=0$ for all $n \geqslant 2$. In other words, the cohomological dimension of $G$ is 1 .

Proof. Let $F[1]$ be an object of $\operatorname{Perv}_{0}$, corresponding to a representation $V$ of $G$. The subspace of fixed vectors $V^{G} \subseteq V$ then corresponds via tannakian duality to the subobject $\pi^{*} \pi_{*} j_{*} j^{*} F[1]$ of $F[1]$, where

$$
\pi: \mathbb{A}^{1} \rightarrow \operatorname{Spec} k \quad \text { and } \quad j: \mathbb{A}^{1} \backslash\{0\} \rightarrow \mathbb{A}^{1}
$$

are the structure morphism and the inclusion. Therefore, since $\pi^{*}$ and $j^{*}$ are exact, the cohomological dimension of the functor $V \longmapsto V^{G}$ is the same as the cohomological dimension of the functor $(\pi \circ j)_{*}$, which is equal to 1 by Artin's vanishing theorem 2.1.10 and because $\mathbb{A}^{1} \backslash\{0\}$ is an affine variety of dimension 1 .

### 2.9.13 (Characters). -

Definition 2.9.14. - We call an object of a neutral tannakian category abelian if its tannakian fundamental group is commutative.

Proposition 2.9.15. - An object of Perv $_{0}$ is abelian if and only if it is isomorphic to a direct sum of objects with only one singularity.

Proof. An object of $\operatorname{Perv}_{0}$ which has only one singularity is of the form $E(z) \otimes A$ where $A$ is an object whose only singularity is at $0 \in \mathbb{C}$. In other words, $A$ is a monodromic vector space, corresponding to a representation $\mathbb{Z} \rightarrow \mathrm{GL}_{n}(\mathbb{Q})$. The fundamental group of $A$ is the Zariski closure of the image of this representation, and hence $A$ is abelian. The fundamental group of $E(z)$ is trivial if $z=0$ and $\mathbb{G}_{m}$ otherwise, and hence $E(z)$ is abelian. One implication of the proposition follows thus from the general observation that in whatever neutral tannakian category, any tensor construction of abelian objects is abelian. Conversely, let $A$ be an abelian object in Perv ${ }_{0}$, so the fundamental group $G=G_{A}$ of $A$ is a commutative algebraic group over $\mathbb{Q}$. Write $S$ for the set of singularities of $A$, and $T_{S}$ for the split torus dual to the group $\mathbb{Z} S$ generated by the set of complex numbers $S$. The object $A$ corresponds to a finite-dimensional, faithful representation $V$ of $G_{A}$. We can decompose the vector space $V$ into eigenspaces

$$
V=\bigoplus_{s \in S} V_{s}
$$

for the action of the torus $T_{S}$. The eigenspace corresponding to an element in $\mathbb{Z} S$ is zero unless it belongs to $S$. Since $G_{A}$ is commutative, and hence in particular $T_{S}$ is central in $G_{A}$, this decomposition is compatible with the action of $G_{A}$ on $V$. In other words, the above eigenspace decomposition is a decomposition

$$
A=\bigoplus_{s \in S} A_{s}
$$

of $A$ into a direct sum, where each summand $A_{s}$ has only one singularity $s \in S$.

Corollary 2.9.16. - A one-dimensional object of $\mathbf{P e r v}_{0}$ is isomorphic to $\left(j_{z}\right)!F_{\lambda}[1]$ where $F_{\lambda}$ is the local system with monodromy $\lambda \in \mathbb{Q}^{\times}$on $\mathbb{A}^{1} \backslash\{z\}$.

Proof. A one-dimensional object of $\mathbf{P e r v}_{0}$ is abelian, irreducible and non-zero, and hence has exactly one singularity, say $z$, with trivial fibre at $z$.

## CHAPTER 3

## Three points of view on rapid decay cohomology

In this chapter, we present three different constructions of rapid decay cohomology. We first repeat the elementary definition from the introduction and give a few examples. Next, we associate with a tuple $[X, Y, f, n]$ an object of the category $\operatorname{Perv}_{0}$ whose nearby fibre at infinity is the rapid decay cohomology group $H_{\mathrm{rd}}^{n}(X, Y, f)$. This enables us to derive the statement analogous to Nori's basic lemma for rapid decay cohomology from Beilinson's basic lemma for perverse sheaves, which will be a key ingredient in the definition of a tensor product for exponential motives in Chapter 4. In the case where $X$ is a smooth variety and $Y$ a normal crossing divisor, we express rapid decay cohomology as usual relative cohomology, without any limiting process, on the real blow-up of a good compactification, a point of view that will be useful to prove the comparison isomorphism between rapid decay and de Rham cohomology in Chapter 7. Finally, we introduce cup products, the Künneth formula, and Poincaré-Verdier duality for rapid decay cohomology.

### 3.1. Elementary construction

For each real number $r$, let $S_{r}$ be the closed complex half-plane $S_{r}=\{z \in \mathbb{C} \mid \operatorname{Re}(z) \geqslant r\}$. Throughout this section, all homology and cohomology groups are understood to be singular homology and cohomology with rational coefficients.

Definition 3.1.1. - Let $X$ be a complex variety, $Y \subseteq X$ a closed subvariety, and $f$ a regular function on $X$. For each integer $n \geqslant 0$, the rapid decay homology group in degree $n$ of the triple $[X, Y, f]$ is defined as the limit

$$
\begin{equation*}
H_{n}^{\mathrm{rd}}(X, Y, f)=\lim _{r \rightarrow+\infty} H_{n}\left(X, Y \cup f^{-1}\left(S_{r}\right)\right) \tag{3.1.1.1}
\end{equation*}
$$

The limit is taken in the category of $\mathbb{Q}$-vector spaces, with respect to the transition maps on relative singular homology induced by the inclusions $f^{-1}\left(S_{t}\right) \subseteq f^{-1}\left(S_{r}\right)$ for $t \geqslant r$. Dually, the rapid decay cohomology group in degree $n$ of the triple $[X, Y, f]$ is the colimit

$$
\begin{equation*}
H_{\mathrm{rd}}^{n}(X, Y, f)=\operatorname{colim}_{r \rightarrow+\infty} H^{n}\left(X, Y \cup f^{-1}\left(S_{r}\right)\right) \tag{3.1.1.2}
\end{equation*}
$$

with respect to the transition maps on relative singular cohomology induced by the same inclusions. Whenever the subvariety $Y$ is empty, we shall drop it from the notation.
3.1.2. - Given $X, Y$, and $f$ as in 3.1.1, there exists a real number $r_{0}$ such that, for all $r \geqslant r_{0}$ and all $z \in S_{r}$, the inclusions

$$
\begin{equation*}
Y \cup f^{-1}(z) \subseteq Y \cup f^{-1}\left(S_{r}\right) \subseteq Y \cup f^{-1}\left(S_{r_{0}}\right) \tag{3.1.2.1}
\end{equation*}
$$

are homotopy equivalences, and hence the transition maps in (3.1.1.1) and (3.1.1.2) are eventually isomorphisms. Indeed, it follows from resolution of singularities and Ehresmann's fibration theorem (see e.g. [87, Prop.9.3]) that, given any morphism of complex algebraic varieties, in our case $f: X \rightarrow \mathbb{A}^{1}$, there exists a non-empty Zariski open subset $U$ of the target space, in our case the complement of a finite subset of $\mathbb{A}^{1}$, such that $f^{-1}(U) \rightarrow U$ is a fibre bundle for the complex topology (see also [86, Cor. 5.1]). Together with the well-known fact that complex algebraic varieties admit a finite triangulation, this implies that rapid decay homology and cohomology groups are finite-dimensional vector spaces dual to each other. They depend naturally on $[X, Y, f]$ for the obvious notion of morphisms of such triples, that is, morphisms $h: X \rightarrow X^{\prime}$ such that $h(Y) \subseteq Y^{\prime}$ and $f^{\prime} \circ h=f$. If $f$ is constant, we recover the usual singular homology of the pair $[X, Y]$, since in that case the set $f^{-1}\left(S_{r}\right)$ is empty for big $r$. For sufficiently large $r$, we obtain isomorphisms

$$
\begin{equation*}
H_{\mathrm{rd}}^{n}(X, Y, f) \cong H^{n}\left(X, Y \cup f^{-1}(z)\right) \tag{3.1.2.2}
\end{equation*}
$$

for any $z \in \mathbb{C}$ with $\operatorname{Re}(z) \geqslant r$. These isomorphisms (3.1.2.2) are natural in the sense that, if a finite family of triples $\left[X_{\alpha}, Y_{\alpha}, f_{\alpha}\right]$ and morphisms between them are given, then for any $z \in \mathbb{C}$ with sufficiently large real part the isomorphism (3.1.2.2) is natural with respect to the given morphisms.

Example 3.1.3. - Let $X=\mathbb{A}^{2}=\operatorname{Spec} \mathbb{C}[x, y]$, together with the regular function $f(x, y)=x^{a} y^{b}$ for two integers $a, b \geqslant 1$. If $a$ and $b$ are coprime, the curve $f^{-1}(r) \subseteq X$ is isomorphic to $\mathbb{G}_{m}$ as long as $r \neq 0$. In general, if $d$ denotes the greatest common divisor of $a$ and $b$, the subvariety $f^{-1}(r)$ is a finite disjoint union of copies of $\mathbb{G}_{m}$ indexed by the group of roots of unity $\mu_{d}(\mathbb{C})$. From the long exact sequence of relative cohomology

$$
0 \rightarrow H^{0}\left(\mathbb{A}^{2}\right) \rightarrow H^{0}\left(f^{-1}(r)\right) \rightarrow H^{1}\left(\mathbb{A}^{2}, f^{-1}(r)\right) \rightarrow H^{1}\left(\mathbb{A}^{2}\right) \rightarrow H^{1}\left(f^{-1}(r)\right) \rightarrow H^{2}\left(\mathbb{A}^{2}, f^{-1}(r)\right) \rightarrow 0
$$

it follows that $\operatorname{dim} H_{\mathrm{rd}}^{1}(X, f)=d-1$ and $\operatorname{dim} H_{\mathrm{rd}}^{2}(X, f)=d$.

Example 3.1.4. - Set $X=\mathbb{G}_{m}^{2}=\operatorname{Spec} \mathbb{C}\left[u^{ \pm 1}, v^{ \pm 1}\right]$, and $f(u, v)=u+v+(u v)^{-1}$. In order to compute the rapid decay homology of $(X, f)$, we need to understand the topology of the hypersurface $f(u, v)=r$ for large real $r$. Let us look at $X$ as the zero locus of $x y z-1$ in $\mathbb{A}^{3}=\operatorname{Spec} \mathbb{C}[x, y, z]$ via $(u, y) \longmapsto\left(u, v,(u v)^{-1}\right)$ and extend $f$ to the function $f(x, y, z)=x+y+z$ on $\mathbb{A}^{3}$. The equation $f(x, y, z)=r$ describes a hypersurface in $\mathbb{A}^{3}$ which has the homotopy type of a honest sphere $S^{2}$, while $x y z=1$ has the homotopy type of a torus $S^{1} \times S^{1}$. Finish this
3.1.5. - Let $X$ be a complex variety, $Z \subseteq Y \subseteq X$ closed subvarieties, and $f$ a regular function on $X$. As for ordinary singular cohomology, there is a long exact sequence

$$
\begin{equation*}
\cdots \longrightarrow H_{\mathrm{rd}}^{n-1}\left(Y, Z, f_{\mid Y}\right) \longrightarrow H_{\mathrm{rd}}^{n}(X, Y, f) \longrightarrow H_{\mathrm{rd}}^{n}(X, Z, f) \longrightarrow \cdots \tag{3.1.5.1}
\end{equation*}
$$

which is functorial in $[X, Y, Z, f]$. It is obtained as follows: for each real number $r$, there is a natural long exact sequence of cohomology groups
$\cdots \longrightarrow H^{n-1}\left(Y \cup f^{-1}\left(S_{r}\right), Z \cup f^{-1}\left(S_{r}\right)\right) \longrightarrow H^{n}\left(X, Y \cup f^{-1}\left(S_{r}\right)\right) \longrightarrow H^{n}\left(X, Z \cup f^{-1}\left(S_{r}\right)\right) \longrightarrow \cdots$ associated with the triple of topological spaces $Z \cup f^{-1}\left(S_{r}\right) \subseteq Y \cup f^{-1}\left(S_{r}\right) \subseteq X$. The inclusion of pairs $\left[Y,\left.f\right|_{Y} ^{-1}\left(S_{r}\right)\right] \rightarrow\left[Y \cup f^{-1}\left(S_{r}\right), Z \cup f^{-1}\left(S_{r}\right)\right]$ induces an isomorphism in relative cohomology by excision. We may hence identify the previous long exact sequence with

$$
\cdots \longrightarrow H^{n-1}\left(Y,\left.Z \cup f\right|_{Y} ^{-1}\left(S_{r}\right)\right) \longrightarrow H^{n}\left(X, Y \cup f^{-1}\left(S_{r}\right)\right) \longrightarrow H^{n}\left(X, Z \cup f^{-1}\left(S_{r}\right)\right) \longrightarrow \cdots
$$

and obtain (3.1.5.1) by taking colimits.

Example 3.1.6. - Let $X$ be a variety, $Y \subseteq X$ a closed subvariety, and $f: X \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ the projection onto the second factor. All the rapid decay cohomology groups vanish:

$$
H_{\mathrm{rd}}^{n}\left(X \times \mathbb{A}^{1}, Y \times \mathbb{A}^{1}, f\right)=0
$$

Indeed, for each real number $r$, the inclusion $f^{-1}\left(S_{r}\right)=X \times S_{r} \subseteq X \times \mathbb{C}$ is a homotopy equivalence, hence the vanishing $H_{\mathrm{rd}}^{n}\left(X \times \mathbb{A}^{1}, f\right)=0$ for all $n$. Similarly, $H_{\mathrm{rd}}^{n}\left(Y \times \mathbb{A}^{1},\left.f\right|_{Y}\right)=0$, and the claim follows from the long exact sequence (3.1.5.1) applied to the triple $\varnothing \subseteq Y \subseteq X$.

### 3.2. Rapid decay cohomology in terms of perverse sheaves

In this section, we present a less elementary construction of rapid decay cohomology in terms of perverse sheaves. It has the advantage of automatically endowing $H_{\mathrm{rd}}^{n}(X, Y, f)$ with rich additional structure and also of being purely based on the six-functors formalism, and hence portable to other contexts. Ultimately, we wish to equip $H_{\mathrm{rd}}^{n}(X, Y, f)$ with the data of an exponential mixed Hodge structure, which is a special kind of mixed Hodge module on the complex affine line.

Definition 3.2.1. - Let $X$ be a complex algebraic variety, $Y \subseteq X$ a closed subvariety, and $f: X \rightarrow \mathbb{A}^{1}$ a regular function on $X$. We call the object

$$
H_{\text {perv }}^{n}(X, Y, f)=\Pi\left({ }^{p} \mathcal{H}^{n}\left(R f_{*} \underline{\mathbb{Q}}_{[X, Y]}\right)\right)
$$

of Perv 0 the perverse cohomology in degree $n$ of the triple $[X, Y, f]$. Here, ${ }^{p} \mathcal{H}^{n}$ means homology in degree $n$ with respect to the perverse $t$-structure, and $\mathbb{Q}_{[X, Y]}$ stands for the sheaf $\beta_{!} \beta^{*} \mathbb{Q}_{X}$ on $X$, where $\beta: X \backslash Y \rightarrow X$ is the inclusion.

Proposition 3.2.2. - Let $X$ be a complex algebraic variety, let $Y \subseteq X$ be a closed subvariety, and let $f: X \rightarrow \mathbb{A}^{1}$ be a regular function on $X$. Let $\Gamma \subseteq X \times \mathbb{A}^{1}$ be the graph of $f$, and let $p: X \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ be the projection. There is a canonical and natural isomorphism in the derived category of constructible sheaves on $\mathbb{A}^{1}$ :

$$
\begin{equation*}
\Pi\left(R f_{*} \underline{\mathbb{Q}}_{[X, Y]}\right) \xrightarrow{\cong} R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]}[1] . \tag{3.2.2.1}
\end{equation*}
$$

Proof. Let $s: X \rightarrow\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma$ be the morphism of algebraic varieties given by $s(x)=$ $(x, f(x))$. It sends $Y$ to $Y \times \mathbb{A}^{1}$ and satisfies $p \circ s=f$, and hence induces a morphism in the derived category of constructible sheaves

$$
\begin{equation*}
R p_{*} \underline{\mathbb{Q}}_{\left[\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma, Y \times \mathbb{A}^{1}\right]} \longrightarrow R f_{*} \underline{\mathbb{Q}}_{[X, Y]} \tag{3.2.2.2}
\end{equation*}
$$

which is natural in $[X, Y, f]$. This morphism is an isomorphism. Indeed, the map $s$ is an isomorphism of $[X, Y]$ onto its image $\left[\Gamma,\left(Y \times \mathbb{A}^{1}\right) \cap \Gamma\right]$, and the cohomology of this pair over any open subset of $\mathbb{A}^{1}(\mathbb{C})$ is isomorphic to that of $\left[\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma, Y \times \mathbb{A}^{1}\right]$ by excision: cut out the open subspace $\left\{(y, t) \in Y \times \mathbb{A}^{1} \mid f(y) \neq t\right\}$. From the triple of spaces $Y \times \mathbb{A}^{1} \subseteq\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma \subseteq X \times \mathbb{A}^{1}$ we obtain the following natural exact triangle.

$$
R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]} \rightarrow R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1}, Y \times \mathbb{A}^{1}\right]} \rightarrow R p_{*} \underline{\mathbb{Q}}_{\left[\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma, Y \times \mathbb{A}^{1}\right]} \rightarrow R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]}[1] .
$$

The object $R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1}, Y \times \mathbb{A}^{1}\right]}$ is the same as $\pi^{*} R \pi_{*} \underline{\mathbb{Q}}_{[X, Y]}$, and hence applying $\Pi$ to it returns the zero object. We find therefore a natural isomorphism

$$
\Pi\left(R f_{*} \underline{\mathbb{Q}}_{[X, Y]}\right) \xrightarrow{\cong} \Pi\left(R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]}[1]\right)
$$

and it remains to show that the adjunction $R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]} \rightarrow \Pi\left(R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]}\right)$ is an isomorphism. The triangle (2.4.3.2) reads

$$
\pi^{*} R \pi_{*} R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]} \rightarrow R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]} \rightarrow \Pi\left(R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]}\right)
$$

and hence we must show that $R(\pi \circ p)_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]}$ is zero. This is just a complicated way of saying that the cohomology groups $H^{n}\left(X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right)$ vanish. The cohomology groups of the pair $\left[X \times \mathbb{A}^{1}, \Gamma\right]$ are zero, because this pair is homotopic to $[X, X]$. The long exact sequence of the triple $\Gamma \subseteq\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma \subseteq X \times \mathbb{A}^{1}$ shows that it is enough to prove that the cohomology groups of the pair $\left[\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma, \Gamma\right]$ vanish. The excision isomorphism shows that the cohomology of $\left[\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma, \Gamma\right]$ is the same as the cohomology of $\left[Y \times \mathbb{A}^{1}, \Gamma \cap\left(Y \times \mathbb{A}^{1}\right)\right]$, and since $\Gamma \cap\left(Y \times \mathbb{A}^{1}\right)$ is just the graph of the restriction of $f$ to $Y$, this cohomology vanishes as we wanted to show.

Corollary 3.2.3. - Let $X$ be a complex variety, $Y \subseteq X$ a closed subvariety, and $f$ a regular function on $X$. There is a canonical and natural isomorphism of $\mathbb{Q}$-vector spaces

$$
\Psi_{\infty}\left(H_{\mathrm{perv}}^{n}(X, Y, f)\right) \cong H_{\mathrm{rd}}^{n}(X, Y, f)
$$

Proof. By Lemma 2.3.3 and part (2) of Proposition 2.4.3, given any object $C$ of the derived category of constructible sheaves, we have canonical and natural isomorphisms in Perv $\mathbf{p}_{0}$

$$
\Pi\left({ }^{p} \mathcal{H}^{n}(C)\right) \cong{ }^{p} \mathcal{H}^{n}(\Pi(C)) \cong \mathcal{H}^{n-1}(\Pi(C))[1]
$$

Proposition 3.2.2 yields therefore an isomorphism

$$
\Pi\left({ }^{p} \mathcal{H}^{n}\left(R f_{*} \underline{\mathbb{Q}}_{[X, Y]}\right)\right) \cong \mathcal{H}^{n-1}\left(\Pi\left(R f_{*} \underline{\mathbb{Q}}_{[X, Y]}\right)\right)[1] \xrightarrow{\cong} \mathcal{H}^{n}\left(R p_{*} \underline{\mathbb{Q}}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]}\right)[1]
$$

in the category $\mathbf{P e r v}_{0}$. The fibre of the constructible sheaf $\mathcal{H}^{n}\left(R p_{*} \mathbb{Q}_{\left[X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma\right]}\right)$ over a large real number $r$ is the cohomology of the pair $\left[X \times\{r\},(Y \times\{r\}) \cup\left(f^{-1}(r) \times\{r\}\right)\right]$ in degree $n$, which is the rapid decay cohomology $H_{\mathrm{rd}}^{n}(X, Y, f)$ as $r$ tends to infinity.
3.2.4. - Corollary 3.2 .3 suggests a way of defining rapid decay cohomology with coefficients in a constructible sheaf. Given a bounded complex of constructible sheaves $A$ on $X$, we consider the object

$$
\begin{equation*}
\Psi_{\infty}\left(\Pi\left(R f_{*} A\right)\right) \tag{3.2.4.1}
\end{equation*}
$$

of $D^{b}\left(\mathbf{V e c}_{\mathbb{Q}}\right)$. Taking for $A$ the complex $\underline{\mathbb{Q}}_{[X, Y]}$ concentrated in degree 0 , for some closed subvariety $Y \subseteq X$, we find

$$
H^{n}\left(\Psi_{\infty}\left(\Pi\left(R f_{*} \underline{\mathbb{Q}}_{[X, Y]}\right)\right)\right)=\Psi_{\infty}{ }^{p} \mathcal{H}^{n}\left(\Pi\left(R f_{*} \underline{\mathbb{Q}}_{[X, Y]}\right)\right)=\Psi_{\infty}\left(H_{\mathrm{perv}}^{n}(X, Y, f)\right)=H_{\mathrm{rd}}^{n}(X, Y, f)
$$

by Corollary 3.2.3 and exactness of $\Psi_{\infty}$ and $\Pi$ for the perverse $t$-structure. Hence the complex (3.2.4.1) computes the rapid decay cohomology of $(X, Y, f)$. For general complexes $A$, we write

$$
H_{\mathrm{rd}}^{n}(X, f ; A)=H^{n}\left(\Psi_{\infty}\left(\Pi\left(R f_{*} A\right)\right)\right)
$$

and refer to these vector spaces as rapid decay cohomology of $X$ with coefficients in $A$.
3.2.5. - The definition of rapid decay cohomology with coefficients allows for the introduction of a Leray spectral sequence. Given an algebraic variety $S$ with potential $f: S \rightarrow \mathbb{A}^{1}$, a morphism $\pi: X \rightarrow S$ of algebraic varieties, and a complex of constructible sheaves $A$ on $X$, there is a first quadrant spectral sequence

$$
E_{2}^{p, q}=H_{\mathrm{rd}}^{p}\left(S, f ; R^{q} \pi_{*} A\right) \Longrightarrow H_{\mathrm{rd}}^{p+q}(X, f \circ \pi ; A)
$$

It is obtained, as is the classical Leray spectral sequence, as the Grothendieck spectral sequence of a composite of derived functors

$$
R f_{*} R \pi_{*}=R(f \circ \pi)_{*}: D^{b}(X) \rightarrow D^{b}\left(\mathbb{A}^{1}\right)
$$

Using the standard $t$-structure to filter $R f_{*}$ and the perverse $t$-structure to filter $R \pi_{*}$, the spectral sequence reads

$$
{ }^{p} R^{p} f_{*}\left(R^{q} \pi_{*}(A)\right) \Longrightarrow{ }^{p} R^{p+q}(f \circ \pi)_{*}(A) .
$$

This is a spectral sequence of perverse sheaves on $\mathbb{A}^{1}$. Applying the exact functors $\Pi$ and $\Psi_{\infty}$ yields the desired spectral sequence of vector spaces.

### 3.3. Cell decomposition and the exponential basic lemma

In this section, we prove the analogue of Nori's basic lemma for rapid decay cohomology. Using its description in terms of perverse sheaves, it will be a more or less straightforward consequence of the most general version of the basic lemma, obtained by Beilinson in [7, Lemma 3.3]. We recall the argument for the convenience of the reader.

Theorem 3.3.1 (Beilinson's basic lemma). - Let $f: X \rightarrow S$ be a morphism of quasi-projective varieties over $k$ and let $F$ be a perverse sheaf on $X$. There exists a dense open subvariety $j: U \hookrightarrow X$ such that the perverse sheaves ${ }^{p} \mathcal{H}^{n}\left(R f_{*} j!j^{*} F\right)$ on $S$ vanish for all $n<0$.

Proof. We will show that the complement in $X$ of a general hyperplane section has the desired property. To alleviate notation, we agree that for the duration of this proof all direct and inverse image functors are between derived categories of sheaves, and write them just as $f_{*}$ and $f_{!}$instead of $R f_{*}$ and $R f_{!}$. Consider a commutative diagram

where $i_{S}$ and $i_{X}$ are open immersions and $\bar{S}$ and $\bar{X}$ are projective. Choose an embedding of $\bar{X}$ into some projective space $\mathbb{P}=\mathbb{P}_{k}^{N}$. Let $\mathbb{P}^{\prime} \simeq \mathbb{P}_{k}^{N}$ be the dual projective space parametrising hyperplanes in $\mathbb{P}$. The family of all hyperplane sections of $\bar{X}$ is the closed subvariety of $\bar{X} \times \mathbb{P}^{\prime}$ defined as

$$
\bar{H}=\left\{(x, L) \in \bar{X} \times \mathbb{P}^{\prime} \mid x \in L\right\}
$$

Setting $H=\bar{H} \cap\left(X \times \mathbb{P}^{\prime}\right)$, we obtain the following commutative diagram of varieties:

In this diagram, all vertical maps are open immersions, whereas the horizontal maps $\kappa$ and $\bar{\kappa}$ are closed immersions. Let $\bar{p}: \bar{X} \times \mathbb{P}^{\prime} \rightarrow \bar{X}$ and $p: X \times \mathbb{P}^{\prime} \rightarrow X$ be the projections, and set $G=p^{*} F$ for the given perverse sheaf $F$ on $X$. The composite morphism

$$
\bar{p} \circ \bar{\kappa}: \bar{H} \rightarrow \bar{X} \times \mathbb{P}^{\prime} \rightarrow \bar{X}
$$

is a projective bundle, in particular is smooth. It follows from the smooth base change theorem, which we recalled in 2.1.6, that the canonical morphism

$$
\begin{equation*}
\bar{\kappa}^{*}\left(i_{X} \times \mathrm{id}\right)_{*} G=(\bar{p} \circ \bar{\kappa})^{*} i_{X *} F \rightarrow i_{H *}(\kappa \circ p)^{*} F=i_{H *} \kappa^{*} G \tag{3.3.1.2}
\end{equation*}
$$

is an isomorphism. Pick a point of $\mathbb{P}^{\prime}$ corresponding to a hyperplane $L \subseteq \mathbb{P}$. The fibre of the diagram (3.3.1.1) over this point is the diagram

where $\bar{H}_{L}=\bar{X} \cap L$ and $H_{L}=X \cap L$ are the hyperplane sections given by $L$. By the smooth base change theorem, there exists a Zariski dense open subvariety of $\mathbb{P}^{\prime}$ such that, for all points $L$ in this open subvariety, the base change morphisms

$$
\begin{equation*}
\left.\left(\bar{\kappa}^{*}\left(i_{X} \times \mathrm{id}\right)_{*} G\right)\right|_{H_{L}} \rightarrow \bar{\kappa}_{L}^{*} i_{X *} F \quad \text { and }\left.\quad\left(i_{H *} \kappa^{*} G\right)\right|_{H_{L}} \rightarrow i_{H_{L *}} \kappa_{L}^{*} F \tag{3.3.1.4}
\end{equation*}
$$

are isomorphisms. Indeed, any point $L$ around which the map $H(\mathbb{C}) \rightarrow \mathbb{P}^{\prime}(\mathbb{C})$ is smooth, or just a topological fibration will do. Fix now an $L$ such that the base change morphisms (3.3.1.4) are
isomorphisms and such that the hyperplane section $H_{L} \subset X$ has codimension $\geqslant 1$, so that its complement is dense. Since (3.3.1.2) and (3.3.1.4) are isomorphisms, the canonical morphism

$$
\begin{equation*}
\bar{\kappa}_{L}^{*} i_{X *} F \rightarrow i_{H_{L *}} \kappa_{L}^{*} F \tag{3.3.1.5}
\end{equation*}
$$

obtained from (3.3.1.2) by base change is an isomorphism as well. Let $\bar{U}$ be the complement of $\bar{H}_{L}$ in $\bar{X}$, set $U=\bar{U} \cap X=X \backslash H_{L}$, and consider the diagram

where $j$ and $\bar{j}$ are the inclusions. The canonical morphism

$$
\begin{equation*}
\overline{j_{!}} i_{U *}\left(j^{*} F\right) \rightarrow\left(i_{X}\right)_{*} j_{!}\left(j^{*} F\right) \tag{3.3.1.6}
\end{equation*}
$$

is an isomorphism. This is indeed a consequence of (3.3.1.5) and the five lemma applied to the commutative diagram with exact rows

where the rightmost isomorphism is obtained by applying $\bar{\kappa}_{L *}$ to the isomorphism (3.3.1.5). Finally, we obtain from (3.3.1.6) an isomorphism
where we used $\bar{f}_{!}=\bar{f}_{*}$ in the last equality. The morphism $\bar{f} \circ \bar{j}: \bar{U} \rightarrow \bar{S}$ is affine, and hence the functor $R(\bar{f} \circ \bar{j})$ ! is $t$-left exact for the perverse $t$-structure by Artin's theorem 2.1.16. It follows that the last term above vanishes for $n<0$, thus concluding the proof.
3.3.2 (Exponential basic lemma). - We now deduce the basic lemma for rapid decay cohomology. Below, we say that a variety has dimension $\leqslant d$ if all its irreducible components do.

Corollary 3.3.3 (Exponential basic lemma). - Let $X$ be an affine variety of dimension $\leqslant d$, together with a regular function $f$, and let $\left(Y_{i} \rightarrow X_{i} \rightarrow X\right)_{i \in I}$ be a finite family of closed immersions. There exists a closed subvariety $Z \subseteq X$ of dimension $\leqslant d-1$ such that, for all $n \neq d$,

$$
H_{\mathrm{rd}}^{n}\left(X_{i}, Y_{i} \cup\left(X_{i} \cap Z\right), f\right)=0 .
$$

Proof. Let $W \subseteq X$ be a closed subvariety of dimension $\leqslant d-1$ such that, for each $i$, the variety $X_{i} \backslash\left(W \cup Y_{i}\right)$ is either empty or smooth and equidimensional of dimension $d$. Set $W_{i}=X_{i} \cap W$.

The complex of constructible sheaves $\underline{\mathbb{Q}}_{\left[X_{i}, W_{i} \cup Y_{i}\right]}[d]$ is a perverse sheaf on $X$ as we have seen in Example 2.1.17. Set

$$
F=\bigoplus_{i \in I} \mathbb{Q}_{\left[X_{i}, W_{i} \cup Y_{i}\right]}[d]
$$

and let us apply Beilinson's Theorem 3.3.1: there exists a dense open subvariety $j: U \hookrightarrow X$ such that ${ }^{p} \mathcal{H}^{n}\left(R f_{*} j_{!} j^{*} F\right)=0$ for $n<0$, in particular

$$
{ }^{p} \mathcal{H}^{n}\left(R f_{*} j!j^{*} \mathbb{Q}_{\left[X_{i}, W_{i} \cup Y_{i}\right]}[d]\right)=0
$$

for each $i$. Let $Z$ be the union of the complement of $U$ and $W$. Since $U$ is dense, $Z$ has dimension $\leqslant d-1$ and since $Z$ contains $W$, we have $j!j^{*} \underline{\mathbb{Q}}_{\left[X_{i}, W_{i} \cup Y_{i}\right]}=\underline{\mathbb{Q}}_{\left[X_{i}, Y_{i} \cup\left(X_{i} \cap Z\right)\right]}$. Hence, Corollary 3.2.3 yields the vanishing

$$
H_{\mathrm{rd}}^{n+d}\left(X_{i}, Y_{i} \cup\left(X_{i} \cap Z\right), f\right) \cong \psi_{\infty}\left(\Pi\left({ }^{p} \mathcal{H}^{n}\left(R f_{*} \underline{\mathbb{Q}}_{\left[X_{i}, Y_{i} \cup\left(X_{i} \cap Z\right)\right]}[d]\right)\right)\right)=0
$$

for all $n<0$, and hence $H_{\mathrm{rd}}^{n}\left(X_{i}, Y_{i} \cup\left(X_{i} \cap Z\right), f\right)=0$ for $n<d$. On the other hand, since $X_{i}$ is affine of dimension $\leqslant d$, Artin's vanishing theorem shows that $H_{\mathrm{rd}}^{n}\left(X_{i}, Y_{i} \cup\left(X_{i} \cap Z\right), f\right)=0$ for $n>d$.

### 3.4. Preliminaries on the oriented real blow-up

The two previous descriptions of rapid decay cohomology involve passing to the limit when the real part of the function goes to infinity. We reinterpret these constructions as the cohomology of a manifold with boundary, where the boundary might not be smooth but have corners. In a sense, the limit is now taken over the ambient space itself.
3.4.1 (The oriented real blow-up of $\mathbb{P}^{1}$ at infinity). - We write $\widetilde{\mathbb{P}}^{1}$ for the compactification of $\mathbb{C}$ by a circle at infinity, that is, the disjoint union $\widetilde{\mathbb{P}}^{1}=\mathbb{C} \sqcup S^{1}$. A system of open neighbourhoods of $z \in S^{1}=\{w \in \mathbb{C}| | w \mid=1\}$ is given by the sets

$$
\{w \in \mathbb{C}||w|>R \text { and }| \arg (w)-\arg (z) \mid<\varepsilon\} \sqcup\left\{z^{\prime} \in S^{1}| | \arg \left(z^{\prime}\right)-\arg (z) \mid<\varepsilon\right\}
$$

for large $R$ and small $\varepsilon$ (see Figure 3.4.1). Given a complex number $z$ of norm 1, we will write $z \infty$ for the element of the boundary $\partial \widetilde{\mathbb{P}}^{1}=S^{1}$ of $\widetilde{\mathbb{P}}^{1}$ with argument $\arg (z)$. There is a canonical map $\pi: \widetilde{\mathbb{P}}^{1} \rightarrow \mathbb{P}^{1}$ sending the circle at infinity to $\infty \in \mathbb{P}^{1}$, which we call the oriented real blow-up of $\mathbb{P}^{1}$ at infinity. For a real number $r$, we let $\widetilde{S}_{r}$ denote the union of the closed half-plane $S_{r}$ and the closed half-circle at infinity $\left\{z \infty \in \partial \widetilde{\mathbb{P}}^{1} \mid \operatorname{Re}(z) \geqslant 0\right\}$, as displayed in Figure 3.4.1.
3.4.2. - Let us recapitulate how the oriented real blow-up of a complex variety $X$ along a subvariety is constructed. We follow the exposition in [39]. Let $\pi: L \rightarrow X$ be a complex line bundle on $X$, and let $s$ be a section of $L$. We consider the subspace $B_{L, s}^{*}$ of $L$ whose elements in a fibre $L_{x}=\pi^{-1}(x)$ are those non-zero $l \in L_{x}$ satisfying $r \cdot l=s(x)$ for some non-negative real number $r$. The fibre of $B_{L, s}^{*}$ over $x \in X$ is thus the set of positive real multiples $\mathbb{R}_{>0} \cdot s(x)$


Figure 3.4.1. A neighbourhood of $z \infty$ (left) and the closed region $\widetilde{S}_{r}$ of $\widetilde{\mathbb{P}}^{1}$ for a sufficiently large real number $r$ (right)
whenever $s(x)$ is non-zero, and the set $L_{x} \backslash\{0\}$ in case $s(x)=0$. Therefore, the action of $\mathbb{R}_{>0}$ on $L$ by fibrewise multiplication leaves $B_{L, s}^{*}$ stable. The quotient topological space

$$
\operatorname{Blo}_{L, s} X=B_{L, s}^{*} / \mathbb{R}_{>0}
$$

is called the oriented real blow-up of $X$ along $(L, s)$. It is a closed, real semialgebraic subspace of the oriented circle bundle $S^{1} L=L^{*} / \mathbb{R}_{>0}=\operatorname{Blo}_{L, 0} X$. Two sections of $L$ that differ by a nowhere vanishing function define the same oriented real blow-up, and hence it makes sense to define the oriented real blow-up of $X$ along an effective Cartier divisor $D$ as

$$
\operatorname{Blo}_{D} X=\operatorname{Blo}_{\mathcal{O}(D), s} X,
$$

where $s$ is a section of $\mathcal{O}(D)$ with $D$ as zero locus. Finally, if $Z \subseteq X$ is an arbitrary closed subvariety, we define the oriented real blow-up of $X$ along $Z$ as

$$
\mathrm{Blo}_{Z} X=\mathrm{Blo}_{E}\left(\mathrm{Bl}_{Z} X\right)
$$

where $E \subseteq \mathrm{Bl}_{Z} X$ is the exceptional divisor in the ordinary blow-up of $X$ along $Z$. The real blow-up comes with a map $\pi: \operatorname{Blo}_{Z} X \rightarrow X$, and we call

$$
\partial \mathrm{Blo}_{Z} X=\pi^{-1}(Z)
$$

the boundary of $\mathrm{Blo}_{Z} X$. If $X$ is smooth and $Z$ a smooth subvariety, then $\mathrm{Blo}_{Z} X$ has canonically the structure of a real manifold with boundary $\partial \operatorname{Blo}_{Z} X$.

Example 3.4.3. - Viewing $\{\infty\} \subseteq \mathbb{P}^{1}$ as an effective Cartier divisor, the above definition agrees with the ad-hoc construction of the oriented real blow-up of $\mathbb{P}^{1}$ at $\{\infty\}$ from 3.4.1. A bit more generally, consider the effective Cartier divisor $D=n \cdot[0]$ on $\mathbb{A}^{1}$ for some integer $n \geqslant 1$, which is the zero locus of the section $z \longmapsto z^{n}$ of the trivial line bundle on $\mathbb{A}^{1}$. The oriented real blow-up map $\pi: \operatorname{Blo}_{D} \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ is an isomorphism above $\mathbb{A}^{1} \backslash\{0\}$, and the fibre over $\{0\}$ is the circle $S^{1}=\mathbb{C}^{\times} / \mathbb{R}_{>0} \cong\left\{z \in \mathbb{C}^{\times}| | z \mid=1\right\}$. In order to understand the topology of $\operatorname{Blo}_{D} \mathbb{A}^{1}$ we need to describe neighbourhoods of points in the boundary. By definition, $\operatorname{Blo}_{D} \mathbb{A}^{1}$ is the quotient of

$$
B_{\mathbb{C}, z^{n}}^{*}=\left\{(z, w) \in \mathbb{C} \times \mathbb{C}^{\times} \mid z^{n} w^{-1} \in \mathbb{R} \geqslant 0\right\}
$$

by the action of $\mathbb{R}_{>0}$ by multiplication on the second coordinate. Equivalently, we can and will regard $\mathrm{Blo}_{D} \mathbb{A}^{1}$ as the subspace

$$
\operatorname{Blo}_{D} \mathbb{A}^{1}=\left\{(z, w) \in \mathbb{C} \times S^{1} \mid z^{n} w^{-1} \in \mathbb{R}_{\geqslant 0}\right\}
$$

of the trivial circle bundle on $\mathbb{A}^{1}$. Elements of the boundary are of the form $\left(0, w_{0}\right)$ with $w_{0}$ a complex number of norm 1. A fundamental system of neighbourhoods of $\left(0, w_{0}\right)$ is then given by

$$
U_{\varepsilon}=\left\{(z, w) \in \mathbb{C} \times S^{1}\left|z^{n} w^{-1} \in \mathbb{R}_{\geqslant 0},|z|<\varepsilon, \arg \left(w w_{0}^{-1}\right)<\varepsilon\right\}\right.
$$

for small $\varepsilon>0$. The intersection of $U_{\varepsilon}$ with the boundary is a small arc of circle around $w_{0} \in S^{1}$. On the other hand, the intersection of $U_{\varepsilon}$ with $\mathbb{A} \backslash\{0\}$ consists of the $n$ small sectors

$$
\left\{z \in \mathbb{C}\left|0<|z|<\varepsilon,\left|\arg \left(z^{n} w_{0}^{-1}\right)\right|<\varepsilon\right\}\right.
$$

around the missing origin in $\mathbb{A} \backslash\{0\}$. We might thus topologically describe the oriented real blow-up of $\mathbb{P}^{1}$ along $D=n \cdot[0]$ as the result of gluing a disk to a circle via the $n$-fold covering map of the circle by itself. In particular, the topological space $\mathrm{Blo}_{D} \mathbb{A}^{1}$ does not admit the structure of a real manifold with boundary unless $n=1$.
3.4.4. - Let $X$ be a complex variety, and let $Z_{1}, Z_{2}, \ldots, Z_{m}$ be closed subvarieties of $X$. We define the oriented real blow-up of $X$ along the centres $\left(Z_{1}, Z_{2}, \ldots, Z_{m}\right)$ to be

$$
\mathrm{Blo}_{\left(Z_{1}, \ldots, Z_{m}\right)} X=\mathrm{Blo}_{Z_{1}} X \times_{X} \mathrm{Blo}_{Z_{2}} X \times_{X} \cdots \times_{X} \mathrm{Blo}_{Z_{m}} X
$$

where the fibre products are taken in the category of topological spaces. We will work with oriented real blow-ups along multiple centres in the particular case where $X$ is a smooth complex variety and the $Z_{i}$ are the irreducible components of a normal crossing divisor. In such a situation, we may describe the oriented real blow-up in terms of local coordinates as follows. Suppose $X=\mathbb{A}^{d}$, with coordinates $x_{1}, \ldots, x_{d}$, and let $D_{p} \subseteq X$ be the effective Cartier divisor given by $x_{p}=0$. As in Example 3.4.3, the oriented real blow-up of $X$ along $\left(D_{1}, \ldots, D_{m}\right)$ is identified with the subspace

$$
\operatorname{Blo}_{\left(D_{1}, \ldots, D_{m}\right)} \mathbb{A}^{d}=\left\{\left(z_{1}, \ldots, z_{d}, w_{1}, \ldots, w_{m}\right) \in \mathbb{C}^{d} \times\left(S^{1}\right)^{m} \mid z_{p} w_{p}^{-1} \in \mathbb{R}_{\geqslant 0} \text { for } 1 \leqslant p \leqslant m\right\}
$$

of the trivial torus bundle $\mathbb{C}^{d} \times\left(S^{1}\right)^{m} \rightarrow \mathbb{C}^{d}$. The fibre over a point $\left(z_{1}, \ldots z_{d}\right) \in \mathbb{A}^{d}$ is a torus whose dimension is the number of zeroes in the tuple $\left(z_{1}, \ldots, z_{m}\right)$. A standard neighbourhood of a point $\left(z_{1}, \ldots, z_{d}, w_{1}, \ldots, w_{m}\right)$ is given by

$$
\prod_{p=1}^{m}\left\{(z, w) \in \mathbb{C} \times S^{1}\left|z w \in \mathbb{R}_{\geqslant 0},\left|z-z_{p}\right|<\varepsilon, \arg \left(w w_{p}^{-1}\right)<\varepsilon\right\} \times \prod_{p=m+1}^{d}\left\{z \in \mathbb{C}| | z-z_{p} \mid<\varepsilon\right\}\right.
$$

where in the first product, for sufficiently small $\varepsilon$, each factor is either an open disk if $z_{p} \neq 0$, or a half-disk with boundary if $z_{p}=0$. From this description, we see that in general, $\operatorname{Blo}_{\left(D_{1}, \ldots, D_{m}\right)} \mathbb{A}^{d}$ does not admit the structure of a real manifold with boundary, at least not in a straightforward way. However, we have seen that for a general $d$-dimensional smooth complex variety $X$ and a normal crossing divisor with irreducible components $D_{1}, \ldots, D_{m}$, the oriented real blow-up $\operatorname{Blo}_{\left(D_{1}, \ldots, D_{m}\right)} X$ is locally diffeomorphic to a product $[0,1)^{a} \times(0,1)^{b}$ with $a+b=2 d$, the diffeomorphisms depending
on the choice of local coordinates. Such a beast is called a manifold with corners ${ }^{1}$. Notice that in Example 3.4.3, the real oriented blow-up of $\mathbb{P}^{1}$ at $n \cdot[0]$ is not a manifold with corners for $n>1$.
3.4 .5 (Manifolds with corners and the collar neighbourhood theorem). - The classical collar neighbourhood theorem for a manifold with boundary $M$ states that the boundary $\partial M$ admits a neighbourhood in $M$ which is diffeomorphic to $\partial M \times[0,1)$, see e.g. [46, §4.6]. For manifolds with corners a similar statement is true, except that one can of course not ask for a diffeomorphism.

ThEOREM 3.4.6 (Collar neighbourhood theorem). - Let $B$ be a real manifold with corners. The boundary $\partial B$ of $B$ admits an open neighbourhood which is homeomorphic to $\partial B \times[0,1)$. In particular, if $C$ is any subset of $\partial B$, then the inclusion $B \backslash C \hookrightarrow B$ is a homotopy equivalence.

We do not know of a reference for this proposition as it is stated. There is an ad-hoc construction of rounding corners: a manifold with corners is homeomorphic to a manifold with boundary via a homeomorphism respecting the boundaries. This procedure is described in the appendix $A r$ rondissement des variétés à coins by Douady and Herault to [15]. Having rounded off the corners, one can apply the classical collar neighbourhood theorem [46, §4.6]. Alternatively, we can avoid the rounding of corners by generalising one of the proofs of the classical collar neighbourhood theorem to manifolds with corners. Let us suppose for simplicity that the boundary $\partial B$ is compact. In a first step we construct an inward pointing vector field $F$ on $B$. Locally, on a chart $[0,1)^{a} \times(0,1)^{b}$ one can make an explicit choice of such a vector field, and using a partition of unity these vector fields can be glued together to a global one. Consider the associated flow $\varphi: \partial B \times \mathbb{R} \rightarrow B$, restricted to the boundary. By definition, this means that $\varphi(b, 0)=b$ and $\frac{\partial \varphi}{\partial t} \varphi(b, t)=F(\varphi(b, t))$. Locally, this flow exists and is unique for small times $0 \leqslant t<\varepsilon$, and since $\partial B$ is compact, we may assume that $\varphi$ is globally well defined for small times. The flow $\varphi: \partial B \times[0, \varepsilon) \rightarrow B$ is then locally a homeomorphism. Again using compactness of the boundary we see that we may choose a smaller $\varepsilon$ is necessary so that $\varphi$ is injective and a homeomorphism onto its image, and thus yields the collar neighbourhood.

### 3.5. Rapid decay cohomology as the cohomology of a real blow-up

3.5.1. - Let $\bar{X}$ be a smooth compact complex manifold of dimension $d$. Let $f: \bar{X} \rightarrow \mathbb{P}^{1}$ be a meromorphic function with pole divisor $P=f^{-1}(\infty)$, and let $H$ (for "horizontal") be another divisor on $X$. Suppose that the union $D=P \cup H$ is a normal crossing divisor and set $X=\bar{X} \backslash D$. Denote by $\pi: B \rightarrow \bar{X}$ the oriented real blow-up of $\bar{X}$ along the irreducible components of $D$, and by $\tilde{\mathbb{P}}^{1}$ the oriented real blow-up of $\mathbb{P}^{1}$ at $\infty$. The function $f$ lifts uniquely to a function $f_{B}: B \rightarrow \widetilde{\mathbb{P}}^{1}$

[^1]making the diagram

commutative. The local description of $f_{B}$ is as follows: around a given point $0 \in \bar{X}$, we can choose local coordinates $x_{1}, \ldots, x_{d}$ in which the divisor $D$ is given by $x_{1} \cdots x_{m}=0$, for some $0 \leqslant m \leqslant d$, and the function $f$ by
$$
f\left(x_{1}, \ldots, x_{d}\right)=\frac{f_{1}\left(x_{1}, \ldots, x_{d}\right)}{x_{1}^{e_{1}} x_{2}^{e_{2}} \cdots x_{m}^{e_{m}}}
$$
for a holomorphic function $f_{1}$ that does not vanishing at 0 and exponents $e_{p} \geqslant 0$. Locally around this point, $P_{\text {red }}($ resp. $H)$ is the zero locus of the product of those $x_{i}$ with $e_{i}>0$ (resp. $e_{i}=0$ ). As described in 3.4.4, the map $\pi: B \rightarrow X$ is locally given by the projection of
$$
\left\{\left(x_{1}, \ldots, x_{d}, w_{1}, \ldots, w_{m}\right) \in \mathbb{C}^{d} \times\left(S^{1}\right)^{m} \mid x_{p} w_{p}^{-1} \in \mathbb{R}_{\geqslant 0} \text { for } 1 \leqslant p \leqslant m\right\}
$$
onto the coordinates $\left(x_{1}, \ldots, x_{d}\right)$. The map $f_{B}: B \rightarrow \widetilde{\mathbb{P}}^{1}$ is given by
\[

f_{B}\left(x_{1}, ···, x_{d}, w_{1}, ···, w_{m}\right)= $$
\begin{cases}\left(x_{1}^{e_{1}} x_{2}^{e_{2}} \cdots x_{m}^{e_{m}}\right)^{-1} & \text { if } x_{1}^{e_{1}} x_{2}^{e_{2}} \cdots x_{m}^{e_{m}} \neq 0 \\ \left(w_{1}^{e_{1}} w_{2}^{e_{2}} \cdots w_{m}^{e_{m}}\right)^{-1} & \text { if } x_{1}^{e_{1}} x_{2}^{e_{2}} \cdots x_{m}^{e_{m}}=0\end{cases}
$$
\]

so that $f_{B}$ maps the pole divisor $P$ to the circle at infinity in $\widetilde{\mathbb{P}}^{1}$. Outside $P$, where the function $f$ is regular, $f_{B}$ is the composite of $f: \bar{X} \backslash P \rightarrow \mathbb{C}$ with the inclusion $\mathbb{C} \subseteq \widetilde{\mathbb{P}}^{1}$. Let us set

$$
\partial^{+} \widetilde{\mathbb{P}}^{1}=\left\{z \infty \in \partial \widetilde{\mathbb{P}}^{1} \mid \operatorname{Re}(z) \geqslant 0\right\} \quad \text { and } \quad \partial^{+} B=f_{B}^{-1}\left(\partial^{+} \widetilde{\mathbb{P}}^{1}\right)
$$

and denote by $B^{\circ} \subseteq B$ the subset

$$
B^{\circ}=B \backslash\left\{b \in \partial B \mid \pi(b) \in H \text { or } \operatorname{Re}\left(f_{B}(b)\right) \leqslant 0\right\}
$$

The boundary $\partial B^{\circ}$ of $B^{\circ}$ is the set of those $b \in \partial B$ such that $f$ has a pole at $\pi(b)$ and $f_{B}(b)$ has strictly positive real part.

Proposition 3.5.2. - In the situation of 3.5.1, the linear maps

$$
H^{n}\left(X, f^{-1}\left(S_{r}\right)\right) \stackrel{\cong}{\longleftarrow} H^{n}\left(B, f_{B}^{-1}\left(\widetilde{S}_{r}\right)\right) \stackrel{\cong}{\leftrightarrows} H^{n}\left(B, \partial^{+} B\right) \stackrel{\cong}{\oiiint} H^{n}\left(B^{\circ}, \partial B^{\circ}\right)
$$

induced by inclusions of pairs of topological spaces are isomorphisms for large enough real $r \gg 0$.
Proof. The inclusions $X=\bar{X} \backslash D \subseteq B$ and $f^{-1}\left(S_{r}\right) \subseteq f_{B}^{-1}\left(\widetilde{S}_{r}\right)$ are homotopy equivalences by the collar neighbourhood theorem, whence the leftmost isomorphism, with no constraint on $r$. That the middle morphism is an isomorphism for large $r$ is essentially a consequence of the proper base change theorem. Indeed, let $[0, \infty]$ be the real half-line completed by a point at infinity and consider the subspace

$$
C=\left\{(b, r) \in B \times[0, \infty] \mid f_{B}(b) \in \widetilde{S}_{r}\right\}
$$

of $B \times[0, \infty]$. The projection map pr : $B \times[0, \infty] \rightarrow[0, \infty]$ is proper because $\bar{X}$, and hence $B$, is compact. Therefore, by the proper base change theorem, the canonical morphism

$$
R^{n} \operatorname{pr}_{*}\left(\underline{\mathbb{Q}}_{[B \times[0, \infty], C]}\right)_{\infty} \longrightarrow H^{n}\left(B, \partial^{+} B\right)
$$

is an isomorphism. On the left-hand side stands the stalk at $\infty$ of the sheaf on $[0, \infty]$ associated with the presheaf $U \longmapsto H^{n}(B \times U,(B \times U) \cap C)$. The sets $[r, \infty]$ for $0 \leqslant r<\infty$ form a fundamental system of closed neighbourhoods of $\infty \in[0, \infty]$, and hence this stalk is by definition the colimit

$$
\underset{r<\infty}{\operatorname{colim}} H^{n}(B \times[r, \infty],(B \times[r, \infty]) \cap C)
$$

as $r$ goes to $\infty$. The pair $\left(B, f_{B}^{-1}\left(\widetilde{S}_{r}\right)\right)$ is a deformation retract of $\left.B \times[r, \infty],(B \times[r, \infty]) \cap C\right)$, and hence this colimit is the same as

$$
\underset{r<\infty}{\operatorname{colim}} H^{n}\left(B, f_{B}^{-1}\left(\widetilde{S}_{r}\right)\right),
$$

which eventually stabilises. Finally, the inclusion of pairs $\left(B^{\circ}, \partial B^{\circ}\right) \rightarrow\left(B, \partial^{+} B\right)$ is a homotopy equivalence, again by the collar neighbourhood theorem.

From Proposition 3.5.2, we immediately derive:

Corollary 3.5.3. - In the situation of 3.5.1, there is a canonical isomorphism of vector spaces

$$
H_{\mathrm{rd}}^{n}(X, f) \cong H^{n}\left(B, \partial^{+} B\right) \cong H^{n}\left(B^{\circ}, \partial B^{\circ}\right)
$$

3.5.4. - Let us again consider the situation of 3.5.1: Proposition 3.5.2 states that the rapid decay cohomology of $(X \backslash D, f)$ is canonically isomorphic to the cohomology of the pair $\left(B, \partial^{+} B\right)$. Let us denote by $\kappa:\left(B \backslash \partial^{+} B\right) \rightarrow B$ the inclusion, and write $\mathbb{Q}_{\left[B, \partial^{+} B\right]}=\kappa!\kappa^{*} \underline{\mathbb{Q}}_{B}$. The cohomology of the pair $\left(B, \partial^{+} B\right)$ is the cohomology of $B$ with coefficients in the sheaf $\mathbb{Q}_{\left[B, \partial^{+} B\right]}$, hence a canonical isomorphism

$$
H^{n}(X \backslash D, f) \cong H^{n}\left(B, \mathbb{Q}_{\left[B, \partial^{+} B\right]}\right) \cong H^{n}\left(X, R \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}\right)
$$

where $\pi: B \rightarrow X$ is the blow-up map. Let us examine the object $R \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}$ in the derived category of sheaves on $X$.
3.5.5. - Here is a topological preparation which will eventually help us to get a better grasp on $R \pi_{*} \underline{\mathbb{Q}}_{\left[B, \partial^{+} B\right]}$. Let $m \geqslant 1$ be an integer, and let $T \subseteq \mathbb{R}^{m} / \mathbb{Z}^{m}$ be the subset defined by

$$
T=\left\{\left(x_{1}, \ldots, x_{m}\right) \mid d e_{1} x_{1}+\cdots+d e_{m} x_{m} \equiv 0 \bmod 1\right\}
$$

for some integer $d \geqslant 1$ and primitive vector $e=\left(e_{1}, \ldots, e_{m}\right) \in \mathbb{Z}^{m}$. Here, primitive means not divisible in $\mathbb{Z}^{m}$ by an integer $\geqslant 2$, and in particular non-zero. We propose ourselves to find an explicit description of the homology groups $H_{p}\left(\mathbb{R}^{m} / \mathbb{Z}^{m}, T\right)$. The subspace $T$ has $d$ connected components, namely, $T$ is the disjoint union of the subtorus

$$
T_{0}=\left\{\left(x_{1}, \ldots, x_{m}\right) \mid e_{1} x_{1}+\cdots+e_{m} x_{m} \equiv 0 \bmod 1\right\}
$$

and its translates $T_{k}=T_{0}+\left(0, \ldots, 0, \frac{k}{d}\right)$. The pair of spaces $\left(\mathbb{R}^{m} / \mathbb{Z}^{m}, T\right)$ is homeomorphic to the product of $T_{0}$, which is a torus of dimension $m-1$, and the circle $\mathbb{R} / \mathbb{Z}$ marked in the $d$ points $\frac{1}{d} \mathbb{Z} / \mathbb{Z}$. The pair $\left(\mathbb{R} / \mathbb{Z}, \frac{1}{d} \mathbb{Z} / \mathbb{Z}\right)$ has homology in degree 1 only, and therefore the cross-product morphism

$$
H_{p-1}\left(T_{0}\right) \times H_{1}\left(\mathbb{R} / \mathbb{Z}, \frac{1}{d} \mathbb{Z} / \mathbb{Z}\right) \rightarrow H_{p}\left(\mathbb{R}^{m} / \mathbb{Z}^{m}, T\right)
$$

is an isomorphism. Fix a $\mathbb{Z}$-basis $a_{1}, \ldots, a_{m-1}$ of the orthogonal complement of $e$ in $\mathbb{Z}^{m}$. For any non-decreasing injective map $f:\{1,2, \ldots, p-1\} \rightarrow\{1,2, \ldots, m-1\}$, and any $k \in\{0,1, \ldots, d-1\}$, the continuous map

$$
c_{f, k}:[0,1]^{p} \rightarrow \mathbb{R}^{m} / \mathbb{Z}^{m} \quad c_{f, k}\left(t_{1}, \ldots, t_{p}\right)=\frac{k}{d} t_{p}+\sum_{i=1}^{p-1} a_{f(i)} t_{i}
$$

represents an element in $H_{p}\left(\mathbb{R}^{m} / \mathbb{Z}^{m}, T\right)$, once we decompose the cube $[0,1]^{p}$ appropriately into a sum of simplicies. Together, these elements form a basis of $H_{p}\left(\mathbb{R}^{m} / \mathbb{Z}^{m}, T\right)$. The dimension of $H_{p}\left(\mathbb{R}^{m} / \mathbb{Z}^{m}, T\right)$ is $d\binom{m-1}{p-1}$.

Proposition 3.5.6. - Set $n=\operatorname{dim} X$. The homology sheaves $R^{p} \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}$ are constructible and vanish for $p>n$. Therefore, $R \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}$ is an object of the derived category of constructible sheaves on $X$. The sheaf

$$
R \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}[n]
$$

is a perverse sheaf on $X$. Its Verdier dual is the sheaf $R \pi_{*} \mathbb{Q}_{\left[B, \partial^{0} B\right]}[n]$ where $\partial^{0} B \subseteq \partial B$ is the closure of the subset $\partial B \backslash \partial^{+} B$ of the boundary.

Proof. The blow-up map $\pi: B \rightarrow X$ is proper, and hence for every $x \in X$ the stalk at $x$ of the sheaf $R^{p} \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}$ is identical to $H^{p}\left(\pi^{-1}(x), \pi^{-1}(x) \cap \partial^{+} B\right)$. This shows that $R^{p} \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}$ is constructible with respect to the stratification given by intersections of the components of $D$. Precisely, if we denote by $D^{(m)} \subseteq X$ the smooth subvariety of codimension $m$ given by the union of all intersections of $m$ distinct components of $D$, then

$$
\varnothing \subseteq D^{(n)} \subseteq D^{(n-1)} \subseteq \cdots \subseteq D^{(2)} \subseteq D \subseteq X
$$

is a stratification for $R^{p} \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}$, for all $p$. The fibre $\pi^{-1}(x)$ is a real torus of real dimension $m$, where $m \leqslant n$ is the number of components of $D$ meeting at $x$, and $\pi^{-1}(x) \cap \partial^{+} B$ is either empty or a finite union of real tori of dimension $m-1$. In view of 3.5 .5 we can be more explicit: If $x$ is in the intersection of components $D_{1}, D_{2}, \ldots, D_{m}$ of $D$, and $f$ has a pole of order $e_{i} \geqslant 0$ on $D_{i}$, then the stalk of $R^{p} \pi_{*} \mathbb{Q}_{\left(B, \partial^{+} B\right)}$ at $x$ has dimension $\operatorname{gcd}\left(e_{1}, \ldots, e_{m}\right)\binom{m-1}{p-1}$ if $f$ has a pole at $x$, i.e. at least one of the $e_{i}$ is non-zero, and dimension $\binom{m}{p}$ if $f$ is regular at $x$. In either case, $H^{p}\left(\pi^{-1}(x), \pi^{-1}(x) \cap \partial^{+} B\right)=0$ if $p>m$, and hence the inclusion

$$
\begin{equation*}
\operatorname{supp}\left(R^{p} \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}\right) \subseteq D^{(p)} \tag{3.5.6.1}
\end{equation*}
$$

holds. Next, we compute the dual of $R^{p} \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}$. Let us denote by $\omega_{B / X}=\pi^{!} \mathbb{Q}_{X}$ the relative dualising sheaf of $\pi: B \rightarrow X$. Local Verdier duality reads

$$
R \mathcal{H o m}\left(R \pi_{*} \mathbb{Q}_{\left[B, \partial^{+} B\right]}, \mathbb{Q}\right) \cong R \pi_{*} R \mathcal{H o m}\left(\mathbb{Q}_{\left[B, \partial^{+} B\right]}, \omega_{B / X}\right)
$$

and hence it suffices to produce a canonical isomorphism

$$
\operatorname{RHom}\left(\mathbb{Q}_{\left[B, \partial^{+} B\right]}, \omega_{B / X}\right)=\mathbb{Q}_{\left[B, \partial^{0} B\right]}
$$

in the derived category of sheaves on $B$. Let us name the inclusions

$$
\alpha: B \backslash \partial B \rightarrow B \quad \kappa: B \backslash \partial^{+} B \rightarrow B \quad \lambda: B \backslash \partial^{0} B \rightarrow B
$$

so that $\mathbb{Q}_{\left(B, \partial^{+} B\right)}=\kappa!\kappa^{*} \mathbb{Q}_{B}$ and $\mathbb{Q}_{\left(B, \partial^{0} B\right)}=\lambda_{!} \lambda^{*} \mathbb{Q}_{B}$. Since $X$ is smooth of real dimension $2 n$, the dualising sheaf on $X$ is $\omega_{X}=\mathbb{Q}_{X}[2 n]$, and hence we can compute the relative dualising sheaf $\omega_{B / X}$ as $\omega_{B}[-2 n]$. We find $\omega_{B / X}=\alpha_{!} \alpha^{*} \mathbb{Q}_{B}$, as we would for any $C^{0}$-manifold with boundary. Notice that for any sheaf $F$ on $B$ there is a natural isomorphism $\mathcal{H o m}\left(\kappa_{!} \kappa^{*} \mathbb{Q}_{B}, F\right)=\kappa_{*} \kappa^{*} F$, and hence we find in particular an isomorphism

$$
R \mathcal{H o m}\left(\kappa!\kappa^{*} \mathbb{Q}_{B}, \omega_{B / X}\right)=R \kappa_{*} \kappa^{*} \alpha!\alpha^{*} \mathbb{Q}_{B}=\kappa_{*} \kappa^{*} \alpha_{!} \alpha^{*} \mathbb{Q}_{B}
$$

in the derived category of sheaves on $B$. The functor $\kappa_{*}$ is exact, hence the equality on the right. Inspecting sections, we find $\kappa_{*} \kappa^{*} \alpha_{!} \alpha^{*} \mathbb{Q}_{B}=\lambda_{!} \lambda^{*} \mathbb{Q}_{B}$ as we wanted to show. For any $p \geqslant 0$, the direct image $R^{p} \pi_{*} \mathbb{Q}_{\left(B, \partial^{0} B\right)}$ is a constructible sheaf on $X$, and since $\pi$ is proper, we can compute its stalks using proper base change: the stalk at $x$ is isomorphic to $H^{p}\left(\pi^{-1}(x), \pi^{-1}(x) \cap \partial^{0} B\right)$. The fibre $\pi^{-1}(x)$ is still a real torus of real dimension equal to the number $m$ of components of $D$ meeting at $x$, and $\pi^{-1}(x) \cap \partial^{0} B$ is either all of $\pi^{-1}(x)$ in case $f$ is regular on one of the components of $D$ meeting at $x$, or else, a finite union of real tori of dimension $m-1$. In either case, $H^{p}\left(\pi^{-1}(x), \pi^{-1}(x) \cap \partial^{0} B\right)=0$ if $p>m$, and hence the inclusion

$$
\begin{equation*}
\operatorname{supp}\left(R^{p} \pi_{*} \mathbb{Q}_{\left[B, \partial^{0} B\right]}\right) \subseteq D^{(p)} \tag{3.5.6.2}
\end{equation*}
$$

holds. Together, the inclusions (3.5.6.1) and (3.5.6.2) show that $\mathbb{Q}_{\left[B, \partial^{+} B\right]}[n]$ is perverse.
3.5.7 (Good compactifications). - In 3.5.1 and Proposition 3.5.2 we started with a smooth and compact complex manifold $X$ and a function $X \rightarrow \mathbb{P}^{1}$, restricting to $X \backslash D \rightarrow \mathbb{A}^{1}$ for some normal crossing divisor $D$. In practice, we usually start with a smooth variety $X$ and a function $f: X \rightarrow \mathbb{A}^{1}$, and seek to compactify $X$ by a normal crossing divisor in such a way that $f$ extends to a function with values in $\mathbb{P}^{1}$ on the compactification.

Definition 3.5.8. - Let $X$ be a smooth variety over $k$, let $Y \subseteq X$ be a normal crossing divisor, and let $f: X \rightarrow \mathbb{A}^{1}$ be a regular function. A good compactification of $(X, Y, f)$ is a triple $(\bar{X}, \bar{Y}, \bar{f})$ consisting of a smooth projective variety $\bar{X}$ over $k$ containing $X$ as the complement of a normal crossing divisor $D$, a divisor $\bar{Y} \subseteq X$ such that $Y=\bar{Y} \cap X$ and that $\bar{Y}+D$ has normal crossings, and a morphism $\bar{f}: \bar{X} \rightarrow \mathbb{P}^{1}$ extending $f$.
3.5.9. - The situation of Definition 3.5.8 one has a commutative diagram

where all horizontal maps are inclusions. A good compactification of $(X, Y, f)$ always exists. Indeed, let $\bar{X}_{0}$ be any smooth compactification of $X$ by a normal crossing divisor $D$, such that also $\bar{Y}_{0}+D$ is a normal crossing divisor, where $\bar{Y}_{0}$ is the closure of $Y$ in $\bar{X}_{0}$. Such a compactification can be "found" using resolutions of singularities. The function $f$ extends to a rational map $\bar{X}_{0} \rightarrow \mathbb{P}^{1}$. By resolution of indeterminacies, there exists a finite tower of blow-ups $\bar{X} \rightarrow \bar{X}_{m-1} \rightarrow \cdots \rightarrow \bar{X}_{0}$ at smooth centers of $D$ such that $f$ extends to a morphism $\bar{f}: \bar{X} \rightarrow \mathbb{P}^{1}$. Define $\bar{Y}$ to be the strict transform of $\bar{Y}_{0}$ in $\bar{X}$.

Example 3.5.10. - Let $X=\mathbb{A}^{2}=\operatorname{Spec} \mathbb{Q}[x, y]$, together with the function $f=x^{2}+y^{2}$. We start with the compactification $\bar{X}_{0}=\mathbb{P}^{1} \times \mathbb{P}^{1}$ and the rational map

$$
\begin{aligned}
& \mathbb{P}^{1} \times \mathbb{P}^{1} \longrightarrow \mathbb{P}^{1} \\
& {[x: a],[y: b] } \longmapsto\left[b^{2} x^{2}+a^{2} y^{2}: a^{2} b^{2}\right],
\end{aligned}
$$

whose only indeterminacy is $(\infty, \infty)$. Let $\bar{X}$ be the blow-up of this point, i.e. the closed subvariety of $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{1}$ given by the equation $a v=b u$, where $[u: v]$ are the coordinates of the last $\mathbb{P}^{1}$. Then $f$ extends to the morphism

$$
\begin{aligned}
& \bar{X} \xrightarrow{\bar{f}} \mathbb{P}^{1} \\
& {[x: a],[y: b],[u: v] \longmapsto\left[v^{2} x^{2}+u^{2} y^{2}: u^{2} v^{2}\right] .}
\end{aligned}
$$

The pole divisor has irreducible components $P_{1}=\infty \times \mathbb{P}^{1} \times[0: 1]$ and $P_{2}=\mathbb{P}^{1} \times \infty \times[1: 0]$, and each of them appears with multiplicity two. The horizontal component is the exceptional divisor.

Corollary 3.5.11. - Let $X$ be a smooth complex algebraic variety with potential $f: X \rightarrow \mathbb{A}^{1}$ and let $Y \subseteq X$ be a normal crossing divisor. Let $(\bar{X}, \bar{Y}, \bar{f})$ be a good compactification of $(X, Y, f)$. Let $\pi: B \rightarrow \bar{X}$ be the real blow-up of $\bar{X}$ along the components of $D=\bar{X} \backslash X$, let $B_{Y} \subseteq B$ be the real blow-up of $Y$ along the components of $Y \cap D$, and let $\bar{f}_{B}: B \rightarrow \widetilde{\mathbb{P}}^{1}$ be the lift of $\bar{f}$ to $B$. There is a canonical isomorphism

$$
H_{\mathrm{rd}}^{n}(X, Y, f) \cong H^{n}\left(B, B_{Y} \cup \partial^{+} B\right)
$$

Proof. If $Y$ is empty, this is the statement of Corollary 3.5.3. If $Y$ has only one (smooth) irreducible component, there is a commutative diagram

we can also apply Proposition 3.5.2 to $Y$ and deduce the statement of the corollary by dévissage. The general case is by induction on the number of irreducible components of $Y$.

### 3.6. The Künneth formula

The classical Künneth formula relates the singular cohomology of reasonable topological spaces $X_{1}$ and $X_{2}$ to the cohomology of the product space $X_{1} \times X_{2}$. In the case of rational coefficients, or indeed coefficients in any field, the Künneth formula simply states that the map

$$
H^{*}\left(X_{1}, \mathbb{Q}\right) \otimes H^{*}\left(X_{2}, \mathbb{Q}\right) \longrightarrow H^{*}\left(X_{1} \times X_{2}, \mathbb{Q}\right)
$$

induced by the cup-product is an isomorphism of graded vector spaces. This works equally well for pairs of spaces: given closed subspaces $Y_{1} \subseteq X_{1}$ and $Y_{2} \subseteq X_{2}$, the cup-product induces an isomorphism of graded vector spaces

$$
H^{*}\left(X_{1}, Y_{1} ; \mathbb{Q}\right) \otimes H^{*}\left(X_{2}, Y_{1} ; \mathbb{Q}\right) \longrightarrow H^{*}\left(X_{1} \times X_{2},\left(X_{1} \times Y_{2}\right) \cup\left(Y_{1} \times X_{2}\right) ; \mathbb{Q}\right)
$$

In this section, we introduce the cup-product for rapid decay cohomology and establish a Künneth formula in this context.

Definition 3.6.1. - Given sets (schemes, topological spaces, ...) $X_{1}$ and $X_{2}$, a commutative group (scheme, ...) $C$ and maps $f_{1}: X_{1} \rightarrow C$ and $f_{2}: X_{2} \rightarrow C$, the Thom-Sebastiani sum $f_{1} \boxplus f_{2}$ is the map $X_{1} \times X_{2} \rightarrow C$ defined by the formula

$$
\left(f_{1} \boxplus f_{2}\right)\left(x_{1}, x_{2}\right)=f_{1}\left(x_{1}\right)+f_{2}\left(x_{2}\right)
$$

3.6.2 (Elementary construction of the cup-product). - Let $\left(X_{1}, Y_{1}\right)$ and ( $X_{2}, Y_{2}$ ) be pairs consisting of a complex variety and a closed subvariety, and let $f_{1}: X_{1} \rightarrow \mathbb{A}^{1}$ and $f_{2}: X_{2} \rightarrow \mathbb{A}^{1}$ be regular functions. The cup product

$$
\begin{equation*}
H^{i}\left(X_{1}, Y_{1}, f_{1}\right) \otimes H^{j}\left(X_{2}, Y_{2}, f_{2}\right) \longrightarrow H^{i+j}\left(X_{1} \times X_{2},\left(Y_{1} \times X_{2}\right) \cup\left(X_{1} \times Y_{2}\right), f_{1} \boxplus f_{2}\right) \tag{3.6.2.1}
\end{equation*}
$$

is the linear map obtained, by passing to the limit $r \rightarrow+\infty$, from the composition

$$
\begin{aligned}
H^{i}\left(X_{1},\right. & \left.Y_{1} \cup f_{1}^{-1}\left(S_{r}\right)\right) \otimes H^{j}\left(X_{2}, Y_{2} \cup f_{2}^{-1}\left(S_{r}\right)\right) \\
& \longrightarrow H^{i+j}\left(X_{1} \times X_{2},\left(Y_{1} \times X_{2}\right) \cup\left(X_{1} \times Y_{2}\right) \cup\left(f_{1}^{-1}\left(S_{r}\right) \times X_{2}\right) \cup\left(X_{1} \times f_{2}^{-1}\left(S_{r}\right)\right)\right) \\
& \longrightarrow H^{i+j}\left(X_{1} \times X_{2},\left(Y_{1} \times X_{2}\right) \cup\left(X_{1} \times Y_{2}\right) \cup\left(f_{1} \boxplus f_{2}\right)^{-1}\left(S_{2 r}\right)\right)
\end{aligned}
$$

where the first map is the usual cup product of relative cohomology, and the second one is induced by the inclusion of closed subsets $\left(f_{1} \boxplus f_{2}\right)^{-1}\left(S_{2 r}\right) \subset\left(f_{1}^{-1}\left(S_{r}\right) \times X_{2}\right) \cup\left(X_{1} \times f_{2}^{-1}\left(S_{r}\right)\right)$.

Proposition 3.6.3 (Künneth formula). - Let $\left(X_{1}, f_{1}\right)$ and $\left(X_{2}, f_{2}\right)$ be complex varieties with potentials and let $Y_{1} \subseteq X_{1}$ and $Y_{2} \subseteq X_{2}$ be closed varieties. The cup product (3.6.2.1) induces an isomorphism of graded vector spaces

$$
H^{*}\left(X_{1}, Y_{1}, f_{1}\right) \otimes H^{*}\left(X_{2}, Y_{2}, f_{2}\right) \xrightarrow{\cong} H^{*}\left(X_{1} \times X_{2},\left(Y_{1} \times X_{2}\right) \cup\left(X_{1} \times Y_{2}\right), f_{1} \boxplus f_{2}\right)
$$

Proof. The Künneth formula for relative topological spaces yields an isomorphism of graded vector spaces

$$
H^{*}\left(X_{1}, f_{1}^{-1}\left(S_{r}\right)\right) \otimes H^{*}\left(X_{2}, f_{2}^{-1}\left(S_{r}\right)\right) \stackrel{\cong}{\longrightarrow} H^{*}\left(X_{1} \times X_{2},\left(f_{1}^{-1}\left(S_{r}\right) \times X_{2}\right) \cup\left(X_{1} \times f_{2}^{-1}\left(S_{r}\right)\right)\right)
$$

induced by cup products. To ease the notation, we left out $Y_{1}$ and $Y_{2}$ from the notation. We need to show that the linear map

$$
\begin{equation*}
H^{n}\left(\left(f_{1}^{-1}\left(S_{r}\right) \times X_{2}\right) \cup\left(X_{1} \times f_{2}^{-1}\left(S_{r}\right)\right)\right) \rightarrow H^{n}\left(\left(f_{1} \boxplus f_{2}\right)^{-1}\left(S_{2 r}\right)\right) \tag{3.6.3.1}
\end{equation*}
$$

induced by the inclusion $\left(f_{1} \boxplus f_{2}\right)^{-1}\left(S_{2 r}\right) \subset\left(f_{1}^{-1}\left(S_{r}\right) \times X_{2}\right) \cup\left(X_{1} \times f_{2}^{-1}\left(S_{r}\right)\right)$ is an isomorphism for sufficiently large real $r$. In terms of constructible sheaves, this amounts to the following: let $F_{i}$ be a constructible sheaf on $X_{i}$, say for example $F_{i}=\mathbb{Q}_{\left[X_{i}, Y_{i}\right]}$, and consider the open sets

$$
U=\left\{\left(z_{1}, z_{2}\right)=\operatorname{Re}\left(z_{1}\right) \geqslant r \text { or } \operatorname{Re}\left(z_{1}\right) \geqslant r \quad \text { and } \quad V=\left\{\left(z_{1}, z_{2}\right) \mid \operatorname{Re}\left(z_{1}+z_{2}\right) \geqslant 2 r\right\}\right.
$$

We must show that the map

$$
H^{n}\left(U,\left.\left(R f_{1 *} F_{1} \boxtimes R f_{1 *} F_{2}\right)\right|_{U}\right) \rightarrow H^{n}\left(V,\left.\left(R f_{1 *} F_{1} \boxtimes R f_{1 *} F_{2}\right)\right|_{V}\right)
$$

induced by the inclusion $V \subseteq U$ is an isomorphism. The homology sheaves of $R f_{1_{*}} F_{1} \boxtimes R f_{1_{*}} F_{2}$ are constructible with respect to a stratification consisting of finitely many horizontal and vertical lines. Let $G$ be any such constructible sheaf, that is, $G$ is a sheaf on $\mathbb{C}^{2}$ constructible with respect to the stratification given by lines $\mathbb{C} \times\{s\}$ or $\{s\} \times \mathbb{C}$ and their intersection points, where $s$ belongs to a finite set of complex numbers $S$. Fix a real $r$ such that $r>\operatorname{Re}(s)$ for all $s \in S$, and let us show that the inclusion $V \subseteq U$ induces an isomorphism $H^{n}\left(V,\left.G\right|_{V}\right) \cong H^{n}\left(U,\left.G\right|_{U}\right)$. To this end, define

$$
B=\left\{\left(z_{1}, z_{2}, t\right) \in \mathbb{C}^{2} \times[0,1] \mid \operatorname{Re}\left(z_{1}+t z_{2}\right) \geqslant r+t r \text { or } \operatorname{Re}\left(t z_{1}+z_{2}\right) \geqslant r+t r\right\}
$$

and consider the sheaf $G_{B}=\left.\left(\operatorname{pr}^{*} G\right)\right|_{B}$ on $B$. The projection $p: B \rightarrow[0,1]$ is a topological fibre bundle, its fibre over 0 is $U$ and its fibre over 1 is $V$. The sheaf $G_{B}$ is constructible with respect to a stratification of $B$ by subvarieties, each of which also is a fibre bundle over $[0,1]$, and hence the sheaf $R^{n} p_{*} G_{B}$ is a local system on $[0,1]$. Parallel transport from the fibre over 0 to the fibre over 1 is the isomorphism we sought.
3.6.4. - Here is an illustration in the real plane of the various sets considered in the proof of the Künneth formula. In this picture the horizontal and vertical lines represent the stratification


Figure 3.6.2. The sets $V \subseteq p^{-1}(t) \subseteq U$
for $G$, so $G$ is a local system outside these lines, and also when restricted to each of the lines except at the intersection points. The whole coloured region is $U$, and the blue region is $V$. The green and blue parts together form $p^{-1}(t)$.
3.6.5. - We defined the cup product for rapid decay cohomology in 3.6.2 in elementary terms. We can also give a construction in terms of sheaf cohomology.

### 3.7. Rapid decay cohomology with support

In this section, we define the rapid decay cohomology with support on some closed subvariety. As one is accustomed, this cohomology with support will fit into a long exact sequence relating it with the rapid decay cohomology of the ambient variety and the rapid decay cohomology of the open complement. We will also define a Gysin map for rapid decay cohomology, and construct the corresponding long exact Gysin sequence.
3.7.1. - Let $X$ be a variety equipped with a potential $f: X \rightarrow \mathbb{A}^{1}$, and let $Y \subseteq X$ be a closed subvariety. Let $\alpha: Z \rightarrow X$ be the inclusion of a closed subvariety with complement $\beta: U \rightarrow X$. We call

$$
\begin{equation*}
H_{\mathrm{rd}, Z}^{n}(X, Y, f)=\Psi_{\infty} \Pi\left({ }^{p} \mathcal{H}^{n}\left(R f_{*} R \alpha_{!} \alpha^{!} \mathbb{Q}_{[X, Y]}\right)\right) \tag{3.7.1.1}
\end{equation*}
$$

the rapid decay cohomology in degree $n$ of $(X, Y, f)$ with support on $Z$. There is an exact triangle

$$
R \alpha_{!} \alpha^{\prime} \mathbb{Q}_{[X, Y]} \rightarrow \mathbb{Q}_{[X, Y]} \rightarrow R \beta_{*} \beta^{*} \mathbb{Q}_{[X, Y]} \rightarrow R \alpha_{!} \alpha^{!} \mathbb{Q}_{[X, Y]}[1]
$$

in the derived category of constructible sheaves on $X$. The sheaf $\beta^{*} \mathbb{Q}_{[X, Y]}$ on $U$ is the same as $\mathbb{Q}_{[U, Y \cap U]}$, and the functors $\Psi_{\infty}$ and $\Pi$ are exact. Hence we obtain a long exact sequence

$$
\begin{equation*}
\cdots \rightarrow H_{\mathrm{rd}, Z}^{n}(X, Y, f) \rightarrow H_{\mathrm{rd}}^{n}(X, Y, f) \rightarrow H_{\mathrm{rd}}^{n}\left(U, Y \cap U,\left.f\right|_{U}\right) \rightarrow H_{\mathrm{rd}, Z}^{n+1}(X, Y, f) \rightarrow \cdots \tag{3.7.1.2}
\end{equation*}
$$

of vector spaces. We call the morphism $H_{\mathrm{rd}, Z}^{n}(X, Y, f) \rightarrow H_{\mathrm{rd}}^{n}(X, Y, f)$ the forget supports map. The morphism following it is the usual restriction morphism, that is, the morphism in rapid decay cohomology induced by the inclusion $U \rightarrow X$.
3.7.2. - Let $X$ be a smooth variety, together with a regular function $f$. Let $i: Z \hookrightarrow X$ be a smooth closed subvariety of pure codimension $c$ with complementary immersion $j: U \hookrightarrow X$. Recall from 2.1.5 that $i_{!}=i_{*}$ and $i^{!}=i^{*}[-2 c]$, so in particular $i_{!}!\mathbb{Q}=i_{*} i^{*} \mathbb{Q}[-2 c]$. The adjunction morphism for $i_{!}$sits in a triangle

$$
i_{*} i^{*} \mathbb{Q}[-2 c] \longrightarrow \mathbb{Q} \longrightarrow R j_{*} j^{*} \mathbb{Q} .
$$

Upon application of $R f_{*}$, this triangle induces a long exact sequence of perverse sheaves

$$
\cdots \longrightarrow{ }^{p} \mathcal{H}^{n-2 c}\left(R f_{*} i_{*} i^{*} \mathbb{Q}\right) \longrightarrow{ }^{p} \mathcal{H}^{n}\left(R f_{*} \mathbb{Q}\right) \longrightarrow{ }^{p} \mathcal{H}^{n}\left(R f_{*} R j_{*} j^{*} \mathbb{Q}\right) \longrightarrow \cdots
$$

Taking the projector $\Pi$ and the nearby fibre at infinity we find a long exact sequence

$$
\begin{equation*}
\cdots \longrightarrow H_{\mathrm{rd}}^{n-2 c}\left(Z, f_{\mid Z}\right) \longrightarrow H_{\mathrm{rd}}^{n}(X, f) \longrightarrow H_{\mathrm{rd}}^{n}\left(U, f_{\mid U}\right) \longrightarrow \cdots \tag{3.7.2.1}
\end{equation*}
$$

of rational vector spaces which is called the Gysin long exact sequence. The morphism

$$
\begin{equation*}
H_{\mathrm{rd}}^{n-2 c}\left(Z, f_{\mid Z}\right) \longrightarrow H_{\mathrm{rd}}^{n}(X, f) \tag{3.7.2.2}
\end{equation*}
$$

is call the Gysin map for rapid decay cohomology.

### 3.8. Poincaré-Verdier duality

The goal of this section is to establish a Poincaré-Verdier duality pairing for rapid decay cohomology. To construct a natural duality pairing such as displayed in (3.8.2.1) below out of local Verdier duality is an exercise in the six functors formalism. However, since later we want to show that the resulting pairing is motivic, in a sense yet to be made precise, a sheaf-theoretic construction is not enough for us. We will rather construct a specific pairing by geometric means, not involving local Verdier duality. Then, we will have to check that the pairing we constructed geometrically is indeed a perfect pairing, by comparing it to the sheaf-theoretic construction.
3.8.1. - To say that a finite-dimensional vector space $V$ is dual to another space $W$ usually means that there is some particular linear map $p: V \otimes W \rightarrow \mathbb{Q}$ called pairing. This pairing has to be perfect, meaning that the induced maps $V \rightarrow \operatorname{Hom}(W, \mathbb{Q})$ and $W \rightarrow \operatorname{Hom}(V, \mathbb{Q})$ are isomorphisms. Less usual, but better suited to our later needs, is the point of view that to exhibit $W$ as the dual of $V$ is to give a linear map $c: \mathbb{Q} \rightarrow W \otimes V$ called copairing. Again, this copairing has to be perfect, that is, the induced $\operatorname{map} \operatorname{Hom}(W, \mathbb{Q}) \rightarrow V$ sending $\varphi$ to $\left(\varphi \otimes \mathrm{id}_{V}\right)(c(1))$ and its companion $\operatorname{Hom}(V, \mathbb{Q}) \rightarrow W$ are both isomorphisms. Given vector spaces $V$ and $W$, there is a canonical bijection between the set of perfect pairings $p: V \otimes W \rightarrow \mathbb{Q}$ and the set of perfect copairings $c: \mathbb{Q} \rightarrow W \otimes V$, as both sets are also in canonical bijection with the set of isomorphisms between $V$ and $\operatorname{Hom}(W, \mathbb{Q})$. A pairing $p$ and a copairing $c$ correspond to each other via this bijection if the composite linear map

$$
\begin{equation*}
V=V \otimes \mathbb{Q} \xrightarrow{\mathrm{id}_{V} \otimes c} V \otimes W \otimes V \xrightarrow{p \otimes \mathrm{id}_{V}} \mathbb{Q} \otimes V=V \tag{3.8.1.1}
\end{equation*}
$$

is the identity on $V$.
3.8.2. - Let $X$ be a smooth connected complex variety of dimension $d$, let $Y \subseteq X$ be a normal crossing divisor, and let $f$ be a regular function on $X$. We choose a good compactification $(\bar{X}, \bar{Y}, \bar{f})$ as in Definition 3.5.8, we let $P$ be the reduced pole divisor of $\bar{f}$, and we decompose the divisor at infinity $D=\bar{X} \backslash X$ as a sum $D=P+H$. We set

$$
X^{\prime}=\bar{X} \backslash(\bar{Y} \cap P), \quad Y^{\prime}=H \backslash(H \cap P)
$$

and denote by $f^{\prime}$ the restriction of $\bar{f}$ to $X^{\prime}$. Our aim is to construct a canonical duality pairing

$$
\begin{equation*}
H_{\mathrm{rd}}^{n}(X, Y, f) \otimes H_{\mathrm{rd}}^{2 d-n}\left(X^{\prime}, Y^{\prime},-f^{\prime}\right) \longrightarrow \mathbb{Q}(-d) . \tag{3.8.2.1}
\end{equation*}
$$

Observe that in the special case where $f=0$ and $Y$ is empty, the space $X^{\prime}$ is just a smooth compactification of $X$ and $H_{\mathrm{rd}}^{2 d-n}\left(X^{\prime}, Y^{\prime},-f^{\prime}\right)$ is the cohomology with compact support $H_{c}^{2 d-n}(X)$. We want to recover from (3.8.2.1) the classical Poincaré-Verdier duality pairing. For non-empty $Y$ but $f$ still zero, the resulting pairing is sometimes called "red-green duality". What we will actually construct is not directly a pairing (3.8.2.1), but rather a copairing. Set $U=X \cap X^{\prime}$ and let $\Delta_{U} \subseteq X \times X^{\prime}$ be the diagonal embedding of $U$. We call Poincaré-Verdier copairing the following composite linear map:

$$
\begin{align*}
\mathbb{Q}(-d)=H^{0}\left(\Delta_{U}\right)(-d) \xrightarrow{\text { Gysin }} & H_{\mathrm{rd}}^{2 d}\left(X \times X^{\prime}, Y \times X^{\prime} \cup X \times Y^{\prime}, f \boxplus-f^{\prime}\right) \\
\downarrow & \downarrow \text { Künneth }  \tag{3.8.2.2}\\
& H_{\mathrm{rd}}^{n}(X, Y, f) \otimes H_{\mathrm{rd}}^{2 d-n}\left(X^{\prime}, Y^{\prime},-f^{\prime}\right) .
\end{align*}
$$

We can recognise this copairing as the fibre at infinity of the similarly defined copairing for perverse cohomology.

Theorem 3.8.3. - The Poincaré-Verdier copairing (3.8.2.2) is perfect.
3.8.4. - Let us explain how the global Verdier duality theorem can be formulated in terms of copairings. Fix an object $F$ in the derived category of constructible sheaves on a complex algebraic variety $X$. We write $\Delta: X \rightarrow X \times X$ for the inclusion of the diagonal, and $\pi$ for the map from $X$ to a point, so the dualising sheaf on $X$ is the complex $\omega=\pi^{!} \mathbb{Q}$. The dual of the evaluation morphism $\varepsilon: \Delta^{*}(F \boxtimes \mathbb{D} F)=F \otimes \mathbb{D} F \rightarrow \omega$ is a morphism $\mathbb{D} \varepsilon: \mathbb{Q} \rightarrow \Delta^{!}(\mathbb{D} F \boxtimes F)$. Writing $\pi_{*}$ as the composition of $\Delta$ and $\pi^{2}=\pi \times \pi$, we obtain the sequence of morphisms

in the derived category of vector spaces. We have used here the fact that $\Delta$ is proper, and hence $\Delta_{*}=\Delta_{!}$. Taking homology in degree 0 and projecting onto some component in the Künneth formula yields a copairing

$$
\mathbb{Q}=H^{0}(X) \rightarrow H^{-n}(X, \mathbb{D} F) \otimes H^{n}(X, F)
$$

which is perfect, and corresponds via the linear algebra operations outlined in 3.8.1 to the usual Verdier duality pairing $H^{n}(X, F) \otimes H^{-n}(X, \mathbb{D} F) \rightarrow \mathbb{Q}$. To verify this fact, which we are not going to do here, one has to check that the composite as in (3.8.1.1) of the pairing and the copairing is
equal to the identity, which amounts to prove that the following diagram commutes:


Our next task is to compare the recipe for the Poincaré-Verdier copairing (3.8.2.2) with the sheaftheoretic description of the Verdier duality copairing (3.8.4.1).

Lemma 3.8.5. - Let $Z$ be a smooth complex manifold and $D, D^{\prime} \subset Z$ closed subvarieties which have no common irreducible component and such that $D \cup D^{\prime}$ is a normal crossing divisor. Consider the diagram of inclusions


There is a canonical isomorphism

$$
\begin{equation*}
\lambda_{!} R \beta_{*} \mathbb{Q}_{Z \backslash\left(D \cup D^{\prime}\right)} \cong R \lambda_{*}^{\prime} \beta_{!}^{\prime} \mathbb{Q}_{Z \backslash\left(D \cup D^{\prime}\right)} \tag{3.8.5.2}
\end{equation*}
$$

in the derived category of constructible sheaves on $Z$.

Proof. A proof can be found e.g. in [13, Lemma 6.1.1]. We recall the argument for the convenience of the reader. Noting that $R \beta_{*} \mathbb{Q}_{Z \backslash\left(D \cup D^{\prime}\right)}=\lambda^{*} R \lambda_{*}^{\prime} \beta_{!}^{\prime} \mathbb{Q}_{Z \backslash\left(D \cup D^{\prime}\right)}$, adjunction yields a canonical morphism in the derived category of constructible sheaves on $Z$,

$$
\begin{equation*}
\lambda_{!} R \beta_{*} \mathbb{Q}_{Z \backslash\left(D \cup D^{\prime}\right)} \longrightarrow R \lambda_{*}^{\prime} \beta_{!}^{\prime} \mathbb{Q}_{Z \backslash\left(D \cup D^{\prime}\right)} \tag{3.8.5.3}
\end{equation*}
$$

extending the identity on $\mathbb{Q}_{Z \backslash\left(D \cup D^{\prime}\right)}$. It suffices to prove that (3.8.5.3) induces an isomorphism on stalks at each $z \in D \cup D^{\prime}$. Since $D$ and $D^{\prime}$ have no common component, both sides are zero unless $z \in D \cap D^{\prime}$ so we may assume that this is the case. Since $D \cup D^{\prime}$ has normal crossings, there exists a polydisk $B$ centered at $z$, a decomposition $B=B_{D} \times B_{D^{\prime}}$ into a product of smaller-dimensional polydisks, and analytic subvarieties $P \subset B_{D}$ and $P^{\prime} \subset B_{D^{\prime}}$ such that $D \cap B=P \times B_{D^{\prime}}$ and
$D^{\prime} \cap B=B_{D} \times P^{\prime}$. Therefore, locally around $z$ for the analytic topology, diagram (3.8.5.1) looks like

where $j_{D}: B_{D} \backslash P \hookrightarrow B_{D}$ and $j_{D^{\prime}}: B_{D^{\prime}} \backslash P^{\prime} \hookrightarrow B_{D^{\prime}}$ stand for the inclusions. Writing $\mathbb{Q}_{\left(B_{D} \backslash P\right) \times\left(B_{D^{\prime}} \backslash P^{\prime}\right)}$ as $\mathbb{Q}_{B_{D} \backslash P} \boxtimes \mathbb{Q}_{B_{D^{\prime}} \backslash P^{\prime}}$, both sides of (3.8.5.3) are canonically isomorphic to

$$
\left(j_{D}\right)!\mathbb{Q}_{B_{D} \backslash P} \boxtimes R\left(j_{D^{\prime}}\right)_{*} \mathbb{Q}_{B_{D^{\prime}} \backslash P^{\prime}}
$$

by a variant of the Künneth formula, thus finishing the proof.

Proposition 3.8.6. - Let $(X, Y, f)$ and $\left(X^{\prime}, Y^{\prime}, f^{\prime}\right)$ be as in 3.8.2. The objects

$$
\Pi\left({ }^{p} \mathcal{H}^{n} R f_{*} \mathbb{Q}_{[X, Y]}\right) \quad \text { and } \quad \Pi\left({ }^{p} \mathcal{H}^{2 d-n} R(-f)_{*} \mathbb{Q}_{\left[X^{\prime}, Y^{\prime}\right]}\right)
$$

of $\mathbf{P e r v}_{0}$ are dual to each other.

Proof. We apply Lemma 3.8.5 to $Z=\bar{X} \backslash P, D=H \backslash(H \cap P)$ and $D^{\prime}=\bar{Y} \backslash(\bar{Y} \cap P)$. Then the diagram of inclusions in loc.cit. becomes

and there are canonical isomorphisms

$$
\begin{equation*}
\mathbb{D}\left(R \lambda_{*} \mathbb{Q}_{[X, Y]}\right)=\mathbb{D}\left(R \lambda_{*} \beta_{!} \beta^{*} \mathbb{Q}\right) \stackrel{\text { Verdier duality }}{=} \lambda_{!} R \beta_{*} \beta^{*} \mathbb{Q}[2 d] \stackrel{(3.8 .5 .2)}{=} R \lambda_{*}^{\prime} \beta_{!}^{\prime}\left(\beta^{\prime}\right)^{*} \mathbb{Q}[2 d] \tag{3.8.6.1}
\end{equation*}
$$

And now begins the fun:

$$
\begin{align*}
\Pi\left({ }^{p} \mathcal{H}^{n} R f_{*} \mathbb{Q}_{[X, Y]}\right)^{\vee} & =\Pi\left([-1]^{*} \mathbb{D}\left(\Pi\left({ }^{p} \mathcal{H}^{n} R f_{*} \mathbb{Q}_{[X, Y]}\right)\right)\right) & & \\
& =\Pi\left([-1]^{*} \mathbb{D}\left({ }^{p} \mathcal{H}^{n} R f_{*} \mathbb{Q}_{[X, Y]}\right)\right) & & \text { Lemma 2.4.8 } \\
& =\Pi\left({ }^{p} \mathcal{H}^{-n}[-1]^{*} \mathbb{D}\left(R f_{*} \mathbb{Q}_{[X, Y]}\right)\right) & & \mathbb{D} \circ{ }^{p} \mathcal{H}^{n}={ }^{p} \mathcal{H}^{-n} \circ \mathbb{D} \\
& =\Pi\left({ }^{p} \mathcal{H}^{-n} \mathbb{D}\left(R(-f)_{*} \mathbb{Q}_{[X, Y]}\right)\right) & & \\
& =\Pi\left({ }^{p} \mathcal{H}^{-n} \mathbb{D}\left(R(-\bar{f})_{*} R \lambda_{*} \mathbb{Q}_{[X, Y]}\right)\right) & & f=\bar{f} \circ \lambda \\
& =\Pi\left(\mathcal{H}^{-n}\left(R(-\bar{f})_{*} \mathbb{D}\left(R \lambda_{*} \mathbb{Q}_{[X, Y]}\right)\right)\right) & & \bar{f} \text { proper } \\
& \left.=\Pi{ }^{p} \mathcal{H}^{-n}\left(R(-\bar{f})_{*} R \lambda_{*}^{\prime} \beta_{!}^{\prime}\left(\beta^{\prime}\right)^{*} \mathbb{Q}[2 d]\right)\right) & & (3.8 .6 .1)  \tag{3.8.6.1}\\
& =\Pi\left({ }^{p} \mathcal{H}^{2 d-n}\left(R\left(-f^{\prime}\right)_{*} \beta_{!}^{\prime}\left(\beta^{\prime}\right)^{*} \mathbb{Q}\right)\right) & & f^{\prime}=\bar{f} \circ \lambda^{\prime} \\
& =\Pi\left({ }^{p} \mathcal{H}^{2 d-n}\left(R\left(-f^{\prime}\right)_{*} \mathbb{Q}_{\left[X^{\prime}, Y^{\prime}\right]}\right)\right) . & &
\end{align*}
$$

This is what we wanted to show.

Proposition 3.8.7. - There is a non-degenerate duality pairing

$$
\begin{equation*}
H_{\mathrm{rd}}^{n}(X, Y, f) \otimes H_{\mathrm{rd}}^{2 d-n}\left(X^{\prime}, Y^{\prime},-f\right) \longrightarrow \mathbb{Q}(-d) \tag{3.8.7.1}
\end{equation*}
$$

Proof. We take the real blow-up point of view on rapid decay cohomology. Let $B$ be the real blow-up of $\bar{X}$ along the components of $D$. By Corollary 3.5.11

$$
\begin{aligned}
H_{\mathrm{rd}}^{n}(X, Y, f) & \cong H^{n}\left(\bar{X}, R \pi_{*} \mathbb{Q}_{\left[B, B_{Y} \cup \partial^{+} B\right]}\right) \\
H_{\mathrm{rd}}^{2 d-n}\left(X^{\prime}, Y^{\prime},-f\right) & \cong H^{-n}\left(\bar{X}, R \pi_{*} \mathbb{Q}_{\left[B, B_{Y^{\prime}} \cup \partial^{-}-B\right]}[2 d]\right)
\end{aligned}
$$

We compute the Verdier dual: since $\pi$ is proper, by local Verdier duality (Theorem 2.1.8), one has

$$
\mathbb{D}\left(R \pi_{*} \mathbb{Q}_{\left[B, B_{Y} \cup \partial^{+} B\right]}\right)=R \pi_{*} R \mathcal{H o m}\left(\mathbb{Q}_{\left[B, B_{Y} \cup \partial^{+} B\right]}, \omega_{B / X}\right)
$$

3.8.8 (Real blow-up point of view). -

Lemma 3.8.9. - Let $B$ be a topological manifold with boundary, of real dimension n, and let $\alpha: B \backslash \partial B \hookrightarrow B$ be the inclusion of the complement of the boundary. The dualising sheaf $\omega_{B}$ on $B$ is isomorphic to $\alpha_{!} \alpha^{*} \mathbb{Q}[n]$.

Proof. The dualising sheaf on a general topological space is not a sheaf properly, but an object in the derived category of sheaves. We have to show that $H^{-n}\left(\omega_{B}\right)=\alpha_{!} \alpha^{*} \mathbb{Q}$ and $H^{-p}\left(\omega_{B}\right)=0$ holds for $p \neq n$. The sheaf $H^{-p}\left(\omega_{B}\right)$ is the sheafification of the presheaf

$$
U \longmapsto H_{p}(\dot{U},\{\cdot\}, \mathbb{Q})
$$

where $\dot{U}$ is the one point compactification of $U$. For opens $V \subseteq U$, the restriction morphism $H_{p}(\dot{U},\{\cdot\}, \mathbb{Q}) \rightarrow H_{p}(\dot{V},\{\cdot\}, \mathbb{Q})$ in this presheaf is given by the morphism in homology induced by the map $\dot{U} \rightarrow \dot{V}$ contracting $U \backslash V$ to the special point $\cdot \in \dot{V}$. A point $b \in B$ which is not in the boundary has a fundamental system of neighbourhoods $U$ which are homeomorphic to an open ball of dimension $n$. The one point compactification of such a ball is a sphere of dimension $n$. We find that $H_{p}(\dot{U},\{\cdot\}, \mathbb{Q})$ is zero for $p \neq n$ and equal to $\mathbb{Q}$ for $p=n$. A point $b \in \partial B$ has a fundamental system of neighbourhoods $U$ which are homeomorphic to a half-ball

$$
\left\{x=\left(x_{1} \ldots x_{n}\right) \in \mathbb{R}^{n} \mid\|x\|<1 \text { and } x_{1} \geqslant 0\right\}
$$

whose one point compactification is a closed ball of dimension $n$. We find that $H_{p}(\dot{U},\{\cdot\}, \mathbb{Q})$ is zero for all $p$.

Lemma 3.8.10. - Let $B$ a topological manifold with boundary, of real dimension n. Assume that the boundary $\partial B$ is the union of two closed subsets $Z_{1}$ and $Z_{2}$ such that $Z_{1} \cap Z_{2}$ has dense complement in $\partial B$. Then the Verdier dual of $\mathbb{Q}_{\left[B, Z_{1}\right]}$ is $\mathbb{Q}_{\left[B, Z_{2}\right]}[n]$.

Proof. Let $\lambda_{i}: B \backslash Z_{i} \hookrightarrow B$ denote the inclusions. By the previous lemma:

$$
\mathbb{D}\left(\mathbb{Q}_{\left[B, Z_{1}\right]}\right)=\operatorname{RHom}\left(\left(\lambda_{1}\right) \mid \lambda_{1}^{*} \mathbb{Q}, \alpha_{!} \alpha^{*} \mathbb{Q}[n]\right) .
$$

Example 3.8.11. - Let us describe the Poincaré-Verdier duality pairing (3.8.2.1) in the case where $X=\mathbb{A}^{1}=\operatorname{Spec} k[t]$ is the affine line, $Y$ is empty, and $f \in k[t]$ is a unitary polynomial of degree $d \geqslant 2$. We start with the linear dual of the copairing (3.8.2.2). This is a pairing

$$
\begin{equation*}
\langle-,-\rangle: H_{1}^{\mathrm{rd}}\left(\mathbb{A}^{1}, f\right) \otimes H_{1}^{\mathrm{rd}}\left(\mathbb{A}^{1},-f\right) \rightarrow \mathbb{Q}(1) \tag{3.8.11.1}
\end{equation*}
$$

which we seek to describe in terms of the usual explicit bases for rapid decay homology of a polynomial on the affine line. Here, $\mathbb{Q}(1)$ should be read as $\mathbb{Q}(1)=H^{1}\left(S^{1}\right) \simeq \mathbb{Q}$. The following picture shows a basis $\gamma_{1}, \gamma_{2}, \ldots$ of the rapid decay homology group $H_{1}^{\text {rd }}\left(\mathbb{A}^{1}, f\right)$ in green, and superposed in red a basis $\gamma_{1}^{\prime}, \gamma_{2}^{\prime}, \ldots$ for the rapid decay homology group $H_{1}^{\text {rd }}\left(\mathbb{A}^{1},-f\right)$, here in the case of a polynomial of degree $d=7$. Importantly, we have chosen the paths $\gamma_{i}$ and $\gamma_{i}^{\prime}$ in such a way that


Figure 3.8.3. Paths $\gamma_{i}$ and $\gamma_{i}^{\prime}$
they intersect at most once, and if so, transversally. The pairing (3.8.11.1) is defined in elementary terms as follows: Choose a sufficiently large real number $r>0$, and an open tubular neighbourhood $N \Delta$ of the diagonal $\Delta \subseteq \mathbb{A}^{2}$, sufficiently thin so that $N \Delta$ and $(f \boxplus-f)^{-1}\left(S_{r}\right)$ do not meet. Write $U \subseteq \mathbb{A}^{2}$ for the complement of the diagonal, and set $S \Delta=N \Delta \cap U$. The rapid decay homology $H_{2}^{\mathrm{rd}}\left(\mathbb{A}^{2}, f \boxplus-f\right)$ contains the cross-product cycles

$$
\gamma_{i j}=\gamma_{i} \cup \gamma_{j}^{\prime}:[0,1]^{2} \rightarrow \mathbb{A}^{2}
$$

defined by $\gamma_{i j}(s, t)=\left(\gamma_{i}(s), \gamma_{j}^{\prime}(t)\right)$, which in fact form a basis. The sought pairing (3.8.11.1) sends $\gamma_{i} \otimes \gamma_{j}^{\prime}$ to the image of $\gamma_{i} \cup \gamma_{j}^{\prime}$ under the connecting morphism

$$
\partial: H_{2}^{\mathrm{rd}}\left(\mathbb{A}^{2}, f \boxtimes-f\right) \rightarrow H_{1}(S \Delta) \cong H_{1}\left(\Delta \times S^{1}\right) \cong \mathbb{Q}(1)
$$

in the Mayer-Vietoris sequence for the covering $\mathbb{A}^{2}=N \Delta \cup U$. Recall how this connecting morphism is made: Using Lebesgue's lemma, we can write $\gamma_{i j}$ up to a boundary in the form

$$
\gamma_{i j}=\alpha+\beta,
$$

where $\alpha$ is a cycle in $N \Delta$ and $\beta$ a cycle in $U$. Then we declare $\partial \gamma_{i j}$ to be the homology class of $d \alpha$. This already shows that, if the paths $\gamma_{i}$ and $\gamma_{j}^{\prime}$ do not cross, then $\left\langle\gamma_{i}, \gamma_{j}^{\prime}\right\rangle=0$, since in that case we can choose $\alpha=0$. If $\gamma_{i}$ and $\gamma_{j}^{\prime}$ do cross, then we subdivide [ 0,1$]$ in rectangles, sufficiently small so that the only rectangle containing the point $\gamma_{i j}^{-1}(\Delta)$ in its interior is sent to $N \Delta$. We may take for $\alpha$ the restriction of $\gamma_{i j}$ to this small rectangle, and see that the image of $\gamma_{i j}$ in $H_{1}(S \Delta) \cong H_{1}\left(S^{1}\right)=\mathbb{Q}(1)$ is +1 if $d \alpha$ winds in the positive direction around the diagonal, and -1 in the opposite case. This in turn depends on whether $\gamma_{i}$ and $\gamma_{j}^{\prime}$ intersect positively or negatively. In summary, we find

$$
\left\langle\gamma_{i}, \gamma_{j}^{\prime}\right\rangle=\text { Intersection number }\left(\gamma_{i}, \gamma_{j}^{\prime}\right)
$$

and we can easily compile a table of these intersection numbers. Here it is.

|  | $\gamma_{1}^{\prime}$ | $\gamma_{2}^{\prime}$ | $\gamma_{3}^{\prime}$ | $\gamma_{4}^{\prime}$ | $\gamma_{5}^{\prime}$ | $\gamma_{6}^{\prime}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\gamma_{1}$ | 0 | 0 | 0 | -1 | 0 | 0 |
| $\gamma_{2}$ | 0 | 0 | 0 | -1 | -1 | 0 |
| $\gamma_{3}$ | 0 | 0 | 0 | -1 | -1 | -1 |
| $\gamma_{4}$ | 1 | 1 | 1 | 0 | 0 | 0 |
| $\gamma_{5}$ | 0 | 1 | 1 | 0 | 0 | 0 |
| $\gamma_{6}$ | 0 | 0 | 1 | 0 | 0 | 0 |

Example 3.8.12. - Let us continue the previous example, but suppose from now on that $f$ is an odd polynomial, so $f(-x)=-f(x)$, of degree $d=2 e+1$. In that case, the object $H_{\text {perv }}^{1}\left(\mathbb{A}^{1}, f\right)$ is self-dual via the isomorphism $\varphi: H_{\text {perv }}^{1}\left(\mathbb{A}^{1}, f\right) \rightarrow H_{\text {perv }}^{1}\left(\mathbb{A}^{1},-f\right)$ induced by the multiplication-by-(-1) map $\mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$. The Poincaré-Verdier duality pairing becomes via this isomorphism a pairing

$$
\begin{equation*}
H_{\text {perv }}^{1}\left(\mathbb{A}^{1}, f\right) \otimes H_{\text {perv }}^{1}\left(\mathbb{A}^{1}, f\right) \rightarrow \mathbb{Q}(-1) \tag{3.8.12.1}
\end{equation*}
$$

which will put some constraints on the tannakian fundamental group $G$ of $H_{\text {perv }}^{1}\left(\mathbb{A}^{1}, f\right)$. Let us make this constraint explicit. The basis for rapid decay homology we have considered above consists of the usual and somewhat arbitrary choice $\gamma_{1}, \ldots, \gamma_{2 e}$ for $H_{1}^{\mathrm{rd}}\left(\mathbb{A}^{1}, f\right)$. However, we have chosen the basis $\gamma_{i}^{\prime}=-\gamma_{i}$ of $H_{1}^{\mathrm{rd}}\left(\mathbb{A}^{1},-f\right)$ in such a way that the isomorphism

$$
H_{1}^{\mathrm{rd}}\left(\mathbb{A}^{1},-f\right) \rightarrow H_{1}^{\mathrm{rd}}\left(\mathbb{A}^{1}, f\right)
$$

dual to $\varphi$ sends the $\gamma_{i}^{\prime}$ to $\gamma_{i}$. The pairing (3.8.11.1) can be seen via this isomorphism as an alternating bilinear form on $H_{1}^{\text {rd }}\left(\mathbb{A}^{1}, f\right)$, which in the basis $\gamma_{1}, \ldots, \gamma_{2 e}$ is given by the skew-symmetric matrix

$$
A=\left(\begin{array}{cc}
0 & -{ }^{t} T \\
T & 0
\end{array}\right)
$$

where $T \in \mathrm{GL}_{e}$ is the upper triangular matrix with 1 's on and above the diagonal. Its coefficients are just the entries of the table of intersection numbers above. The same matrix $A$ also characterises
the bilinear form in rapid decay cohomology

$$
H_{\mathrm{rd}}^{1}\left(\mathbb{A}^{1}, f\right) \otimes H_{\mathrm{rd}}^{1}\left(\mathbb{A}^{1},-f\right) \rightarrow \mathbb{Q}(-1)
$$

with respect to the dual bases. This pairing is the fibre at infinity of (3.8.12.1), and hence the tannakian fundamental group $G \subseteq \mathrm{GL}_{d-1}$ of $H_{\text {perv }}^{1}\left(\mathbb{A}^{1}, f\right)$ must consist of matrices $g$ satisfying

$$
{ }^{t} g \cdot A \cdot g=A
$$

or in other words, $G$ must be contained in the symplectic group $\mathrm{Sp}_{A} \subseteq \mathrm{GL}_{2 e}$.

### 3.9. Hard Lefschetz theorem

3.9.1. - Let $f: X \rightarrow \mathbb{A}$ be a regular function. Unlike the classical case, the total cohomology $H_{\mathrm{rd}}^{*}(X, f)$ does not have a ring structure, because the map obtained from the cup product (3.6.2.1) and pullback by the diagonal

$$
H_{\mathrm{rd}}^{m}(X, f) \otimes H_{\mathrm{rd}}^{n}(X, f) \rightarrow H_{\mathrm{rd}}^{m+n}(X, 2 f)
$$

lands in the cohomology of the pair $(X, 2 f)$ rather than in the cohomology of $(X, f)$. However, the total cohomology $H_{\mathrm{rd}}^{*}(X, f)$ has the structure of a graded module over the graded cohomology ring $H^{*}(X)$. This module structure is given degreewise by

$$
\begin{equation*}
H^{m}(X) \otimes H^{n}(X, f) \xrightarrow{\text { cup }} H^{m+n}(X \times X, 0 \boxplus f) \xrightarrow{\Delta^{*}} H^{m+n}(X, f) \tag{3.9.1.1}
\end{equation*}
$$

where $\Delta: X \rightarrow X \times X$ is the diagonal. We refer to the morphism (3.9.1.1) just as cup product. The following statement is a simple application of a Hard Lefschez theorem for perverse sheaves.

Theorem 3.9.2 (Hard Lefschetz). - Let $X$ be a smooth variety of dimension d and $f: X \rightarrow \mathbb{A}^{1} a$ proper morphism. Let $\eta \in H^{2}(X, \mathbb{Q})$ be the class of a hyperplane section. For every $i=0,1,2, \ldots, d$, the cup product by $\eta^{i} \in H^{2 i}(X)$

$$
\eta^{i}: H_{\mathrm{rd}}^{d-i}(X, f) \rightarrow H_{\mathrm{rd}}^{d+i}(X, f)
$$

is an isomorphism.

Proof. The proof relies on Theorem 1.6.3[20] stating the following: Let $f: X \rightarrow Y$ be a proper morphism, and let $\eta$ be the first Chern class of a hyperplane section of $X$. Then, the $i$-fold cup-product with $\eta$

$$
\eta^{i}:{ }^{p} \mathcal{H}^{-i}\left(R f_{*} I C_{X}\right) \rightarrow{ }^{p} \mathcal{H}^{+i}\left(R f_{*} I C_{X}\right)
$$

is an isomorphism of perverse sheaves on $Y$. Here, $I C_{X}$ stands for the so-called intersection complex on $X$, which is a certain perverse sheaf on $X$. In the case where $X$ is smooth of dimension $d$, the intersection complex $I C_{X}$ is the constant sheaf $\underline{\mathbb{Q}}_{X}[d]$. We deduce that in our situation, the $i$-fold cup-product with $\eta$ induces an isomorphism

$$
\eta^{i}:{ }^{p} \mathcal{H}^{d-i}\left(R f_{*} \underline{\mathbb{Q}}_{X}\right) \rightarrow^{p} \mathcal{H}^{d+i}\left(R f_{*} \underline{\mathbb{Q}}_{X}\right)
$$

of perverse sheaves on $\mathbb{A}^{1}$. Applying the projector $\Pi$ : $\operatorname{Perv}\left(\mathbb{A}^{1}\right) \rightarrow \operatorname{Perv}_{0}$ yields an isomorphism in $\operatorname{Perv}_{0}$, and taking nearby fibres at infinity proves the theorem.
3.9.3. - Let $X$ be a smooth variety of dimension $d$ and let $f: X \rightarrow \mathbb{A}^{1}$ be a proper morphism. Let $n$ be an integer between 0 and $d$. The primitive part of $H_{\mathrm{rd}}^{n}(X, f)$ is the subspace

$$
P^{n}(X, f)=\operatorname{ker}\left(\eta^{d-n+1}: H_{\mathrm{rd}}^{n}(X, f) \rightarrow H_{\mathrm{rd}}^{2 d-n+2}(X, f)\right)
$$

of $H_{\mathrm{rd}}^{n}(X, f)$. This subspace has a canonical complement, namely the image of the injective map $\eta: H_{\mathrm{rd}}^{n-2}(X, f) \rightarrow H_{\mathrm{rd}}^{n}(X, f)$. From this decomposition results inductively the Lefschetz decomposition:

$$
\begin{equation*}
H_{\mathrm{rd}}^{n}(X, f)=\bigoplus_{i=0}^{n / 2} \eta^{i} P^{n-i}(X, f) . \tag{3.9.3.1}
\end{equation*}
$$

## CHAPTER 4

## Exponential motives

This chapter contains the technical core of our work, namely the construction of the $\mathbb{Q}$-linear neutral tannakian category $\mathbf{M}^{\exp }(k)$ of exponential motives over a subfield $k$ of $\mathbb{C}$. To this end, we first recall in Section 4.1 the basics of Nori's formalism, which attaches to a quiver representation $\rho: Q \rightarrow \mathrm{Vec}_{\mathbb{Q}}$ a $\mathbb{Q}$-linear abelian category $\langle Q, \rho\rangle$. We then apply this construction to a quiver consisting of tuples $[X, Y, f, n, i]$ and to the representation given by rapid decay cohomology.

### 4.1. Reminder and complements to Nori's formalism

In this section, we recall the notions of quivers and quiver representations. For us, this will just be a handy terminology to speak about categories without composition law.

Definition 4.1.1. - A quiver is the data $Q=(\operatorname{Ob}(Q), \operatorname{Mor}(Q), s, t, i)$ of two classes $\operatorname{Ob}(Q)$ and $\operatorname{Mor}(Q)$, together with maps

such that $s \circ i$ and $s \circ t$ are the identity on $\mathrm{Ob}(Q)$ and that, for each pair of elements $p, q \in \mathrm{Ob}(C)$, the class $\{f \in \operatorname{Mor}(Q) \mid s(f)=p, t(f)=q\}$ is a set. We say that a quiver $Q$ is finite if $\operatorname{Ob}(C)$ and $\operatorname{Mor}(Q)$ are both finite sets.

As the notation suggests, one thinks of $\operatorname{Ob}(Q)$ as a class of objects and of $\operatorname{Mor}(Q)$ as morphisms between them. The maps $s$ and $t$ associate with a morphism its source and target, and each object is equipped with an identity morphism. From this point of view, a quiver is a category, except that there is no composition law specified. We will freely adopt the terminology from category theory; for example, we will call functor from $Q$ to a category $\mathbf{C}$ the assignment of an object of $\mathbf{C}$ to each object of $Q$ and, to each morphism in $Q$, of a morphism between the corresponding objects in $\mathbf{C}$ in such a way that identities are mapped to identities.

Definition 4.1.2. - A representation of a quiver $Q$ in a category $\mathbf{C}$ is a morphism of quivers $\rho$ from $Q$ to $\mathbf{C}$. A morphism of quiver representations $(Q \stackrel{\rho}{\longrightarrow} \mathbf{C}) \rightarrow\left(Q^{\prime} \xrightarrow{\rho^{\prime}} \mathbf{C}\right)$ consists of a quiver morphism $\varphi: Q \rightarrow Q^{\prime}$ and an isomorphism (a natural transformation) of quiver morphisms $\rho^{\prime} \circ \psi \cong \rho$.
4.1.3. - Let $Q$ be a finite quiver, and let $\rho: Q \rightarrow \mathbf{C}$ be a quiver representation of $Q$ in a monoidal closed abelian category $\mathbf{C}$. The endomorphism ring $\operatorname{End}(\rho)$ is the algebra object in $\mathbf{C}$ given by

$$
\begin{equation*}
\operatorname{End}(\rho)=\text { equaliser }\left(\prod_{q \in Q} \operatorname{End}(\rho(q)) \Longrightarrow \prod_{p \rightarrow q} \operatorname{Hom}(\rho(p), \rho(q))\right) \tag{4.1.3.1}
\end{equation*}
$$

where $\operatorname{End}(\rho(q))$ and $\operatorname{Hom}(\rho(p), \rho(q))$ are the internal homomorphism objects in C. Typically, the category $\mathbf{C}$ at the receiving end of a quiver representation is the category of finite-dimensional rational vector spaces $\mathbf{V e c}_{\mathbb{Q}}$. In that case, the $\mathbb{Q}$-algebra $\operatorname{End}(\rho)$ consists of tuples $\left(e_{q}\right)_{q \in Q}$ of $\mathbb{Q}$-linear endomorphisms $e_{q}: \rho(q) \rightarrow \rho(q)$ such that the squares

commute for all morphisms $f: p \rightarrow q$ in $Q$. We may recognise (4.1.3.1) as part of a certain Hochschild simplicial complex. In particular, if $Q$ has only one object, we recognise a part of the Hochschild complex of the free $R$-algebra generated by the morphisms of $Q$ acting on the bimodule $\operatorname{End}(\rho(q))$. The Hochschild cohomology vanishes from $H^{2}$ on, and the first Hochschild cohomology group, whose elements have the interpretation of derivations modulo inner derivations, is the coequaliser of (4.1.3.1).

Given an arbitrary quiver $Q$, a representation $\rho: Q \rightarrow \mathbf{C}$ in a closed monoidal category $\mathbf{C}$ and a finite subquiver $Q_{0} \subseteq Q$, we can consider the algebra of endomorphisms $E_{0}=\operatorname{End}_{\mathbb{Q}}\left(\left.\rho\right|_{Q_{0}}\right)$ as before. It is an algebra object in $\mathbf{C}$. The endomorphism algebra $\operatorname{End}(\rho)$ is the formal limit of algebra objects

$$
\operatorname{End}(\rho)=\lim _{Q_{0} \subseteq Q} \operatorname{End}\left(\left.\rho\right|_{Q_{0}}\right)
$$

as $Q_{0}$ runs over the finite subquivers of $Q$ and transition maps are restrictions. Thus, $\operatorname{End}(\rho)$ is a pro-object in the category of algebra objects in $\mathbf{C}$. The following lemma tells us that in the case $\mathbf{C}=\operatorname{Vec}_{\mathbb{Q}}$ case we don't have to worry about the distinction between formal pro-objects the category of finite-dimensional algebras and infinite-dimensional algebras equipped with a topology.

LEMMA 4.1.4. - Let $I$ be a partially ordered set (where for every two elements $i, j \in I$ there exists $k \in I$ with $k \geqslant i$ and $k \geqslant j$ ), and let $\left(E_{i}\right)_{i \in I}$ be a collection of finite-dimensional $\mathbb{Q}$-algebras, together with algebra morphisms $r_{j i}: E_{j} \rightarrow E_{i}$ for $j \geqslant i$ satisfying $r_{j i} \circ r_{k j}=r_{k i}$ for $k \geqslant j \geqslant i$. Set

$$
E=\lim _{i \in I} E_{i}=\left\{\left(e_{i}\right)_{i \in I} \in \prod_{i \in I} E_{i} \mid r_{i j}\left(e_{j}\right)=e_{i} \text { for all } j>i\right\}
$$

and denote by $p_{i}: E \rightarrow E_{i}$ the canonical projections. For every finite-dimensional $\mathbb{Q}$-algebra $F$, the canonical map

$$
\underset{i \in I}{\operatorname{colim}_{i}} \operatorname{Hom}_{\mathbb{Q}-\operatorname{alg}}\left(E_{i}, F\right) \xrightarrow{(*)} \operatorname{colim}_{i \in I} \operatorname{Hom}_{\mathbb{Q}-\operatorname{alg}}\left(E / \operatorname{ker}\left(p_{i}\right), F\right)
$$

is bijective.
Proof. Any element of the left hand set is represented by an algebra morphism $h: E_{i} \rightarrow F$ for some $i \in I$, and the map labelled $(*)$ sends this element to the class of the composite of $h$ with the canonical injection $u_{i}: E / \operatorname{ker}\left(p_{i}\right) \rightarrow E_{i}$.

Injectivity. Any two elements of the left hand set can be represented by algebra morphisms $g: E_{i} \rightarrow F$ and $h: E_{i} \rightarrow F$ for some large enough $i \in I$. To say that $h$ and $g$ are mapped to the same element by $(*)$ is to say that there exists an element $j \geqslant i$ such that the two composite maps

$$
E / \operatorname{ker}\left(p_{j}\right) \rightarrow E / \operatorname{ker}\left(p_{i}\right) \xrightarrow{u_{i}} E_{i} \xrightarrow{g, h} F
$$

coincide. Here $E / \operatorname{ker}\left(p_{j}\right) \rightarrow E / \operatorname{ker}\left(p_{i}\right)$ is the canonical projection obtained from $\operatorname{ker}\left(p_{j}\right) \subseteq \operatorname{ker}\left(p_{i}\right)$. These maps are the same as the composite maps

$$
E / \operatorname{ker}\left(p_{j}\right) \xrightarrow{u_{i}} E_{j} \xrightarrow{r_{j i}} E_{i} \xrightarrow{g, h} F
$$

which means that $g \circ r_{j i}$ coincides with $g \circ r_{j i}$ on the image of the projection $E \rightarrow E_{j}$. Since $E_{j}$ is finite-dimensional as a $\mathbb{Q}$-vector space, the image of the projection $E \rightarrow E_{j}$ is equal to the image of $r_{k j}: E_{k} \rightarrow E_{j}$ for some $k \geqslant j$. Hence the maps $g \circ r_{k i}$ and $h \circ r_{k i}$ from $E_{k}$ to $F$ are equal, which means that $g$ and $h$ represent the same element.

Surjectivity. Pick an algebra homomorphism $h: E / \operatorname{ker}\left(p_{i}\right) \rightarrow F$ representing an element of the right hand set. Since $E_{i}$ is finite-dimensional as a $\mathbb{Q}$-vector space, there exists $j \geqslant i$ such that the image of $p_{i}: E \rightarrow E_{i}$ is equal to the image of $r_{j i}: E_{j} \rightarrow E_{i}$. The composite

$$
E_{j} \rightarrow E_{j} / \operatorname{ker}\left(r_{j i}\right) \cong E / \operatorname{ker}\left(p_{i}\right) \xrightarrow{h} F
$$

represents a preimage by $(*)$ of the class of $h$.
4.1.5. - Let us keep the notation from Lemma 4.1.4. The collection of the algebras $E_{i}$ and morphisms $r_{j i}$ describes a pro-object in the category of finite-dimensional algebras. Elements of the set colim $\operatorname{Hom}_{\mathbb{Q}-\text { alg }}\left(E_{i}, F\right)$ are morphisms of pro-objects from $\left(E_{i}, r_{j i}\right)$ to $F$. On the other hand, we can define a topology on $E$ by declaring the ideals $\operatorname{ker}\left(p_{i}\right)$ to be a fundamental system of open neighbourhoods of 0 . Elements of the set $\operatorname{colim} \operatorname{Hom}_{\mathbb{Q}-\mathrm{alg}}\left(E / \operatorname{ker}\left(p_{i}\right), F\right)$ are then the same as continuous algebra morphisms $E \rightarrow F$ for the discrete topology on $F$. A consequence of the lemma is that the category of finite-dimensional $\left(E_{i}, r_{j i}\right)$-modules is the same as the category of finite-dimensional continuous $E$-modules. The statement of the lemma, as well as the latter consequence of it, are false if instead of finite-dimensional algebras over a field one takes finite $R$-algebras over a coherent ring $R$, even for $R=\mathbb{Z}$.

Definition 4.1.6. - Let $Q$ be a quiver and $\rho: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ a quiver representation. We call linear hull of $(Q, \rho)$ the category $\langle Q, \rho\rangle$ defined as follows:
(1) Objects of $\langle Q, \rho\rangle$ are triples $\left(M, Q_{0}, \alpha\right)$ consisting of a finite-dimensional $\mathbb{Q}$-vector space $M$, a finite subquiver $Q_{0}$ of $Q$, and a $\mathbb{Q}$-linear action $\alpha$ of the algebra $\operatorname{End}\left(\left.\rho\right|_{Q_{0}}\right)$ on $M$.
(2) Morphisms $\left(M_{1}, Q_{1}, \alpha_{1}\right) \rightarrow\left(M_{2}, Q_{2}, \alpha_{2}\right)$ in $\langle Q, \rho\rangle$ are linear maps $f: M_{1} \rightarrow M_{2}$ with the property that there exists a finite subquiver $Q_{3}$ of $Q$ containing $Q_{1}$ and $Q_{2}$ such that $f$ is $\operatorname{End}\left(\left.\rho\right|_{Q_{3}}\right)$-linear. The action of $\operatorname{End}\left(\left.\rho\right|_{Q_{3}}\right)$ on $M_{i}$ is obtained via $\alpha_{i}$ and the restriction $\operatorname{End}\left(\left.\rho\right|_{Q_{3}}\right) \rightarrow \operatorname{End}\left(\left.\rho\right|_{Q_{i}}\right)$.
(3) Composition of morphisms in $\langle Q, \rho\rangle$ is composition of linear maps.

Equivalently, in light of Lemma 4.1.4, the linear hull $\langle Q, \rho\rangle$ is the category of continuous $\operatorname{End}(\rho)$ modules which are finite-dimensional as vector spaces. It is therefore a $\mathbb{Q}$-linear abelian category. We call canonical lift the representation

$$
\widetilde{\rho}: Q \rightarrow\langle Q, \rho\rangle
$$

sending an object $q \in Q$ to the triple $\widetilde{\rho}(q)=(\rho(q),\{q\}$, id) and a morphism $p \rightarrow q$ to the linear $\operatorname{map} \rho(f): \rho(p) \rightarrow \rho(q)$.

Proposition 4.1.7. - Let $\rho: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be a quiver representation. Every object of the abelian category $\langle Q, \rho\rangle$ is isomorphic to a subquotient of a finite sum of objects of the form $\tilde{\rho}(q)$ for $q$ in $Q$.

Proof. Let $M$ be an object of $\langle Q, \rho\rangle$, that is, a finite-dimensional vector space together with an $E_{0}$-module structure for some finite subquiver $Q_{0} \subseteq Q$ and $E_{0}=\operatorname{End}\left(\left.\rho\right|_{Q_{0}}\right)$. We can regard $E_{0}$ with its left $E_{0}$-module structure as an object of $\langle Q, \rho\rangle$ too. Since $M$ is finite-dimensional, there is a surjection of $E_{0}$-modules $E_{0}^{n} \rightarrow M$ for some integer $n \geqslant 0$, and hence it suffices to prove the proposition in the case $M=E_{0}$. There is an exact sequence of $E_{0}$-modules

$$
\begin{equation*}
0 \longrightarrow E_{0} \longrightarrow \prod_{q \in Q_{0}} \operatorname{End}(\rho(q)) \longrightarrow \prod_{p \rightarrow q} \operatorname{Hom}(\rho(p), \rho(q)) \tag{4.1.7.1}
\end{equation*}
$$

which shows that $E_{0}$, seen as a left $E_{0}$-module, is indeed isomorphic to a subobject of a product of modules of the form $\rho(q)$. Notice that $\operatorname{End}(\rho(q))$ is isomorphic as an $E_{0}$-module to $\rho(q)^{d}$ for $d=\operatorname{dim}_{\mathbb{Q}}(\rho(q))$.

Example 4.1.8. - Let $G$ be a finite (or profinite) group, and let $Q$ be the category of finite $G$-sets, viewed as a quiver. Let $\rho: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be the quiver representation sending a finite $G$-set $X$ to the vector space $\rho(X)=\mathbb{Q}^{X}$ generated by $X$, and morphisms of $G$-sets $X \rightarrow Y$ to the induced $\mathbb{Q}$-linear map $\mathbb{Q}^{X} \rightarrow \mathbb{Q}^{Y}$. We can consider $G$ as a (pro) $G$-set, and hence obtain a morphism $\operatorname{End}(\rho) \rightarrow \operatorname{End}_{\mathbb{Q}}\left(\mathbb{Q}^{G}\right)$ whose image must commute with all maps $\mathbb{Q}^{G} \rightarrow \mathbb{Q}^{G}$ induced by right multiplication by an element of $G$. This morphism is thus a morphism

$$
\operatorname{End}(\rho) \rightarrow \mathbb{Q}[G]
$$

where $\mathbb{Q}[G]$ stands for the group algebra of $G$. It is not hard to check that this morphism is indeed an isomorphism, and hence the linear hull $\langle Q, \rho\rangle$ is the category of finite-dimensional $\mathbb{Q}$-linear representations of $G$.
4.1.9. - An important feature of linear hulls of quiver representations is that they are functorial in the following sense: Given a morphism of quiver representations, that is, a triangle of quiver morphisms together with a natural transform


$$
\begin{aligned}
s: & \rho^{\prime} \circ \varphi \xrightarrow{\cong} \rho \\
s_{q}: & \rho^{\prime}(\varphi(q)) \xrightarrow{\cong} \rho(q)
\end{aligned}
$$

we obtain a functor $\Phi:\langle Q, \rho\rangle \rightarrow\left\langle Q^{\prime}, \rho^{\prime}\right\rangle$ by setting $\Phi\left(M, Q_{0}, \alpha\right)=\left(M, \varphi\left(Q_{0}\right), \alpha \circ \sigma\right)$, where $\varphi\left(Q_{0}\right)$ is the image of the finite subquiver $Q_{0} \subseteq Q$ in $Q^{\prime}$ under $\varphi$, and $\sigma$ the morphism of algebras $\operatorname{End}\left(\left.\rho^{\prime}\right|_{\varphi\left(Q_{0}\right)}\right) \rightarrow \operatorname{End}\left(\left.\rho\right|_{Q_{0}}\right)$ obtained from $s$. In terms of 4.1.3, the morphism $\sigma$ sends the tuple $\left(e_{q^{\prime}}\right)_{q^{\prime} \in \varphi\left(Q_{0}\right)}$ to the tuple $\left(s_{q} \circ e_{\varphi(q)} \circ s_{q}^{-1}\right)_{q \in Q_{0}}$. We notice that the functor $\Phi$ is faithful and exact, and that it commutes with the forgetful functors and up to natural isomorphisms with the canonical lifts.
4.1.10. - The induced functor $\Phi$ in the previous paragraph depends naturally on the morphism of quiver representations $(\varphi, s)$ in the following sense. Let $\rho: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ and $\rho^{\prime}: Q^{\prime} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be quiver representations, and let


$$
\begin{aligned}
& s: \rho^{\prime} \circ \varphi \rightarrow \rho \\
& t: \rho^{\prime} \circ \psi \rightarrow \rho
\end{aligned}
$$

be two morphisms of quiver representations. Denote by $\Phi$ and $\Psi$ the induced functors between linear hulls $\langle Q, \rho\rangle \rightarrow\left\langle Q^{\prime}, \rho^{\prime}\right\rangle$. We call 2-morphism from $(\varphi, s)$ to $(\psi, t)$ every natural transform $\eta: \tilde{\rho} \circ \varphi \rightarrow \tilde{\rho} \circ \psi$ with the property that for every $q \in Q$ the diagram of $R$-modules

commutes. Such a 2-morphism $\eta$ indeed induces a morphism of functors $E: \Phi \rightarrow \Psi$, namely, for every object $X=(M, F, \alpha)$ in $\langle Q, \rho\rangle$, the morphism

$$
E_{X}: \Phi(X)=(M, \varphi(F), \alpha \circ \sigma) \rightarrow \Psi(X)=(M, \psi(F), \alpha \circ \tau)
$$

in $\left\langle Q^{\prime}, \rho^{\prime}\right\rangle$ given by the identity $\mathrm{id}_{M}$. Let us check that $\mathrm{id}_{M}: \Phi(X) \rightarrow \Psi(X)$ is indeed a morphism in $\left\langle Q^{\prime}, \rho^{\prime}\right\rangle$. We can without loss of generality suppose that $Q$ and $Q^{\prime}$ are finite quivers. What has to be shown is that the two actions of $\operatorname{End}\left(\rho^{\prime}\right)$ on $M$, one induced by $s$ and the other by $t$, agree. Indeed, already the two algebra morphisms

$$
\sigma, \tau: \operatorname{End}\left(\rho^{\prime}\right) \rightarrow \operatorname{End}(\rho)
$$

are the same: given an element $\left(e_{q^{\prime}}\right)_{q^{\prime} \in Q^{\prime}}$ of $\operatorname{End}\left(\rho^{\prime}\right)$ and $q \in Q$, the diagram

commutes because $\eta_{q}$ is not just an arbitrary morphism of modules, but comes from a morphism $\widetilde{\rho}^{\prime}(\varphi(q)) \rightarrow \widetilde{\rho}^{\prime}(\psi(q))$ in $\left\langle Q^{\prime}, \rho^{\prime}\right\rangle$ and hence is $\operatorname{End}\left(\rho^{\prime}\right)$-linear.

Theorem 4.1.11. - Let $\mathbf{A}$ be an abelian, $\mathbb{Q}$-linear category, and let $h: \mathbf{A} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be a faithful, linear and exact functor. Regard $h$ as a quiver representation. The canonical lift $\widetilde{h}: \mathbf{A} \rightarrow\langle\mathbf{A}, h\rangle$ is an equivalence of categories.

References. In a slightly different form, the statement goes back to Freyd and Mitchell, who proved their embedding theorem for abelian categories in 1964. In the form presented here, Theorem 4.1 .11 was originally shown by Nori in [65]. There are accounts by Bruguières, Levine, and Huber and Müller-Stach $([17,61,47])$. Ivorra deduces in $[49]$ the result from a more general construction.

Theorem 4.1.12 (Nori's universal property). - Let A be a $\mathbb{Q}$-linear abelian category, together with a functor $\sigma: Q \rightarrow \mathbf{A}$, and let $h: \mathbf{A} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be an exact, $\mathbb{Q}$-linear, and faithful functor such that the following diagram of solid arrows commutes:


Then the above dashed arrow, rendering the whole diagram commutative, exists and is unique up to a unique isomorphism.

Proof. We can then regard $\sigma$ as a morphism of quiver representations from $\rho: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ to $h: \mathbf{A} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$. By naturality of the linear hull construction it gives a functor $\langle Q, \rho\rangle \rightarrow\langle\mathbf{A}, h\rangle$, or, in view of theorem 4.1.11, a functor $\langle Q, \rho\rangle \rightarrow \mathbf{A}$ which renders the whole diagram commutative up to natural isomorphisms.

LEMmA 4.1.13. - Let $\psi:\left(Q \xrightarrow{\rho} \operatorname{Vec}_{\mathbb{Q}}\right) \rightarrow\left(Q^{\prime} \xrightarrow{\rho^{\prime}} \mathbf{V e c}_{\mathbb{Q}}\right)$ be a morphism of quiver representations. The induced functor $\Psi:\langle Q, \rho\rangle \rightarrow\left\langle Q^{\prime}, \rho^{\prime}\right\rangle$ is an equivalence of categories if and only if
there exists a quiver representation $\lambda: Q^{\prime} \rightarrow\langle Q, \rho\rangle$ such that the following diagram commutes up to natural isomorphisms:


Proof. If $\Psi$ is an equivalence of categories, then there exists a functor $\Phi:\left\langle Q^{\prime}, \rho^{\prime}\right\rangle \rightarrow\langle Q, \rho\rangle$ and isomorphisms $\Phi \circ \Psi \cong \mathrm{id}$ and $\Psi \circ \Phi \cong \mathrm{id}$. A possible choice for $\lambda$ is then $\lambda=\Phi \circ \widetilde{\rho}$, indeed, since the outer square in (4.1.13.1) commutes up to an isomorphism, we have isomorphisms

$$
\Psi \circ \lambda \cong \Psi \circ \Phi \circ \widetilde{\rho} \cong \widetilde{\rho} \quad \text { and } \quad \lambda \circ \psi=\Phi \circ \widetilde{\rho} \circ \psi \cong \Phi \circ \Psi \circ \widetilde{\rho} \cong \widetilde{\rho}
$$

as required.
On the other hand, suppose that a representation $\lambda$ as in the statement of the lemma exists. We extend the diagram (4.1.13.1) to a diagram

with arrows as follows: let $f:\langle Q, \rho\rangle \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ and $f^{\prime}:\left\langle Q^{\prime}, \rho^{\prime}\right\rangle \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be the forgetful functors. We have an isomorphism $f^{\prime} \circ \Psi \cong f$, and hence an isomorphism $\rho^{\prime}=f^{\prime} \circ \widetilde{\rho}^{\prime} \cong f^{\prime} \circ \Psi \circ \lambda \cong f \circ \lambda$, and can thus view $\lambda$ as a morphism of quiver representations from $\rho^{\prime}$ to $f$. The arrow $\Lambda$ is the corresponding functor. The functor $P$ is the canonical lift of $\widetilde{\rho}$ regarded as a morphism of quiver representations from $\rho$ to $f$; by Theorem 4.1.11 it is an equivalence of categories. Let $\iota$ be a quasi-inverse to $P$. We claim that the functor $\Phi=\iota \circ \Lambda$ is a quasi-inverse to $\Psi$.

To get an isomorphism $\Phi \circ \Psi \cong$ id it suffices to get an isomorphism $\Lambda \circ \Psi \cong P$. Let us apply 4.1.10 to the representations


$$
\begin{aligned}
& f \circ \lambda \circ \psi \cong f \circ \widetilde{\rho}=\rho \\
& f \circ \widetilde{\rho}=\rho
\end{aligned}
$$

where we use an isomorphism $\lambda \circ \psi \cong \widetilde{\rho}$ which makes (4.1.13.1) commute. This isomorphism induces an isomorphism $\eta: \widetilde{f} \circ \lambda \circ \psi \cong \widetilde{f} \circ \widetilde{\rho}$ which makes the diagrams corresponding to (4.1.10.1) commute, and hence we obtain an isomorphism of functors $\Lambda \circ \Psi \cong P$. It remains to construct an isomorphism $\Psi \circ \Phi \cong \mathrm{id}$. This is done by replacing the diagram (4.1.13.1) in the statement of the Lemma with the bottom half of (4.1.13.2), and the same application of 4.1.10.
4.1.14 (Caveat). - In the situation of Lemma 4.1.13, it will not do to just produce a representation $\lambda$ as in diagram (4.1.13.1) and natural isomorphisms of $R$-modules $\rho(q) \cong f(\lambda(q))$ in order to show that $\Psi$ is an equivalence. Such a $\lambda$ will produce some functor $\Phi:\left\langle Q^{\prime}, \rho^{\prime}\right\rangle \rightarrow\langle Q, \rho\rangle$ which, in general, is not a quasi-inverse to $\Psi$. Each time we apply 4.1.13, the hard part is not to define $\lambda$, but to check commutativity of the diagram. The point seems to have been overlooked at several places ${ }^{1}$. Consider for example a homomorphism of finite groups $G^{\prime} \rightarrow G$, the quivers $Q$ and $Q^{\prime}$ of finite $G$-sets, respectively $G^{\prime}$-sets, and the quiver representations $\rho$ and $\rho^{\prime}$ which associate with a set $X$ the vector space generated by $X$. The linear hulls identify with the categories of $\mathbb{Q}$-linear group representations, and the restriction functor $Q \rightarrow Q^{\prime}$ is a morphism of quiver representations which induces the restriction functor between representation categories. For any $G^{\prime}$-set $X^{\prime}$ write $\lambda(X)$ for the trivial $G$-representation on the vector space generated by the set $X$. We obtain a diagram

which does in not commute except in trivial cases, but commutes after forgetting the group actions. The functor $\Psi$ is not an equivalence, trivial cases excepted, and the functor induced by $\lambda$ sends a $G^{\prime}$-representation $V$ to the constant $G$-representation with underlying module $V$.

Definition 4.1.15. - Let $\rho: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ and $\rho^{\prime}: Q^{\prime} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be quiver representations. We denote by

$$
\rho \boxtimes \rho^{\prime}: Q \boxtimes Q^{\prime} \rightarrow \mathbf{V e c}_{\mathbb{Q}}
$$

the following quiver representation. Objects of the quiver $Q \boxtimes Q^{\prime}$ are pairs $\left(q, q^{\prime}\right)$ consisting of an object $q$ of $Q$ and an object $q^{\prime}$ of $Q^{\prime}$, and morphisms are either of the form (id $\left.{ }_{q}, f^{\prime}\right):\left(q, q^{\prime}\right) \rightarrow\left(p, p^{\prime}\right)$ for some morphism $f^{\prime}: q^{\prime} \rightarrow p^{\prime}$ in $Q^{\prime}$, or of the form $\left(f, \mathrm{id}_{q^{\prime}}\right):\left(q, q^{\prime}\right) \rightarrow\left(p, q^{\prime}\right)$ for some morphism $f: p \rightarrow q$ in $Q$. The representation $\rho \boxtimes \rho^{\prime}$ is defined by

$$
\left(\rho \boxtimes \rho^{\prime}\right)\left(q, q^{\prime}\right)=\rho(q) \otimes_{R} \rho^{\prime}\left(q^{\prime}\right)
$$

on objects, and by $\left(\rho \boxtimes \rho^{\prime}\right)\left(\mathrm{id}_{q}, f^{\prime}\right)=\operatorname{id}_{\rho(q)} \otimes \rho\left(f^{\prime}\right)$ and $\left(\rho \boxtimes \rho^{\prime}\right)\left(f, \mathrm{id}_{q^{\prime}}\right)=\rho(f) \otimes \mathrm{id}_{\rho^{\prime}\left(q^{\prime}\right)}$ on morphisms.
4.1.16. - Our next proposition relates the linear hull of a quiver representation $\rho \boxtimes \rho^{\prime}$ with the tensor product of the linear hulls of $\rho$ and $\rho^{\prime}$. A tensor product $\mathbf{A} \otimes \mathbf{B}$ of abelian $\mathbb{Q}$-linear categories $\mathbf{A}$ and $\mathbf{B}$, as introduced in [23], is a $\mathbb{Q}$-linear category characterised up to equivalence by a universal property. It does not exist in general as is shown in [62], however, it exists and has good properties as soon as one works in an appropriate enriched setting, as is shown in [41]. We only need to know the following fact: If $\mathbf{A}$ is the category of continuous, finite-dimensional

[^2]$A$-modules and $\mathbf{B}$ the category of continuous, finite-dimensional $B$-modules for some $\mathbb{Q}$-profinite algebras
$$
A=\lim _{i} A_{i} \quad \text { and } \quad B=\lim _{j} B_{j}
$$
then $\mathbf{A} \otimes \mathbf{B}$ exists and is given by the category of continuous $A \widehat{\otimes} B$-modules, where
$$
A \widehat{\otimes} B=\lim _{i, j} A_{i} \otimes B_{j}
$$
stands for the completed tensor product. This follows from $\S 5.1$ and Proposition 5.3 of [23].

Proposition 4.1.17. - There is a canonical faithful and exact functor

$$
\left\langle Q \boxtimes Q^{\prime}, \rho \boxtimes \rho^{\prime}\right\rangle \rightarrow\langle Q, \rho\rangle \otimes\left\langle Q^{\prime}, \rho^{\prime}\right\rangle
$$

which commutes with the forgetful functors to $\mathbf{V e c}_{\mathbb{Q}}$, and is natural in $\rho$ and $\rho^{\prime}$ for morphisms of quiver representations. This functor is an equivalence of categories.

Proof. It suffices to construct a functor in the case where $Q$ and $Q^{\prime}$ are finite quivers. Set

$$
V=\bigoplus_{q \in Q} \rho(q)
$$

and write $X \subseteq \operatorname{End}_{R}(V)$ for the finite set of compositions of the form $V \xrightarrow{\mathrm{pr}} \rho(p) \xrightarrow{\rho(f)} \rho(q) \xrightarrow{\subseteq} V$ for some morphism $f$ in $Q$, and write $E_{X}=\operatorname{End}(\rho) \subseteq \operatorname{End}(V)$ for the commutator of $X$. Define $E_{X^{\prime}} \subseteq \operatorname{End}\left(V^{\prime}\right)$ and $E_{X \boxtimes X^{\prime}} \subseteq \operatorname{End}\left(V \otimes V^{\prime}\right)$ similarly. We want to show that the canonical, natural morphism of $\mathbb{Q}$-algebras $E_{X} \otimes_{\mathbb{Q}} E_{X^{\prime}} \rightarrow E_{X \boxtimes X^{\prime}}$ given in the diagram

is an isomorphism. All morphisms in this diagram are injective, and $\alpha$ is an isomorphism. We want to show that the top horizontal map is surjective. Let $f \in \operatorname{End}\left(V \otimes V^{\prime}\right)$ be an endomorphism that commutes with $X \boxtimes X^{\prime}$. We write $f$ as $f=\alpha\left(f_{1} \otimes f_{1}^{\prime}+\cdots+f_{n} \otimes f_{n}^{\prime}\right)$ with $f_{i} \in \operatorname{End}(V)$ and linearly independent $f_{i}^{\prime} \in \operatorname{End}\left(V^{\prime}\right)$. For all $x \in X$ we have $(x \otimes 1) \circ f=f \circ(x \otimes 1)$, that is,

$$
\sum_{i=1}^{n}\left(x \circ f_{i}-f_{i} \circ x\right) \otimes f_{i}^{\prime}=0
$$

and hence $f_{i} \in E_{X}$. In other words, $f$ comes via $\alpha$ from an element of $E_{X} \otimes \operatorname{End}\left(V^{\prime}\right)$, and symmetrically, $f$ comes from an element of $\operatorname{End}(V) \otimes E_{X^{\prime}}$. Finally, again since $E_{X}$ and $E_{X^{\prime}}$ are direct factors of $\operatorname{End}(V)$ and $\operatorname{End}\left(V^{\prime}\right)$, we have

$$
\left(E_{X} \otimes \operatorname{End}\left(V^{\prime}\right)\right) \cap\left(\operatorname{End}(V) \otimes E_{X^{\prime}}\right)=E_{X} \otimes_{R} E_{X^{\prime}}
$$

so $\alpha^{-1}(f)$ is indeed an element of $E_{X} \otimes_{R} E_{X^{\prime}}$ as we wanted to show.
4.1.18. - All statements presented in this section hold verbatim for $R$-linear quiver representations when $R$ is a field. With the exception of Lemma 4.1.4 and Proposition 4.1.17 one can even take for $R$ a commutative coherent ring, and replace categories of finite-dimensional vector spaces by categories of finitely presented modules. If in Proposition 4.1 .17 we choose to work with a coherent ring of coefficients $R$, the exact and faithful functor still exists, but it is in general not an equivalence of categories. A sufficient condition for this functor to be an equivalence of categories is that $R$ is a hereditary ring, and $\rho(q)$ and $\rho^{\prime}\left(q^{\prime}\right)$ are projective $R$-modules for all $q \in Q$ and $q^{\prime} \in Q^{\prime}$. Hereditary means that every ideal of $R$ is projective, or equivalently, every submodule of a projective module is projective. Fields, finite products of fields, Dedekind rings and finite rings are examples. A commutative, coherent and hereditary ring which has no zero divisors is either a field or a Dedekind ring.

One might be tempted to replace the category $\mathbf{V e c}_{\mathbb{Q}}$ in Definition 4.1 .6 by an arbitrary abelian monoidal closed category. However, this will not result in a useful definition, since Theorem 4.1.11 and the universal property described in 4.1.12 do not hold in this generality. The point is the following: Let $\rho: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be a quiver representation, and regard the forgetful functor $f:\langle Q, \rho\rangle \rightarrow \operatorname{Vec}_{\mathbb{Q}}$ as a quiver representation. The the key turn in the proof of 4.1.11 is to show that the canonical lift $\widetilde{f}$ of $f$, and the functor $P$ induced by $\widetilde{\rho}: Q \rightarrow\langle Q, \rho\rangle$ viewed as a morphism of quiver representations

$$
\tilde{f}, P:\langle Q, \rho\rangle \Longrightarrow\langle\langle Q, \rho\rangle, f\rangle
$$

are isomorphic functors. This relies on the fact that the neutral object for the tensor product in $\operatorname{Vec}_{\mathbb{Q}}$ is a projective generator, which is particular to categories of modules. In the case where we replace $\operatorname{Vec}_{\mathbb{Q}}$ by a tannakian category, a correct abelian hull which satisfies Ivorra's universal property (it is the initial object in a certain strict 2-category, see [49], Definition 2.2) is given by the equaliser category of $\widetilde{f}$ and $P$.

### 4.2. Exponential motives

We fix for this section a field $k$ endowed with a complex embedding $\sigma: k \hookrightarrow \mathbb{C}$. All varieties and morphisms of varieties are understood to be defined over $k$. Given a variety $X$, a closed subvariety $Y$ of $X$, and a regular function $f$ on $X$, when there is no risk of confusion, we will still denote by $X, Y, f$ the associated complex analytic varieties $X(\mathbb{C}), Y(\mathbb{C})$, and the holomorphic function $f_{\mathbb{C}}: X(\mathbb{C}) \rightarrow \mathbb{C}$. We set $S_{r}=\{z \in \mathbb{C} \mid \operatorname{Re}(z) \geqslant r\}$ and write

$$
H^{n}(X, Y, f)=\underset{r \rightarrow \infty}{\operatorname{colim}} H^{n}\left(X, Y \cup f^{-1}\left(S_{r}\right), \mathbb{Q}\right)
$$

for rapid decay cohomology with rational coefficients.

Definition 4.2.1. - The quiver of exponential relative varieties over $k$ is the quiver $\mathrm{Q}^{\exp }(k)$ consisting of the following objects and morphisms:
(1) Objects are tuples $[X, Y, f, n, i]$, where $X$ is a variety over $k, Y \subseteq X$ is a closed subvariety, $f$ is a regular function on $X$, and $n$ and $i$ are integers.
(2) Morphisms with target $[X, Y, f, n, i]$ are given by either (a), (b) or (c) as follows:
(a) for each morphism of varieties $h: X \rightarrow X^{\prime}$ satisfying $h(Y) \subseteq Y^{\prime}$ and $f^{\prime} \circ h=f$, a morphism

$$
h^{*}:\left[X^{\prime}, Y^{\prime}, f^{\prime}, n, i\right] \longrightarrow[X, Y, f, n, i] ;
$$

(b) for each pair of closed immersions $Z \subseteq Y \subseteq X$, a morphism

$$
\partial:\left[Y, Z,\left.f\right|_{Y}, n-1, i\right] \longrightarrow[X, Y, f, n, i] ;
$$

(c) a morphism

$$
\left[X \times \mathbb{G}_{m},\left(Y \times \mathbb{G}_{m}\right) \cup(X \times\{1\}), f \boxplus 0, n+1, i+1\right] \longrightarrow[X, Y, f, n, i]
$$

We refer to the integer $n$ as cohomological degree of just degree, and to the integer $i$ as twist.

Definition 4.2.2. - The Betti representation of the quiver of exponential relative varieties over $k$ is the functor $\rho: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ defined on objects by

$$
\rho([X, Y, f, n, i])=H^{n}(X, Y, f)(i)
$$

where $(i)$ denotes the tensor product with the $(-i)$-fold tensor power of the one-dimensional vector space $H^{1}\left(\mathbb{G}_{m}, \mathbb{Q}\right)$, and as follows on morphisms:
(a) a morphism of type (a) given by a morphism of varieties $h: X \rightarrow X^{\prime}$ is sent to the linear map

$$
H^{n}\left(X^{\prime}, Y^{\prime}, f^{\prime}\right)(i) \rightarrow H^{n}(X, Y, f)(i)
$$

obtained by functoriality of rapid decay cohomology;
(b) a morphism of type (b) is sent to the map

$$
H_{\mathrm{rd}}^{n-1}\left(Y, Z, f_{\mid Y}\right)(i) \rightarrow H_{\mathrm{rd}}^{n}(X, Y, f)(i)
$$

induced, by passing to the limit $r \rightarrow+\infty$, from the composition

where the horizontal map is the connecting morphism in the long exact sequence associated with the triple $Z \cup f^{-1}\left(S_{r}\right) \subseteq Y \cup f^{-1}\left(S_{r}\right) \subseteq X$, and the vertical morphism is the inverse of the map induced by the obvious morphism of pairs, which is an isomorphism by excision;
(c) a morphism of type (c) is sent to the map

$$
H^{n+1}\left(X \times \mathbb{G}_{m},\left(Y \times \mathbb{G}_{m}\right) \cup(X \times\{1\}), f \boxplus 0\right)(i+1) \rightarrow H^{n}(X, Y, f)(i)
$$

induced by the Künneth isomorphism (Proposition 3.1.1)

$$
\begin{aligned}
H^{n+1}\left(X \times \mathbb{G}_{m},\right. & \left.\left(Y \times \mathbb{G}_{m}\right) \cup(X \times\{1\}), f \boxplus 0\right) \\
& \xrightarrow{\sim} H^{n}(X, Y, f) \otimes H^{1}\left(\mathbb{G}_{m},\{1\}\right)=H^{n}(X, Y, f)(-1) .
\end{aligned}
$$

Definition 4.2.3. - The category of exponential motives over $k$ is the linear hull

$$
\mathbf{M}^{\exp }(k)=\left\langle\mathrm{Q}^{\exp }(k), \rho\right\rangle
$$

that is, the category whose objects are triples $(M, Q, \alpha)$, where $M$ is a finite-dimensional rational vector space, $\mathrm{Q} \subseteq \mathrm{Q}^{\exp }(k)$ a finite subquiver, and $\alpha$ a linear action of $\operatorname{End}\left(\left.\rho\right|_{Q}\right)$ on $M$. We write

$$
R_{B}: \mathbf{M}^{\exp }(k) \longrightarrow \mathbf{V e c}_{\mathbb{Q}}
$$

for the forgetful functor, and call it Betti realisation. Given an object $[X, Y, f, n, i]$ of the quiver $\mathrm{Q}^{\exp }(k)$, we denote by $H^{n}(X, Y, f)(i)$ the exponential motive $\widetilde{\rho}([X, Y, f, n, i])$. Whenever $Y=\varnothing$ or $i=0$, we shall usually omit them from the notation.
4.2.4. - Let us list for future reference a few conspicuous properties of the category $\mathbf{M}^{\exp }(k)$. First of all, $\mathbf{M}^{\exp }(k)$ is an abelian and $\mathbb{Q}$-linear category, and there is by definition a commutative diagram

where $\rho$ and its canonical lift $\widetilde{\rho}$ are quiver representations, and where $R_{B}$ is a faithful, exact and conservative functor. Conservative means that a morphism $f$ in $\mathbf{M}^{\exp }(k)$ is an isomorphism if and only if its Betti realisation $R_{B}(f)$ is an isomorphism of vector spaces. From Proposition 4.1.7 we know that every object in $\mathbf{M}^{\exp }(k)$ is isomorphic to a subquotient of a sum of objects of the form $H^{n}(X, Y, f)(i)$. Morphisms in the quiver $\mathrm{Q}^{\exp }(k)$ lift to morphisms in $\mathbf{M}^{\exp }(k)$. In particular we have morphisms of motives

$$
\begin{equation*}
h^{*}: H^{n}\left(X^{\prime}, Y^{\prime}, f^{\prime}\right)(i) \rightarrow H^{n}(X, Y, f)(i) \tag{4.2.4.1}
\end{equation*}
$$

induced by morphisms of varieties $h^{\prime}: X \rightarrow X^{\prime}$ compatible with subvarieties and potentials. The Betti realisation of this morphism is the corresponding morphisms of rapid decay cohomology groups. Let $Z \subseteq Y \subseteq X$ be a pair of closed immersions and $f$ a regular function on $X$. There is a long exact sequence of exponential motives

$$
\begin{equation*}
\cdots \longrightarrow H^{n}(X, Y, f) \longrightarrow H^{n}(X, Z, f) \longrightarrow H^{n}\left(Y, Z,\left.f\right|_{Y}\right) \longrightarrow H^{n+1}(X, Y, f) \longrightarrow \cdots \tag{4.2.4.2}
\end{equation*}
$$

realising to the corresponding long exact sequence in rapid decay cohomology. Indeed, all morphisms in the sequence (4.2.4.2) are morphisms of motives because they come from morphisms in
the quiver $\mathrm{Q}^{\exp }(k)$, and the sequence is also exact because the corresponding sequence of vector spaces is so. Finally, there are isomorphisms

$$
\begin{equation*}
H^{n+1}\left(X \times \mathbb{G}_{m},\left(Y \times \mathbb{G}_{m}\right) \cup(X \times\{1\}), f \boxplus 0\right)(i+1) \rightarrow H^{n}(X, Y, f)(i) \tag{4.2.4.3}
\end{equation*}
$$

in $\mathbf{M}^{\exp }(k)$ realising to the Künneth isomorphisms. Of course, the above are not all morphisms in the category $\mathbf{M}^{\exp }(k)$. Taking compositions and $\mathbb{Q}$-linear combinations produces many other morphisms which are not of the elementary shapes (4.2.4.1), (4.2.4.2) or (4.2.4.3). It is not clear whether by any means all morphisms in $\mathbf{M}^{\exp }(k)$ are obtained from those elementary ones.

LEMMA 4.2.5. - For every pair of varieties $Y \subseteq X$ and every regular function $f: X \rightarrow \mathbb{A}^{1}$, there is a canonical isomorphism of motives

$$
H^{n}(X, Y, f) \xrightarrow{\cong} H^{n+1}\left(X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma, p\right)
$$

where $\Gamma \subseteq X \times \mathbb{A}^{1}$ is the graph of $f$ and $p: X \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ is the projection.
Proof. This follows essentially from the previous remarks and elements of the proof of Proposition 3.2.2. Associated with the triple $\left(Y \times \mathbb{A}^{1}\right) \subseteq\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma \subseteq\left(X \times \mathbb{A}^{1}\right)$ is a long exact sequence in $\mathbf{M}^{\exp }(k)$. The motives $H^{n}\left(X \times \mathbb{A}^{1}, Y \times \mathbb{A}^{1}, p\right)$ appearing in this sequence are zero for all $n$, so the sequence breaks down to isomorphisms

$$
\begin{equation*}
H^{n}\left(\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma, Y \times \mathbb{A}^{1}, p\right) \xrightarrow{\cong} H^{n+1}\left(X \times \mathbb{A}^{1},\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma, p\right) \tag{4.2.5.1}
\end{equation*}
$$

The inclusion $h: X \rightarrow\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma$ given by $h(x)=(x, f(x))$ sends $Y \subseteq X$ to $Y \times \mathbb{A}^{1}$ and satisfies $p \circ h=f$, hence a morphism

$$
\begin{equation*}
h^{*}: H^{n}\left(\left(Y \times \mathbb{A}^{1}\right) \cup \Gamma, Y \times \mathbb{A}^{1}, p\right) \rightarrow H^{n}(X, Y, f) \tag{4.2.5.2}
\end{equation*}
$$

in $\mathbf{M}^{\exp }(k)$. By excision, this morphism induces an isomorphism in rapid decay cohomology, and hence is an isomorphism of motives. The composite of (4.2.5.1) and (4.2.5.2) is what we sought.
4.2.6. - Let us now show how Nori's universal property is used to construct realisation functors. Let $\mathbf{A}$ be an abelian $\mathbb{Q}$-linear category equipped with a faithful exact functor $h: \mathbf{A} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$, and suppose that we are given a cohomology theory for triples $(X, Y, f)$ with values in $\mathbf{A}$ which is comparable to rapid decay cohomology. Precisely, that means we have a quiver representation

$$
\begin{equation*}
\sigma: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{A} \quad[X, Y, f, n, i] \longmapsto H_{\mathbf{A}}^{n}(X, Y)(i) \tag{4.2.6.1}
\end{equation*}
$$

and an isomorphism between $h \circ \sigma$ and the Betti representation $\rho$. Nori's universal property as stated in Theorem 4.1.12 applies, yielding a faithful and exact functor

$$
\begin{equation*}
\mathbf{R}_{\mathbf{A}}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{A} \tag{4.2.6.2}
\end{equation*}
$$

which we call realisation functor. A typical examples of such a cohomology theory is the representation associating with $[X, Y, f, n, i]$ the object $H_{\text {perv }}^{n}(X, Y)(i)$ of $\mathbf{P e r v}_{0}$, in which case we choose for $\mathbf{A}$ the category $\mathbf{P e r v}_{0}$ and for $h$ the nearby fibre at infinity. It can and will happen that we want to study cohomology theories and realisation functors with values in a category which is not $\mathbb{Q}$-linear but $F$-linear for some field of characteristic zero, typically $F=k$ or $F=\mathbb{Q}_{\ell}$ or $F=\mathbb{C}$.

In that case we can not use Theorem 4.1.12 directly, but have to use the following trick. Suppose we have a cohomology theory such as (4.2.6.1) where now $\mathbf{A}$ is $F$-linear with a faithful and exact functor $\mathbf{A} \rightarrow \mathbf{V e c}_{F}$ and a natural isomorphism

$$
\begin{equation*}
H_{\mathbf{A}}^{n}(X, Y)(i) \otimes_{F} B \cong H^{n}(X, Y)(i) \otimes_{\mathbb{Q}} B \tag{4.2.6.3}
\end{equation*}
$$

of $B$-vector spaces for some large field $B$ containing $F$. Let $\mathbf{A}^{+}$be the category whose objects are triples $(A, V, \alpha)$ consisting of an object $A$ of $\mathbf{A}$, a rational vector space $V$, and an isomorphism of $B$-vector spaces $h(A) \otimes_{F} B \cong V \otimes_{\mathbb{Q}} B$. The category $\mathbf{A}^{+}$is $\mathbb{Q}$-linear, with a faithful and exact functor $h: \mathbf{A} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ sending $(A, V, \alpha)$ to $V$. Combining the given representation $\sigma: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{A}$ with (4.2.6.3), we obtain a representation $\sigma^{+}: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{A}^{+}$such that the equality $h \circ \sigma^{+}=\rho$ holds, and hence from Nori's universal property a realisation functor $\mathbf{R}_{\mathbf{A}^{+}}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{A}^{+}$. We obtain a functor (4.2.6.2) by composing $\mathrm{R}_{\mathbf{A}^{+}}$with the forgetful functor $\mathbf{A}^{+} \rightarrow \mathbf{A}$.
4.2.7. - Much of the strength of Nori's theories of motives, among which we count our category of exponential motives, stems from the fact that there are many variants of the Betti representation $\rho: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ which produce the same category of exponential motives.

Proposition 4.2.8. - Let $\mathrm{Q}_{\mathrm{aff}}^{\exp }(k)$ be the full subquiver of $\mathrm{Q}^{\exp }(k)$ whose objects are those tuples $[X, Y, f, n, i]$ where $X$ is an affine variety. The functor

$$
\left\langle\mathrm{Q}_{\mathrm{aff}}^{\exp }(k), \rho\right\rangle \rightarrow\left\langle\mathrm{Q}^{\exp }(k), \rho\right\rangle=\mathbf{M}^{\exp }(k)
$$

induced by the inclusion $\mathrm{Q}_{\mathrm{aff}}^{\exp }(k) \rightarrow \mathrm{Q}^{\exp }(k)$ is an equivalence of categories.

### 4.3. The derived category of exponential motives

In a wide range of contexts, spectral sequences associated with simplicial or filtered spaces are a powerful tool to compute cohomology. We would like to use these techniques to compute exponential motives as well. The difficulty in doing so stems from the fact that $H^{n}(X, Y, f)$ is not defined as the homology in degree $n$ of a complex, as it is the case for most familiar cohomology theories. Our goal in this section is to fabricate adequately functorial complexes which compute exponential motives, as it is done for usual motives in [65].

Definition 4.3.1. - A triple $[X, Y, f]$ consisting of a variety $X$ over $k$, a closed subvariety $Y$ of $X$, and a regular function $f$ is said to be cellular in degree $n$ if $H^{p}(X, Y, f)=0$ for all $p \neq n$.

We write $\mathrm{Q}_{\mathrm{c}}^{\exp }(k)$ for the full subquiver of $\mathrm{Q}^{\exp }(k)$ of those objects $[X, Y, f, n, i]$ such that $X$ is affine of dimension $\leqslant n$ and $[X, Y, f]$ is cellular in degree $n$.

We equip $\mathrm{Q}_{\mathrm{c}}^{\exp }(k)$ with the restriction of the Betti representation $\rho$ from 4.2.2, so that the inclusion $\mathrm{Q}_{\mathrm{c}}^{\exp }(k) \subseteq \mathrm{Q}^{\exp }(k)$ can be seen as a morphism of quiver representations. We set $\mathbf{M}_{\mathrm{c}}^{\exp }(k)=$
$\left\langle\mathrm{Q}_{\mathrm{c}}^{\exp }(k), \rho\right\rangle$ and call canonical the functor

$$
\begin{equation*}
\mathbf{M}_{\mathrm{c}}^{\exp }(k) \longrightarrow \mathbf{M}^{\exp }(k) \tag{4.3.1.1}
\end{equation*}
$$

induced by the inclusion $\mathrm{Q}_{\mathrm{c}}^{\exp }(k) \rightarrow \mathrm{Q}^{\exp }(k)$.

THEOREM 4.3.2. - There exists a quiver representation $\lambda: \mathrm{Q}^{\exp }(k) \rightarrow D^{b}\left(\mathbf{M}_{\mathrm{c}}^{\exp }(k)\right)$ with the following three properties:
(1) The following diagram commutes up to natural isomorphisms:

(2) The equalities $\lambda([X, Y, f, n, i])=\lambda([X, Y, f, 0,0])[-n](i)$ hold.
(3) For all tuples $[X, Y, Z, f]$, the triangles

$$
\lambda([X, Y, f, n, i]) \rightarrow \lambda([X, Z, f, n, i]) \rightarrow \lambda\left(\left[Y, Z,\left.f\right|_{Y}, n, i\right]\right) \rightarrow \lambda([X, Y, f, n+1, i])
$$

are exact, where morphisms are the images under $\lambda$ of the corresponding morphisms of type (a) for inclusions and of type (b) for the triple.

In particular, the canonical functor (4.3.1.1) is an equivalence of categories.
4.3.3. - The construction of the representation $\lambda$ that we present below shows that it is characterised by the three properties of the statement essentially in a unique way. In view of property (2), the construction of $\lambda$ amounts to the construction of complexes

$$
C^{\bullet}(X, Y, f)=\lambda([X, Y, f, 0,0])
$$

in the category $\mathbf{M}_{\mathrm{c}}^{\exp }(k)$, which uses two essential ingredients: One is the Basic Lemma 3.3.3 which we use to define a complex for every object $[X, Y, f, n, i]$ where $X$ is affine, and the other is Jouanolou's trick, which permits us to replace a general variety with an affine one which is homotopic to it. Having done so, we obtain a complex in $\mathbf{M}_{\mathrm{c}}^{\exp }$ which is our candidate for $\lambda([X, Y, f, n, i])$, but depends on several choices. Once we look at the complex as an object in the derived category $D^{b}\left(\mathbf{M}_{\mathrm{c}}^{\exp }\right)$, we get rid of all dependence on these choices.
4.3.4. - Let us recall the following observation from [51, Lemme 1.5], which is commonly referred to as Jouanolou's trick:

Lemma 4.3.5. - For every quasi-projective variety $X$ over $k$, there exists an affine variety $\tilde{X}$ and a morphism $p: \widetilde{X} \rightarrow X$ such that each fibre $p^{-1}(x)$ is isomorphic to $\mathbb{A}^{d}$ for some $d \geqslant 0$ (but there is no such thing as a zero-section $X \rightarrow \widetilde{X})$. In particular, the induced continuous map $\widetilde{X}(\mathbb{C}) \rightarrow X(\mathbb{C})$ is a homotopy equivalence.

The proof is simple: For $X=\mathbb{P}^{n}$ take for $\widetilde{X}$ the variety of $(n+1) \times(n+1)$ matrices of rank 1 up to scalars with its obvious map to $\mathbb{P}^{n}$, and for general $X$ choose a projective embedding and do a pullback. Let us call such a morphism $p: \widetilde{X} \rightarrow X$ an affine homotopy replacement.

Jouanoulou's trick does not give a functorial homotopy replacement of varieties $X$ by affine $\tilde{X}$, but nearly so. Given a morphism of varieties $Y \rightarrow X$, we can replace first $X$ with an affine $\widetilde{X} \rightarrow X$, and then $Y$ with an affine homotopy replacement $\widetilde{Y}$ of the fibre product $Y \times_{X} \widetilde{X}$. The map $\widetilde{Y} \rightarrow Y$ is an affine homotopy replacement, and we obtain a morphism $\widetilde{Y} \rightarrow \widetilde{X}$ which lifts the given morphism $Y \rightarrow X$. This procedure can be generalised to the case of several morphisms from $Y \rightarrow X$, but not to arbitrary diagrams of varieties.

Definition 4.3.6. - Let $X$ be an affine variety over $k$, let $Z \subseteq Y \subseteq X$ be closed subvarieties and let $f$ be a regular function on $X$. A cellular filtration of $[X, Y, Z, f]$ is a chain of closed immersions

$$
\begin{equation*}
\varnothing \subseteq X_{0} \subseteq X_{1} \subseteq \cdots \subseteq X_{p-1} \subseteq X_{p} \subseteq \cdots \subseteq X_{d}=X \tag{4.3.6.1}
\end{equation*}
$$

where each $X_{p}$ is of dimension $\leqslant p$, such that the triples
$\left[X_{p}, X_{p-1},\left.f\right|_{X_{p}}\right], \quad\left[Y_{p}, Y_{p-1},\left.f\right|_{Y_{p}}\right], \quad\left[Z_{p}, Z_{p-1},\left.f\right|_{Z_{p}}\right], \quad\left[X_{p}, Y_{p} \cup X_{p-1},\left.f\right|_{X_{p}}\right], \quad\left[Y_{p}, Z_{p} \cup Y_{p-1},\left.f\right|_{Y_{p}}\right]$
are cellular in degree $p$, for $Y_{p}=X_{p} \cap Y$ and $Z_{p}=X_{p} \cap Z$. By a cellular filtration of $[X, Y, f]$ we understand a cellular filtration of $[X, Y, \varnothing, f]$.

Proposition 4.3.7. - Let $X$ be an affine variety over $k$, let $Z \subseteq Y \subseteq X$ be closed subvarieties and let $f$ be a regular function on $X$. There exist cellular filtrations of $[X, Y, Z, f]$, and every filtration of $X$ by closed subvarieties $X_{p}$ of dimension $\leqslant p$ is contained in a cellular filtration.

Proof. This is a direct consequence of the basic lemma 3.3.3: Suppose we are given a filtration of the form (4.3.6.1), which satisfies the cellularity condition for $j \geqslant p+1$. By 3.3.3 there exists a closed subvariety $Z$ of dimension $\leqslant p-1$ of $X_{p}$ such that the triples

$$
\begin{gathered}
{\left[X_{p}, X_{p-1} \cup Z,\left.f\right|_{X_{p}}\right], \quad\left[Y_{p}, Y_{p-1} \cup\left(Y_{p} \cap Z\right),\left.f\right|_{Y_{p}}\right], \quad\left[Z_{p}, Z_{p-1} \cup\left(Z_{p} \cap Z\right),\left.f\right|_{Z_{p}}\right]} \\
{\left[X_{p},\left(Y_{p} \cup X_{p-1}\right) \cup Z,\left.f\right|_{X_{p}}\right], \quad\left[Y_{p},\left(Z_{p} \cup Y_{p-1}\right) \cup\left(Z \cap Y_{p}\right),\left.f\right|_{Y_{p}}\right]}
\end{gathered}
$$

are cellular in degree $p$. Replace then $X_{p-1}$ with $X_{p-1} \cup Z$ and continue by induction on $p$.
4.3.8. - Let $X$ be an affine variety over $k$, together with a regular function $f$, and let $Y \subseteq X$ be a closed subvariety. Choose a cellular filtration $X_{*}$ of $[X, Y, f]$ and set $Y_{p}=X_{p} \cap Y$. We will consider the complex

$$
\begin{equation*}
C^{*}\left(X_{*}, Y_{*}, f\right)=\left[\cdots \rightarrow H^{p}\left(X_{p}, Y_{p} \cup X_{p-1},\left.f\right|_{X_{p}}\right) \xrightarrow{d_{p}} H^{p+1}\left(X_{p+1}, Y_{p+1} \cup X_{p},\left.f\right|_{X_{p+1}}\right) \rightarrow \cdots\right] \tag{4.3.8.1}
\end{equation*}
$$

in the category $\mathbf{M}_{\mathrm{c}}^{\exp }(k)$, where the differential $d_{p}$ is the connecting morphism in the long exact sequence associated with the triple $X_{p-1} \subseteq X_{p} \subseteq X_{p+1}$ and the sheaf on $X$ which computes the cohomology of the pair $[X, Y]$. By this we mean the following: For every constructible sheaf $F$ on
$X$ and every triple $X_{p-1} \subseteq X_{p} \subseteq X_{p+1}$ there is a short exact sequence of sheaves on $X$

$$
0 \rightarrow F_{\left[X_{p+1}, X_{p-1}\right]} \rightarrow F_{\left[X_{p}, X_{p-1}\right]} \rightarrow F_{\left[X_{p+1}, X_{p}\right]} \rightarrow 0
$$

and hence a long exact sequence

$$
\cdots \rightarrow{ }^{p} \mathcal{H}^{n}\left(R f_{*} F_{\left[X_{p+1}, X_{p-1}\right]}\right) \rightarrow{ }^{p} \mathcal{H}^{n}\left(R f_{*} F_{\left[X_{p}, X_{p-1}\right]}\right) \stackrel{\partial}{\longrightarrow}{ }^{p} \mathcal{H}^{n+1}\left(R f_{*} F_{\left[X_{p+1}, X_{p}\right]}\right) \rightarrow \cdots
$$

of perverse sheaves on the affine line. Applying the functor $\Pi$ and taking fibres at infinity, this yields the exact sequence of vector spaces

$$
\begin{equation*}
\cdots \rightarrow H_{\mathrm{rd}}^{n}\left(X_{p+1}, X_{p-1}, f ; F\right) \rightarrow H_{\mathrm{rd}}^{n}\left(X_{p}, X_{p-1}, f ; F\right) \xrightarrow{\partial} H_{\mathrm{rd}}^{n+1}\left(X_{p+1}, X_{p}, f ; F\right) \rightarrow \cdots \tag{4.3.8.2}
\end{equation*}
$$

by definition of rapid decay cohomology with coefficients in a constructible sheaf. We consider (4.3.8.2) for the terms of the standard short exact sequence of sheaves on $X$

$$
0 \rightarrow \underline{\mathbb{Q}}_{[X, Y]} \rightarrow \underline{\mathbb{Q}}_{X} \rightarrow \underline{\mathbb{Q}}_{Y} \rightarrow 0
$$

Taking into account that

$$
\begin{aligned}
H_{\mathrm{rd}}^{n}\left(X_{p}, X_{p-1}, f ; \mathbb{Q}_{[X, Y]}\right) & =H_{\mathrm{rd}}^{n}\left(X_{p}, Y_{p} \cup X_{p-1},\left.f\right|_{X_{p}}\right) \\
H_{\mathrm{rd}}^{n}\left(X_{p}, X_{p-1}, f ; \underline{\mathbb{Q}}_{X}\right) & =H_{\mathrm{rd}}^{n}\left(X_{p}, X_{p-1},\left.f\right|_{X_{p}}\right) \\
H_{\mathrm{rd}}^{n}\left(X_{p}, X_{p-1}, f ; \mathbb{Q}_{Y}\right) & =H_{\mathrm{rd}}^{n}\left(Y_{p}, Y_{p-1},\left.f\right|_{Y_{p}}\right),
\end{aligned}
$$

the long exact sequence (4.3.8.2) and the cellularity assumptions yield a morphism of short exact sequences of vector spaces

$$
\begin{gathered}
0 \rightarrow H_{\mathrm{rd}}^{p}\left(X_{p}, Y_{p} \cup X_{p-1}, f_{\mid X_{p}}\right) \longrightarrow H_{\mathrm{rd}}^{p}\left(X_{p}, X_{p-1}, f_{\mid X_{p}}\right) \longrightarrow H_{\mathrm{rd}}^{p}\left(Y_{p}, Y_{p-1}, f_{\mid Y_{p}}\right) \rightarrow 0 \\
d_{p} \downarrow \\
0 \rightarrow H_{\mathrm{rd}}^{p+1}\left(X_{p+1}, Y_{p+1} \cup X_{p}, f_{\mid X_{p+1}}\right) \longrightarrow H_{\mathrm{rd}}^{p+1}\left(X_{p+1}, X_{p}, f_{\mid X_{p+1}}\right) \longrightarrow H_{\mathrm{rd}}^{p+1}\left(Y_{p+1}, Y_{p}, f_{\mid Y_{p+1}}\right) \rightarrow 0
\end{gathered}
$$

in which the differential of (4.3.8.1) appears. All vector spaces in this diagram underlie objects of $\mathbf{M}_{\mathrm{c}}^{\exp }(k)$. This diagram shows as well that $d_{p}$ is a morphism in $\mathbf{M}_{\mathrm{c}}^{\exp }(k)$ rather than just a morphism of vector spaces, indeed, all other morphisms in the diagram are morphisms in $\mathbf{M}_{\mathrm{c}}^{\exp }(k)$ since they either are given by inclusions of pairs or by connecting morphisms of triples, and hence come from morphisms in $\mathrm{Q}_{\mathrm{c}}^{\exp }(k)$. That the composite $d_{p-1} \circ d_{p}$ is zero follows from the fact that for any chain of closed subvarieties $X_{p-2} \subseteq X_{p-1} \subseteq X_{p} \subseteq X_{p+1}$ of $X$ and any sheaf $F$ on $X$, the composite

$$
H_{\mathrm{rd}}^{p-1}\left(\left[X_{p-1}, X_{p-2}\right], F\right) \rightarrow H_{\mathrm{rd}}^{p}\left(\left[X_{p}, X_{p-1}\right], F\right) \rightarrow H_{\mathrm{rd}}^{p+1}\left(\left[X_{p+1}, X_{p}\right], F\right)
$$

is zero. The complex $C^{*}\left(X_{*}, Y_{*}, f\right)$ is functorial in the obvious way for morphisms of filtered pairs: let $h: X^{\prime} \rightarrow X$ be a morphism of affine varieties over $k$, restricting to a morphism $Y^{\prime} \rightarrow Y$ between closed subvarieties, set $f^{\prime}=f \circ h$, and let $X_{*}$ and $X_{*}^{\prime}$ be cellular filtrations for $[X, Y, f]$ and [ $\left.X^{\prime}, Y^{\prime}, f^{\prime}\right]$ such that $h\left(X_{p}^{\prime}\right)$ is contained in $X_{p}$ and $h\left(Y_{p}^{\prime}\right)$ in $Y_{p}$ for all $p \geqslant 0$. The morphism

$$
\begin{equation*}
C^{*}(h): C^{*}\left(X_{*}, Y_{*}, f\right) \rightarrow C^{*}\left(X_{*}^{\prime}, Y_{*}^{\prime}, f^{\prime}\right) \tag{4.3.8.3}
\end{equation*}
$$

shall be the one induced by the morphism $H^{p}\left(X_{p}, Y_{p} \cup X_{p-1},\left.f\right|_{X_{p}}\right) \rightarrow H^{p}\left(X_{p}^{\prime}, Y_{p}^{\prime} \cup X_{p-1}^{\prime},\left.f^{\prime}\right|_{X_{p}^{\prime}}\right)$ given by the restriction of $h$ to $X_{p}^{\prime}$.
4.3.9. - We now turn to the proof that the cohomology of the complex $C^{*}\left(X_{*}, Y_{*}, f\right)$ computes the exponential motives $H^{n}(X, Y, f)$. Recall from (??) that $\Gamma_{f}: \operatorname{Sh}(X) \rightarrow \operatorname{Vec}_{\mathbb{Q}}$ is the left exact functor obtained by composing in that order: the direct image functor $f_{*}$, taking the tensor product $-\boxtimes j!\underline{\mathbb{Q}}_{\mathbb{G}_{m}}$ on $\mathbb{A}^{2}$, the direct image functor $\operatorname{sum}_{*}$ and the fibre functor $\Psi_{\infty}$.

Lemma 4.3.10. - Let $X$ be an affine variety over $k$, together with a regular function $f$, and let $Y \subseteq X$ be a closed subvariety. Choose a cellular filtration $X_{*}$ of $[X, Y, f]$. There is a natural isomorphism in the derived category of vector spaces

$$
\begin{equation*}
C^{*}\left(X_{*}, Y_{*}, f\right) \cong R \Gamma_{f}\left(\mathbb{Q}_{[X, Y]}\right) \tag{4.3.10.1}
\end{equation*}
$$

Proof. That the complex $R \Gamma_{f}\left(\underline{\mathbb{Q}}_{[X, Y]}\right)$ computes rapid decay cohomology was explained in Proposition ??. The complex on the right hand is calculated by choosing an injective resolution

$$
I_{*}=\left[I_{0} \rightarrow I_{1} \rightarrow I_{2} \rightarrow \cdots\right]
$$

of the sheaf $\underline{\mathbb{Q}}_{[X, Y]}$ and applying to this resolution the functor $\Gamma_{f}$. On the left-hand side we have a complex of motives, which has an underlying complex of vector spaces. It is given in degree $p$ by the vector space $H^{p}\left(X_{p}, X_{p-1} \cup Y_{p}, f\right)$ which is the same as $R \Gamma_{f}\left(\underline{\mathbb{Q}}_{\left[X_{p}, X_{p-1} \cup Y_{p}\right]}\right)$ by Proposition ??. Thus, the claim of the Lemma is the following:

Claim: Let $F$ be a sheaf on $X$, and let $X_{*}$ be a finite exhaustive filtration of $X$ by closed subspaces $X_{p}$ such that $H^{n}\left(X_{p}, X_{p-1},\left.f\right|_{X_{p}}, F\right)$ is zero for $n \neq p$. Then the complex of vector spaces

$$
\begin{equation*}
\cdots \rightarrow H^{p-1}\left(X_{p-1}, X_{p-2}, f, F\right) \rightarrow H^{p}\left(X_{p}, X_{p-1}, f, F\right) \rightarrow H^{p+1}\left(X_{p+1}, X_{p}, f, F\right) \rightarrow \cdots \tag{4.3.10.2}
\end{equation*}
$$

is isomorphic to $R \Gamma_{f}(F)$ in the derived category of vector spaces.
To see this, choose an injective resolution $F \rightarrow I_{*}$ of $F$. The long exact sequence (4.3.10.2) is natural in $F$, so if we apply it to $I_{*}$ we obtain a double complex, and hence a spectral sequence

$$
E_{1}^{p, q}=H^{p+q}\left(X_{p}, X_{p-1}, f ; F\right) \Longrightarrow H^{p+q}(X, f, F)
$$

By the assumption that the filtration is cellular, the term $E_{1}^{p, q}$ vanishes for $q \neq 0$, and hence the spectral sequence degenerates at the second page, yielding the desired quasi-isomorphism of complexes.

Naturality of the isomorphism (4.3.10.1) for morphisms of filtered pairs follows from naturality of (4.3.10.2) in $X_{*}$ and $F$.

Proposition 4.3.11. - Let $X$ be an affine variety over $k$, together with a regular function $f$, and let $Y \subseteq X$ be a closed subvariety. Choose a cellular filtration $X_{*}$ of $[X, Y, f]$. There is a canonical isomorphism in $\mathbf{M}^{\exp }(k)$

$$
H^{p}\left(C^{*}\left(X_{*}, Y_{*}, f\right)\right) \cong H^{p}(X, Y, f)
$$

which is natural for morphisms of filtered pairs. If $X$ is of dimension $\leqslant n$ and $[X, Y, f]$ is cellular in degree $n$, then this isomorphism is an isomorphism in $\mathbf{M}_{\mathrm{c}}^{\exp }(k)$.

Proof. The cohomology of $C^{*}\left(X_{*}, Y_{*}, f\right)$ in degree $p$ is the object

$$
\begin{equation*}
H^{p}\left(C^{*}\left(X_{*}, Y_{*}, f\right)\right)=\frac{\operatorname{ker}\left(H^{p}\left(X_{p}, Y_{p} \cup X_{p-1}, f_{p}\right) \rightarrow H^{p+1}\left(X_{p+1}, Y_{p+1} \cup X_{p}, f_{p+1}\right)\right)}{\operatorname{im}\left(H^{p-1}\left(X_{p-1}, Y_{p-1} \cup X_{p-2}, f_{p-1}\right) \rightarrow H^{p}\left(X_{p}, Y_{p} \cup X_{p-1}, f_{p}\right)\right)} \tag{4.3.11.1}
\end{equation*}
$$

in $\mathbf{M}_{\mathrm{c}}^{\exp }(k)$ and we wish to show that this object is naturally isomorphic to $H^{p}(X, Y, f)$ in $\mathbf{M}^{\exp }(k)$, and even in $\mathbf{M}_{\mathrm{c}}^{\exp (k)}$ whenever $[X, Y, f]$ is cellular. To treat cases uniformly, pick any finite subquiver $Q$ of $\mathrm{Q}^{\exp }(k)$ or of $\mathrm{Q}_{\mathrm{c}}^{\exp }(k)$ which contains at least $[X, Y, f, p, 0]$, the $\left[X_{p}, Y_{p} \cup X_{p-1}, f_{p}, p, 0\right]$, the morphisms coming from inclusions, and the connecting morphisms of triples, subject to future enlargement. Set $E=\operatorname{End}\left(\left.\rho\right|_{Q}\right)$. For all integers $q<p$ and $n$, we have $H^{n}\left(X_{p}, Y_{p} \cup X_{q}, f_{p}\right)=0$ unless $q<n \leqslant p$. Indeed, this is true by definition if $q=p-1$, and follows in general by induction on $p-q$ using the long exact sequence associated with the triple $X_{q} \subseteq X_{p-1} \subseteq X_{p}$. This explains why the morphisms

$$
\begin{equation*}
H^{p}(X, Y, f) \rightarrow H^{p}\left(X_{p+1}, Y_{p+1}, f_{p+1}\right) \longleftarrow H^{p}\left(X_{p+1}, Y_{p+1} \cup X_{p-2}, f_{p+1}\right) \tag{4.3.11.2}
\end{equation*}
$$

are isomorphisms of vector spaces, and also explains the surjections and injections in the following diagram, whose exact rows and columns are pieces of the long exact sequences associated with triples out of the quadruple $X_{p-2} \subseteq \cdots \subseteq X_{p+1}$.


This diagram is a diagram of vector spaces where all morphisms labelled with a $*$ are morphisms of $E$-modules between $E$-modules. But then the whole diagram is a diagram of $E$-modules, in only one possible way. Now we have an $E$-module structure on $H^{p}(X, Y, f)$ and on $H^{p}\left(X_{p+1}, Y_{p+1} \cup\right.$ $X_{p-2}, f_{p+1}$ ), and we need to show that the isomorphisms (4.3.11.2) are isomorphisms of $E$-modules after possibly enlarging $Q$. In the case where we work with subquivers of $\mathrm{Q}(k)$ we add to $Q$ the two morphisms of pairs needed to define (4.3.11.2) and are done. If we work with cellular pairs only, then $X$ has dimension $\leqslant p$ and $[X, Y]$ is cellular in degree $p$, and we enlarge $Q$ as follows: By the Basic Lemma 3.3.3, there exists a closed $Z \subseteq X$ of dimension $\leqslant p-1$ such that $H^{p}\left(X, Y^{\prime}, f\right)$ is cellular in degree $p$ for $Y^{\prime}=Y \cup X_{p-1} \cup Z$. Add the morphism $\left[X, Y^{\prime}, f, n, 0\right] \rightarrow[X, Y, f, n, 0]$ to $Q$ so that $H^{p}\left(X, Y^{\prime}, f\right) \rightarrow H^{p}(X, Y, f)$ is an $E$-linear morphism. It is surjective for dimension reasons, and the diagram of $E$-modules and linear maps

commutes, where the isomorphism $u$ is induced by (4.3.11.2). All morphisms labelled $*$ are $E$-linear and hence so is $u$. Altogether, we conclude that the homology in the middle of

$$
H^{p-1}\left(X_{p-1}, Y_{p-1} \cup X_{p-2}, f_{p-1}\right) \rightarrow H^{p}\left(X_{p}, Y_{p} \cup X_{p-1}, f_{p}\right) \rightarrow H^{p+1}\left(X_{p+1}, Y_{p+1} \cup X_{p}, f_{p+1}\right)
$$

is indeed canonically isomorphic to $H^{p}(X, Y, f)$ as an $E$-module, which is what we had to show. Naturality of the isomorphism for morphisms of filtered pairs follows from functoriality of (4.3.11.2).

Corollary 4.3.12. - Let $X$ and $X^{\prime}$ be affine varieties and $h:[X, Y, f] \rightarrow\left[X^{\prime}, Y^{\prime}, f^{\prime}\right]$ be a morphism in $Q^{\exp }(k)$. Let $X_{*}$ and $X_{*}^{\prime}$ be cellular filtrations of $\left[X_{1}, Y^{\prime}, f^{\prime}\right]$ and $\left[X^{\prime}, Y^{\prime}, f^{\prime}\right]$. If $h$ induces an isomorphism in rapid decay cohomology, then the morphism of complexes $C^{*}\left(\left[X_{*}, Y_{*}, f\right]\right) \rightarrow$ $C^{*}\left(\left[X_{*}^{\prime}, Y_{*}^{\prime}, f\right]\right)$ defined in (4.3.8.3) is a quasi-isomorphism.

Proof. This follows from the conservativity of the forgetful functor $\mathbf{M}^{\exp }(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ and Proposition 4.3.11.

Proposition 4.3.13. - Let $X$ be an affine variety over $k$, let $f$ be a regular function on $X$ and let $Z \subseteq Y \subseteq X$ and $Z \subseteq Y$ be closed subvarieties. Let $X_{*}$ be a cellular filtration of $[X, Y, Z, f]$. The sequence of complexes with morphisms given by (4.3.8.3) for inclusions

$$
\begin{equation*}
0 \rightarrow C^{*}\left(\left[X_{*}, Y_{*}\right]\right) \rightarrow C^{*}\left(\left[X_{*}, Z_{*}\right]\right) \rightarrow C^{*}\left(\left[Y_{*}, Z_{*}\right]\right) \rightarrow 0 \tag{4.3.13.1}
\end{equation*}
$$

is degreewise exact.
Proof. The sequences in question are sequences in $\mathbf{M}_{\mathrm{c}}^{\exp }(k)$, but in order to show that they are exact it suffices to show that the underlying sequence in Vec are exact. But that immediately follows from the definition of cellular filtrations and a diagram chase.

We have now all the ingredients needed for the proof of the main result of this section.
Proof of Theorem 4.3.2. For each object $[X, Y, f, n, i]$ of the quiver $\mathrm{Q}^{\exp }(k)$ and each cellular filtration $X_{*}$ of $[X, Y, f]$, we consider the complex $C^{*}\left(X_{*}, Y_{*}, f\right)[-n](i)$ obtained from (4.3.8.1) by shifting and twisting degree by degree. Let us define $\lambda$ as follows:

$$
\lambda([X, Y, f, n, i])=\underset{\widetilde{X} \rightarrow X}{\operatorname{colim}} \lim _{\widetilde{X}_{*}} C^{*}\left(\widetilde{X}_{*}, \widetilde{Y}_{*}, \tilde{f}\right)[-n](i)
$$

on objects [ $X, Y, f, n, i]$ of $\mathrm{Q}^{\exp }(k)$, where the limit runs over all cellular filtrations of the triple $[\widetilde{X}, \widetilde{Y}, f]$ and the colimit over all affine homotopy replacements $\Pi: \widetilde{X} \rightarrow X$, setting $\widetilde{Y}=\widetilde{X} \times{ }_{X} Y$ and $\tilde{f}=f \circ p$. These colimits and limits exist in the derived category $D^{b}\left(\mathbf{M}_{\mathrm{c}}^{\exp }(k)\right)$. Indeed, all transition maps are isomorphisms by Corollary 4.3.12. From the practical point of view, $\lambda([X, Y, f, n, i])$ is isomorphic to any of the complexes $C^{*}\left(\widetilde{X}_{*}, \widetilde{Y}_{*}, \widetilde{f}\right)[-n](i)$ up to a unique isomorphism in $D^{b}\left(\mathbf{M}_{\mathrm{c}}^{\exp }(k)\right)$, and the use of the limiting processes is only an artifact to get rid of choices ${ }^{2}$. We define $\lambda$ on morphisms as follows:

Type (a): Let $h:[X, Y, f, n, i] \rightarrow\left[X^{\prime}, Y^{\prime}, f^{\prime}, n, i\right]$ be given by a morphism of varieties $h: X^{\prime} \rightarrow X$ such that $h\left(Y^{\prime}\right) \subseteq Y$ and $f^{\prime}=f \circ h$. From (4.3.8.3) we obtain a morphism

$$
C^{\bullet}(h): C^{\bullet}\left(\widetilde{X}_{*}^{\prime}, \tilde{Y}_{*}^{\prime}, f^{\prime}\right) \longrightarrow C^{\bullet}\left(\widetilde{X}_{*}, \widetilde{Y}_{*}, f\right)
$$

for suitable affine homotopy replacements and cellular filtrations, and set $\lambda(f)=C^{\bullet}(f)[-n](i)$.

[^3]Type (b): Let $d:[Y, Z, n, i] \rightarrow[X, Y, n+1, i]$ be given by closed immersions $Z \subseteq Y \subseteq X$ between affine varieties. Choose an affine homotopy replacement $\widetilde{X} \rightarrow X$, set $\widetilde{Y}=\widetilde{X} \times{ }_{X} Y$ and $\widetilde{Z}=\widetilde{X} \times_{X} Z$ and cellular filtration of the triple $[\widetilde{X}, \widetilde{Y}, \widetilde{Z}]$. From Proposition 4.3 .13 we obtain an degreewise exact sequence of complexes

$$
\begin{equation*}
0 \rightarrow C^{*}\left(\left[\widetilde{X}_{*}, \widetilde{Y}_{*}\right]\right) \xrightarrow{r} C^{*}\left(\left[\widetilde{X}_{*}, \widetilde{Z}_{*}\right]\right) \xrightarrow{s} C^{*}\left(\left[\widetilde{Y}_{*}, \widetilde{Z}_{*}\right] \rightarrow 0\right. \tag{4.3.13.2}
\end{equation*}
$$

where $r$ and $s$ induced by inclusions, and hence a morphism in $D^{b}\left(\mathbf{M}_{\mathrm{c}}\right)$ given by the hat

and define $\lambda(d)=\partial[-n](i)$.
Type (c): If $\tilde{X} \rightarrow X$ is an affine homotopy replacement, then so is $\tilde{X} \times \mathbb{G}_{m} \rightarrow X \times \mathbb{G}_{m}$. If $\widetilde{X_{*}}$ a cellular filtration of $[\widetilde{X}, \widetilde{Y}]$, then the $\widetilde{X}_{p} \times \mathbb{G}_{m} \subseteq \widetilde{X} \times \mathbb{G}_{m}$ form a cellular filtration of $\left[\widetilde{X} \times \mathbb{G}_{m}, \widetilde{Y} \times \mathbb{G}_{m} \cup \widetilde{X} \times\{1\}\right]$. Hence there is a canonical isomorphism of complexes

$$
\begin{equation*}
C^{*}\left(\left[\widetilde{X}_{*} \times \mathbb{G}_{m}, \widetilde{Y}_{*} \times \mathbb{G}_{m} \cup \widetilde{X}_{*} \times\{1\}\right]\right)(1) \rightarrow C^{*}\left(\left[\widetilde{X}_{*}, \widetilde{Y}_{*}\right]\right) \tag{4.3.13.3}
\end{equation*}
$$

obtained from the corresponding isomorphisms degree-by-degree, and we declare this morphism shifted and twisted by $[-n](i)$ to be the image under $\lambda$ of the morphism of type (c) with target [ $X, Y, n, i]$.

Now that we have defined $\lambda$, it remains to show that the diagram in the statement of Theorem 4.3.2 indeed commutes up to natural isomorphisms. All other statement hold by construction. The isomorphisms we seek

$$
\lambda([X, Y, n, i]) \cong H^{n}([X, Y])(i)
$$

are those of Proposition 4.3 .11 with a twist. Naturality of these isomorphisms for morphisms in $\mathrm{Q}(k)$ is a question on the level of modules, and follows from the fact that the isomorphisms in Proposition 4.3.11 are induced, as morphisms of modules, by the isomorphisms of complexes of Lemma 4.3.10

Corollary 4.3.14. - Each object of the category $\mathbf{M}^{\exp }(k)$ is a subquotient of a sum of objects of the form $H^{n}(X, Y, f)(i)$, where $X=\bar{X} \backslash Y_{\infty}$ and $Y=Y_{0} \backslash\left(Y_{0} \cap Y_{\infty}\right)$ for a smooth projective variety $\bar{X}$ of dimension n and two normal crossing divisors $Y_{0}$ and $Y_{\infty}$ without common components, such that the union $Y_{0} \cup Y_{\infty}$ has normal crossings as well.

Proof. The combination of Theorem 4.3.2 and Proposition 4.1.7 shows that every object $M$ of the category $\mathbf{M}^{\exp }(k)$ is a subquotient of a sum of exponential motives

where each variety $X_{\alpha}$ is affine of dimension $n_{\alpha}$ and $Y_{\alpha} \subseteq X_{\alpha}$ is a closed subvariety of dimension $\leqslant n-1$, such that the triple $\left[X_{\alpha}, Y_{\alpha}, f_{\alpha}\right]$ is cellular in degree $n_{\alpha}$. Let $Y_{\alpha}^{\prime} \subseteq X_{\alpha}$ be a closed subvariety
of dimension $\leqslant n_{\alpha}-1$ containing $Y_{\alpha}$ and the singular locus of $X_{\alpha}$. Since $H^{n_{\alpha}}\left(Y_{\alpha}^{\prime}, Y_{\alpha}, f_{\alpha}\right)=0$ by Artin vanishing, the long exact sequence (4.2.4.2) shows that the morphism

$$
H^{n_{\alpha}}\left(X_{\alpha}, Y_{\alpha}^{\prime}, f_{\alpha}\right) \rightarrow H^{n_{\alpha}}\left(X_{\alpha}, Y_{\alpha}, f_{\alpha}\right)
$$

is surjective. Up to replacing $Y_{\alpha}$ with $Y_{\alpha}^{\prime}$, we may therefore assume that the open complement $X_{\alpha} \backslash Y_{\alpha}$ is smooth. We claim that he motive $H^{n_{\alpha}}\left(X_{\alpha}, Y_{\alpha}^{\prime}, f_{\alpha}\right)$ is isomorphic to a motive of the desired shape. Indeed, by resolution of singularities there exists a smooth projective variety $\bar{X}$ with normal crossing divisors $Y_{0}$ and $Y_{\infty}$ as claimed, together with a morphism $\pi: X \rightarrow X_{\alpha}$ inducing an isomorphism $X \backslash Y \rightarrow X_{\alpha} \backslash Y_{\alpha}$ on open complements. Here, $X=\bar{X} \backslash Y_{\infty}$ and $Y=Y_{0} \backslash\left(Y_{0} \cap Y_{\infty}\right)$ as in the statement. Thus, $Y$ is equal to $\pi^{-1}\left(Y_{\alpha}\right)$ and the morphism of motives

$$
H^{n_{\alpha}}\left(X_{\alpha}, Y_{\alpha}, f_{\alpha}\right) \rightarrow H^{n}(X, Y, f)
$$

induced by $\pi$ is an isomorphism for $n=n_{\alpha}$ and $f=f_{\alpha} \circ \pi$.

### 4.4. Tensor products

In this section, we introduce a tensor product on the category of exponential motives, following Nori's ideas. We shall prove later that this tensor product endows $\mathbf{M}^{\exp }(k)$ with the structure of a neutral tannakian category, with $\mathrm{R}_{\mathrm{B}}$ as fibre functor.

Theorem 4.4.1. - The category $\mathbf{M}^{\exp }(k)$ admits a unique $\mathbb{Q}$-linear monoidal structure that satisfies the following properties.
(1) The forgetful functor $\mathrm{R}_{\mathrm{B}}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ is strictly monoidal.
(2) Künneth morphisms are morphisms of motives.

With respect to this monoidal structure and $R_{B}$ as fibre functor, $\mathbf{M}^{\exp }(k)$ is a neutral tannakian category.

By a symmetric monoidal structure on $\mathbf{M}^{\exp }$ we understand a functor

$$
\otimes: \mathbf{M}^{\exp }(k) \times \mathbf{M}^{\exp }(k) \rightarrow \mathbf{M}^{\exp }(k)
$$

which we call tensor product together with isomorphisms of functors expressing associativity and commutativity of the tensor product, and the properties of $\mathbb{Q}(0)=H^{0}(\operatorname{Spec} k)$ playing the role of a neutral object. That the forgetful functor or Betti realisation $\mathrm{R}_{\mathrm{B}}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ is strictly monoidal means that there exist natural isomorphisms

$$
\begin{equation*}
\mathrm{R}_{\mathrm{B}}(M \otimes N) \cong \mathrm{R}_{\mathrm{B}}(M) \otimes \mathrm{R}_{\mathrm{B}}(N) \tag{4.4.1.1}
\end{equation*}
$$

that are compatible with the associativity and commutativity constraints. These isomorphisms will come directly from the construction of the tensor product, and will be equalities. Given relative varieties with functions $[X, Y, f]$ and $\left[X^{\prime}, Y^{\prime}, f^{\prime}\right]$, the Künneth morphisms are maps of vector spaces

$$
H^{n}(X, Y, f) \otimes H^{n^{\prime}}\left(X^{\prime}, Y^{\prime}, f^{\prime}\right) \rightarrow H^{n+n^{\prime}}\left(X \times X^{\prime},\left(Y \times X^{\prime}\right) \cup\left(X \times Y^{\prime}\right), f \boxplus f^{\prime}\right)
$$

and property (2) states that these morphisms are compatible with the motivic structures.
What remains to be done to get the tannakian structure on $\mathbf{M}^{\exp }(k)$ is to prove that, for each object $M$, the functor $-\otimes M$ has a natural right adjoint, which we denote by Hom $(M,-)$, so that the usual adjunction formula holds:

$$
\operatorname{Hom}(X \otimes M, Y) \cong \operatorname{Hom}(X, \underline{\operatorname{Hom}}(M, Y)) .
$$

4.4.2 (Construction of the tensor product). - To ease notation, let us denote by $\mathrm{Q}_{c}=\mathrm{Q}_{c}^{\exp }(k)$ the cellular quiver from Definition 4.3.1. We consider the quiver morphism

$$
\text { prod: } \mathrm{Q}_{c} \boxtimes \mathrm{Q}_{c} \rightarrow \mathrm{Q}_{c}
$$

given on objects by

$$
\operatorname{prod}\left([X, Y, f, n, i] \boxtimes\left[X^{\prime}, Y^{\prime}, f^{\prime}, n, i^{\prime}\right]\right)=\left[X \times X^{\prime}, Y \times X^{\prime} \cup X \times Y^{\prime}, f \boxplus f^{\prime}, n+n^{\prime}, i+i^{\prime}\right],
$$

and with the evident definition on morphisms. The Künneth formula provides a natural isomorphism

$$
H^{n}(X, Y, f)(i) \otimes H^{n^{\prime}}\left(X^{\prime}, Y^{\prime}, f^{\prime}\right)\left(i^{\prime}\right) \cong H^{n+n^{\prime}}\left(X \times X^{\prime}, Y \times X^{\prime} \cup X \times Y^{\prime}, f \boxplus f^{\prime}\right)\left(i+i^{\prime}\right)
$$

since all other terms in the Künneth formula are zero for dimension reasons. We obtain hence a morphism of quiver representations

which induces a functor

$$
\mathbf{M}^{\exp }(k) \times \mathbf{M}^{\exp }(k) \longrightarrow \mathbf{M}^{\exp }(k)
$$

4.4.3 (Construction of the commutativity constraint). -

### 4.5. Intermezzo: Simplicial spaces and hypercoverings

We collect in this section some facts about simplicial topological spaces and their cohomology. These are also summarised in $[22, \S 5]$. We denote by $\Delta$ the category whose objects are the finite ordered sets $[n]=\{0,1, \ldots, n\}$, for $n \geqslant 0$, and whose morphisms are order-preserving maps. Given an integer $N$, let $\Delta_{N}$ denote the full subcategory of $\Delta$ consisting of objects [ $n$ ] with $n \leqslant N$.
4.5.1. - A simplicial object in a category $\mathbf{C}$ is a contravariant functor $X_{\bullet}: \Delta \rightarrow \mathbf{C}$. A morphism between simplicial objects is a morphism of functors. Any object of $\mathbf{C}$ can be seen as a constant simplicial object. It is custom to picture a simplicial object as a diagram of the shape

$$
\cdots X_{3} \underset{\rightleftarrows}{\rightleftarrows} X_{2} \underset{ }{\rightleftarrows} X_{1} \rightleftarrows X_{0}
$$

where the object $X_{n}$ is the image of $[n]$ under the given functor $\Delta \rightarrow \mathbf{C}$, and the arrows symbolise the images of the $n-1$ surjective morphisms $[n] \rightarrow[n-1]$ and the $n$ injective morphisms $[n-1] \rightarrow[n]$ in $\Delta$. A morphism $\varepsilon: X_{0} \rightarrow S$ coequalising the two maps $X_{1} \rightarrow X_{0}$ is called an augmentation. We may view it as a morphism from $X_{\bullet}$ to the constant simplicial object $S$, or we may view it as a simplicial object in the slice category $\mathbf{C} / S$.
4.5.2. - Starting from an augmented simplicial object $X_{\bullet}$, one finds by restriction a functor $\Delta_{N} \rightarrow \mathbf{C}$ which is denoted by $\mathrm{sk}_{N} X_{\bullet}$ and called the $N$-skeleton. If finite limits exist in the category $\mathbf{C}$, then the functor

$$
\operatorname{sk}_{N}: \operatorname{Functors}\left(\Delta^{\mathrm{op}}, \mathbf{C} / S\right) \rightarrow \operatorname{Functors}\left(\Delta_{N}^{\mathrm{op}}, \mathbf{C} / S\right)
$$

admits a right adjoint $\operatorname{cosk}_{N}$, called coskeleton. For instance, the coskeleton of the 0 -skeleton $X_{0} \rightarrow S$ is given by

$$
\operatorname{cosk}_{0}\left(X_{0} \rightarrow S\right)_{n}=\underbrace{X_{0} \times_{S} X_{0} \times_{S} \cdots \times_{S} X_{0}}_{n+1 \text { copies }}=\operatorname{Maps}\left([n],\left(X_{0} / S\right)\right)
$$

for all $n \geqslant 0$, with projections and diagonal embeddings as face and coface maps. There is also a relative version of coskeleta: given a morphism of augmented simplicial objects $X_{\bullet} \rightarrow Y_{\bullet}$, the restriction functor

$$
\operatorname{sk}_{N}^{Y_{\bullet}}: \operatorname{Functors}\left(\Delta^{\mathrm{op}}, \mathbf{C} / S\right) / Y_{\bullet} \rightarrow \operatorname{Functors}\left(\Delta_{N}^{\mathrm{op}}, \mathbf{C} / S\right) /\left(\mathrm{sk}_{N} Y_{\bullet}\right)
$$

admits a right adjoint $\operatorname{cosk}_{N}^{Y_{\bullet}}$, which we call the relative coskeleton over $Y_{\bullet}$. It is related to the absolute version by

$$
\operatorname{cosk}_{N}^{Y_{\bullet}}\left(X_{\bullet}\right)=\operatorname{cosk}_{N}\left(X_{\bullet}\right) \times_{\operatorname{cosk}_{N}\left(\operatorname{sk}_{N}\left(Y_{\bullet}\right)\right)} Y_{\bullet}
$$

where the fibre product is taken in the category of simplicial objects in $\mathbf{C} / S$. For the record, the adjunction morphism reads

and should be interpreted as a morphism in the category of simplicial objects over $Y_{\bullet}$.
4.5.3. - A sheaf on a simplicial topological space $X_{\bullet}$ is the data of a sheaf $F_{n}$ on each $X_{n}$ and a natural morphism of sheaves

$$
\begin{equation*}
f_{X}^{*} F_{n} \rightarrow F_{m} \tag{4.5.3.1}
\end{equation*}
$$

on $X_{m}$ for each morphism $f:[n] \rightarrow[m]$ in $\Delta$ inducing $f_{X}: X_{m} \rightarrow X_{n}$. One can regard sheaves $F_{\bullet}$ on $X_{\bullet}$ as sheaves on some convenient site, and using that point of view, the cohomology $H^{n}\left(X_{\bullet}, F_{\bullet}\right)$ is defined. Global sections of $F_{\bullet}$ are given by

$$
\Gamma\left(X_{\bullet}, F_{\bullet}\right)=\operatorname{ker}\left(\Gamma\left(X_{0}, f_{0}\right) \Longrightarrow \Gamma\left(X_{1}, F_{1}\right)\right)
$$

where the two morphisms are those induced by the morphisms of sheaves (4.5.3.1) corresponding to the two face maps $d_{0}, d_{1}: X_{1} \rightarrow X_{0}$. Given a sheaf of commutative groups $F_{\bullet}$ on a simplicial topological space $X_{\bullet}$, there is a spectral sequence of groups

$$
\begin{equation*}
E_{1}^{p, q}=H^{q}\left(X_{p}, F_{p}\right) \Longrightarrow H^{p+q}\left(X_{\bullet}, F_{\bullet}\right) \tag{4.5.3.2}
\end{equation*}
$$

constructed as follows. An injective resolution of $F_{\bullet}$ gives an injective resolution of each $F_{n}$. Applying global sections gives a simplicial complex of groups, hence a double complex. Its horizontal differentials (say) are alternating sums of face maps, and its vertical differentials are induced by the differentials of the resolution. The spectral sequence (4.5.3.2) is the one associated with this double complex for the filtration by columns, see e.g. Definition 5.6.1 in [89].

Example 4.5.4. - Let $Y_{\bullet} \rightarrow X_{\bullet}$ be a morphism of simplicial topological spaces, such that for each $n \geqslant 0$ the map $Y_{n} \rightarrow X_{n}$ is a cofibration (closed immersion and neighbourhood deformation retract). On each space $X_{n}$, consider the sheaf

$$
\mathbb{Q}_{\left[X_{n}, Y_{n}\right]}=\beta_{!} \beta^{*} \underline{\mathbb{Q}}
$$

where $\beta$ is the inclusion of the open complement of $Y_{n}$ in $X_{n}$. These sheaves form together a sheaf $\mathbb{Q}_{\left[X_{\bullet}, Y_{\bullet}\right]}$ on $X_{\bullet}$, hence a spectral sequence

$$
E_{1}^{p, q}=H^{q}\left(X_{p}, Y_{p} ; \mathbb{Q}\right) \Longrightarrow H^{p+q}\left(X_{\bullet}, Y_{\bullet} ; \mathbb{Q}\right)
$$

Example 4.5.5. - Let $X \bullet$ be a simplicial object in the category of smooth manifolds, and let $p \geqslant 0$ be an integer. On each manifold $X_{n}$, consider the sheaf $\Omega_{X_{n}}^{p}$ of differential $p$-forms. For every order-preserving map $f:[n] \rightarrow[m]$, inducing a $C^{\infty}-\operatorname{map} f_{X}: X_{n} \rightarrow X_{m}$, there is a natural morphism

$$
\iota_{f}^{p}: f_{X}^{*} \Omega_{X_{m}}^{p} \rightarrow \Omega_{X_{n}}^{p}
$$

of sheaves on $X$. Thus, the sheaves $\Omega_{X_{n}}^{p}$ fit together into a sheaf $\Omega_{X_{\bullet}}^{p}$ on $X_{\bullet}$. The exterior derivatives for the individual sheaves $d: \Omega_{X_{n}}^{p} \rightarrow \Omega_{X_{n}}^{p+1}$ commute with the morphisms $\iota_{f}^{p}$, hence yield a morphism $d: \Omega_{X_{\bullet}}^{p} \rightarrow \Omega_{X_{\bullet}}^{p+1}$. The resulting complex of sheaves

$$
0 \rightarrow \Omega_{X_{\bullet}}^{0} \xrightarrow{d} \Omega_{X_{\bullet}}^{1} \xrightarrow{d} \Omega_{X_{\bullet}}^{2} \xrightarrow{d} \cdots
$$

is called the de Rham complex on $X_{\bullet}$.
4.5.6. - Let $X_{\bullet}$ be a simplicial topological space, and let $\varepsilon: X_{\bullet} \rightarrow S$ be an augmentation. There is a pair of adjoint functors $\left(\varepsilon^{*}, \varepsilon_{*}\right)$ between the categories of sheaves on $X_{\bullet}$ and on $S$. The $\operatorname{map} \varepsilon$ is said to have the property of cohomological descent if the adjunction transform

$$
\operatorname{id}_{D(S)} \rightarrow R \varepsilon_{*} \varepsilon^{*}
$$

is an isomorphism of functors from $D(S)$, the derived category of the category of sheaves on $S$, to itself (cf. SGA 4, Exp. V-bis, Définition 2.2.4). Equivalently, $\varepsilon$ has cohomological descent if and only if for every sheaf $F$ on $S$, the morphism of sheaves on $S$

$$
F \rightarrow \operatorname{ker}\left(\varepsilon_{0 *} \varepsilon_{0}^{*} F \rightarrow \varepsilon_{1 *} \varepsilon_{1}^{*} F\right)
$$

is an isomorphism. Here, $\varepsilon_{0}: X_{0} \rightarrow S$ is the augmentation map, and $\varepsilon_{1}: X_{1} \rightarrow S$ is the composite of a face map $X_{1} \rightarrow X_{0}$ with the augmentation. If $\varepsilon: X_{\bullet} \rightarrow S$ has cohomological descent, there is an isomorphism $H^{n}\left(X_{\bullet}, \varepsilon^{*} F\right) \rightarrow H^{n}(S, F)$ for every sheaf $F$ on $S$, and the spectral sequence (4.5.3.2) translates into a spectral sequence

$$
\begin{equation*}
E_{1}^{p, q}=H^{q}\left(X_{p}, \varepsilon^{*} F\right) \Longrightarrow H^{p+q}(S, F) \tag{4.5.6.1}
\end{equation*}
$$

where the induced filtration on the cohomology groups $H^{n}(S, F)$ is the one coming from the $q$-skeletons of $X \bullet$ for $q=0,1,2, \ldots$ A continuous map $X \rightarrow S$ is said to have the property of cohomological descent if the induced augmentation $\operatorname{cosk}_{0}(X) \rightarrow S$ has cohomological descent. A continuous map $X \rightarrow S$ is said to be of universal cohomological descent if for every continuous $T \rightarrow S$ the map $X \times{ }_{S} T \rightarrow T$ has cohomological descent.

Example 4.5.7. - Let $S$ be a topological space, and let $\left(U_{i}\right)_{i \in I}$ be an open cover of $S$. Let $X$ be the disjoint union of the open subsets $U_{i}$, and denote by $\varepsilon: X \rightarrow S$ the covering map. The coskeleton of $(\varepsilon: X \rightarrow S)$ is a simplicial topological space given by $X_{0}=X$ and

$$
X_{1}=X_{0} \times_{S} X_{0}=\coprod_{i, j \in I} U_{i} \cap U_{j}
$$

The face maps $X_{1} \rightarrow X_{0}$ are the two projections $X_{0} \times_{S} X_{0} \rightarrow X_{0}$, which in the present case are open immersions. One checks on stalks that, for every sheaf $F$ on $S$, the morphism $F \rightarrow$ $\operatorname{ker}\left(\varepsilon_{0 *} \varepsilon_{0}^{*} F \rightarrow \varepsilon_{1 *} \varepsilon_{1}^{*} F\right)$ is an isomorphism. Therefore, $\varepsilon$ has cohomological descent. The associated spectral sequence (4.5.6.1) is the classical Leray spectral sequence of the open covering. For every continuous $T \rightarrow S$, the map $X \times_{S} T \rightarrow T$ is an open cover of $T$ and hence has cohomological descent too. Therefore, $\varepsilon: X \rightarrow S$ has universal cohomological descent.

THEOREM 4.5.8. - Continuous maps of universal cohomological descent form a Grothendieck topology on the category of topological spaces. The following types of continuous maps are of universal cohomological descent:
(1) Proper surjective maps, in particular locally finite closed covers.
(2) Maps $X \rightarrow S$ which locally on $S$ admit a section, in particular all open covers.

Proof. This is shown in SGA 4, Exposé V-bis, see in particular Proposition 3.3.1 for the statement that morphisms of universal cohomological descent form a Grothendieck topology, Corollaire 4.1.6, Corollaire 4.1.7 for statement (1), and Proposition 4.1.8 for statement (2).
4.5.9. - Let $f_{\bullet}: X_{\bullet} \rightarrow Y_{\bullet}$ be a morphism of simplicial topological spaces augmented to $S:$


The morphism $f_{\bullet}$ is called a hypercovering if, for every integer $N \geqslant 0$, the continuous map

$$
X_{N+1} \longrightarrow\left(\operatorname{cosk}_{N}^{Y_{\bullet}} \operatorname{sk}_{N}^{Y} X_{\bullet}\right)_{N+1}
$$

obtained from the adjunction (4.5.2.1) is of universal cohomological descent.

### 4.6. Motives of simplicial varieties

Let $X$ be a variety together with a regular function $f$. If $X$ is covered by two open subvarieties $U$ and $V$, the rapid decay cohomology groups of $X, U, V$, and $U \cap V$ with respect to the function $f$ are related by a Mayer-Vietoris long exact sequence:

$$
\cdots \longrightarrow H_{\mathrm{rd}}^{n}(X, f) \longrightarrow H_{\mathrm{rd}}^{n}\left(U,\left.f\right|_{U}\right) \oplus H_{\mathrm{rd}}^{n}\left(V,\left.f\right|_{V}\right) \longrightarrow H_{\mathrm{rd}}^{n}\left(U \cap V,\left.f\right|_{U \cap V}\right) \xrightarrow{\partial} H_{\mathrm{rd}}^{n+1}(X, f) \longrightarrow \cdots
$$

All terms in this sequence are motives, and differentials that connect cohomology groups of the same degree are morphisms of motives since they are induced by inclusions of varieties. It turns out that the connecting morphism $\partial$ is also a morphism of motives, but this is not clear a priori. We shall prove it in this section in the slightly more general setting where a closed subvariety is also allowed (Corollary 4.6.5). As an easy consequence, we establish a projective bundle formula (Proposition 4.7.5) and a sphere bundle formula (Proposition 4.7.7) for exponential motives.
4.6.1. - Let $\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)$ be a simplicial object in the category of pairs of varieties with potential. Fix an integer $N \gg 1$. Choose cellular filtrations of affine homotopy replacements of ( $X_{n}, Y_{n}, f_{n}$ ) for every $n \leqslant N$, which are compatible with the face maps $\left(X_{n}, Y_{n}, f_{n}\right) \rightarrow\left(X_{n-1}, Y_{n-1}, f_{n-1}\right)$. We obtain the following double complex of motives.


The vertical differentials are those of the complexes $C^{*}\left(X_{n}, Y_{n}, f_{n}\right)$. The horizontal differentials are alternating sums of morphisms induced by face maps. For every integer $n<N$, we define the motive $H^{n}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)$ to be the homology in degree $n$ of the associated total complex. Two remarks about this constructions are in order.
(1) The reason why we bound the double complex horizontally has to do with the choice of affine homotopy replacements and cellular filtrations. We can start by choosing an affine homotopy replacement and a cellular filtration of $X_{N}$, and proceed by choosing a filtration of $X_{N-1}$ which is sufficiently fine, so that it is compatible with the finitely many face maps $X_{N} \rightarrow X_{N-1}$, and so on. It is not clear that we can choose compatible cellular filtrations
on all $X_{n}$ simultaneously. As $N$ can be arbitrarily large, $H^{n}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)$ is well defined for all integers $n \geqslant 0$.
(2) The vertical complex $C^{*}\left(X_{n}, Y_{n}, f_{n}\right)$ is, as a complex of vector spaces, naturally isomorphic to $R \Gamma_{f}\left(\underline{\mathbb{Q}}_{[X, Y]}\right)=\Psi_{\infty} \Pi\left(R f_{*}\left(\underline{\mathbb{Q}}_{\left[X_{\bullet}, Y_{\bullet}\right]}\right)\right)$. The vector space underlying $H^{n}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)$ is therefore the rapid decay cohomology $H^{n}\left(X_{\bullet}(\mathbb{C}), Y_{\bullet}(\mathbb{C}), f_{\bullet}\right)$.

Theorem 4.6.2. - Let $\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)$ be an simplicial object in the category of pairs of varieties with potential, together with an augmentation $\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right) \rightarrow(X, Y, f)$. There is a natural spectral sequence of exponential motives

$$
E_{1}^{p, q}=H^{q}\left(X_{p}, Y_{p}, f_{p}\right) \Longrightarrow H^{p+q}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)
$$

and the augmentation induces a morphism of exponential motives $H^{n}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right) \rightarrow H^{n}(X, Y, f)$.

Proof. The spectral sequence in question, in the region $E_{r}^{p, q}$ with $p+q<N$, is the one associated with the double complex (4.6.1.1). The spectral sequence is independent of the choice of cellular filtrations. Given a morphism of simplicial objects $\varphi:\left(X_{\bullet}^{\prime}, Y_{\bullet}^{\prime}, f_{\bullet}^{\prime}\right) \rightarrow\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)$, we can choose cellular filtrations on $X_{0}, \ldots X_{N}$ and $X_{0}^{\prime}, \ldots X_{N}^{\prime}$ which are compatible with face maps and the morphism $\varphi$. Such a choice induces a morphism of double complexes (4.6.1.1), hence a morphism of spectral sequences. An augmentation is just a morphism of simplicial objects whose target is constant, hence the last claim.
4.6.3. - Let $\varepsilon: X_{\bullet} \rightarrow X$ be an augmented simplicial variety and let $f: X \rightarrow \mathbb{A}^{1}$ be a regular function. Set $f_{\bullet}=f \circ \varepsilon$. For every real number $r$, the sets $f_{n}^{-1}\left(S_{r}\right)=\left\{x \in X_{n}(\mathbb{C}) \mid \operatorname{Re}\left(f_{n}(x)\right) \geqslant r\right\}$ form a closed simplicial subspace of $X_{\bullet}$, augmented to $f^{-1}\left(S_{r}\right)$.

Proposition 4.6.4. - Let $\varepsilon:\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right) \rightarrow(X, Y, f)$ be an augmented simplicial object as above. Suppose that the augmentations of simplicial topological spaces $X_{\bullet}(\mathbb{C}) \rightarrow X(\mathbb{C})$ and $Y_{\bullet}(\mathbb{C}) \rightarrow$ $Y(\mathbb{C})$ both have cohomological descent. Then, the morphism

$$
\varepsilon^{*}: H^{n}(X, Y, f) \rightarrow H^{n}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)
$$

induced by the augmentation is an isomorphism.

Proof. We recall that for a topological space $X$ and a closed subspace $A \subseteq X$, we write $\underline{\mathbb{Q}}_{[X, A]}$ for the sheaf $\beta_{!} \beta^{*} \underline{\mathbb{Q}}_{X}$, where $\beta$ is the inclusion of the open complement of $A$ into $X$. Singular cohomology of the pair $(X, A)$ can be computed as sheaf cohomology of the sheaf $\mathbb{Q}_{[X, A]}$ on $X$.

Let us fix an integer $n \geqslant 0$ and choose a large real number $r$. Consider the sheaves

$$
F=\mathbb{Q}_{\left[X, f^{-1}\left(S_{r}\right)\right]} \quad \text { and } \quad F_{p}=\mathbb{Q}_{\left[X_{p}, f_{p}^{-1}\left(S_{r}\right)\right]}
$$

and notice that the sheaves $F_{p}$ fit together to a sheaf $F_{\bullet}$ on the simplicial space $X_{\bullet}$, namely $F_{\bullet}=\varepsilon^{*} F$. Since the augmentation $\varepsilon: X_{\bullet}(\mathbb{C}) \rightarrow X(\mathbb{C})$ has cohomological descent, the adjunction map

$$
H^{n}(X, F) \cong H^{n}\left(X, R \varepsilon_{*} F_{\bullet}\right) \rightarrow H^{n}\left(X_{\bullet}, F_{\bullet}\right)
$$

is an isomorphism. If $r$ is large enough, the canonical morphisms $H^{n}(X, f) \rightarrow H^{n}(X, F)$ and $H^{p}\left(X_{q}, f_{q}\right) \rightarrow H^{p}\left(X_{q}, F_{q}\right)$ for $p+q \leqslant n$ are isomorphisms. It follows that $H^{n}(X, f) \rightarrow H^{n}\left(X_{\bullet}, f_{\bullet}\right)$ is an isomorphism. The same argument shows that $H^{n}(Y, f) \rightarrow H^{n}\left(Y_{\bullet}, f_{\bullet}\right)$ is an isomorphism, and the proposition follows from the five lemma.

Corollary 4.6.5 (Mayer-Vietoris). - Let $X$ be a variety over $k$, let $Y \subseteq X$ be a closed subvariety, and let $f$ be a regular function on $X$. Assume $X$ is covered by two open subvarieties $U_{a}$ and $U_{b}$, and set $U_{a b}=U_{a} \cap U_{b}$ and $V_{(-)}=U_{(-)} \cap Y$. All morphisms, in particular the connecting morphism $\partial$, in the Mayer-Vietoris sequence in rapid decay cohomology
$\cdots \rightarrow H^{n}(X, Y, f) \longrightarrow H^{n}\left(U_{a}, V_{a}, f\right) \oplus H^{n}\left(U_{b}, V_{b}, f\right) \longrightarrow H^{n}\left(U_{a b}, V_{a b}, f\right) \xrightarrow{\partial} H^{n+1}(X, Y, f) \rightarrow \cdots$
are morphisms of motives.
Proof. Associated with the open covering $U_{a} \sqcup U_{b} \rightarrow X$ is the simplicial object $U_{\bullet}$ defined by

$$
U_{n}=\coprod_{\alpha \in\{a, b\}^{n+1}}\left(U_{\alpha_{0}} \cap U_{\alpha_{1}} \cap \cdots \cap U_{\alpha_{n}}\right)
$$

with $n+1$ face maps $U_{n} \rightarrow U_{n-1}$ given by the covering obtained by deleting one of the $n+1$ coordinates of the index $\alpha \in\{a, b\}^{n+1}$, and $n$ coface maps $U_{n-1} \rightarrow U_{n}$ given by the inclusion obtained by duplicating one of the $n$ coordinates of the index $\alpha \in\{a, b\}^{n}$. Contained in $U_{\mathbf{0}}$ is the simplicial subobject $V_{\bullet}=U_{\bullet} \times_{X} Y$. The $E_{1}$-page of the spectral sequence calculating $H^{n}\left(U_{\bullet}, V_{\bullet}, f_{\bullet}\right)$ reads

$$
\begin{aligned}
& \vdots \\
& H^{2}\left(U_{0}, V_{0}, f_{0}\right) \longrightarrow \cdots \\
& H^{1}\left(U_{0}, V_{0}, f_{0}\right) \longrightarrow H^{1}\left(U_{1}, V_{1}, f_{1}\right) \longrightarrow H^{1}\left(U_{2}, V_{2}, f_{2}\right) \longrightarrow \cdots \\
& H^{0}\left(U_{0}, V_{0}, f_{0}\right) \longrightarrow H^{0}\left(U_{1}, V_{1}, f_{1}\right) \longrightarrow H^{0}\left(U_{2}, V_{2}, f_{2}\right) \longrightarrow \cdots
\end{aligned}
$$

where differentials are alternating sums of maps induced by face maps. These horizontal complexes are exact, except for their 0 -th and 1 -st homology groups, which may be non-zero. The kernel of the differential $H^{n}\left(U_{1}, V_{1}, f_{1}\right) \rightarrow H^{n}\left(U_{2}, V_{2}, f_{2}\right)$ is equal to $H^{n}\left(U_{a b}, V_{a b}, f\right)$. In particular, the spectral sequence degenerates on the second page, and there are exact sequences

$$
0 \rightarrow E_{\infty}^{n, 0} \rightarrow E_{1}^{n, 0} \rightarrow E_{1}^{n, 1} \rightarrow E_{\infty}^{n, 1} \rightarrow 0 .
$$

These sequences correspond to the pieces of the Mayer-Vietoris sequence displayed in the statement of the corollary.

Lemma 4.6.6. - There are canonical isomorphisms of motives:

$$
\text { (1) } H^{n}\left(\mathbb{P}_{k}^{d}\right)= \begin{cases}\mathbb{Q}(-i) & n=2 i \leqslant 2 d \\ 0 & n \text { odd or } n>2 d\end{cases}
$$

(2) $H^{n}\left(\mathbb{A}_{k}^{d} \backslash\{0\}\right)= \begin{cases}\mathbb{Q}(0) & n=0 \\ \mathbb{Q}(-d) & n=2 d-1 \\ 0 & \text { else. }\end{cases}$
(3) $H^{2 d}(X)=\mathbb{Q}(-d)$ for every projective and geometrically connected variety $X$ of dimension $d$ over $k$.

Proof. Morphisms of type (c) in the standard quiver representation of $\mathrm{Q}^{\exp }(k)$ induce isomorphisms $H^{d}\left(\mathbb{G}_{m}^{d}\right)(0) \cong H^{d-1}\left(\mathbb{G}_{m}^{d-1}\right)(-1) \cong \ldots \cong H^{0}(\operatorname{Spec} k)(-d)=\mathbb{Q}(-d)$. From the standard covering of the projective space $\mathbb{P}_{k}^{d}$ by $d+1$ affine spaces we obtain a spectral sequence (the MayerVietoris sequence if $d=1$ ), in which the connecting morphism

$$
H^{d}\left(\mathbb{G}_{m}^{d}\right) \rightarrow H^{2 d}\left(\mathbb{P}_{k}^{d}\right)
$$

appears. This connecting morphism is an isomorphism of vector spaces because $\mathbb{A}_{k}^{d}$ is contractible as well as a morphism of motives by Theorem 4.6.2 (or Corollary 4.6 .5 for $d=1$ ), hence it is an isomorphism of motives. The first statement of the proposition follows from this isomorphism, induction on $d$, and the fact that the inclusion of a hyperplane $\mathbb{P}_{k}^{d-1} \rightarrow \mathbb{P}_{k}^{d}$ induces an isomorphism on cohomology in degrees up to $2 d-2$. The argument for the second statement is similar-one covers $\mathbb{A}_{k}^{d} \backslash\{0\}$ by affine opens $\mathbb{A}^{a} \times \mathbb{G}_{m} \times \mathbb{A}^{b}$ with $a+b=d-1$. To show the third statement of the proposition, we choose a projective embedding $\iota: X \rightarrow \mathbb{P}_{k}^{N}$ and consider the induced morphism of motives $\mathbb{Q}(-d) \cong H^{2 d}\left(\mathbb{P}_{k}^{N}\right) \rightarrow H^{2 d}(X)$ which is an isomorphism of vector spaces, hence an isomorphism of motives.

### 4.7. The Leray spectral sequence

Let $S$ be an algebraic variety endowed with a potential $f: S \rightarrow \mathbb{A}^{1}$, and let $\pi: X \rightarrow S$ be a morphism of algebraic varieties. In this section, we define a spectral sequence of motives

$$
E_{2}^{a, b}=H^{a}\left(S, f ; R^{b} \pi_{*} \underline{\mathbb{Q}}\right) \Longrightarrow H^{a+b}(X, f \circ \pi)
$$

whose underlying spectral sequence of vector spaces is the Leray spectral sequence in rapid decay cohomology introduced in Section 3.2.5. In particular, we will equip the vector space $H_{\mathrm{rd}}^{a}\left(S, f ; R \pi_{*} \mathbb{\mathbb { Q }}\right)$ with the structure of a motive. The overall strategy for doing this is an idea of Arapura, who shows in [3] that the classical Leray spectral sequence (when $f=0$ ) is motivic. It will suffice to treat the case where $S$ is affine. Following Arapura, we construct in that case a filtration $X_{0} \subseteq X_{1} \subseteq \cdots$ of $X$ whose associated spectral sequence

$$
E_{1}^{p, q}=H^{p+q}\left(X_{p}, X_{p-1}, f \circ \pi\right) \Longrightarrow H^{p+q}(X, f \circ \pi),
$$

can be identified with the Leray spectral sequence of vector spaces from the second page on. This identification is canonical, and we use it to transport motivic structures.
4.7.1. - Let $S$ be a topological space with an exhaustive filtration $\varnothing \subseteq S_{0} \subseteq S_{1} \subseteq \cdots \subseteq S_{n}=S$ by closed subspaces. Setting $U_{p}=S \backslash S_{p-1}$ we have a decreasing filtration by open subsets

$$
S=U_{0} \supseteq U_{1} \supseteq U_{2} \supseteq \cdots \supseteq U_{n} \supseteq \varnothing .
$$

From the given filtration of $S$, one obtains a natural filtration on all complexes of sheaves $A$ on $S$ in the following way. Let $\beta_{p}: U_{p} \rightarrow S$ be the inclusion and define

$$
A^{(p)}=\left(\beta_{p}\right)!\left(\beta_{p}\right)^{*} A .
$$

These complexes of sheaves define the filtration

$$
A=A^{(0)} \supseteq A^{(1)} \supseteq \cdots \supseteq A^{(n)} \supseteq 0
$$

of $A$. Here, the symbol $\supseteq$ means degreewise injective morphism of complexes of sheaves. By construction,

$$
H^{n}\left(S, \operatorname{gr}^{p} A\right)=H^{n}\left(S_{p}, S_{p-1} ; A\right) .
$$

Lemma 4.7.2. - Let $A$ be a bounded complex of sheaves on $S$ such that, for all integers $a, p, b$ with $a \neq p$, the equality

$$
H^{a}\left(S_{p}, S_{p-1} ; \mathcal{H}^{b}(A)\right)=0
$$

holds. Then, there is a canonical isomorphism between the spectral sequences

$$
\begin{align*}
E_{1}^{p, q}=H^{p+q}\left(S_{p}, S_{p-1} ; A\right) & \Longrightarrow \quad H^{p+q}(S, A)  \tag{4.7.2.1}\\
E_{2}^{a, b}=H^{a}\left(S ; \mathcal{H}^{b}(A)\right) & \Longrightarrow \quad H^{a+b}(S, A) \tag{4.7.2.2}
\end{align*}
$$

from the second page on.
4.7.3 (Reduction to the affine case). - The first observation is that we can reduce to the case where $S$ is affine. Indeed, using Jouanolou's trick (Lemma 4.3.5) we can find an affine homotopy replacement $h: S^{\prime} \rightarrow S$ and set $X^{\prime}=X \times_{S} S^{\prime}$. Set $g=f \circ \pi$
4.7.4. - According to the Basic Lemma 3.3.1 (Variant!), there exists a filtration $S_{0} \subseteq S_{1} \subseteq \ldots$ of $S$ such that, for all $b \geqslant 0$, the perverse sheaves

$$
{ }^{p} \mathcal{H}^{a}\left(S_{p}, S_{p-1} ; R^{b} \pi_{*} F\right)
$$

vanish for $a \neq p$. Choose such a filtration, and define a filtration $X_{0} \subseteq X_{1} \subseteq \cdots$ of the space $X$ by setting $X_{p}=\pi^{-1}\left(S_{p}\right)$. From this filtration results a spectral sequence

$$
E_{1}^{p, q}=H_{\mathrm{rd}}^{p+q}\left(X_{p}, X_{p-1}, g ; F\right) \Longrightarrow H^{p+q}(X, g ; F)
$$

with differentials $E_{1}^{p, q} \rightarrow E_{1}^{p+1, q}$ given by the connecting morphisms in the long exact sequence for the triple $X_{p-1} \subseteq X_{p} \subseteq X_{p+1}$. By design, the Leray spectral sequence associated with each map $\pi: X_{p} \rightarrow S_{p}$

$$
E_{2}^{a, b}=H^{a}\left(S_{p}, S_{p-1}, f ; R^{b} \pi_{*} F\right) \Longrightarrow H^{a+b}\left(X_{p}, X_{p-1}, g ; F\right)
$$

degenerates to isomorphisms $H^{p}\left(S_{p}, S_{p-1}, f ; R^{b} \pi_{*} F\right) \cong H^{p+b}\left(X_{p}, X_{p-1}, g ; F\right) \cong E_{1}^{p, b}$ for all $b$.

$$
\begin{array}{r}
\cdots \longrightarrow H^{p+b}\left(X_{p}, X_{p-1}, g ; F\right) \longrightarrow H^{p+b+1}\left(X_{p+1}, X_{p}, g ; F\right) \longrightarrow \cdots \\
\| \\
\cdots \longrightarrow H^{p}\left(S_{p}, S_{p-1}, f ; R^{b} \pi_{*} F\right) \longrightarrow H^{p+1}\left(S_{p+1}, S_{p}, f ; R^{b} \pi_{*} F\right) \longrightarrow \cdots
\end{array}
$$

The homology of the complex below is precisely $H^{p}\left(S, f ; R^{b} \pi_{*} F\right)$, the second page of the Leray spectral sequence.

Proposition 4.7 .5 (Projective bundle formula). - Let $X$ be a variety over $k$ equipped with a regular function $f: X \rightarrow \mathbb{A}^{1}$, let $Y \subseteq X$ be a closed subvariety and let $E \rightarrow X$ be a vector bundle of rank $r$ over $X$, with projectivisation $\pi: \mathbb{P}(E) \rightarrow X$. There is an isomorphism in $\mathbf{M}^{\exp }(k)$ as follows:

$$
H^{n}\left(\mathbb{P}(E),\left.\mathbb{P}(E)\right|_{Y}, f \circ \pi\right) \cong \bigoplus_{i=0}^{r-1} H^{n-2 i}(X, Y, f)(i)
$$

Proof.

Corollary 4.7.6. - Let $X$ be a variety over $k$, let $f: X \rightarrow \mathbb{A}^{1}$ be a regular function and let $\pi: E \rightarrow X$ be a vector bundle of rank $r$ over $X$. Let $Y \subseteq X$ be a closed subvariety. There is an isomorphism in $\mathbf{M}^{\exp }(k)$ as follows:

$$
H^{n}\left(\mathbb{P}\left(E \oplus \mathcal{O}_{X}\right),\left.\mathbb{P}(E) \cup \mathbb{P}\left(E \oplus \mathcal{O}_{X}\right)\right|_{Y}, f \circ \pi\right)=H^{n-2 r}(X, Y, f)(-r)
$$

Proof. The projective bundle formula established in Proposition 4.7.5 can be seen as an isomorphism

$$
C^{*}(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r-1} C^{*}(X, f)[2 i](i)
$$

in the derived category of $\mathbf{M}^{\exp }(k)$. The inclusion $\mathbb{P}(E) \rightarrow \mathbb{P}\left(E \oplus \mathcal{O}_{X}\right)$ induces a morphism

$$
\bigoplus_{i=0}^{r} C^{*}(X, f)[2 i](i) \rightarrow \bigoplus_{i=0}^{r-1} C^{*}(X, f)[2 i](i)
$$

which is indeed just the obvious projection, hence an isomorphism

$$
C^{*}\left(\mathbb{P}\left(E \oplus \mathcal{O}_{X}\right), \mathbb{P}(E), f \circ \pi\right)=C^{*}(X, f)[2 r](r)
$$

as we wanted, in the case where $Y$ is empty. For the general case, observe that the following diagram the columns are exact triangles, and the second and third horizontal map are isomorphisms in the derived category of $\mathbf{M}^{\exp }(k)$.


The top horizontal morphism is thus an isomorphism in the derived category as well, and yields the sought isomorphism of motives by taking homology in degree $n$.

Proposition 4.7.7 (Sphere bundle formula). - Let $X$ be a smooth variety over $k$, let $f: X \rightarrow \mathbb{A}^{1}$ be a regular function, and let $\pi: E \rightarrow X$ be a vector bundle of rank $r$ whose Euler class is zero. Denote by $E_{0}$ the complement of the zero section in $E$. There is an isomorphism in $\mathbf{M}^{\exp }(k)$ as follows:

$$
H^{n}\left(E_{0}, f \circ \pi\right) \cong H^{n}(X, f) \oplus H^{n-2 r+1}(X, f)(r)
$$

Proof.

### 4.8. Motives with support, Gysin morphism, and proper pushforward

In this section, we show that the Gysin map in rapid decay cohomology (3.7.2.2) is a morphism of motives. This will enable us to construct a duality pairing in the next section, thus completing the proof that exponential motives form a tannakian category.
4.8.1. - Let $(X, f)$ be a variety with a potential and $Y \subseteq X$ a closed subvariety. Let $Z \subseteq X$ be another closed subvariety with open complement $U \subseteq X$. The inclusion $U \subseteq X$ induces a morphism

$$
\begin{equation*}
C^{\bullet}(X, Y, f) \longrightarrow C^{\bullet}\left(U, U \cap Y, f_{\mid U}\right) \tag{4.8.1.1}
\end{equation*}
$$

in $D^{b}\left(\mathbf{M}^{\exp }(k)\right)$. Concretely, choosing cellular filtrations we can see this as an actual morphism of chain complexes in $\mathbf{M}^{\exp }(k)$. We set

$$
\begin{equation*}
C_{Z}^{\bullet}(X, Y, f)=\operatorname{cone}\left(C^{\bullet}(X, Y, f) \longrightarrow C^{\bullet}\left(U, U \cap Y, f_{\mid U}\right)\right) \tag{4.8.1.2}
\end{equation*}
$$

Definition 4.8.2. - The exponential motive of $(X, Y, f)$ with support on $Z$ is the homology of the cone of the morphism (4.8.1.1), namely:

$$
\begin{equation*}
H_{Z}^{n}(X, Y, f)=H^{n}\left(C_{Z}^{\bullet}(X, Y, f)\right) \tag{4.8.2.1}
\end{equation*}
$$

By definition, $H_{Z}^{n}(X, Y, f)$ fits into a long exact sequence of motives

$$
\begin{equation*}
\cdots \rightarrow H_{Z}^{n}(X, Y, f) \rightarrow H^{n}(X, Y, f) \rightarrow H^{n}\left(U, Y \cap U,\left.f\right|_{U}\right) \rightarrow H_{Z}^{n+1}(X, Y, f) \rightarrow \cdots \tag{4.8.2.2}
\end{equation*}
$$

Theorem 4.8.3. - Let $X$ be a smooth, irreducible variety over $k$, together with a regular function $f$, and let $Y$ be a closed subvariety of $X$. For any smooth closed subvariety $Z \subseteq X$ of pure codimension $c$, with open complement $U$, there is a canonical isomorphism of motives

$$
\begin{equation*}
H_{Z}^{n}(X, Y, f) \xrightarrow{\sim} H^{n-2 c}\left(Z, Y \cap Z, f_{\mid Z}\right)(-c) . \tag{4.8.3.1}
\end{equation*}
$$

Under this isomorphism, the long exact sequence (4.8.2.2) becomes a long exact sequence of motives

$$
\cdots \rightarrow H^{n}(X, f) \rightarrow H^{n}\left(U,\left.f\right|_{U}\right) \rightarrow H^{n-2 c+1}\left(Z,\left.f\right|_{Z}\right)(-c) \rightarrow H^{n+1}(X, f) \rightarrow \cdots
$$

whose underlying long exact sequence of vector spaces is the Gysin sequence (3.7.2.1).
Proof. We use deformation to the normal cone as in Chapter 5 of [37]. Let $\widetilde{X}$ be the blow-up of $Z \times\{0\}$ in $X \times \mathbb{A}^{1}$, and equip $\widetilde{X}$ with the potential

$$
\tilde{f}: \widetilde{X} \rightarrow X \times \mathbb{A}^{1} \rightarrow X \xrightarrow{f} \mathbb{A}^{1}
$$

obtained by composing the blow-up map, the projection to $X$ and $f$. Let $\pi: \widetilde{X} \rightarrow \mathbb{A}^{1}$ denote the composition of the blow-up map $\widetilde{X} \rightarrow X \times \mathbb{A}^{1}$ and the projection to $\mathbb{A}^{1}$. The fibre of $\pi$ over any non-zero point of $\mathbb{A}^{1}$ is isomorphic to $X$. The fibre $\pi^{-1}(0)$ has two irreducible components, namely $\mathbb{P}\left(N_{Z} \oplus \mathcal{O}_{Z}\right)$, the projective completion of the normal bundle $N_{Z}=T_{X} Z / T_{Z} Z$ of $Z$ in $X$, and $\mathrm{Bl}_{Z} X$, the blow-up of $Z$ in $X$. These two components intersect in $\mathbb{P}\left(N_{Z}\right)$, seen as the infinity section in $\mathbb{P}\left(N_{Z} \oplus \mathcal{O}_{Z}\right)$ and as the exceptional divisor in $\mathrm{Bl}_{Z} X$.

Let $\tilde{Y}$ be the strict transform of $Y \times \mathbb{A}^{1}$ in $\widetilde{X}$. The intersection of $\tilde{Y}$ with $\pi^{-1}(0)$ also has two components, namely $\widetilde{Y} \cap \mathbb{P}\left(N_{Z} \oplus \mathcal{O}_{Z}\right)=\left.\mathbb{P}\left(N_{Z} \oplus \mathcal{O}_{Z}\right)\right|_{Y \cap Z}$ and $\widetilde{Y} \cap \mathrm{Bl}_{Z} X$, the strict transform of $Y$ in $\mathrm{Bl}_{Z} X$.

The inclusions $\pi^{-1}(0) \rightarrow \widetilde{X}$ and $X \cong \pi^{-1}(1) \rightarrow \widetilde{X}$ induce morphisms of motives as follows:

$$
H^{n}\left(\pi^{-1}(0),\left.\mathrm{Bl}_{Z} X \cup \mathbb{P}\left(N_{Z} \oplus \mathcal{O}_{Z}\right)\right|_{Y \cap Z},\left.\widetilde{f}\right|_{\pi^{-1}(0)}\right) \stackrel{(*)}{\longleftarrow} H^{n}\left(\widetilde{X}, \widetilde{Y} \cup \mathrm{Bl}_{Z} X, \widetilde{f}\right) \rightarrow H^{n}(X, Y, f)
$$

The map labelled $(*)$ is an isomorphism (of vector spaces, hence of motives) because $\left[\tilde{X}, \pi^{-1}(0), \tilde{f}\right]$ has trivial cohomology. The reason for this is that the quotient space $\widetilde{X} / \pi^{-1}(0)$ is the same as $\left(X \times \mathbb{A}^{1}\right) /(X \times 0)$, which is a cone, hence contractible. We have a morphism

$$
\begin{aligned}
H^{n}\left(\pi^{-1}(0), \mathrm{Bl}_{Z} X \cup \mathbb{P}\left(N_{Z} \oplus \mathcal{O}_{Z}\right)\right. & \left.\left.\right|_{Y \cap Z},\left.\widetilde{f}\right|_{\pi^{-1}(0)}\right) \\
& \cong \\
& \cong H^{n}\left(\mathbb{P}\left(N_{Z} \oplus \mathcal{O}_{Z}\right),\left.\mathbb{P}\left(N_{Z}\right) \cup \mathbb{P}\left(N_{Z} \oplus \mathcal{O}_{Z}\right)\right|_{Y \cap Z},\left.\widetilde{f}\right|_{\mathbb{P}\left(N_{Z} \oplus \mathcal{O}_{Z}\right)}\right)
\end{aligned}
$$

induced by inclusion, which is an isomorphism by excision. The right-hand side is isomorphic to $H^{n-2 c}\left(Z, Y \cap Z,\left.f\right|_{Z}\right)(-c)$ by Corollary 4.7 .6 to the projective bundle formula. We obtain a morphism of motives

$$
\begin{equation*}
H^{n-2 c}(Z, Y \cap Z, f)(-c) \rightarrow H^{n}(X, Y, f) \tag{4.8.3.2}
\end{equation*}
$$

whose underlying morphism of vector spaces is the Gysin map.
The rest of the argument is formal. The projective bundle formula can be seen as an isomorphism

$$
C^{*}(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r-1} C^{*}(X)[2 i](i)
$$

in the derived category $D^{b}(\mathbf{M}(k))$. The morphism of pairs $\left[\pi^{-1}(0), \mathrm{Bl}_{Z} X\right] \rightarrow\left[\widetilde{X}, \mathrm{Bl}_{Z} X\right]$ induces therefore a morphism

$$
\begin{equation*}
C^{*}(Z)[2 c](c) \rightarrow C^{*}(X) \tag{4.8.3.3}
\end{equation*}
$$

in $D^{b}(\mathbf{M}(k))$ inducing (4.8.3.2). Its composition with $C^{*}(X) \rightarrow C^{*}(U)$ is zero, hence a morphism

$$
C^{*}(Z)[2 c](c) \rightarrow C_{Z}^{*}(X)
$$

in $D^{b}(\mathbf{M}(k))$. This morphism is indeed an isomorphism, because the underlying morphism in the derived category of $\mathbb{Q}$-vector spaces is so. The Gysin sequence in the statement of the theorem is, via this isomorphism, the sequence of cohomology with support (4.8.2.2).
4.8.4. - Let $(X, f)$ be a smooth variety together with a regular function, and let $Z \subseteq X$ be a smooth subvariety of pure codimension $c$.

Proposition 4.8.5. - Let $\left(X, f_{X}\right)$ and $\left(Z, f_{Z}\right)$ be smooth varieties, together with regular functions, and $\pi: Z \rightarrow X$ a proper morphism such that $f_{Z}=f_{X} \circ \pi$. Set $c=\operatorname{dim} X-\operatorname{dim} Z$. The proper pushforward morphism

$$
\begin{equation*}
\pi_{*}: H^{n}\left(Z, f_{Z}\right) \longrightarrow H^{n+2 c}\left(X, f_{X}\right)(c) \tag{4.8.5.1}
\end{equation*}
$$

is a morphism of exponential motives.
Proof. It suffices to treat the case where $\pi$ is a closed immersion. Indeed, since $X$ is quasi-projective, choosing a locally closed embedding $Z \hookrightarrow \mathbb{P}^{m}$, we can factor the morphism $\pi$ into the composite $Z \stackrel{\iota}{\hookrightarrow} X \times \mathbb{P}^{m} \xrightarrow{p} X$, where $\iota$ is a closed embedding and $p$ is the projection. If we endow $X \times \mathbb{P}^{m}$ with the function $f_{X} \boxplus 0$, then both maps are compatible with the functions. Assume that the pushforward $\iota_{*}$ is a morphism of motives. Then $\pi_{*}$ is given by the composition

$$
H^{n}\left(Z, f_{Z}\right) \xrightarrow{\iota_{*}} H^{n+2 c+2 m}\left(X \times \mathbb{P}^{m}, f_{X} \boxplus 0\right)(c+m) \longrightarrow H^{n+2 c}\left(X, f_{X}\right)(c),
$$

where the second morphism is the projection onto the component

$$
H^{n+2 c}\left(X, f_{X}\right)(c) \otimes H^{2 m}\left(\mathbb{P}^{m}\right)(m)=H^{n+2 c}\left(X, f_{X}\right)(c)
$$

of the Künneth formula.

### 4.9. Duality

Let $X$ be a smooth connected variety of dimension $d$, together with a regular function $f$, and $Y \subseteq X$ a normal crossing divisor. We choose a good compactification $(\bar{X}, \bar{Y}, \bar{f})$ of the triple $(X, Y, f)$ in the sense of Definition 3.5.8. We let $D$ denote the complement of $X$ in $\bar{X}, P$ the reduced pole divisor of $\bar{f}$, and we write $D=P+H$. We set

$$
X^{\prime}=\bar{X} \backslash(\bar{Y} \cup P), \quad Y^{\prime}=H \backslash(H \cap P)
$$

and denote by $f^{\prime}$ the restriction of $\bar{f}$ to $X^{\prime}$.

$$
\begin{equation*}
H_{\mathrm{rd}}^{n}(X, Y, f) \otimes H_{\mathrm{rd}}^{2 d-n}\left(X^{\prime}, Y^{\prime},-f^{\prime}\right) \longrightarrow \mathbb{Q} \tag{4.9.0.1}
\end{equation*}
$$

Proposition 4.9.1. - There is a unique morphism of exponential motives

$$
\begin{equation*}
H^{n}(X, Y, f) \otimes H^{2 d-n}\left(X^{\prime}, Y^{\prime},-f^{\prime}\right) \longrightarrow \mathbb{Q}(-d) \tag{4.9.1.1}
\end{equation*}
$$

whose perverse realisation is the duality pairing (4.9.0.1).
Proof. We first construct a morphism in the opposite direction. For this, let $\Delta=X \cap X^{\prime}$ embedded diagonally in $X \times X^{\prime}$. By construction, $\Delta$ does not intersect $\left(Y \times X^{\prime}\right) \cup\left(X \times Y^{\prime}\right)$ and the function $f \boxplus\left(-f^{\prime}\right)$ is identically zero on $\Delta$. Thus, Theorem 4.8.3 yields an isomorphism of motives

$$
\mathbb{Q}(-d)=H^{0}(\Delta)(-d) \xrightarrow{\sim} H_{\Delta}^{2 d}\left(X \times X^{\prime},\left(Y \times X^{\prime}\right) \cup\left(X \times Y^{\prime}\right), f \boxplus\left(-f^{\prime}\right)\right) .
$$

Composing with the natural "forget support" map and with the projection to the $H^{n}(X, Y, f) \otimes$ $H^{2 d-n}\left(X^{\prime}, Y^{\prime},-f^{\prime}\right)$ component of the Künneth isomorphism, we obtain a morphism of motives

$$
\mathbb{Q}(-d) \longrightarrow H^{n}(X, Y, f) \otimes H^{2 d-n}\left(X^{\prime}, Y^{\prime},-f^{\prime}\right)
$$

Observe that, when $f$ is constant, we recover the usual duality between cohomology and cohomology with compact support. More generally, this suggests to introduce the following definition:

Definition 4.9.2. - Let $X$ be a smooth variety and $f: X \rightarrow \mathbb{A}^{1}$ a regular function. We choose a good relative compactification of $X$ over $\mathbb{A}^{1}$, i.e. a smooth variety $X^{\text {rel }}$ such that $H=X^{\text {rel }} \backslash X$ is a normal crossing divisor and $f$ extends to a proper morphism $f^{\mathrm{rel}}: X^{\mathrm{rel}} \rightarrow \mathbb{A}^{1}$. The motive with compact support of the pair $(X, f)$ is $H^{n}\left(X^{\mathrm{rel}}, H, f^{\mathrm{rel}}\right)$.

### 4.10. The motivic Galois group

Let us summarise what we did so far. We first constructed $\mathbf{M}^{\exp }(k)$ as an abelian category. Using the basic lemma, we proved that this category is equivalent to the one obtained from the quiver of cellular objects. This enabled us to define a tensor product. We then show that each objects admits a dual, completing the proof that $\mathbf{M}^{\exp }(k)$ is a tannakian category.

Definition 4.10.1. - The exponential motivic Galois group $\mathrm{G}^{\exp }(k)$ is the affine group scheme over $\mathbb{Q}$ such that

$$
\mathbf{M}^{\exp }(k)=\boldsymbol{\operatorname { R e p }}\left(G^{\exp }(k)\right)
$$

Given an exponential motive $M$, the smallest tannakian subcategory $\langle M\rangle^{\otimes}$ of $\mathbf{M}^{\exp }(k)$ containing $M$ is equivalent to $\operatorname{Rep}\left(G_{M}\right)$ for a linear algebraic group $G_{M}$. We shall call it the Galois group of $M$

## CHAPTER 5

## Relation with other theories of motives

### 5.1. Relation with classical Nori motives

Let $\mathrm{Q}(k)$ denote the full subquiver of $\mathrm{Q}^{\exp }(k)$ consisting of those objets $[X, Y, f, n, i]$ where $f$ is the zero function. The restriction of the Betti representation $\rho$ to $\mathrm{Q}(k)$ is given by ordinary relative cohomology:

$$
\rho([X, Y, 0, n, i])=H^{n}(X(\mathbb{C}), Y(\mathbb{C}))(i) .
$$

The cohomological, non-effective variant of Nori's category of mixed motives over $k$ may be defined as the category $\mathbf{M}(k)=\langle Q(k), \rho\rangle$. This is not Nori's original construction, but the one Ayoub sketches in [6, p.6]. The inclusion $\mathrm{Q}(k) \rightarrow \mathrm{Q}^{\exp }(k)$ can be seen as a morphism of quiver representations, hence induces a faithful and exact functor

$$
\iota: \mathbf{M}(k) \rightarrow \mathbf{M}^{\exp }(k)
$$

which permits us to regard classical Nori motives as exponential motives.

Theorem 5.1.1. - The functor $\iota: \mathbf{M}(k) \rightarrow \mathbf{M}^{\exp }(k)$ is fully faithful and exact.
Proof. We only need to prove that $\iota$ is full. For this, it suffices to show that, for each object $M$ of $\mathbf{M}(k)$, the following map is surjective.

$$
\operatorname{End}_{\mathbf{M}(k)}(M) \rightarrow \operatorname{End}_{\mathbf{M}^{\mathbf{e x p}}(k)}(\iota(M))
$$

Let $M$ be an object of $\mathbf{M}(k)$ and let $h: \iota(M) \rightarrow \iota(M)$ be an endomorphism in $\mathbf{M}^{\exp }(k)$. Recall from 4.1 that $M$ consists of the data $(V, Q, \alpha)$, where $V$ is a finite-dimensional $\mathbb{Q}$-vector space, $Q$ is a finite subquiver of $Q(k)$ which we suppose to be non-empty to rule out degenerate cases, and $\alpha: \operatorname{End}\left(\left.\rho\right|_{Q}\right) \rightarrow \operatorname{End}(V)$ a morphism of $\mathbb{Q}$-algebras. The exponential motive $\iota(M)$ is given by the same triple $(V, Q, \alpha)$, with $Q$ now regarded as a finite subquiver of $\mathrm{Q}^{\exp }(k)$. The morphism $h$ is a linear map $V \rightarrow V$ such that there exists a finite subquiver $P \subseteq \mathrm{Q}^{\exp }(k)$ containing Q and such that $h$ is $\operatorname{End}\left(\left.\rho\right|_{P}\right)$-linear. We need to find a subquiver $\mathrm{Q}^{\prime}$ of $\mathrm{Q}(k)$ containing Q such that $h$ is $\operatorname{End}\left(\left.\rho\right|_{\mathrm{Q}^{\prime}}\right)$-linear.

Let $S \subset \mathbb{A}^{1}(\mathbb{C})$ be the union of the singularities of all perverse realisations of objects in $P$. As Q is non-empty, this set contains $0 \in \mathbb{C}$. We choose $z \in \mathbb{A}^{1}(k)$ such that $\operatorname{Re}(z)>\operatorname{Re}(s)$ for all
$s \in S$. Consider the functor $\lambda_{z}: P \rightarrow \mathrm{Q}(k)$ given by

$$
\lambda_{z}:[X, Y, f, n, i] \longmapsto\left[X, Y \cup f^{-1}(z), 0, n, i\right]
$$

on objects and by the obvious rules on morphisms. After enlarging Q and $P$ by adding all objects and morphisms in the image of $\lambda_{z}$, we may see $\lambda_{z}$ as a functor from $P$ to Q . As we have seen in (3.1.2.2), there are isomorphisms of vector spaces

$$
\begin{equation*}
H_{\mathrm{rd}}^{n}(X, Y, f)(i) \cong H^{n}\left(X, Y \cup f^{-1}(z)\right)(i) \tag{5.1.1.1}
\end{equation*}
$$

which are functorial for morphisms in $P$. Together with these isomorphisms, $\lambda_{z}$ is a morphism of quiver representations. It fits into a commutative diagram

where the composition of the horizontal arrows is the identity on $Q$. The left hand triangle commutes, and the right hand triangle commutes up to the natural isomorphisms (5.1.1.1). We obtain morphisms of $\mathbb{Q}$-algebras

$$
\operatorname{End}\left(\left.\rho\right|_{\mathrm{Q}}\right) \xrightarrow{\text { via (5.1.1.1) }} \operatorname{End}\left(\left.\rho\right|_{P}\right) \xrightarrow{\text { res }} \operatorname{End}\left(\left.\rho\right|_{\mathrm{Q}}\right)
$$

whose composite is the identity. The restriction homomorphism $\operatorname{End}\left(\left.\rho\right|_{P}\right) \rightarrow \operatorname{End}\left(\left.\rho\right|_{Q}\right)$ is thus surjective, and the induced functor from the category of $\operatorname{End}\left(\left.\rho\right|_{Q}\right)$ modules to the category of $\operatorname{End}\left(\left.\rho\right|_{P}\right)$-modules is full. In particular, the given $\operatorname{End}\left(\left.\rho\right|_{P}\right)$-linear morphism $h: V \rightarrow V$ is also $\operatorname{End}\left(\left.\rho\right|_{Q}\right)$-linear.
5.1.2. - From now on, we identify the category of classical Nori motives with its image in the category of exponential motives via the fully faithful functor $\iota$. In the course of the proof of Theorem 5.1.1 we have shown that the morphism of proalgebras

$$
\operatorname{End}(\rho) \rightarrow \operatorname{End}\left(\left.\rho\right|_{Q(k)}\right)
$$

given by restriction is surjective and, invoking Zorn's lemma, we even see that it has sections. This tells us more than just fullness of the canonical functor $\iota: \mathbf{M}(k) \rightarrow \mathbf{M}^{\exp }(k)$.

Proposition 5.1.3. - The category of classical motives $\mathbf{M}(k)$ is stable under taking subobjects and quotients in $\mathbf{M}^{\exp }(k)$.

Proof. Let $M$ be an object of $\mathbf{M}(k)$ and let $M^{\prime}$ be a subobject of $M$ in $\mathbf{M}^{\exp }(k)$. We represent $M$ by a triple $(V, Q, \alpha)$, where $V$ is a finite-dimensional $\mathbb{Q}$-vector space, $Q$ is a finite subquiver of $Q(k)$, and $\alpha: \operatorname{End}\left(\left.\rho\right|_{Q}\right) \rightarrow \operatorname{End}(V)$ is a morphism of $\mathbb{Q}$-algebras. Then, $M^{\prime}$ is given by a subspace $V^{\prime}$ of $V$ which is stable under $\operatorname{End}\left(\left.\rho\right|_{P}\right)$ for some finite $P \subseteq Q^{\exp }(k)$ containing $Q$. As in the proof of Theorem 5.1.1, we may again enlarge $P$ and $Q$ in such a way that the restriction morphism $\operatorname{End}\left(\left.\rho\right|_{P}\right) \rightarrow \operatorname{End}\left(\left.\rho\right|_{Q}\right)$ is surjective. But then, $V^{\prime}$ is stable under $\operatorname{End}\left(\left.\rho\right|_{Q}\right)$ as we wanted. The same argument works for quotients.

From Theorem 5.1.1 and Proposition 5.1.3 we immediately derive:

Corollary 5.1.4. - The morphism of affine group schemes $G^{\exp }(k) \rightarrow G(k)$ induced by the canonical functor $\iota: \mathbf{M}(k) \rightarrow \mathbf{M}^{\exp }(k)$ is faithfully flat.

### 5.2. Artin motives and Galois descent

Artin motives are motives which correspond to cohomology in degree zero. They exist in different frameworks - one can isolate a class of Artin motives within the category of Chow motives modulo any adequate equivalence relation for cycles, within the triangulated categories of motives, and of course also within Nori's category. From a tannakian point of view, Artin motives over a field $k$ with algebraic closure $\bar{k}$ correspond to finite-dimensional, continuous representations of the Galois group $\operatorname{Gal}(\bar{k} \mid k)$. This stipulates that there is a morphism of group schemes

$$
G_{\mathrm{mot}, k} \rightarrow \operatorname{Gal}(\bar{k} \mid k)
$$

where the Galois group is seen as a constant group scheme. The classical fullness conjectures of Hodge, Tate, Ogus, and others all imply that the kernel of this map is the group $G_{\text {mot }, \bar{k}}$, and that the latter is connected, see [82], statement 6.3. The main result of this section is Theorem 5.2.4. It is more a statement about classical Nori motives rather than about exponential motives.

We fix a subfield $k$ of $\mathbb{C}$ and denote by $\bar{k}$ the algebraic closure of $k$ in $\mathbb{C}$. Throughout this section, we regard $\mathbf{M}(k)$ as a full subcategory of $\mathbf{M}^{\exp }(k)$ via the canonical functor of Theorem 5.1.1.
5.2.1. - Let $K \subseteq \mathbb{C}$ be a subfield containing $k$. There is a canonical base change functor

$$
\begin{equation*}
\operatorname{res}_{K}^{k}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{M}^{\exp }(K) \tag{5.2.1.1}
\end{equation*}
$$

which we can describe in terms of quiver representations as follows. There is a morphism of quiver representations given by the commutative triangle of quiver morphisms

where $\psi$ is defined by $\psi(X, Y, f, n, i)=\left(X_{K}, Y_{K}, f_{K}, n, i\right)$ on objects, and in the obvious way on morphisms. The representations $\rho_{k}$ and $\rho_{K}$ are the standard Betti-representations 4.2.2. We define the base change functor (5.2.1.1) to be the functor induced on linear hulls by the morphism of quiver representations, as explained in 4.1.9. It is straightforward to check that the base change functor (5.2.1.1) is exact, faithful, and compatible with tensor products, duals and fibre functors. It thus induces a morphism of tannakian fundamental groups

$$
G_{\mathrm{mot}, K} \rightarrow G_{\mathrm{mot}, k}
$$

which, in general, is neither injective nor surjective. The base change functor (5.2.1.1) sends classical motives to classical motives.

Definition 5.2.2. - Let $\mathrm{Q}^{\text {Art }}(k)$ denote the full subquiver of $\mathrm{Q}^{\exp }(k)$ consisting of objects of the form $[X, \varnothing, 0,0,0]$ for some finite and étale $k$-scheme $X$. We call category of Artin motives the linear hull

$$
\mathbf{M}^{\text {Art }}(k)=\left\langle\mathrm{Q}^{\mathrm{Art}}(k), \rho\right\rangle
$$

where $\rho: \mathrm{Q}^{\text {Art }}(k) \rightarrow$ Vec is the quiver representation obtained by restricting the standard representation on $\mathrm{Q}^{\exp }(k)$. The canonical functor $\mathbf{M}^{\text {Art }}(k) \rightarrow \mathbf{M}^{\exp }(k)$ is the functor induced by the inclusion $\mathrm{Q}^{\text {Art }}(k) \subseteq \mathrm{Q}^{\exp }(k)$ viewed as a morphism of quiver representations.
5.2.3. - The quiver $\mathrm{Q}^{\text {Art }}(k)$ is contained in the quiver $\mathrm{Q}(k)$ whose linear hull is the category of classical Nori motives. The canonical functor $\mathbf{M}^{\text {Art }}(k) \rightarrow \mathbf{M}^{\exp }(k)$ factors therefore through the category of classical motives. The quiver $\mathrm{Q}^{\operatorname{Art}}(k)$ is also contained in the quiver $\mathrm{Q}_{\mathrm{c}}(k)$ of cellular objects, and is closed under the product used in 4.4.2 to construct the tensor product in $\mathbf{M}^{\exp }(k)$. The category $\mathrm{Q}^{\text {Art }}(k)$ comes thus equipped with a symmetric tensor product, compatible with the canonical functor $\mathbf{M}^{\text {Art }}(k) \rightarrow \mathbf{M}^{\exp }(k)$.

THEOREM 5.2.4. - The tannakian fundamental group of $\mathbf{M}^{\text {Art }}(k)$ is canonically isomorphic to the Galois group $\operatorname{Gal}(\bar{k} \mid k)$. The sequences of affine group schemes

are exact. Here, the injections are induced by base change (5.2.1.1) from $k$ to $\bar{k}$, and the surjections are induced by the inclusion of $\mathbf{M}^{\mathrm{Art}}(k)$ into $\mathbf{M}(k)$, and into $\mathbf{M}^{\exp }(k)$.

We split the proof of Theorem 5.2.4 into manageable portions, verifying one by one the necessary conditions in order to apply the exactness criteria for sequences of tannakian fundamental groups summarised in A.3.
5.2.5. - The base change functor (5.2.1.1) is exact. It admits thus formally a right adjoint which takes values in the ind-category of $\mathbf{M}^{\exp }(k)$. In the situation of a finite extension $k^{\prime}$ of $k$ there is no need for ind-objects, so we have a functor

$$
\operatorname{ind}_{k}^{k^{\prime}}: \mathbf{M}^{\exp }\left(k^{\prime}\right) \rightarrow \mathbf{M}^{\exp }(k)
$$

which is right adjoint to $\operatorname{res}_{k^{\prime}}^{k}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{M}^{\exp }\left(k^{\prime}\right)$. The functor $\operatorname{ind}_{k}^{k^{\prime}}$ is exact, and we can construct it from a morphism of quiver representations as follows. Consider the morphism of quivers

$$
\operatorname{ind}_{k}^{k^{\prime}}: \mathrm{Q}^{\exp }\left(k^{\prime}\right) \rightarrow \mathrm{Q}^{\exp }(k)
$$

given on objects by sending $[X, Y, f, n, i]$ to itself, by regarding a variety $X$ over $k^{\prime}$ as a variety over $k$ via the morphism Spec $k^{\prime} \rightarrow \operatorname{Spec} k$ induced by the inclusion $k \subseteq k^{\prime}$. We obtain a morphism of quiver representations


Lemma 5.2.6. -

$$
\mathbf{M}^{\exp }\left(k^{\prime}\right)=\left\langle\mathrm{Q}^{\exp }\left(k^{\prime}\right), \rho_{k^{\prime}}\right\rangle \cong\left\langle\mathrm{Q}^{\exp }\left(k^{\prime}\right), \rho_{k} \circ \operatorname{ind}_{k}^{k^{\prime}}\right\rangle
$$

LEMmA 5.2.7. - The morphisms of affine group schemes $\pi_{1}\left(\mathbf{M}^{\exp }(\bar{k})\right) \rightarrow \pi_{1}\left(\mathbf{M}^{\exp }(k)\right)$ and $\pi_{1}\left(\mathbf{M}^{\exp }(\bar{k})\right) \rightarrow \pi_{1}\left(\mathbf{M}^{\exp }(k)\right)$ are both closed immersions.

Proof. Lemma A.3.2 states that in order to show that the morphism of affine group schemes $\pi_{1}\left(\mathbf{M}^{\exp }(\bar{k})\right) \rightarrow \pi_{1}\left(\mathbf{M}^{\exp }(k)\right)$ is a closed immersion, we have to show that every motive $M$ over $\bar{k}$ is isomorphic to a subquotient of a motive coming via base change from a motive over $k$. In view of Proposition 4.1.7, it suffices to show this for a motive of the form

$$
M=H^{n}(X, Y, f)
$$

over $\bar{k}$, so $X$ any $Y$ are varieties over $\bar{k}$. There is a finite extension $k^{\prime} \subseteq \bar{k}$ of $k$ of degree $d$, such that $X, Y$ and $f$ are defined over $k^{\prime}$, and it suffices to show that the corresponding object $H^{n}(X, Y, f)$ of $\mathbf{M}^{\exp }\left(k^{\prime}\right)$ is isomorphic to a subquotient of a motive coming from $k$. Let us denote by $X_{k}$ the scheme $X$ viewed as a $k$-variety via the structural morphism $X \rightarrow \operatorname{Spec} k^{\prime} \rightarrow \operatorname{Spec} k$, and consider the object

$$
M_{k}=H^{n}\left(X_{k}, Y_{k}, f\right)
$$

of $\mathbf{M}^{\exp }(k)$. Applying the base change functor $\mathbf{M}^{\exp }(k) \rightarrow \mathbf{M}^{\exp }\left(k^{\prime}\right)$ to $M_{k}$ yields the motive

$$
H^{n}\left(X_{k} \times_{k} k^{\prime}, Y_{k} \times_{k} k^{\prime}, f \times \mathrm{id}\right)=H^{n}(X, Y, f) \otimes H^{0}\left(\operatorname{Spec}\left(k^{\prime} \otimes_{k} k^{\prime}\right), \varnothing, 0\right) \simeq H^{n}(X, Y, f)^{\oplus d}
$$

which contains $M$ as a direct factor.

Lemma 5.2.8. - For every finite étale $k$-scheme $X$, let the Galois group $G=\operatorname{Gal}(\bar{k} \mid k)$ act on the finite-dimensional vector space $H^{0}(X, \varnothing, 0)(0)=H^{0}(X(\mathbb{C}), \mathbb{Q})$ via its canonical action on the finite set $X(\mathbb{C})=X(\bar{k})$. The functor induced by this action and the universal property of linear hulls 4.1.12

$$
\mathbf{M}^{\operatorname{Art}}(k) \rightarrow \boldsymbol{\operatorname { R e p }}(G)
$$

is an equivalence of categories, compatible with tensor product. Here, $\boldsymbol{\operatorname { R e p }}(G)$ stands for the category of $\mathbb{Q}$-linear, finite-dimensional and continuous representations of the profinite group $G$.

Proof. Note that $\mathrm{Q}^{\text {Art }}(k)$ does not contain morphisms of type (b) or type (c), hence is equivalent as a quiver to the category of finite étale $k$-schemes, which in turn is equivalent to the category of finite $G$-sets. The quiver $\mathrm{Q}^{\text {Art }}(k)$ with its standard representation is thus isomorphic to the quiver of finite $G$-sets with the representation sending a finite $G$-set $S$ to the vector space $\mathbb{Q}[S]$
with basis $S$. From this point of view, the statement of the Lemma holds for any profinite group $G$, as we have already seen in Example 4.1.8.

LEMMA 5.2.9. - The morphism of affine group schemes $\pi_{1}(\mathbf{M}(k)) \rightarrow \operatorname{Gal}(\bar{k} \mid k)$ induced by the canonical functor $\mathbf{M}^{\text {Art }}(k) \rightarrow \mathbf{M}(k)$ is faithfully flat. In other words, this canonical functor is fully faithful, and its essential image is stable under taking subobjects and quotients.

Proof. Fix a prime number $\ell$. A morphism of affine group schemes over $\mathbb{Q}$ is faithfully flat if and only if its base change to $\mathbb{Q}_{\ell}$ is so, hence we can switch to motives with $\mathbb{Q}_{\ell}$-coefficients. We will use the $\ell$-adic realisation functor on the category of classical Nori motives. Let us explain first how this works. As usual $\mathrm{Q}(k)$ denotes the quiver of relative pairs $[X, Y, 0, n, i]$, and we consider the representation

$$
\rho \otimes \mathbb{Q}_{\ell}: \mathrm{Q}(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}_{\ell}}
$$

sending $[X, Y, 0, n, i]$ to $\rho(X, Y, 0, n, i) \otimes \mathbb{Q}_{\ell}=H^{n}(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})(i) \otimes \mathbb{Q}_{\ell}$. Notice that since $\mathbb{Q}_{\ell}$ is flat as a $\mathbb{Q}$-algebra, the natural morphism of proalgebras $\operatorname{End}(\rho) \otimes \mathbb{Q}_{\ell} \rightarrow \operatorname{End}\left(\rho \otimes \mathbb{Q}_{\ell}\right)$ is an isomorphism. Let $\operatorname{Rep}_{c}\left(k, \mathbb{Q}_{\ell}\right)$ denote the tannakian category of continuous $\ell$-adic representations of $\operatorname{Gal}(\bar{k} \mid k)$. Étale $\ell$-adic cohomology furnishes a representation

$$
\rho_{\ell}: \mathbf{M}(k)_{\mathbb{Q}_{\ell}} \rightarrow \boldsymbol{\operatorname { R e p }}_{c}\left(k, \mathbb{Q}_{\ell}\right)
$$

sending $[X, Y, 0, n, i]$ to $H_{\text {et }}^{n}\left(X_{\bar{k}}, Y_{\bar{k}}, \mathbb{Q}_{\ell}\right)(i)$, and the comparison theorem of Artin and Grothendieck yields a $\mathbb{Q}_{\ell}$-linear isomorphism $H^{n}(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})(i) \otimes \mathbb{Q}_{\ell} \cong H_{\text {ét }}^{n}\left(X_{\bar{k}}, Y_{\bar{k}}, \mathbb{Q}_{\ell}\right)(i)$ which is functorial for morphisms in $\mathrm{Q}(k)$. From Nori's universal property we obtain the functor $R_{\ell}$ which renders the following diagram commutative up to isomorphisms of functors:


We call $R_{\ell}: \mathbf{M}(k)_{\mathbb{Q}_{\ell}} \rightarrow \mathbf{R e p}_{c}\left(k, \mathbb{Q}_{\ell}\right)$ the $\ell$-adic realisation functor. The $\ell$-adic realisation functor is faithful and exact. The verification that it is also compatible with tensor products and duals is straightforward, we will exemplarily do the checking in the case of the perverse realisation in Section 6.1. Given a motive $M$ with motivic fundamental group $G_{M}$, we obtain from $R_{\ell}$ a continuous morphism of topological groups

$$
\operatorname{Gal}(\bar{k} \mid k) \rightarrow G_{M}\left(\mathbb{Q}_{\ell}\right)
$$

whose image is believed to be open.

Let us now return to the proof of the lemma at hand, and call a motive Weil finite if its motivic fundamental group is finite. Thus, if $M$ is Weil finite, then its associated Galois representation $R_{\ell}(M)$ has finite image, and it is also clear that the functor $\mathbf{M}^{\text {Art }}(k) \rightarrow \mathbf{M}(k)$ factors over the full subcategory of Weil finite objects in $\mathbf{M}(k)_{\mathbb{Q}_{\ell}}$ which we denote by $\mathbf{M}^{\mathrm{Wf}}(k)_{\mathbb{Q}_{\ell}}$. Notice that $\mathbf{M}^{\mathrm{Wf}}(k)_{\mathbb{Q}_{\ell}}$ is stable under tensor products, duals, sums and subquotients, hence constitutes a tannakian subcategory. Its tannakian fundamental group is the group of connected components of $\pi_{1}\left(\mathbf{M}(k)_{\mathbb{Q}_{\ell}}\right)$. Let us write $\operatorname{Rep}_{f}\left(k, \mathbb{Q}_{\ell}\right)$ for the full subcategory of $\boldsymbol{\operatorname { R e p }}_{c}\left(k, \mathbb{Q}_{\ell}\right)$ whose objects are those representations with finite image. It follows from Lemma 5.2.9 that the composite of faithful functors

$$
\mathbf{M}^{\mathrm{Art}}(k)_{\mathbb{Q}_{\ell}} \longrightarrow \mathbf{M}^{\mathrm{Wf}}(k)_{\mathbb{Q}_{\ell}} \xrightarrow{R_{\ell}} \operatorname{Rep}_{f}\left(G, \mathbb{Q}_{\ell}\right)
$$

is an equivalence of categories, which proves the lemma.
5.2.10. - Let $\rho: Q \rightarrow$ Vec be a quiver representation, and let $V_{0}$ be a finite-dimensional vector space. Denote by $\rho \otimes V_{0}$ the representation of $Q$ defined by

$$
\left(\rho \otimes V_{0}\right)(q)=\rho(q) \otimes V_{0}
$$

on objects, and in the evident way on morphisms. There is a canonical morphisms of proalgebras

$$
\begin{equation*}
\operatorname{End}(\rho) \otimes \operatorname{End}\left(V_{0}\right) \rightarrow \operatorname{End}\left(\rho \otimes V_{0}\right) \tag{5.2.10.1}
\end{equation*}
$$

given as follows: An element of $\operatorname{End}(\rho)$ is a collection of linear endomorphisms $e_{q}: \rho(q) \rightarrow \rho(q)$, one for every object of $Q$, compatible with morphisms in $Q$. The canonical morphism (5.2.10.1) sends $\left(\left(e_{q}\right)_{q \in Q} \otimes e\right)$ to the collection $\left(e_{q} \otimes e\right)_{q \in Q}$, and is actually an isomorphism as we have seen in Proposition 4.1.17. To apply said proposition, we regard $V_{0}$ as a representation of the quiver with only one object and only the identity morphism. The category of $\operatorname{End}\left(V_{0}\right)$-modules is equivalent to the category of vector spaces via the functor that sends a vector space $V$ to the $\operatorname{End}\left(V_{0}\right)$-module $V \otimes V_{0}$, hence the functor given in Proposition 4.1.17 is actually an equivalence of categories:

$$
\left\langle Q, \rho \otimes V_{0}\right\rangle \xrightarrow{\simeq}\langle Q, \rho\rangle
$$

5.2.11. - Let $k^{\prime} \subseteq \mathbb{C}$ be a finite extension of $k$. Associated with this extension are quiver morphisms

$$
\mathrm{Q}(k) \rightarrow \mathrm{Q}\left(k^{\prime}\right) \quad \text { and } \quad \mathrm{Q}\left(k^{\prime}\right) \rightarrow \mathrm{Q}(k)
$$

the left-hand one given by base change, and the right-hand one given by regarding varieties as varieties over $k$ via the morphism $\operatorname{Spec} k^{\prime} \rightarrow \operatorname{Spec} k$.
5.2.12. - We end this section with a remark about how much information about a finite field extension is contained in the corresponding Artin motive. To simplify the discussion, we work with motives over $\mathbb{Q}$. Recall that two number fields $k_{1}$ and $k_{2}$ are said to be arithmetically equivalent if the associated Dedekind zeta functions are equal. The point here is of course that arithmetically equivalent number fields need not be isomorphic, for example if $a$ is an integer such that none of
$-2 a,-a, a, 2 a$ is a square, then

$$
k_{1}=\mathbb{Q}(\sqrt[8]{a}) \quad k_{2}=\mathbb{Q}(\sqrt[8]{16 a})
$$

are arithmetically equivalent, nonisomorphic number fields. Such number fields can even have different class numbers for certain values of $a$ (Bart de Smit, Robert Perlis, 1994, Zeta functions do not determine...).

Proposition 5.2.13. - Two number fields define isomorphic Artin motives if and only if they are arithmetically equivalent.

Proof. Denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $k_{1}$ and $k_{2}$ be number fields, and let $V_{1}=\mathbb{Q}\left[\operatorname{Hom}\left(k_{1}, \mathbb{C}\right)\right]$ and $V_{2}$ be the corresponding linear representations of the absolute Galois group of $\mathbb{Q}$. It is a consequence of Chebotarev's Density Theorem that the zeta functions of $k_{1}$ and $k_{2}$ are equal if and only if the $\operatorname{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q})$-representations $V_{1}$ and $V_{2}$ are isomorphic, which in turn is equivalent to saying that the Artin motives $M_{1}=H^{0}\left(\operatorname{Spec} k_{1}\right)$ and $M_{2}=H^{0}\left(\operatorname{Spec} k_{2}\right)$ are isomorphic.

### 5.3. Conjectural relation with triangulated categories of motives

Let $k$ be a subfield of $\mathbb{C}$. We denote by $\pi: \mathbb{A}_{k}^{1} \rightarrow \operatorname{Spec}(k)$ the structure morphism and by $j: \mathbb{A}_{k}^{1} \backslash\{0\} \rightarrow \mathbb{A}_{k}^{1}$ the inclusion.
5.3.1. - Let $\operatorname{DExp}(k)$ be the full subcategory of $\mathrm{DM}_{\mathrm{gm}}\left(\mathbb{A}^{1}\right)$ consisting of those objects $M$ satisfying $\pi_{*} M=0$.

COnJECTURE 5.3.2. - The canonical functor $\operatorname{DExp}(k) \rightarrow D^{b}\left(\mathbf{M}^{\exp }(k)\right)$ is an equivalence of categories.
5.3.3. - The category $\operatorname{DExp}(k)$ contains the category $\mathrm{DM}_{\mathrm{gm}}\left(\mathbb{A}^{1}\right)$ via the functor sending a motive $M$ over $k$ to the motive $j!j^{*} \pi^{*} M[1]$ over $\mathbb{A}^{1}$. In particular, $\operatorname{DExp}(k)$ contains Tate motives $j!j^{*} \pi^{*} \mathbb{Q}(n)$, which are sent to Tate motives in $D^{b}\left(\mathbf{M}^{\exp }(k)\right)$ by the canonical functor. Let us see
what Conjecture 5.3 .2 predicts for extensions.

$$
\begin{array}{rll} 
& \operatorname{Ext}_{\mathbf{M}^{\exp }(k)}^{1}(\mathbb{Q}, \mathbb{Q}(n)) & \\
= & \operatorname{Hom}_{\operatorname{DExp}(k)}\left(j!j^{*} \pi^{*} \mathbb{Q}, j!j^{*} \pi^{*} \mathbb{Q}(n)[1]\right) & \text { by Conjecture } 5.3 .2 \\
= & \operatorname{Hom}_{\operatorname{DM}_{\mathrm{gm}}\left(\mathbb{G}_{m}\right)}\left(\pi^{*} \mathbb{Q}, \pi^{*} \mathbb{Q}(n)[1]\right) & j!\text { fully faithful, renaming } \pi \circ j \text { as } \pi \\
= & \operatorname{Hom}_{\operatorname{DM}}(k)\left(\pi_{\# m} \pi^{*} \mathbb{Q}, \mathbb{Q}(n)[1]\right) & \pi_{\#} \text { left adjoint to } \pi^{*} \\
= & \operatorname{Hom}_{\operatorname{DMgm}}(k)(\mathbb{Q} \oplus \mathbb{Q}(1)[1], \mathbb{Q}(n)[1]) & \text { because } \pi_{\#} \pi^{*} \mathbb{Q}(0)=M\left(\mathbb{G}_{m}\right) \\
= & \left\{\begin{array}{lll}
\operatorname{Ext}_{k}^{1}(\mathbb{Q}, \mathbb{Q}(n)) & n \neq 1 & \text { by Conjecture } 5.3 .2 \\
\operatorname{Ext}_{k}^{1}(\mathbb{Q}, \mathbb{Q}(n)) \oplus \mathbb{Q} & n=1 &
\end{array}\right.
\end{array}
$$

Conjecture 5.3 .2 predicts thus, that extensions of $\mathbb{Q}$ by $\mathbb{Q}(n)$ in the category of exponential motives all come from extensions of ordinary motives, except in the case $n=1$, where we should find essentially one additional nonsplit extension

$$
0 \rightarrow \mathbb{Q}(1) \rightarrow M(\gamma) \rightarrow \mathbb{Q} \rightarrow 0
$$

which explains the additional summand $\mathbb{Q}$ in the last line of the computation. We will produce this extension in section 12.8 and call it Euler-Mascheroni motive, because (spoiler alert) among its periods is the Euler-Mascheroni constant. A similar computation, noting the fact that $\operatorname{Ext}_{k}^{q}(\mathbb{Q}, \mathbb{Q}(n))$ is zero for $q \neq 0,1$ shows

$$
\operatorname{Ext}_{\mathbf{M}^{\exp }(k)}^{2}(\mathbb{Q}, \mathbb{Q}(n))=\operatorname{Ext}_{k}^{1}(\mathbb{Q}, \mathbb{Q}(n-1))
$$

and $\operatorname{Ext}_{\mathbf{M}^{\exp (k)}}^{q}(\mathbb{Q}, \mathbb{Q}(n))=0$ for $q \neq 0,1,2$. As it turns out $(? ?)$, this isomorphism can be described explicitly as follows: Given an extension $0 \rightarrow \mathbb{Q}(n-1) \rightarrow M(\nu) \rightarrow \mathbb{Q} \rightarrow 0$, we twist it by $\mathbb{Q}(1)$ and take the Yoneda cup-product with the Euler-Mascheroni motive $M(\gamma)$. We get a four term exact sequence in $\mathbf{M}^{\exp }(k)$

representing a class in $\operatorname{Ext}^{2}(\mathbb{Q}, \mathbb{Q}(n))$.

### 5.4. The Grothendieck ring of varieties with potential

Definition 5.4.1. - The Grothendieck group of varieties with exponential is the abelian group $K_{0}\left(\operatorname{Var}_{k}^{\exp }\right)$ defined by the following generators and relations:

- generators are pairs $(X, f)$ consisting of a $k$-variety and a regular function $f: X \rightarrow \mathbb{A}_{k}^{1}$,
- relations are of the following three types:
(a) $(X, f)=(Y, f \circ h)$ for each isomorphism $h: Y \rightarrow X$;
(b) $(X, f)=\left(Y, f_{\mid Y}\right)+\left(U, f_{\mid Y}\right)$ for each closed subvariety $Y \subseteq X$ with complement $U$;
(c) $\left(X \times \mathbb{A}^{1}, \mathrm{pr}_{\mathbb{A}^{1}}\right)=0$.

ThEOREM 5.4.2. - There is a unique ring morphism

$$
\chi: K_{0}\left(\operatorname{Var}_{k}^{\exp }\right) \longrightarrow K_{0}\left(\mathbf{M}^{\exp }(k)\right)
$$

such that, for each pair $(X, f)$, one has

$$
\begin{equation*}
\chi((X, f))=\sum_{n=0}^{2 \operatorname{dim} X}(-1)^{n} H_{c}^{n}(X, f) \tag{5.4.2.1}
\end{equation*}
$$

Proof. If such a morphism exists, then it is unique since we prescribe it on a set of generators. In order to show its existence, we need to check that (5.4.2.1) is compatible with the relations (a), (b), and (c). For the relations (a) this is clear, and for (c) notice that

$$
H_{c}^{n}\left(X \times \mathbb{A}^{1}, \operatorname{pr}_{\mathbb{A}^{1}}\right)=0
$$

for all $n$. It remains to prove in the situation of relation (b) the equality

$$
\sum_{n=0}^{2 \operatorname{dim} X}(-1)^{n} H_{c}^{n}(X, f)=\sum_{n=0}^{2 \operatorname{dim} Y}(-1)^{n} H_{c}^{n}\left(Y,\left.f\right|_{Y}\right)+\sum_{n=0}^{2 \operatorname{dim} U}(-1)^{n} H_{c}^{n}\left(U,\left.f\right|_{U}\right)
$$

in $K_{0}\left(\mathbf{M}^{\exp }(k)\right)$. Suppose first that $X$ and $Y \subseteq X$ are smooth. We have then an exact sequence

$$
0 \rightarrow H_{c}^{0}(U, f) \rightarrow H_{c}^{0}(X, f) \rightarrow H_{c}^{0}(Z, f) \rightarrow H_{c}^{1}(U, f) \rightarrow \cdots
$$

in $\mathbf{M}^{\exp }(k)$ which gives the desired relation.

## CHAPTER 6

## The perverse realisation

In this chapter, we connect exponential motives and the category Perv $_{0}$ introduced in Chapter 2 by constructing and studying a functor

$$
R_{\text {perv }}: \mathbf{M}^{\exp }(k) \rightarrow \operatorname{Perv}_{0}
$$

which we call perverse realisation. The main result of this chapter is Theorem 6.5.1, stating that an exponential motive is a conventional Nori motive if and only if its perverse realisation is trivial. Its proof hinges on Theorem 6.4 .1 which shows that every exponential motive can be embedded into an exponential motive of the form $H^{n}(X, Y, f)$. Theorem 6.4.1 is the raison d'être for the paper [36], where we prove the analogous statement in the framework of conventional Nori motives.

### 6.1. Construction and compatibility with tensor products

In this section, we construct the perverse realisation functor $R_{\text {perv }}$ : ${ }^{\exp }(k) \rightarrow \operatorname{Perv}_{0}$ using Nori's universal property, and show that this functor is compatible with tannakian structures. This means, among other things, that there is a natural isomorphism

$$
R_{\text {perv }}\left(M_{1} \otimes M_{2}\right) \cong R_{\text {perv }}\left(M_{1}\right) \otimes R_{\text {perv }}\left(M_{2}\right)
$$

in $\mathbf{P e r v}_{0}$ for all objects $M_{1}$ and $M_{2}$ of $\mathbf{M}^{\exp }(k)$, and that this isomorphism is moreover compatible with the unit, the commutativity, and the associativity constraints. The verifications are mostly straightforward and not particularly inspiring, and we will not repeat them for other realisations to come later.
6.1.1. - Denote by $\mathrm{Q}^{\exp }(k)$ the quiver of exponential relative pairs $[X, Y, f, n, i]$ over $k$, and denote by

$$
\rho: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}}
$$

the Betti representation, as defined in 4.2.1 and 4.2.2. Let us call perverse representation the representation

$$
\begin{gather*}
\rho_{\text {perv }}: \mathrm{Q}^{\exp }(k) \rightarrow \operatorname{Perv}_{0}  \tag{6.1.1.1}\\
165
\end{gather*}
$$

defined as follows. On objects $[X, Y, f, n, i]$ we define $\rho_{\text {perv }}$ as perverse cohomology

$$
\rho_{\text {perv }}([X, Y, f, n, i])=H_{\text {perv }}^{n}(X, Y, f)(i)
$$

as defined in 3.2.1. The twist $(i)$ means that we tensor with the $(-i)$-fold tensor power of the one-dimensional vector space $H^{1}\left(\mathbb{G}_{m}, \mathbb{Q}\right)$. On morphisms, we define the representation $\rho_{\text {perv }}$ in the obvious way: Morphisms of type (a) by functoriality of perverse cohomology for morphisms of pairs, morphisms of type (b) are sent to connecting morphisms in long exact sequences for triples, and morphisms of type (c) are sent to Künneth isomorphisms. For the latter, we need to observe that there is a canonical isomorphism $H_{\text {perv }}^{1}\left(\mathbb{G}_{m}\right)=j!j^{*} \underline{\mathbb{Q}}[1](-1)$ in $\operatorname{Perv}_{0}$, where $j: \mathbb{A}^{1} \backslash\{0\} \rightarrow \mathbb{A}^{1}$ is the inclusion, so $j!j^{*} \underline{\mathbb{Q}}[1]$ is the neutral object of the tannakian category $\operatorname{Perv}_{0}$.

Definition 6.1.2. - The perverse realisation functor $R_{\text {perv }}: \mathbf{M}^{\exp }(k) \rightarrow \operatorname{Perv}_{0}$ is the unique functor for which the diagram

commutes.
6.1.3. - Let $\mathbf{A}$ be a $\mathbb{Q}$-linear neutral tannakian category with fibre functor $f: \mathbf{A} \rightarrow \mathbf{V e c}_{\mathbb{Q}}$. We may regard $\mathbf{A}$ as a quiver and $f$ as a quiver representation. Let us write $E$ for the proalgebra $\operatorname{End}(f)$. By Nori's theorem 4.1.11, the canonical functor

$$
\begin{equation*}
\tilde{f}: \mathbf{A} \rightarrow\langle\mathbf{A}, f\rangle=\text { finite-dimensional, continuous } E \text {-modules } \tag{6.1.3.1}
\end{equation*}
$$

is an equivalence of categories. The tensor product in the tannakian category $\mathbf{A}$, together with the natural isomorphism $u_{A, B}: f(A \otimes B) \cong f(A) \otimes f(B)$ can be viewed as a morphism of representations

which then induces a morphism of proalgebras $\mu: E \rightarrow E \otimes E$. The associativity and commutativity constraints of $\mathbf{A}$ show that $\mu$ is a cocommutative comultiplication, and from duality in $\mathbf{A}$ we obtain an involution $E \rightarrow E$ which altogether constitute a cocommutative Hopf algebra structure on $E$. This Hopf algebra structure defines a closed, symmetric monoidal structure on the category $\langle\mathbf{A}, f\rangle$ of finite-dimensional, continuous $E$-modules. The tensor product of two $E$-modules is obtained by
letting $E$ act via the comultiplication on the tensor product of the underlying vector spaces. Thus, the forgetful functor

$$
\omega:\langle\mathbf{A}, f\rangle \rightarrow \mathbf{V e c}_{\mathbb{Q}}
$$

is strictly monoidal, in the sense that the equality $\omega(A \otimes B)=\omega(A) \otimes \omega(B)$ holds, and associativity and commutativity constraints in the category $\langle\mathbf{A}, f\rangle$ are the same as those of the category of vector spaces.

Proposition 6.1.4. - The functor (6.1.3.1) is monoidal. More precisely, the natural isomorphism $u_{A, B}$ is E-linear, hence constitutes a natural isomorphism

$$
u_{A, B}: \widetilde{f}(A \otimes B) \cong \widetilde{f}(A) \otimes \widetilde{f}(B)
$$

in the category $\langle\mathbf{A}, f\rangle$ which is compatible with associativity and commutativity constraints.

### 6.2. Elementary homotopy theory for pairs of varieties with potential

In this section, we review material from [36] and adapt it to the framework of exponential motives. We fix a subfield $k \subseteq \mathbb{C}$ and convene that all varieties and morphisms of varieties are defined over $k$. We denote by Ho the category whose objects are pointed CW-complexes, and whose morphisms are continuous maps respecting base points up to homotopies preserving base points. By the cohomology of an object $\left(X, x_{0}\right)$ in Ho we mean the reduced singular cohomology with rational coefficients, that is, the singular cohomology $H^{n}\left(X, x_{0} ; \mathbb{Q}\right)$ of the pair $\left(X, x_{0}\right)$.
6.2.1. - We call exponential affine pair any triple $(X, Y, f)$ consisting of an affine variety $X$ over $k$, a closed, non-empty subvariety $Y$ of $X$, and a regular function $f$ on $X$. With the obvious notion of morphisms and compositions, these exponential affine pairs form a category $\mathbf{A f f}_{2}^{\exp }(k)$. Since $k$ is embedded in the complex numbers, there is a well-defined functor

$$
\begin{align*}
(-)^{\text {top }}: \mathbf{A f f}_{2}^{\exp }(k) & \rightarrow \mathbf{H o} \\
(X, Y, f) & \longmapsto \lim _{r \rightarrow \infty} X(\mathbb{C}) /\left(Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)\right) \tag{6.2.1.1}
\end{align*}
$$

where the base point of the quotient space $X(\mathbb{C}) /\left(Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)\right)$ is chosen to be the class of any point in $Y(\mathbb{C})$. As in Section 3.1, we write $S_{r}$ for the complex half-plane $\{z \in \mathbb{C} \mid \operatorname{Re}(z) \geqslant r\}$. The limit (6.2.1.1) in the category Ho exists, since for sufficiently big real $r^{\prime} \geqslant r$, the transition maps

$$
X(\mathbb{C}) /\left(Y(\mathbb{C}) \cup f^{-1}\left(S_{r^{\prime}}\right)\right) \rightarrow X(\mathbb{C}) /\left(Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)\right)
$$

are isomorphisms in Ho. This functor (6.2.1.1) is compatible with cohomology, in the sense that there is a canonical and natural isomorphism of vector spaces

$$
\begin{equation*}
H_{\mathrm{rd}}^{n}(X, Y, f) \cong H^{n}\left(X(\mathbb{C}) /\left(Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)\right)\right) \tag{6.2.1.2}
\end{equation*}
$$

for each pair $(X, Y, f)$. Indeed, for any $r \in \mathbb{R}$, one can define the singular cohomology of the pair of topological spaces $\left(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)\right)$ to be the cohomology of the mapping cone of
the inclusion $i: Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right) \rightarrow X(\mathbb{C})$. Since $i$ is the inclusion of a closed real semialgebraic subvariety, it is with respect to an appropriate CW-structure the inclusion of a finite, closed CWcomplex into a finite CW-complex, hence is in particular a cofibration, and we know that the mapping cone of a cofibration is homotopic to the corresponding quotient space.
6.2.2. - We say that a morphism $g$ in $\operatorname{Aff}_{2}^{\exp }(k)$ is a homotopy equivalence if the induced continuous map $g^{\text {top }}$ between pointed topological spaces is, and we say that two morphisms $g$ and $h$ with same source and target are homotopic to each other if the continuous maps $g^{\text {top }}$ and $h^{\text {top }}$ are. Being homotopic is an equivalence relation on morphisms in $\mathbf{A f f}_{2}^{\exp }(k)$ which is compatible with compositions, so there is a well-defined category $\mathbf{A f f}_{2}^{\exp }(k) \simeq$ whose objects are exponential affine pairs, and whose morphisms are homotopy classes of morphisms of exponential affine pairs. By definition, the functor (6.2.1.1) induces a functor

$$
\begin{equation*}
(-)^{\mathrm{top}}: \mathbf{A f f}_{2}^{\exp }(k)_{\simeq} \rightarrow \mathbf{H o} \tag{6.2.2.1}
\end{equation*}
$$

which sends homotopy equivalences to isomorphisms. The functor (6.2.2.1) is not conservative: it sends morphisms which are not isomorphisms to isomorphisms. Forcing (6.2.2.1) to be conservative leads to the following definition:

Definition 6.2.3. - We call category of exponential algebraic homotopy types over $k$ and denote by $\mathbf{H o}^{\exp }(k)$ the localisation in the class of homotopy equivalences of the category $\mathbf{A f f}_{2}^{\exp }(k) \simeq$.
6.2.4. - The category of exponential affine pairs contains as a full subcategory the category whose objects are those $(X, Y, f)$ where $f$ is the zero function. In [36], this category was denoted by $\mathbf{A f f}_{2}(k)$. Localising $\mathbf{A f f}_{2}(k)$ in the class of homotopy equivalences led to the category of algebraic homotopy types $\mathbf{H o}(k)$ over $k$. There is an obvious functor

$$
\mathbf{H o}(k) \rightarrow \mathbf{H o}^{\exp }(k)
$$

induced by the inclusion $\mathbf{A f f}_{2}(k) \rightarrow \mathbf{A f f}_{2}^{\exp }(k)$, but there is no à priori reason why this functor should be full or faithful. The object (Spec $k, \operatorname{Spec} k, 0)$ of $\mathbf{H o}^{\exp }(k)$ is initial and final, indeed, there are unique morphisms

$$
\begin{equation*}
(\operatorname{Spec} k, \operatorname{Spec} k, 0) \rightarrow(X, Y, f) \quad(X, Y, f) \rightarrow(\operatorname{Spec} k, \operatorname{Spec} k, 0) \tag{6.2.4.1}
\end{equation*}
$$

for every object $(X, Y, f)$ in $\mathbf{H o}^{\mathbf{e x p}}(k)$. The initial morphism, that is, the left-hand side in (6.2.4.1), is the composite

$$
(\operatorname{Spec} k, \operatorname{Spec} k, 0) \stackrel{\simeq}{\longleftarrow}\left(Y, Y,\left.f\right|_{Y}\right) \stackrel{\subseteq}{\hookrightarrow}(X, Y, f)
$$

and the final morphism is the composite

$$
(X, Y, f) \xrightarrow{f}\left(\mathbb{A}^{1}, \mathbb{A}^{1}, \mathrm{id}\right) \stackrel{\simeq}{\longleftarrow}(\operatorname{Spec} k, \operatorname{Spec} k, 0)
$$

where Spec $k$ is mapped to the origin of $\mathbb{A}^{1}$, and the symbol $\simeq$ indicates homotopy equivalences. It follows in particular that homomorphism sets in $\mathbf{H o}^{\exp }(k)$ contain a distinguished element.

Definition 6.2.5. - We call standard algebraic circle the object $S^{1}(k)=\left(\mathbb{A}^{1},\{0,1\}\right)$ of $\mathbf{H o}(k)$, as well as its image $\left(\mathbb{A}^{1},\{0,1\}, 0\right)$ in $\mathbf{H o}^{\exp }(k)$.
6.2.6. - There is a canonical isomorphism $S^{1}(k)^{\text {top }} \cong[0,1]_{0 \sim 1}=S^{1}$ in Ho. Marking any other two distinct rational points on $\mathbb{A}^{1}$ results in an isomorphic object, but the isomorphism becomes canonical only once an ordering of the two marked points is chosen. In a similar spirit, we may define the standard algebraic interval as the object $\left(\mathbb{A}^{1},\{0\}\right)$ of $\mathbf{H o}(k)$, corresponding to the object $\left(\mathbb{A}^{1},\{0\}, 0\right)$ in $\mathbf{H o}{ }^{\exp }(k)$.

Definition 6.2.7. - Let $(X, Y, f)$ and $\left(X^{\prime}, Y^{\prime}, f^{\prime}\right)$ be exponential affine pairs. We call

$$
(X, Y, f) \vee\left(X^{\prime}, Y^{\prime}, f^{\prime}\right)=\left(X \sqcup X^{\prime}, Y \sqcup Y^{\prime}, f \sqcup f^{\prime}\right)
$$

the wedge, and

$$
(X, Y, f) \wedge\left(X^{\prime}, Y^{\prime}, f\right)=\left(X \times X^{\prime},\left(X \times Y^{\prime}\right) \cup\left(Y \times X^{\prime}\right), f \boxplus f^{\prime}\right)
$$

the smash product of $(X, Y, f)$ and $\left(X^{\prime}, Y^{\prime}, f^{\prime}\right)$. We call the pairs

$$
\begin{aligned}
& C(X, Y, f)=(X, Y) \wedge\left(\mathbb{A}^{1},\{0\}, 0\right)=\left(X \times \mathbb{A}^{1},(X \times\{0\}) \cup\left(Y \times \mathbb{A}^{1}\right), f \boxplus 0\right) \\
& \Sigma(X, Y, f)=(X, Y) \wedge\left(\mathbb{A}^{1},\{0,1\}, 0\right)=\left(X \times \mathbb{A}^{1},(X \times\{0,1\}) \cup\left(Y \times \mathbb{A}^{1}\right), f \boxplus 0\right)
\end{aligned}
$$

the cone and the suspension of $(X, Y, f)$.
6.2.8. - Cones and suspensions do what they ought to do. They are compatible with the functor $(-)^{\text {top }}$ in the obvious way. There is a natural morphism $(X, Y, f) \rightarrow C(X, Y, f)$, namely the one sending $x$ to $(x, 1)$. The cone $C(X, Y, f)$ is contractible, in the sense that the unique morphism $C(X, Y, f) \rightarrow(\operatorname{Spec} k, \operatorname{Spec} k, 0)$ in $\mathbf{H o}^{\exp }(k)$ is an isomorphism. The triple of pairs

$$
(X, Y, f) \rightarrow C(X, Y, f) \rightarrow \Sigma(X, Y, f)
$$

induces a long exact sequence of cohomology groups, which, since $C(X, Y, f)$ is contractible, degenerates to isomorphisms $H^{n}(\Sigma(X, Y, f)) \cong H_{\mathrm{rd}}^{n+1}(X, Y, f)$. Indeed, the morphism of pairs

$$
\begin{equation*}
(X, Y, f) \rightarrow\left((X \times\{0,1\}) \cup\left(Y \times \mathbb{A}^{1}\right),(X \times\{0\}) \cup\left(Y \times \mathbb{A}^{1}\right), f \boxplus 0\right) \tag{6.2.8.1}
\end{equation*}
$$

sending $x$ to $(x, 1)$ is a homotopy equivalence, hence an isomorphism in $\mathbf{H o}(k)$. The long exact sequence of cone and suspension can be identified with the the long exact sequence of the following triple of spaces.

$$
\begin{equation*}
(X \times\{0\}) \cup\left(Y \times \mathbb{A}^{1}\right) \subseteq(X \times\{0,1\}) \cup\left(Y \times \mathbb{A}^{1}\right) \subseteq X \times \mathbb{A}^{1} \tag{6.2.8.2}
\end{equation*}
$$

Given a triple of nonempty varieties $Z \subseteq Y \subseteq X$, we can think of $\left(Y, Z,\left.f\right|_{Y}\right) \rightarrow(X, Z, f) \rightarrow$ $(X, Y, f)$ as a cofibre sequence. Associated with it comes the long Puppe-sequence

$$
\begin{equation*}
\left(Y, Z,\left.f\right|_{Y}\right) \rightarrow(X, Z, f) \rightarrow(X, Y, f) \rightarrow \Sigma\left(Y, Z,\left.f\right|_{Y}\right) \rightarrow \Sigma(X, Z, f) \rightarrow \cdots \tag{6.2.8.3}
\end{equation*}
$$

where only the connecting morphism $(X, Y, f) \rightarrow \Sigma\left(Y, Z,\left.f\right|_{Y}\right)$ needs some explanation. It is a morphism in the category $\mathbf{H o}^{\exp }(k)$, given by the roof

where $X \cup_{Z} Y$ stands for the union $(X \times\{1\}) \cup\left(Z \times \mathbb{A}^{1}\right) \cup(Y \times\{0\})$, in which the topologist may recognise a homotopy pushout, as we explain in the proof of Proposition 6.2.11. The map $h$ in (6.2.8.4) sends $x$ to ( $x, 0$ ), and the homotopy equivalence $s$ is defined by the inclusion $Y \times \mathbb{A}^{1} \subseteq$ $X \times \mathbb{A}^{1}$. For later reference, we observe that the diagram of vector spaces

commutes, so the connecting morphism $\partial$ in the long exact sequence is the composite of the suspension isomorphism and the morphism induced in cohomology by $s^{-1} h$.

Lemma 6.2.9. - Any morphism of pairs of affine algebraic varieties $f:(X, Y) \rightarrow\left(X_{1}, Y_{1}\right)$ can be factorised as

where $g$ is a closed immersion, and s is a homotopy equivalence which admits an algebraic homotopy inverse.

Proof. Since $X$ is affine, there exists a closed immersion $e: X \rightarrow \mathbb{A}^{N}$ for some sufficiently large $N$. Consider the pair $\left(\widetilde{X}_{1}, \widetilde{Y}_{1}\right)=\left(\mathbb{A}^{N} \times X_{1}, \mathbb{A}^{N} \times Y_{1}\right)$, and define the morphisms $g$ and $s$ by $g(x)=(e(x), f(x))$ and $s\left(t, x_{1}\right)=x_{1}$. The morphism $g$ is a closed immersion because $e$ is, the morphism $s$ is a homotopy equivalence because $\mathbb{C}^{N}$ is contractible, and $f=s g$ holds by construction. An algebraic homotopy inverse to $s$ is the morphism $\left(X_{1}, Y_{1}\right) \rightarrow\left(\widetilde{X}_{1}, \widetilde{Y}_{1}\right)$ sending $x_{1}$ to $\left(0, x_{1}\right)$.

Lemma 6.2.10. - Let $f_{1}:(X, Y) \rightarrow\left(X_{1}, Y_{1}\right)$ and $f_{2}:(X, Y) \rightarrow\left(X_{2}, Y_{2}\right)$ be morphisms of pairs of algebraic varieties, both of them given by closed immersions. The pushout

exists in the category of pairs of varieties. If $X_{1}$ and $X_{2}$ are affine, then so is $X^{\prime}$.

Proof. We define $X^{\prime}$ to be the variety obtained by gluing $X_{1}$ and $X_{2}$ along $X$. There is an obvious way of gluing ringed spaces, and one can check that gluing varieties along a common closed subvariety results is a variety, as is done in [80, Theorem 3.3 and Corollary 3.7]. If $X=\operatorname{Spec}(A)$ and $X_{i}=\operatorname{Spec}\left(A_{i}\right)$ are all affine, then $X^{\prime}$ is affine, and indeed the spectrum of the ring

$$
A^{\prime}=\left\{\left(a_{1}, a_{2}\right) \in A_{1} \times A_{2} \mid \varphi_{1}\left(a_{1}\right)=\varphi_{2}\left(a_{2}\right)\right\}
$$

where $\varphi_{i}: A_{i} \rightarrow A$ are the surjective ring morphisms corresponding to the inclusions $X \rightarrow X_{i}$. There are canonical closed immersions $g_{i}: X_{i} \rightarrow X^{\prime}$, and with these, the diagram

commutes and is a push-out diagram in the category of varieties. As a closed subvariety $Y^{\prime} \subseteq X^{\prime}$ we choose the union $g_{1}\left(Y_{1}\right) \cup g_{2}\left(Y_{2}\right)$. With this choice for $Y^{\prime}$ we obtain a diagram of the shape (6.2.10.1), and the universal property of the pair $\left(X^{\prime}, Y^{\prime}\right)$ in the category of pairs of varieties is a direct consequence of the universal property of $X^{\prime}$ in the category of varieties.

Proposition 6.2.11. - Let $s:(X, Y) \rightarrow\left(X_{1}, Y_{1}\right)$ be a homotopy equivalence between affine pairs and let $f:(X, Y) \rightarrow\left(X_{2}, Y_{2}\right)$ be an arbitrary morphism of affine pairs of varieties. There exists a diagram of affine pairs of varieties

which commutes up to homotopy, and where $\widetilde{s}$ is a homotopy equivalence.

Proof. We will construct the homotopy pushout of $f$ and $s$. Let us first recall how this is done classically in algebraic topology, see [85, §4.2]: Given a diagram of pointed CW-complexes
and continuous maps

the homotopy pushout is the space $\widetilde{X}_{1}$ obtained by quotienting $X_{1} \sqcup(X \wedge[0,1]) \sqcup \widetilde{X}$ by the relations $(x, 0) \sim f(x)$ and $(x, 1) \sim s(x)$ for $x \in X$. If $f$ and $s$ are both cofibrations the homotopy pushout is an ordinary pushout, but not in general: the homotopy pushout remembers the space of relations $X$. For example, if $X_{1}$ and $X_{2}$ are reduced to a point, then the homotopy pushout is the suspension of $X$. If the morphism $s: X \rightarrow \widetilde{X}$ in (6.2.11.2) is a homotopy equivalence, then so is the canonical morphism $\widetilde{s}: X_{1} \rightarrow \widetilde{X}_{1}$.

We now model the homotopy pushout with pairs of algebraic varieties. By Lemma 6.2 .9 we can and will suppose without loss of generality that the morphisms $s: X \rightarrow \widetilde{X}$ and $f: X \rightarrow X_{1}$ in (6.2.11.1) are closed immersions. Define the pair $\left(\widetilde{X}_{1}, \widetilde{Y}_{1}\right)$ to be the ordinary pushout of the diagram

where $\left(i_{0}, i_{1}\right)$ is the map sending one copy of $X$ to $X \times\{0\}$ and the other to $X \times\{1\}$. Since both, $\left(i_{0}, i_{1}\right)$ and $f \vee s$ are closed immersions, this pushout exists as we have seen in Lemma 6.2.10. We define $\widetilde{s}$ and to be the restriction of the map $u$ to the component $\left(X_{1}, Y_{1}\right)$, and $\widetilde{f}$ to be the restriction of $u$ to $(\widetilde{X}, \tilde{Y})$. In other terms, $\left(\widetilde{X}_{1}, \widetilde{Y}_{1}\right)$ is the quotient space

$$
\left(\widetilde{X}_{1}, \widetilde{Y}_{1}\right)=\left(\left(X_{1}, Y_{1}\right) \sqcup\left(X \times \mathbb{A}^{1}, Y \times \mathbb{A}^{1}\right) \sqcup(\tilde{X}, \tilde{Y})\right) /_{(x, 0)=s(x),(x, 1) \sim f(x)}
$$

and the morphisms $\widetilde{s}$ and $\widetilde{f}$ are the ones induced by inclusions. The functor $(-)^{\text {top }}: \mathbf{H o}(k) \rightarrow \mathbf{H o}$ sends the diagram (6.2.11.1) to a homotopy pushout. In particular, the morphism $\widetilde{s}$ is a homotopy equivalence since $s$ is so by hypothesis.

Corollary 6.2.12. - The class of homotopy equivalences in $\mathbf{A f f}_{2}(k) / \simeq$ admits a calculus of left fractions. In the terminology of [38, III, §2, Def. 6], the class of homotopy equivalences is localising.

Proof. All conditions of [38, III, $\S 2$, Def. 6] are trivially satisfied by homotopy equivalences in $\operatorname{Aff}_{2}(k) / \simeq$, except condition (b) which is the content of Proposition 6.2.11.

### 6.3. A homotopy variant of the quiver of exponential pairs

6.3.1. - Let $\mathrm{Q}_{\mathrm{aff}}(k) \subseteq \mathrm{Q}(k)$ be the full subquiver of $\mathrm{Q}(k)$ whose objects are those tuples $[X, Y, n, i]$ where $X$ is affine and $Y$ is non-empty. Let $\mathrm{Q}_{h}(k)$ be the quiver whose objects are the same as those of $\mathrm{Q}_{\mathrm{aff}}(k)$ but whose morphisms of type (a) are given by morphisms in the category $\mathbf{H o}(k)$ instead. We can restrict the quiver representation $\rho: \mathrm{Q}(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ to a quiver representation $\rho_{\text {aff }}$ of $\mathrm{Q}_{\mathrm{aff}}(k)$, which then factorises over a representation $\rho_{\mathrm{h}}$ of $\mathrm{Q}_{\mathrm{h}}(k)$. The situation is summarised in the following commutative diagram of quiver morphisms and representations:


PROPOSITION 6.3.2. - The morphisms of proalgebras $\operatorname{End}(\rho) \longrightarrow \operatorname{End}\left(\rho_{\text {aff }}\right) \longleftarrow \operatorname{End}\left(\rho_{\mathrm{h}}\right)$ obtained from the commutative diagram (6.3.1.1) are isomorphisms.

Proof. Let $e$ be an endomorphism of $\rho$. Concretely, $e$ is a collection of linear endomorphisms $e_{q}: \rho(q) \rightarrow \rho(q)$, one for every object $q=[X, Y, n, i]$ of $\mathrm{Q}(k)$, which are compatible in that, for every morphism $f: p \rightarrow q$ in $\mathrm{Q}(k)$, the diagram of vector spaces

commutes. In order to show that the morphism

$$
\begin{equation*}
\operatorname{End}(\rho) \longrightarrow \operatorname{End}\left(\rho_{\text {aff }}\right) \tag{6.3.2.2}
\end{equation*}
$$

is injective, we may assume that $e_{q}=0$ for all $q=[X, Y, n, i]$ where $X$ is affine, and have to show that $e$ is zero. This is an immediate consequence of Jouanolou's trick: given an arbitrary object $[X, Y, n, i]$ of $\mathrm{Q}(k)$, there exists an affine variety $\widetilde{X}$ and a homotopy equivalence $f: \widetilde{X} \rightarrow X$. Setting $\widetilde{Y}=f^{-1}(Y)$ and $\widetilde{q}=[\widetilde{X}, \widetilde{Y}, n, i]$, the morphism $f: \widetilde{q} \rightarrow q$ of type (a) induces an isomorphism of vector spaces $\rho(f): \rho(\widetilde{q}) \rightarrow \rho(q)$. In case $\widetilde{Y}$ is empty, add to $\widetilde{X}$ and isolated point $*$ and set $\widetilde{Y}=*$. The diagram (6.3.2.1) for this particular morphism $f$, and hence $e_{q}$ is indeed zero. Let us for now refer to $\widetilde{q} \rightarrow q$ as an affine homotopy replacement. In order to show that the morphism of proalgebras (6.3.2.2) is surjective as well, consider an element $e$ of $\operatorname{End}\left(\rho_{\text {aff }}\right)$. Choosing arbitrary affine homotopy replacements as before, we can extend $e$ to a well defined collection of endomorphisms $e_{q}$ for all objects $q$ of $\mathrm{Q}(k)$. We must check that the square (6.3.2.1) commutes, knowing that such squares commute for morphisms in $\mathrm{Q}_{\mathrm{aff}}(k)$. This can be checked by observing that for every morphism
$f: p \rightarrow q$ in $\mathrm{Q}(k)$, we can choose affine homotopy replacements $\widetilde{p} \rightarrow p$ and $\widetilde{q} \rightarrow q$ such that there is a morphism $\widetilde{f}: \widetilde{p} \rightarrow \widetilde{q}$ in $\mathrm{Q}_{\mathrm{aff}}(b)$ for which the diagram

commutes. Attaching this square to (6.3.2.1) yields a cube where all faces other than (6.3.2.1) commute, so (6.3.2.1) commutes as well. Thus, the collection $\left(e_{q}\right)_{q \in \mathrm{Q}(k)}$ is an element of $\operatorname{End}(\rho)$ as we wanted to show.

The isomorphism $\operatorname{End}\left(\rho_{\text {aff }}\right) \cong \operatorname{End}\left(\rho_{\mathrm{h}}\right)$ is a formality: since the quivers $\mathrm{Q}_{\mathrm{aff}}(k)$ and $\mathrm{Q}_{\mathrm{h}}(k)$ have the same objects, we just need to check that a collection of endomorphisms $e_{q}$ commutes with morphisms in $\mathrm{Q}_{\mathrm{aff}}(k)$ if and only if it commutes with morphisms in $\mathrm{Q}_{\mathrm{h}}(k)$, but that is obvious.

Definition 6.3.3. - Let $Q_{0}$ and $Q_{1}$ be finite subquivers of $\mathrm{Q}_{\mathrm{h}}(k)$. We say that $Q_{0}$ and $Q_{1}$ are equivalent if there exists a finite subquiver $Q^{+}$of $\mathrm{Q}_{\mathrm{h}}(k)$ containing $Q_{0}$ and $Q_{1}$ and an isomorphism of $\operatorname{End}\left(\left.\rho\right|_{Q^{+}}\right)$algebras

$$
\operatorname{End}\left(\left.\rho\right|_{Q_{0}}\right) \stackrel{\cong}{\cong} \operatorname{End}\left(\left.\rho\right|_{Q_{1}}\right)
$$

Proposition 6.3.4. - Every finite subquiver of $\mathrm{Q}_{\mathrm{h}}(k)$ is equivalent to a quiver containing only one object.

Proof. This follows from Lemma 6.3.8 and Lemma 6.3.9.
6.3.5. - Here is how we will use Proposition 6.3.4: given a motive $M$, that is, a finite-dimensional continuous $\operatorname{End}(\rho)$-module, we may by definition of continuity find some finite subquiver $Q$ of $\mathrm{Q}_{\mathrm{h}}(k)$ such that the structural map $\operatorname{End}(\rho) \rightarrow \operatorname{End}_{\mathbb{Q}}(M)$ factors through the finite-dimensional algebra $E=\operatorname{End}\left(\left.\rho\right|_{Q}\right)$. By Proposition 6.3.4, we may arrange $Q$ to contain only one object, say $q=[X, Y, n, i]$. As an $E$-module, $M$ is a quotient of $E^{n}$ for some $n \geqslant 0$, so in order to prove that $M$ is a quotient of a motive of the form $H^{n_{0}}\left(X_{0}, Y_{0}\right)\left(i_{0}\right)$ it suffices to show that $E$ is. As an $E$-module, $E$ is a submodule of

$$
\operatorname{End}(\rho(q)) \cong H_{n}(X, Y)(-i) \otimes H^{n}(X, Y)(i)
$$

where in the tensor product $E$ acts via $e(u \otimes v)=u \otimes e v$. The commutator of an endomorphism $f$ of $q$ in $\operatorname{End}(\rho(q))$ is the kernel of the map $f_{*} \otimes \mathrm{id}-\mathrm{id} \otimes f^{*}$, and we will then use cogroup structures on suspensions to show that the intersection of finitely many such kernels has the desired form.
6.3.6. - Let $\left.\rho\right|_{Q}: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be the standard representation on some subquiver $Q$ of $\mathrm{Q}_{\mathrm{h}}(k)$, and suppose that we dislike certain objects in $Q$, and want to replace them with more likeable objects, without changing the endomorphism algebra $\operatorname{End}\left(\left.\rho\right|_{Q}\right)$. In other words, we wish to find a quiver $Q_{0}$ which is equivalent to $Q$ and contains only likeable objects. That may be possible in theory, as follows. Write $Q_{\mathrm{b}}$ for the full subquiver of bad objects and $Q_{\mathrm{g}}$ for the full subquiver of good objects of $Q$, and let us enlarge $Q$ to a quiver $Q^{+}$in three steps.

Step 1: Start with setting $Q^{+}=Q$. Then, find for each bad object $q$ of $Q_{\mathrm{b}}$ a finite, connected quiver $L(q)$ containing $q$ and also containing a non-empty connected subquiver $L_{\mathrm{g}}(q)$ consisting of good objects, such that the diagram of vector spaces $\rho(L(q))$ is a commutative diagram of isomorphisms. For an object $q^{\prime}$ in $L(q)$, denote by $\lambda\left(q^{\prime}\right)$ the isomorphism $\rho(q) \rightarrow \rho\left(q^{\prime}\right)$ appearing in $\rho(L(q))$. We add to $Q^{+}$these quivers $L(q)$. We understand here that we have made sure that the only object common to $L(q)$ and $Q$ is $q$, and that for different objects $p$ and $q$ in $Q_{\mathrm{b}}$, the quivers $L(p)$ and $L(q)$ are disjoint.

Step 2: Next, for every morphism $f: p \rightarrow q$ in $Q_{\mathrm{b}}$, find and add to $Q^{+}$a morphism $f^{\prime}: p^{\prime} \rightarrow q^{\prime}$, where $p^{\prime}$ and $q^{\prime}$ are objects in $L_{\mathrm{g}}(p)$ and $L_{\mathrm{g}}(q)$, such that the diagram

commutes.
Step 3: Finally, for every morphism $f: p \rightarrow q$ or $f: q \rightarrow p$ between an object $q$ of $Q_{\mathrm{b}}$ and an object $p$ of $Q_{\mathrm{g}}$, find and add to $Q^{+}$a morphism $f^{\prime}: p \rightarrow q^{\prime}$ or $f^{\prime}: q^{\prime} \rightarrow p$, where $q^{\prime}$ is an object in $L_{\mathrm{g}}(q)$, such that the corresponding diagram

commutes.
Denote now by $Q_{\mathrm{g}}^{+} \subseteq Q^{+}$the full subquiver of good objects, obtained from $Q^{+}$by deleting all bad objects and all morphisms to and from bad objects. It is straightforward to check, as we will in 6.3.7, that the restriction morphisms

$$
\begin{equation*}
\operatorname{End}\left(\left.\rho\right|_{Q_{\mathrm{g}}^{+}}\right) \leftarrow \operatorname{End}\left(\left.\rho\right|_{Q^{+}}\right) \rightarrow \operatorname{End}\left(\left.\rho\right|_{Q}\right) \tag{6.3.6.3}
\end{equation*}
$$

are isomorphisms of algebras. In particular, the quivers $Q_{\mathrm{g}}^{+}$and $Q$ are equivalent, and the quiver we were looking for at the beginning of this discussion is $Q_{0}=Q_{\mathrm{g}}^{+}$.

LEMMA 6.3.7. - The quiver $Q_{0}$, as constructed in 6.3.6, is equivalent to $Q$.

Proof. We check that the restriction morphisms (6.3.6.3) are isomorphisms. Elements of $E^{+}=\operatorname{End}\left(\left.\rho\right|_{Q^{+}}\right)$are collections of linear endomorphisms $\left(e_{q}\right)_{q \in Q^{+}}$indexed by objects of $Q^{+}$, with $e_{q} \in \operatorname{End}(\rho(q))$, satisfying

$$
\begin{equation*}
e_{q} \circ \rho(f)=\rho(f) \circ e_{p} \tag{6.3.7.1}
\end{equation*}
$$

for each morphism $f: p \rightarrow q$ in $Q^{+}$. Elements of $E=\operatorname{End}\left(\left.\rho\right|_{Q}\right)$ and $E_{0}=\operatorname{End}\left(\left.\rho\right|_{Q_{0}}\right)$ are described similarly. In order to prove that the restriction map $E^{+} \rightarrow E$ is injective, consider an element $e=\left(e_{q}\right)_{q \in Q^{+}}$of $E^{+}$such that $e_{q}=0$ for all $q \in Q$, fix an object $q^{\prime} \in Q^{+}$, and let us show that $e_{q^{\prime}}$ is zero. If $q^{\prime}$ is not already an object of $Q$, then $q^{\prime}$ is an object of $L(q)$ for some unique $q \in Q_{\mathrm{b}}$. By definition, the diagram of vector spaces and linear maps

commutes, and since $e_{q}=0$ we have indeed $e_{q^{\prime}}=0$. This settles injectivity of the map $E^{+} \rightarrow E$, and injectivity of $E^{+} \rightarrow E_{0}$ is shown similarly. In order to prove that the restriction map $E^{+} \rightarrow E$ is also surjective, fix an element $\left(e_{q}\right)_{q \in Q}$ of $E$. We construct a tuple $\left(e_{q}\right)_{q \in Q^{+}}$by considering as before for each $q^{\prime} \in Q^{+}$which is not already in $Q$ the unique map $\lambda\left(q^{\prime}\right): \rho(q) \rightarrow \rho\left(q^{\prime}\right)$ and take for $e_{q^{\prime}}$ the unique endomorphism of $\rho\left(q^{\prime}\right)$ for which the square (6.3.7.2) commutes. It remains to pick a morphism $f$ in $Q^{+}$and check that the relation (6.3.7.1) holds. If the target and the source of $f$ both belong to $Q$ then $f$ is a morphism in $Q$ and (6.3.7.1) holds by definition. If neither target nor source of $f$ belong to $Q$, then (6.3.7.1) holds because the square (6.3.6.1) in Step 2 is supposed to commute, and if the target of $f$ but not the source belongs to $Q$, or the other way around, then the commutativity of (6.3.6.2) implies (6.3.7.1). Surjectivity of the restriction morphism $E^{+} \rightarrow E_{0}$ is shown similarly.

Lemma 6.3.8. - Let $Q$ be a finite subquiver of $\mathrm{Q}_{\mathrm{h}}(k)$. There exists a quiver $Q_{0}$ which is equivalent to $Q$ and such that there exist integers $n_{0}$ and $i_{0}$ such that all objects in $Q_{0}$ are of degree $n_{0}$ and twist $i_{0}$.

Proof. Recall that we refer to the integers $n$ and $i$ in an object $[X, Y, n, i]$ of $\mathrm{Q}_{\mathrm{h}}(k)$ as degree and twist respectively. Given integers $n$ and $i$ and a quiver $Q \subseteq \mathrm{Q}_{\mathrm{h}}(k)$, let us denote by $Q[n, i]$ the full subquiver of $Q$ consisting of objects with degree $n$ and twist $i$. Notice that $Q[n, i]$ only contains morphisms of type (a).

Let $Q \subseteq \mathrm{Q}_{\mathrm{h}}(k)$ be a finite quiver containing objects with different twists and degrees. Following the process outlined in 6.3 .6 , we will show that there exists a finite quiver $Q_{0}$ which is equivalent to $Q$ and contains fewer different twists, and then continue inductively until there is only one twist left. We then proceed with a different construction, reducing the number of different degrees and not adding any new twists. This will eventually lead to a quiver which is equivalent to $Q$ and has only one twist and one degree.

Since $Q$ is finite, only finitely many of the quivers $Q[n, i]$ are non-empty. Choose $\left(n_{0}, i_{0}\right)$ large enough, such that whenever $Q[n, i]$ non-empty, then $\left(n_{0}, i_{0}\right)=(n+d+t, i+t)$ for non-negative integers $d$ and $t$. The quiver $Q$ can be drawn schematically as a finite diagram of the shape

where the vertical arrows symbolise (many) morphisms of type (b) and the diagonal arrows symbolise morphisms of type (c), and where all nodes are finite quivers with internal morphisms only of type (a).

Claim: Let $i_{1}$ be the smallest integer such that $Q\left[n, i_{1}\right]$ is non-empty for some $n \leqslant n_{0}$, and suppose $i_{1}<i_{0}$. There exists a finite quiver $Q_{0}$ which is equivalent to $Q$ and such that if $Q_{0}[n, i]$ is non-empty, then $i_{1}<i \leqslant i_{0}$ and $n \leqslant n_{0}$.

To prove this claim, let us denote by $Q\left[i_{1}\right]$ the full subquiver of $Q$ of objects with twist $i_{1}$ and declare these to be the bad objects. We will construct $Q^{+}$as outlined in 6.3.6. As for the first step, let $Q^{+}$be the quiver, subject to further enlargement, obtained from $Q$ by adding for every $q=\left[X, Y, n, i_{1}\right]$ of $Q\left[i_{1}\right]$ the quiver $L(q)$ consisting of the two objects $q$ and

$$
T q=\left[X \times \mathbb{G}_{m},\left(Y \times \mathbb{G}_{m}\right) \cup(X \times\{1\}), n+1, i_{1}+1\right]
$$

and the morphism $\kappa_{q}: T q \rightarrow q$ of type (c). The induced linear map $\rho\left(\kappa_{q}\right): \rho(T q) \rightarrow \rho(q)$ is an isomorphism. Objects in $Q^{+}$have twists between $i_{1}$ and $i_{0}$, and degrees at most $n_{0}$. The construction of $T q \rightarrow q$ is functorial in the evident way for morphisms $f: p \rightarrow q$ in $Q\left[i_{1}\right]$ of type (a) and (b), so that the following diagram of vector spaces, corresponding to (6.3.6.1) in the abstract setting, commutes.


As for the second step in the process, add the morphisms $T f$ to $Q^{+}$. For the final step, whenever there is a morphism in $Q$ between an object $q$ of $Q\left[i_{1}\right]$ and an object $p$ not in $Q\left[i_{1}\right]$, this morphism must be a morphism $p \rightarrow q$ of type (c). Thus $p$ is a copy of $T q$, and we add the canonical isomorphism $p=T q$ to $Q^{+}$. Now we can define $Q_{0} \subseteq Q^{+}$to be the full subquiver obtained by deleting objects in $Q\left[i_{1}\right]$. As we have checked in Lemma 6.3.7, the quivers $Q$ and $Q_{0}$ are equivalent,
and by construction $Q_{0}$ contains only objects with twist $i_{1}<i \leqslant i_{0}$ and degrees $n \leqslant n_{0}$. This proves the claim. Arguing by induction, we can continue the proof of the lemma under the assumption that $Q$ contains only objects with twist $i_{0}$.

Claim: Let $n_{1}$ be the smallest integer such that $Q\left[n_{1}, i_{0}\right]$ is non-empty, and suppose $n_{1}<n_{0}$. There exists a finite quiver $Q_{0}$ which is equivalent to $Q$ and such that, if $Q_{0}[n, i]$ is non-empty, then $i=i_{0}$ and $n_{1}<n \leqslant n_{0}$.

Let us denote by $Q\left[n_{1}\right]=Q\left[n_{1}, i_{0}\right]$ the full subquiver of $Q$ whose objects are those of degree $n_{1}$, and declare these to be the bad objects to be replaced. Given an object $q=\left[Y, Z, n_{1}, i_{0}\right]$ of $Q\left[n_{1}\right]$ let us denote by $H q$ and $\Sigma q$ the objects

$$
\begin{aligned}
H q & =\left[(Y \times\{0,1\}) \cup\left(Z \times \mathbb{A}^{1}\right),(Y \times\{0\}) \cup\left(Z \times \mathbb{A}^{1}\right), n_{1}, i_{0}\right] \\
\Sigma q & =\left[Y \times \mathbb{A}^{1},(Y \times\{0,1\}) \cup\left(Z \times \mathbb{A}^{1}\right), n_{1}+1, i_{0}\right]
\end{aligned}
$$

and let us write $\iota_{q}: H q \rightarrow q$ for the morphism of type (a), given by the inclusion of $Y=Y \times\{1\}$ into $Y \times\{0,1\} \cup\left(Z \times \mathbb{A}^{1}\right)$ and $\delta_{q}: H q \rightarrow \Sigma q$ for the unique morphism of type (b). The morphisms $\rho\left(\iota_{q}\right): \rho(H q) \rightarrow \rho(q)$ and $\rho\left(\delta_{q}\right): \rho(H q) \rightarrow \rho(\Sigma q)$ are isomorphisms, and their composite is the canonical isomorphism $H^{n_{1}}(Y, Z)\left(i_{0}\right) \cong H^{n_{1}+1}(\Sigma(Y, Z))\left(i_{0}\right)$ as we explained in 6.2.8. Let $Q^{+}$be the quiver, subject to further enlargement, obtained by adding to $Q$ the objects and morphisms

$$
L(q)=\left[\Sigma q \stackrel{\delta_{q}}{\longleftarrow} H q \xrightarrow{\iota_{q}} q\right]
$$

for $q \in Q\left[n_{1}\right]$. The construction of the objects $H q$ and $\Sigma q$ and morphisms $\delta_{q}$ and $\iota_{q}$ is in the obvious way functorial for morphisms $f: p \rightarrow q$ in $Q\left[n_{1}\right]$, which are all of type (a), and the following diagram of vector spaces corresponding to (6.3.6.1) in the abstract setting commutes:

$$
\begin{array}{rll}
\rho(\Sigma p) & \cong \rho(H p) & \cong \rho(p) \\
\rho(\Sigma f) \downarrow & \rho(H f) \downarrow & \rho(f) \downarrow \\
\rho(\Sigma q) & \cong \rho(H q) & \cong \rho(q)
\end{array}
$$

As for the second step in 6.3.6, add for every morphism $f$ in $Q\left[n_{1}\right]$ the morphism $\Sigma f$ to $Q^{+}$. For the third and final step, whenever there is a morphism in $Q$ between an object of $Q\left[n_{1}\right]$ and an object not in $Q\left[n_{1}\right]$, this morphism must be a morphism $q \rightarrow p$ of type (b), say

$$
d:\left[Y, Z, n_{1}, i_{0}\right] \rightarrow\left[X, Y, n_{1}+1, i_{0}\right]
$$

of type (b). Add then to $Q^{+}$the morphism $s^{-1} f: \Sigma q \rightarrow p$ of type (a) as given in (6.2.8.4). The commutative diagram (6.2.8.5) cast in different notation is the following commutative triangle:


This completes step 3 in 6.3 .6 , and hence proves the claim. Arguing by induction on the number of different degrees in $Q$ finishes the proof of the lemma.

Lemma 6.3.9. - Let $Q$ be a finite subquiver of $\mathrm{Q}_{\mathrm{h}}(k)$ and suppose that there exist integers $n_{0}$ and $i_{0}$ such that all objects in $Q$ are of degree $n_{0}$ and twist $i_{0}$. There exists a quiver $Q_{0}$ which is equivalent to $Q$ and consists of only one object $q_{0}$ and endomorphisms.

Proof. For notational convenience, we index objects of $Q$ by a finite set, $\operatorname{Obj}(Q)=\left(q_{\alpha}\right)_{\alpha \in A}$, and write $q_{\alpha}=\left[X_{\alpha}, Y_{\alpha}, n_{0}, i_{0}\right]$ for every $\alpha \in A$. Define $q_{0}=\left[X_{0}, Y_{0}, n_{0}, i_{0}\right]$ to be the object obtained from the pair of varieties

$$
\left(X_{0}, Y_{0}\right)=\bigvee_{\alpha \in A}\left(X_{\alpha}, Y_{\alpha}\right)
$$

and let us construct a quiver $Q^{+}$by adding to $Q$ the object $q_{0}$ and the following morphisms:
(1) For each $\alpha \in A$, the morphism $q_{0} \rightarrow q_{\alpha}$ given by the inclusion $\iota_{\alpha}: X_{\alpha} \rightarrow X_{0}$.
(2) For each $\alpha \in A$, the morphism $q_{\alpha} \rightarrow q_{0}$ given by the morphism $\pi_{\alpha}: X \rightarrow X_{\alpha}$ which is the identity on $X_{\alpha}$ and the zero map on all other components.
(3) For each morphism $h: q_{\alpha} \rightarrow q_{\beta}$ in $Q$ the endomorphism $q_{0} \rightarrow q_{0}$ given by the composite $\iota_{\beta} \circ h \circ \pi_{\alpha}$.

The vector space $\rho\left(q_{0}\right)$ is the direct sum

$$
\rho\left(q_{0}\right)=\bigoplus_{\alpha \in A} \rho\left(q_{\alpha}\right)
$$

the morphisms $\rho\left(\iota_{\alpha}\right): \rho\left(q_{0}\right) \rightarrow \rho\left(q_{\alpha}\right)$ are projections and the morphisms $\rho\left(\pi_{\alpha}\right): \rho\left(q_{\alpha}\right) \rightarrow \rho\left(q_{0}\right)$ are the inclusions. The endomorphisms of the object $q_{0}$ induce, besides the identity, the linear endomorphisms

$$
\rho\left(q_{0}\right) \xrightarrow{\text { proj. }} \rho\left(q_{\alpha}\right) \xrightarrow{\rho(h)} \rho\left(q_{\beta}\right) \xrightarrow{\text { incl. }} \rho\left(q_{0}\right)
$$

for every morphism $h$ of $Q$, in particular projectors are obtained from identity morphisms $\mathrm{id}_{q_{\alpha}}$. It is clear that to give an endomorphism of $\rho\left(q_{0}\right)$ which commutes with all these linear endomorphisms is the same as to give an endomorphism of the representation $\left.\rho\right|_{Q}: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$, or more precisely, that the algebra morphisms

$$
\operatorname{End}\left(\left.\rho\right|_{Q}\right) \leftarrow \operatorname{End}\left(\left.\rho\right|_{Q^{+}}\right) \rightarrow \operatorname{End}\left(\left.\rho\right|_{Q_{0}}\right)
$$

are isomorphisms, which is what we wanted to show.

The conjunction of the statement of Lemma 6.3.8 and Lemma 6.3.9 yields Proposition 6.3.4, stating that every finite subquiver of $\mathrm{Q}_{\mathrm{h}}(k)$ is equivalent to a quiver containing only one object. We now follow the outline 6.3 .5 towards the proof of our main theorem.

### 6.4. The subquotient question for exponential motives

Theorem 6.4.1. - Let $M$ be an exponential motive.
(1) There exists an exponential motive of the form $M_{1}=H^{n_{1}}\left(X_{1}, Y_{1}, f_{1}\right)\left(i_{1}\right)$ and a surjective morphism $M_{1} \rightarrow M$.
(2) There exists an exponential motive of the form $M_{2}=H^{n_{2}}\left(X_{2}, Y_{2}, f_{2}\right)\left(i_{2}\right)$ and an injective morphism $M \rightarrow M_{2}$.

### 6.5. The theorem of the fixed part

The inclusion $\mathbf{M}(k) \rightarrow \mathbf{M}^{\exp }(k)$ of the category of ordinary motives into the category of exponential motives has a right adjoint

$$
\Gamma: \mathbf{M}^{\exp }(k) \longrightarrow \mathbf{M}(k)
$$

associating with an exponential motive $M$ the largest ordinary submotive $M_{0} \subseteq M$. The functor $\Gamma$ is left exact. Similarly, the inclusion of the category of vector spaces into $\mathbf{P e r v}_{0}$ has a right adjoint

$$
\Gamma: \operatorname{Perv}_{0} \longrightarrow \operatorname{Vec}_{\mathbb{Q}}
$$

associating with an object $V$ the largest constant subobject $V_{0}$, that is, invariants under the tannakian fundamental group. Explicitly, if $j: \mathbb{G}_{m} \hookrightarrow \mathbb{A}^{1}$ denotes the inclusion and $\pi: \mathbb{G}_{m} \rightarrow \operatorname{Spec}(\mathbb{C})$ for the structure morphism, this largest constant subobject is given by

$$
V_{0}=j!\pi^{*} \pi_{*} j^{*} V \hookrightarrow V
$$

The arrow above is induced by adjunction, on noting that $j^{*}=j^{!}$since $j$ is an open immersion.
The perverse realisation of $M_{0}$ is contained in the invariants of the perverse realisation of $M$, hence a natural, injective map $\tau_{M}: R_{\mathrm{B}}(\Gamma(M)) \rightarrow \Gamma\left(R_{\text {perv }}(M)\right)$. We can regard this map as a morphism of functors $\tau: R_{\mathrm{B}} \circ \Gamma \rightarrow \Gamma \circ R_{\text {perv }}$ from $\mathbf{M}^{\exp }(k)$ to $\mathbf{V e c}_{\mathbb{Q}}$. The theorem of the fixed part states that $\tau$ is an isomorphism, so the square diagram of categories and functors

commutes up to an isomorphism of functors $\tau$. This amounts to showing that the map $\tau_{M}$ is surjective for all exponential motives $M$, which is the content of the following theorem:

TheOrem 6.5.1. - Let $M$ be an exponential motive with perverse realisation $V$, and denote by $V_{0} \subseteq V$ the largest trivial subobject of $V$. There exists an ordinary motive $M_{0}$ and an injection $M_{0} \rightarrow M$ such that the image of the perverse realisation of $M_{0}$ in $V$ is equal to $V_{0}$.
6.5.2 (Caveat). - We will show in a first step that the statement of Theorem 6.5.1 holds for exponential motives of the form $M=H^{n}(X, Y, f)(i)$. The theorem in its full generality does not follow from this particular case. We know that every exponential motive is isomorphic to a
subquotient of a sum of motives of that particular shape. It is also easy to see that if the statement of Theorem 6.5.1 holds for an exponential motive $M$, then it holds for every subobject of $M$, and if the statement holds for $M_{1}$ and $M_{2}$, then it holds for $M_{1} \oplus M_{2}$. Quotients are the problemthere is no easy relation between the largest ordinary submotive $M_{0}$ of $M$ and the largest ordinary submotive of a quotient of $M$. Duals won't help. The problem is taken care of by Theorem 6.4.1.

Lemma 6.5.3. - Let $X$ be an algebraic variety over $k$, let $Y \subseteq X$ be a closed subvariety and let $f: X \rightarrow \mathbb{A}^{1}$ be a regular function. Define $g: X \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ to be the function $g(z, x)=z f(x)$, where $z$ stands for the coordinate of $\mathbb{A}^{1}$. The exponential motive

$$
H^{n}\left(X \times \mathbb{A}^{1}, Y \times \mathbb{A}^{1}, g\right)(i)
$$

is isomorphic to a conventional Nori motive.
Proof. We can ignore the twist. According to Lemma 4.2.5, there is a canonical isomorphism of motives

$$
M=H^{n}\left(X \times \mathbb{A}^{1}, Y \times \mathbb{A}^{1}, g\right) \xrightarrow{\cong} H^{n+1}\left(X \times \mathbb{A}^{2},\{(x, z, t) \mid x \in Y \text { or } z f(x)=t\}, p\right)
$$

where $p$ is the projection given by $p(x, z, t)=t$. We fabricate an ordinary motive $M_{0}$ from the fibre of $p$ over 1 , setting

$$
M_{0}=H^{n}\left(X \times \mathbb{A}^{1},\{(x, z) \mid x \in Y \text { or } z f(x)=1\}, 0\right)
$$

Lemma 4.2.5 applied to the motive $M_{0}$ yields the isomorphism

$$
M_{0} \cong H^{n+1}\left(X \times \mathbb{A}^{2},\{(x, z, t) \mid x \in Y \text { or } z f(x)=1 \text { or } t=0\}, p\right)
$$

With this presentation of $M_{0}$, a morphism $\varphi^{*}: M \rightarrow M_{0}$ is given by the map

$$
\begin{aligned}
X \times \mathbb{A}^{2} & \xrightarrow{\varphi} X \times \mathbb{A}^{2} \\
\{(x, z, t) \mid x \in Y \text { or } z f(x)=1 \text { or } t=0\} & \longrightarrow\{(x, z, t) \mid x \in Y \text { or } z f(x)=t\}
\end{aligned}
$$

given by $\varphi(x, z, t)=(x, z t, t)$. We claim that $\varphi^{*}: M \rightarrow M_{0}$ is an isomorphism. This can be checked on perverse realisations.

REmARK 6.5.4. - Let us keep the notation from above but assume moreover that $X$ is smooth and that the zero locus of $f$ is a smooth subvariety $Z \subseteq X$. Let $U$ denote its complement. Then one can see that $H^{n}\left(X \times \mathbb{A}^{1}, g\right)$ is an ordinary motive as follows: the Gysin long exact sequence of motives (Theorem 4.8.3) associated with the smooth divisor $Z \times \mathbb{A}^{1}$ of $X \times \mathbb{A}^{1}$ reads

$$
H^{n-1}\left(U \times \mathbb{A}^{1},\left.g\right|_{U \times \mathbb{A}^{1}}\right) \longrightarrow H^{n-2}\left(Z \times \mathbb{A}^{1}\right)(-1) \longrightarrow H^{n}\left(X \times \mathbb{A}^{1}, g\right) \longrightarrow H^{n}\left(U \times \mathbb{A}^{1},\left.g\right|_{U \times \mathbb{A}^{1}}\right)
$$

We claim that $H^{m}\left(U \times \mathbb{A}^{1},\left.g\right|_{U \times \mathbb{A}^{1}}\right)=0$ in all degrees $m$. Indeed, since the function $f$ is invertible on $U$, the map $h: U \times \mathbb{A}^{1} \rightarrow U \times \mathbb{A}^{1}$ sending $(x, z)$ to $(x, z f(x))$ is an isomorphism compatible with the function $g$ on the source and the function $0 \boxplus \mathrm{id}$ on the target. Therefore,

$$
H^{i}\left(U \times \mathbb{A}^{1},\left.g\right|_{U \times \mathbb{A}^{1}}\right) \simeq H^{i}\left(U \times \mathbb{A}^{1}, 0 \boxplus \mathrm{id}\right)=0
$$

by the Künneth formula and the vanishing of $H^{*}\left(\mathbb{A}^{1}, i d\right)$. We thus have an isomorphism

$$
H^{n}\left(X \times \mathbb{A}^{1}, g\right) \simeq H^{n-2}(Z)(-1)
$$

Using resolution of singularities, this observation can be used to give an alternative proof of Lemma 6.5.3.

Proof of Theorem 6.5.1. In a first step, we prove the theorem in the case where $M$ is of the form $M=H^{n}(X, Y, f)(i)$. Without loss of generality we suppose $i=0$ and suppress the twist from the notation. We give a geometric construction of a morphism $\varphi: M_{0} \rightarrow M$ with the required properties. We do not care whether this morphism is injective, because we can always render $\varphi$ injective by replacing $M_{0}$ by $\varphi\left(M_{0}\right)$, and by Proposition 5.1.3 $\varphi\left(M_{0}\right)$ is still ordinary. Set $M_{0}=H^{n}\left(X \times \mathbb{A}^{1}, Y \times \mathbb{A}^{1}, g\right)$ with $g(x, z)=z f(x)$, and define

$$
M_{0}=H^{n}\left(X \times \mathbb{A}^{1}, Y \times \mathbb{A}^{1}, g\right) \xrightarrow{\varphi} H^{n}(X, Y, f)=M
$$

to be the morphism induced by the inclusion $x \longmapsto(x, 1)$ of $X$ into $X \times \mathbb{A}^{1}$. By Lemma 6.5.3 we know that $M_{0}$ is an ordinary motive. Let $\varphi: V_{0} \rightarrow V$ be the perverse realisation of $\varphi: M_{0} \rightarrow M$. We have to prove that the image of $\varphi: V_{0} \rightarrow V$ is the largest trivial subobject of $V$. This amounts to showing that the inclusion $x \longmapsto(x, 1)$ induces an isomorphism

$$
R m_{*}\left(\Pi R f_{*} \underline{\mathbb{Q}}_{[X, Y]} \boxtimes \underline{\mathbb{Q}}_{\mathbb{A}^{1}}\right) \cong \Pi R g_{*} \operatorname{pr}_{X}^{*} \underline{\mathbb{Q}}_{X, Y} \rightarrow j!\pi^{*} R \pi_{*} j^{*}\left(\Pi R f_{*} \underline{\mathbb{Q}}_{[X, Y]}\right)
$$

in $\operatorname{Perv}_{0}$, where $j: \mathbb{A}^{1} \backslash\{0\} \rightarrow \mathbb{A}^{1}$ is the inclusion and $m: \mathbb{A}^{1} \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ is the multiplication map. Indeed, for any object $F$ of $\mathbf{P e r v}_{0}$, there is a canonical and natural isomorphism

$$
R m_{*}\left(F \boxtimes \mathbb{Q}_{\mathbb{A}^{1}}\right) \cong j!\pi^{*} R \pi_{*} j^{*} F
$$



ThEOREM 6.5.5. - The canonical functor $\iota: \mathbf{M}(k) \rightarrow \mathbf{M}^{\exp }(k)$ and the perverse realisation $R_{\text {perv }}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{P e r v}_{0}$ induce an exact sequence

$$
\begin{equation*}
\pi_{1}\left(\operatorname{Perv}_{0}\right) \xrightarrow{R_{\text {perv }}^{*}} \pi_{1}\left(\mathbf{M}^{\exp }(k)\right) \xrightarrow{\iota^{*}} \pi_{1}(\mathbf{M}(k)) \longrightarrow 1 \tag{6.5.5.1}
\end{equation*}
$$

of affine group schemes over $\mathbb{Q}$.

Remark 6.5.6. - The morphism $\pi_{1}\left(\mathbf{P e r v}_{0}\right) \rightarrow \pi_{1}\left(\mathbf{M}^{\exp }(k)\right)$ is not a closed immersion since there are objects in Perv ${ }_{0}$ which are not isomorphic to a subquotient of the perverse realisation of an exponential motive. For example, any object of $\operatorname{Perv}_{0}$ whose restriction to an open of $\mathbb{C}$ is a local system with non-quasi-unipotent local monodromies at infinity has this property.

Proof of Theorem 6.5.5. The proof of Theorem 6.5.5 relies on a general exactness criterion for fundamental groups of tannakian categories. We have already seen in Corollary 5.1.4 that the morphism of affine group schemes $\pi_{1}\left(\mathbf{M}^{\exp }(k)\right) \longrightarrow \pi_{1}(\mathbf{M}(k))$ is faithfully flat. The composite
$\iota^{*} \circ R_{\text {perv }}^{*}$ is the trivial morphism, because $R_{\text {perv }}(\iota M)$ is isomorphic to a sum of copies of the unit object of $\mathbf{P e r v}_{0}$ for every classical Nori motive $M$. In order to show that the sequence (6.5.5.1) is exact in the middle, we can without loss of generality assume that $k$ is algebraically closed. Indeed, if $\bar{k}$ denotes the algebraic closure of $k$ in $\mathbb{C}$, the commutative diagram of affine group schemes

has exact rows and columns.
morphism

$$
R_{\text {perv }}^{*}: \pi_{1}\left(\mathbf{P e r v}_{0}\right) \rightarrow \operatorname{ker}\left(\iota^{*}\right)
$$

is surjective. To this end, we verify conditions (1) and (2) of Proposition A.3.4, starting with (1). Let $M$ be an exponential motive, corresponding to a representation $V$ of the fundamental group $\pi_{1}\left(\mathbf{M}^{\exp }(k)\right)$. Taking the invariants of $V$ under the action of $\pi_{1}\left(\mathbf{P e r v}_{0}\right)$ amounts to taking the largest trivial subobject of $R_{\text {perv }}(M)$ in $\operatorname{Perv}_{0}$. Taking invariants under the action of ker $\left(\iota^{*}\right)$ amounts to extracting the largest ordinaty submotive of $M$. Hence, the sought after equality

$$
V^{\pi_{1}\left(\operatorname{Perv}_{0}\right)}=V^{\operatorname{ker}\left(\iota^{*}\right)}
$$

is but the statement of Theorem 6.5.1. In order to verify condition (2), recall from Corollary 2.9.16 that one-dimensional objects of $\operatorname{Perv}_{0}$ are of the form $\left(j_{z}\right)!F$ where $j: \mathbb{C} \backslash\{z\} \rightarrow \mathbb{C}$ is the inclusion, and $F$ a one-dimensional local system on $\mathbb{C} \backslash\{z\}$ which either constant or has monodromy ( -1 ) around $z$. If $\left(j_{z}\right)!F$ is a subquotient of an object of the

### 6.6. Applications of Gabber's torus trick

Right at the beginning of the book [55], Katz lists several fairly general-yet extremely powerfulresults from representation theory which later on become the main tools to determine monodromy groups. It is not surprising that these results are useful to understand the fundamental groups in the tannakian category $\left(\mathbf{P e r v}_{0}, \Phi\right)$. As we shall see later (where?), there is a direct link between the monodromy groups of differential equations computed in [55] and the tannakian fundamental groups of objects of $\mathbf{P e r v}_{0}$.
6.6.1 (Results from representation theory). -

Theorem 6.6.2 (Gabber's torus trick, [55, Theorem 1.0]). - Let $\mathfrak{g} \subseteq \mathfrak{g l}{ }_{n}$ be a semisimple Lie algebra acting irreducibly on $\mathbb{C}^{n}$. Let $K$ be a torus and let $\chi_{1}, \ldots, \chi_{n}: K \rightarrow \mathbb{G}_{m}$ be characters
of $K$ corresponding to a homomorphism $\chi: K \rightarrow \mathrm{GL}_{n}$ to the diagonal of $\mathrm{GL}_{n}$. Suppose that the conjugation action of $K$ on $\mathfrak{g l}_{n}$ leaves $\mathfrak{g}$ invariant. Let $\mathfrak{t} \subseteq \mathfrak{g l}_{n}$ be the subspace of those diagonal matrices whose diagonal entries $t_{1}, \ldots, t_{n}$ satisfy
(1) $t_{1}+t_{2}+\cdots+t_{n}=0$
(2) $t_{i}+t_{j}=t_{k}+t_{m}$ whenever $\chi_{i} \chi_{j}=\chi_{k} \chi_{m}$.

Then $\mathfrak{t}$ is contained in $\mathfrak{g}$.

THEOREM 6.6.3 (Kostant). - Let $\mathfrak{g} \subseteq \mathfrak{g l}_{n}$ be a semisimple Lie algebra acting irreducibly on $\mathbb{C}^{n}$. If $\mathfrak{g}$ contains the diagonal matrix $\operatorname{diag}(n-1,-1, \ldots,-1)$, then $\mathfrak{g}$ is $\mathfrak{s l}_{n}$.

Proposition 6.6.4. - Let $A$ be a Lie-simple object of $\mathbf{P e r v}_{0}$ of rank $n$ whose set of singularities $S$ has cardinality $n$ and whose determinant has finite monodromy. Suppose that for all elements (not necessarily distinct) $a, b, c, d \in S$, the relation $a+b=c+d$ implies $\{a, b\}=\{c, d\}$. Then, the Lie algebra of the tannakian fundamental group of $A$ contains $\mathfrak{s l}_{n}$. It is equal to $\mathfrak{s l}_{n}$ if and only if the singularities of $A$ sum to zero.

Proof. Let us enumerate the singularities of $A$ as $S=\left\{s_{1}, \ldots, s_{n}\right\}$ and set

$$
V=\Phi(A)=\bigoplus_{s \in S} \Phi_{s}(A)
$$

Since $S$ contains as many elements as the dimension of $V$, each space of vanishing cycles $\Phi_{s}(A)$ is one-dimensional. Let us choose a basis $e_{1}, \ldots, e_{n}$ of $V$ adapted to this decomposition and identify the fundamental group $G$ of $A$ with a subgroup of $\mathrm{GL}_{n}$ through this choice of basis.

Set $r=s_{1}+\cdots s_{n}$. The determinant of $A^{\prime}=A \otimes E(-r / n)$ is a rank-one object with singularity at 0 and finite monodromy, and hence has finite fundamental group by Lemma 2.9.2. Since the total vanishing cycles of $A^{\prime}$ are $V \otimes \Phi_{-r / n}(E(-r / n))$, the fundamental group $G^{\prime}$ of $A^{\prime}$ can be canonically identified with a subgroup of $\mathrm{GL}_{V}=\mathrm{GL}_{n}$. Its connected component of the identity is then contained in $\mathrm{SL}_{n}$, and the equality

$$
G= \begin{cases}G^{\prime} \cdot \mathbb{G}_{m} & \text { if } r \neq 0 \\ G^{\prime} & \text { if } r=0\end{cases}
$$

holds. The object $A^{\prime}$ is still Lie-simple, so we may from now on assume that $r=0$ and that the Lie algebra $\mathfrak{g}$ of $G$ is contained in $\mathfrak{s l}_{n}$. By assumption, the standard $n$-dimensional representation of $\mathfrak{g}$ is simple, and hence $\mathfrak{g}$ is a semisimple Lie algebra (ref?).

The torus of singularities of $A$ is the torus $K=\underline{\operatorname{Hom}}\left(\mathbb{Z}[S], \mathbb{G}_{m}\right)$ with character group $\mathbb{Z}[S]$, seen inside $G \subseteq \mathrm{GL}_{n}$ as the subgroup of diagonal matrices $\operatorname{diag}\left(\varphi\left(s_{1}\right), \ldots, \varphi\left(s_{n}\right)\right)$ for $\varphi \in K$. Write $\chi_{i}: K \rightarrow \mathbb{G}_{m}$ for the character defined by $\chi_{i}(\varphi)=\varphi\left(s_{i}\right)$. Since $K$ is contained in $G$, the conjugation action of $K$ on $\mathfrak{g l}_{n}$ leaves $\mathfrak{g}$ invariant. By the assumption on the singularities of $A$, the relation $\chi_{i} \chi_{j}^{-1}=\chi_{k} \chi_{m}^{-1}$ implies $\{i, m\}=\{j, k\}$. Therefore, the Lie algebra $\mathfrak{g}$ contains the subspace of diagonal matrices of trace zero, and in particular the matrix $\operatorname{diag}(n-1,-1, \ldots,-1)$ by Theorem 6.6.2, and hence $\mathfrak{g}=\mathfrak{s l}_{n}$ by Konstant's Theorem 6.6.3.
6.6.5 (The generic Galois group of the motive associated with a polynomial). - As a first application of Proposition 6.6.4, we show that the Galois group of the exponential motive $H^{1}\left(\mathbb{A}^{1}, f\right)$ associated with a generic polynomial $f$ is as large as possible.

Theorem 6.6.6. - Let $f \in k[x]$ be a polynomial of degree $n+1$. Assume that the following two conditions hold:
(i) The derivative $f^{\prime}$ has no multiple roots.
(ii) Given four roots $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}$ of $f^{\prime}$ in $\mathbb{C}$, not necessarily distinct, the equality of complex numbers $f\left(\alpha_{1}\right)+f\left(\alpha_{2}\right)=f\left(\alpha_{3}\right)+f\left(\alpha_{4}\right)$ implies $\left\{\alpha_{1}, \alpha_{2}\right\}=\left\{\alpha_{3}, \alpha_{4}\right\}$.

Then the motivic Galois group of $H^{1}\left(\mathbb{A}^{1}, f\right)$ equals $\mathrm{GL}_{n}$.
Proof. The perverse realisation

$$
A=R_{\operatorname{perv}}\left(H^{1}\left(\mathbb{A}^{1}, f\right)\right)=\left(f_{*} \mathbb{Q} / \mathbb{Q}\right)[1]
$$

of the motive $H^{1}\left(\mathbb{A}^{1}, f\right)$ has dimension $n$. The set of singularities of $A$ is the set $S=\left\{f(\alpha) \mid f^{\prime}(\alpha)=\right.$ $0\}$ of critical values of $f$. Conditions (i) and (ii) imply that $S$ contains $n$ elements.

Proposition 6.6 .4 shows that the Galois group $A$ contains $\mathrm{SL}_{n}$, hence also the motivic Galois group of $H^{1}\left(\mathbb{A}^{1}, f\right)$ contains $\mathrm{SL}_{n}$.

Example 6.6.7. - We end this section with a complete classification of the Galois groups of perverse sheaves $A=H^{1}\left(\mathbb{A}^{1}, f\right)$ for polynomials $f$ of degree 2 and 3 . If $f$ is of degree 2 , then $A$ has dimension 1 and hence the Galois group $G$ of $A$ is a subgroup of $\mathbb{G}_{m}=\mathrm{GL}_{1}$. It is given by

$$
G= \begin{cases}\mathbb{G}_{m} & \text { if the critical value of } f \text { is non-zero } \\ \{ \pm 1\} & \text { if the critical value of } f \text { is zero }\end{cases}
$$

Suppose now that $f$ is of degree 3 , with critical values $s_{1}$ and $s_{2}$. The object $A$ is of dimension 2 , and $G$ a subgroup of $\mathrm{GL}_{2}$.

$$
G= \begin{cases}\mathrm{GL}_{2} & \text { if } s_{1} \neq s_{2} \text { and } s_{1}+s_{2} \neq 0 \\ \mathrm{SL}_{2} & \text { if } s_{1} \neq s_{2} \text { and } s_{1}+s_{2}=0 \\ \mathbb{Z} / 3 \mathbb{Z} \cdot \mathbb{G}_{m} & \text { if } s_{1}=s_{2} \text { and } s_{1}+s_{2} \neq 0 \\ \mathbb{Z} / 3 \mathbb{Z} & \text { if } s_{1}=s_{2}=0\end{cases}
$$

The group $\mathbb{Z} / 3 \mathbb{Z} \subseteq \mathrm{GL}_{2}$ is, up to conjugation, generated by the matrix $\left(\begin{array}{ll}0 & -1 \\ 1 & -1\end{array}\right)$ and $\mathbb{G}_{m} \subseteq \mathrm{GL}_{2}$ is the group of scalar matrices.

Set $r=s_{1}+s_{2}$. In the proof of Proposition 6.6.4 we have seen that it suffices to treat the cases where $r=0$, to which we come by replacing $f$ by $f-\frac{r}{2}$. If $s_{1}$ and $s_{2}$ are distinct, then Proposition 6.6 .4 shows that $G$ contains $\mathrm{SL}_{2}$. On the other hand, $\operatorname{det}(A)$ is the trivial object, hence $G=\mathrm{SL}_{2}$.

If $s_{1}=s_{2}=0$, then $A=j_{!} F[1]$ where $F$ is the local system of rank 2 on $\mathbb{C}^{\times}$with finite monodromy $\mathbb{Z} / 3 \mathbb{Z}$.

## CHAPTER 7

## The comparison isomorphism revisited

Let $X$ be a variety defined over a field $k \subseteq \mathbb{C}$, let $Y \subseteq X$ be a closed subvariety, and let $f$ be a regular function on $X$. In this chapter, we introduce the de Rham cohomology $H_{\mathrm{dR}}^{n}(X, Y, f)$ and define a period pairing

$$
\begin{equation*}
H_{n}^{\mathrm{rd}}(X, Y, f) \times H_{\mathrm{dR}}^{n}(X, Y, f) \longrightarrow \mathbb{C} \tag{7.0.0.1}
\end{equation*}
$$

of which we have already given examples in the introduction using the elementary point of view on rapid decay homology. In the case where $f=0$, this pairing is the same as the usual period pairing between singular homology and de Rham cohomology. Neither rapid decay homology nor de Rham cohomology changes when we replace $f$ by $f+c$ for some constant $c$. The period pairing will change! The main result of this section is Theorem 7.6 .1 which states that the period pairing (7.0.0.1) is non-degenerate, in the sense that it induces an isomorphism of complex vector spaces

$$
\begin{equation*}
H_{\mathrm{dR}}^{n}(X, Y, f) \otimes_{k} \mathbb{C} \xrightarrow{\cong} H_{\mathrm{rd}}^{n}(X, Y, f) \otimes_{\mathbb{Q}} \mathbb{C} . \tag{7.0.0.2}
\end{equation*}
$$

which we call comparison isomorphism. The result in this form, at least in the essential case when $X$ is smooth and $Y$ empty, is due to Hien and Roucairol, see [45, Theorem 2.7]. A notable earlier version by Esnault and Bloch [12] deals with arbitrary irregular connections on a curve. The overall structure of the proof, of which we give a simplified and self-contained version here, is similar to that of the classical proof in the case where $f=0$. It relies on a Poincaré Lemma which we state as Theorem 7.5.11, and a GAGA argument which permits to compare algebraic with analytic de Rham cohomology.

### 7.1. Algebraic de Rham cohomology of varieties with a potential

In this section, we introduce algebraic de Rham cohomology of pairs of varieties with a potential. In the case of smooth varieties, the definition is straightforward and was already given in (1.1.1.3) in the introduction. As for ordinary de Rham cohomology, there are several ways of extending it to singular varieties, which all lead to the same result [48, Chapter 3]. Following Deligne, we adopt here the point of view of hypercoverings. Throughout, $k$ denotes a field of characteristic zero, all varieties and morphisms are tacitly supposed to be defined over $k$, and we write $\mathbb{A}^{1}$ for the affine line Spec $k[x]$.
7.1.1. - Let $X$ be a smooth algebraic variety and $f: X \rightarrow \mathbb{A}^{1}$ a regular function. We denote by $d_{f}$ the integrable connection on the rank one trivial vector bundle $\mathcal{O}_{X}$ defined by $d_{f}(1)=-d f$
or, equivalently, by

$$
\begin{aligned}
d_{f}: \mathcal{O}_{X} & \longrightarrow \Omega_{X}^{1} \\
g & \longmapsto d_{f}(g)=d g-g d f
\end{aligned}
$$

on local sections $g$ of $\mathcal{O}_{X}$. Note that $d_{f}$ depends only on $d f$, and agrees with the usual exterior derivative whenever $f$ is constant. The connection $d_{f}: \mathcal{O}_{X} \rightarrow \Omega_{X}^{1}$ canonically extends to a complex

$$
\mathcal{O}_{X} \xrightarrow{d_{f}} \Omega_{X}^{1} \xrightarrow{d_{f}} \cdots \xrightarrow{d_{f}} \Omega_{X}^{\operatorname{dim} X},
$$

where $d_{f}: \Omega_{X}^{p} \rightarrow \Omega_{X}^{p+1}$ is given by $d_{f}(\omega)=d \omega-d f \wedge \omega$ on local sections $\omega$ of $\Omega_{X}^{p}$.

Definition 7.1.2. - Let $X$ be a smooth variety. The de Rham cohomology of the pair ( $X, f$ ) is the Zariski cohomology of the complex $\operatorname{DR}\left(\mathcal{E}^{f}\right)$. It will be denoted it by

$$
H_{\mathrm{dR}}^{n}(X, f)=H^{n}\left(X,\left(\Omega_{X}^{\bullet}, d_{f}\right)\right) .
$$

7.1.3. - If $f$ is constant, we recover from Definition 7.1.2 the usual algebraic de Rham cohomology of $X$. In general, there is a spectral sequence

$$
E_{1}^{p, q}=H^{p}\left(X, \Omega_{X}^{q}\right) \Longrightarrow H_{\mathrm{dR}}^{p+q}(X, f)
$$

which degenerates at the second page if $X$ is affine, since coherent sheaves on affine varieties have trivial cohomology in degrees $>0$. Therefore, if $X$ is affine, the de Rham cohomology of $(X, f)$ is the cohomology of the complex of global section $\left(\Omega_{X}^{\bullet}(X), d_{f}\right)$. For not necessarily affine $X$, this yields a way to compute $H_{\mathrm{dR}}^{*}(X, f)$ as follows. Given a covering $\left(U_{i}\right)_{i \in I}$ of $X$ by open affine subvarieties, define $X_{n}$ for integers $n \geqslant 0$ to be the disjoint union of the opens $U_{\sigma(0)} \cap \cdots \cap U_{\sigma(n)}$ as $\sigma$ ranges over all maps $\sigma:[n] \rightarrow I$. Together with the inclusions obtained from non-decreasing maps $[m] \rightarrow[n]$, the $X_{n}$ form a simplicial scheme $X_{\bullet}$. Denote by $f_{n}$ the restriction of $f$ to each component of $X_{n}$. We obtain a double complex of vector spaces

whose vertical differentials are defined to be alternating sums of face maps $X_{n+1} \rightarrow X_{n}$. The associated total complex computes the de Rham cohomology $H^{n}(X, f)$. This total complex depends naturally on the chosen affine covering, but not in a serious way. To get rid of any dependence on the affine covering, we take the limit over all affine coverings, and denote the resulting total complex by $R \Gamma\left(X,\left(\Omega_{X}^{\bullet}, d_{f}\right)\right)$. This complex depends functorially on the pair $(X, f)$. Seen as an object in the derived category of vector spaces, it is the object obtained by applying the derived functor $R \Gamma(X,-)$ to the object $\left(\Omega_{X}^{\bullet}, d_{f}\right)$ in the bounded derived category of coherent sheaves on
$X$, so there is no conflict in notation. Yet, we prefer for the moment the point of view where $R \Gamma\left(X,\left(\Omega_{X}^{\bullet}, d_{f}\right)\right)$ is an object in the category of complexes of vector spaces.
7.1.4. - Let $X$ be a possibly singular variety, together with a regular function $f: X \rightarrow \mathbb{A}^{1}$. Let $X_{\bullet} \rightarrow X$ be a smooth proper hypercovering of $X$ and let $f_{n}$ be the function induced on each $X_{n}$. Recall from Section 4.5 that this means that $X_{\bullet}$ is a simplicial variety where each $X_{n}$ smooth, and that the augmentation $X_{0} \rightarrow X$ as well as the adjunction morphisms

$$
X_{n} \rightarrow\left(\operatorname{cosk}_{n-1}\left(\mathrm{sk}_{n-1} X_{\bullet}\right)_{n}\right.
$$

are proper. Such hypercoverings exist thanks to resolution of singularities, a construction is sketched in $[\mathbf{2 2}, \S 6.2]$. We say that $\left(X_{\bullet}, f_{\bullet}\right)$ is a smooth proper hypercovering of $(X, f)$. Each face $\delta_{i}: X_{n+1} \rightarrow$ $X_{n}$ induces by functoriality morphisms of coherent sheaves $\delta_{i}^{*} \Omega_{X_{n}}^{p} \rightarrow \Omega_{X_{n+1}}^{p}$ that commute with $d_{f}$. Together, these sheaves form a complex of sheaves $\left(\Omega_{X_{\bullet}}^{\bullet}, d_{f_{\bullet}}\right)$ on the simplicial scheme $X_{\bullet}$. Face maps induce morphisms of complexes

$$
\delta_{i}^{*}: R \Gamma\left(X_{n},\left(\Omega_{X_{n}}^{\bullet}, d_{f}\right)\right) \rightarrow R \Gamma\left(X_{n-1},\left(\Omega_{X_{n+1}}^{\bullet}, d_{f}\right)\right)
$$

By considering the alternating sums of these morphisms, we obtain a double complex in the category of $k$-vector spaces. We denote by $R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, f_{\bullet}\right)$ the associated total complex. Let $Y \subseteq X$ be a closed subvariety and let $Y_{\bullet}$ be a smooth proper hypercovering of $Y$ mapping to $X_{\bullet}$, compatible with the inclusion $Y \subseteq X$. Again, such a hypercovering exists as explained in [22, §6.2]. The complex of vector spaces

$$
\begin{equation*}
R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)=\operatorname{cone}\left(R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, f_{\bullet}\right) \rightarrow R \Gamma_{\mathrm{dR}}\left(Y_{\bullet},\left(\left.f\right|_{Y}\right)_{\bullet}\right)\right) \tag{7.1.4.1}
\end{equation*}
$$

depends functorially on the triple $\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)$. It is a consequence of Lemma 7.1.6 below that, once we regard this complex as an object in the derived category of vector spaces, it only depends on $(X, Y, f)$ up to a unique isomorphism, and not on the chosen hypercoverings. Therefore we can use it to define the de Rham cohomology of $(X, Y, f)$.
7.1.5. - The construction of the complex $R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)$ is compatible with extensions of scalars in the following sense. Let $k^{\prime}$ a field extension of $k$ and set $X_{\bullet}^{\prime}=X_{\bullet} \times{ }_{\operatorname{Spec} k} \operatorname{Spec} k^{\prime}$, and similarly for $Y_{\bullet}$ and $f_{\bullet}$. There is a canonical and natural isomorphism

$$
\begin{equation*}
R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right) \otimes_{k} k^{\prime}=R \Gamma_{\mathrm{dR}}\left(X_{\bullet}^{\prime}, Y_{\bullet}^{\prime}, f_{\bullet}^{\prime}\right) \tag{7.1.5.1}
\end{equation*}
$$

in the derived category of $k^{\prime}$-vector spaces. Indeed, if $X$ is smooth and affine, there is a canonical and natural isomorphism

$$
\Omega_{X}^{\bullet}(X) \otimes_{k} k^{\prime}=\Omega_{X}^{\bullet}\left(X^{\prime}\right)
$$

of complexes of $k^{\prime}$-vector spaces by definition of base change and universal property of differentials. This isomorphism induces along all construction steps the isomorphism (7.1.5.1).

Lemma 7.1.6. - Let $X$ be a variety together with a regular function $f$. Let $\left(X_{\bullet}, f_{\bullet}\right) \rightarrow(X, f)$ and $\left(X_{\bullet}^{\prime}, f_{\bullet}^{\prime}\right) \rightarrow(X, f)$ be smooth proper hypercoverings of $X$, and let $h:\left(X_{\bullet}, f_{\bullet}\right) \rightarrow\left(X_{\bullet}^{\prime}, f_{\bullet}^{\prime}\right)$ be a
morphism of hypercoverings. The morphism of complexes of $k$-vector spaces induced by $h$

$$
\begin{equation*}
h^{*}: R \Gamma_{\mathrm{dR}}\left(X_{\bullet}^{\prime}, f_{\bullet}\right) \rightarrow R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, f_{\bullet}\right) \tag{7.1.6.1}
\end{equation*}
$$

is a quasi-isomorphism, and is independent of $h$ up to homotopy. In other words, the class of $h^{*}$ in the derived category of $k$-vector spaces is independent of $h$.

Proof. This follows from the work of Du Bois [31] - let us explain how. By compatibility with extension of scalars and the Lefschetz principle, can without loss of generality assume that $k=\mathbb{C}$. Let us write $\varepsilon: X_{\bullet} \rightarrow X$ and $\varepsilon^{\prime}: X_{\bullet}^{\prime} \rightarrow X$ for the augmentation morphisms. Theorem 3.11 of [31] states that, for $k=\mathbb{C}$, the natural morphism

$$
\begin{equation*}
R \varepsilon_{*}^{\prime}\left(\Omega_{X_{\bullet}^{\prime}}^{p}\right) \rightarrow R \varepsilon_{*}\left(\Omega_{X_{\bullet}}^{p}\right) \tag{7.1.6.2}
\end{equation*}
$$

induced by $h$ is an isomorphism in the derived category of coherent sheaves on $X$. This is a highly non-formal result, which is ultimately proven in the case of proper varieties using Hodge theory and induction on the dimension of $X$. By dévissage, it follows that the morphism

$$
\begin{equation*}
h^{*}: R \varepsilon_{*}^{\prime}\left(\Omega_{X_{\bullet}^{\prime}}^{\bullet}, d_{f}\right) \rightarrow R \varepsilon_{*}\left(\Omega_{X_{\bullet}}^{\bullet}, d_{f}\right) \tag{7.1.6.3}
\end{equation*}
$$

is an isomorphism as well. The isomorphism (7.1.6.1) is deduced from this by applying the functor $R \Gamma(X,-)$. That the morphisms (7.1.6.2), and hence (7.1.6.3), as morphisms in the derived category of coherent sheaves on $X$, are independent of $h$ follows from [31, Theorem 2.4]. In contrast to the previous result, the proof of the latter is a formal argument using the notion of homotopies between morphisms of hypercoverings. In particular, the hypothesis in loc.cit. that $X$ is projective is superfluous. The general theory of homotopies between hypercoverings is explained in SGA 4, Exposé V.
7.1.7. - Using Du Bois's results, we can define for every possibly singular variety $X$ and function $f: X \rightarrow \mathbb{A}^{1}$ an object

$$
\left(\underline{\underline{\Omega}}_{X}^{\bullet}, d_{f}\right)=\operatorname{colim}_{\varepsilon: X_{\bullet} \rightarrow X} R \varepsilon_{*}\left(\Omega_{X_{\bullet}}^{\bullet}, d_{f}\right)
$$

in the filtered derived category of coherent sheaves on $X$. We call it Du Bois complex. In this definition, the colimit runs over all smooth proper hypercoverings of $X$, and the filtration is induced by the filtration bête on the de Rham complexes $\Omega_{X_{\bullet}}^{\bullet}$. Despite the twisted differential, it enjoys all the functoriality properties given in $[31, \S 4]$. It comes by design with isomorphisms

$$
H^{p}\left(X,\left(\underline{\underline{\Omega}}_{X}^{\bullet}, d_{f}\right)\right)=H_{\mathrm{dR}}^{n}(X, f)
$$

and, if $X$ is smooth, we get back the usual twisted de Rham complex $\left(\Omega_{X}^{\bullet}, d_{f}\right)$ with the filtration bête. It follows that, in order to compute the de Rham cohomology of a singular variety, we can use any hypercovering, and need not necessarily choose a proper one. In particular, smooth affine hypercoverings will do.

Definition 7.1.8. - Let $X$ be a variety together with a closed subvariety $Y \subseteq X$ and a regular function $f$. We call the $k$-vector spaces

$$
\begin{equation*}
H_{\mathrm{dR}}^{n}(X, Y, f)=H^{n}\left(R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)\right) . \tag{7.1.8.1}
\end{equation*}
$$

the de Rham cohomology of $(X, Y, f)$.
7.1.9. - De Rham cohomology is functorial for morphisms of pairs in the obvious way. It admits by construction long exact Mayer-Vietoris sequences and, more generally, a Leray spectral sequence for open covers, and also for locally finite closed covers. Moreover, there is a natural long exact sequence

$$
\begin{equation*}
\cdots \longrightarrow H_{\mathrm{dR}}^{n-1}\left(Y, Z,\left.f\right|_{Y}\right) \longrightarrow H_{\mathrm{dR}}^{n}(X, Y, f) \longrightarrow H_{\mathrm{dR}}^{n}(X, Z, f) \longrightarrow H_{\mathrm{dR}}^{n}(Y, Z, f) \longrightarrow \cdots \tag{7.1.9.1}
\end{equation*}
$$

for each triple of closed subvarieties $Z \subseteq Y \subseteq X$. It is obtained by choosing compatible smooth proper hypercoverings of $X, Y$ and $Z$, and considering the following commutative diagram of complexes of vector spaces.


The cone of the morphism $(*)$ is canonically isomorphic to the cone of $(* *)$, hence a distinguished triangle

$$
R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right) \longrightarrow R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, Z_{\bullet}, f_{\bullet}\right) \longrightarrow R \Gamma_{\mathrm{dR}}\left(Y_{\bullet}, Z_{\bullet},\left(\left.f\right|_{Y}\right)_{\bullet}\right)
$$

in the derived category of $k$-vector spaces. It is independent of the chosen hypercoverings, and the long exact sequence (7.1.9.1) is obtained by taking cohomology.
7.1.10. - Let $X$ and $X^{\prime}$ be smooth varieties equipped with regular functions $f$ and $f^{\prime}$. There is a canonical isomorphism of complexes of sheaves

$$
\begin{equation*}
\left(\Omega_{X}^{\bullet}, d_{f}\right) \boxtimes\left(\Omega_{X^{\prime}}^{\bullet}, d_{f^{\prime}}\right) \longrightarrow\left(\Omega_{X \times X^{\prime}}^{\bullet}, d_{f \boxplus f^{\prime}}\right) \tag{7.1.10.1}
\end{equation*}
$$

on $X \times X^{\prime}$ given by $\omega \boxtimes \omega^{\prime} \longmapsto \omega \wedge \omega^{\prime}$ on local sections. Indeed, these maps are compatible with differentials by the following calculation

$$
d_{f \boxplus f^{\prime}}\left(\omega \wedge \omega^{\prime}\right)=d\left(\omega \wedge \omega^{\prime}\right)-d\left(f \boxplus f^{\prime}\right) \wedge \omega \wedge \omega^{\prime}=d_{f} \omega \wedge \omega^{\prime}+(-1)^{p} \omega \wedge d_{f^{\prime}}\left(\omega^{\prime}\right)
$$

for local sections $\omega$ of $\Omega_{X}^{p}$ and $\omega^{\prime}$ of $\Omega_{X^{\prime}}^{q}$, and (7.1.10.1) is an isomorphism degree by degree by the usual Künneth formula for algebraic de Rham complexes. For the general case, we observe that, if $\varepsilon: X_{\bullet} \rightarrow X$ and $\varepsilon^{\prime}: X_{\bullet}^{\prime} \rightarrow X$ are smooth proper hypercoverings, then so is $\left(\varepsilon, \varepsilon^{\prime}\right): X \bullet \times X_{\bullet}^{\prime} \rightarrow X \times X^{\prime}$, and we obtain an isomorphism analogous to (7.1.10.1) for the Du Bois complexes. If, in addition, subvarieties $Y \subseteq X$ and $Y^{\prime} \subseteq X^{\prime}$ are given, we observe that the tensor product of the complexes $R \Gamma_{\mathrm{dR}}\left(X_{\bullet}, Y_{\bullet}, f_{\bullet}\right)$ and $R \Gamma_{\mathrm{dR}}\left(X_{\bullet}^{\prime}, Y_{\bullet}^{\prime}, f_{\bullet}^{\prime}\right)$ is equal to the total complex of the bottom row in the
following diagram of complexes (functions omitted from the notation).


Each row in this diagram can be interpreted as a double complex, and the vertical maps induce a quasi-isomorphism between associated double complexes by Mayer-Vietoris. The cup-product

$$
H_{\mathrm{dR}}^{*}(X, Y, f) \otimes_{k} H_{\mathrm{dR}}^{*}\left(X^{\prime}, Y^{\prime}, f^{\prime}\right) \rightarrow H_{\mathrm{dR}}^{*}\left(X \times X^{\prime}, Y \times X^{\prime} \cup X \times Y^{\prime}, f \boxplus f^{\prime}\right)
$$

is the map induced in cohomology. A particular case which we shall use in the next definition is the Künneth formula for the product with $\left(\mathbb{G}_{m},\{1\}, 0\right)$. The de Rham cohomology $H^{*}\left(\mathbb{G}_{m},\{1\}, 0\right)$ is one-dimensional concentrated in cohomological degree 1 . For a $k$-vector space $V$ and an integer $i$, we write

$$
V(i)=V \otimes_{k} H_{\mathrm{dR}}^{1}\left(\mathbb{G}_{m}\right)^{\otimes(-i)}
$$

so that the Künneth formula for a product with $\left(\mathbb{G}_{m},\{1\}, 0\right)$ can be cast as an isomorphism

$$
\begin{equation*}
H_{\mathrm{dR}}^{n}(X, Y, f) \rightarrow H_{\mathrm{dR}}^{n+1}\left(X \times \mathbb{G}_{m}, Y \times \mathbb{G}_{m} \cup X \times\{1\}, f \boxplus 0\right)(1) \tag{7.1.10.2}
\end{equation*}
$$

of vector spaces.

Definition 7.1.11. - The de Rham representation $\rho_{\mathrm{dR}}: Q^{\exp }(k) \rightarrow \mathbf{V e c}_{k}$ is given on objects by

$$
\rho_{\mathrm{dR}}([X, Y, f, n, i])=H_{\mathrm{dR}}^{n}(X, Y, f)(i)
$$

and on morphisms as follows:
(a) a morphism of type (a) is sent to the morphism given by functoriality of de Rham cohomology for morphisms of pairs;
(b) a morphism of type (b) is sent to the connecting morphism in the long exact sequence (7.1.9.1);
(c) a morphism of type (c) is sent to the isomorphism (7.1.10.2) obtained from the Künneth formula.
7.1.12. - In fairness, the most complicated pair of varieties $(X, Y)$ of which we need to actually calculate the de Rham cohomology consists of a smooth variety $X$ and a normal crossing divisor $Y$, for which there is an elementary recipe. With regard to concrete calculations, the machinery of hypercoverings will merely ensure that the recipe produces a well defined outcome.

### 7.2. Construction of the comparison isomorphism

In this section, we construct the period isomorphism (7.0.0.2). Of course we don't know yet that it is indeed an isomorphism. We will in fact construct something slightly better than just a natural morphism between cohomology groups, namely a natural morphism between chain complexes which compute de Rham and rapid decay cohomology of $(X, Y, f)$ respectively. The advantage of such a construction is that it suffices essentially to work out the case where $X$ is smooth and $Y$ is empty, which is what we will do. For smooth affine varieties, the construction of the period pairing can be given by an elementary recipe, as we have seen in the examples given in the introduction. We will check that it agrees with the general, sheaf-theoretic construction in this section. Since algebraic de Rham cohomology is compatible with extension of scalars, as explained in 7.1 .5 , we may work without loss of generality with complex varieties.
7.2.1. - Let $X$ be a smooth projective complex variety. Every Zariski open subset of $X$ is open for the analytic topology, and every regular function on a Zariski open set of $X$ is analytic, hence a continuous map $s: X^{\text {an }} \rightarrow X$ and a morphism of sheaves of rings $s^{*} \mathcal{O}_{X} \rightarrow \mathcal{O}_{X}^{\text {an }}$. Serre's GAGA theorems [81, Theorems $1,2,3$ ] state that the analytification functor

$$
(-)^{\text {an }}:\left\{\text { Coherent } \mathcal{O}_{X} \text {-modules }\right\} \rightarrow\left\{\text { Coherent } \mathcal{O}_{X}^{\text {an }} \text {-modules }\right\}
$$

sending a coherent $\mathcal{O}_{X}$-module $F$ to $F^{\text {an }}=s^{*} F \otimes_{s^{*}} \mathcal{O}_{X} \mathcal{O}_{X}^{\text {an }}$ is an equivalence of categories. A particular aspect of this is that, for any coherent sheaf $F$ on $X$, the canonical morphisms

$$
H^{n}(X, F) \rightarrow H^{n}\left(X^{\mathrm{an}}, s^{*} F\right) \rightarrow H^{n}\left(X^{\mathrm{an}}, F^{\mathrm{an}}\right)
$$

obtained from $s$ are isomorphisms, and this continues to hold when in place of a single coherent sheaf $F$ we put a complex of coherent sheaves. The differentials in such a complex need not be $\mathcal{O}_{X}$-linear. The most important example for this situation is the algebraic de Rham complex $\Omega_{X}^{\bullet}$ with its usual exterior differential and its analytification $\Omega_{X}^{\text {an, } \bullet}$. We obtain the canonical isomorphisms

$$
H_{\mathrm{dR}}^{n}(X)=H^{n}\left(X, \Omega_{X}^{\bullet}\right) \rightarrow H^{n}\left(X^{\mathrm{an}}, \Omega_{X}^{\mathrm{an}, \bullet}\right)
$$

comparing algebraic and analytic de Rham cohomology.
The GAGA theorems fail catastrophically if $X$ is not projective. We can still obtain an easy comparison isomorphism between algebraic and analytic de Rham cohomology for smooth quasi-projective varieties, at the price of choosing a smooth compactification $\bar{X}$ of $X$. We can compute the algebraic de Rham cohomology of $X=\bar{X} \backslash D$ as the cohomology on $\bar{X}$ of the de Rham complex $\Omega_{\bar{X}, D}^{\bullet}$ of rational differential forms with poles of arbitrary order along the divisor at infinity $D$. This complex, which is also often denoted by $\Omega_{\bar{X}}^{\bullet}[* D]$ in the literature, is the direct image by the inclusion $j: X \hookrightarrow \bar{X}$ of the algebraic de Rham complex of $X$ and computes the de Rham cohomology of $X$ since $j$ is an affine morphism. The analytification of $\Omega_{\bar{X}, D}^{\bullet}$ is the complex $\Omega_{\bar{X}, D}^{a n, \bullet}$ of meromorphic differential forms on $\bar{X}$ with poles of arbitrary order along $D$. It is not the direct image of the analytic de Rham complex of $X$, but rather the subcomplex of forms of moderate growth. We obtain isomorphisms

$$
\begin{equation*}
H_{\mathrm{dR}}^{n}(X)=H^{n}\left(\bar{X}, \Omega_{\bar{X}, D}^{\bullet}\right) \rightarrow H^{n}\left(\bar{X}^{\mathrm{an}}, \Omega_{\bar{X}, D}^{\mathrm{an}, \bullet}\right) \tag{7.2.1.1}
\end{equation*}
$$

using the fact that sheaf cohomology commutes with colimits, and writing

$$
\left.\Omega_{\bar{X}, D}^{p}=\Omega_{\frac{\bar{X}}{p}}^{p} * D\right]=\operatorname{colim}_{m} \Omega_{\frac{\bar{X}}{p}}[m D]
$$

as a colimit of coherent sheaves. Grothendieck's theorem [42, Theorem 1'] comparing algebraic and analytic de Rham cohomology of $X$ relies then on resolution of singularities in order to reduce to a normal crossing divisor $D$, and an explicit computation by Atiyah and Hodge of the local cohomology of $\Omega_{\bar{X}, D}^{\text {an }}$.

When working with a variety $X$ equipped with a potential $f: X \rightarrow \mathbb{A}^{1}$, we content ourselves for the moment with the isomorphism (7.2.1.1) obtained from GAGA. The differential operator $d_{f}$ is well defined on rational or meromorphic differential forms on $\bar{X}$ with poles in $D=\bar{X} \backslash X$, hence the following proposition is immediate.

Proposition 7.2.2. - Let $X$ be a smooth complex variety, and let $f$ be a regular function on $X$. Let $\bar{X}$ be a smooth compactification of $X$ with complementary divisor $D=\bar{X} \backslash X$. The analytification functor induces natural isomorphisms of complex vector spaces

$$
\begin{equation*}
H_{\mathrm{dR}}^{n}(X, f)=H^{n}\left(\bar{X},\left(\Omega_{\bar{X}, D}^{\bullet}, d_{f}\right)\right) \rightarrow H^{n}\left(\bar{X}^{\mathrm{an}},\left(\Omega_{\bar{X}, D}^{\mathrm{an}, \bullet}, d_{f}\right)\right) \tag{7.2.2.1}
\end{equation*}
$$

7.2.3. - In Proposition 7.2 .2 we expressed the de Rham cohomology of $(X, f)$ as the cohomology of a complex of sheaves on the compact topological space $\bar{X}^{\text {an }}$. We do the same for rapid decay cohomology, using the real blow-up point of view explained in Section 3.5. Let $X$ be a smooth complex algebraic variety, and let $f: X \rightarrow \mathbb{A}^{1}$ be a regular function on $X$. Recall from Section 3.5 that a good compactification of $(X, f)$ is a compactification $\bar{X}$ of $X$ by a strict normal crossing divisor $D$ such that $f$ extends to a meromorphic function $\bar{f}: \bar{X} \rightarrow \mathbb{P}^{1}$. Choose such a good compactification, let $\pi: B \rightarrow \bar{X}$ be the real blow-up of $X$ along $D$, and let $f_{B}: B \rightarrow \widetilde{\mathbb{P}}^{1}$ be the extension of $f$ to $B$. Here, $\widetilde{\mathbb{P}}^{1}$ stands for the real blow-up of $\mathbb{P}^{1}$ at infinity, which we describe as the complex plane $\mathbb{C}$ completed by a circle at infinity. For $b \in \partial B$, either $f_{B}(b) \in \mathbb{C}$ or $f_{B}(b)$ lies on the circle at infinity, in which case its real part can either be negative, zero or positive. Set

$$
B^{0}=B \backslash\left\{b \in \partial B \mid f_{B}(b) \in \mathbb{C} \text { or } \operatorname{Re}\left(f_{B}(b)\right) \leqslant 0\right\}
$$

so that $\partial B^{0}$ consists of those $b \in \partial B$ where $f_{B}$ takes an infinite value with strictly positive real part. The rapid decay cohomology of $(X, f)$ is the cohomology of the pair $\left(B^{0}, \partial B^{0}\right)$, which can be computed as the cohomology of the singular cochain complex $C_{\left(B^{0}, \partial B^{0}\right)}^{\bullet}$ on $B$ by Proposition 3.5.2, or alternatively as

$$
\begin{equation*}
\left.H_{\mathrm{rd}}^{n}(X, f)=H^{n}\left(\bar{X}^{\mathrm{an}}, \pi_{*} C_{\left(B^{0}, \partial B^{0}\right)}^{\bullet}\right)\right) . \tag{7.2.3.1}
\end{equation*}
$$

Note that since $C_{\left(B^{0}, \partial B^{0}\right)}^{\bullet}$ is a complex of flasque sheaves, there is no difference between its direct image and its derived direct image on $\bar{X}^{\text {an }}$.
7.2.4. - In order to compare the complexes $\left(\Omega_{\bar{X}, D}^{\text {an }, \bullet}, d_{f}\right)$ and $\pi_{*} C_{\left(B^{0}, \partial B^{0}\right)}$ on $\bar{X}^{\text {an }}$, let us recall our conventions for singular cochain complexes. For this purpose, let $X$ be a manifold with boundary, possibly with corners. The group $C_{p}(X)$ of singular $p$-chains on $X$ is the $\mathbb{Q}$-linear vector space
generated by piecewise smooth ${ }^{1}$ maps $T: \Delta^{p} \rightarrow X$ where $\Delta^{p} \subseteq \mathbb{R}^{p+1}$ is the standard $p$-simplex, defined as the convex hull of the set of canonical basis vectors $e_{0}, e_{1}, \ldots, e_{p}$. The differential $d: C_{p}(X) \rightarrow C_{p-1}(X)$ is given by linearity and

$$
d T=\sum_{i=0}^{p}(-1)^{i}\left(T \circ d^{i}\right)
$$

where $d^{i}: \Delta^{p-1} \rightarrow \Delta^{p}$ is induced by the linear map given by $d^{i}\left(e_{j}\right)=e_{j}$ for $j<i$ and $d^{i}\left(e_{j}\right)=e_{j+1}$ for $j \geqslant i$. The resulting complex of rational vector spaces $C \bullet(X)$ is the singular chain complex associated with $X$, and we call

$$
C^{\bullet}(X)=\operatorname{Hom}\left(C_{\bullet}(X), \mathbb{Q}\right)
$$

the singular cochain complex of $X$. It computes by definition the singular cohomology of $X$ with rational coefficients. The complex $C^{\bullet}(X)$ depends contravariantly functorially on $X$. In particular, the assignment of the complex $C^{\bullet}(U)$ to any open $U \subseteq X$ defines a complex of presheaves on $X$, whose sheafification we denote by $C_{X}^{\bullet}$. Since $X$ is locally contractible, this complex of sheaves is a flasque resolution of the constant sheaf $\mathbb{Q}_{X}$ on $X$, hence $C_{X}^{\bullet}(X)$ computes the sheaf cohomology of $X$ with coefficients in $\mathbb{Q}_{X}$. One can show, using barycentric subdivision, that the canonical map $C^{\bullet}(X) \rightarrow C_{X}^{\bullet}(X)$ is a quasi-isomorphism, hence the canonical isomorphism between singular and sheaf cohomology. Given the inclusion of a subspace $\alpha: Y \rightarrow X$, there is a canonical morphism of sheaves $C_{X}^{\bullet} \rightarrow \alpha_{*} C_{Y}^{\bullet}$, ultimately induced by sending a simplex $T: \Delta^{p} \rightarrow Y$ to $\alpha \circ T$. We denote by

$$
C_{(X, Y)}^{\bullet}=\operatorname{cone}\left(C_{X}^{\bullet} \rightarrow \alpha_{*} C_{Y}^{\bullet}\right)
$$

its cone, and observe that $C_{(X, Y)}^{\bullet}$ is a flasque resolution of $\mathbb{Q}_{(X, Y)}=\operatorname{ker}\left(\mathbb{Q}_{X} \rightarrow \alpha_{*} \mathbb{Q}_{Y}\right)$, hence computes the cohomology of the pair of spaces $(X, Y)$.
7.2.5. - Let us now return to the situation where $\bar{X}$ is a good compactification of $(X, f)$. On $\bar{X}^{\text {an }}$ we have the analytic de Rham complex $\left(\Omega_{\bar{X}, D}^{\mathrm{an}, \bullet}, d_{f}\right)$ of meromorphic differential forms with poles contained in $D$, and the complex of singular cochains $\pi_{*} C_{\left(B^{0}, \partial B^{0}\right)}^{\bullet}$. A morphism of complexes of sheaves

$$
\begin{equation*}
I:\left(\Omega_{\bar{X}, D}^{\bullet}, d_{f}\right) \longrightarrow \pi_{*} C_{\left(B^{0}, \partial B^{0}\right)}^{\bullet} \otimes \mathbb{C} \tag{7.2.5.1}
\end{equation*}
$$

[^4]is specified by the following data: for every open $U \subseteq \bar{X}$ in the analytic topology and every meromorphic $p$-form $\omega \in \Omega_{\bar{X}, D}^{p}(U)$, a linear map
\[

$$
\begin{equation*}
I_{U}(\omega): C_{p}\left(\pi^{-1}(U)\right) \oplus C_{p-1}\left(\pi^{-1}(U) \cap \partial B^{0}\right) \longrightarrow \mathbb{C} \tag{7.2.5.2}
\end{equation*}
$$

\]

needs to be given. This data must be compatible with inclusions of open subsets and with differentials. Given a piecewise smooth $p$-simplex $T: \Delta^{p} \rightarrow U$ and a piecewise smooth $(p-1)$-simplex $T^{\prime}: \Delta^{p-1} \rightarrow U \cap \partial B^{0}$, we set

$$
\begin{equation*}
I_{U}(\omega)\left(T, T^{\prime}\right)=\int_{T} e^{-f} \omega=\int_{\Delta^{p}} e^{-f \circ T} T^{*} \omega \tag{7.2.5.3}
\end{equation*}
$$

This makes sense, since $e^{-f} \omega$ is everywhere defined on $B^{0}$, and so $T^{*}\left(e^{-f} \omega\right)=e^{-f \circ T} T^{*} \omega$ is a piecewise smooth differential form on $\Delta^{p}$. That the right-hand side of (7.2.5.3) does not depend on $T^{\prime}$ is intentional. Compatibility of the maps $I_{U}(\omega)$ with inclusions of open subsets is tautological. Compatibility with differentials is essentially a consequence of Stokes's formula as we show in the following Lemma.

Lemma 7.2.6 (Twisted Stokes formula). - With notation as in 7.2.5, the equality

$$
I_{U}\left(d_{f} \omega\right)\left(T, T^{\prime}\right)=I_{U}(\omega)\left(d T+T^{\prime}, d T^{\prime}\right)
$$

holds.
Proof. This can be verified by a straightforward computation. Here it is:

$$
\begin{array}{rlr}
I_{U}\left(d_{f} \omega\right)\left(T, T^{\prime}\right) & =\int_{T} e^{-f}\left(d_{f} \omega\right) \\
& =\int_{T} d\left(e^{-f} \omega\right) & \\
& =\int_{d T} e^{-f} \omega &  \tag{Stokes}\\
& =\int_{d T} e^{-f} \omega+\int_{T^{\prime}} e^{-f} \omega & \\
& =I_{U}(\omega)\left(d T+T^{\prime}, d T^{\prime}\right) &
\end{array}
$$

7.2.7. - It follows from Lemma 7.2.6 that the integration map (7.2.5.1) is a well defined morphism of complexes of sheaves of complex vector spaces on $X$. Taking cohomology yields morphisms of complex vector spaces

$$
\begin{equation*}
H^{n}\left(\bar{X}^{\mathrm{an}},\left(\Omega_{\bar{X}, D}^{\bullet}, d_{f}\right)\right) \longrightarrow H^{n}\left(\bar{X}^{\mathrm{an}}, \pi_{*} C_{\left(B^{0}, \partial B^{0}\right)}^{\bullet} \otimes \mathbb{C}\right) \tag{7.2.7.1}
\end{equation*}
$$

or alternatively

$$
\begin{equation*}
H_{\mathrm{dR}}^{n}(X, f) \longrightarrow H_{\mathrm{rd}}^{n}(X, f) \otimes \mathbb{C} \tag{7.2.7.2}
\end{equation*}
$$

via the isomorphisms (7.2.2.1) and (7.2.3.1). This is the sought period map for $(X, f)$. It is independent of the choice of a good compactification of $(X, f)$. We will prove that this map is
an isomorphism by proving that the morphism of sheaves on $\bar{X}^{\text {an }}$ given by (7.2.5.3) is a quasiisomorphism.

Definition 7.2.8. - Let $X$ be a smooth complex variety, and let $f$ be a regular function on $X$. We call the map (7.2.7.2) period map or also comparison morphism for the pair $(X, f)$.
7.2.9. - Let $X$ be a smooth and affine complex variety.

### 7.3. The comparison isomorphism for curves

The proof of the comparison theorem 7.6.1, affirming that the map from de Rham cohomology to rapid decay cohomology constructed in the previous section is an isomorphism, relies on a Poincaré lemma, stated as Theorem 7.5.11. This Poincaré lemma relates the twisted de Rham complex of differential forms with moderate growth on the real blow-up with a singular chain complex, and in turn relies on a theorem of Hien [43, Theorem A.1] about the growth behaviour of solutions of certain systems of linear partial differential equations. In loc. cit, this theorem is stated and proven in a more general setup than what we need here, which makes its proof substantially more involved, but only in the two-dimensional case. The case of arbitrary dimension is a straightforward generalisation. For the readers convenience, we shall reformulate and prove the case we need here, which is Theorem 7.5.5. In this section, we treat the one-dimensional case which is much lighter in terms of notation, yet contains all essential ideas. We convene that singular cochain complexes are taken with complex coefficients.
7.3.1. - We write $\bar{X}$ for the open complex unit disk, $D=\{0\}$, and $X=\bar{X} \backslash D$. Let $\mathcal{O}_{\bar{X}, D}$ denote the sheaf of meromorphic functions on $\bar{X}$ which are holomorphic on $X$. The real blow-up $B=\operatorname{Blo}_{D} \bar{X}$ is an annulus, with the inner circle as its boundary $\partial B$. For a complex number $w$


Figure 7.3.1. The real blow-up of the disk at its centre. Sectors map to sectors.
of norm 1, we write $w \cdot 0$ for the corresponding boundary point of $\partial B$. We fix a meromorphic function $f \in \mathcal{O}_{\bar{X}, D}(\bar{X})$ and write it as $f(x)=x^{-e} f_{1}(x)$, where $e \geqslant 0$ is the order of the pole and $f_{1}$ is a holomorphic function on the disk. The canonical extension $f_{B}: B \rightarrow \widetilde{\mathbb{P}}^{1}$ of $f$ is given by
$f_{B}(x)=f(\pi(x))$ for $x \notin \partial B$ and by

$$
f_{B}(w \cdot 0)= \begin{cases}w^{-e} f_{1}(0) \cdot \infty & \text { if } e>0 \\ f(0) & \text { if } e=0\end{cases}
$$

on the boundary. The positive part of the boundary $\partial^{+} B$ consists of those elements $w \cdot 0 \in \partial B$ where $f_{B}(w \cdot 0)=z \cdot \infty$ for some complex number $z$ of norm 1 with non-negative real part. The set $\partial^{+} B$ consists of $e$ equally spaced closed arcs on the circle $\partial B$.
7.3.2. - Set $\Omega_{\bar{X}, D}=\mathcal{O}_{\bar{X}, D} d x$, and define

$$
d_{f}: \mathcal{O}_{\bar{X}, D} \rightarrow \Omega_{\bar{X}, D}^{1}
$$

by $d_{f}(g)=d g-g d f=\left(g^{\prime}-g f^{\prime}\right) d x$.

Proposition 7.3.3. - If $e=0$, then $d_{f}$ is surjective, and its kernel is one-dimensional. If $e>0$, then the map

$$
d_{f}: \mathcal{O}_{\bar{X}, D}(\bar{X}) \rightarrow \mathcal{O}_{\bar{X}, D}(\bar{X}) d x
$$

is injective, and the classes $\left\{x^{-m} d x \mid 0<m \leqslant e\right\}$ form a basis of coker $\left(d_{f}\right)$.
Proof. Suppose first that $e>0$. Let $g$ be a meromorphic function on $\bar{X}$ such that $d_{f}(g)=0$. We can write $g$ and $f$ as power series

$$
f(x)=\sum_{n=0}^{\infty} a_{n} x^{n-e} \quad \text { and } \quad g(x)=\sum_{n=N}^{\infty} b_{n} x^{n}
$$

with $a_{0} \neq 0$. We need to show that $b_{N}=0$ holds. To this end, we calculate

$$
0=g^{\prime}(x)-g(x) f^{\prime}(x)=\sum_{n=N}^{\infty} n b_{n} x^{n-1}-\left(\sum_{p=0}^{\infty}(p-e) a_{p} x^{p-e-1}\right)\left(\sum_{q=N}^{\infty} b_{q} x^{q}\right)
$$

and see that the coefficient of $x^{N-e-1}$ equals $-e a_{0} b_{N}$. Since $-e a_{0}$ is nonzero, $b_{N}=0$ follows, hence injectivity of $d_{f}$.
7.3.4. - Given a meromorphic function $h$ on $\bar{X}$, we are interested in the inhomogeneous linear differential equation

$$
\begin{equation*}
u^{\prime}-f^{\prime} u=h \tag{7.3.4.1}
\end{equation*}
$$

in the unknown meromorphic function $u$. Locally, say in a neighbourhood of $x_{0} \in X$, the equation has a one-dimensional space of solutions, namely

$$
\begin{equation*}
u(x)=e^{f(x)} \int_{x_{0}}^{x} h(t) e^{-f(t)} d t+A e^{f(x)} \tag{7.3.4.2}
\end{equation*}
$$

where $A \in \mathbb{C}$ is a constant. The problem with this is of course that (7.3.4.2) might not define a global function on $X$, and even if, it might have an essential singularity at 0 .
7.3.5. - Let $U \subseteq B$ be a simply connected, open subset, such that $U \cap \partial^{+} B$ is simply connected or empty.

Suppose now that $h$ has moderate growth as $x \rightarrow 0$, that means, there exists an integer $N$ such that $|h(x)|=O\left(|x|^{-N}\right)$ holds for small $x$. Our question is whether the function $u$ has moderate growth as $x \rightarrow 0$ for an appropriate choice of the constant $A$. We shall show that this is indeed the case. Writing $f$ as a Laurent series

$$
f(x)=a x^{-d}+(\text { terms of degree }>-d)
$$

for some integer $d$ and non-zero $a \in \mathbb{C}$, we distinguish four cases. First case: $d \leqslant 0$, so $f$ is holomorphic. In that case, $f$ is bounded around $x=0$ and $u$ has moderate growth for any choice of $A$. Second case: $d>0$ and $\operatorname{Re}(a)>0$. In that case, a special solution of the differential equation is given by the improper integral

$$
\begin{equation*}
u(x)=e^{f(x)} \int_{0}^{x} h(t) e^{-f(t)} d t \tag{7.3.5.1}
\end{equation*}
$$

which converges, since $e^{-\operatorname{Re}(f(x))}$ decreases exponentially as $x \rightarrow 0$. We can use L'Hôpital's rule to examine the behaviour of $u$ near zero: For small $x$ we have

$$
u(x)=\frac{\int_{0}^{x} h(t) e^{-f(t)} d t}{e^{-f(x)}} \sim \frac{h(x)}{f^{\prime}(x)}
$$

hence $|u(x)|=O\left(|x|^{-N+d-1}\right) \leqslant O\left(|x|^{-N}\right)$ as $x \rightarrow 0$. Third case: $d>0$ and $\operatorname{Re}(a)<0$. In this case $e^{f(x)}$ converges to 0 as $x \rightarrow 0$, hence if $u$ has moderate growth for one choice of $A$, then so it does for any other. We use again L'Hôpital's rule to see that $u(x)$ grows as $\frac{h(x)}{f^{\prime}(x)}$ as $x$ approaches 0 . The difference between this case and the previous one is that now the indeterminacy has the shape $\frac{\infty}{\infty}$ no matter where the integration starts, whereas before it was $\frac{0}{0}$ only because the integration started at $t=0$. Fourth and last case: $d>0$ and $\operatorname{Re}(a)=0$. Assume $a=s i$ with real $s>0$, the case $s<0$ being similar. A special solution to (7.3.4.1) is again given by the integral formula (7.3.5.1), where the integration path approaches zero from a positive angle $0<\delta<\varepsilon$. The integral


Figure 7.3.2. An integration path approaching 0 from a positive angle
converges, since for sufficiently small $\delta$

$$
\lim _{r \rightarrow 0} r^{d} \operatorname{Re}\left(f\left(r e^{\delta i}\right)\right)=\lim _{r \rightarrow 0}\left(r^{d} \cdot s i \cdot r^{-d} \cdot e^{-d \delta i}\right)=s e^{(\pi / 2-d \delta) i}
$$

has positive real part. This special solution $u(x)$ has moderate growth of order at most $O\left(|x|^{-N}\right)$ along any angle above and below the real line as we have seen in the previous cases, hence again $|u(x)|=O\left(|x|^{-N}\right)$.

Suppose now that instead of solutions of (7.3.4.1) on a sector, we are interested in solutions of defined on the whole pointed disk. Now $f$ and $h$ are both meromorphic functions on the unit disk, with only pole at the origin. As we have seen presently, there exists on each sufficiently small sector around any angle $\alpha$ a solution of moderate growth, and in fact a unique one given by the improper integral (7.3.5.1) if the real part of $f(z)$ tends to $+\infty$ as $z$ approaches zero in the direction of $\alpha$. These local solutions glue together to a global solution if any only if for any two such angles $\alpha$ and $\beta$ the integral

$$
\begin{equation*}
\int_{0 \cdot \alpha}^{0 \cdot \beta} h(t) e^{-f(t)} d t \tag{7.3.5.2}
\end{equation*}
$$

vanishes. The notation means that we integrate along a path in the punctured unit disk starting at 0 in the direction of $\alpha$ and ending at 0 from the direction $\beta$.

We can reformulate our findings in terms of sheaves as follows: Let $\bar{X}$ be the open complex unit disk, set $X=\bar{X} \backslash\{0\}$, and let $\pi: B \rightarrow \bar{X}$ be the real blow-up of the origin. Let $f: X \rightarrow \mathbb{C}$ be a meromorphic function with only pole at zero, and denote by $f_{B}: B \rightarrow \widetilde{\mathbb{P}}^{1}$ its extension to the real blow-up. As we have shown in Section 3.5, the rapid decay cohomology of $(X, f)$ is the cohomology of the pair of spaces $\left[B^{\circ}, \partial B^{\circ}\right]$, where $B^{\circ}$ is the union (inside $B$ ) of $X$ and those elements $b$ in $\partial B$ with $f_{B}(b) \in \partial \widetilde{\mathbb{P}}^{1}$ with positive real part. The cohomology of the pair $\left[B^{\circ}, \partial B^{\circ}\right]$ is the cohomology of $B^{\circ}$ with coefficients in the sheaf $\mathbb{Q}_{\left[B^{\circ}, \partial B^{\circ}\right]}$. This sheaf admits as a flasque resolution the complex of sheaves $C_{\left[B^{\circ}, \partial B^{\circ}\right]}^{\bullet} \otimes \mathbb{C}$ given in degree $p$ by the sheaf of singular cochains on $B^{\circ}$ with boundary in $\partial B^{\circ}$. Let $\mathcal{O}_{B^{\circ}}^{\text {an }}$ denote the sheaf of holomorphic functions on $X=B^{\circ} \backslash \partial B^{\circ}$ with moderate growth near $\partial B^{\circ}$, set $\Omega_{B^{\circ}}^{\text {an, } 1}=\mathcal{O}_{B^{\circ}}^{\text {an }} d x$ and consider the connection $d_{f}: \mathcal{O}_{B^{\circ}}^{\text {an }} \rightarrow \Omega_{B^{\circ}}^{\text {an, } 1}$ sending $u$ to $\left(u^{\prime}-f^{\prime} u\right) d x$. Integration on chains defines a morphism of complexes of sheaves

as follows: A local section $u$ of $\mathcal{O}_{B^{\circ}}^{\text {an }}$ is sent by $I_{0}$ to the map which sends a 0 -simplex $T: \Delta^{0} \rightarrow B^{\circ}$ to the complex number $e^{-f\left(T\left(e_{0}\right)\right)} u\left(T\left(e_{0}\right)\right)$, and a local section $h d x$ of $\Omega_{B^{\circ}}^{\text {an, } 1}$ is sent $I_{0}$ to the map which sends a 1 -simplex $T: \Delta^{1} \rightarrow B^{\circ}$ to the integral

$$
I_{1}(h d x)(T)=\int_{T} e^{-f} h d x=\int_{0}^{1} e^{-f\left(T\left(t e_{0}+(1-t) e_{1}\right)\right)} h\left(T\left(t e_{0}+(1-t) e_{1}\right)\right) d t
$$

with the convention that the standard $n$-simplex is the convex hull in $\mathbb{R}^{n+1}$ of the canonical basis $e_{0}, e_{1}, \ldots, e_{n}$. The kernel of $d_{f}$ is generated by the function $e^{f}$ on opens which are disjoint from $\partial B^{\circ}$ and is zero on opens meeting $\partial B^{\circ}$, hence the morphism $I_{0}$ induces an isomorphism of sheaves $\operatorname{ker}\left(d_{f}\right) \rightarrow \mathcal{H}^{0}\left(C_{\left[B^{\circ}, \partial B^{\circ}\right]}^{\bullet}\right) \otimes \mathbb{C}$. We have shown that also $I_{1}$ induces an isomorphism $\operatorname{coker}\left(d_{f}\right) \rightarrow$ $\mathcal{H}^{1}\left(C_{\left[B^{\circ}, \partial B^{\circ}\right]}^{\bullet}\right) \otimes \mathbb{C}$. Therefore (7.3.5.3) is an isomorphism in the derived category of complexes of sheaves

$$
I:\left(\Omega_{B}^{\mathrm{an}, \bullet}, d_{f}\right) \xrightarrow{\cong} C_{\left[B, \partial^{+} B\right]}^{\bullet}
$$

on $B^{\circ}$. This is our first local Poincaré Lemma. On $X$, we look at the de Rham complex $\left(\Omega_{X}^{\text {an } \bullet}[* D], d_{f}\right)$ of meromorphic differential forms with a pole of any order at $D=\{0\}$, and the
integration morphism

$$
I: \Omega_{X}^{\mathrm{an}, \bullet}[* D] \rightarrow \pi_{*} C_{\left[B^{\circ}, \partial B^{\circ}\right]}^{\bullet} \otimes \mathbb{C}
$$

given by the same formula. Also this morphism is an isomorphism in the derived category. This is our second local Poincaré Lemma. An interesting thing to notice here is that since $C_{\left[B^{\circ}, \partial B^{\circ}\right]}^{\bullet}$ is flasque, we could also place the derived direct image $R \pi_{*} C_{\left[B^{\circ}, \partial B^{\circ}\right]}^{\bullet} \otimes \mathbb{C}$ in the above map, and still get an isomorphism. Hence, the canonical morphism

$$
\Omega_{X}^{\mathrm{an}, \bullet}[* D]=\pi_{*} \Omega_{B^{\circ}}^{\mathrm{an}, \bullet} \rightarrow R \pi_{*} \Omega_{B^{\circ}}^{\mathrm{an}, \bullet}
$$

is an isomorphism too. This is shown in much greater generality in [69, Corollary II, 1.1.8].

Proposition 7.3.6. - The morphism of complexes of sheaves on $\bar{X}$

given by integration of forms on chains is a quasi-isomorphism.

Proof. The case where $\bar{f}$ is holomorphic is easily settled, so we suppose that $f$ has a pole of positive order at 0 , and hence that $\partial^{+} B$ is nonempty. Let $U$ be a simply connected open subset of $\bar{X}$. It suffices to show that the morphism of complexes of vector spaces obtained from (7.3.6.1) by taking global sections on $U$ is a quasi-isomorphism. If $0 \notin U$, this is clear, since $d_{f}$ is surjective on sections with kernel of dimension 1 over $U$, and $\pi: \pi^{-1}(U) \rightarrow U$ is a homeomorphism. If $0 \in U$ we can as well assume $U=\bar{X}$. The differentials $d_{f}$ and $d$ are both injective on global sections, so all that's left is to show that the map

$$
\begin{equation*}
H^{1}\left(\Omega_{\bar{X}, D}^{\mathrm{an}, \bullet}(\bar{X})\right) \rightarrow H^{1}\left(C_{\left(B, \partial^{+} B\right)}^{\bullet}(\bar{X})\right) \tag{7.3.6.2}
\end{equation*}
$$

given by integration is an isomorphism. This morphism sends the class of a meromorphic differential form $\omega=h d x$ to to the cochain sending a 1-simplex, say given by a path $\gamma:[0,1] \rightarrow B$ with endpoints in $\partial^{+} B$, to the integral

$$
I_{1}(\omega)(\gamma)=\int_{0}^{1} e^{-f(\gamma(t))} h(\gamma(t)) d t
$$

It follows from ?? that $\omega$ is exact if and only if $I_{1}(\omega)$ is zero, which is the same as to say that $I_{1}(\omega)$ is exact because $C_{\left(B, \partial^{+} B\right)}^{\bullet}$ has no nontrivial global section. This shows that (7.3.6.2) is injective.

### 7.4. A Dolbeault-Grothendieck lemma on the real blow-up

The goal of this section is to understand the cohomology of the sheaf of analytic functions with moderate growth on the real blow-up at a normal crossing divisor of a variety. In particular, we wish to show that locally on the basis, say when we blow up a complex polydisk, this cohomology vanishes. The main result of the section is Theorem 7.4 .9 , originally due to Sabbah [69, Corollary II, 1.1.8]. We deduce it from a Dolbeault-Grothendieck Lemma on the real blow-up B. In the following Lemma 7.4 .1 and Proposition 7.4 .2 we recall the $\bar{\partial}$-Poincaré Lemma in one variable, essentially [87, Théorème 1.28], except that we formulate it for bounded functions on sectors.

Lemma 7.4.1 (Cauchy Representation). - Let Be the oriented real blow-up of the open complex unit disk at its centre, and let $U \subseteq B$ be open. Let $f$ be a bounded, smooth function on $U$ and let $V \subseteq U$ be a closed disk or sector. The equality

$$
f(z)=\frac{1}{2 \pi i} \int_{\partial V} \frac{f(\xi)}{\xi-z} d \xi+\frac{1}{2 \pi i} \iint_{V} \frac{\partial f}{\partial \bar{z}}(\xi) \frac{1}{\xi-z} d \xi d \bar{\xi}
$$

holds for every $z$ in the interior of $V$.

Proof. This is essentially a consequence of Stokes' Theorem for a domain in the complex plane with piecewise smooth boundary. We suppose $V$ is a sector, the case of a disk is well known and proven similarly. Let $V_{\varepsilon}$ be the closed subset $V$ obtained by removing from $V$ what is contained in the open disks around $z$ and around 0 , with small radius $\varepsilon>0$. The domain $V_{\varepsilon} \subseteq \mathbb{C}$ has a piecewise


Figure 7.4.3. The domain $V_{\varepsilon}$, pictured in the blow-up (left), and in the complex plane (right). Lenghts and areas are faithfully represented only on the right.
smooth boundary, consisting of two components: the positively oriented outer boundary, and the small circle around $z$ with negative orientation. The differential form

$$
\omega=\frac{f(\xi)}{\xi-z} d \xi
$$

is a well-defined and smooth in a neighbourhood of $V_{\varepsilon}$. Applying Stokes' theorem and taking limits as $\varepsilon \rightarrow 0$ yields

$$
\iint_{V} \frac{\partial f}{\partial \bar{\xi}}(\xi) d \bar{\xi} d \xi=\lim _{\varepsilon \rightarrow 0} \iint_{V_{\varepsilon}} d \omega=\lim _{\varepsilon \rightarrow 0} \int_{\partial V_{\varepsilon}} \omega=\int_{\partial V} \frac{f(\xi)}{\xi-z} d \xi-\lim _{\varepsilon \rightarrow 0} \int_{0}^{2 \pi} f\left(z+\varepsilon e^{i \vartheta}\right) i d \vartheta
$$

which is the desired formula up to the factor $2 \pi i$.

Proposition 7.4.2. - Let $B$ be the oriented real blow-up of the open complex unit disk at its centre, and let $U \subseteq B$ be open. Let $f$ be a bounded, smooth function on $U$ and let $V \subseteq U$ be $a$ closed disk or sector. The function $g$ defined on the interior of $V$ by

$$
\begin{equation*}
g(z)=\frac{1}{2 \pi i} \iint_{V} \frac{f(\xi)}{\xi-z} d \xi d \bar{\xi} \tag{7.4.2.1}
\end{equation*}
$$

is smooth, bounded, and satisfies $\partial g / \partial \bar{z}=f$.
Proof. The surface integral is improper because of the apparent pole, but converges absolutely since indeed $\xi \longmapsto(\xi-z)^{-1}$ is of class $L^{1}$ on the unit disk with $L^{1}$-norm at most 1 . This already shows that $g$ is bounded, in fact $\|g\|_{\infty} \leqslant\|f\|_{\infty}$ holds, and since integration and differentiation can be interchanged, $g$ is smooth since $f$ is so. It remains to verify the differential equation $\partial g / \partial \bar{z}=f$. Fix a point $z \in V \backslash(V \cap \partial B)$ and decompose $f$ as a sum $f=f_{1}+f_{2}$ of smooth functions, where $f_{1}$ is identically 0 in a neighbourhood of $z$ and agrees with $f$ in a heighbourhood of the boundary of $V$. For the function $f_{2}$, this means that it has compact support contained in the interior of $V \backslash(V \cap \partial B)$ and $f_{2}$ is equal to $f$ in a neighbourhood of $z$. It suffices to verify equation (7.4.2.1) for $f_{1}$ and $f_{2}$ separately. Defining $g_{1}$ and $g_{2}$ accordingly, we find

$$
\frac{\partial g_{1}}{\partial \bar{z}}(z)=\frac{1}{2 \pi i} \iint_{V} \frac{\partial}{\partial \bar{z}}\left(\frac{f_{1}(\xi)}{\xi-z}\right) d \xi d \bar{\xi}=0=f_{1}(z)
$$

because $f_{1}(\xi)(\xi-z)^{-1}$ is a smooth, well defined function of $\xi$ on $V$, so exchanging integration and differentiation is justified, and the same expression is a holomorphic function of $z$. Since $f_{2}$ has compact support away from $\partial B$, we find

$$
\frac{\partial g_{2}}{\partial \bar{z}}(z)=\frac{1}{2 \pi i} \frac{\partial}{\partial \bar{z}} \iint_{\mathbb{C}} \frac{f_{1}(\xi)}{\xi-z} d \xi d \bar{\xi}=\frac{1}{2 \pi i} \iint_{\mathbb{C}} \frac{\partial f_{1}}{\partial \bar{\xi}}(\xi) \frac{1}{\xi-z} d \xi d \bar{\xi}=f(z)
$$

where we used Lemma 7.4.1 for the last equality. The exchange of differentiation and integral can be justified, for example, by using polar coordinates.
7.4.3. - In this section, $\bar{X}$ denotes the $n$-dimensional open complex polydisk with coordinate functions $z_{1}, \ldots z_{n}$, and $D \subseteq \bar{X}$ stands for the normal crossing divisor given by the equation $z_{1} z_{2} \cdots z_{m}=0$. Let $\pi: B \rightarrow \bar{X}$ be the real blow-up of $\bar{X}$ at $D$. In the sequel, if we speak of smooth or holomorphic functions on an open subset $U$ of $B$, when we really mean complex valued functions with domain $U \backslash(U \cap \partial B)$. A smooth function $f$ on $U$ is said to have moderate growth near $x \in \partial B \cap U$ if for some large enough exponent $N \geqslant 0$ the estimate

$$
\left|f\left(z_{1}, z_{2}, \ldots, z_{n}\right)\right| \leqslant\left|z_{1} z_{2} \cdots z_{n}\right|^{-N}
$$

holds in a neighbourhood of $x$. We say for short that the function $f$ has moderate growth if it has moderate growth near every point $x \in \partial B \cap U$, and denote by $\mathcal{O}_{B, \partial B}^{\mathrm{sm}}$ the sheaf of smooth functions of moderate growth on $B$. This sheaf contains the sheaf $\mathcal{O}_{B, \partial B}^{\text {an }}$ of analytic functions of moderate growth.
7.4.4. - Let $p, q \geqslant 0$ be integers. We denote by $\mathcal{A}_{B, \partial B}^{p, q}$ the sheaf on $B$ whose sections on an open set $U \subseteq B$ are smooth differntial forms

$$
\sum_{I, J} f_{I, J}(z) d z_{I} \wedge d \bar{z}_{J}
$$

of type $(p, q)$ on $U \backslash(U \cap \partial B)$, with smooth coefficients $f_{I, J}$ of moderate growth. The sheaf $\mathcal{A}_{B, \partial B}^{p, q}$ on $B$ is soft. Indeed, there exists for every closed subset $K \subseteq B$ and every open neighbourhood $U$ of $K$ a smooth, bounded function $\eta$ on $B$ with support contained in $U$, and which is identically 1 in a neighbourhood of $K$. Such functions can be used to extend sections of $\mathcal{A}_{B, \partial B}^{p, q}$ on $K$ to global sections. In particular, the sheaf $\mathcal{A}_{B, \partial B}^{p, q}$ is acyclic, that means

$$
H^{n}\left(B, \mathcal{A}_{B, \partial B}^{p, q}\right)=0
$$

holds for all $n>0$.

Lemma 7.4.5 ( $\bar{\partial}$-Poincaré Lemma). - Let $U \subseteq B$ be open, and let $f$ be a smooth function of moderate growth on $U$ and let $k \in\{1,2, \ldots, n\}$. There exists locally on $U$ a smooth function $g$ of moderate growth satisfying

$$
\frac{\partial g\left(z_{1}, \ldots, z_{n}\right)}{\partial \bar{z}_{k}}=f\left(z_{1}, \ldots, z_{n}\right)
$$

If moreover $f$ is holomorphic with respect to variables $z_{j}, j \neq k$, then $g$ can be chosen holomorphic in the same variables.

Proof. If not for the growth condition, this would be the classical Dolbeault-Grothendieck Lemma on the open set $U \backslash(\partial B \cap U)$. We will thus work in a neighbourhood of a boundary point $x \in \partial B \cap U$ and walk through the classical proof, checking at each step that the construction of $g$ can be made compatible with growth conditions. We start with the essential case where the dimension $n$ is one. We can write $f$ in the form $z^{-N} f_{1}$ for some smooth, complex valued function $f_{1}$ on $U$ which is bounded in a neighbourhood of the boundary point $x=w \cdot 0$. Choose $0<r<1$ sufficiently small, so that the closure of the open sector

$$
S(w, r)=\left\{s e^{i \vartheta} \mid 0<s<r,-r<\vartheta<r\right\}
$$

is contained in $U$, and so that $f$ is bounded on $S(w, r)$. A function $g_{1}$ with $\bar{\partial} g=f_{1}$ is given by

$$
g_{1}\left(z_{0}\right)=\frac{1}{2 \pi i} \iint_{S(w, r)} \frac{f_{1}(z)}{z-z_{0}} d z d \bar{z}
$$

for $z_{0} \in S(w, r)$.
The so defined function $g_{1}$ is bounded by the same bound as $f_{1}$ is, and satisfies $\partial g_{1} / \partial \bar{z}=f_{1}$ as shown in Proposition 7.4.2. Setting $g(z)=z^{-Z} g_{1}(z)$ We find

$$
\frac{\partial g(z)}{\partial \bar{z}}=z^{-N} \frac{\partial g_{1}(z)}{\partial \bar{z}}=z^{-N} f_{1}(z)=f(z)
$$

by the chain rule and the fact that $z \longmapsto z^{-N}$ is holomorphic.
We introduce the notation

$$
D_{\varepsilon}=\{z \in \mathbb{C}| | z \mid<r\}
$$

for open disks, and

$$
S_{\varepsilon}=\left\{r w e^{2 \pi i \vartheta} \in \mathbb{C} \mid 0<r \leqslant \varepsilon,-\varepsilon \leqslant \vartheta \leqslant \varepsilon\right\}
$$

for open sectors. A neighbourhood of a point of $U$

Proposition 7.4.6 (Dolbeault-Grothendieck Lemma). - Let $U \subseteq B$ be open, $x \in U$ and $q>0$. For every $\beta \in \mathcal{A}_{B, \partial B}^{p, q}(U)$ with $\bar{\partial} \beta=0$ there exists an open neighbourhood $V$ of $x$ and a form $\alpha \in \mathcal{A}_{B, \partial B}^{p, q-1}(V)$ with $\bar{\partial} \alpha=\left.\beta\right|_{V}$.

Proof. Consider now the general case where $X$ has dimension $n \geqslant 1$, and $\beta$ is a $\bar{\partial}$-closed form of type $(p, q)$, with $q>0$. We can write $\beta$ as

$$
\beta=\sum_{\# I=p} \beta_{I} \wedge d z_{I}
$$

where each $\beta_{I}$ is a $\bar{\partial}$-closed $(0, q)$ form. It suffices to show that each $\beta_{I}$ is $\bar{\partial}$-exact. We may thus assume $p=0$, so $\beta$ can be written as

$$
\begin{equation*}
\beta=\sum_{\# J=q} f_{J} d \bar{z}_{J} \tag{7.4.6.1}
\end{equation*}
$$

and argue by induction on the smallest integer $k \in\{0,1,2, \ldots, n\}$ such that the sum in (7.4.6.1) can be indexed by sets $J \subseteq\{1,2, \ldots, k\}$, that is, if $J$ contains an index larger than $k$, then $f_{J}=0$. For $k=0$ we find $\beta=0$ and there is nothing to be proved. For $k \geqslant 1$, we can write $\beta$ as

$$
\beta=\beta_{0}+\beta_{1} \wedge d \bar{z}_{k}
$$

where $\beta_{0}$ and $\beta_{1}$ are forms of type $(0, q-1)$, in the ideal spanned by $d \bar{z}_{1}, \ldots d \bar{z}_{k-1}$.

Corollary 7.4.7. - For every $p \geqslant 0$, the complex of sheaves on $B$

$$
0 \rightarrow \Omega_{B, \partial B}^{p, \mathrm{an}} \rightarrow \mathcal{A}_{B, \partial B}^{0, q} \rightarrow \mathcal{A}_{B, \partial B}^{1, q} \rightarrow \cdots
$$

is exact.

Lemma 7.4.8. - Let $U \subseteq X$ be an open subset. A function $h$ on $U$ has moderate growth if and only if the composite $h \circ \pi$ on $\pi^{-1}(U)$ has moderate growth. In particular, the following equalities hold for all $p, q \geqslant 0$.

$$
\pi_{*} \Omega_{B, \partial B}^{p, \text { an }}=\Omega_{X, D}^{p, \text { an }} \quad \text { and } \quad \pi_{*} \mathcal{A}_{B, \partial B}^{p, q}=\mathcal{A}_{X, D}^{p, q}
$$

Proof. This follows form the fact that $\pi$ is a proper map.

Theorem 7.4.9. - For every $p \geqslant 0$, the canonical morphism

$$
\Omega_{X, D}^{p, \text { an }}=\pi_{*} \Omega_{B, \partial B}^{p, \text { an }} \rightarrow R \pi_{*} \Omega_{B, \partial B}^{p, \text { an }}
$$

is an isomorphism in the derived category of sheaves on $X$.

Proof. The sheaves $\mathcal{A}_{B, \partial B}^{p, q}$ are soft, hence acyclic. We can thus compute the derived direct image $R \pi_{*} \Omega_{B, \partial B}^{p, \text { an }}$ using the resolution given in Corollary 7.4.7. We find

$$
R \pi_{*} \Omega_{B, \partial B}^{p, \mathrm{an}}=\left(\pi_{*} \mathcal{A}_{B, \partial B}^{p, \bullet}, \bar{\partial}\right)=\left(\mathcal{A}_{X, D}^{p, \bullet}, \bar{\partial}\right)=\Omega_{X, D}^{p, \text { an }}
$$

by Lemma 7.4.8, and the classical Dolbeault-Grothendieck Lemma for meromorphic functions on a polydisk.

### 7.5. The Poincaré Lemmas on a polydisk

7.5.1. - Our goal is to generalise the discussion of the preceding section to several variables. To this end, we fix the following notation and terminology: pick integers $n>0$ and $0 \leqslant m \leqslant n$, consider the open unit polydisk $\bar{X} \subseteq \mathbb{C}^{n}$, the divisor $D$ of $X$ given by $x_{1} x_{2} \cdots x_{m}=0$, and the real blow-up

$$
\pi: B \rightarrow \bar{X}
$$

of $\bar{X}$ along the components of $D$. We consider the holomorphic function $f: X \rightarrow \mathbb{P}^{1}$ given in projective coordinates by $f(x)=\left[f_{0}(x): f_{1}(x)\right]$, with $f_{0}(x)=x_{1}^{e_{1}} x_{2}^{e_{2}} \cdots x_{m}^{e_{m}}$ for some non-negative integers $e_{1}, e_{2}, \ldots, e_{m}$. The poles $f^{-1}(\infty)$ are contained in $D$, and we denote by $f_{B}: B \rightarrow \widetilde{\mathbb{P}}^{1}$ the continuous extension of $f$ to $B$. Recall that $B$ is the space

$$
\left\{\left(x_{1}, \ldots, x_{n}, w_{1}, \ldots, w_{m}\right) \in \bar{X} \times\left(S^{1}\right)^{m} \mid x_{i} w_{i}^{-1} \in \mathbb{R}_{\geqslant 0} \text { for } 1 \leqslant i \leqslant m\right\}
$$

and that the boundary $\partial B$ of $B$ is the set $\pi^{-1}(D)$. We identify the interior of $B$ with $X=\bar{X} \backslash D$ via the projection map $\pi$. The oriented real blow-up $\widetilde{\mathbb{P}}^{1}$ of $\mathbb{P}^{1}$ at $\{\infty\}$ is the complex plane to which a circle at infinity $\partial \widetilde{\mathbb{P}}^{1}=S^{1}$ has been glued. As discussed in 3.5.1, the function $f_{B}: B \rightarrow \widetilde{\mathbb{P}}^{1}$ is given by

$$
f_{B}(x, w)= \begin{cases}\frac{f_{1}(x)}{f_{0}(x)} & \in \mathbb{C}, \text { if } f_{0}(x) \neq 0 \\ \frac{f_{1}(x)}{\left|f_{1}(x)\right|} f_{0}(w)^{-1} & \in \partial \widetilde{\mathbb{P}}^{1}, \text { if } f_{0}(x)=0, \text { hence }(x, w) \in \partial B\end{cases}
$$

In what follows, we are interested in the behaviour of local solutions of differential equations near a point $b \in \partial B$. For our purposes, we may choose a point $b$ in $\pi^{-1}(0)$, thus of the form

$$
b=\left(0, \ldots, 0, e^{2 \pi i \beta_{1}}, \ldots, e^{2 \pi i \beta_{m}}\right)
$$

and consider the open neighbourhoods

$$
U=U(\varepsilon)=\left\{(x, w)| | x_{i} \mid<\varepsilon \text { for } 1 \leqslant i \leqslant n \text { and } \arg \left(e^{2 \pi i \beta_{p}} w_{p}^{-1}\right)<\varepsilon \text { for } 1 \leqslant p \leqslant m\right\}
$$

of $b$. As a manifold with corners, $U$ is diffeomorphic to $(-1,1)^{n} \times[0,1)^{n}$. Concretely, a local chart around $b \in \pi^{-1}(0)$ is given by

$$
\begin{equation*}
(-\varepsilon, \varepsilon)^{n} \times[0, \varepsilon)^{n} \xrightarrow{\simeq} U \tag{7.5.1.1}
\end{equation*}
$$

sending $\left(\alpha_{1}, \ldots, \alpha_{n}, r_{1}, \ldots, r_{n}\right)$ to $(x, w) \in B$ with $x_{k}=r_{k} e^{2 \pi i\left(\alpha_{p}+\beta_{p}\right)}$ and $w_{k}=e^{2 \pi i\left(\alpha_{p}+\beta_{p}\right)}$.

Lemma 7.5.2. - Suppose that $f_{B}(b)=i \infty \in \partial \mathbb{P}^{1}$ holds. For sufficiently small $\varepsilon>0$, the set $\left\{x \in U \mid \operatorname{Re}\left(f_{B}(x)\right)=0\right\}$ divides $U$ into two simply connected components.

Proof. First of all, notice that the integers $e_{p}$ are not all zero - if so, $0 \in \bar{X}$ would not be a pole of $f$, and hence $f_{B}(b)=f(0)$ would not be an element of the boundary of $\widetilde{\mathbb{P}}^{1}$. In terms of the coordinates given in (7.5.1.1), the argument of the function $f_{B}$ is given by

$$
\arg \left(f_{B}(x, w)\right)=\arg \left(f_{1}(x)\right)-\left(e_{1}\left(\alpha_{1}+\beta_{1}\right)+\cdots+e_{m}\left(\alpha_{m}+\beta_{m}\right)\right) \quad \in \mathbb{R} / 2 \pi \mathbb{Z}
$$

noting that since $f_{1}(0) \neq 0$, the argument of $f_{1}$ is a well defined real analytic function in a neighbourhood of 0 , taking values in $\mathbb{R} / 2 \pi \mathbb{Z}$. Set $f_{1}(x)=f_{1}(0) \cdot e^{g(x)}$, where $g$ is a holomorphic function satisfying $g(0)=0$, so that the argument of $f_{1}$ is given by $\arg \left(f_{1}(\alpha, r)\right)=\arg \left(f_{1}(0)\right)+$ $\operatorname{im}(g(x))$. Writing $(x, w)$ in coordinates $(r, \alpha)$, the Taylor expansion of the function $\operatorname{im}(g(r, \alpha))$ has no linear terms in $\alpha$. We have $\arg \left(f_{B}(b)\right)=\frac{\pi}{2}$ by hypothesis, so we can write $\arg \left(f_{B}(x, w)\right)$ as

$$
\arg \left(f_{B}(x, w)\right)=\frac{\pi}{2}+L(\alpha)+\text { higher order terms }
$$

where $L$ is a non-zero linear form, and higher order terms mean terms which contain a factor which is quadratic in $\alpha$ or a factor which is linear in $r$. The cube $(-\varepsilon, \varepsilon)^{n} \times[0, \varepsilon)^{n}$ is divided in two halves by the hyperplane $L(\alpha)=0$, and we deduce from the implicit function theorem that for small enough $\varepsilon>0$, the cube $(-\varepsilon, \varepsilon)^{n} \times[0, \varepsilon)^{n}$ is divided in two halves by the hyperplane $\arg \left(f_{B}(x, w)\right)=\frac{\pi}{2}$.

Definition 7.5.3. - We say that a holomorphic function $h: U \backslash \partial U \rightarrow \mathbb{C}$ has moderate growth near $b$ if there exists a neighbourhood $V \subseteq U$ of $b$ and a Laurent polynomial $g \in \mathbb{C}\left[x_{1}, \ldots, x_{n}, x_{1}^{-1}, \ldots, x_{m}^{-1}\right]$ such that the inequality $|h(x)| \leqslant|g(x)|$ holds for $x \in V \backslash \partial V$.
7.5.4. - Sums and products of functions with moderate growth again have moderate growth, and, in particular, the function $f$ has moderate growth near $b$. Let us introduce the linear differential operators

$$
D_{i}(u)=\frac{\partial u}{\partial x_{i}} \quad \text { and } \quad Q_{i}(u)=\frac{\partial u}{\partial x_{i}}-\frac{\partial f}{\partial x_{i}} u
$$

for $1 \leqslant i<n$. If $h$ has moderate growth near $b$, then so do $D_{i}(h)$ and $Q_{i}(h)$.

Theorem 7.5.5. - Let $h$ be a holomorphic function on $U \backslash \partial U$ and let $1 \leqslant r \leqslant n$ be an integer. If $h$ satisfies the integrability condition $Q_{s}(h)=0$ for all $1 \leqslant s<r$, then the system of partial differential equations

$$
\left\{\begin{array}{l}
Q_{s}(u)=0 \quad \text { for } 1 \leqslant s<r  \tag{f}\\
Q_{r}(u)=h
\end{array}\right.
$$

admits a holomorphic solution. If moreover $h$ has moderate growth near $b \in \pi^{-1}(0)$, then there exists a holomorphic solution defined in a neighbourhood of $b$, with moderate growth near $b$.

Proof. The difference of any two solutions of $\left(\Sigma_{f}(h)\right)$ is a solution of the corresponding homogeneous system $\left(\Sigma_{0}(h)\right)$, whose holomorphic solutions form the vector space of functions of the form $A e^{f}$, where $A$ is a holomorphic function in the variables $x_{r+1}, \ldots, x_{n}$. Let us set

$$
u(x)=w(x) e^{f(x)} \quad \text { and } \quad g=h e^{-f}
$$

where $w$ stands for a holomorphic function to be determined. We have $Q_{s}(u)=D_{s}(w) e^{f}$, hence must solve the new system

$$
\left\{\begin{array}{l}
D_{s}(w)=0 \quad \text { for } 1 \leqslant s<r  \tag{0}\\
D_{r}(w)=g
\end{array}\right.
$$

in the unknown function $w$. The integrability condition on $h$ translates to

$$
D_{s}(g)=\frac{\partial h}{\partial x_{s}} e^{-f}-h \frac{\partial f}{\partial x_{s}} e^{-f}=Q_{s}(h) e^{-f}=0
$$

for $1 \leqslant s<r$. The differential system $\left(\Sigma_{0}(g)\right)$ together with the integrability condition on $g$ is precisely what has to be solved in the proof of the classical Poincaré Lemma. Indeed, the integrability condition on $g$ means that $g$ is constant with respect to the variables $x_{1}, \ldots, x_{r-1}$, and we can set

$$
w\left(x_{1}, \ldots, x_{n}\right)=\int_{\frac{\varepsilon}{2}}^{x_{r}} g\left(x_{1}, \ldots, x_{r-1}, z, x_{r+1}, \ldots, x_{n}\right) d z
$$

where the integration path from $\frac{\varepsilon}{2}$ to $x_{r}$ may be chosen to be a straight line. The general solution $u$ to $\left(\Sigma_{f}(h)\right)$ is therefore given by

$$
\begin{equation*}
u(x)=e^{f} \cdot \int_{\frac{\varepsilon}{2}}^{x} h e^{-f} d z+A e^{f} \tag{7.5.5.1}
\end{equation*}
$$

with the same integration path and some holomorphic function $A$ in the variables $x_{r+1}, \ldots, x_{n}$. The function $u$ is holomorphic, and all that's left to show is that for some appropriate choice of $A$ the solution $u$ has moderate growth near $b$ if $h$ has so. Let us suppose that this is the case, and choose $\varepsilon<1$ small enough so that there exists an integer $N \geqslant 0$ for which the inequality

$$
|h(x)| \leqslant\left|x_{1} x_{2} \cdots x_{m}\right|^{-N}
$$

holds for $x \in U \backslash \partial U$. We distinguish four possible regimes for $f_{B}(b) \in \widetilde{\mathbb{P}}^{1}$, namely $f_{B}(b)$ can be:
(1) An element of the interior of $\widetilde{\mathbb{P}}^{1}$. So $f_{B}(b)$ is a complex number.
(2) An element in the boundary $\partial \widetilde{\mathbb{P}}^{1}$ with positive real part.
(3) An element in the boundary $\partial \widetilde{\mathbb{P}}^{1}$ with negative real part.
(4) Either $+i \infty$ or $-i \infty$.

In the first case, the meromorphic function $f=\frac{f_{1}}{f_{0}}: \bar{X} \rightarrow \mathbb{C}$ is holomorphic and its extension to $B$ is the composite of $f$ with the blow-up map $\pi: B \rightarrow \bar{X}$. We may hence assume $f$ is bounded on $U$, say

$$
\left|e^{f(x)}\right| \leqslant M \quad \text { and } \quad\left|e^{-f(x)}\right| \leqslant M
$$

hold. The function defined by (7.5.5.1) has moderate growth if we choose for $A$ any function of moderate growth, for example a constant. In case (2), the function $e^{-f}$ decays exponentially in a
neighbourhood of $b$, and therefore, since $h$ has moderate growth, the integral

$$
\int_{0}^{\frac{\varepsilon}{2}} h e^{-f} d z
$$

converges. Put differently, we may choose 0 in place of $\frac{\varepsilon}{2}$ as starting point of the integral in (7.5.5.1) even if this new starting point is now on the boundary and not in the interior if $U$. Let us show that the function

$$
\begin{equation*}
u(x)=e^{f(x)} \cdot x_{r} \cdot \int_{0}^{1} h\left(x_{1}, \ldots, t x_{r}, \ldots, x_{n}\right) e^{-f\left(x_{1}, \ldots, t x_{r}, \ldots, x_{n}\right)} d t \tag{7.5.5.2}
\end{equation*}
$$

has moderate growth near $b$. Set $\Phi(x, t)=f\left(x_{1}, \ldots, x_{r}, \ldots, x_{n}\right)-f\left(x_{1}, \ldots, t x_{r}, \ldots, x_{n}\right)$, and notice that for $0<t<1$ and all $x \in U$ we have $\operatorname{Re}(\Phi(x, t)<0$. We may estimate (7.5.5.2) by

$$
\begin{aligned}
|u(x)| & =\left|x_{r} \cdot \int_{0}^{1} h\left(x_{1}, \ldots, t x_{r}, \ldots, x_{n}\right) e^{\Phi(x, t)} d t\right| \\
& \leqslant\left|x_{r}\right| \cdot \int_{0}^{1} t \cdot\left|x_{1} x_{2} \cdots x_{m}\right|^{-N} e^{\operatorname{Re}(\Phi(x, t))} d t \\
& \leqslant\left|x_{1} \cdots x_{m}\right|^{-N+1}
\end{aligned}
$$

which shows that $u$ has moderate growth near $b$ as claimed. Case (3) is similar to case (1): Since $\operatorname{Re}(f)$ tends to $-\infty$ near $b$, the function $e^{f}$ is bounded near $B$, hence the function defined by (7.5.5.1) has moderate growth near $b$ if we choose for $A$ a constant. The last case which remains to discuss is case $(4)$, in which $f_{B}(b)= \pm i \infty \in \partial \widetilde{\mathbb{P}}^{1}$. The boundary of the open $U$ contains the set $\partial^{+} U$ given by

$$
\partial^{+} U=\left\{(x, w) \in \partial B \mid f_{0}(x)=0 \text { and } \operatorname{Re}\left(f_{B}(x, w)\right)>0\right\}
$$

and we will show that $\partial^{+} U$ is connected for sufficiently small $\varepsilon$. The rest of the argument will be similar to the case (2).
7.5.6. - Our next goal is to reinterpret Theorem 7.5.5 in terms of sheaves of differential forms having moderate growth on a smooth complex variety $X$ together with a potential $f$. For the remainder of this section, we work with a fixed smooth and proper complex algebraic variety $\bar{X}$, a normal crossing divisor $D \subseteq \bar{X}$ and a potential $f: \bar{X} \rightarrow \mathbb{P}^{1}$ satisfying $f^{-1}(\infty) \subseteq D$. In other words, writing also $f$ for the restriction of $f$ to $X=\bar{X} \backslash D \rightarrow \mathbb{A}^{1}$, the pair $(\bar{X}, f)$ is a good compactification of $(X, f)$. Let us denote by $\pi: B \rightarrow \bar{X}$ the real oriented blow-up of $\bar{X}$ in the components of $D$, and by $\partial B=\pi^{-1}(D)$ the boundary of $B$. The function $f: \bar{X} \rightarrow \mathbb{P}^{1}$ lifts to a function $f_{B}: B \rightarrow \widetilde{\mathbb{P}}^{1}$ as was shown in 3.5.1. As usual, we set $\partial^{+} B=\left\{b \in \partial B \mid f_{B}(b) \in \partial \widetilde{\mathbb{P}}^{1}\right.$ and $\left.\operatorname{Re}\left(f_{B}(b)\right) \geqslant 0\right\}$ and

$$
B^{\circ}=\left\{b \in B \mid \pi(b) \notin D, \text { or } f_{B}(b) \in \partial \widetilde{\mathbb{P}}^{1} \text { and } \operatorname{Re}\left(f_{B}(b)\right)>0\right\}
$$

and $\partial B^{\circ}=B^{\circ} \cap \partial B$. By Proposition 3.5.2 and its Corollary 3.5.3, the cohomology of the pairs $\left(B, \partial^{+} B\right)$ and $\left(B^{\circ}, \partial B^{\circ}\right)$ is canonically isomorphic to the rapid decay cohomology of $(X, f)$. The
following diagram summarises the situation.


Since $\bar{X}$ is compact, so is the real blow-up $B$ and its boundary $\partial B$, and the blow-up map $\pi: B \rightarrow X$ is proper. Let $U \subseteq B^{\circ}$ be an open subset, with boundary $\partial U=U \cap \partial B$. Since $U \backslash \partial U$ is an open subset of the complex algebraic variety $B^{\circ} \backslash \partial B^{\circ}=X$, it makes sense to speak about algebraic, rational, holomorphic or meromorphic functions on $U \backslash \partial U$. Informally, a function on an open of $B$ has moderate growth if it grows with at most polynomial speed near the boundary. It need not be defined on the boundary but can have a pole there. (Compare with §II, Definition, 2.6 in [21], or Section 9.2 in [75]).

Definition 7.5.7. - Let $U \subseteq \bar{X}$ be an open subset. We say that a function $h: U \backslash(U \cap D) \rightarrow \mathbb{C}$ has moderate growth on $U$ if for every point $x_{0} \in D \cap U$ there exists a neighbourhood $V$ of $x_{0}$ and a rational function $g$ on $V$ whose poles are contained in $D \cap V$, such that for some open $W \subseteq U \cap V$ the inequality $|h(x)| \leqslant|g(x)|$ holds for all $x \in W \backslash(W \cap D)$.

Let $U \subseteq B$ be an open subset. We say that a function $h: U \backslash(U \cap \partial B) \rightarrow \mathbb{C}$ has moderate growth if for every point $b_{0} \in \partial B \cap U$ there exists a neighbourhood $V$ of $\pi\left(b_{0}\right)$ and a rational function $g$ on $V$ whose poles are contained in $D \cap V$, such that for some open $W \subseteq U \cap \pi^{-1}(V)$ the inequality $|h(x)| \leqslant|g(x)|$ holds for $x \in W \backslash(W \cap \partial B)$.
7.5.8. - In the first part of the definition we could replace $V$ by $W$, hence assume $W=U \cap V$, but not so in the second part. As it is custom for meromorphic functions too, we will speak about functions of moderate growth on open subsets $U \subseteq X$ or $U \subseteq B$, when we really mean functions on $U \backslash(U \cap D)$ or $U \backslash(U \cap \partial B)$. Meromorphic functions on $X$ with poles on $D$ have moderate growth. Finite sums and products of functions of moderately growing functions grow moderately. Having moderate growth is a local condition, hence the presheaves on $X$ and on $B$ given by

$$
\begin{aligned}
\mathcal{O}_{X, D}^{\text {an }}(U) & =\text { holomorphic functions on } U \backslash \partial U \text { with moderate growth on } U \subseteq \bar{X} \\
\mathcal{O}_{B, \partial B}^{\text {an }}(U) & =\text { holomorphic functions on } U \backslash \partial U \text { with moderate growth on } U \subseteq B
\end{aligned}
$$

are indeed sheaves. For any open $U \subseteq X$, holomorphic functions on $U \backslash D$ with moderate growth are in fact meromorphic functions with poles in $D$. The sheaf we call $\mathcal{O}_{X, D}^{\text {an }}$ is more commonly denoted $\mathcal{O}_{X}^{\mathrm{an}}[* D]$.
7.5.9. - We now extend the definitions of sheaves of functions with moderate growth to differential forms with moderate growth. Following Hien and Roucairol, we define the sheaf of analytic
differential p-forms with moderate growth as the sheaves

$$
\begin{align*}
\Omega_{X, D}^{\mathrm{an}, p} & =\mathcal{O}_{X, D}^{\mathrm{an}} \otimes_{\mathcal{O}_{X}^{\mathrm{an}}} \Omega_{X}^{\mathrm{an}, p}  \tag{7.5.9.1}\\
\Omega_{B, \partial B}^{\mathrm{an}, p} & =\mathcal{O}_{B, \partial B}^{\mathrm{an}} \otimes_{\pi^{*}} \mathcal{O}_{X}^{\mathrm{an}} \pi^{*} \Omega_{X}^{\mathrm{an}, p} \tag{7.5.9.2}
\end{align*}
$$

on $X$, respectively on the real blow-up $B$. This looks more difficult than it is. On an open, say $U \subseteq B$, a section of $\Omega_{B, \partial B}^{\mathrm{an}, p}$ is a finite linear combination of expressions of the form $h \otimes \omega$ or just $h \omega$, where $h$ is a holomorphic function with moderate growth on $U$, and $\omega$ is a holomorphic $p$-form defined in a neighbourhood of $\pi(U)$. The usual rules of computation apply.

These sheaves of differential forms with moderate growth come equipped with obvious differential maps, which we all denote by $d_{f}$. Let us give the local description of the differential $\Omega_{B, \partial B}^{\mathrm{an}, p} \rightarrow \Omega_{B, \partial B}^{\mathrm{an}, p+1}$. Fix a point $b \in B$, set $x=\pi(b)$, and choose local coordinate functions $x_{1}, \ldots, x_{n}$ around $x \in X$ such that $D$ is given by the equation $x_{1} x_{2} \cdots x_{m}=0$ for some $0 \leqslant m \leqslant n$. If $m=0$, then $x$ lies not on $D$ and $b$ not on the boundary $\partial B$. For a subset $I \subseteq\{1,2, \ldots, n\}$, say with elements $i_{1}<i_{2}<\cdots<i_{p}$, set

$$
d x_{I}=d x_{i_{1}} d x_{i_{2}} \cdots d x_{i_{p}}
$$

so that in a neighbourhood of $x$ the $p$-forms $d x_{I}$ form a $\mathcal{O}_{X}^{\text {an }}$-basis of $\Omega_{X}^{\text {an,p }}$ as $I$ runs through the subsets of $\{1,2, \ldots, n\}$ of cardinality $p$. A moderate $p$-form $\eta$ can be written, in a sufficiently small neighbourhood of $b$, as

$$
\begin{equation*}
\eta=\sum_{\# I=p} u_{I} d x_{I} \tag{7.5.9.3}
\end{equation*}
$$

where the coefficients $u_{I}$ are holomorphic functions with moderate growth. The differential of $\eta$ is given by

$$
d_{f}(\eta)=\sum_{\# I=p}\left(\sum_{j \notin I} \frac{\partial u_{I}}{\partial x_{j}}+\frac{\partial f}{\partial x_{j}} u_{I}\right) d x_{j} d x_{I}
$$

where the inner sum could as well run over all $j \in\{1,2, \ldots, n\}$, only that the terms with $j \in I$ are zero. The description of the differential for smooth forms on $B$ is similar, only that this time we need to choose $2 n$ real coordinate functions on around $x=\pi(b)$.
7.5.10. - The real blow-up $B$ comes with the function $f: B \rightarrow \widetilde{\mathbb{P}}^{1}$. We denote by $\partial^{+} \widetilde{\mathbb{P}}^{1} \subseteq \widetilde{\mathbb{P}}^{1}$ the half-circle of nonnegative real part, and set $\partial^{+} B=f^{-1}\left(\partial^{+} \widetilde{\mathbb{P}}^{1}\right)$. Let us denote by $\kappa$ the inclusion of the open complement of $\partial^{+} B$ into $B$, so that cohomology on $B$ of the constructible sheaf

$$
\mathbb{C}_{\left[B, \partial^{+} B\right]}=\kappa_{!} \kappa^{*} \underline{\mathbb{C}}_{B}
$$

is the cohomology of the pair of spaces $\left[B, \partial^{+} B\right]$. The cohomology of the pair is indeed the rapid decay cohomology of $(X, f)$ with complex coefficients. We define a morphism of sheaves

$$
\varepsilon: \underline{\mathbb{C}}_{\left[B, \partial^{+} B\right]} \rightarrow \mathcal{O}_{B, \partial B}^{\mathrm{an}}
$$

on $B$ as follows: Given a connected open subset $U$ of $B$, we have either $U \cap \partial^{+} B \neq \varnothing$ in which case $\mathbb{C}_{\left[B, \partial^{+} B\right]}(U)=0$, or we have $U \cap \partial^{+} B=\varnothing$ in which case $\mathbb{C}_{\left[B, \partial^{+} B\right]}(U)=\mathbb{C}$ and we send $\lambda \in \mathbb{C}$ to the function $x \longmapsto \lambda e^{f(x)}$ on $U \backslash \partial U$, which indeed has moderate growth.

Theorem 7.5.11 (Poincaré Lemma 1). - The integration map $\left(\Omega_{B, \partial B}^{\mathrm{an},}, d_{f}\right) \rightarrow C_{B, \partial B}^{\bullet}$ is a quasiisomorphism of complexes of sheaves on $B$. More precisely, for every simply connected open subset $U \subseteq B$, the morphism of complexes of vector spaces given by integration

$$
I_{U}:\left(\Omega_{B, \partial B}^{\mathrm{an}, \stackrel{ }{c}}(U), d_{f}\right) \rightarrow C_{B, \partial B}^{\bullet}(U)
$$

is a quasi-isomorphism.
Proof. We must show that for every simply connected open set $U \subseteq B$ the complex of vector spaces

$$
0 \longrightarrow \mathbb{C}_{\left(B, \partial^{+} B\right)}(U) \xrightarrow{\varepsilon} \mathcal{O}_{B, \partial B}^{\mathrm{an}}(U) \xrightarrow{d_{f}} \Omega_{B, \partial B}^{\mathrm{an}, 1}(U) \xrightarrow{d_{f}} \Omega_{B, \partial B}^{\mathrm{an}, 2}(U) \xrightarrow{d_{f}} \cdots
$$

is exact. For notational convenience, let us introduce for $1 \leqslant i \leqslant n$ the linear differential operator

$$
Q_{i}(u)=\frac{\partial u}{\partial x_{i}}+\frac{\partial f}{\partial x_{i}} u
$$

and for a subset $J \subseteq\{1,2, \ldots, n\}$ and $j \in J$, let us write $\operatorname{sgn}_{J}(j)=(-1)^{\#\{i \in J \mid i<j\}}$ so that $d x_{j} d x_{J \backslash\{j\}}=\operatorname{sgn}_{J}(j) d x_{J}$ holds. With this notation, the differential of a moderate $p$-form $\eta$ on $U$ as in (7.5.9.3) is given by

$$
d_{f}(\eta)=\sum_{\# J=p+1}\left(\sum_{j \in J} \operatorname{sgn}_{J}(j) Q_{j}\left(u_{J \backslash\{j\}}\right)\right) d x_{J}
$$

Let $\omega$ be a moderate $(p+1)$-form on $U$ with $d_{f}(\omega)=0$, and let us show that $\omega=d_{f}(\eta)$ for some moderate $p$-form $\eta$ on $U$. We can write $\omega$ as

$$
\begin{equation*}
\omega=\sum_{\# J=p+1} h_{J} d x_{J} \quad 0=d_{f}(\omega)=\sum_{\# K=p+2}\left(\sum_{k \in K} \operatorname{sgn}_{K}(k) Q_{k}\left(h_{K \backslash\{k\}}\right)\right) d x_{K} \tag{7.5.11.1}
\end{equation*}
$$

and consider the largest integer $r \geqslant 1$ for which the implication $\{1,2, \ldots, r-1\} \cap J \neq \varnothing \Longrightarrow h_{J}=0$ holds. If $r=n+1$ then $\omega=0$ and there is nothing to prove. Reasoning by induction on $r$, we only need to show that there exists a $p$-form $\eta$, say as given by (7.5.9.3), such that the coefficient of $d x_{J}$ in $\omega-d_{f}(\eta)$ is zero whenever $\{1,2, \ldots, r\} \cap J \neq \varnothing$. This amounts to solving a system of linear partial differential equations in the unknown functions $u_{I}$. Concretely, this system is given by

$$
\begin{equation*}
0=h_{J}-\sum_{j \in J} \operatorname{sgn}_{J}(j) Q_{j}\left(u_{J \backslash\{j\}}\right) \tag{7.5.11.2}
\end{equation*}
$$

with one equation for every subset $J \subseteq\{1,2, \ldots, n\}$ with $p+1$ elements, containing at least one element $j \leqslant r$. Pick any $k \leqslant r-1$ and $J \subseteq\{1, \ldots, n\}$ of cardinality $p+1$ with $h_{J} \neq 0$, and set $K=J \cup\{k\}$. The term in $d_{f}(\omega)=0$ corresponding to $K$ just reads $Q_{k}\left(h_{J}\right)=0$ because for any other $s \in K$ we have $k \in K \backslash\{s\}$, hence $h_{K \backslash\{s\}}=0$ and hence $Q_{s}\left(h_{K \backslash\{s\}}\right)=0$. For a similar reason, we will suppose that $u_{I}=0$ as soon as $I$ contains an element $i<r$. One way of solving (7.5.11.2) is to produce for every subset $I \subseteq\{r, r+1, \ldots, n\}$ a solution $u_{I}$ of the partial differential equation

$$
\left(\Sigma_{I}\right):\left\{\begin{array}{l}
Q_{s}(u)=0 \quad \text { for } 1 \leqslant s<r  \tag{7.5.11.3}\\
Q_{r}(u)=h
\end{array}\right.
$$

with the given $h=h_{I \cup\{r\}}$, knowing that $h$ is holomorphic and has moderate growth and that the integrability condition

$$
\begin{equation*}
Q_{s}(h)=0 \quad \text { for } 1 \leqslant s<r \tag{7.5.11.4}
\end{equation*}
$$

holds. If these solutions $u_{I}$ are holomorphic and have moderate growth, then the form $\eta$ as given in (7.5.9.3) has the desired properties and the proof is done. The existence of the solutions $u_{I}$ is precisely what Theorem 7.5.5 provides.

### 7.6. Proof of the comparison isomorphism

We keep the notation and assumptions of the previous section as presented in 7.5.6.

Theorem 7.6.1 (Comparison isomorphism). - Let $k \subseteq \mathbb{C}$ be a subfield, let $X$ be a variety over $k$, let $Y \subseteq X$ a closed subvariety, and $f$ a regular function on $X$. The morphism of complex vector spaces

$$
\alpha_{[X, Y, f, n, i]}: H_{\mathrm{dR}}^{n}(X, Y, f)(i) \otimes_{k} \mathbb{C} \longrightarrow H_{\mathrm{rd}}^{n}(X, Y, f)(i) \otimes_{\mathbb{Q}} \mathbb{C}
$$

induced by the period pairing (7.0.0.1) is an isomorphism. Moreover, this isomorphism is functorial with respect to morphisms of type (a), (b), and (c) from Definition 4.2.1, and compatible with cup products.

## Corollary 7.6.2 (Poincaré Lemma 2). - The integration morphism

$$
\left(\Omega_{\bar{X}, D}^{\mathrm{an}, \bullet}, d_{f}\right) \longrightarrow \pi_{*} C_{\left(B, \partial^{+} B\right)}^{\bullet}
$$

is a quasi-isomorphism of complexes of sheaves on $\bar{X}$.

Proof. For every integer $p \geqslant 0$, using the definition of the sheaves of differential forms with moderate growth (7.5.9.2) and the fact that $\pi$ is a proper morphism, we get

$$
\pi_{*} \Omega_{B, \partial B}^{\text {an,p }}=\pi_{*}\left(\mathcal{O}_{B, \partial B}^{\text {an }} \otimes_{\pi^{*}} \mathcal{O}_{X}^{\text {an }} \pi^{*} \Omega_{X}^{\text {an }, p}\right)=\pi_{*} \mathcal{O}_{B, \partial B}^{\mathrm{an}} \otimes \otimes_{\bar{X}} \Omega_{\bar{X}}^{p}=\Omega_{\bar{X}, D}^{\text {an,p }} .
$$

Therefore, it suffices to show that, for every polydisk $U \subset \bar{X}$, the morphism of chain complexes

$$
\Omega_{B, \partial B}^{\bullet}\left(\pi^{-1}(U)\right) \longrightarrow C_{B, \partial+B}^{\bullet}\left(\pi^{-1}(U)\right)
$$

given by integration is a quasi-isomorphism. To this end, we choose a covering of $\pi^{-1}(U)$ by simply connected open subsets such that each intersection is simply connected or empty (this exists). The statement then follows from the Poincaré lemma 7.5.11 and...

Proof of Theorem 7.6.1. We can suppose that $X$ is smooth and $Y$ empty. Since de Rham cohomology and rapid decay cohomology are compatible with extension of scalars, we can as well
assume that $k=\mathbb{C}$ and work with complex coefficients. We then have

$$
\begin{aligned}
H_{\mathrm{dR}}^{n}(X, f) & \left.\simeq H^{n}\left(\bar{X},\left(\Omega_{\bar{X}, D}^{\mathrm{an}, \bullet}, d_{f}\right)\right) \quad \text { (Proposition } 7.2 .2\right) \\
& \simeq H^{n}\left(\bar{X}, R \pi_{*} C_{B, \partial^{+} B}^{\bullet}\right) \quad(\text { Corollary } 7.6 .2) \\
& \simeq H^{n}\left(B, C_{B, \partial^{+} B}^{\bullet}\right) \\
& \simeq H^{n}\left(B, \partial^{+} B\right) \\
& \simeq H_{\mathrm{rd}}(X, f) \quad \quad(\text { Corollary } 3.5 .3)
\end{aligned}
$$

## CHAPTER 8

## The period realisation

In this chapter, building on the comparison isomorphism from Chapter 7, we construct a realisation functor from $\mathbf{M}^{\exp }(k)$ to the category $\mathbf{P S}(k)$ of period structures over $k$.

### 8.1. Period structures

In this section, we introduce a tannakian formalism of period structures to which are associated period algebras, which permits us to deal abstractly with the situation where we are given vector spaces over $\mathbb{Q}$ (rapid decay cohomology) and over $k$ (de Rham cohomology) and a period isomorphism between their complexifications. We fix for the whole section a subfield $k$ of $\mathbb{C}$.

## Definition 8.1.1. -

(1) A period structure over $k$ is a triple ( $V, W, \alpha$ ) consisting of a finite-dimensional $\mathbb{Q}$-vector space $V$, a finite-dimensional $k$-vector space $W$, and an isomorphism of complex vector spaces $\alpha: V \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow W \otimes_{k} \mathbb{C}$.
(2) A morphism of period structures $(V, W, \alpha) \rightarrow\left(V^{\prime}, W^{\prime}, \alpha^{\prime}\right)$ is a pair $\left(f_{V}, g_{W}\right)$ consisting of a $\mathbb{Q}$-linear map $f_{V}: V \rightarrow V^{\prime}$ and a $k$-linear map $f_{W}: W \rightarrow W$ such that the following diagram of complex vector spaces commutes:


Denote the resulting category by $\operatorname{PS}(k)$. We equip it with the evident $\mathbb{Q}$-linear monoidal structure, and regard it as a neutral $\mathbb{Q}$-linear tannakian category with the forgetful functor

$$
(V, W, \alpha) \longmapsto V
$$

as fibre functor.

Definition 8.1.2. - Let $P=(V, W, \alpha)$ be a period structure, let $v_{1}, \ldots, v_{n}$ be a basis of $V$ and let $w_{1}, \ldots, w_{n}$ be a basis of $W$. Let $\underline{\alpha}$ be the matrix of $\alpha$ with respect to the bases $v_{1} \otimes 1, \ldots, v_{n} \otimes 1$ of $V \otimes_{\mathbb{Q}} \mathbb{C}$ and $w_{1} \otimes 1, \ldots, w_{n} \otimes 1$ of $W \otimes_{k} \mathbb{C}$. The period algebra associated with $P$ is the $k$-algebra $A$ generated by the coefficients of $\underline{\alpha}$ and $\operatorname{det}(\underline{\alpha})^{-1}$. The period! field of $P$ is the fraction field of $A$.
8.1.3. - Let $P=(V, W, \alpha)$ be a period structure. We call Galois group of $P$ the tannakian fundamental group $G$ of the full tannakian subcategory $\langle P\rangle$ of $\mathbf{P S}(k)$ generated by $P$. It is a linear algebraic group over $\mathbb{Q}$. Let $A$ be the period algebra of $P$. There is a canonical $G_{k}$-torsor $T$, called torsor of formal periods, and a canonical morphism $\operatorname{from} \operatorname{Spec}(A)$ to $T$, which we shall now construct.

Every object of $\langle P\rangle$ can be obtained from $P$ by tensor constructions and extracting subquotients. The category $\langle P\rangle$ comes equipped with two canonical functors: the fibre functor $\sigma$ with values in rational vector spaces given by $\sigma\left(V^{\prime}, W^{\prime}, \alpha^{\prime}\right)=V^{\prime}$, and the other one with values in $k$-vector spaces given by $\tau\left(V^{\prime}, W^{\prime}, \alpha^{\prime}\right)=W^{\prime}$. The group $G$ is the affine group scheme over $\mathbb{Q}$ which represents the following functor:

$$
G:\{\text { commutative } \mathbb{Q} \text {-algebras }\} \rightarrow\{\text { Groups }\} \quad G(R)=\operatorname{Aut}_{R}^{\otimes}(\sigma \otimes R) .
$$

To give an element of $G(R)$ is to give for every period structure ( $V^{\prime}, W^{\prime}, \alpha^{\prime}$ ) in $\langle P\rangle$ an $R$-linear automorphism $g_{\left(V^{\prime}, W^{\prime}, \alpha^{\prime}\right)}: V^{\prime} \otimes R \rightarrow V^{\prime} \otimes R$, and these automorphisms are required to be compatible with morphisms of period structures and tensor products. In particular, $g_{(V, W, \alpha)}$ determines $g_{\left(V^{\prime}, W^{\prime}, \alpha^{\prime}\right)}$ for every other object $\left(V^{\prime}, W^{\prime}, \alpha^{\prime}\right)$ of $\langle P\rangle$, hence $G$ can be viewed as a subgroup of $\mathrm{GL}_{V}$. The group $G_{k}=G \times_{\mathbb{Q}} k$ over $k$ is given by the "same" functor, but now viewed as a functor from $k$-algebras to groups.

Next, we wish to understand the torsor of formal periods $T$. This shall be a $G_{k}$-torsor (aka. principal homogeneous space), which we first describe as a functor:

$$
T:\{\text { commutative } k \text {-algebras }\} \rightarrow\{\text { Sets }\} \quad T(R)=\operatorname{Isom}_{R}^{\otimes}\left(\tau \otimes_{\mathbb{Q}} R, \sigma \otimes_{k} R\right) .
$$

The group $G_{k}(R)$ acts simply transitively on the set $T(R)$ on the left, for as long as $T(R)$ is not empty. Notice that $T(\mathbb{C})$ contains a canonical element given by $(V, W, \alpha) \longmapsto \alpha$, hence $T$ is not the empty functor. By representability of torsors under affine group schemes [63, Chapter III, Theorem 4.3a)], the functor $T$ is representable by an affine scheme of finite type $T$ over $k$.

Proposition 8.1.4. - Let $P$ be a period structure with torsor of formal periods $T$ and period algebra $A$. There exists a canonical closed immersion of $k$-schemes $\varepsilon: \operatorname{Spec}(A) \rightarrow T$. Its image is the Zariski closure of $\alpha \in T(\mathbb{C})$.

Proof. Set $T=\operatorname{Spec}(B)$. The complex point $\alpha$ on $T$ corresponds to a morphism of $k$-algebras $B \rightarrow \mathbb{C}$, namely the evaluation at $\alpha$. We claim that the image in $\mathbb{C}$ of this evaluation morphism is the period algebra $A$. Once this claim is proven, we define $\varepsilon: \operatorname{Spec}(A) \rightarrow T$ to be the corresponding morphism of affine schemes. This morphism $\varepsilon$ is then indeed a closed immersion since $A$ is an integral ring, and its image is the Zariski closure of $\alpha \in T(\mathbb{C})$ by construction.

A regular function on $T$ is uniquely determined by a regular function on the variety of $k$-linear isomorphisms from $V \otimes_{k} k$ to $W$, which is affine and contains $T$ as a closed subvariety. Thus, given bases $v_{1}, \ldots, v_{n}$ of $V$ and $w_{1}, \ldots, w_{n}$ of $W$, the algebra $B$ is generated by elements $b_{i j}$ and $\operatorname{det}\left(\left(b_{i j}\right)_{1 \leqslant i, j \leqslant n}\right)^{-1}$. An $R$-valued point of $t \in T(R)$ is an isomorphism $t: V \otimes_{\mathbb{Q}} R \rightarrow W \otimes_{k} R$ and the evaluation of $b_{i j}$ at $t$ is determined by the formula

$$
t\left(v_{i} \otimes 1\right)=\sum_{j=1}^{n} w_{j} \otimes b_{i j}(t)
$$

which in the case $R=\mathbb{C}$ and $t=\alpha$ shows the desired equality.
8.1.5. - Here is an alternative, equivalent definition of $\varepsilon$ as a morphism of representable functors $\varepsilon: \operatorname{Spec}(A) \rightarrow T$. Fix bases of $V$ and $W$ as in the proof of the proposition. For every morphism of $k$-algebras $f: A \rightarrow R$, we obtain an $R$-linear isomorphism $V \otimes_{\mathbb{Q}} R \rightarrow W \otimes_{k} R$ given by

$$
\varepsilon(f)\left(v_{i} \otimes 1\right)=\sum_{j=1}^{n} w_{i} \otimes f\left(a_{i j}\right)
$$

which is independent of the choice of bases and defines an element of $T(R)$. If $g: A \rightarrow R$ is another algebra morphism, then $\varepsilon(f)=\varepsilon(g)$ implies $f\left(a_{i j}\right)=g\left(a_{i j}\right)$ for all $1 \leqslant i, j \leqslant n$, hence $f=g$. Therefore $\varepsilon$ is injective.
8.1.6. - Let $P=(V, W, \alpha)$ be a period structure with Galois group $G \subseteq \mathrm{GL}_{V}$. We can characterise the group $G$ as follows: it is the smallest algebraic subgroup of $\mathrm{GL}_{V}$ such that there is a closed $k$-subscheme $T \subseteq \operatorname{Isom}(V \otimes k, W)$ which is a $G$-torsor and such that $\alpha$ is a complex point of $T$.

Definition 8.1.7. - Let $P$ be a period structure with torsor of formal periods $T$ and period algebra $A$. We say that $P$ is normal if the canonical morphism $\varepsilon: \operatorname{Spec}(A) \rightarrow T$ is an isomorphism.
8.1.8. - If two period structures $P$ and $P^{\prime}$ generate the same tannakian subcategory of $\mathbf{P S}(k)$, then $P$ and $P^{\prime}$ have canonically isomorphic Galois groups and period torsors, and their period algebras are equal. Hence $P$ is normal if and only if $P^{\prime}$ is. It is not hard to show that any substructure, quotient structure or tensor construction of a normal period structure is again normal. However, the sum of two normal structures might not be normal (see Example (2) below)

Example 8.1.9. - It is not hard to give examples of normal and non-normal period structures.
(1) Consider the case $k=\mathbb{Q}$ and $V=W=\mathbb{Q}$, so that $\alpha$ is just a complex number. The period structure $(\mathbb{Q}, \mathbb{Q}, \alpha)$ is normal if and only if $\alpha$ is transcendental or an $n$-th root of a rational number.
(2) By the previous example, both $(\mathbb{Q}, \mathbb{Q}, \pi)$ and $(\mathbb{Q}, \mathbb{Q}, \pi+1)$ are normal period structures. Their sum is however not normal, as the corresponding period algebra is $A=$ $\mathbb{Q}\left[x, x^{-1},(x+1)^{-1}\right]$ and $\operatorname{Spec}(A)=\mathbb{A}^{1} \backslash\{0,-1\}$ is not a torsor under an algebraic group.

Assuming the formal period conjecture, this means that $\pi+1$ is not a period of a onedimensional motive (although it is of course a period of a two-dimensional motive).
(3) Let $F$ be a finite field extension of $k$. Let $V$ be the rational vector space with basis the complex embeddings $\varphi_{1}, \ldots, \varphi_{n}$ of $F$, let $w_{1} \ldots, w_{n}$ be a $k$-basis of $W=F$ and set

$$
\alpha\left(\varphi_{i} \otimes 1\right)=\sum_{j=1}^{n} w_{j} \otimes \varphi_{i}\left(w_{j}\right) .
$$

The period structure $(V, W, \alpha)$ is normal. The period algebra of $(V, W, \alpha)$ is the normalisation of $F$ in $\mathbb{C}$.

Proposition 8.1.10. - Let $P_{0}$ be normal period structure. The following holds:
(0) The unit structure $(\mathbb{Q}, k, 1)$ is normal.
(1) Every substructure, quotient and tensor construction of $P_{0}$ is normal.

Proof. Statement (0) is trivial. To prove statement (1), pick any substructure $P=(V, W, \alpha)$ of $P_{0}$. The Galois group $G$ of $P$ is a quotient of the Galois group $G_{0}$ of $P_{0}$, and there is a corresponding surjective morphism of formal period torsors $T_{0} \rightarrow T$. On $R$-points, the map $T_{0}(R) \rightarrow T(R)$ is given by restriction. The period algebra $A$ of $P$ is contained in the period algebra $A_{0}$ of $P_{0}$, and the diagram

commutes, hence $\varepsilon: \operatorname{Spec}(A) \rightarrow T$ is surjective, hence an isomorphism. The same argument settles the case where $P$ is a quotient or a tensor construction of $P_{0}$, or in fact any object in the tannakian category $\left\langle P_{0}\right\rangle$ generated by $P_{0}$, hence statement (1) is proven.

### 8.2. The period realisation and the de Rham realisation

In this section, we construct a realisation functor

$$
\mathrm{R}_{\mathrm{PS}}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{P S}(k)
$$

from the category of exponential motives to the category of period structures over $k$, which we call the period realisation. As a byproduct, composing with the forgetful functor $\mathbf{P S}(k) \rightarrow \mathbf{V e c}_{k}$, we shall obtain a fibre functor

$$
\mathrm{R}_{\mathrm{dR}}: \mathbf{M}^{\exp }(k) \longrightarrow \mathbf{V e c}_{k},
$$

which we call the de Rham realisation, as well as a canonical isomorphism $\mathrm{R}_{\mathrm{dR}} \otimes_{k} \mathbb{C} \simeq \mathrm{R}_{B} \otimes_{\mathbb{Q}} \mathbb{C}$ of fibre functors on the category of exponential motives.
8.2.1. - The period realisation functor will be constructed by means of Nori's universal property (Theorem 4.1.12). We thus need to define a functor from the quiver of exponential relative varieties $\mathrm{Q}^{\exp }(k)$ to the category of period structures $\mathbf{P S}(k)$ which is compatible, upon application of the forgetful functor $\mathbf{P S}(k) \rightarrow \mathbf{V e c}_{\mathbb{Q}}$, with the standard rapid decay representation, in the sense that the exterior triangle in diagram (8.2.2.1) below commutes. With every object $q=[X, Y, f, n, i]$ in $\mathrm{Q}^{\exp }(k)$ we associate the period structure

$$
\sigma(q)=\left(H_{\mathrm{dR}}^{n}(X, Y, f)(i), H_{\mathrm{rd}}^{n}(X, Y, f)(i), \alpha_{[X, Y, f, n, i]}\right)
$$

where $\alpha_{[X, Y, f, n, i]}$ is the comparison isomorphism from Theorem 7.6.1. Since these isomorphisms are natural with respect to morphisms of types (a), (b), and (c) by loc.cit., we obtain in this way a functor $\sigma: \mathrm{Q}^{\exp }(k) \rightarrow \mathbf{P S}(k)$ with values in the category of period structures.

Definition 8.2.2. - The period realisation functor $\mathrm{R}_{\mathrm{PS}}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{P S}(k)$ is the unique functor which renders the following diagram commutative:


The de Rham realisation $\mathrm{R}_{\mathrm{dR}}: \mathbf{M}{ }^{\exp }(k) \rightarrow \mathbf{V e c}_{k}$ is the composite of the period realisation functor and the forgetful functor $\mathbf{P S}(k) \rightarrow \mathbf{V e c}_{k}$.
8.2.3. - The period realisation functor is compatible with tannakian structures. Therefore, $\mathrm{R}_{\mathrm{dR}}: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{V e c}_{k}$ is a fibre functor. The scheme of tensor isomorphisms $\underline{\mathrm{Isom}}^{\otimes}\left(R_{\mathrm{dR}}, R_{B} \otimes_{\mathbb{Q}} k\right)$ forms a torsor under the motivic exponential Galois group.

Given an object $M$ of $\mathbf{M}^{\exp }(k)$, we denote by $\langle M\rangle$ the smallest tannakian subcategory containing $M$. Let $G_{M}$ be the tannakian fundamental group of $M$ and set

$$
T_{M}=\underline{\mathrm{Isom}}^{\otimes}\left(\left.R_{\mathrm{dR}}\right|_{\langle M\rangle},\left.R_{\mathrm{dR}}\right|_{\langle M\rangle} \otimes_{\mathbb{Q}} k\right)
$$

It is a torsor under $G_{M}$ defined over $k$ and comes equipped a canonical complex point

$$
\alpha_{M}: \operatorname{Spec}(\mathbb{C}) \longrightarrow T_{M}
$$

Conjecture 8.2.4. - The $k$-variety $T_{M}$ is irreducible and $\alpha_{M}$ is its generic point.
Since $T_{M}$ it is a torsor under $G_{M}$, it is a smooth variety, and hence $T_{M}$ is irreducible if and only if $T_{M}$ is connected. Assuming that $T_{M}$ is connected, the conjecture amounts to the equality
of dimensions

$$
\operatorname{dim}_{k}{\overline{\alpha_{M}}}^{\mathrm{Zar}}=\operatorname{dim}_{k} T_{M}=\operatorname{dim}_{\mathbb{Q}} G_{M} .
$$

The Zariski closure of $\alpha_{M}$ is the spectrum of the period algebra and has dimension the transcendence degree of the field obtained by adjoining to $\overline{\mathbb{Q}}$ the periods of $M$.

Conjecture 8.2.5. - $T_{M}$ is connected

$$
\begin{gathered}
\operatorname{trdeg} \overline{\mathbb{Q}}(\text { periods of } M)=\operatorname{dim} G_{M} . \\
{\overline{\alpha_{M}}}^{\mathrm{Zar}} \subset T_{P} \subset T_{M}
\end{gathered}
$$

Consequence: inclusion $G_{P} \hookrightarrow G_{M}$ is an equality. In tannakian terms, the period realisation functor $\langle M\rangle \rightarrow\langle P\rangle$ is an equivalence of categories, which amounts to say that it is full and the essential image is stable under subobjects.

Conjecture 8.2.6 (Exponential period conjecture). - The period realisation functor is fully faithful and stable under subobjects. For every motive $M$, the associated period structure $\mathrm{R}_{\mathrm{PS}}(M)$ is normal.
8.2.7. - The period conjecture 8.2.6 consists of two statements. The full faithfulness of the period realisation functor is sometimes referred to as formal period conjecture. Given a motive $M$ with period structure $P=\mathrm{R}_{\mathbf{P S}}(M)$ and writing $G_{M}$ and $G_{P}$ for the tannakian fundamental groups of $M$ and $P$, the formal part of the period conjecture states that the inclusion of algebraic groups

$$
G_{P} \xrightarrow{\subseteq} G_{M}
$$

is an equality. This equality of groups can be verified in many examples, often by some trickery with algebraic groups and very limited information about the involved periods. The second statement of Conjecture 8.2.6 is that the period structure $P$ of $M$ is normal.

This leads to the following numerical variant of the exponential period conjecture.

Conjecture 8.2.8. - Let $M$ be an exponential motive over $\overline{\mathbb{Q}}$ with motivic Galois group $G_{M}$. Then

$$
\operatorname{trdeg} \overline{\mathbb{Q}}(\text { periods of } M)=\operatorname{dim} G_{M} .
$$

The following theorem is nothing else but a restatement of Theorem 7.6.1.

Theorem 8.2.9. - Let $\mathrm{Q}^{\exp }(k)$ denote the quiver of exponential relative varieties over $k$ from Definition 4.2.1. There exists a canonical isomorphism of quiver representations

$$
\operatorname{comp}_{B, \mathrm{dR}}: \mathrm{R}_{\mathrm{dR}} \otimes_{k} \mathbb{C} \xrightarrow{\sim} \mathrm{R}_{B} \otimes_{\mathbb{Q}} \mathbb{C} .
$$

### 8.3. Motivic exponential periods

Proposition 8.3.1. - The scheme of tensor isomorphisms $\underline{\operatorname{Isom}}^{\otimes}\left(R_{\mathrm{dR}}, R_{B}\right)$ forms a torsor under the motivic exponential Galois group.

Definition 8.3.2. - The ring of motivic exponential periods is

$$
\begin{equation*}
\mathcal{P}_{\exp }^{\mathfrak{m}}=\mathcal{O}\left(\underline{\operatorname{Isom}}^{\otimes}\left(\mathrm{R}_{\mathrm{dR}}, \mathrm{R}_{B}\right)\right) . \tag{8.3.2.1}
\end{equation*}
$$

A typical object of $\mathcal{P}_{\text {exp }}^{\mathfrak{m}}$ is a triple $[M, \omega, \gamma]^{\mathfrak{m}}$ consisting of an exponential motive $M$ in $\mathbf{M}^{\exp }(\mathbb{Q})$, together with elements $\omega \in \mathrm{R}_{\mathrm{dR}}(M)$ and $\sigma \in \mathrm{R}_{B}(M)^{\vee}$. Such a triple is called a matrix coefficient and defines a regular function on the scheme of tensor isomorphisms via

$$
\underline{\mathrm{Isom}}^{\otimes}\left(R_{\mathrm{dR}}, R_{B}\right) \rightarrow \mathbb{A}_{\mathbb{Q}}^{1}, \quad \varphi \longmapsto\langle\varphi(\omega), \sigma\rangle .
$$

Indeed, one can show that $\mathcal{P}_{\exp }^{\mathfrak{m}}$ is the $\mathbb{Q}$-algebra generated by the matrix coefficients $[M, \omega, \sigma]^{\mathfrak{m}}$ modulo the following two relations:
(i) Bilinearity: for all $\lambda_{1}, \lambda_{2}, \mu_{1}, \mu_{2} \in \mathbb{Q}$ :

$$
\begin{aligned}
& {\left[M, \lambda_{1} \omega_{1}+\lambda_{2} \omega_{2}, \sigma\right]^{\mathfrak{m}}=\lambda_{1}\left[M, \omega_{1}, \sigma\right]^{\mathfrak{m}}+\lambda_{2}\left[M, \omega_{2}, \sigma\right]^{\mathfrak{m}}} \\
& {\left[M, \omega, \mu_{1} \sigma_{1}+\mu_{2} \sigma_{2}\right]^{\mathfrak{m}}=\mu_{1}\left[M, \omega, \sigma_{1}\right]^{\mathfrak{m}}+\mu_{2}\left[M, \omega, \sigma_{2}\right]^{\mathfrak{m}}}
\end{aligned}
$$

(ii) Functoriality: if $f: M_{1} \rightarrow M_{2}$ is a morphism in $\mathbf{M}^{\exp }(\mathbb{Q})$ such that $\omega_{2}=R_{\mathrm{dR}}(f)\left(\omega_{1}\right)$ and $\sigma_{1}=R_{B}(f)^{\vee}\left(\sigma_{2}\right)$, then

$$
\left[M_{1}, \omega_{1}, \sigma_{1}\right]^{\mathrm{m}}=\left[M_{2}, \omega_{2}, \sigma_{2}\right]^{\mathrm{m}}
$$

The product is defined as

$$
\left[M_{1}, \omega_{1}, \sigma_{1}\right]^{\mathrm{m}}\left[M_{2}, \omega_{2}, \sigma_{2}\right]^{\mathrm{m}}=\left[M_{1} \otimes M_{2}, \omega_{1} \otimes \omega_{2}, \sigma_{1} \otimes \sigma_{2}\right]^{\mathrm{m}} .
$$

Evaluation at comp $\in \mathcal{P}_{\exp }^{\mathrm{m}}(\mathbb{C})$ yields a map

$$
\text { per : } \mathcal{P}_{\exp }^{\mathfrak{m}} \longrightarrow \mathbb{C} .
$$

The main reason to consider motivic exponential periods is that they come with a new structure, invisible at the level of numbers:

$$
\begin{equation*}
\Delta: \mathcal{P}_{\exp }^{\mathrm{m}} \longrightarrow \mathcal{P}_{\exp }^{\mathrm{m}} \otimes_{\mathbb{Q}} \mathcal{O}(G) . \tag{8.3.2.2}
\end{equation*}
$$

Let $e_{1}, \ldots, e_{n}$ be a basis of $\mathrm{R}_{B}(M)$. Then:

$$
\begin{equation*}
\Delta[M, \omega, \gamma]^{\mathfrak{m}}=\sum_{i=1}^{n}\left[M, \omega, e_{i}^{\vee}\right] \otimes\left[M, e_{i}, \gamma\right] \tag{8.3.2.3}
\end{equation*}
$$

## CHAPTER 9

## The $\mathscr{D}$-module realisation

In this chapter, we construct a realisation functor from the category of exponential motives to a tannakian category of $\mathscr{D}$-modules over the affine line that is the de Rham counterpart of $\operatorname{Perv}_{0}$. Throughout, $k$ denotes a subfield of the complex numbers and we set $\mathbb{A}^{1}=\operatorname{Spec} k[x]$. References: [55, Chapter 12], [58, 4.2]

### 9.1. Preliminaries on $\mathscr{D}$-modules

We start by recollecting some basic facts about $\mathscr{D}$-modules on algebraic varieties. Standard references for this section are [14] and [52]. Let $X$ be a variety of dimension $d$ over $k$. Let $\mathscr{D}_{X}$ be the sheaf of differential operators on $X$.

Definition 9.1.1. - A holonomic $\mathscr{D}_{X}$-module is a coherent $\mathscr{D}_{X}$-module $\mathscr{M}$ satisfying the following two properties:
(1) $\operatorname{Ext}_{\mathscr{D}_{X}}^{i}\left(\mathscr{M}, \mathscr{D}_{X}\right)=0$ for all $i<d$
(2) the characteristic variety $\operatorname{Char}(\mathscr{M})$ has dimension $d$.

Six operations formalism: $f_{+}, f^{+}, f_{\dagger}, f^{\dagger}$
Introduce regular singular holonomic $\mathscr{D}$-modules. We denote by $\operatorname{Mod}_{\text {rh }}\left(\mathscr{D}_{X}\right)$ the abelian category of regular singular holonomic $\mathscr{D}$-modules on the variety $X$.

Example 9.1.2. - With each regular function $f: X \rightarrow \mathbb{A}^{1}$ on an algebraic variety $X$ is associated the locally free $\mathcal{O}_{X}$-module with connection

$$
\mathcal{E}^{f}=\left(\mathcal{O}_{X}, d-d f\right) .
$$

If $f$ is the composition of morphisms $g: X \rightarrow Y$ and $h: Y \rightarrow \mathbb{A}^{1}$, then this $\mathscr{D}_{X}$-module is equal to $g^{+} \mathcal{E}^{h}$. In particular, if $t$ denotes a coordinate on $\mathbb{A}^{1}$, the equality $\mathcal{E}^{f}=f^{+} \mathcal{E}^{t}$ holds.
9.1.3. - Let $k=\mathbb{C}$. Introduce the de Rham functor

$$
\begin{gathered}
\mathrm{DR}_{X}(\mathscr{M})=\left[\mathscr{M}^{\text {an }} \longrightarrow \Omega_{X^{\text {an }}}^{1} \otimes_{\Omega_{X^{\text {an }}}^{1}} \mathscr{M}^{\text {an }} \longrightarrow \cdots \longrightarrow \Omega_{X^{\text {an }}}^{d} \otimes_{\Omega_{X^{\text {an }}}^{1}} \mathscr{M}^{\text {an }}\right],
\end{gathered}
$$

where $\mathscr{M}^{\text {an }}$ sits in degree $-d$.

Theorem 9.1.4 (Riemann-Hilbert correspondence). - Let $X$ be a smooth complex algebraic variety. The de Rham functor induces an equivalence of categories

$$
\mathrm{DR}_{X}: \operatorname{Mod}_{\mathrm{rh}}\left(\mathscr{D}_{X}\right) \longrightarrow \operatorname{Perv}(X(\mathbb{C}), \mathbb{C})
$$

For example, if $\mathscr{M}$ is a vector bundle with connection on $X$ with associated local system $F$, the corresponding perverse sheaf is $F[\operatorname{dim} X]$.

### 9.2. Holonomic $\mathscr{D}$-modules on the affine line

In what follows, we will mainly deal with $\mathscr{D}$-modules on the affine line. In this case, the ring of differential operators $\mathscr{D}_{\mathbb{A}^{1}}$ is the Weyl algebra $k[x]\left\langle\partial_{x}\right\rangle$, that is, the non-commutative $k$-algebra obtained by quotienting the free algebra generated by the polynomial algebras $k[x]$ and $k\left[\partial_{x}\right]$ by the relation $\left[\partial_{x}, x\right]=1$. Each element $L$ of the Weyl algebra can be uniquely written as

$$
L=a_{d}(x) \partial_{x}^{d}+a_{d-1}(x) \partial_{x}^{d-1}+\cdots+a_{0}(x)
$$

where $a_{i} \in k[x]$ are polynomials and $a_{d}$ is non-zero; the integer $d$ is called the degree of $L$. A $\mathscr{D}$-module on the affine line is a left $k[x]\left\langle\partial_{x}\right\rangle$-module of finite type $\mathscr{M}$. This amounts to the data of a $k[x]$-module $\mathscr{M}$ together with a connection, that is, a $k$-linear map $\partial_{x}: \mathscr{M} \rightarrow \mathscr{M}$ that satisfies the Leibniz rule $\partial_{x}(f m)=f \partial_{x}(m)+\partial_{x}(f) m$. A $\mathscr{D}$-module on the affine line $\mathscr{M}$ is said to be holonomic if every element of $\mathscr{M}$ is annihilated by a non-zero element of the Weyl algebra.
9.2.1 (Regular and irregular singularities). - Let $L\left(x, \partial_{x}\right)=\sum a_{i}(x) \partial_{x}^{i}$ be a differential operator of degree $d$. The singularities at finite distance of $L$ are the roots of the polynomial $a_{d}$. If $\alpha$ is such a root, we say that $L$ has a regular singularity at $\alpha$ if the so-called Fuchs criterium

$$
i-\operatorname{ord}_{\alpha}\left(a_{i}\right) \leqslant d-\operatorname{ord}_{\alpha}\left(a_{d}\right)
$$

holds for all $i=1, \ldots, d-1$, where $\operatorname{ord}_{\alpha}\left(a_{i}\right)$ stands for the order of vanishing at $\alpha$. Otherwise, we say that $L$ has an irregular singularity at $\alpha$, and we call the integer

$$
\operatorname{irr}_{\alpha}(L)=\max _{i=1, \ldots, d-1}\left\{i-d+\operatorname{ord}_{\alpha}\left(a_{d}\right)-\operatorname{ord}_{\alpha}\left(a_{i}\right)\right\}>0
$$

the irregularity of $L$ at $\alpha$. To study the behaviour of $L$ at infinity, we make the change of variables $x=1 / t$ and $\partial_{x}=-t^{2} \partial_{t}$ and consider the operator $L_{\infty}\left(t, \partial_{t}\right)=\sum a_{i}(1 / t)\left(-t^{2} \partial_{t}\right)^{i}$. We say that $L$ has a regular (resp. irregular) singularity at infinity if $L_{\infty}$ has a regular (resp. irregular) singularity at $t=0$. Finally, a holonomic $\mathcal{D}$-module $\mathscr{M}$ is said to have regular singularities if every element of $\mathscr{M}$ is annihilated by a non-zero differential operator with regular singularities.

The Newton polygon of the differential operator $L\left(x, \partial_{x}\right)=\sum_{i=0}^{d} a_{i}(x) \partial_{x}^{i}$ is the convex hull of the set of points

$$
\left\{\left(-i, \operatorname{ord}\left(a_{i}\right)\right) \mid i=0, \ldots, d, a_{i} \neq 0\right\}
$$

in $\mathbb{R}^{2}$. The slopes of $L$ are the slopes of this Newton polygon.

Example 9.2.2. - The trivial $\mathscr{D}_{\mathbb{A}^{1}}$-module $\mathcal{O}_{\mathbb{A}^{1}}$ is $k[x]$ on which $\partial_{x}$ acts by derivation. Let $j: \mathbb{A}^{1} \backslash\{0\} \hookrightarrow \mathbb{A}^{1}$ be the inclusion. The functor $j_{+}$Then:

$$
j_{+} j^{+} \mathcal{O}_{\mathbb{A}^{1}}=\mathscr{D} / \mathscr{D} \partial_{x} x, \quad j_{+} j^{+} \mathcal{O}_{\mathbb{A}^{1}}=\mathscr{D} / \mathscr{D} x \partial_{x}
$$

The Riemann-Hilbert correspondence sends the $\mathscr{D}$-module $j_{\dagger} j^{+} \mathcal{O}_{\mathbb{A}^{1}}$ to the perverse sheaf $j!j^{*} \mathbb{C}[1]$.

### 9.3. Additive convolution

9.3.1. - Let $\pi: \mathbb{A}^{1} \rightarrow \operatorname{Spec}(k)$ be the structural morphism.

Lemma 9.3.2. -
9.3.3. - Let $\operatorname{Hol}_{\mathrm{rs}}\left(\mathbb{A}^{1}\right)_{0}$ be the full subcategory of $\operatorname{Hol}_{\mathrm{rs}}\left(\mathbb{A}^{1}\right)$ consisting of those holonomic $\mathscr{D}_{\mathbb{A}^{1}}$-modules $\mathscr{M}$ such that the operator $\partial_{x}: \mathscr{M} \rightarrow \mathscr{M}$ is invertible.

Definition 9.3.4. - Let $\mathscr{M}$ and $\mathscr{N}$ be objects of $D\left(\mathscr{D}_{\mathbb{A}^{1}}\right)$. The additive convolution of $\mathscr{M}$ and $\mathscr{N}$ is the object

$$
\mathscr{M} * \mathscr{N}=\operatorname{sum}_{+}(\mathscr{M} \boxtimes \mathscr{N})
$$

of $D\left(\mathscr{D}_{\mathbb{A}^{1}}\right)$.

### 9.4. Fourier transform

Since $\left[\partial_{x}, x\right]=\left[-x, \partial_{x}\right]$, the map

$$
\text { FT: } \begin{align*}
k[x]\left\langle\partial_{x}\right\rangle & \longrightarrow k[y]\left\langle\partial_{y}\right\rangle  \tag{9.4.0.1}\\
x & \longmapsto \partial_{y} \\
\partial_{x} & \longmapsto-y
\end{align*}
$$

is an isomorphism of $k$-algebras.

Definition 9.4.1. - Let $\mathscr{M}$ be a $\mathscr{D}$-module on the affine line $\mathbb{A}_{x}^{1}=\operatorname{Spec} k[x]$. The Fourier transform of $\mathscr{M}$ is the same $\mathscr{M}$ viewed as a $\mathscr{D}$-module on the dual affine line $\mathbb{A}_{y}^{1}=\operatorname{Spec} k[y]$ through the isomorphism (9.4.0.1). We shall denote it by $\mathrm{FT}(\mathscr{M})$. Formally,

$$
\operatorname{FT}(\mathscr{M})=k[y]\left\langle\partial_{y}\right\rangle \otimes_{k[x]\left\langle\partial_{x}\right\rangle} \mathscr{M} .
$$

It follows immediately from the definition that Fourier transform preserves holonomicity.

Example 9.4.2. - Let $L \in k[x]\left\langle\partial_{x}\right\rangle$ be a differential operator. The Fourier transform of the $\mathscr{D}$-module $k[x]\left\langle\partial_{x}\right\rangle / L$ is the $\mathscr{D}$-module $k[y]\left\langle\partial_{y}\right\rangle / \mathrm{FT}(L)$. For example, the Fourier transform of $k[x]\left\langle\partial_{x}\right\rangle /$ is
9.4.3. - Let $\widehat{\mathbb{A}}^{1}=\operatorname{Spec} k[y]$ and consider the diagram


Proposition 9.4.4. - The Fourier transform of a $\mathscr{D}$-module $\mathscr{M}$ is defined as

$$
\mathrm{FT}(\mathscr{M})=\left(p_{y}\right)_{+}\left(p_{x}^{+} \mathscr{M} \otimes \mathcal{E}^{x y}\right) .
$$

Proof.
9.4.5. - The Fourier transform FT and the projector $\Pi$ are compatible with each other in that there is a canonical isomorphism of functors

$$
\mathrm{FT} \circ \Pi \cong j_{+} j^{+} \mathrm{FT}
$$

9.4.6. - Let $\operatorname{Hol}_{\mathrm{rs}}\left(\mathbb{A}^{1}\right)$ be the abelian category of holonomic $\mathscr{D}$-modules with regular singularities on the affine line.

Proposition 9.4.7. - Via Fourier transform, the category $\operatorname{Hol}_{\mathrm{rs}}\left(\mathbb{A}^{1}\right)$ is equivalent to the category of holonomic $\mathscr{D}$-modules $\mathscr{M}$ on the dual affine line $\mathbb{A}_{y}^{1}$ which are smooth on $\mathbb{A}^{1} \backslash\{0\}$, have a regular singularity at 0 , and possibly a irregular singularity of exponential type at infinity.
9.4.8. - Let $X$ be a smooth variety together with a regular function $f: X \rightarrow \mathbb{A}^{1}$. Consider $\mathbb{G}_{m}$ with coordinate $z$ and $\mathbb{A}^{1}$ with coordinate $t$, and let $q: X \times \mathbb{G}_{m} \rightarrow \mathbb{G}_{m}$ and $p: \mathbb{A}^{1} \times \mathbb{G}_{m} \rightarrow \mathbb{G}_{m}$ denote the projection, so that the diagram

commutes. Let $M$ be a $\mathscr{D}$-module on $X$ and consider the $\mathscr{D}$-module $N=M \boxtimes \mathcal{O}_{\mathbb{G}_{m}}$ on $X \times \mathbb{G}_{m}$. There are isomorphisms

$$
\begin{array}{rlcl}
q_{+}\left(N \otimes \mathcal{E}^{z f}\right) & \cong & p_{+}\left((f \times \mathrm{Id})_{+}\left(N \otimes \mathcal{E}^{z f}\right)\right) & \\
& \cong & p_{+}\left((f \times \mathrm{Id})_{+}\left(N \otimes(f \times \mathrm{Id})^{+} \mathcal{E}^{z t}\right)\right) & \\
& \cong & p_{+}\left((f \times \mathrm{Id})_{+} N \otimes \mathcal{E}^{z t}\right) & \\
& \cong \text { Prample 9.1.2 } \\
& \cong & p_{+}\left(\left(f_{+} M \boxtimes \mathcal{O}_{\mathbb{G}_{m}}\right) \otimes \mathcal{E}^{z t}\right) & \\
& \cong & p_{+}\left(p^{+} f_{+} M \otimes \mathcal{E}^{z t}\right) & \text { Künneth formula } \\
& \cong & j^{+} \mathrm{FT}\left(f_{+} M\right) & \\
& \text { Künneth formula } \\
& & \text { Proposition 9.4.4 }
\end{array}
$$

### 9.5. The $\mathscr{D}$-module realisation

Let $\mathbf{P S}\left(\mathbb{A}^{1}\right)$ be the category whose objects are triples $(\mathscr{M}, C, \alpha)$ consisting of a regular holonomic $\mathscr{D}$-module on $\mathbb{A}^{1}$, a $\mathbb{Q}$-perverse sheaf $C$ on $\mathbb{A}^{1}(\mathbb{C})$, and an isomorphism $\alpha: \operatorname{DR}(\mathscr{M}) \xrightarrow{\sim} C \otimes_{\mathbb{Q}} \mathbb{C}$.

Proposition 9.5.1. - Let $X$ be a smooth variety, let $Y \subset X$ be a smooth closed subvariety with open complement $\beta: X \backslash Y \hookrightarrow X$, and let $f: X \rightarrow \mathbb{A}^{1}$ be a regular function. For each integer $n \geqslant 0$, the fibre at $y=1$ of the Fourier transform of the $\mathscr{D}_{\mathbb{A}^{1}}$-module $\mathscr{H}^{n}\left(f_{+}\left(\beta_{\dagger} \beta^{+} \mathcal{O}_{X}\right)\right)$ is canonically isomorphic to the de Rham cohomology $H_{\mathrm{dR}}^{n}(X, Y, f)$.

Proof.
9.6.

## CHAPTER 10

## The $\ell$-adic realisation

### 10.1. The perverse $\ell$-adic realisation

### 10.2. Reduction modulo $p$ via nearby fibres

Let $p$ be a prime number, $q$ a power of $p$, and $k$ a finite field with $q$ elements.
10.2.1 (Fourier transform). - Let $k=\mathbb{F}_{q}$ be the field with $q$ elements. With an additive character $\psi: \mathbb{F}_{q} \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$, one associates a rank one lisse sheaf $\mathcal{L}_{\psi}$ on $\mathbb{A}_{k}^{1}$ called the Artin-Schreier sheaf. It is constructed out of the map $x \longmapsto x^{q}-x$, which defines a finite étale morphism $\pi: \mathbb{A}_{k}^{1} \rightarrow \mathbb{A}_{k}^{1}$ with Galois group $\mathbb{F}_{q}$. Thereofore, the étale fundamental group $\pi_{1}^{\text {ét }}\left(\mathbb{A}_{k}^{1}\right)$ surjects onto $\mathbb{F}_{q}$. Composing with the character $\psi$ gives the corresponding $\ell$-adic representation $\pi_{1}^{\text {et }}\left(\mathbb{A}_{k}^{1}\right) \rightarrow$ $\overline{\mathbb{Q}}_{\ell} \times$. More geometrically, $\mathcal{L}_{\psi}$ is the isotypical component associated with $\psi$ in the direct sum decomposition $\pi_{*} \overline{\mathbb{Q}}_{\ell}=\bigoplus_{\psi} \mathcal{L}_{\psi}$. In particular, if $\psi$ is the trivial character, then $\mathcal{L}_{\psi}=\overline{\mathbb{Q}}_{\ell}$ is the trivial sheaf.

$$
\operatorname{FT}_{\psi}(C)=R p_{2 *}\left(p_{1}^{*} C \otimes \mathcal{L}_{\psi(x y)}\right)
$$

Theorem 10.2.2 (Laumon). - If $C$ is tamely ramified, then $\mathrm{FT}_{\psi}(C)$ is a lisse sheaf on $\mathbb{G}_{m}$.
10.2.3 (Specialisation to characteristic $p$ ). -


Definition 10.2.4. - The nearby cycles at $p$ is

$$
R \Psi_{p} C=\bar{\iota}^{*} R \bar{j}_{*} \kappa^{*} C
$$

Definition 10.2.5 (Sawin). - A perverse sheaf $C$ on $\mathbb{A}_{\mathbb{Q}}^{1}$ has good reduction at a prime number $p$ if the following three conditions hold:
(a) the generic rank of $R \Psi_{p} C$ is equal to the generic rank of $C$,
(b) the singularities of $C$ lie in $\overline{\mathbb{Z}}_{p}$,
(c) the inertia subgroup $I_{p} \subset \operatorname{Gal}\left(\overline{\mathbb{Q}}_{p} / \mathbb{Q}_{p}\right)$ acts trivially on $R \Psi_{p} C$.

Example 10.2.6. - Let $r \in \mathbb{Q}$ and let $\delta_{r}$ be the skyscraper sheaf supported at the point $r \in \mathbb{A}_{\mathbb{Q}}^{1}$. If $r=0$, then $\delta_{0}$ has good reduction everywhere. If $r$ is non-zero, we write $r=a / b$ with $a$ and $b$ coprime integers. Then $\delta_{r}$ has good reduction at $p$ if and only if $p$ does not divide $b$.

Example 10.2.7. - The perverse sheaf $j_{!} \mathcal{L}_{\chi_{2}}[1]$ has bad reduction at $p=2$.

$$
R \Psi_{p}\left(\pi_{*} \overline{\mathbb{Q}}_{\ell}[1]\right)=\overline{\mathbb{Q}}_{\ell}[1] \oplus R \Psi_{p}\left(j_{!} \mathcal{L}_{\chi_{2}}[1]\right)
$$

Theorem 10.2.8 (Sawin). - If $C_{1}$ and $C_{2}$ have good reduction at $p$, then

$$
R \Psi_{p}\left(C_{1} * C_{2}\right)=R \Psi_{p}\left(C_{1}\right) * R \Psi_{p}\left(C_{2}\right)
$$

From this, we immediately derive that, if $C$ has good reduction at $p$, then

$$
\Pi\left(R \Psi_{p}(C)\right)=R \Psi_{p}(C) * j!j^{*} \overline{\mathbb{Q}}_{\ell}[1]=R \Psi_{p}\left(C * j!j^{*} \overline{\mathbb{Q}}_{\ell}[1]\right)=R \Psi_{p}(\Pi(C))
$$

Therefore, $R \Psi_{p}$ restricts to a functor from $\operatorname{Perv}_{0}\left(\mathbb{A}_{\mathbb{Q}}^{1}, \overline{\mathbb{Q}}_{\ell}\right)$ to $\operatorname{Perv}_{0}\left(\mathbb{A}_{\mathbb{F}_{p}}, \overline{\mathbb{Q}}_{\ell}\right)$

THEOREM 10.2.9 (Sawin). - Let $S$ be a finite set of prime numbers including $\ell$ and let $\mathbf{C C}_{S}$ be the full subcategory of $\operatorname{Perv}_{0}\left(\mathbb{A}_{\mathbb{Q}}^{1}, \overline{\mathbb{Q}}_{\ell}\right)$ consisting of those objects with good reduction outside $S$. For each prime $p \notin S$ and each non-trivial additive character $\psi$, the functor

$$
C \longmapsto H^{0}\left(\mathbb{A}_{\mathbb{F}_{p}}^{1}, R \Psi_{p} C \otimes \mathcal{L}_{\psi}\right)
$$

is a fibre functor $\mathbf{C C}_{S} \rightarrow \mathbf{V e c}_{\overline{\mathbb{Q}}_{\ell}}$. The Frobenius at $p$ is an automorphism of this functor.

## 10.3. $L$-functions of exponential motives

## CHAPTER 11

## Exponential Hodge theory

In this chapter, we construct a Hodge realisation functor from the category of exponential motives to a subcategory of mixed Hodge modules over the complex affine line - parallel to Perv $\mathbf{p}_{0}$ that Kontsevich and Soibelman call exponential mixed Hodge structures. Throughout, "Hodge structure" means rational mixed Hodge structure. We always suppose them to be graded polarisable, that is, each pure subquotient admits a polarisation. We denote the category of Hodge structures by MHS. It is a $\mathbb{Q}$-linear neutral tannakian category, with respect to the forgetful functor

$$
f: \text { MHS } \rightarrow \text { Vec }_{\mathbb{Q}} .
$$

### 11.1. Reminder on mixed Hodge modules

The theory of Hodge modules is a long story-we will recite here a few essential properties of categories of mixed Hodge modules, and give a brief description of their construction. For a thorough introduction see $[77,72]$.

Definition 11.1.1. - Let $X$ be a complex algebraic variety. A pre-mixed Hodge module on $X$ consists of the following data:

- A rational perverse sheaf $L$, together with an increasing filtration $W_{\bullet} L$ by perverse subsheaves.
- A regular holonomic $\mathscr{D}_{X}$-module $\mathcal{M}$, together with an increasing filtration $W_{\bullet} \mathcal{M}$ and a good filtration $F^{\bullet} \mathcal{M}$.
- An isomorphism $\alpha: D R(\mathcal{M}) \simeq L \otimes_{\mathbb{Q}} \mathbb{C}$ under which $W_{\bullet} \mathcal{M}$ corresponds to $W_{\bullet} L \otimes_{\mathbb{Q}} \mathbb{C}$.

Pre-mixed Hodge modules form a category and mixed Hodge modules are defined inductively as a subcategory of them.
11.1.2. - For every complex algebraic variety $X$, there is an abelian category $\mathbf{M H M}(X)$ of mixed Hodge modules on $X$, and a functor

$$
\text { rat }: \operatorname{MHM}(X) \rightarrow \operatorname{Perv}(X)
$$

which is exact and faithful, so we may look at mixed Hodge modules as perverse sheaves with extra data, though the functor rat is not essentially surjective. Categories of mixed Hodge modules (or better: their bounded derived categories) enjoy a six functors formalism, which is compatible with the functor rat. If $X$ is a point, then the category $\mathbf{M H M}(X)$ is the category of mixed Hodge structures (recall the proviso that they are assumed to be graded polarisable).
11.1.3. - Let $X$ be a smooth, connected algebraic variety of dimension $n$, and let $V$ be a variation of mixed Hodge structures on $X$. There is a mixed Hodge module on $X$ naturally associated with $V$, which we shall denote by $V[n]$. As the notation suggests, its underlying perverse sheaf is the local system underlying $V$ shifted to degree $-n$.

Mixed Hodge modules come with a functorial, exact weight filtration.
A formal consequence of the six functors formalism for mixed Hodge modules is that we can define additive convolution on $\mathbf{M H M}(\mathbb{C})$ as we did for perverse sheaves in 2.4.1.

When $X$ is a point, $\mathbf{M H M}(X)$ is nothing but the category of mixed Hodge structures (recall the proviso that they are assumed to be graded polarisable).

### 11.2. Exponential mixed Hodge structures

Definition 11.2.1 (Kontsevich-Soibelman). - An exponential mixed Hodge structure is a mixed Hodge module on the complex affine line $\mathbb{C}$ whose underlying perverse sheaf belongs to Perv $\mathbf{v}_{0}$. We denote the corresponding full subcategory by EMHS.

Example 11.2.2. - Of particular interest are the exponential mixed Hodge structures

$$
E(s)=j(s)!j(s)^{*} \pi^{*} \underline{\mathbb{Q}}[1],
$$

where $\Pi: \mathbb{A}_{\mathbb{C}}^{1} \rightarrow \operatorname{Spec} \mathbb{C}$ is the structure morphism, $j(s): \mathbb{C} \backslash\{s\} \rightarrow \mathbb{C}$ the inclusion, and $\mathbb{Q}=\mathbb{Q}(0)$ stands for the one-dimensional Hodge structure of weight 0 , regarded as a Hodge module on the point. The perverse sheaf underlying $E(0)$ was introduced under the same name in Example 2.3.4.

The inclusion of EMHS into $\mathbf{M H M}(\mathbb{C})$ admits as a left adjoint the exact idempotent functor

$$
\begin{aligned}
& \Pi: \mathbf{M H M}(\mathbb{C}) \longrightarrow \\
& M \text { EMHS } \\
& M * E(0)
\end{aligned}
$$

Definition 11.2.3. - We call canonical the functor $\iota$ : MHS $\rightarrow$ EMHS which sends $H$ to the exponential mixed Hodge structure $\iota(H)=\Pi\left(i_{*} H\right)$, where $i:\{0\} \hookrightarrow \mathbb{C}$ is the inclusion. Explicitly,

$$
\iota(H)=j!j^{*} \pi^{*} H[1] .
$$

Observe that $\iota(H)$ has singularities only at 0 and trivial monodromy.

Lemma 11.2.4. - The canonical functor $\iota:$ MHS $\rightarrow$ EMHS is fully faithful, and its essential image is stable under taking quotients and subobjects.

Proof. The canonical functor $\iota:$ MHS $\rightarrow$ EMHS is exact and faithful, because the functors $j!, j^{*}$ and $\pi^{*}$ are so. To check that the canonical functor is full, let $V$ and $W$ be Hodge structures, and let

$$
f: j!j^{*} \pi^{*} V[1](-1) \rightarrow j!j^{*} \pi^{*} W[1](-1)
$$

be a morphism of Hodge modules. The perverse sheaf underlying $j!j^{*} \pi^{*} V[1](-1)$ is a constant local system on $\mathbb{C}^{*}$ given by the rational vector space underlying $V(-1)$ in degree -1 , and its fibre over $0 \in \mathbb{C}$ is zero. The same holds for $W$. Therefore, if $f$ induces the zero morphism on the fibre over any $z \neq 0$, then $f$ is the zero morphism. Let $i_{1}:\{1\} \rightarrow \mathbb{C}$ be the inclusion. The fibre of $f$ over 1 is the morphism

$$
i_{1}^{*}(f): i_{1}^{*} j^{*} \pi^{*} V[1](-1) \rightarrow i_{1}^{*} j!j^{*} \pi^{*} W[1](-1)
$$

induced by $f$. The fibre $i_{1}^{*} j^{*} \pi^{*} V[1](-1)$ is the Hodge structure $V(-1)$ put in degree -1 . After twisting and shifting we obtain thus a morphism of Hodge structures $f_{1}: V \rightarrow W$. The difference $f-\iota\left(f_{1}\right)$ is then a morphism of Hodge modules and its fibre over 1 is zero, hence $f=\iota\left(f_{1}\right)$. Let us now check that the essential image of the canonical functor is stable under taking subobjects. Let $V$ be a Hodge structure, and let $M \subseteq j!j^{*} \pi^{*} V[1](-1)$ be a subobject of in the category EMHS. Applying the left exact functor $\pi_{*} j_{*} j^{*}(-)[-1](1)$ we obtain a subobject

$$
W=\pi_{*} j_{*} j^{*} M[-1](1) \subseteq \pi_{*} j_{*} j^{*} \pi^{*} V=V
$$

in the category of Hodge structures. Applying $j^{*} \pi^{*}(-)[1](-1)$, using adjunction and applying $j$ ! yields a morphism

$$
j!j^{*} \pi^{*} W[1](-1) \rightarrow M \subseteq j!j^{*} \pi^{*} V[1](-1)
$$

and we need to show that the morphism of Hodge modules $j!j^{*} \pi^{*} W[1](-1) \rightarrow M$ is an isomorphism. This is indeed the case, since the morphism of underlying perverse sheaves one obtains by applying the functor rat is an isomorphism. This shows that the essential image of the canonical functor is stable under taking subobjects, hence also under taking quotients.

REmark 11.2.5. - Contrary to what is claimed in [58, p.262], the image of the canonical functor does not form a Serre subcategory of EMHS, i.e. is not stable under extension. Here is an example. Every graded polarisable variation of mixed Hodge structures $V$ on $\mathbb{C}^{*}$ determines a mixed Hodge module $V[1]$ on $\mathbb{C}^{*}$ with the evident underlying perverse sheaf. For example we may consider the variation of mixed Hodge structure whose fibre over $z \in \mathbb{C}$ is the Hodge realisation of the 1-motive $\left[\mathbb{Z} \xrightarrow{u} \mathbb{C}^{*}\right]$ given by $u(1)=z$. This variation $V$ sits in a short exact sequence

$$
0 \rightarrow \mathbb{Q}(1) \rightarrow V \rightarrow \mathbb{Q} \rightarrow 0
$$

and applying $j_{!}(-)[1]$ yields an exact sequence in EMHS. While the first and last term in this sequence come from Hodge structures via the canonical functor, the object in the middle does not, as the underlying perverse sheaf has a non-trivial monodromy around 0 .

Proposition 11.2.6.-
(1) Exponential mixed Hodge structures form a $\mathbb{Q}$-linear tannakian category. A fibre functor is given by the composite of the forgetful functor EMHS $\rightarrow \mathbf{P e r v}_{0}$ and the fibre functor $\Psi_{\infty}: \operatorname{Perv}_{0} \rightarrow \operatorname{Vec}_{\mathbb{Q}}$.
(2) The functors $\mathbf{M H S} \rightarrow$ EMHS $\rightarrow \mathbf{P e r v}_{0}$ are functors of tannakian categories, compatible with the given fibre functors. Their composite is the trivial functor, which sends a mixed Hodge structure $V$ to the perverse sheaf $j!j^{*} f(V)[1]$.

## Proof.

### 11.3. Intermezzo: Extensions of groups from the tannakian point of view

Let $F$ and $H$ be groups. By an extension of $F$ by $H$ one understands a group $G$ sitting in an exact sequence $1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1$. The problem of classifying all extensions of $F$ by $H$ is a classical problem in group theory, systematically studied by Schreier, Zassenhaus, Schur, Eilenberg, Mac Lane and many others. Two types of extensions are particularly well understood: semidirect products and extensions by abelian groups. A semidirect product or also split extension is an extension such that the quotient map $G \rightarrow F$ admits a section $F \rightarrow G$. The group $F$ acts via this section on $H$ by conjugation, and reciprocally, any action $\alpha: F \rightarrow \operatorname{Aut}(H)$ defines a split extension of $F$ by $H$ by considering on the set $G=H \times F$ the group law

$$
(h, f)\left(h^{\prime}, f^{\prime}\right)=\left(h \alpha(f)\left(h^{\prime}\right), f f^{\prime}\right)
$$

Central extensions are those where $H$ is contained in the centre of $G$, hence in particular is commutative. Central extensions of $F$ by $H$ up to equivalence form a commutative group $\operatorname{Ext}^{1}(F, H)$, with the Baer sum as group law. This group is naturally isomorphic to the group cohomology $H^{2}(F, H)$, where $H$ is regarded as an $F$-module with trivial $F$-action. Given a central extension $1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1$, the corresponding cohomology class is represented by the cocycle $c: F \times F \rightarrow H$ given by

$$
c\left(f, f^{\prime}\right)=s(f)^{-1} s\left(f^{\prime}\right)^{-1} s\left(f f^{\prime}\right)
$$

where $s: F \rightarrow G$ is any map, not necessarily a group homomorphism, whose composition with the quotient map $G \rightarrow F$ is the identity on $F$. The generalisation to not necessarily central extensions of $F$ by an abelian group $H$ is not difficult. Such extensions are also classified by group cohomology $H^{2}(F, H)$, but now with the possibly nontrivial action of $F$ on $H$ corresponding to the conjugation action. The even more general case where $F$ is not necessarily commutative was worked out by Schreier [78] and Eilenberg-Mac Lane [32]. It inevitably leads to non-commutative group cohomology.

More generally, one would like to classify group extensions in a topos. A complete geometric solution to this problem was given by Grothendieck and Giraud [40], and later a cohomological interpretation was given by Breen [16]. A new problem that arises in this generality which was not seen in the elementary case of extensions of abstract groups is that in a general topos, an extension
$1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1$ defines an $H$-torsor over $F$ which need not be trivial. So, unlike in the case of abstract groups, there is not always a morphism $s: F \rightarrow G$ which splits the surjection $G \rightarrow F$.

We are interested in certain extensions of affine group schemes $1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1$, namely those where the Hopf algebra underlying $G$ is, as a coalgebra, isomorphic to the tensor product of the coalgebras associated with $H$ and $F$. In other words, we are interested in certain extensions of commutative group objects in the category of not necessarily commutative coalgebras. Such extensions arise naturally when one tries to turn the vanishing cyles functor for exponential Hodgestructures into a tensor functor. Indeed, this vanishing cyles functor takes values in the category of what Scherk and Steenbrink call $\widehat{\mu}$-Hodge structure in [76], that is, mixed Hodge structures with an automorphism of finite order. The category MHS ${ }^{\widehat{\mu}}$ of $\widehat{\mu}$-Hodge structures comes equipped with a symmetric tensor product, not the obvious one, which turns it into a tannakian category. The tannakian fundamental group sits in an extension

$$
0 \rightarrow \widehat{\mathbb{Z}} \rightarrow \pi_{1}\left(\mathbf{M H S}^{\widehat{\mu}}\right) \rightarrow \pi_{1}(\mathbf{M H S}) \rightarrow 1
$$

which is exactly of the nature described above: as an abelian category $\mathbf{M H S}{ }^{\widehat{\mu}}$ is the obvious thing, morphisms are morphisms of Hodge structures compatible with the automorphisms, so the coalgebra underlying the affine group scheme $\pi_{1}\left(\mathbf{M H S}{ }^{\widehat{\mu}}\right)$ is the tautological one. The commutative multiplication turning this coalgebra into a commutative Hopf algebra corresponds to the special tensor product we are considering.

The plan for this section is as follows: after fixing conventions, we start by describing extensions of group schemes in terms of tannakian categories and in terms of Hopf algebras. That done, we translate classical constructions from group theory such as semidirect products and the classification of extensions, in particular commutative extensions, by group cohomology into the language of coalgebras. In particular, we show how to use 2-cocycles to describe extensions of Hopf algebras.

Proposition 11.3.1. - Let $K \rightarrow A \xrightarrow{p} E \xrightarrow{i} B \rightarrow K$ be morphisms of commutative Hopf algebras. The corresponding sequence of affine group schemes $1 \rightarrow \operatorname{Spec} B \rightarrow \operatorname{Spec} E \rightarrow \operatorname{Spec} A \rightarrow 1$ is exact if and only if the morphism $p: A \rightarrow E$ is injective, $i: E \rightarrow B$ is surjective, and

$$
\operatorname{ker}(i)=E \cdot p\left(A^{+}\right)
$$

where $A^{+}=\operatorname{ker}\left(\varepsilon_{A}: A \rightarrow K\right)$ is the augmentation ideal of $A$.
11.3.2. - Let $A$ and $B$ be commutative Hopf algebras and set $F=\operatorname{Spec} A$ and $H=\operatorname{Spec} B$. By an extension of $B$ by $A$ we understand a sequence of (not necessarily commutative) Hopf algebras

$$
K \rightarrow A \xrightarrow{p} E \xrightarrow{i} B \rightarrow K
$$

where $p: A \rightarrow E$ is injective, $i: E \rightarrow B$ is surjective, and $\operatorname{ker}(i)=E \cdot p\left(A^{+}\right)$, up to the usual notion of equivalence. Commutative extensions, that means those where the multiplication on $E$ is commutative, are in one to one correspondence with extensions of the group scheme $F$ by the group scheme $H$. Let us denote by $\operatorname{EXT}(B, A)$ the set of all (equivalence classes of) extensions of $B$ by $A$ and by

$$
\operatorname{CEXT}(B, A) \subseteq \operatorname{EXT}(B, A)
$$

the subset of commutative extensions. These are just pointed sets, with the trivial extension $A \otimes B$ as distinguished element. Every commutative extension $E$ of $B$ by $A$ defines an $H$-torsor $G=\operatorname{Spec} E$ over the scheme $F$, corresponding to an element $t_{G} \in H_{\mathrm{fppf}}^{1}(F, H)$. If $G$ is trivial as an $H$-torsor, or in other words if $t_{G}=0$, then $G$ is isomorphic as a scheme to $F \times H$. Let us denote by

$$
\operatorname{EXT}_{m}(A, B) \subseteq \operatorname{CEXT}(A, B)
$$

the subset of $\operatorname{EXT}(A, B)$ consisting of those extensions whose underlying algebra is $A \otimes B$ with the commutative multiplication $m_{A} \otimes m_{B}$, obtained from the multiplication $m_{A}$ on $A$ and $m_{B}$ on $B$. To give an element of $\operatorname{EXT}_{m}(A, B)$ is to give a group structure on the scheme $H \times F$ which is compatible with the inclusion $H \rightarrow H \times F$ and the projection $H \times F \rightarrow F$, or else, a comultiplication on the commutative algebra $\left(A \otimes B, m_{A} \otimes m_{B}\right)$ compatible with the morphisms $A \otimes B \rightarrow A$ and $B \rightarrow A \otimes B$. The following bijection is tautological:

$$
\operatorname{EXT}_{m}(A, B) \stackrel{\cong}{\longleftrightarrow}\left\{\begin{array}{l}
\text { Comultiplications on the algebra }\left(A \otimes B, m_{A} \otimes m_{B}\right) \text { which are } \\
\text { compatible with the morphisms } A \otimes B \rightarrow A \text { and } B \rightarrow A \otimes B
\end{array}\right\}
$$

Instead of considering extensions with fixed underlying scheme $H \times F$, that is, keeping the algebra structure $m_{A} \otimes m_{B}$ on $A \otimes B$ and modifying the comultiplication, we can also consider extensions which arise by keeping the coalgebra structure $\mu_{A} \otimes \mu_{B}$ on $A \otimes B$ and letting the algebra structure vary. Let us denote by

$$
\operatorname{EXT}_{\mu}(A, B) \subseteq \operatorname{EXT}(A, B)
$$

the subset consisting of those extensions whose underlying coalgebra is $\left(A \otimes B, \mu_{A} \otimes \mu_{B}\right)$. The following bijection is tautological:

$$
\operatorname{EXT}_{\mu}(A, B) \stackrel{\cong}{\longleftrightarrow}\left\{\begin{array}{l}
\text { Multiplications on the coalgebra }\left(A \otimes B, \mu_{A} \otimes \mu_{B}\right) \text { which are } \\
\text { compatible with the morphisms } A \otimes B \rightarrow A \text { and } B \rightarrow A \otimes B
\end{array}\right\}
$$

In categorical terms, this means we consider $\operatorname{Rep}(H \times F)=\boldsymbol{\operatorname { C o m o d }}(A \otimes B)$ as an abelian category, and seek to modify the tensor product on it. The situation is not completely symmetric, since in our setup we require $A$ and $B$ to be commutative, but not necessarily cocommutative. Let

$$
\operatorname{CEXT}_{\mu}(A, B) \subseteq \operatorname{CEXT}(A, B)
$$

be the subset of commutative extensions of $B$ by $A$ with underlying coalgebra $\left(A \otimes B, \mu_{A} \otimes \mu_{B}\right)$. Again we have a tautological bijection

$$
\operatorname{CEXT}_{\mu}(A, B) \stackrel{\cong}{\longleftrightarrow}\left\{\begin{array}{l}
\text { Commutative multiplications on the coalgebra }\left(A \otimes B, \mu_{A} \otimes \mu_{B}\right) \text { which } \\
\text { are compatible with the morphisms } A \otimes B \rightarrow A \text { and } B \rightarrow A \otimes B
\end{array}\right\}
$$

Any such commutative extension of Hopf algebras gives rise to an extension of affine group schemes $1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1$. If the corresponding torsor class $t_{G} \in H_{\mathrm{fppf}}^{1}(F, H)$ is zero, then the multiplication on $A \otimes B$ is $m_{A} \otimes m_{B}$ and the comultiplication is $\mu_{A} \otimes \mu_{B}$, so the extension is trivial. In other words, the following map of pointed sets has trivial kernel:

$$
\operatorname{CEXT}_{\mu}(A, B) \rightarrow H_{\mathrm{fppf}}^{1}(F, H)
$$

Example 11.3.3. - Let $\mathbf{C}$ be the tannakian category of $\mathbb{Z}$-graded rational vector spaces, with its usual tensor product and the forgetful functor as fibre functor. Its tannakian fundamental group is the multiplicative group $\mathbb{G}_{m}$. Let $\mathbf{C}^{\widehat{\mu}}$ denote the category of pairs $(V, T)$ consisting of a graded
vector space $V$ and a finite order automorphism $T$ of $V$ respecting the grading. The category $\mathbf{C}^{\widehat{\mu}}$ is abelian and semisimple, and as such equivalent to the category of representations of $\widehat{\mathbb{Z}} \times \mathbb{G}_{m}$. The simple objects are those $(V, T)$ where $V$ is pure for the given grading and has no proper $T$-invariant subspaces. If $T$ has order exactly $n$, then $V$ has dimension $\varphi(n)$ and the characteristic polynomial of $T$ is the cyclotomic polynomial $\Phi_{n}(X)$. Let us denote by

$$
\mathbb{Q}(k, n) \quad k \in \mathbb{Z}, \quad n \in \mathbb{Z}_{\geqslant 1},
$$

the simple object $(V, T)$ where $V$ has degree $k$ and $T$ has order $n$. Simple objects of $\mathbf{C}$ are those of the form $\mathbb{Q}(k, 1)$. They are of dimension 1 . For $\alpha \in \mathbb{Q}$, set $V^{\alpha}=\operatorname{ker}(T-\exp (-2 \pi i \alpha)) \subseteq V \otimes \mathbb{C}$, so that we have an eigenspace decomposition

$$
V \otimes \mathbb{C}=\bigoplus_{\alpha \in \mathbb{Q} \cap(-1,0]} V^{\alpha} .
$$

Note that each $V^{\alpha}$ inherits a grading from $V$. We define the tensor product of two objects $(V, T)$ and $\left(V^{\prime}, T^{\prime}\right)$ of $\mathbf{C}^{\widehat{\mu}}$ by

$$
(V, T) \otimes\left(V^{\prime}, T^{\prime}\right)=\left(V \otimes V^{\prime}, T \otimes T^{\prime}\right)
$$

where $V \otimes V^{\prime}$ has the following grading:

$$
\operatorname{gr}_{k}\left(V \otimes V^{\prime}\right)=\left(\bigoplus_{\alpha, \beta} \bigoplus_{i, j}\left(\operatorname{gr}_{i}\left(V^{\alpha}\right) \otimes \operatorname{gr}_{j}\left(V^{\prime \beta}\right)\right)\right) \cap\left(V \otimes V^{\prime}\right)
$$

where, as before, the sums run over all $\alpha, \beta \in \mathbb{Q} \cap(-1,0]$ and all integres $i, j$ satisfying

$$
i+j= \begin{cases}k & \text { if } \alpha=0 \text { or } \beta=0 \\ k-2 & \text { if } \alpha+\beta=-1 \\ k-1 & \text { else }\end{cases}
$$

Let us see what happens with simple objects. If either $n_{1}$ or $n_{2}$ is equal to 1 , say $n_{2}=1$, then we have

$$
\mathbb{Q}\left(k_{1}, n_{1}\right) \otimes \mathbb{Q}\left(k_{2}, 1\right)=\mathbb{Q}\left(k_{1}+k_{2}, n_{1}\right)
$$

for all $k_{1}, k_{2}, n_{1}$. Suppose now that $n_{1} \neq 1$ and $n_{2} \neq 1$, and let $N$ be the least common multiple of $n_{1}$ and $n_{2}$. We have

$$
\mathbb{Q}\left(k_{1}, n_{1}\right) \otimes \mathbb{Q}\left(k_{2}, n_{2}\right)=\mathbb{Q}\left(k_{1}+k_{2}+2,1\right)^{\eta\left(n_{1}, n_{2}, 1\right)} \oplus \bigoplus_{d \mid N, d \neq 1} \mathbb{Q}\left(k_{1}+k_{2}+1, d\right)^{\eta\left(n_{1}, n_{2}, d\right)}
$$

where $\eta\left(n_{1}, n_{2}, d\right) \varphi(d)$ is the number of pairs $\left(a_{1}, a_{2}\right) \in(\mathbb{Z} / N \mathbb{Z})^{2}$ where $a_{1}$ has order $n_{1}$, $a_{2}$ has order $n_{2}$ and $a_{1}+a_{2}$ has order $d$. For example

$$
\mathbb{Q}(0,100) \otimes \mathbb{Q}(0,100)=\mathbb{Q}(2,1)^{40} \oplus \mathbb{Q}(1,2)^{40} \oplus \mathbb{Q}(1,5)^{40} \oplus \mathbb{Q}(1,10)^{40} \oplus \mathbb{Q}(1,25)^{30} \oplus \mathbb{Q}(1,50)^{30}
$$

which is $1600=\varphi(100)^{2}=40 \varphi(0)+40 \varphi(1)+40 \varphi(5)+40 \varphi(10)+30 \varphi(25)+30 \varphi(50)$ on the level of dimensions. If $\left(n_{1}, n_{2}\right)=1$, then

$$
\mathbb{Q}\left(k_{1}, n_{1}\right) \otimes \mathbb{Q}\left(k_{2}, n_{2}\right)=\mathbb{Q}\left(k_{1}+k_{2}+1, n_{1} n_{2}\right)
$$

holds, and if $p$ is a prime, then

$$
\mathbb{Q}\left(k_{1}, p\right) \otimes \mathbb{Q}\left(k_{2}, p\right)=\mathbb{Q}\left(k_{1}+k_{2}+1, p\right)^{p-2} \oplus \mathbb{Q}\left(k_{1}+k_{2}+2,1\right)^{p-1}
$$

11.3.4. - We now start adapting the theory of group extensions à la Schreier to the framework of Hopf algebras. More precisely, we replace groups with group objects in the category of $K$-coalgebras. Ultimately we are only concerned with commutative group objects, that is, commutative Hopf algebras, yet we need to start with semidirect products.

Definition 11.3.5. - Let $A$ and $B$ be Hopf algebras. An action of $B$ on $A$ is a linear map $\tau: B \otimes A \rightarrow A$ such that the following diagrams commute.


We call trivial action the action defined by $\tau(b \otimes a)=\varepsilon_{B}(b) a$.
11.3.6. - Let $A$ and $B$ be Hopf algebras, and let $\tau: B \otimes A \rightarrow A$ be an action of $B$ on $A$. We can use $\tau$ to define a multiplication $m_{\tau}$ on the coalgebra $A \otimes B$ as the following composite.

$$
\begin{align*}
& A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes \mu_{B} \otimes 1 \otimes 1} A \otimes B \otimes B \otimes A \otimes B \\
& \text { 11×1 }  \tag{11.3.6.1}\\
& A \otimes B \otimes A \otimes B \otimes B \xrightarrow{1 \otimes \tau \otimes 1 \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_{A} \otimes m_{B}} A \otimes B
\end{align*}
$$

It is straightforward to check that the so defined map $m_{\tau}: A \otimes B \otimes A \otimes B \rightarrow A \otimes B$ is indeed a multiplication on $A \otimes B$, compatible with the comultiplication $\mu_{A} \otimes \mu_{B}$, so that together they combine to a Hopf algebra structure on $A \otimes B$. We call this a semidirect product. The trivial action induces this way the multiplication $m_{A} \otimes m_{B}$. Reciprocally, given an extension of Hopf algebras of the form

$$
K \rightarrow A \xrightarrow{1 \otimes e_{B}}(A \otimes B, m) \xrightarrow{\varepsilon_{A} \otimes 1} B \rightarrow K
$$

where on $A \otimes B$ the comultiplication is $\mu_{A} \otimes \mu_{B}$, we obtain an action $\tau_{m}$ of $B$ on $A$ as follows.


It is straightforward to check that the so defined map $\tau_{m}: A \otimes B \otimes A \otimes B \rightarrow A \otimes B$ is indeed an action of $B$ on $A$. If the multiplication $m=m_{\tau}$ is obtained from a given action $\tau: B \otimes A \rightarrow A$, so that $\left(A \otimes B, m_{\tau}\right)$ is a semidirect coproduct, we recover $\tau$ from $m_{\tau}$ - this is the content of Lemma
11.3.7. On the other hand, a multiplication $m$ on $A \otimes B$ can in general not be recovered from its induced action $\tau_{m}$. In particular, we notice that if the multiplication $m$ on $A \otimes B$ is commutative, then the induced action $\tau_{m}$ is trivial, and the trivial action induces the multiplication $m_{A} \otimes m_{B}$ on $A \otimes B$.

Lemma 11.3.7. - Let $\tau: B \otimes A \rightarrow A$ be an action of $B$ on $A$, and let $m_{\tau}:(A \otimes B)^{2} \rightarrow A \otimes B$ be the multiplication of the corresponding semidirect product, as defined by (11.3.6.1). The action of $B$ on $A$ defined by means of (11.3.6.2) is equal to $\tau$.

Proof. The product $m_{\tau}$ is expressed by

$$
m_{\tau}\left(a \otimes b \otimes a^{\prime} \otimes b^{\prime}\right)=\sum_{\bullet} a \tau\left(b_{1} \otimes a^{\prime}\right) \otimes b_{2} b^{\prime}
$$

Let us pick an element $a \otimes b$ of $A \otimes B$ and check that it gets sent to $\tau(b \otimes a)$ by the composite (11.3.6.2). The element $a \otimes b$ is mapped to

$$
\sum_{\bullet}\left(1 \otimes b_{1}\right) \otimes(a \otimes 1) \otimes\left(1 \otimes i_{B}\left(b_{2}\right)\right)
$$

in $(A \otimes B)^{3}$. Multiplying the three terms together with $m_{\tau}$ we obtain the element

$$
\sum_{\bullet \bullet} \tau\left(b_{1} \otimes a\right) \otimes b_{2} i_{B}\left(b_{3}\right)=\sum_{\bullet} \tau\left(b_{1} \otimes a\right) \otimes b_{2}
$$

of $A \otimes B$. Here we used the coassociativity of $\mu_{B}$ and property $b=\sum b_{1} i_{B}\left(b_{2}\right)$ of the antipode. Finally, applying $1 \otimes \varepsilon_{B}$ yields the element

$$
\sum_{\bullet} \varepsilon\left(b_{2}\right) \tau\left(b_{1} \otimes a\right)=\tau(b \otimes a)
$$

of $A$ as desired. In this last step, we used the property $b=\sum \varepsilon\left(b_{2}\right) b_{1}$ of the counit and bilinearity of $\tau$.
11.3.8. - We call an extension $K \rightarrow A \xrightarrow{p} E \xrightarrow{i} B \rightarrow K$, where the coalgebra underlying $E$ is $\left(A \otimes B, \mu_{A} \otimes \mu_{B}\right)$, central if the action of $B$ on $A$ is trivial.

$$
\operatorname{CEXT}_{\mu}(A, B) \subseteq \operatorname{ZEXT}_{\mu}(A, B) \subseteq \operatorname{EXT}_{\mu}(A, B)
$$

Definition 11.3.9. - Let $A$ and $B$ be commutative Hopf algebras. A 2-cocycle of $B$ with coefficients in $A$ (for the trivial action of $B$ on $A$ ) is a morphism of coalgebras $c: B \otimes B \rightarrow A$ such that the following diagram commutes

$$
\begin{gather*}
B^{3} \xrightarrow{\mu_{B}^{3}} B^{6} \xrightarrow{\varepsilon_{B} \otimes 1 \otimes 1 \otimes 1 \otimes m_{B}} B^{4} \xrightarrow{c \otimes c} A^{2}  \tag{11.3.9.1}\\
\mu_{B}^{3} \\
\downarrow \\
B^{6} \xrightarrow{m_{B} \otimes 1 \otimes 1 \otimes 1 \otimes \varepsilon_{B}} B^{4} \xrightarrow{c \otimes c} A^{2} \xrightarrow{m_{A}} \downarrow^{\downarrow} A
\end{gather*}
$$

The multiplication induced by $c$ on $A \otimes B$ is the map $m_{c}:(A \otimes B)^{2} \rightarrow A \otimes B$ defined by linearity and

$$
m_{c}\left(a \otimes b \otimes a^{\prime} \otimes b^{\prime}\right)=\sum_{\bullet \bullet} a a^{\prime} c\left(b_{1} \otimes b_{1}^{\prime}\right) \otimes b_{2} b_{2}^{\prime}
$$

for all $a \otimes b \otimes a^{\prime} \otimes b^{\prime} \in(A \otimes B)^{2}$.
11.3.10. - Let $A$ be a commutative Hopf algebra over $K$ with unit $e: K \rightarrow A$, counit $\varepsilon: A \rightarrow K$, multiplication $m: A \otimes A \rightarrow A$, comultiplication $\mu: A \rightarrow A \otimes A$, and antipode $i: A \rightarrow A$. As is customary, we write the multiplication of two elements $m(a \otimes b)$ just as $a b$, and the comultiplication of an element $a$ as

$$
\mu(a)=\sum_{\bullet} a_{1} \otimes a_{2} \quad(1 \otimes \mu)(\mu(a))=(\mu \otimes 1)(\mu(a))=\sum_{\bullet} a_{1} \otimes a_{2} \otimes a_{3}
$$

for as long as no confusion seems to arise (but maybe it's already too late for that concern). The category of representations of the affine group scheme $\operatorname{Spec} A$ is canonically equivalent to the category of $A$-comodules. As an abelian category, it only depends on $A$ as a coalgebra. The algebra structure on $A$ corresponds to the tensor product, and the existence of the antipode is equivalent to the existence of duals. We now seek to produce multiplications $m_{\tau}: A \otimes A \rightarrow A$ such that $\left(A, e, \varepsilon, \mu, m_{\tau}, i_{\tau}\right)$ is a commutative Hopf algebra for an antipode $i_{\tau}: A \rightarrow A$. Let us call symmetric 2-cocycle any symmetric bilinear map

$$
\tau: A \otimes A \rightarrow K
$$

which, seen as an element of the algebra $(A \otimes A)^{\vee}$, is invertible with inverse $\tau^{-1}$, and satisfies the following cocycle condition:

$$
\sum_{\bullet \bullet} \tau\left(a_{1} \otimes b_{1}\right) \tau\left(a_{2} b_{2} \otimes c\right)=\sum_{\bullet \bullet} \tau\left(a \otimes b_{2} c_{2}\right) \tau\left(b_{1} \otimes c_{1}\right)
$$

As the nontion suggests, we can use such a cocycle in order to twist the originally given multiplication $m$ to a new multiplication $m_{\tau}$. It is defined by

$$
m_{\tau}(a \otimes b)=\sum_{\bullet \bullet} \tau\left(a_{1} \otimes b_{1}\right) a_{2} b_{2} \tau^{-1}\left(a_{3} \otimes b_{3}\right)
$$

for all $a, b \in A$, and we call it twisted multiplication. It will turn out that $A$, equipped with this twisted multiplication instead of the original one, is again a Hopf algebra. The new antipode will be given by

$$
i_{\tau}(a)=\sum_{\bullet} \tau\left(a_{1} \otimes i\left(a_{2}\right)\right) a_{3} \tau^{-1}\left(i\left(a_{4}\right) \otimes a_{5}\right)
$$

for $a \in A$, and we call it twisted antipode.

Proposition 11.3.11. - Let $A=(A, e, \varepsilon, \mu, m, i)$ be a commutative Hopf algebra and let $\tau: A \otimes$ $A \rightarrow K$ be a symmetric 2-cocycle. With the twisted multiplication $m_{\tau}$, the twisted antipode $i_{\tau}$ and the original unit, counit and comultiplication, $A$ is a commutative Hopf algebra $\left(A, e, \varepsilon, \mu, m_{\tau}, i_{\tau}\right)$.

Proof. All required properties of $m_{\tau}$ and $i_{\tau}$ are straightforward to verify. Before we start checking a few of them, we notice that the map $\tau^{-1}: A \otimes A \rightarrow K$ is in general not a cocycle, but is symmetric and satisfies

$$
\sum_{\bullet \bullet} \tau^{-1}\left(a_{1} b_{1} \otimes c\right) \tau^{-1}\left(a_{2} \otimes b_{2}\right)=\sum_{\bullet \bullet} \tau^{-1}\left(a \otimes b_{1} c_{1}\right) \tau^{-1}\left(b_{2} \otimes c_{2}\right)
$$

for all $a, b, c \in A$. With this relation in hand, we verify associativity of $m_{\tau}$.

$$
\begin{aligned}
m_{\tau}\left(m_{\tau}(a \otimes b) \otimes c\right) & =\sum_{\bullet \bullet} m_{\tau}\left(\tau\left(a_{1} \otimes b_{1}\right) a_{2} b_{2} \tau^{-1}\left(a_{3} \otimes b_{3}\right) \otimes c\right) \\
& =\sum_{\bullet \bullet \bullet} \tau\left(a_{1} \otimes b_{1}\right) \tau\left(a_{2} b_{2} \otimes c_{2}\right) a_{3} b_{3} c_{3} \tau^{-1}\left(a_{4} b_{4} \otimes c_{4}\right) \tau^{-1}\left(a_{5} \otimes b_{5}\right) \\
& =\sum_{\bullet \bullet \bullet} \tau\left(a_{2} \otimes b_{2} c_{2}\right) \tau\left(b_{1} \otimes c_{1}\right) a_{3} b_{3} c_{3} \tau^{-1}\left(a_{4} \otimes b_{4} c_{4}\right) \tau^{-1}\left(b_{5} \otimes c_{5}\right) \\
& =\sum_{\bullet \bullet} m_{\tau}\left(a \otimes \tau\left(b_{1} \otimes c_{1}\right) b_{2} c_{2} \tau^{-1}\left(b_{3} \otimes c_{3}\right)\right) \\
& =m_{\tau}\left(a \otimes m_{\tau}(b \otimes c)\right)
\end{aligned}
$$

That $m_{\tau}$ is commutative is an immediate consequence of the requirement that $\tau$ is symmetric.
11.3.12. - Let $A=(A, e, \varepsilon, \mu, m, i)$ be a commutative Hopf algebra, and let $n: A \otimes A \rightarrow A$ be a symmetric bilinear map such that $A^{\prime}=(A, e, \varepsilon, \mu, n, j)$ is a Hopf algebra, for some antipode $j$. Recall that if a bialgebra admits an antipode, it is unique. We want to fabricate a symmetric 2-cocycle $\tau$ such that $n=m_{\tau}$ holds.

Proposition 11.3.13. - Let $H \xrightarrow{i} G \stackrel{p}{\longrightarrow} F$ be morphisms of profinite groups.
(1) The morphism $i$ is injective if and only if for every finite $H$-set $S$ there exists a finite $G$-set $T$ and an injective map of $H$-sets $S \rightarrow T$.
(2) The morphism $p$ is surjective if and only if the functor $p^{*}: \operatorname{Set}(H) \rightarrow \boldsymbol{\operatorname { S e t }}(G)$ is full.

## Proof.

Proposition 11.3.14. - Let $G$ be a profinite group. Two closed subgroups $H$ and $N$ of $G$ are equal if and only if for every finite $G$-set $S$, the equality $S^{H}=S^{N}$ holds.

Proof. If the closed subgroups $H$ and $N$ of $G$ are distinct, there exists an open normal subgroup $U$ of $G$ such that $H /(H \cap U)$ and $N /(N \cap U)$ are distinct in $G / U$. Up to replacing $G$ by $G / U$, we may thus assume without loss of generality that $G$ is finite. Let $S$ be the set of all subsets of $G$, on which $G$ acts by left translation: $g X=\{g x \mid x \in X\}$ for $X \in S$ a subset of $G$. The set $H \in S$ is a fixed point for the restricted action of $H$ on $X$, hence by assumption it is a fixed point for the action of $N$ on $G$. In other words, the equality $N H=H$ holds, whence $N \subseteq H$, and thus $N=H$ by symmetry.

### 11.4. A fundamental exact sequence

We have introduced two canonical functors relating exponential Hodge structures to more benign objects. The first one is the inclusion $e:$ MHS $\rightarrow$ EMHS sending an ordinary Hodge structure to corresponding constant exponential Hodge structure, and the second one is the functor $r:$ EMHS $\rightarrow$ Perv $_{0}$ associating with an exponential Hodge structure its underlying perverse sheaf. The functors

$$
\text { MHS } \xrightarrow{e} \text { EMHS } \xrightarrow{r} \operatorname{Perv}_{0}
$$

are compatible with tensor products and with fibre functors. The composite of these functors is the trivial functor. From the point of view of tannakian fundamental groups, this means that the two functors induce morphisms of group schemes $i: \pi_{1}\left(\right.$ Perv $\left._{0}\right) \longrightarrow \pi_{1}(\mathbf{E M H S})$ and $p: \pi_{1}(\mathbf{E M H S}) \longrightarrow$ $\pi_{1}$ (MHS) whose composite is the trivial morphism. The following theorem answers the question at hand.

Theorem 11.4.1. - The sequence of group schemes over $\mathbb{Q}$

$$
\begin{equation*}
\pi_{1}\left(\operatorname{Perv}_{0}\right) \xrightarrow{i} \pi_{1}(\mathbf{E M H S}) \xrightarrow{p} \pi_{1}(\mathbf{M H S}) \longrightarrow 1 \tag{11.4.1.1}
\end{equation*}
$$

induced by the canonical functors $e$ : MHS $\rightarrow$ EMHS and $r$ : EMHS $\rightarrow \mathbf{P e r v}_{0}$ is exact.
11.4.2. - Before going into the proof, let us make a few comments. First, the morphism $i: \pi_{1}\left(\mathbf{P e r v}_{0}\right) \longrightarrow \pi_{1}(\mathbf{E M H S})$ is not a closed immersion since there are objects in $\mathbf{P e r v}_{0}$ which are not isomorphic to a subquotient of an object underlying a mixed Hodge module. However, if one starts with an object $M$ in EMHS, the fundamental group fits into an exact sequence

$$
1 \longrightarrow \pi_{1}\left(\left\langle R_{B}(M)\right\rangle^{\otimes}\right) \longrightarrow \pi_{1}\left(\langle M\rangle^{\otimes}\right) \longrightarrow \pi_{1}\left(\langle M\rangle^{\otimes} \cap \mathbf{M H S}\right) \longrightarrow 1 .
$$

where $\langle-\rangle^{\otimes}$ stands for "tannakian category generated by". Indeed, we can understand the image of $i$ as the tannakian fundamental group of the tannakian subcategory of Perv $\mathbf{v}_{0}$ generated by all objects which underlie an exponential Hodge structure. Our second comment is that the surjective morphism $p$ has no section. Indeed, a section of $p$ would provide a functor of tannakian categories EMHS $\rightarrow$ MHS such that the composition with the canonical functor $c:$ MHS $\rightarrow$ EMHS is isomorphic to the identity. But this is not possible, since in EMHS one has a square root of $\mathbb{Q}(-1)$ which does not exist in the category of mixed Hodge structures. However, as we will see in the next section, the corresponding exact sequence of Lie algebras is split.

Proof of Theorem 11.4.1. A morphism of affine group schemes $G \rightarrow F$ is surjective if and only if the corresponding functor $\operatorname{Rep}(F) \rightarrow \operatorname{Rep}(G)$ is fully faithful, with essential image stable under taking subobjects and quotients. Surjectivity of the morphism $p$ in the statement of the theorem follows thus from Lemma 11.2.4. It remains to show exactness in the middle. In order to apply the exactness criterion given in Proposition A.3.4 we need to interpret categorically what invariants under the kernel of $p$ are. Let $G \rightarrow F$ be a surjective morphism of affine group schemes with kernel $N$. The functor $V \longmapsto V^{N}$ from representations of $G$ to representations of $F$ is right adjoint to the functor $\operatorname{Rep}(F) \rightarrow \boldsymbol{\operatorname { R e p }}(G)$. Let thus $c:$ EMHS $\rightarrow$ MHS be the right adjoint of
the canonical functor $e$, and denote by $E(0)$ the unit object in $\mathbf{P e r v}_{0}$. We need to verify that the two following statements are true.
(1) Let $M$ be an object of EMHS. The morphism

$$
\operatorname{Hom}_{\operatorname{Perv}_{0}}(E(0), \operatorname{rec}(M)) \rightarrow \operatorname{Hom}_{\operatorname{Perv}_{0}}(E(0), r(M))
$$

induced by the adjunction map $e c(M) \rightarrow M$ is an isomorphism.
(2) Let $E$ be a one-dimensional object of $\operatorname{Perv}_{0}$ obtained as a subquotient of an object underlying an exponential Hodge structure. Then $E$ itself underlies an exponential Hodge structure.

Statement (1) follows formally from the existence of a six operations formalism for Hodge modules which is compatible with the six operations for perverse sheaves via the forgetful functors $r$ associating with a Hodge module on a variety its underlying perverse sheaf on the same variety. Let $\Pi: \mathbb{A}^{1} \backslash\{0\} \rightarrow$ Spec $k$ be the structral morphism and $j: \mathbb{A}^{1} \backslash\{0\} \rightarrow \mathbb{A}^{1}$ be the inclusion. The functor $e$ is the functor $j!\pi^{*}$ from Hodge modules on the point to Hodge modules on the affine line. Its right adjoint is the functor $c=\pi_{*} j^{!}$. The functors $c$ and $e$ commut with $r$, so we find

$$
\begin{aligned}
\operatorname{Hom}_{\operatorname{Perv}_{0}}(E(0), r(M)) & =\operatorname{Hom}_{\text {Perv }_{0}}(e \mathbb{Q}, r(M)) \\
& =\operatorname{Hom}_{\mathbf{V e c}}(\mathbb{Q}, \operatorname{cr}(M)) \\
& =\operatorname{Hom}_{\text {Perv }_{0}}(e \mathbb{Q}, \operatorname{ecr}(M)) \\
& =\operatorname{Hom}_{\text {Perv }_{0}}(E(0), \operatorname{rec}(M))
\end{aligned}
$$

using that $E(0)=e \mathbb{Q}$ and that $e$ is fully faithful. As for statement (2), recall that a one-dimensional object of $\operatorname{Perv}_{0}$ is determined up to isomorphism by the data of its only singularity $s \in k \subseteq \mathbb{C}$, and by the eigenvalue $\lambda \in \mathbb{Q}^{\times}$of the local monodromy operator near $s$. The local monodromy operators of any Hodge module on $\mathbb{A}^{1}$ are quasi-unipotent. Hence if a one-dimensional object of $\mathbf{P e r v}_{0}$ is a subquotient of an object underlying an exponential Hodge structure, then its local monodromy is either the identity, in which case it underlies the exponential Hodge structure $E(s)$, or its local monodromy is multiplication by -1 , in which case it underlies the exponential Hodge structure $E(s) \otimes \mathbb{Q}\left(\frac{1}{2}\right)$.

### 11.5. The Hodge realisation of exponential motives

$$
\begin{equation*}
\mathrm{R}_{\mathrm{Hdg}}: \mathbf{M}^{\exp }(k) \longrightarrow \mathbf{E M H S} \tag{11.5.0.1}
\end{equation*}
$$

Conjecture 11.5.1. - The Hodge realisation functor $\mathbf{M}^{\exp }(k) \longrightarrow \mathbf{E M H S}$ is full.
11.5.2. - Conjecture 11.5.1 enables us to control to a certain extent extension groups of exponential motives. For example, assuming the conjecture, the morphism of vector spaces

$$
\operatorname{Ext}_{\mathbf{M}^{\exp }(k)}^{1}\left(M_{1}, M_{2}\right) \rightarrow \operatorname{Ext}_{\mathbf{E M H S}}^{1}\left(\mathrm{R}_{\mathrm{Hdg}}\left(M_{1}\right), \mathrm{R}_{\mathrm{Hdg}}\left(M_{2}\right)\right)
$$

is injective for all exponential motives $M_{1}$ and $M_{2}$. We can use this to gain some heuristics about the nature of the extension groups $\operatorname{Ext}_{\mathbf{M}^{\exp }(k)}^{1}(\mathbb{Q}(0), \mathbb{Q}(a))$ for integers $a$.

### 11.6. The vanishing cycles functor

Let $\mathbb{C}$ denote the complex affine line with coordinate $x$. For each $z \in \mathbb{C}$ and each mixed Hodge module $M$ on $\mathbb{A}^{1}$, the vanishing cycles $\varphi_{x-z} M$ form a mixed Hodge module on the point $\{z\}$, hence a mixed Hodge structure. We consider the functor:

$$
\begin{align*}
\Phi: \text { EMHS } & \longrightarrow \text { MHS } \\
M & \longmapsto \bigoplus_{z \in \mathbb{C}} \varphi_{x-z} M . \tag{11.6.0.1}
\end{align*}
$$

Observe that the sum is finite, since $\varphi_{x-z} M=0$ unless $z$ is a singular point of $M$.

Proposition 11.6.1. - The functor $\Phi$ is compatible with the fibre functors.
Proof.
The composition of $\Phi$ with the canonical functor MHS $\rightarrow$ EMHS is the identity. Observe that this refrains $\Phi$ from being a tensor functor, since EMHS contains a square root of the object $\Pi\left(i_{*} \mathbb{Q}(-1)\right)$. To remedy this, we shall rather consider $\Phi$ with values in an enriched category, which takes into account the monodromy of vanishing cycles as well.

### 11.6.1. $\widehat{\mu}$-mixed Hodge structures.

Definition 11.6.2. - A $\widehat{\mu}$-mixed Hodge structure is a pair $(H, T)$ consisting of a mixed Hodge structure and a finite order automorphism of mixed Hodge structures $T: H \rightarrow H$. Together with the obvious morphisms, $\mu$-mixed Hodge structures form a category which will be denoted by $\mathbf{M H S}{ }^{\widehat{\mu}}$.

For each rational number $\alpha \in \mathbb{Q}$, let $H^{\alpha}=\operatorname{ker}(T-\exp (-2 \pi i \alpha)) \subseteq H \otimes_{\mathbb{Q}} \mathbb{C}$, so there is a direct sum decomposition

$$
H \otimes_{\mathbb{Q}} \mathbb{C}=\bigoplus_{\alpha \in \mathbb{Q} \cap(-1,0]} H^{\alpha}
$$

Following [76, p.661], we define the tensor product ${ }^{1}$

$$
H_{1} \otimes^{\mu} H_{2}
$$

of two $\widehat{\mu}$-mixed Hodge structures $\left(H_{1}, T_{1}\right)$ and $\left(H_{2}, T_{2}\right)$ as follows:

[^5](i) the underlying rational vector space is the tensor product of the underlying vector spaces $H_{1} \otimes H_{2}$, together with the automorphism $T_{1} \otimes T_{2} ;$
(ii) the weight filtration is given by
$$
W_{k}\left(H_{1} \otimes^{\widehat{\mu}} H_{2}\right)=\left(\bigoplus_{\alpha, \beta} \sum_{i, j} W_{i} H_{1}^{\alpha} \otimes W_{j} H_{2}^{\beta}\right) \cap\left(H_{1} \otimes H_{2}\right)
$$
where the sum is over pairs of integers $(i, j)$ such that
\[

i+j= $$
\begin{cases}k & \text { if } \alpha=0 \text { or } \beta=0 \\ k-2 & \text { if } \alpha+\beta=-1 \\ k-1 & \text { else }\end{cases}
$$
\]

(iii) the Hodge filtration is given by

$$
F^{p}\left(H_{1} \otimes^{\widehat{\mu}} H_{2}\right)=\bigoplus_{\alpha, \beta} \sum_{k, \ell} F^{k} H_{1}^{\alpha} \otimes F^{\ell} H_{2}^{\beta}
$$

where the sum is over pairs of integers $(k, \ell)$ such that

$$
k+\ell= \begin{cases}p & \text { if } \alpha+\beta>-1 \\ p-1 & \text { if } \alpha+\beta \leqslant-1\end{cases}
$$

One checks that, equipped with these new filtrations, $H_{1} \otimes^{\widehat{\mu}} H_{2}$ is again a mixed Hodge structure. Note that the inclusion MHS $\rightarrow \mathbf{M H S}^{\widehat{\mu}}$ sending a Hodge structure $H$ to ( $H$, id) is a tensor functor, but the forgetful functor $\mathbf{M H S}^{\widehat{\mu}} \rightarrow \mathbf{M H S}$ is not.
11.6.2. The enriched vanishing cycles functor. Recall that each $\varphi_{x-z} M$ comes together with a monodromy operator $T$. If $T_{s}$ denotes its semisimple part, the pair ( $\varphi_{x-z} M, T_{s}$ ) defines a $\widehat{\mu}$-mixed Hodge structure. We get thus a functor with values in MHS ${ }^{\widehat{\mu}}$. The following theorem of Saito [74] asserts that it is compatible with the tensor structures on both sides:

THEOREM 11.6.3 (Saito). - The functor $\phi^{\widehat{\mu}}:$ EMHS $\rightarrow \mathbf{M H S}^{\widehat{\mu}}$ is a tensor functor.

REmARK 11.6.4. - Let $M$ be the square root of $\Pi\left(i_{*} \mathbb{Q}(-1)\right)$ in EMHS. Then $\varphi^{\widehat{\mu}}(M)$ is the Hodge structure $\mathbb{Q}(0)$ equipped with the automorphism -Id. Its tensor square is $\mathbb{Q}(-1)$ together with the trivial automorphism, which solves the problem we encountered before.

ThEOREM 11.6.5. - The corresponding exact sequence of Lie algebras is split, and a splitting is given by the vanishing cycles functor $\mathbf{E M H S} \rightarrow$ MHS.

### 11.7. Monodromic exponential Hodge structures

Definition 11.7.1 (Kontsevich). - We call an exponential Hodge structure $M$ monodromic if $0 \in \mathbb{C}$ is its only singularity.
11.7.2. - A monodromic exponential Hodge structure is thus a Hodge module on the affine line whose fibre over $0 \in \mathbb{C}$ is trivial, and which is given by a variation of mixed Hodge structures on $\mathbb{C} \backslash\{0\}$. In other words, monodromic exponential Hodge structures are precisely those Hodge modules of the form $j_{!} V[1]$, where $j: \mathbb{C} \backslash\{0\} \rightarrow \mathbb{C}$ is the inclusion and $V$ is a variation of mixed Hodge structures on $\mathbb{C} \backslash\{0\}$. The category of monodromic exponential Hodge structures is, as an abelian category, equivalent to the category of variations of Hodge structures on $\mathbb{C} \backslash\{0\}$.

### 11.8. The vanishing cycles functor

The vanishing cycles functor

$$
\Psi: \text { EMHS } \rightarrow\{\mathbb{C} \text {-graded monodromic exponential Hodge structures }\}
$$

Theorem 11.8.1 (Kontsevich-Soibelmann). - There exists a (non-canonical) natural isomorphism

$$
\Psi(M \otimes N) \cong \Psi(M) \otimes \Psi(N) .
$$

### 11.9. The weight filtration

In this section, we show that exponential motives carry a canonical weight filtration with respect to which pure objects are semisimple. We first define the filtration at the level of exponential mixed Hodge structures following Kontsevich-Soibelman [58, 4.4], then we prove in Theorem 11.9.7 that it is motivic.

Definition 11.9.1 (Kontsevich-Soibelman). - The weight filtration of an exponential mixed Hodge structure $M$ is defined by

$$
W_{n} M=\Pi\left(W_{n}(M)\right),
$$

where on the right-hand side we regard $M$ as an object of $\operatorname{MHM}(\mathbb{C})$ and $W_{n} M$ denotes the weight filtration of mixed Hodge modules over the complex affine line.

Example 11.9.2. - It is instructive to examine the exponential Hodge structures $E(s)$ from Example 11.2.2. They are simple objects of EMHS, hence pure of some weight. However, regarded as objects of the bigger category $\operatorname{MHM}(\mathbb{C})$, the $E(s)$ are not simple, for they fit into an extension

$$
0 \rightarrow \mathbb{Q}_{s} \rightarrow E(s) \rightarrow \mathbb{Q}[1] \rightarrow 0,
$$

where $\mathbb{Q}_{s}$ denotes the skyscraper mixed Hodge module supported on $s$ with stalk $\mathbb{Q}(0)$. The above exact sequence describes the weight filtration of $E(s)$ as an object of $\mathbf{M H M}(\mathbb{C})$ as well:

$$
W_{0} E(s)=\mathbb{Q}_{s} \subseteq W_{1} E(s)=E(s) .
$$

Since the graded piece $\operatorname{gr}_{1}^{W} E(s)=\mathbb{Q}[1]$ is constant, it is killed by the projector $\Pi$ and one has $W_{0} E(s)=E(s)$ inside EMHS. We conclude that $E(s)$ is pure of weight 0 .
11.9.3. - The weight filtration is functorial, and the functor $W_{n}$ : EMHS $\rightarrow$ EMHS is exact, because the inclusion of EMHS into the category of mixed Hodge modules and the functor $\Pi$ are exact, and because the weight filtration is exact on mixed Hodge modules. We will use exactness of the weight filtration in the following way: Given an exponential Hodge structure $H$ and a substructure $H_{0} \subseteq H$, we can recover the weight filtration on $H_{0}$ and on the quotient $H_{1}=H / H_{0}$ by

$$
W_{n} H_{0}=H_{0} \cap W_{n} H \quad \text { and } \quad W_{n} H_{1}=W_{n} H /\left(H_{0} \cap W_{n} H\right)
$$

from the weight filtration on $H$. Thus, the weight filtration on $H$ determines the weight filtration on any subquotient of $H$.

Proposition 11.9.4. - The canonical functor $\iota:$ MHS $\rightarrow$ EMHS is strictly compatible with the weight filtration. In other words, for every Hodge structure $H$ and every integer $n$, the subobjects $W_{n}(\iota(H))$ and $\iota\left(W_{n} H\right)$ of $\iota(H)$ are the same.

Proof. Fix a Hodge structure $H$ and a weight $n$, and denote by $\alpha:\{0\} \rightarrow \mathbb{C}$ the inclusion and by $\pi: \mathbb{C} \rightarrow\{0\}$ the map to a point. Regarding $H$ as a Hodge module on the point $\{0\}$, there is a short exact sequence

$$
0 \rightarrow \alpha_{*} H \rightarrow \iota(H) \rightarrow \pi^{*} H[1] \rightarrow 0
$$

in $\operatorname{MHM}(\mathbb{C})$. In this sequence, the Hodge module $\pi^{*} H_{\mathbb{C}}[1]$ is the one defined by the constant variation of Hodge structures with fibre $H$, put in homological degree -1 . We apply the exact functor $W_{n}$, and obtain the exact sequence

$$
0 \rightarrow \alpha_{*} W_{n} H \rightarrow W_{n} \iota H \rightarrow \pi^{*} W_{n+1} H[1] \rightarrow 0
$$

in $\operatorname{MHM}(\mathbb{C})$. Applying the exact functor $\Pi$ yields an isomorphism $\Pi\left(\alpha_{*} W_{n} H\right) \rightarrow \Pi\left(W_{n} \iota H\right)$. The functor $\iota$ is the composite $\Pi \circ \alpha_{*}$, so we find $\iota\left(W_{n} H\right) \xrightarrow{=} W_{n}(\iota(H))$ where the weight filtration $W_{n}$ is now that in the category of exponential motives.

Let $X$ be smooth. If the canonical map $H_{c}(X, f) \longrightarrow H(X, f)$ is an isomorphism, then the exponential mixed Hodge structure $H(X, f)$ is pure.

Example 11.9.5. -
(1) If $X$ is smooth and $f: X \longrightarrow \mathbb{A}^{1}$ is proper, then $H^{n}(X, f)$ is pure of weight $n$.
(2) More generally, it suffices to assume that the function $f$ is cohomologically tame in the sense of Katz [55, Prop. 14.13.3, item (2)]. This means that all the cohomology sheaves of the cone of the "forget supports" morphism $R f_{!} \mathbb{Q}_{X} \rightarrow R f_{*} \mathbb{Q}_{X}$ are lisse over $\mathbb{A}^{1}$ hence trivial. A more geometric condition, used by Sabbah [70, §8], is that there exists an embedding $j: X \hookrightarrow Y$ into a smooth variety $Y$ and a proper map $\bar{f}: Y \rightarrow \mathbb{A}^{1}$ extending $f$ such that, for all $z \in \mathbb{C}$, the vanishing cycles complex $\varphi_{\bar{f}-z}\left(R j_{*} \mathbb{Q}_{X}\right)$ is supported at a finite number of points lying in $X$ (as opposed to $Y$ ).
11.9.6 (The weight filtration is motivic). - Let $M$ be an exponential motive. The Hodge realisation of $M$, and hence its perverse realisation and its Betti realisation come equipped with a weight filtration. A natural question to ask is whether this filtration comes from a filtration of $M$ by submotives. If such a filtration exists, it is necessarily unique.

THEOREM 11.9.7. - Every object $M$ of $\mathbf{M}^{\exp }(k)$ is equipped with an increasing and exhaustive filtration $W_{\bullet} M$ which maps to the weight filtration under the Hodge realisation functor.

Proof. We start with the following observation: Suppose the motive $M$ admits an exhaustive filtration $F$, such that for each graded piece $M_{n}=\operatorname{gr}_{n}^{F} M$ the statement of the theorem holds, that is, $M_{n}$ admits a filtration by submotives whose Hodge realisation is the weight filtration. Then, the statement of the theorem holds for $M$ too.

We prove the theorem for motives $M$ of increasing generality. The cases we consider are, in summary, the following:
(1) $M=H^{n}(X, Y, f)$, where $f: X \rightarrow \mathbb{A}^{1}$ is proper.
(2) $M=H^{n}(X, Y, f)$ for arbitrary $X, Y$ and $f$
(3) $M$ an arbitrary exponential motive

Case 1: Let $X$ be a variety of dimension $\leqslant d$ with a proper morphism $f: X \rightarrow \mathbb{A}^{1}$, and let $Y \subseteq X$ be a subvariety of dimension $\leqslant d-1$. If $d=0$, then $X$ is a collection of points and $Y$ is empty, and hence $H^{n}(X, Y, f)$ is pure of weight 0 . Arguing by induction on dimension, we may suppose that the weight filtration on $H^{n-1}(Y)$ is motivic, with weights $0,1, \ldots, n-1$. By resolution of singularities, there is a smooth variety $\widetilde{X}$ of dimension $d$ mapping to $X$ with a normal crossing divisor $\widetilde{Y}$ mapping to $Y$ such that

$$
H^{n}(X, Y, f) \rightarrow H^{n}(\widetilde{X}, \tilde{Y}, \widetilde{f})
$$

is an isomorphism. We may thus suppose without loss of generality that $X$ is smooth and $Y$ a normal crossing divisor. From the long exact sequence

$$
\cdots \longrightarrow H^{n-1}\left(Y,\left.f\right|_{Y}\right) \longrightarrow H^{n}(X, Y, f) \longrightarrow H^{n}(X, f) \longrightarrow \cdots
$$

and the fact that $H^{n}(X, f)$ is pure of weight $n$, we see that the weight filtration on $H^{n}(X, Y, f)$ is given by

$$
\begin{array}{ll}
W_{s} H^{n}(X, Y, f)=\operatorname{im}\left(W_{s} H^{n-1}\left(Y,\left.f\right|_{Y}\right) \rightarrow H^{n}(X, Y, f)\right) & \text { for } s<n \\
W_{s} H^{n}(X, Y, f)=H^{n}(X, Y, f) & \text { for } s \geqslant n
\end{array}
$$

hence is motivic. In particular, the weights of $H^{n}(X, Y, f)$ are $0,1, \ldots, n$.
Case 2: We now treat the case of a motive $M$ of the form $M=H^{n}(X, Y, f)$ for a smooth, not necessarily proper variety $X$ with a function $f: X \rightarrow \mathbb{A}^{1}$, and a smooth subvariety $Y \subseteq X$. We choose a smooth relative compactification $\bar{f}: \bar{X} \rightarrow \mathbb{A}^{1}$. That means the following: $X$ is an open subvariety of a smooth variety $\bar{X}$ with complement a normal crossing divisor $D$, the closure $\bar{Y}$ of $Y$ in $\bar{X}$ is smooth and $D+\bar{Y}$ has normal crossings, and $\bar{f}: \bar{X} \rightarrow \mathbb{A}^{1}$ is a proper map extending $f$. Let $D_{1}, D_{2}, \ldots, D_{N}$ be the smooth components of the divisor $D$, and set

$$
X^{(p)}=\bigsqcup_{1 \leqslant i_{1}<\cdots<i_{p} \leqslant N} D_{i_{1}} \cap \cdots \cap D_{i_{p}}
$$

for $p=0,1, \ldots, N$, and $Y^{(p)}=X^{(p)} \cap Y$. In particular we set $X^{(0)}=\bar{X}$. The varieties $X^{(p)}$ and $Y^{(p)}$ are smooth, and there are inclusions maps $\iota_{s}:\left(X^{(p)}, Y^{(p)}\right) \rightarrow\left(X^{(p-1)}, Y^{(p)}\right)$ for $s=1,2, \ldots, p$. We use alternating sums of the induced Gysin morphisms (4.8.3.3) to get a double complex

$$
\cdots \rightarrow C^{*}\left(X^{(p)}, Y^{(p)}, f\right)[2 p](p) \rightarrow C^{*}\left(X^{(p-1)}, Y^{(p-1)}, f\right)[2 p-2](p-1) \rightarrow \cdots \rightarrow C^{*}(\bar{X}, \bar{Y}, \bar{f})
$$

The total complex of this double complex computes the cohomology of $(X, Y, f)$. This is where the spectral sequence

$$
E_{2}^{p, q}=H^{2 p+q}\left(X^{(p)}, Y^{(p)}, f\right)(p) \Longrightarrow H^{p+q}(X, Y, f)
$$

comes from.

## Case 3:

Theorem 11.9.8. - Pure objects for the weight filtration are semisimple.
Proof. Explain how to deduce it from Theorem 11.12.1.

### 11.10. The irregular Hodge filtration

In this section, we recall that the de Rham cohomology $H_{\mathrm{dR}}^{*}(X, f)$ is equipped with an irregular Hodge filtration which is indexed by rational numbers and has finitely many jumps. It was first introduced by Deligne [24] in the case of curves, then generalized to higher dimensional varieties by Yu [90]. Further properties - especially the degeneration of the corresponding spectral sequencewere studied by Sabbah, Esnault and Yu in [71] and [35].
11.10.1 (The Kontsevich complex). - Let $X$ be a smooth variety of dimension $n$ over $k$, together with a regular function $f$, and let $\bar{X}$ be a good compactification of $(X, f)$ as in Definition 3.5.8. We keep the same notation from loc. cit., so $D=\bar{X} \backslash X$ is the normal crossing divisor at infinity and
$P$ the pole divisor of $f$. We write $P=\sum e_{i} P_{i}$ with $P_{i}$ the irreducible components. The connection $\mathcal{E}^{f}$ on $X$ extends to an integrable meromorphic connection on $\bar{X}$ with associated de Rham complex

$$
\left(\Omega_{\bar{X}}^{\bullet}(* D), d_{f}\right) .
$$

However, the subsheaves $\Omega \frac{p}{X}(\log D) \subseteq \Omega_{\bar{X}}^{p}(* D)$ of logarithmic differentials do not form a subcomplex, so one cannot naively imitate the constructions from Hodge theory.

A possible way to circumvent this problem, after Kontsevich, is as follows: given a rational number $\alpha \in[0,1) \cap \mathbb{Q}$, we set $[\alpha P]=\sum\left[\alpha e_{i}\right] P_{i}$, where $[\cdot]$ stands for the integral part, and

$$
\begin{equation*}
\Omega_{\bar{X}}^{p}(\log D)([\alpha P])=\Omega_{\bar{X}}^{p}(\log D) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{\bar{X}}([\alpha P]) \tag{11.10.1.1}
\end{equation*}
$$

We then define a subsheaf $\Omega_{f}^{p}(\alpha)$ of (11.10.1.1) by asking that, for every open subset $U \subseteq \bar{X}$,

$$
\Omega_{f}^{p}(\alpha)(U)=\left\{\omega \in \Omega_{\bar{X}}^{p}(\log D)([\alpha P])(U) \mid d f \wedge \omega \in \Omega_{\bar{X}}^{p+1}(\log D)([\alpha P])(U)\right\}
$$

In particular, one has:

$$
\Omega_{f}^{0}(\alpha)=\mathcal{O}_{\bar{X}}([\alpha-1] P), \quad \Omega_{f}^{n}(\alpha)=\Omega_{\bar{X}}^{n}(\log D)([\alpha P])
$$

The sheaves $\Omega_{f}^{p}(\alpha)$ are stable under $d_{f}$ and form a complex which computes the de Rham cohomology of the pair $(X, f)$ :

Proposition 11.10.2. - The inclusion $\left(\Omega_{f}^{\bullet}(\alpha), d_{f}\right) \hookrightarrow\left(\Omega_{\bar{X}}^{\bullet}(* D), d_{f}\right)$ is a quasi-isomorphism for each $\alpha \in[0,1) \cap \mathbb{Q}$. In particular, there are canonical isomorphisms

$$
\begin{equation*}
H_{\mathrm{dR}}^{n}(X, f) \cong H^{n}\left(\bar{X},\left(\Omega_{f}^{\bullet}(\alpha), d_{f}\right)\right) \tag{11.10.2.1}
\end{equation*}
$$

11.10.3. -

Definition 11.10.4. - The irregular Hodge filtration is given by

$$
\begin{equation*}
F^{p-\alpha} H_{\mathrm{dR}}^{n}(X, f)=\operatorname{Im}\left(\mathbb{H}^{n}\left(\bar{X},\left(\Omega_{f}^{\bullet \geqslant p}(\alpha), d_{f}\right)\right) \longrightarrow \mathbb{H}^{n}\left(\bar{X},\left(\Omega_{f}^{\bullet}(\alpha), d_{f}\right)\right)\right) . \tag{11.10.4.1}
\end{equation*}
$$

In fact, the relevant $\alpha$ will be those of the form $\alpha=\frac{\ell}{m}$ where $m$ is the multiplicity of an irreducible component of $P$ and $\ell=1, \ldots, m-1$.
11.10.5. - Let us compute a few examples of irregular Hodge filtrations:
11.10.6 (Compatibility with the Künneth formula). - We now assume that we are given two pairs $\left(X_{1}, f_{1}\right)$ and $\left(X_{2}, f_{2}\right)$ consisting of smooth varieties over $k$ and regular functions. As usual, we consider the cartesian product $X_{1} \times X_{2}$ together with the Thom-Sebastiani sum $f_{1} \boxplus f_{2}$. By the Künneth formula, cup-product induces an isomorphism of $k$-vector spaces

$$
\begin{equation*}
\bigoplus_{i+j=n} H_{\mathrm{dR}}^{i}\left(X_{1}, f_{1}\right) \otimes H_{\mathrm{dR}}^{j}\left(X_{2}, f_{2}\right) \longrightarrow H_{\mathrm{dR}}^{n}\left(X_{1} \times X_{2}, f_{1} \boxplus f_{2}\right) \tag{11.10.6.1}
\end{equation*}
$$

We equip the left-hand side of (11.10.6.1) with the product filtration, that is

$$
\begin{equation*}
\bigoplus_{i+j=n}\left(\sum_{a+b=\lambda} F^{a} H_{\mathrm{dR}}^{i}\left(X_{1}, f_{1}\right) \otimes F_{\mathrm{dR}}^{b}\left(X_{2}, f_{2}\right)\right) \tag{11.10.6.2}
\end{equation*}
$$

Theorem 11.10.7 (Chen-Yu, [18]). - The map (11.10.6.1) is an isomorphism of filtered vector spaces.

### 11.11. Twistors

A twistor is a holomorphic vector bundle $\mathcal{T}$ on $\mathbb{P}^{1}=\mathbb{P}^{1}(\mathbb{C})$. We typically equip $\mathcal{T}$ with a real structure $\kappa$ and a connection $\nabla$, compatible with each other. In that case, we call

$$
\mathcal{T}=(\mathcal{T}, \kappa, \nabla)
$$

a real integrable twistor. In this section, we explain some details about twistors and why they are useful, in particular how to associate twistors with exponential Hodge structures. Short introductions to twistors can be found in [73], the full story is told in [64].
11.11.1. - Denote by $\sigma, \gamma$ and $\iota$ the automorphisms

$$
\begin{array}{lll}
\sigma: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1} & \sigma(z)=-\bar{z}^{-1} & 0 \longleftrightarrow \infty \\
\gamma: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1} & \gamma(z)=\bar{z}^{-1} & 0 \longleftrightarrow \infty \\
\iota: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1} & \iota(z)=-z & 0 \rightarrow 0 ; \infty \rightarrow \infty
\end{array}
$$

of $\mathbb{P}^{1}$. The equalities $\gamma=\iota \circ \sigma=\sigma \circ \iota$ hold, $\gamma$ induces the identity on the unit circle, and $\sigma$ and $\iota$ are equal on the unit circle. For a vector bundle $\mathcal{T}$ on $\mathbb{P}^{1}$, set

$$
\begin{array}{ll}
\mathcal{T}^{\vee}=\mathcal{H o m}\left(\mathcal{T}, \mathcal{O}_{\mathbb{P}^{1}}\right) & \text { the dual bundle } \\
\mathcal{T}^{*}=\sigma^{*} \overline{\mathcal{T}}^{\vee} & \text { the Hermitian dual bundle } \\
\mathcal{T}^{c}=\gamma^{*} \overline{\mathcal{T}} & \text { the conjugate bundle. }
\end{array}
$$

A real structure on $\mathcal{T}$ is an isomorphism of vector bundles
such that
is equal to $\kappa^{-1}$.

### 11.12. Semisimplicity

In classical frameworks, motives of the form $H^{n}(X)(i)$ are expected to be semisimple if $X$ is a smooth and proper variety. For Nori motives, this is well known. In the framework of exponential motives, this fact generalises to the statement that exponential motives of the form $H^{n}(X, f)(i)$ are semisimple, provided $X$ is smooth and $f: X \rightarrow \mathbb{A}^{1}$ is a proper morphism. The main objective of this section is to prove this assertion, stated as Theorem 11.12 .1 below. Again in the classical setup, semisimplicity is proven in two steps: First one establishes the hard Lefschetz theorem, in order to decompose the cohomology of $X$ into primitive pieces, and to obtain a pairing

$$
\begin{equation*}
H^{n}(X) \otimes H^{n}(X) \xrightarrow{\text { id } \otimes \eta^{d-n}} H^{n}(X) \otimes H^{2 d-n}(X) \xrightarrow{\text { cup }} \mathbb{Q}(-d) \tag{11.12.0.1}
\end{equation*}
$$

for $n \leqslant d=\operatorname{dim} X$. Using this pairing, one can associate with every subobject $M \subseteq H^{n}(X)$ an orthogonal subobject $M^{\perp} \subseteq H^{n}(X)$. In a second step, one uses the fact that the pairing (11.12.0.1) induces a polarisation of Hodge structures in order to show that the canonical morphism

$$
M \oplus M^{\perp} \rightarrow H^{n}(X)
$$

is an isomorphism. To prove Theorem 11.12.1, we will essentially follow this strategy, but using twistors in place of Hodge structures. There is one additional issue which arises with exponential motives: Hard Lefschetz gives us an isomorphism $H^{n}(X, f) \cong H^{2 d-n}(X, f)$, but the duality pairing is a pairing between $H^{n}(X,-f)$ and $H^{2 d-n}(X, f)$, and hence given a subobject $M \subseteq H^{n}(X, f)$ its orthogonal $M^{\perp}$ is contained in $H^{n}(X,-f)$. This seems to be a problem, since in general $H^{n}(X, f)$ and $H^{n}(X,-f)$ are non-isomorphic motives. We will take care of this by constructing an autoequivalence $T: \mathbf{M}^{\exp }(k) \rightarrow \mathbf{M}^{\exp }(k)$ sending $H^{n}(X, f)$ to $H^{n}(X,-f)$, so we can use it to transport $M^{\perp}$ to a subobject of $H^{n}(X, f)$. On the other hand, it will turn out that this sign is essential in the construction of polarisations of twistors.

Theorem 11.12.1 (Semisimplicity Theorem). - Let $X$ be a smooth variety and $f: X \rightarrow \mathbb{A}^{1}$ be a proper morphism. For all $n \geqslant 0$ the object $H^{n}(X, f)(i)$ of $\mathbf{M}^{\exp }(k)$ is semisimple.
11.12.2. - Let $k$ be a subfield of $\mathbb{C}$ and let $(a, \gamma)$ be a pair consisting of an element $a \in k^{*}$ and a path $\gamma$ from $1 \infty$ to $a^{-1} \infty$ in the boundary of the real blowup of $\mathbb{P}^{1}(\mathbb{C})$ at infinity. We can associate with $(a, \gamma)$ an automorphism $T=T(a, \gamma)$ of the category $\mathbf{M}^{\exp }(k)$, compatible with the tensor product, duals, and the Betti realisation, as follows: Let $\mathrm{Q}=\mathrm{Q}^{\exp }(k)$ be the quiver of exponential relative varieties over $k$, introduced in 4.2.1, and let $\rho: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$ be the Betti representation. A quiver morphism $T_{Q}: Q \rightarrow Q$ is given by

$$
T_{Q}([X, Y, f, n, i])=[X, Y, a f, n, i]
$$

on objects, and in the straightforward way on morphisms. In order to turn the quiver endomorphism $T_{Q}$ into an endomorphism of the representation $(Q, \rho)$ we must, according to 4.1 .9 , specify an isomorphism $\rho \circ T_{Q} \cong \rho$, that is, a natural isomorphism of vector spaces

$$
H_{\mathrm{rd}}^{n}(X, Y, a f)(i) \cong H_{\mathrm{rd}}^{n}(X, Y, f)(i)
$$

for every object $[X, Y, f, n, i]$ of the quiver $Q$. Such an isomorphism is obtained from the chosen path $\gamma$. Indeed, we can think of these vector spaces as the fibres at $1 \infty$ and at $a^{-1} \infty$ of the sheaf $H_{\text {perv }}^{n}(X, Y, f)(i)$, which are isomorphic via parallel transport along $\gamma$.
11.12.3. -

## CHAPTER 12

## Examples and implications of the period conjecture

In this chapter, we present a number of examples of exponential motives and their periods. These include notably exponentials of algebraic numbers, the square root of $\pi$, special values of certain $E$-functions such as the Bessel function, and the Euler-Mascheroni constant. In each case, we compute the Galois group of the corresponding exponential motive and we investigate the implications of the exponential period conjecture.

### 12.1. Exponentials of algebraic numbers

Arguably, the most elementary exponential period that is not expected to be a period in the classical sense is the base of the natural $\operatorname{logarithm} e$. That $e$ is an irrational number was known to Euler, and its transcendence was proved by Hermite in 1873. The Lindemann-Weierstrass theorem, which we recall below, generalises Hermite's transcendence theorem. We will show that it is a consequence of the exponential period conjecture, and hence serves as an illustration of it. We will also show that the period conjecture implies that $e$ is not a period in the classical sense, and in fact even that it is algebraically independent from all classical periods.

Theorem 12.1.1 (Lindemann-Weierstrass). - Let $\alpha_{1}, \ldots, \alpha_{n}$ be algebraic numbers, and denote by $\left\langle\alpha_{1}, \ldots, \alpha_{n}\right\rangle$ the $\mathbb{Q}$-subvector space of $\overline{\mathbb{Q}}$ generated by them. Then the equality

$$
\operatorname{trdeg} \overline{\mathbb{Q}}\left(e^{\alpha_{1}}, \ldots, e^{\alpha_{n}}\right)=\operatorname{dim}_{\mathbb{Q}}\left\langle\alpha_{1}, \ldots, \alpha_{n}\right\rangle
$$

holds. In particular, if $\alpha_{1}, \ldots, \alpha_{n}$ are $\mathbb{Q}$-linearly independent, then their exponentials $e^{\alpha_{1}}, \ldots, e^{\alpha_{n}}$ are algebraically independent.
12.1.2. - Let us now explain how one can see the Lindemann-Weierstrass theorem as an instance of the exponential period conjecture. Given algebraic numbers $\alpha_{1}, \ldots, \alpha_{n}$, set

$$
E\left(\alpha_{1}, \ldots, \alpha_{n}\right)=E\left(\alpha_{1}\right) \oplus \cdots \oplus E\left(\alpha_{n}\right),
$$

where $E\left(\alpha_{i}\right)$ denotes the one-dimensional exponential motive over $\overline{\mathbb{Q}}$ defined by

$$
E\left(\alpha_{i}\right)=H^{0}\left(\operatorname{Spec}(\overline{\mathbb{Q}}),-\alpha_{i}\right)
$$

In particular, $E(0)=\mathbb{Q}(0)$ is the unit motive. The period algebra of the motive $E\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ is generated by the exponentials $e^{\alpha_{1}}, \ldots, e^{\alpha_{n}}, e^{-\left(\alpha_{1}+\cdots+\alpha_{n}\right)}$, and the period conjecture predicts that its transcendence degree over $\overline{\mathbb{Q}}$ is the dimension of the motivic Galois group of $E\left(\alpha_{1}, \ldots, \alpha_{n}\right)$.

Proposition 12.1.3. - Let $\alpha_{1}, \ldots, \alpha_{n}$ be algebraic numbers. The Galois group of the exponential motive $E\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ is a split torus of dimension $\operatorname{dim}_{\mathbb{Q}}\left\langle\alpha_{1}, \ldots, \alpha_{n}\right\rangle_{\mathbb{Q}}$.

Proof. For every $\alpha \in \overline{\mathbb{Q}}$, the motive $E(\alpha)$ is one-dimensional. Its tannakian fundamental group is thus canonically isomorphic to a subgroup of $\mathbb{G}_{m}$. This allows us to canonically identify the fundamental group $G$ of $E\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ with a subgroup of $\mathbb{G}_{m}^{n}$. We will show that $G \subseteq \mathbb{G}_{m}^{n}$ is equal to the subtorus $T \subseteq \mathbb{G}_{m}^{n}$ whose group of characters is the subgroup of $\overline{\mathbb{Q}}$ generated by $\alpha_{1}, \ldots, \alpha_{n}$, which we view as a quotient of $\mathbb{Z}^{n}$.

There is a canonical isomorphism of motives $E(\alpha) \otimes E(\beta) \cong E(\alpha+\beta)$ for all algebraic numbers $\alpha$ and $\beta$. In particular, the $\otimes$-inverse of $E(\alpha)$ is $E(-\alpha)$. By induction, every $\mathbb{Z}$-linear relation $c_{1} \alpha_{1}+\cdots+c_{n} \alpha_{n}=0$ yields an isomorphism of motives:

$$
E\left(\alpha_{1}\right)^{\otimes c_{1}} \otimes \cdots \otimes E\left(\alpha_{n}\right)^{\otimes c_{n}} \cong \mathbb{Q}(0)
$$

The action of the Galois group on the right-hand side is trivial, and hence it must be trivial on the left-hand side as well. Thus, if $\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{G}_{m}^{n}$ lies in $G$, then $z_{1}^{c_{1}} z_{2}^{c_{2}} \cdots z_{n}^{c_{n}}=1$. This yields the inclusion $G \subseteq T$.

In order to establish the inclusion $T \subseteq G$, recall that the Galois group of a motive $M$ contains the Galois group of its perverse realisation. Set $F(\alpha)=\mathrm{R}_{\text {perv }}(E(\alpha))$, and let us show that the Galois group of $F\left(\alpha_{1}, \ldots, \alpha_{n}\right)=F\left(\alpha_{1}\right) \oplus \cdots \oplus F\left(\alpha_{n}\right)$ in $\operatorname{Perv}_{0}$ is already the full $T$. All objects in the tannakian category generated by $F$ are semisimple, and simple objects are precisely those one-dimensional objects of the form

$$
F(\alpha)=F\left(\alpha_{1}\right)^{\otimes c_{1}} \otimes \cdots \otimes F\left(\alpha_{n}\right)^{\otimes c_{n}}
$$

where $\alpha=c_{1} \alpha_{1}+\cdots+c_{n} \alpha_{n}$ is a linear combination of the algebraic numbers $\alpha_{1}, \ldots, \alpha_{n}$. The claim now follows from the fact that for any two complex numbers $\alpha$ and $\beta$ we have $\operatorname{Hom}(F(\alpha), F(\beta))=0$ unless $\alpha=\beta$. In other words, the tannakian category generated by $F\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ is equivalent to the category of rational vector spaces with a grading indexed by the finitely generated group $\left\langle\alpha_{1}, \ldots, \alpha_{n}\right\rangle_{\mathbb{Z}}$.

Proposition 12.1.4. - Assume that the exponential period conjecture 8.2.6 holds. Then the exponential of a non-zero algebraic number is transcendental over the field of usual periods.

Proof. Let $\mathbb{P}$ be the field generated by the periods of usual motives. We need to show that, given a non-zero algebraic number $\alpha$ and a polynomial $f \in \mathbb{P}[x]$, the relation $f\left(e^{\alpha}\right)=0$ implies $f=0$. We choose a usual motive $M$ over $\overline{\mathbb{Q}}$ such that all the coefficients of $f$ lie in the field generated by the periods of $M$ and we consider the exponential motive $M^{+}=M \oplus E(\alpha)$. Its field of periods is generated by all the periods of $M$ together with $e^{\alpha}$. Let $G_{M}$ and $G_{M^{+}}$denote the corresponding motivic Galois groups. Since $\langle M\rangle^{\otimes}$ is a subcategory of $\left\langle M^{+}\right\rangle^{\otimes}$, there is a canonical
surjection $G_{M^{+}} \rightarrow G_{M}$. Assuming the exponential period conjecture, it suffices to prove that the inequality $\operatorname{dim} G_{M^{+}}>\operatorname{dim} G_{M}$ holds.

Let $F$ and $F^{+}$be the perverse realisations of $M$ and $M^{+}$, and denote by $G_{F+}$ and $G_{F}$ their Galois groups. The group $G_{F}$ is trivial, since $F$ comes from a usual motive, and hence is isomorphic to a sum of copies of the neutral object in the tannakian category Perv $\mathbf{v}_{0}$. The group $G_{F^{+}}$is the same as $G_{E(\alpha)}$, and hence is isomorphic to $\mathbb{G}_{m}$ since $\alpha \neq 0$. The diagram

shows that the surjection $G_{M^{+}} \rightarrow G_{M}$ contains a copy of $\mathbb{G}_{m}$ in its kernel, hence the sought inequality of dimensions.

Proposition 12.1.5. - Assume that the exponential period conjecture 8.2.6 holds. Then, the algebraic closure of the ring of classical periods inside the ring of exponential periods is generated by special values of the gamma function $\Gamma(q)$ with $q \in \mathbb{Q} \backslash \mathbb{Z}_{\leqslant 0}$.

Proof. Let $w$ be an exponential period which is algebraic over the ring of classical periods. There exist classical periods $c_{0}, \ldots, c_{n}$ such that

$$
c_{0}=c_{1} w \cdots+c_{n} w^{n}
$$

holds. Let $M$ be an exponential motive with $w$ as a period, and let $M_{i}$ be a classical motive with $c_{i}$ as a period.

$$
M_{0} \rightarrow \bigoplus_{i=1}^{n}\left(M_{i} \otimes M^{\otimes i}\right)
$$

### 12.2. The motive $\mathbb{Q}\left(-\frac{1}{2}\right)$

There has already been some speculation about motives $\mathbb{Q}\left(-\frac{1}{2}\right)$ and $\mathbb{Q}\left(-\frac{1}{4}\right)$ over finite fields ${ }^{1}$. Specifically, if $E$ is a supersingular elliptic curve over a finite field and $F$ a field of coefficients splitting the quaternion algebra $\operatorname{End}(E)$, then the 2-dimensional motive $H^{1}(E)$ decomposes as a sum $M \oplus M$ where $M \otimes M$ is isomorphic to the Tate motive $F(-1)=H^{2}\left(\mathbb{P}^{1}\right)$.

Over a field of characteristic zero, motives $M$ with $M \otimes M \simeq \mathbb{Q}(-1)$ should not exist, at least not in the classical sense, since the Hodge realisation of such an $M$ would necessarily be a onedimensional Hodge structure of weight 1. This is why it is not expected that $\sqrt{2 \pi i}$ is a period in

[^6]the classical sense. However, we can easily write $\sqrt{2 \pi i}$ and $\sqrt{\pi}$ as periods of exponential motives over $\mathbb{Q}(i)$ and over $\mathbb{Q}$ respectively:
\[

$$
\begin{aligned}
\sqrt{2 \pi i} & =\int_{(i+1) \mathbb{R}} e^{\frac{-1}{2 i} x^{2}} d x \\
\sqrt{\pi} & =\Gamma\left(\frac{1}{2}\right)=\int_{\mathbb{R}} e^{-x^{2}} d x
\end{aligned}
$$
\]

The corresponding exponential motives are $H^{1}\left(\mathbb{A}_{\mathbb{Q}(i)}^{1}, \frac{1}{2 i} x^{2}\right)$ and $H^{1}\left(\mathbb{A}_{\mathbb{Q}}^{1}, x^{2}\right)$ respectively, where $x$ is the coordinate of the affine line. This suggests that the motive $H^{1}\left(\mathbb{A}_{k}^{1}, \frac{1}{2 i} x^{2}\right)$ is a reasonable candidate for $\mathbb{Q}\left(-\frac{1}{2}\right)$. We will show that indeed for any field $k \subseteq \mathbb{C}$ and non-zero element $a \in k$ such that $a=\frac{1}{2 i} c^{2}$ for some $c \in k$, there is an isomorphism

$$
H^{1}\left(\mathbb{A}_{k}^{1}, a x^{2}\right) \otimes H^{1}\left(\mathbb{A}_{k}^{1}, a x^{2}\right) \cong \mathbb{Q}(-1)
$$

of exponential motives over $k$. Given any two non-zero elements $a$ and $b$ of $k$, the motives $H^{1}\left(\mathbb{A}_{k}^{1}, a x^{2}\right)$ and $H^{1}\left(\mathbb{A}_{k}^{1}, b x^{2}\right)$ are isomorphic if and only if $a=c^{2} b$ for some $c \in k^{*}$. It follows that if $k$ contains $i$, there exist motives over $k$ whose tensor square is $\mathbb{Q}(-1)$, and indeed many of them unless $k$ is quadratically closed. Let us fix

$$
M(\sqrt{\pi})=H^{1}\left(\mathbb{A}^{1}, x^{2}\right)
$$

as a particular exponential motive over $k$ with period $\sqrt{\pi}$.

Lemma 12.2.1. - Let $a, b \in k^{\times}$and let $C_{a, b}$ be the affine conic over $k$ defined by the equation $a s^{2}+b t^{2}=1$. There exists an isomorphism of exponential motives

$$
H^{1}\left(\mathbb{A}^{1}, a x^{2}\right) \otimes H^{1}\left(\mathbb{A}^{1}, b x^{2}\right) \simeq H^{1}\left(C_{a, b}\right)
$$

Proof. By the Künneth formula, it suffices to show that $H^{2}\left(\mathbb{A}^{2}, a x^{2}+b y^{2}\right)$ and $H^{1}\left(C_{a, b}\right)$ are isomorphic. At the level of periods, this is reflected by the identity

$$
\int_{e^{i \arg (a)} \mathbb{R} \times e^{i \arg (b)} \mathbb{R}} e^{-a x^{2}-b y^{2}} d x d y=\frac{\pi}{\sqrt{a b}}
$$

which follows from the change of coordinates $x=r \cos \theta$ and $y=r \sin \theta$. Inspired by this, we consider the morphism $h: C_{a, b} \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{2}$ given by $h((s, t), r)=(r s, r t)$. Since $h$ sends the subvariety $C_{a, b} \times\{0\}$ to $\{(0,0)\}$ and commutes with the functions $0 \boxplus r^{2}$ on the source and $a x^{2}+b y^{2}$ on the target, it induces a morphism of exponential motives

$$
h: H^{2}\left(\mathbb{A}^{2},\{(0,0)\}, a x^{2}+b y^{2}\right) \longrightarrow H^{2}\left(C_{a, b} \times \mathbb{A}^{1}, C_{a, b} \times\{0\}, 0 \boxplus r^{2}\right)
$$

Noting that the left-hand side is isomorphic to $H^{2}\left(\mathbb{A}^{2}, a x^{2}+b y^{2}\right)$ by the exact sequence (4.2.4.2) associated with the immersions $\emptyset \subseteq\{(0,0)\} \subseteq \mathbb{A}^{2}$ and applying the Künneth formula again, we get

$$
h^{\prime}: H^{2}\left(\mathbb{A}^{2}, a x^{2}+b y^{2}\right) \longrightarrow H^{1}\left(C_{a, b}\right) \otimes H^{1}\left(\mathbb{A}^{1},\{0\}, r^{2}\right)
$$

Now the last factor fits into an exact sequence of motives

$$
0 \rightarrow \mathbb{Q}(0) \rightarrow H^{1}\left(\mathbb{A}^{1},\{0\}, r^{2}\right) \rightarrow H^{1}\left(\mathbb{A}^{1}, r^{2}\right) \rightarrow 0
$$

We will show that the second component of $h^{\prime}$ vanishes in $H^{1}\left(\mathbb{A}^{1}, r^{2}\right)$, and that the induced map $H^{2}\left(\mathbb{A}^{2}, a x^{2}+b y^{2}\right) \rightarrow H^{1}\left(C_{a, b}\right)$ is an isomorphism. For this it suffices to work in a realisation: for
instance, de Rham cohomology. There $h^{\prime}$ sends the generator $d x d y$ to $(t d s-s d t) \otimes r d r$. The first factor is a generator of $H_{\mathrm{dR}}^{1}\left(C_{a, b}\right)$ and the second vanishes in $H_{\mathrm{dR}}^{1}\left(\mathbb{A}^{1}, r^{2}\right)$ since it is equal to $\frac{1}{2} d_{r^{2}}(1)$. However, it is non-zero in $H_{\mathrm{dR}}^{1}\left(\mathbb{A}^{1},\{0\}, r^{2}\right)$, as one can see from the integral $\int_{0}^{+\infty} e^{-r^{2}} r d r=1$.

In particular, $M(\sqrt{\pi})^{\otimes 2}=H^{1}\left(s^{2}+t^{2}=1\right)$.
12.2.2. - If the base field $k$ contains a square root of -1 , the conic $C$ is isomorphic to $\mathbb{G}_{m}$ by the change of coordinates $u=s+i t, v=s-i t$, and therefore $M(\sqrt{\pi})$ is a genuine tensor square root of $\mathbb{Q}(-1)$. We can generalise Lemma 12.2 .1 to the case where in place of $M(\sqrt{\pi})$ we consider a motive of the form $H^{n}\left(\mathbb{A}^{n}, q\right)$ for a quadratic form $q$ in $n$ variables $x_{1}, \ldots, x_{n}$, seen as a regular function on $\mathbb{A}^{n}=\operatorname{Spec} k\left[x_{1}, \ldots, x_{n}\right]$. Given a non-zero element $c \in k$, we define

$$
M(\sqrt{c})= \begin{cases}H^{0}(\operatorname{Spec} k(\sqrt{c})) / \mathbb{Q}(0) & \text { if } c \text { is not a square in } k  \tag{12.2.2.1}\\ \mathbb{Q}(0) & \text { if } c \text { is a square in } k\end{cases}
$$

The motive $M(\sqrt{c})$ is one-dimensional, and only depends on the class of $c$ modulo squares.

Proposition 12.2.3. - Let $q=q\left(x_{1}, \ldots, x_{n}\right)$ be a non-degenerate quadratic form, seen as a regular function on $\mathbb{A}^{n}=\operatorname{Spec} k\left[x_{1}, \ldots, x_{n}\right]$. Then $H^{m}\left(\mathbb{A}^{n}, q\right)=0$ for $m \neq n$ and

$$
H^{n}\left(\mathbb{A}^{n}, q\right) \cong M(\sqrt{\operatorname{det} q}) \otimes M(\sqrt{\pi})^{\otimes n}
$$

Proof. It is a standard fact that there exists a linear automorphism of $\mathbb{A}^{n}$ transforming any given quadratic form into a diagonal one. Thus, we may assume that $q$ is of the form

$$
q\left(x_{1}, \ldots, x_{n}\right)=a_{1} x_{1}^{2}+a_{2} x_{2}^{2}+\cdots+a_{n} x_{n}^{2}
$$

for some non-zero elements $a_{1}, \ldots, a_{n} \in k$. The discriminant of $q$ is the product $a_{1} a_{2} \cdots a_{n}$, which does not depend on the diagonalization modulo $\left(k^{\times}\right)^{2}$. The Künneth formula yields

$$
H^{n}\left(\mathbb{A}^{n}, q\right) \cong H^{1}\left(\mathbb{A}^{1}, a_{1} x^{2}\right) \otimes H^{1}\left(\mathbb{A}^{1}, a_{2} x^{2}\right) \otimes \cdots \otimes H^{1}\left(\mathbb{A}^{1}, a_{n} x^{2}\right)
$$

and $H^{m}\left(\mathbb{A}^{n}, q\right)=0$ for $m \neq n$. The result then follows from $H^{1}\left(\mathbb{A}^{1}, a_{i} x^{2}\right) \cong M(\sqrt{a}) \otimes M(\sqrt{\pi})$.
12.2.4 (The $\ell$-adic realisation). - Let $\chi_{2}: \mathbb{F}_{q}^{\times} \rightarrow\{ \pm 1\}$ be the non-trivial quadratic character on $\mathbb{F}_{q}^{\times}$. Given an additive character $\psi$, one defines the Gauss sum

$$
\begin{equation*}
G\left(\chi_{2}, \psi\right)=\sum_{x \in \mathbb{F}_{q}^{\times}} \chi_{2}(x) \psi(x) \tag{12.2.4.1}
\end{equation*}
$$

Lemma 12.2.5. - The exponential motive $H^{1}\left(\mathbb{A}^{1}, x^{2}\right)$ has good ramification outside $p=2$ and its $\ell$-adic realisation is the one-dimensional $\overline{\mathbb{Q}}_{\ell}$-vector space with Frobenius action given by multiplication by $G\left(\chi_{2}, \psi\right)$.

Proof. The $\ell$-adic perverse realisation of $H^{1}\left(\mathbb{A}^{1}, x^{2}\right)$ is $j_{!} \mathcal{L}_{\chi_{2}}[1]$.

### 12.3. Exponential periods on the affine line

Set $\mathbb{A}^{1}=$ Spec $k[x]$ and let $f \in k[x]$ be a polynomial of degree at least two. In this section, we study the motive $H^{1}\left(\mathbb{A}^{1}, f\right)$ and its motivic Galois group. In particular, we want to understand the determinant of $H^{1}\left(\mathbb{A}^{1}, f\right)$.
12.3.1. - In tannakian terms, exterior powers are constructed as follows. For any object $M$ of a tannakian category and any integer $n \geqslant 1$, the symmetric group $\mathfrak{S}_{n}$ acts on the $n$-fold tensor power $M^{\otimes n}$ by permutation of factors. The $n$-fold exterior power of $M$ is the eigenspace in $M^{\otimes n}$ of the signature character $\varepsilon: \mathfrak{S}_{n} \rightarrow\{ \pm 1\}$. Given a non-constant polynomial $f$, the $n$-fold tensor power of the exponential motive $H^{1}\left(\mathbb{A}^{1}, f\right)$ can be identified with $H^{n}\left(\mathbb{A}^{n}, f f^{\boxplus n}\right)$ via the Künneth isomorphism

$$
\begin{equation*}
\kappa: H^{1}\left(\mathbb{A}^{1}, f\right)^{\otimes n} \xrightarrow{\cong} H^{n}\left(\mathbb{A}^{n}, f^{\boxplus n}\right) \tag{12.3.1.1}
\end{equation*}
$$

because $H^{q}\left(\mathbb{A}^{1}, f\right)=0$ for $q \neq 1$. The symmetric group $\mathfrak{S}_{n}$ acts on $\mathbb{A}^{n}$ by permutation of coordinates, and this action commutes with the Thom-Sebastiani sum $f^{\boxplus n}=f \boxplus \cdots \boxplus f$, hence an action of $\mathfrak{S}_{n}$ on the motive $H^{n}\left(\mathbb{A}^{n}, f^{\boxplus n}\right)$. The Künneth isomorphism is not compatible with the actions of $\mathfrak{S}_{n}$, but we rather have

$$
\kappa \circ \sigma=\varepsilon(\sigma) \cdot(\sigma \circ \kappa)
$$

for $\sigma \in \mathfrak{S}_{n}$. In particular, $\kappa$ sends the $\varepsilon$-eigenspace in $H^{1}\left(\mathbb{A}^{1}, f\right)^{\otimes n}$ to the space of invariants, and we can thus identify the $n$-fold exterior power of $H^{1}\left(\mathbb{A}^{1}, f\right)$ with

$$
\begin{equation*}
\bigwedge^{n} H^{1}\left(\mathbb{A}^{1}, f\right)=H^{n}\left(\mathbb{A}^{n}, f^{\boxplus n}\right)^{\mathfrak{G}_{n}} \tag{12.3.1.2}
\end{equation*}
$$

where on the right-hand side we really mean invariants. If we look at the action of $\mathfrak{S}_{n}$ as a $\mathbb{Q}\left[\mathfrak{S}_{n}\right]$-module structure, the space of invariants is the image of the projector

$$
\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_{n}} \sigma
$$

seen as an idempotent endomorphism of the motive $H^{n}\left(\mathbb{A}^{n}, f^{\boxplus n}\right)$.

Theorem 12.3.2. - Let $n \geqslant 1$ be an integer and $f \in k[x]$ a polynomial of degree $n+1$ with leading term a. Define numbers $b, c \in k$ by

$$
b=\sum_{f^{\prime}(\alpha)=0} f(\alpha), \quad c= \begin{cases}(-1)^{\frac{n(n-1)}{2}} \frac{2 a}{n+1} & \text { if } n \text { is odd } \\ (-1)^{\frac{n(n-1)}{2}} & \text { if } n \text { is even },\end{cases}
$$

where the sum runs over all $\alpha \in \mathbb{C}$ with $f^{\prime}(\alpha)=0$ counted with multiplicity. Let $M(\sqrt{c})$ be the one-dimensional Artin motive with period $\sqrt{c}$, as in (12.2.2.1), $M(\sqrt{\pi})=H^{1}\left(\mathbb{A}^{1}, x^{2}\right)$ and $E(b)=H^{0}(\operatorname{Spec} k, b)$. There is an isomorphism of exponential motives over $k$

$$
\operatorname{det} H^{1}\left(\mathbb{A}^{1}, f\right) \simeq M(\sqrt{c}) \otimes M(\sqrt{\pi})^{\otimes n} \otimes E(b) .
$$

12.3.3. - The following proof of Theorem 12.3 .2 is in large parts copied from ${ }^{2}[11, \S 5]$. We write the polynomial $f \in k[x]$ as

$$
f(x)=a_{n+1} x^{n+1}+a_{n} x^{n}+\cdots+a_{1} x+a_{0}
$$

with $a=a_{n+1} \neq 0$. Since, for any $u \in k^{*}$, the motives $M(\sqrt{u}) \otimes M(\sqrt{\pi})$ and $H^{1}\left(\mathbb{A}^{1}, u x^{2}\right)$ are isomorphic, the theorem claims that there is an isomorphism of exponential motives over $k$

$$
\operatorname{det} H^{1}\left(\mathbb{A}^{1}, f\right) \simeq H^{n}\left(\mathbb{A}^{n}, q\right)
$$

where $q\left(x_{1}, \ldots, x_{n}\right)=b+c x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+\cdots+x_{n}^{2}$.
12.3.4. - The symmetric group $\mathfrak{S}_{n}$ acts on $\mathbb{A}_{x}^{n}=\operatorname{Spec}\left(k\left[x_{1}, \ldots, x_{n}\right]\right)$, and leaves the function $f^{\boxplus n}(\underline{x})=f\left(x_{1}\right)+f\left(x_{2}\right)+\cdots+f\left(x_{n}\right)$ invariant. We start with writing down the quotient variety and the induced function on it. For $1 \leqslant i \leqslant n$, let us write $S_{i}(\underline{x})$ for the $i$-th symmetric polynomial in the variables $x_{1} \ldots x_{n}$, so $S_{1}(\underline{x})=x_{1}+\cdots+x_{n}, S_{2}(\underline{x})=x_{1} x_{2}+x_{1} x_{3}+\cdots$ and so on $u p$ to $S_{n}(\underline{x})=x_{1} x_{2} \cdots x_{n}$. Let $s_{1}, \ldots, s_{n}$ denote another set of indeterminates. The morphism of affine varieties

$$
\mathbb{A}_{x}^{n}=\operatorname{Spec}\left(k\left[x_{1}, \ldots, x_{n}\right]\right) \xrightarrow{\pi} \mathbb{A}_{s}^{n}=\operatorname{Spec}\left(k\left[s_{1}, \ldots, s_{n}\right]\right)
$$

given by the algebra morphism $s_{i} \longmapsto S_{i}(\underline{x})$ identifies $\mathbb{A}_{s}^{n}$ as the quotient $\mathbb{A}_{x}^{n} / \mathfrak{S}_{n}$. Since $f^{\boxplus n}$ is a symmetric polynomial we have $f^{\boxplus n}=F \circ \pi$ for some unique $F \in k[\underline{s}]$. The morphism $\pi$ induces a morphism of motives

$$
H^{n}\left(\mathbb{A}_{s}^{n}, F\right) \xrightarrow{\pi^{*}} H^{n}\left(\mathbb{A}_{x}^{n}, f^{\boxplus n}\right)^{\mathfrak{S}_{n}}=\operatorname{det} H^{n}\left(\mathbb{A}^{1}, f\right)
$$

which will eventually turn ou to be an isomorphism. The key part of the proof is now to produce an automorphism of $\mathbb{A}_{s}^{n}$, that is, a change of variables, which turns $F$ into a quadratic form.
12.3.5. - For each integer $i \geqslant 0$, consider the Newton polynomial $P_{i}(\underline{x})=x_{1}^{i}+x_{2}^{i}+\ldots+x_{n}^{i}$. Each of the $P_{i}$ can be written in a unique way as a polynomial in the elementary symmetric polynomials $S_{i}$. Let us define $Q_{i} \in k[\underline{s}]$ by

$$
Q_{i}\left(S_{1}(\underline{x}), S_{2}(\underline{x}), \ldots, S_{n}(\underline{x})\right)=P_{i}(\underline{x}),
$$

so that we have

$$
Q_{0}(\underline{s})=n, \quad Q_{1}(\underline{s})=s_{1}, \quad Q_{2}(\underline{s})=s_{1}^{2}-2 s_{2}, \quad Q_{3}(\underline{s})=s_{1}^{3}-3 s_{1} s_{2}+3 s_{3}
$$

and, in general,

$$
Q_{i}(\underline{s})=\sum_{\substack{r_{1}+2 r_{2}+\ldots+i r_{i}=i \\ r_{1}, \ldots, r_{i} \geqslant 0}}(-1)^{i} \frac{i\left(r_{1}+\cdots+r_{i}-1\right)!}{r_{1}!\cdots r_{i}!} \prod_{j=1}^{i}\left(-s_{j}\right)^{r_{j}} .
$$

The polynomial $Q_{i}$ has degree $i$ and only contains the variables $s_{1}, \ldots, s_{i}$. We do not add a variable $s_{n+1}$ to $Q_{n+1}$. If we declare that $s_{i}$ has weighted degree $i$, then $Q_{i}$ is homogeneous of weighted degree $i$. For $i \geqslant 1$ the polynomial $Q_{i}$ has no constant part, and the linear part of $Q_{i}$ is

[^7]$s_{i}$ for $1 \leqslant i \leqslant n$ and zero otherwise. For $k+l=i$, the monomial $s_{k} s_{l}$ appears in $Q_{i}$ with coefficient $(-1)^{i} i$ if $k \neq l$, and with coefficient $\frac{1}{2} i$ if $k=l$.
12.3.6. - Let us express the numbers $b$ and $c$ in the statement of Theorem 12.3.2 in terms of the coefficients of $f$ and $F$. The polynomial $f^{\boxplus n} \in k[\underline{x}]$ is the polynomial $f^{\boxplus n}(\underline{x})=a_{0} P_{0}(\underline{x})+$ $a_{1} P_{1}(\underline{x})+\cdots+a_{n} P_{n}(\underline{x})+a_{n+1} P_{n+1}(\underline{x})$, hence
$$
F(\underline{s})=a_{0} Q_{0}(\underline{s})+a_{1} Q_{1}(\underline{s})+\cdots+a_{n} Q_{n}(\underline{s})+a_{n+1} Q_{n+1}(\underline{s})
$$
by definition. Setting
$$
\underline{a}=\frac{1}{(n+1) a_{n+1}}\left(-n a_{n},(n-1) a_{n-1}, \ldots,(-1)^{n} a_{1}\right)
$$
we have $b=F(\underline{a})$ by straightforward computation. The constant term of $F$ is $c=n a_{0}$, its linear homogeneous part is $a_{1} s_{1}+2 a_{2} s_{2}+\cdots+n a_{n} s_{n}$. In the homogeneous quadratic part of $F$, we find the terms $s_{k} s_{l}$ appear with coefficient $(-1)^{i} i a_{i}$ for $k+l=i$ and $k \neq l$, and with coefficient $\frac{1}{2} i a_{i}$ for $k=l$. If we think of the homogeneous quadratic part of $g$ as the quadratic form associated with a symmetric bilinear form, then the matrix of this form is
\[

B=\left(\nabla^{2} F\right)(0)=\frac{1}{2}\left($$
\begin{array}{ccccc}
2 a_{2} & -3 a_{3} & \cdots & (-1)^{n} n a_{n} & (-1)^{n+1}(n+1) a_{n+1} \\
-3 a_{3} & 4 a_{4} & \cdots & (-1)^{n+1}(n+1) a_{n+1} & 0 \\
\vdots & \vdots & & & \vdots \\
(-1)^{n} n a_{n} & (-1)^{n+1}(n+1) a_{n+1} & 0 & 0 \\
(-1)^{n+1}(n+1) a_{n+1} & 0 & \cdots & 0 & 0
\end{array}
$$\right)
\]

and we notice that its determinant is equal to

$$
\operatorname{det} B=(-1)^{\frac{n(n-1)}{2}}\left(\frac{n+1}{2} a_{n+1}\right)^{n}
$$

The sign $(-1)^{\frac{n(n-1)}{2}}$ comes from taking the product of the antidiagonal entries, while the signs we pick up from the matrix entries themselves cancel to $(-1)^{(n-1) n}=1$. In particular, viewed modulo squares, $\operatorname{det}(B)$ takes the value

$$
\operatorname{det}(B)= \begin{cases}(-1)^{\frac{n(n-1)}{2}} \in k^{*} / k^{* 2} & \text { if } n \text { is even } \\ (-1)^{\frac{n(n-1)}{2}} \frac{2 a_{n+1}}{n+1} \in k^{*} / k^{* 2} & \text { if } n \text { is odd }\end{cases}
$$

or $\operatorname{det}(B)=c$ for short, with the notation of Theorem 12.3.2.

Lemma 12.3.7. - The differential form $d F$ on $\mathbb{A}_{s}^{n}$ (or equivalently, the gradient $\nabla F$ of $F$ ) vanishes at the point $\underline{a}$ and nowhere else.

## Proof.

Proof of Theorem 12.3.2. We start with an affine change of variables, setting $G(\underline{s})=F(\underline{s}+$ $\underline{a})-b$. The polynomial $G(\underline{s})$ satisfies $G(0)=0$ and its gradient $\nabla G$ only vanishes at $0 \in \mathbb{A}_{s}^{n}$. Thus, $G$ contains no constant and no linear terms, and we may write $G$ uniquely as

$$
G(\underline{s})=Q(\underline{s})+R(\underline{s})+H(\underline{s})
$$

where $Q$ and $R$ are homogeneous quadratic polynomials, $Q$ containing the monomials of weight $n+1$ and $R$ containing monomials of weight $\leqslant n$ and each monomial in $H$ has degree $\geqslant 3$. This
makes sense, since indeed all monomials in $F$ are of weight $\leqslant n+1$, hence all monomials in $G, Q$, $R$ and $H$ are so too. If a monomial of highest possible weight $n+1$ appears in $F$, then the same monomial appears in $G$, with the same coefficient. In particular, the matrix form of $\left(\nabla^{2} G\right)(0)$ is upper left triangular, with the same (non-zero!) antidiagonal coefficients as $B$. In other words, we have

$$
Q(\underline{s})=\lambda \sum_{i=1}^{n} s_{i} s_{n+1-i}
$$

with $\lambda=(-1)^{n+1}(n+1) a_{n+1}$. We will show that there exists, and in fact construct, an automorphism $\Phi: k[\underline{s}] \rightarrow k[\underline{s}]$ such that

$$
\begin{equation*}
\Phi(G(\underline{s}))=Q(\underline{s}) \tag{12.3.7.1}
\end{equation*}
$$

holds. To do so, we prove by induction on $j \geqslant 1$ the following:
Claim. There exists an automorphism $\Phi: k[\underline{s}] \rightarrow k[\underline{s}]$ such that $\Phi(G) \in k[\underline{s}]$ has the form

$$
\Phi(G(\underline{s}))=Q(\underline{s})+H^{\prime}(\underline{s})
$$

where $H^{\prime} \in k[\underline{s}]$ is a polynomial in the variables $s_{j}, s_{j+1}, \ldots$ where all monomials are of degree $\geqslant 3$ and of weight $\leqslant n+1$.

For $j=1$, a linear unipotent automorphism does the job of $\Phi$. Indeed, setting $\Phi\left(s_{i}\right)=s_{i}+L_{i}(\underline{s})$ where $L_{i}$ is a suitable linear polynomial in the variables $s_{1}, \ldots, s_{i-1}$ yields $\Phi(Q(\underline{s})+R(\underline{s}))=Q(\underline{s})$, hence

$$
\Phi(G(\underline{s}))=Q(\underline{s})+H^{\prime}(\underline{s})
$$

where all monomials in $H^{\prime}$ are of degree $\geqslant 3$ and of weight $\leqslant n+1$. Now fix $j \geqslant 1$, and suppose that we have found an automorphism $\Phi$ of $k[\underline{s}]$ satisfying the conditions in the claim. The monomial of lowest weight which can possibly occur in $H^{\prime}$ is $s_{j}^{3}$. Hence if $j>\frac{n+1}{3}$, then $H^{\prime}=0$ and we are done. Let us suppose thus that $j<\frac{n+1}{2}$. The variable $s_{n+1-j}$ does not appear in $H^{\prime}(\underline{s})$ again for weight reasons. Indeed, if $s_{n+1-j}$ appears in a monomial of $H^{\prime}$, then this monomial must have degree $\geqslant 3$, hence would have weight at least $(n+1-j)+2 j>n+1$. Let us write

$$
H^{\prime}\left(s_{j}, \ldots, s_{n-j}\right)=H^{\prime}\left(0, s_{j+1}, \ldots, s_{n-j}\right)+s_{j} \psi\left(s_{j}, \ldots, s_{n-j}\right)
$$

and define an automorphism $\Psi$ of $k[\underline{s}]$ by $\Psi\left(s_{n+1-j}\right)=s_{n+1-j}-(2 \lambda)^{-1} \psi\left(s_{j}, \ldots, s_{n-j}\right)$ and $\Psi\left(s_{i}\right)=$ $s_{i}$ for $i \neq n+1-j$. We notice that monomials in $\psi$ have degree at least 2 . We find

$$
\begin{aligned}
\Psi(\Phi(G(\underline{s}))) & =\Psi(\Phi(Q(\underline{s})))+\Psi(\Phi(R(\underline{s})))+\Phi(H(\underline{s})) \\
& =\Phi(Q(\underline{s}))+-s_{j} \psi(\underline{s})+H^{\prime}\left(0, s_{j+1}, \ldots, s_{n-j}\right)+s_{j}(\underline{s}) \\
& =Q(\underline{s})+H^{\prime \prime}(\underline{s})
\end{aligned}
$$

where $H^{\prime \prime}(\underline{s})=H^{\prime}\left(0, s_{j+1}, \ldots, s_{n}\right)$ has the property that all of its terms are of degree $\geqslant 3$ and weight $\leqslant n+1$. The composite $\Psi \circ \Phi$ satisfies thus the property of the claim for $j+1$.

Let us now fix an automorphism $\Phi$ of $k[\underline{s}]$ satisfying (12.3.7.1) and interpret it as an automorphism of $\mathbb{A}_{s}^{n}=$ Spec $k[\underline{s}]$. The diagram

commutes, hence induces the sought after isomorphism of motives

$$
H^{n}\left(\mathbb{A}_{s}^{n}, F\right) \rightarrow H^{n}\left(\mathbb{A}_{s}^{n}, Q+b\right)
$$

Corollary 12.3.8. - Let $G \subseteq \mathrm{GL}_{n}$ be the motivic Galois group of $H^{1}\left(\mathbb{A}^{1}, f\right)$. The determinant induces a surjective group morphism det: $G \rightarrow \mathbb{G}_{m}$.

Proof. The determinant $\operatorname{det} H^{1}\left(\mathbb{A}^{1}, f\right)$ is a rank one object of the category $\mathbf{M}^{\exp }(k)$, hence its motivic Galois group is either $\mathbb{G}_{m}$ or a group of roots of unity. To exclude the second case, we observe that the isomorphism in Theorem 12.3.2 implies that no tensor power of $\operatorname{det} H^{1}\left(\mathbb{A}^{1}, f\right)$ becomes the unit object, for example because this motive has weight $n \geqslant 1$.

Corollary 12.3.9. - We keep the notation from Theorem 12.3.2. Up to multiplication by a non-zero element of $k$, the determinant of a period matrix of the motive $H^{1}\left(\mathbb{A}^{1}, f\right)$ is equal to

$$
\begin{equation*}
\sqrt{c} \cdot \pi^{\frac{n}{2}} \cdot e^{b} \tag{12.3.9.1}
\end{equation*}
$$

Example 12.3.10. - Let $d \geqslant 2$ be an integer and $f=x^{d}$. According to Example 1.1.4 from the introduction, the period matrix of the exponential motive $H^{1}\left(\mathbb{A}^{1}, x^{d}\right)$ with respect to suitable bases of the de Rham and Betti realisations reads

$$
P=\left(\frac{\xi^{a b}-1}{d} \Gamma\left(\frac{a}{d}\right)\right)_{1 \leqslant a, b \leqslant d-1} .
$$

Therefore, viewed as an element of $\mathbb{C}^{\times} / \mathbb{Q}^{\times}$, the determinant is equal to

$$
\operatorname{det} P=\frac{\operatorname{det}\left(\xi^{a b}-1\right)}{d^{d-1}} \prod_{a=1}^{d-1} \Gamma\left(\frac{a}{d}\right)
$$

Lemma 12.3.11. - The equality $\operatorname{det}\left(\xi^{a b}-1\right)_{1 \leqslant a, b \leqslant d-1}=i \frac{(3 d-2)(d-1)}{2} d^{\frac{d}{2}}$ holds.

Proof. Let $\Delta$ denote the determinant on the left-hand side. Subtracting the first column from each other column of the Vandermonde matrix $\left(\xi^{a b}\right)_{0 \leqslant a, b \leqslant d-1}$ yields the expression

$$
\Delta=\operatorname{det}\left(\xi^{a b}\right)_{0 \leqslant a, b \leqslant d-1}=\prod_{0 \leqslant a<b \leqslant d-1}\left(\xi^{b}-\xi^{a}\right) .
$$

Noting that, for fixed $b$, the product $\prod_{a \neq b}\left(\xi^{b}-\xi^{a}\right)$ is the derivative of the polynomial $x^{d}-1$ evaluated at $x=\xi^{b}$, one computes the absolute value

$$
|\Delta|=\prod_{0 \leqslant a \neq b \leqslant d-1}\left|\xi^{b}-\xi^{a}\right|^{\frac{1}{2}}=d^{\frac{d}{2}}
$$

We are thus left to determine the argument of $\Delta$. In terms of the notation $\mathbf{e}(x)=\exp (2 \pi i x)$, dear to analytic number theorists, one has

$$
\xi^{b}-\xi^{a}=\mathbf{e}\left(\frac{a+b}{2 d}\right)\left(\mathbf{e}\left(\frac{b-a}{2 d}\right)-\mathbf{e}\left(\frac{a-b}{2 d}\right)\right)=2 \mathbf{e}\left(\frac{a+b}{2 d}\right) i \sin \left(\frac{\pi(b-a)}{d}\right)
$$

and the sine is positive when $a$ and $b$ satisfy $0 \leqslant a<b \leqslant d-1$. Then a straightforward computation allows one to conclude:

$$
\frac{\Delta}{|\Delta|}=\prod_{0 \leqslant a<b \leqslant d-1} \mathbf{e}\left(\frac{a+b}{2 d}\right) i=\mathbf{e}\left(\frac{(d-1)^{2}}{4}\right) i \frac{d(d-1)}{2}=i^{\frac{(3 d-2)(d-1)}{2}}
$$

REMARK 12.3.12. -
Putting everything together, we get the expression $\operatorname{det} P=d^{1-\frac{d}{2}} i^{\frac{(3 d-2)(d-1)}{2}} \prod_{a=1}^{d-1} \Gamma\left(\frac{a}{d}\right)$. Besides, Corollary 12.3 .9 specialises to the equality $\operatorname{det} P=\sqrt{c} \cdot \pi^{\frac{d-1}{2}}$ in $\mathbb{C}^{\times} / \mathbb{Q}^{\times}$. Combined with the previous calculation, this implies

$$
\begin{equation*}
\prod_{j=0}^{d-1} \Gamma\left(\frac{j}{d}\right) \sim_{\mathbb{Q}^{\times}} \frac{(2 \pi)^{\frac{d-1}{2}}}{\sqrt{d}} \tag{12.3.12.1}
\end{equation*}
$$

thus showing that the multiplication formula for the gamma values has motivic origin. Both sides of (12.3.12.1) are actually equal.
12.3.13 (Computation of the epsilon factor). - Let $F=\mathbb{F}_{q}$ be a finite field with $q$ elements. Given a smooth variety $X$ over $F$ and a $\overline{\mathbb{Q}}_{\ell}$-sheaf $\mathscr{F}$ on $X$, the epsilon factor is defined as

$$
\varepsilon(X, \mathscr{F})=\prod_{j \geqslant 0} \operatorname{det}\left(-\varphi_{F} \mid H_{c}^{j}\left(X_{\bar{F}}, \mathscr{F}\right)\right)^{(-1)^{j}} \in \overline{\mathbb{Q}}_{\ell}^{\times},
$$

where $\varphi_{F}$ stands for the geometric Frobenius.

Theorem 12.3.14. - Let $p$ be a prime number and $f \in \mathbb{F}_{p}[x]$ a polynomial of degree $n+1$.

$$
\varepsilon\left(\mathbb{A}_{\mathbb{F}_{p}}^{1}, f^{*} \mathcal{L}_{\psi}\right)= \begin{cases}\psi(b) q^{\frac{n}{2}} & n \text { is even } \\ -\psi(b) G\left(\chi_{2}, \psi\right) \chi_{2}(c) q^{\frac{n-1}{2}} & n \text { is odd }\end{cases}
$$

### 12.4. Bessel motives and moments of Bessel functions

We have already encountered in the introduction, Example 1.1.5, a two-dimensional exponential motive whose periods are special values of the modified Bessel functions. Namely, one considers the variety $\mathbb{G}_{m}=\operatorname{Spec} \overline{\mathbb{Q}}\left[x, x^{-1}\right]$ and the function $f_{\lambda}=-\frac{\lambda}{2}\left(x-\frac{1}{x}\right)$, where $\lambda$ is a non-zero algebraic number, say a non-zero element of a number field $k \subseteq \mathbb{C}$. The Bessel motive associated with $\lambda$ is

$$
B(\lambda)=H^{1}\left(\mathbb{G}_{m}, f_{\lambda}\right)(1)
$$

seen as an object of $\mathbf{M}^{\exp }(k)$. It is a two-dimensional motive. The rapid decay homology $H_{1}\left(X, f_{\lambda}\right)$ has a basis consisting of a simple loop around 0 and a path joining the two connected components of $f_{\lambda}^{-1}\left(S_{r}\right)$ for large $r>0$. Having chosen such a basis, we can identify the motivic Galois group of $B(\lambda)$ with a closed subgroup of $\mathrm{GL}_{2}$. We will in this section compute various realisations and the motivic fundamental group of $B(\lambda)$. After that, we examine what happens if $\lambda$ is thought of not as a fixed parameter, but as an additional variable. We will consider the function

$$
f(t, x)=-\frac{t^{m}}{2}\left(x_{1}+x_{2}+\cdots+x_{n}-\frac{1}{x_{1}}-\frac{1}{x_{2}}-\cdots+\frac{1}{x_{n}}\right)
$$

on $\mathbb{A}^{1} \times \mathbb{G}_{m}^{n}$ for some integers $m \geqslant 1$ and $n \geqslant 1$, and calculate periods of the exponential motive $H^{n+1}\left(\mathbb{A}^{1} \times \mathbb{G}_{m}^{n}, f\right)$. By design, these periods are moments of Bessel functions. Typical examples of such moments are

$$
\int_{0}^{\infty} t K_{0}^{4}(t) d t=\frac{7}{8} \zeta(3) \quad \int_{0}^{\infty} t^{3} K_{0}^{3}(t) d t=L_{-3}(2)-\frac{2}{3}
$$

where $\zeta$ is the Riemann Zeta function, and $L_{-3}$ is the Dirichlet $L$-function associated with the Legendre symbol $n \longmapsto(-3 \mid n)$. Using the theorem of the fixed part 6.5.1, we will show that moments of Bessel functions are polynomial expressions in classical periods and special values of the gamma function.

Proposition 12.4.1. $-\operatorname{det} B(\lambda)=\mathbb{Q}(1)$
Proof. The determinant of $B(\lambda)$ is the one-dimensional motive

$$
\operatorname{det} B(\lambda)=H^{2}(X \times X, f \boxplus f)^{\mathfrak{S}_{2}}(2)
$$

Consider the morphism $X \times X \rightarrow \mathbb{A}^{2}$ given by the algebra morphism $\varphi: k[s, t] \rightarrow k\left[x, x^{-1}, y, y^{-1}\right]$ sending $s$ to $x+y$ and $t$ to $(x y)^{-1}$. Setting $g(s, t)=s+\frac{\lambda^{2}}{4} s t$ we have $\varphi(g(s, t))=f(x)+f(y)$, hence a morphism of motives

$$
\begin{equation*}
H^{2}\left(\mathbb{A}^{2}, g\right) \rightarrow H^{2}\left(\mathbb{A}^{2}, f \boxplus f\right)^{\mathfrak{S}_{2}} \tag{12.4.1.1}
\end{equation*}
$$

induced by $\varphi$. Since $\frac{\lambda^{2}}{4}$ is a square in $k^{*}$, the motive $H^{2}\left(\mathbb{A}^{2}, g\right)$ is that of the quadratic form $(s, t) \longmapsto s t$ which has determinant -1 , hence $H^{2}\left(\mathbb{A}^{2}, g\right)=\mathbb{Q}(-1)$. It remains to check that the morphism (12.4.1.1) is non-zero.

Proposition 12.4.2. - Let $F[1]=\mathrm{R}_{\text {perv }} B(\lambda)$ be the perverse realisation of $B(\lambda)$. The singularities of the constructible sheaf $F$ are at the points $\{i \lambda,-i \lambda\}$. With respect to an appropriate basis of $F_{0}$, the local monodromy operators of the local system on $\mathbb{C} \backslash\{ \pm i \lambda\}$ defined by $F$ are

$$
\rho_{+}=\left(\begin{array}{cc}
1 & 0 \\
-2 & -1
\end{array}\right) \quad \text { and } \quad \rho_{-}=\left(\begin{array}{cc}
1 & 0 \\
2 & -1
\end{array}\right)
$$

and the fibres of $F$ at the singular points are the local invariants.
Proof. Let $p: \mathbb{G}_{m} \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}$ be the projection and write $\mathbb{Q}$ for the constant sheaf with value $\mathbb{Q}$ on $\mathbb{G}_{m} \times \mathbb{A}^{1}$. Let $j$ be the inclusion into $\mathbb{G}_{m} \times \mathbb{A}^{1}$ of the complement of the closed subvariety

$$
\Gamma=\left\{(x, z) \mid f_{\lambda}(x)=z\right\}
$$

of $\mathbb{G}_{m} \times \mathbb{A}^{1}$. The sheaf $F$ is $R^{1} p_{*}\left(j!j^{*} \underline{\mathbb{Q}}\right)$. Rewriting the equation $f_{\lambda}(x)=z$ as

$$
x^{2}+\frac{2 z}{\lambda} x-1=0
$$

shows that the singularities of $F$ are located at those points $z \in \mathbb{C}$ where the discriminant of the quadratic polynomial $x^{2}+\frac{2 z}{\lambda} x-1$ vanishes, and this discriminant equals $4\left(z^{2} \lambda^{-2}+1\right)$. In order to compute the monodromy of $F$ around the singularities $\pm i \lambda$, consider the basis of

$$
V_{0}=F_{0}^{\vee}=H_{1}\left(\mathbb{C}^{\times},\{ \pm 1\}\right)
$$

given by a standard loop $\varphi$ around 0 , and the sum $\gamma=\gamma_{+}+\gamma_{-}$, where $\gamma_{+}$is an arc from -1 to 1 in the upper half-plane, and $\gamma_{-}$is an arc from -1 to 1 in the lower half-plane. As $z$ runs over a loop $\rho_{+}$around $i \lambda$, say

$$
\rho_{+}: t \longmapsto i \lambda+\lambda e^{2 \pi i(t-1 / 4)}
$$

the roots of the polynomial

$$
x^{2}+\frac{2 z}{\lambda} x-1=x^{2}+2\left(i+e^{2 \pi i(t-1 / 4)}\right) x-1
$$

exchange positions, moving in the lower half-plane. The monodromy action $\rho_{+}$is accordingly given by $\rho_{+}(\varphi)=\varphi, \rho_{+}\left(\gamma_{+}\right)=-\gamma_{-}-\varphi$ and $\rho_{+}\left(\gamma_{-}\right)=-\gamma_{-}$. With respect to the basis $\varphi, \gamma$, the monodromy operator for the loop $\rho_{+}$acts on $V_{0}$ as the matrix

$$
\left(\begin{array}{ll}
1 & -2 \\
0 & -1
\end{array}\right)
$$

so the matrix of $\rho_{+}$on the dual space $F_{0}=V_{0}^{\vee}$ is given by the transposed matrix. The computation of the matrix of $\rho_{-}$with respect to the same basis is similar. Finally, since $F$ is an object of $\mathbf{P e r v}_{0}$, the dimensions of the fibres $\operatorname{dim} F_{i \lambda}$ and $\operatorname{dim} F_{-i \lambda}$ must add up to $2=\operatorname{dim} F_{0}$, hence must consist of all the local invariants. In terms of the basis dual to $\varphi, \gamma$, the invariants are the one-dimensional subspaces generated by the vectors $\binom{1}{-1}$ for $\rho_{+}$and by $\binom{1}{1}$ for $\rho_{-}$.

Proposition 12.4.3. - The motivic Galois group of the Bessel motive $B(\lambda)$ is $\mathrm{GL}_{2}$.
Proof. Let $G \subseteq \mathrm{GL}_{2}$ denote the motivic Galois group of $B(\lambda)$ and let $H \subseteq G$ be the tannakian fundamental group of the perverse realisation $F[1]=\mathrm{R}_{\text {perv }} B(\lambda)$ of $B(\lambda)$. We first notice that $H$ and $G$ are both reductive. Indeed, the perverse sheaf $F[1]$ is a simple object in the category $\mathbf{P e r v}_{0}$ since already the local system defined by $F$ is simple. In follows that $B(\lambda)$ itself is simple too, and in any tannakian category of characteristic zero the fundamental group of any simple or semisimple object is reductive. By Proposition 12.4.1, the group $G$ surjects to $\mathbb{G}_{m}$ via the determinant map, and since the perverse realisation of $\mathbb{Q}(1)$ is trivial, the group $H$ is contained in $\mathrm{SL}_{2}$. Again, since $F[1]$ is simple, the tautological two-dimensional representation of $H$ as a subgroup of $\mathrm{SL}_{2}$ is irreducible, but the only algebraic subgroup of $\mathrm{SL}_{2}$ with this property is $\mathrm{SL}_{2}$ itself. It follows that $H=\mathrm{SL}_{2}$ and $G=\mathrm{GL}_{2}$ as claimed.

As a consequence of Proposition 12.4.3, the period conjecture specialises to the following statement:

Conjecture 12.4.4. - For every non-zero algebraic number $\lambda \in \mathbb{C}$, the following complex numbers are algebraically independent:

$$
I_{0}(\lambda), \quad I_{1}(\lambda), \frac{1}{2 \pi i} K_{0}(\lambda), \frac{1}{2 \pi i} K_{1}(\lambda)
$$

12.4.5. - Let $m \geqslant 1$ and $n \geqslant 0$ be integers. We consider the affine variety $X=\mathbb{A}^{1} \times \mathbb{G}_{m}^{n}$ and the regular function $f: X \rightarrow \mathbb{A}^{1}$ given by

$$
f(t, x)=-\frac{1}{2} t^{m} \sum_{q=1}^{n}\left(x_{q}-\frac{1}{x_{q}}\right)
$$

for coordinate functions $t$ and $x=\left(x_{1}, \ldots, x_{n}\right)$ on $X$. The triple $(X, \varnothing, f)$ is cellular in degree $n+1$, and we are interested in the motive

$$
B(m, n)=H^{n+1}\left(\mathbb{A}^{1} \times \mathbb{G}_{m}^{n}, f\right)
$$

which has dimension (?).

Theorem 12.4.6. - The motive $B(m, n)$ belongs to the tannakian category generated by classical motives and $B(m, 0)=H^{1}\left(\mathbb{A}^{1}, t^{m}\right)$. In particular, $B(1, n)$ is a classical motive.

Proof. The only singularity of the perverse realisation of $B(m, n)$ is $0 \in \mathbb{C}$, and the monodromy around 0 of the underlying local system is of order $m$.

### 12.5. Special values of $E$-functions

Definition 12.5.1 (Siegel). - Let $f$ be an entire function given by a power series

$$
f(z)=\sum_{n=0}^{\infty} \frac{a_{n}}{n!} z^{n}
$$

with algebraic coefficients $a_{n}$. For each $n \geqslant 1$, let $\left\|a_{n}\right\|$ denote the largest absolute value of all Galois conjugates of $a_{n}$, and let $d_{n} \in \mathbb{Z} \geqslant 1$ be the smallest integer such that $d_{n} a_{1}, d_{n} a_{2}, \ldots, d_{n} a_{n}$ are all algebraic integers. The function $f$ is called an $E$-function if

- it satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$,
- there exists a constant $C>0$ such that $\left\|a_{n}\right\| \leqslant C^{n}$ and $d_{n} \leqslant C^{n}$ for all $n \geqslant 1$.
12.5.2. - Instead of $d_{n} \leqslant C^{n}$, Siegel [83] asks for the seemingly less stringent condition that, for every $\varepsilon>0$, there is a constant $C_{\varepsilon}>0$ such that $d_{n} \leqslant C_{\varepsilon}(n!)^{\varepsilon}$ holds for all $n$. However, no
examples of functions satisfying the latter condition but not the former one are known ${ }^{3}$. An elegant alternative way to formulate the growth condition on the coefficients is to ask for

$$
h\left(\left[a_{0}: a_{1}: a_{2}: \cdots: a_{n}\right]\right)=O(n)
$$

where $h$ stands for logarithmic height on $\mathbb{P}^{n}$. Standard examples of $E$-functions include polynomials, the exponential function, and the Bessel function $J_{0}\left(z^{2}\right)$. The exponential integral functions $E_{n}$ (see 12.6.1 below) are not $E$-functions, already because they have a singularity at 0 .

Theorem 12.5.3 (Siegel-Shidlovskii). - Let $f=\left(f_{1}, \ldots, f_{n}\right)$ be E-functions which satisfy a linear differential equation $f^{\prime}=A f$ for some $n$ by $n$ matrix $A$ with coefficients in $\overline{\mathbb{Q}}(z)$. The equality

$$
\operatorname{trdeg}_{\mathbb{Q}}\left(f_{1}(\alpha), \ldots, f_{n}(\alpha)\right)=\operatorname{trdeg}_{\mathbb{C}(z)}\left(f_{1}(z), \ldots, f_{n}(z)\right)
$$

holds for any non-zero $\alpha \in \overline{\mathbb{Q}}$ which is not a pole of any of the coefficients of $A$.

### 12.6. Special values of exponential integral functions

In this section, we introduce exponential motives whose periods contain special values of the exponential integral functions $E_{1}, E_{2}, \ldots$ The theorem of Siegel-Shidlovskii about special values of $E$-functions shows that a small part of the period conjecture holds for these motives.
12.6.1. - Recall that, for each integer $n$, the exponential integral function $E_{n}$ is defined, in the half-plane $\operatorname{Re}(s)>0$, by the convergent integral

$$
E_{n}(s)=\int_{1}^{\infty} e^{-s x} \frac{d x}{x^{n}}
$$

In particular, $E_{0}(s)=\frac{e^{-s}}{s}$. As a function of $s$, this integral defines a holomorphic function on the right half complex plane, which extends to a holomorphic function on $\mathbb{C} \backslash[-\infty, 0]$. The function $E_{n}$ is closely related to the incomplete gamma function

$$
\Gamma(s, x)=\int_{x}^{\infty} t^{s-1} e^{-t} d t
$$

namely by $E_{n}(s)=s^{n-1} \Gamma(n-1, s)$. Integration by parts shows the recurrence relation

$$
n E_{n+1}(s)=e^{-s}-s E_{n}(s)
$$

which allows us to calculate $E_{n}$ for $n \leqslant 0$ from $E_{0}$, and $E_{n}$ for $n \geqslant 1$ from $E_{1}$. In particular we see that for $n \leqslant 0$ the function $E_{n}(s)$ is a rational function of $e^{s}$ and $s$, whereas for $n \geqslant 1$ the function $E_{n}(s)$ is a rational function of $s, e^{s}$ and $E_{1}(s)$. One can show that the field extension of $\mathbb{C}(s)$ generated by $\left\{E_{n}(s) \mid n \in \mathbb{Z}\right\}$ has transcendence degree 2 . In other words, the functions $e^{s}$ and $E_{1}(s)$ are algebraically independent over $\mathbb{C}(s)$.

[^8]12.6.2. - Special values of exponential integral functions $E_{n}$ are not very much studied, with the notable exception of the so-called Gompertz constant
$$
G=e \cdot E_{1}(1)=0.596347362323194074341 \ldots
$$
which is in several ways related to the Euler-Mascheroni constant, as we will see in section 12.8. It admits two intriguing continued fraction representations
attributed to Stieltjes. The resulting rational approximations are however not good enough in order to deduce that $G$ is irrational. Irrationality of $G$ is, as far as we know, still an open problem.
12.6.3. - Let $k \subseteq \mathbb{C}$ be a number field, and pick $\alpha \in k^{\times}$. The integral representation of $E_{1}(\alpha)$ suggests that the exponential motive
$$
M=H^{1}\left(\mathbb{G}_{m},\{1\}, \alpha x\right)
$$
contains $E_{1}(\alpha)$ among its periods. By the exact sequence for triples (4.2.4.2), the motive $M$ fits into a short exact sequence
$$
0 \longrightarrow E(-\alpha) \longrightarrow M \longrightarrow H^{1}\left(\mathbb{G}_{m}, \alpha x\right) \longrightarrow 0
$$

We claim that $H^{1}\left(\mathbb{G}_{m}, \alpha x\right)$ is isomorphic to $\mathbb{Q}(-1)=H^{1}\left(\mathbb{G}_{m}\right)$ as exponential motive. To see this, consider $X=\mathbb{G}_{m} \times \mathbb{A}^{1}=\operatorname{Spec} k\left[x, x^{-1}, t\right]$, together with the function $f(x, t)=x t$. The inclusions of $\mathbb{G}_{m}$ into $X$ as $\mathbb{G}_{m} \times\{0\}$, respectively as $\mathbb{G}_{m} \times\{\alpha\}$, yield morphisms of motives $H^{1}\left(\mathbb{A}^{1}, f\right) \rightarrow H^{1}\left(\mathbb{G}_{m}\right)$, respectively $H^{1}\left(\mathbb{A}^{1}, f\right) \rightarrow H^{1}\left(\mathbb{G}_{m}, \alpha x\right)$, which are isomorphisms. Therefore, we get an extension

$$
\begin{equation*}
0 \longrightarrow E(-\alpha) \longrightarrow M \longrightarrow \mathbb{Q}(-1) \longrightarrow 0 \tag{12.6.3.1}
\end{equation*}
$$

A basis for rapid decay homology is given by the cycles $\gamma_{\circ}$ and $\gamma_{-}$, defined as

$$
\gamma_{\circ}(t)=e^{2 \pi i t} \quad \text { and } \quad \gamma_{-}(t)=1+r \alpha^{-1} t
$$

According to the elementary definition of rapid decay homology, $\gamma-$ should be seen as a family of cycles indexed by $r \gg 1$. A basis for the de Rham cohomology consists of the 1 -form $\omega=x^{-1} d x$ on $\mathbb{G}_{m}$, and the 0-form $\delta_{0}$ supported on the marked point $\{1\} \subseteq \mathbb{G}_{m}$. With respect to these bases, the period matrix of $M$ reads:

|  | $\delta_{0}$ | $\omega$ |
| :---: | :---: | :---: |
| $\gamma_{-}$ | $e^{-\alpha}$ | $E_{1}(\alpha)$ |
| $\gamma_{0}$ | 0 | $2 \pi i$ |

Proposition 12.6.4. - The extension of motives (12.6.3.1) is not split. With respect to the basis dual to the basis $\left\{\gamma_{0}, \gamma_{-}\right\}$of $H_{1}\left(\mathbb{G}_{m},\{1\}, \alpha x\right)$, the motivic fundamental group of $M$ is the three-dimensional group

$$
G_{M}=\left\{\left.\left(\begin{array}{ll}
a & b \\
0 & d
\end{array}\right) \right\rvert\, a, d \in \mathbb{G}_{m}, b \in \mathbb{G}_{a}\right\} .
$$

Proof. The semisimplification of $M$ is the motive $E(-\alpha) \oplus \mathbb{Q}(-1)$, whose motivic fundamental group is the two-dimensional diagonal torus in $\mathrm{GL}_{2}$, since $\alpha \neq 0$. It only remains to show that the extension (12.6.3.1) is not split. To do so, it suffices to show that the perverse realisation of this sequence, which reads

$$
0 \longrightarrow E(-\alpha) \longrightarrow R_{\text {perv }}(M) \longrightarrow E(0) \longrightarrow 0
$$

is not split. The object $A=R_{\text {perv }}(M)$ of $\operatorname{Perv}_{0}$ has two singularities $\{0, \alpha\}$. The fibre of $A[-1]$ over a point $z \notin\{1, \alpha\}$ is the linear dual of the vector space $H_{1}\left(\mathbb{C}^{\times},\left\{1, \alpha^{-1} z\right\}\right)$. A basis of this space is given by a positively oriented simple loop $\gamma_{0}$ around 0 , and a path $\gamma_{z}$ from 1 to $\alpha^{-1} z$. As $z$ moves in a loop based at $r \gg 0$ around the singular point $\alpha$, the path $\gamma_{r}$ gets deformed into itself. Hence the local monodromy around $\alpha$ is trivial. As $z$ moves in a loop around the singular point 0 on the other hand, $\gamma_{r}$ gets deformed into a path whose homology class is $\gamma_{0}+\gamma_{r}$. It follows that with respect to the dual basis $\left\{\gamma_{-}^{\vee}, \gamma_{0}^{\vee}\right\}$ of $H_{\mathrm{rd}}^{1}\left(\mathbb{G}_{m},\{1\}, \alpha x\right\}$, the global monodromy of the local system underlying $A$ is given by

$$
\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right) \text { around } 0 \quad \text { and } \quad\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \text { around } \alpha .
$$

Global invariants are the 1 -dimensional subspace generated by $\gamma_{-}^{\vee}=\binom{1}{0}$, which is the subspace $E(-\alpha)$ of $A$. From the monodromy around 0 we see that the extension is not split, and more precisely, that the tannakian fundamental group of $A$ is equal to

$$
G_{A}=\left\{\left.\left(\begin{array}{ll}
a & b \\
0 & 1
\end{array}\right) \right\rvert\, a \in \mathbb{G}_{m}, b \in \mathbb{G}_{a}\right\}
$$

with torus of singularities given by the cocharacter $a \longmapsto\left(\begin{array}{cc}a & 0 \\ 0 & 1\end{array}\right)$.

Conjecture 12.6.5. - For every non-zero algebraic number $\alpha$, the complex numbers $2 i \pi, e^{-\alpha}$, and $E_{1}(\alpha)$ are algebraically independent.

Lemma 12.6.6. - The power series $f(z)=\sum_{n=1}^{\infty} \frac{1}{n \cdot n!} z^{n}$ is a transcendental E-function.
Proof. It is clear that $f(z)$ is an entire function, whose coefficients $a_{n}=1 / n$ are bounded. We only have to check that, for some constant $C$, the inequality

$$
d_{n}=\operatorname{lcm}(1,2,3,4,5, \ldots, n) \leqslant C^{n}
$$

holds. This least common multiple is conveniently expressed using the summatory von-Mangoldt function or secondary Chebyshev function $\psi(n)$ : We have

$$
d_{n}=\exp (\psi(n))
$$

for all $n \geqslant 1$. The function $\psi$ grows asymptotically as $\psi(x) \sim x$ - this is equivalent to the prime number theorem. In particular $\psi(x)<c x$ for sufficiently large $c>1$, hence $d_{n}<e^{c n}=C^{n}$. In order to show that $f$ is a $E$-function, it remains to find a linear differential equation for $f$. Indeed, we have $1+z f^{\prime}(z)=e^{z}$ by inspection of the power series, hence

$$
\left(1+z f^{\prime}(z)\right)^{\prime}=1+z f^{\prime}(z)
$$

is the differential equation $\left(e^{z}\right)^{\prime}=e^{z}$. Rearranging terms yields the differential equation

$$
z f^{\prime \prime}(z)+(1-z) f^{\prime}(z)=1
$$

which is only affine and not linear, but we can always derive once more.
The general solution of $z u^{\prime \prime}+(1-z) u^{\prime}=1$ is $A+B f(z)-\log (z)$ for constants $A$ and $B$. Unfortunately, $\log (z)$ is not an $E$-function. The Siegel-Shidlovski theorem thus only shows that special values of the function $f$, such as

$$
f(-1)=\int_{0}^{1} \int_{0}^{1} e^{-x y} d x d y
$$

are transcendental. Once we show that $f(z)$ and $e^{z}$ are algebraically independent over $\mathbb{C}(z)$, we will obtain algebraic independence over $\overline{\mathbb{Q}}$ of, say, $f(-1)$ and $e$.

### 12.7. Laurent polynomials and special values of $E$-functions

We fix a number field $k \subseteq \mathbb{C}$. Regular functions on $\mathbb{G}_{m}=\mathbb{A}^{1} \backslash\{0\}$ are Laurent polynomials with coefficients in $k$, so we obtain a motive $M=H^{1}\left(\mathbb{G}_{m}, f\right)$ from every Laurent polynomial $f$. In this section, we show how to relate some of the periods of $M$ to special values of $E$-functions. The Siegel-Shidlovskii theorem allows us then to prove some transcendence results.
12.7.1. - Let $f \in k\left[x, x^{-1}\right]$ be a Laurent polynomial of the form

$$
f(x)=\frac{1}{d}\left(c_{-r} x^{-r}+\cdots+c_{s} x^{s}\right)
$$

where $d>0$ is an integer and the $c_{i} \in \mathcal{O}_{k}$ are algebraic integers. We assume that $r$ and $s$ are both positive and the coefficients $c_{-r}$ and $c_{s}$ non-zero. The motive $M=H^{1}\left(\mathbb{G}_{m}, f\right)$ has dimension $r+s$. A particular element in the rapid decay homology of $\left(\mathbb{G}_{m}, f\right)$ is the standard loop $\gamma$ winding once counterclockwise around 0 . Given another Laurent polynomial $g \in k\left[x, x^{-1}\right]$, we set

$$
E(g, z)=\frac{1}{2 \pi i} \oint e^{-z f(x)} g(x) d x
$$

where $z$ is a complex variable and the integral sign means integration along the loop $\gamma$. The function $E(g, z)$ is entire and satisfies the following three relations:

$$
\begin{align*}
a E(g, z)+b E(h, z) & =E(a g+b h, z),  \tag{12.7.1.1}\\
\frac{\partial}{\partial z} E(g, z) & =-E(f g, z),  \tag{12.7.1.2}\\
E\left(g^{\prime}, z\right) & =z E\left(f^{\prime} g, z\right) . \tag{12.7.1.3}
\end{align*}
$$

In the first one, $a$ and $b$ are scalars in $k$ and $h$ is another Laurent polynomial. The second one is obtained by differentiating under the integral sign, which is allowed since the cycle $\gamma$ is compact. Finally, the third one follows from Stokes' formula and could be rewritten as $E\left(d_{z f}(g), z\right)=0$.

Proposition 12.7.2. - The function $E(g, z)$ is an $E$-function.

Proof. We have to verify that $E(g, z)$ satisfies a non-zero linear differential equation, and that the coefficients $a_{n}$ of the Taylor expansion

$$
E(g, z)=\sum_{n=0}^{\infty} \frac{a_{n}}{n!} z^{n}
$$

lie in a common number field and their logarithmic height has at most linear growth. By (12.7.1.1) and the fact that linear combinations of $E$-functions are again $E$-functions, we may assume that $g$ is a monomial, say $g(x)=x^{d}$ for some integer $d \in \mathbb{Z}$.

Let us start with bounding the coefficients. By (12.7.1.2), they are equal to

$$
a_{n}=(-1)^{n} E\left(f^{n} g, 0\right)=\frac{(-1)^{n}}{2 \pi i} \oint f(x)^{n} g(x) d x
$$

which is, by Cauchy's formula, the coefficient of $x^{-1}$ in the Laurent polynomial $f(x)^{n} g(x)$. Since we already assume $g(x)=x^{d}$, the coefficient $a_{n}$ is the coefficient of $x^{1-d}$ in $f(x)^{n}$. It is thus clear that $a_{n}$ belongs to $k$. Moreover, we can write $a_{n}$ in terms of the coefficients of $f$ as

$$
a_{n}=\frac{1}{d^{n}} \sum c_{i_{1}} c_{i_{2}} \cdots c_{i_{n}}
$$

where the sum runs over all $n$-tuples of integers $\left(i_{1}, \ldots, i_{n}\right) \in[-r, s]^{n}$ satisfying $i_{1}+\cdots+i_{n}=1-d$. Define $C=\max \left\{\left\|c_{-r}\right\|, \ldots,\left\|c_{n}\right\|\right\}$. The estimate

$$
\left\|a_{n}\right\| \leqslant(r+s)^{n} \max \left\{\left\|c_{i_{1}} c_{i_{2}} \cdots c_{i_{n}}\right\| \mid-r \leqslant i_{1}, \ldots, i_{n} \leqslant s\right\} \leqslant(r+s)^{n} \cdot C^{n}
$$

is what was needed in Definition 12.5.1. It remains to show that $E(g, z)$ satisfies a non-trivial differential equation. This is a straightforward consequence of the relations (12.7.1.1), (12.7.1.2) and (12.7.1.3). Indeed, the functional equation tells us that the $\mathbb{C}(z)$-linear space of entire functions spanned by $\left\{E(g, z) \mid g \in k\left[x, x^{-1}\right]\right\}$ is finite-dimensional, of dimension at most $r+s$. Therefore, for any fixed $g$, the functions $\frac{\partial^{i}}{\partial z^{i}}\{E(g, z) \mid 0 \leqslant i \leqslant r+s\}$ are $\mathbb{C}(z)$-linearly dependent.
12.7.3. - In the proof of proposition 12.7.2, we have explained why the function $E(g, z)$ satisfies a differential equation of order $\leqslant r+s$. Let us now describe an explicit construction of this differential equation, in the form of a system of first order linear differential equations. Set $E_{p}(z)=E\left(x^{p}, z\right)$. Our goal is to produce an equation

$$
\frac{\partial}{\partial z} E=L E
$$

where $E$ is the vector of functions $\left(E_{0}, \ldots, E_{r+s-1}\right)$ and $L$ is a matrix with coefficients in $k(z)$. The functional equation (12.7.1.3) applied to $g(x)=x^{p+1}$ reads

$$
\begin{equation*}
(p+1) E_{p}(z)=z E\left(x^{p} f^{\prime}(x), z\right)=\frac{z}{d} \sum_{q=-r}^{s} q c_{q} E_{p+q}(z) \tag{12.7.3.1}
\end{equation*}
$$

For each $p \in \mathbb{Z}$, we can determine uniquely $a_{p q} \in k\left[z, z^{-1}\right]$ such that

$$
\begin{equation*}
E_{p}(z)=\sum_{q=0}^{r+s-1} a_{p q}(z) E_{q}(z) \tag{12.7.3.2}
\end{equation*}
$$

holds. This is indeed possible, trivially so for $0 \leqslant p<r+s$ and inductively on $p$ for $p<0$ and $p \geqslant r+s$. For our needs, we need to determine the coefficients $a_{p q}$ for $-r \leqslant p<0$ and $r+s \leqslant p<r+2 s$. The differential relation (12.7.1.2) in the case $g(x)=x^{p}$ reads

$$
\begin{equation*}
E_{p}^{\prime}(z)=-E\left(x^{p} f(x), z\right)=-\sum_{q=-r}^{s} c_{q} E_{p+q}(z) \tag{12.7.3.3}
\end{equation*}
$$

For each $0 \leqslant p<r+s$, we can substitute the relations (12.7.3.2) into the right-hand side of (12.7.3.3), and obtain so the sought system of differential equations. The coefficients of the matrix $L$ are linear combinations of the $a_{p q} \in k\left[z, z^{-1}\right]$, hence are themselves elements of $k\left[z, z^{-1}\right]$.

Example 12.7.4. - Consider the Laurent polynomial $f(x)=x^{-3}+x^{-1}+x+x^{3}$. Since $f$ is odd, we expect that the resulting motive $M=H^{1}\left(\mathbb{G}_{m}, f\right)$ has some extra symmetries. The dimension of $M$ is $3+3-1=5$. The diferential forms $d x, x d x, \ldots, x^{5} d x$ represent a basis of the de Rham cohomology $H_{\mathrm{dR}}^{1}\left(\mathbb{G}_{m}, z f\right)$. Here are the equations (12.7.3.1) for $p=0,1,2,3,4,5$.

$$
\begin{aligned}
E_{0}(z) & =-3 z E_{-3}(z)-z E_{-1}(z)+z E_{1}(z)+3 z E_{3}(z) \\
2 E_{1}(z) & =-3 z E_{-2}(z)-z E_{0}(z)+z E_{2}(z)+3 z E_{4}(z) \\
3 E_{2}(z) & =-3 z E_{-1}(z)-z E_{1}(z)+z E_{3}(z)+3 z E_{5}(z) \\
4 E_{3}(z) & =-3 z E_{0}(z)-z E_{2}(z)+z E_{4}(z)+3 z E_{6}(z) \\
5 E_{4}(z) & =-3 z E_{1}(z)-z E_{3}(z)+z E_{5}(z)+3 z E_{7}(z) \\
6 E_{5}(z) & =-3 z E_{2}(z)-z E_{4}(z)+z E_{6}(z)+3 z E_{8}(z)
\end{aligned}
$$

The linear relations (12.7.3.2) for $p=-1,-2,-3$ and $p=6,7,8$ are obtained from these. Here they are.

$$
\begin{aligned}
E_{-1}(z) & =-\frac{1}{3} E_{1}(z)-\frac{1}{z} E_{2}(z)+\frac{1}{3} E_{3}(z)+E_{5}(z) \\
E_{-2}(z) & =-\frac{1}{3} E_{0}(z)-\frac{2}{3 z} E_{1}(z)+\frac{1}{3} E_{2}(z)+E_{4}(z) \\
E_{-3}(z) & =-\frac{1}{3} E_{-1}(z)-\frac{1}{3 z} E_{0}(z)+\frac{1}{3} E_{1}(z)+E_{3}(z) \\
& =-\frac{1}{3 z} E_{0}(z)+\frac{4}{9} E_{1}(z)+\frac{1}{3 z} E_{2}(z)+\frac{8}{9} E_{3}(z)-\frac{1}{3} E_{5}(z) \\
E_{6}(z) & =E_{0}(z)+\frac{1}{3} E_{2}(z)+\frac{4}{3 z} E_{3}(z)-\frac{1}{3} E_{4}(z) \\
E_{7}(z) & =E_{1}(z)+\frac{1}{3} E_{3}(z)+\frac{5}{3 z} E_{4}(z)-\frac{1}{3} E_{5}(z) \\
E_{8}(z) & =E_{2}(z)+\frac{1}{3} E_{4}(z)+\frac{2}{z} E_{5}(z)-\frac{1}{3} E_{6}(z) \\
& =-\frac{1}{3} E_{0}(z)+\frac{8}{9} E_{2}(z)-\frac{4}{9 z} E_{3}(z)+\frac{4}{9} E_{4}(z)+\frac{2}{z} E_{5}(z)
\end{aligned}
$$

Next, let us write the differential relations (12.7.3.3) for $0 \leqslant p<r+s$.

$$
\begin{aligned}
E_{0}^{\prime}(z) & =-E_{-3}(z)-E_{-1}(z)-E_{1}(z)-E_{3}(z) \\
E_{1}^{\prime}(z) & =-E_{-2}(z)-E_{0}(z)-E_{2}(z)-E_{4}(z) \\
E_{2}^{\prime}(z) & =-E_{-1}(z)-E_{1}(z)-E_{3}(z)-E_{5}(z) \\
E_{3}^{\prime}(z) & =-E_{0}(z)-E_{2}(z)-E_{4}(z)-E_{6}(z) \\
E_{4}^{\prime}(z) & =-E_{1}(z)-E_{3}(z)-E_{5}(z)-E_{7}(z) \\
E_{5}^{\prime}(z) & =-E_{2}(z)-E_{4}(z)-E_{6}(z)-E_{8}(z)
\end{aligned}
$$

Substituting $E_{-3}, E_{-2}, E_{-1}$ and $E_{6}, E_{7}, E_{8}$, we obtain:

$$
\begin{aligned}
& E_{0}^{\prime}(z)=\frac{1}{3 z} E_{0}(z)-\frac{10}{9} E_{1}(z)+\frac{2}{3 z} E_{2}(z)-\frac{20}{9} E_{3}(z)-\frac{2}{3} E_{5}(z) \\
& E_{1}^{\prime}(z)=-\frac{2}{3} E_{0}(z)+\frac{2}{3 z} E_{1}(z)-\frac{4}{3} E_{2}(z)-2 E_{4}(z) \\
& E_{2}^{\prime}(z)=-\frac{2}{3} E_{1}(z)+\frac{1}{z} E_{2}(z)-\frac{4}{3} E_{3}(z)-2 E_{5}(z) \\
& E_{3}^{\prime}(z)=-2 E_{0}(z)-\frac{4}{3} E_{2}(z)-\frac{4}{3 z} E_{3}(z)-\frac{2}{3} E_{4}(z) \\
& E_{4}^{\prime}(z)=-2 E_{1}(z)-\frac{4}{3} E_{3}(z)-\frac{5}{3 z} E_{4}(z)-\frac{2}{3} E_{5}(z) \\
& E_{5}^{\prime}(z)=-\frac{2}{3} E_{0}(z)-\frac{20}{9} E_{2}(z)-\frac{8}{9 z} E_{3}(z)-\frac{10}{9} E_{4}(z)-\frac{2}{z} E_{5}(z)
\end{aligned}
$$

From this system we can read off the matrix $L$.

$$
L=\left(\begin{array}{cccccc}
\frac{1}{3 z} & -\frac{10}{9} & \frac{2}{3 z} & -\frac{20}{9} & 0 & -\frac{2}{3} \\
-\frac{2}{3} & \frac{2}{3 z} & -\frac{4}{3} & 0 & -2 & 0 \\
0 & -\frac{2}{3} & \frac{1}{z} & -\frac{4}{3} & 0 & -2 \\
-2 & 0 & -\frac{4}{3} & -\frac{4}{3 z} & -\frac{2}{3} & 0 \\
0 & -2 & 0 & -\frac{4}{3} & -\frac{5}{3 z} & -\frac{2}{3} \\
-\frac{2}{3} & 0 & -\frac{20}{9} & -\frac{8}{9 z} & -\frac{10}{9} & -\frac{2}{z}
\end{array}\right)
$$

### 12.8. The Euler-Mascheroni constant

In this final section, we describe a two-dimensional exponential motive over the field of rational numbers that is a non-classical extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$ containing the Euler-Mascheroni constant $\gamma$ among its periods. We refer the reader to [60] for a beautiful survey of the role this constant played in Euler's work and subsequent developments. That $\gamma$ is an exponential period was observed by Belkale and Brosnan in [9] using the integral representation

$$
\gamma=-\int_{0}^{\infty} \log (x) e^{-x} d x=-\int_{0}^{\infty} e^{-x} \int_{1}^{x} \frac{d y}{y} d x=-\int_{0}^{\infty} \int_{0}^{1} e^{-x} \frac{x-1}{(x-1) y+1} d y d x
$$

which follows from the fact that $-\gamma$ is the value at 1 of the derivative of the gamma function. Although the integrand has a pole at the point $(0,1)$ which lies on the boundary of the integration domain, the integral converges absolutely. However, this makes it difficult to write down an exponential motive out of this integral representation.

To get rid of the pole of the integrand, we resort to blowing up the affine plane at the origin. In terms of the integral, this just means that we change variables from $(x, y)$ to $(x y, y)$, thus obtaining:

$$
\begin{equation*}
\gamma=-\int_{0}^{\infty} \int_{1}^{x} \frac{1}{y} e^{-x y} d y d(x y)=\int_{0}^{1} \int_{0}^{1} e^{-x y} d x d y-\int_{1}^{\infty} \int_{1}^{\infty} e^{-x y} d x d y \tag{12.8.0.1}
\end{equation*}
$$

The first of these two integrals is a special value of the $E$-function from Lemma 12.6.6, and it is a transcendental number by the Siegel-Shidlovskii theorem. The second integral is the special value $E_{1}(1)$ of the exponential integral function that we already studied in Section 12.6.
12.8.1 (The Euler-Mascheroni motive). - The integral representation (12.8.0.1) suggests the following geometric picture: let $X=\operatorname{Spec} \mathbb{Q}[x, y]$ be the affine plane, $Y$ the union of four lines given by the equation $x y(x-1)(y-1)=0$, and $f$ the regular function $f(x, y)=x y$ on $X$. The exponential motive $H^{2}(X, Y, f)$ has $\gamma$ among its periods by design. As we will see below, it turns out to be three-dimensional. In the course of the computation of its period matrix, we observed that $H^{2}(X, Y, f)$ should admit a two-dimensional quotient $M(\gamma)$, still containing $\gamma$ among its periods, and which sits in an non-split extension

$$
0 \longrightarrow \mathbb{Q}(0) \longrightarrow M(\gamma) \longrightarrow \mathbb{Q}(-1) \longrightarrow 0
$$

To define $M(\gamma)$, we consider the blow-up $\pi: \widetilde{X} \rightarrow X$ of the affine plane at the point $(1,1)$. Let $\widetilde{Y}$ denote the strict transform of $Y$, let $E$ be the exceptional divisor, and $\tilde{f}=f \circ \pi$ the induced function on $\tilde{X}$. The motive $H^{2}(\tilde{X}, \tilde{Y}, \tilde{f})$ is again three-dimensional but the blow-up map $\pi$ yields a rank-two morphism of exponential motives

$$
\pi^{*}: H^{2}(X, Y, f) \longrightarrow H^{2}(\tilde{X}, \tilde{Y}, \tilde{f})
$$

Definition 12.8.2. - The Euler-Mascheroni motive $M(\gamma) \subseteq H^{2}(\tilde{X}, \tilde{Y}, \tilde{f})$ is the image of $\pi^{*}$.

In what follows, we will first compute all realisations of the whole motive $H^{2}(X, Y, f)$, then identify the realisations of the quotient $M(\gamma)$.
12.8.3 (Computation of rapid decay homology). - The topological picture is the following: The topological space $X(\mathbb{C})=\mathbb{C}^{2}$ has the homotopy type of a point. The subspace $Y(\mathbb{C})$ consists of four copies of the complex plane glued to a square, and hence has the homotopy type of a circle. The set $f^{-1}\left(S_{r}\right)=\left\{(x, y) \in \mathbb{C}^{2} \mid \operatorname{Re}(x y) \geqslant r\right\}$ is homeomorphic to $\mathbb{C}^{*} \times \mathbb{R} \times \mathbb{R}_{\geqslant 0}$, and is for $r>1$ glued to $Y(\mathbb{C})$ in the adjacent lines $y=1$ and $x=1$. Here is the real picture.

The space $Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)$ has the homotopy type of a wedge of three circles, which bound three 2-cells forming a basis of $H_{2}\left(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)\right)$. In the picture on the left-hand side, two of these 2 -cells are visible - the square shaped cell $\gamma_{\square}$ and the triangle shaped cell $\gamma_{\triangle}$. In the blow-up pictured on the right-hand side, these two cells merge to a single cell, which maps via the blow-up map $\pi$ to $\gamma_{\square}-\gamma_{\triangle}$. The boundary of the third cell is the circle in $\left\{(x, y) \in \mathbb{C}^{2} \mid x y=r\right\}$ given by the simple loop $t \longmapsto\left(e^{2 \pi i t}, r e^{-2 \pi i t}\right)$. There is another subtlety which is invisible in the picture: the three sides of the triangle shaped cell are paths from $(1,1)$ to $(1, r)$ in the $x=1$ plane, from $(1,1)$ to $(r, 1)$ in the $y=1$ plane, and from $(r, 1)$ to $(1, r)$ in the $x y=r$ plane. For the first


Figure 12.8.1
two paths, any choice is homotopic to any other, but not so for the third since in the $x y=r$ plane a point is missing. Two choices for the boundary of $\gamma_{\Delta}$ differ by a class in $H_{1}\left(f^{-1}\left(S_{r}\right), \mathbb{Q}\right) \simeq \mathbb{Q}$. One evident choice for $\gamma_{\Delta}$ is the cell which is contained in $\mathbb{R}^{2} \subseteq \mathbb{C}^{2}$. The boundary morphism

$$
\partial: H_{2}\left(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)\right) \longrightarrow H_{1}\left(Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)\right)
$$

is an isomorphism. Therefore, $H_{2}\left(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}\left(S_{r}\right)\right)$ has dimension 3, a basis being given by the two cells $\gamma_{\square}$ and $\gamma_{\Delta}$ in the picture, and a disk $\gamma_{\circ}$ filling the loop $t \longmapsto\left(e^{2 \pi i t}, r e^{-2 \pi i t}\right)$. It follows in particular that the motive $H^{2}(X, Y, f)$ is of dimension 3, as claimed. The quotient $H^{2}(X, Y, f) \rightarrow M(\gamma)$ corresponds to the two-dimensional subspace

$$
\left\langle\gamma_{\square}-\gamma_{\Delta}, \gamma_{0}\right\rangle .
$$

Hence, $M(\gamma)$ is two-dimensional.
12.8.4 (Computation of de Rham cohomology). - Let $Y^{(1)}$ be the normalisation of $Y$, i.e. the disjoint union of the four irreducible components of $Y$, and let $Y^{(2)} \subseteq Y$ be the four singular points of $Y$. Write $\iota: Y^{(1)} \rightarrow X$ for the composite of the normalisation map with the inclusion, $\alpha: Y^{(2)} \rightarrow Y^{(1)}$ for the inclusion of $Y^{(2)}$ into the two vertical lines, and $\beta: Y^{(2)} \rightarrow Y^{(1)}$ for the inclusion of $Y^{(2)}$ into the two horizontal lines. The de Rham complex associated with $(X, Y, f)$ is the total complex of the double complex

which we describe now explicitly. The de Rham complex $\left(\Omega^{*}(X), d_{f}\right)$ is the complex

$$
\mathbb{Q}[x, y] \longrightarrow \mathbb{Q}[x, y] d x \oplus \mathbb{Q}[x, y] d y \longrightarrow \mathbb{Q}[x, y] d x d y
$$

with differentials given by

$$
\begin{aligned}
d_{f}(g) & =\left(y g+\frac{\partial g}{\partial x}\right) d x+\left(x g+\frac{\partial g}{\partial y}\right) d y \\
d_{f}(g d x+h d y) & =\left(-x g+y h+\frac{\partial g}{\partial y}-\frac{\partial h}{\partial x}\right) d x d y
\end{aligned}
$$

and its homology is concentrated in degree 2 of dimension 1 , represented by the form $d x d y$.
The variety $Y^{(1)}$ is the union of four affine lines, say the spectrum of $\mathbb{Q}\left[x_{0}\right] \oplus \mathbb{Q}\left[x_{1}\right] \oplus \mathbb{Q}\left[y_{0}\right] \oplus \mathbb{Q}\left[y_{1}\right]$ where we name coordinates in such a way that $x_{0}$ is the coordinate on the line $y=0, x_{1}$ the coordinate on the line $y=1, y_{0}$ the coordinate on the line $x=0$, and $y_{1}$ the coordinate on the line $x=1$. A regular function $g=g(x, y)$ on the plane restricts thus to

$$
\iota^{*}(g)=\left(g\left(x_{0}, 0\right), g\left(x_{1}, 1\right), g\left(0, y_{0}\right), g\left(1, y_{1}\right)\right)
$$

and a 1-form $g d x+h d y$ on the plane restricts to

$$
\iota^{*}(g d x+h d y)=\left(g\left(x_{0}, 0\right) d x_{0}, g\left(x_{1}, 1\right) d x_{1}, h\left(0, y_{0}\right) d y_{0}, h\left(1, y_{1}\right) d y_{1}\right)
$$

In particular, the function $f(x, y)=x y$ restricts to $\left(0, x_{1}, 0, y_{1}\right)$. The differential $d_{f}: \Omega^{0}\left(Y^{(1)}\right) \rightarrow$ $\Omega^{1}\left(Y^{(1)}\right)$ is given by

$$
\left.d_{f}\left(g_{0}, g_{1}, h_{0}, h_{1}\right)=\left(g_{0}^{\prime} d x_{0},\left(g_{1}^{\prime}+g_{1}\right) d x_{1}, h_{0}^{\prime} d y_{0},\left(h_{1}^{\prime}+h_{1}\right) d y_{1}\right)\right)
$$

The homology of $\left(\Omega^{*}\left(Y^{(1)}\right), d_{f}\right)$ is concentrated in degree 0 of dimension 2 , generated by the constant functions

$$
\begin{equation*}
(1,0,0,0) \quad \text { and } \quad(0,0,1,0) \tag{12.8.4.1}
\end{equation*}
$$

Elements of $\Omega^{0}\left(Y^{(2)}\right)$ are quadruples of rational numbers which we arrange in a matrix in the evident way. The map $\alpha^{*}-\beta^{*}: \Omega^{0}\left(Y^{(1)}\right) \rightarrow \Omega^{0}\left(Y^{(2)}\right)$ is given by

$$
\left(\alpha^{*}-\beta^{*}\right)\left(g_{0}, g_{1}, h_{0}, h_{1}\right)=\left(\begin{array}{ll}
-g_{0}(1)+h_{1}(0) & -g_{1}(1)+h_{1}(1) \\
-g_{0}(0)+h_{0}(0) & -g_{0}(1)+h_{1}(0)
\end{array}\right)
$$

A particular basis of $\Omega^{0}\left(Y^{(2)}\right)$ is given by the four elements

$$
\left(\begin{array}{cc}
1 & 0 \\
1 & 0
\end{array}\right), \quad\left(\begin{array}{cc}
0 & 0 \\
-1 & -1
\end{array}\right), \quad \delta_{00}=\left(\begin{array}{cc}
0 & 0 \\
1 & 0
\end{array}\right), \quad \delta_{11}=\left(\begin{array}{cc}
0 & 1 \\
0 & 0
\end{array}\right)
$$

where the first two are the images under $\alpha^{*}-\beta^{*}$ of the basis of $\operatorname{ker}\left(d_{f}\right)$ given in (12.8.4.1). From this explicit description, it is straightforward to check that $H_{\mathrm{dR}}^{2}(X, Y, f)$ has dimension 3 , and a basis is represented by the triples $(d x d y, 0,0),\left(0,0, \delta_{00}\right)$, and $\left(0,0, \delta_{11}\right)$ in $\Omega^{2}(X) \oplus \Omega^{1}\left(Y^{(1)}\right) \oplus \Omega^{0}\left(Y^{(2)}\right)$.
12.8.5 (The period matrix). - In order to compute a period matrix, we need represent rapid cycles in a way which is compatible with de Rham complex. Keeping the notation of the previous paragraph, the rapid decay homology of $[X, Y, f]$ can also be computed as the homology of the
total complex associated with the following double complex of singular chains:


The complexes in the top and middle row are the reduced singular chain complexes of the quotient spaces $X(\mathbb{C}) / f^{-1}\left(S_{r}\right)$ and $Y^{(1)}(\mathbb{C}) /(f \circ \iota)^{-1}\left(S_{r}\right)$ for some large real $r>1$. Up to homotopy, this choice of $r$ is unimportant. In this setup, a cycle $\gamma$ in $H_{2}(X, Y, f)$ is represented by a triple

$$
(T, L, P) \in C_{2}(X, f) \oplus C_{1}\left(Y^{(1)}, f\right) \oplus C_{0}\left(Y^{(2)}\right)
$$

where $T$ represents a preimage of $\gamma$ under the map $H_{2}(X, f) \rightarrow H_{2}(X, Y, f)$, and $L$ and $P$ satisfy $d T=\iota_{*} L$ and $d L=\alpha_{*} P-\beta_{*} P$. The three cycles $\gamma_{\square}, \gamma_{\triangle}$ and $\gamma_{\circ}$ correspond to the following triples:

$$
\begin{array}{ll}
\gamma_{\square} & \left(T_{1}+T_{2},-L_{1}+L_{2}-L_{3}+L_{4}, P_{00}-P_{01}-P_{10}+P_{11}\right) \\
\gamma_{\triangle} & \left(T_{3},-L_{5}+L_{6}, P_{11}\right) \\
\gamma_{\circ} & \left(T_{0}, 0,0\right)
\end{array}
$$

The cycles $T_{1}, T_{2}, \ldots$ are drawn in Figure 12.8.2 except for the cycle $T_{0}$ which is just $\gamma_{\circ}$ seen as an element in $C_{2}(X, f)$.




Figure 12.8.2. Cycles on $X, \tilde{Y}$ and $Z$

We can now turn to the computation of the integrals of $d x d y, \delta_{00}$, and $\delta_{11}$, each over the three topological cycles $\gamma_{\square}, \gamma_{\triangle}$, and $\gamma_{0}$. The integrals over $\gamma_{\square}$ are

$$
\begin{gathered}
\int_{\gamma_{\square}} e^{-f} \delta_{00}=\int_{P_{00}-P_{01}-P_{10}+P_{11}} e^{-f} \delta_{00}=1, \\
\int_{\gamma_{\square}} e^{-f} \delta_{11}=\int_{P_{00}-P_{01}-P_{10}+P_{11}} e^{-f} \delta_{11}=e^{-1}, \\
\int_{\gamma_{\square}} e^{-f} d x d y=\int_{T_{1}+T_{2}} e^{-f} d x d y=\int_{0}^{1} \int_{0}^{1} e^{-x y} d x d y=\gamma+E_{1}(1),
\end{gathered}
$$

and the integrals over $\gamma_{\triangle}$ are

$$
\int_{\gamma_{\Delta}} e^{-f} \delta_{00}=\int_{P_{11}} e^{-f} \delta_{00}=0
$$

$$
\begin{gathered}
\int_{\gamma_{\Delta}} e^{-f} \delta_{11}=\int_{P_{11}} e^{-f} \delta_{11}=e^{-1} \\
\int_{\gamma_{\Delta}} e^{-f} d x d y=\int_{T_{3}} e^{-f} d x d y=\int_{1}^{\infty} \int_{1}^{\infty} e^{-x y} d x d y=E_{1}(1)
\end{gathered}
$$

Over $\gamma_{\circ}$, only the integral over $e^{-f} d x d y$ is non-zero. Setting $x=e^{2 \pi i t}$ and $y=r e^{-2 \pi i t}$ we find

$$
\int_{\gamma_{0}} e^{-f} d x d y=2 \pi i \int_{0}^{1} \int_{0}^{1} e^{-r} e^{2 \pi i t} e^{-2 \pi i t} d t d r=2 \pi i\left(1-e^{-r}\right)
$$

and this quantity converges to $2 \pi i$ as $r \rightarrow \infty$. The following table (aka. period matrix) summarises the results:

| $\int$ | $\delta_{00}$ | $\delta_{11}$ | $d x d y$ |
| :--- | :---: | :---: | :---: |
| $\gamma_{\square}-\gamma_{\triangle}$ | 1 | 0 | $\gamma$ |
| $\gamma_{\triangle}$ | 0 | $e^{-1}$ | $E_{1}(1)$ |
| $\gamma_{\circ}$ | 0 | 0 | $2 \pi i$ |

12.8.6 (The Euler-Mascheroni motive as an extension). - Let us examine the structure of the motive $M(\gamma)$ in detail. We keep the notation from 12.8 .1 and denote by $E$ the exceptional divisor of the blow-up $\pi: \widetilde{X} \rightarrow X$. Let $Z \subseteq Y$ be the union of two lines defined by $(x-1)(y-1)=0$, and denote by $\widetilde{Z} \subseteq \widetilde{X}$ the strict transform of $Z$. We consider the following commutative diagram of exponential motives with exact rows and columns:


The middle column comes from the long exact sequence of the pair of immersions $\widetilde{Y} \subseteq \widetilde{Y} \cup E \subseteq \widetilde{X}$ and the excision isomorphism $H^{n}\left(\widetilde{Y} \cup E, \widetilde{Y}, \tilde{f}_{\mid \widetilde{Y} \cup E}\right) \cong H^{n}(E, E \cap \widetilde{Y}, 1)$. The right-hand column is obtained in the same way replacing $Y$ by $Z$, and the top and bottom isomorphisms follow from the equality $E \cap \widetilde{Y}=E \cap \widetilde{Z}$. The horizontal short exact sequences are part of the long exact sequences associated with the triples $Z \subseteq Y \subseteq X$ and $\widetilde{Z} \subseteq Y \subseteq X$ respectively. The zeroes on the right-hand side are explained by cohomological dimension. The zeroes on the left-hand side can be obtained by writing out the long exact sequences of the pairs $Z \subseteq X$ and $\widetilde{Z} \subseteq \widetilde{X}$.

The motive $H^{1}(\widetilde{Y}, \widetilde{Z}, \widetilde{f})$ is isomorphic to $\mathbb{Q}(0)$. The motives on top and bottom of the vertical sequences are all of dimension 1 , and hence $\pi^{*}$ is of rank $2=3-1$. It follows that $M(\gamma)$ is of dimension 2 , and sits in a short exact sequence

$$
0 \rightarrow \mathbb{Q}(0) \rightarrow M(\gamma) \rightarrow \operatorname{im}\left(\pi_{0}^{*}\right) \rightarrow 0
$$

The sequence $0 \rightarrow H^{1}(E, E \cap \widetilde{Z}, 1) \rightarrow H^{2}(X, Z, f) \rightarrow H^{2}(X, f) \rightarrow 0$ is exact, and the motive $H^{2}(X, f)$ is isomorphic to $\mathbb{Q}(-1)$ by Proposition 12.2.3. Hence the image of the map $\pi_{0}^{*}$ is $\mathbb{Q}(-1)$.

Proposition 12.8.7. - The Euler-Mascheroni motive is a non-trivial extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$. In other words, there is a short exact sequence

$$
0 \rightarrow \mathbb{Q}(0) \rightarrow M(\gamma) \rightarrow \mathbb{Q}(-1) \rightarrow 0
$$

of exponential motives, and the vanishing $\operatorname{Hom}(M(\gamma), \mathbb{Q}(0))=0$ holds.
Proof. In order to show that $M(\gamma)$ is a non-trivial extension, it suffices to check that some realisation of $M(\gamma)$ is a non-trivial extension. Let us look at the Hodge realisation. The Hodge realisation of the exact sequence is a sequence of mixed Hodge modules whose fibre over $z \neq 0,1$ is the sequence of mixed Hodge structures presented in the lower row of the following diagram.


The vertical maps are morphisms induced by triples, and the top horizontal morphism is the induced by the blow-up map $\pi$ restricted to $\widetilde{Y} \cup f^{-1}(z) \cup E$. The morphism labelled $(*)$ is also induced by the blow-up map, and hence the diagram commutes. The image of the morphism $(*)$ is the fibre over $z$ of the Hodge realisation of $M(\gamma)$. The top horizontal morphism is an isomorphism of Hodge structures, so the fibre over $z$ of the Hodge realisation of $M(\gamma)$ is the Hodge structure

$$
H^{1}\left(\mathbb{G}_{m},\left\{1, z^{2}\right\}\right) \cong H^{1}\left(f^{-1}(z),\{(1, z),(z, 1)\}\right)
$$

which is an extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}$, non-split unless $z$ is a root of unity.

Corollary 12.8.8. - The motivic fundamental group of $M(\gamma)$ with respect to an appropriate basis is equal to the group

$$
G_{M(\gamma)}=\left\{\left.\left(\begin{array}{cc}
1 & b \\
0 & a
\end{array}\right) \right\rvert\, a \in \mathbb{G}_{m}, b \in \mathbb{G}_{a}\right\} .
$$

Therefore, the exponential period conjecture implies that $\gamma$ and $2 \pi i$ are algebraically independent.
Proof. The semisimplification of the motive $M(\gamma)$ is isomorphic to $\mathbb{Q}(0) \oplus \mathbb{Q}(-1)$, and hence the reductive quotient of $G_{M(\gamma)}$ is the group

$$
G_{\mathbb{Q}(0) \oplus \mathbb{Q}(-1)}=\left\{\left.\left(\begin{array}{ll}
1 & 0 \\
0 & a
\end{array}\right) \right\rvert\, a \in \mathbb{G}_{m}\right\} .
$$

Since $M(\gamma)$ is a non-trivial extension, the unipotent radical of $G_{M(\gamma)}$ is a non-zero subgroup of $\mathbb{G}_{a}$, and hence equal to $\mathbb{G}_{a}$.

Corollary 12.8.9. - The motivic fundamental group of $H^{2}(X, Y, f)$ with respect to an appropriate basis is equal to the group

$$
G_{H^{2}(X, Y, f)}=\left\{\left.\left(\begin{array}{lll}
1 & 0 & b \\
0 & a & c \\
0 & 0 & d
\end{array}\right) \right\rvert\, b, c \in \mathbb{G}_{a}, a, d \in \mathbb{G}_{m}\right\}
$$

Therefore, the exponential period conjecture implies that the numbers $e^{-1}, E_{1}(1), \gamma$, and $2 \pi i$ are algebraically independent.

REMARK 12.8.10. - The shape of the period matrix suggests that $H^{2}(X, Y, f)$ has a subobject or a quotient isomorphic to the motive associated with $E_{1}(1)$, as introduced in the previous section. Indeed, this is the case. Let $Z$ be the union of the lines $x=1$ and $y=1$. The exact sequence (4.2.4.2) for the pair of inclusions $Z \subseteq Y \subseteq X$ yields an exact sequence

$$
0 \longrightarrow H^{1}\left(Y, Z,\left.f\right|_{Y}\right) \longrightarrow H^{2}(X, Y, f) \longrightarrow H^{2}(X, Z, f) \longrightarrow 0
$$

12.8.11 (Computation of Hodge realisation). - The perverse sheaf underlying the exponential Hodge realisation of $M$ has two singularities, $S=\{0,1\}$.

Corollary 12.8.12. - The exponential period conjecture implies that $\gamma$ is transcendental over the field of usual periods.

Proof. Arguing as in the proof of Proposition 12.1.4, we are reduced to showing that, for each usual motive $M$ over $\overline{\mathbb{Q}}$, the dimension of the motivic Galois group of $M^{+}=M \oplus M(\gamma)$ is bigger than that of the motivic Galois group of $M$. For this, we use that the perverse realisation $\mathrm{R}_{\text {perv }}\left(M^{+}\right)$has fundamental group $\mathbb{G}_{a}$. Letting $G_{M^{+}}$and $G_{M}$ denote the motivic Galois groups of $M^{+}$and $M$ and $G_{F^{+}}$and $G_{F}$ the fundamental groups of their perverse realisations, there is a commutative diagram


The one-dimensional group $\mathbb{G}_{a}$ is thus contained in the kernel of the surjection $G_{M^{+}} \rightarrow G_{M}$, hence the sought-after inequality $\operatorname{dim} G_{M^{+}}>\operatorname{dim} G_{M}$.

## Gamma motives and the abelianisation of the motivic exponential Galois group

At the outset of this monograph stands Lang's conjecture 1.3.4 about the transcendence degree of the field generated by the values of the gamma function at rational numbers with a fixed denominator. As we saw in Example 1.1.4 from the introduction, they all appear as periods of the exponential motives

$$
M_{n}=H^{1}\left(\mathbb{A}_{\mathbb{Q}}^{1}, x^{n}\right) .
$$

Note that, if $n$ divides $m$, the map $x \longmapsto x^{m / n}$ induces an inclusion $M_{n} \subset M_{m}$. We call gamma motive the ind-exponential motive $\operatorname{colim}_{n} M_{n}$.

### 13.1. The Serre tori

We convene that all Hodge structures are polarisable, rational Hodge structures. We say that a Hodge structure is of CM-type if its Mumford-Tate group is commutative.

By a CM field we understand a subfield $k \subseteq \mathbb{C}$ which is
With this definition, $\mathbb{Q} \subseteq \mathbb{C}$ is a CM field. It is important to us that CM fields are actual subfields of $\mathbb{C}$, and hence come with a distinguished embedding $k \rightarrow \mathbb{C}$.
13.1.1. - Tori over $\mathbb{Q}$ are in one to one correspondence with finitely generated free $\mathbb{Z}$-modules equipped with a continuous action of $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$. To a torus $T$ corresponds its group of characters $X_{T}=\operatorname{Hom}\left(T_{\overline{\mathbb{Q}}}, \mathbb{G}_{m, \overline{\mathbb{Q}}}\right)$, and to a $\mathbb{Z}$-module $X$ with Galois action corresponds the torus $T_{X}=$ $\underline{\operatorname{Hom}}\left(X, \mathbb{G}_{m}\right)$. Let $T$ be a torus over $\mathbb{Q}$ with character group $X$. To give a representation $T \rightarrow \mathrm{GL}_{V}$ of $T$ on a vector space $V$ is to give a decomposition of $\overline{\mathbb{Q}}$-vector spaces

$$
V \otimes \overline{\mathbb{Q}}=\bigoplus_{\chi \in X} W_{\chi}
$$

which is compatible with Galois actions, in the sense that $g\left(W_{\chi}\right)=W_{g \chi}$ holds for all $g \in \operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$. To such a decomposition corresponds the representation $\rho$ defined by

$$
\rho(t)(w)=t(\chi) w
$$

for all elements $t: X \rightarrow \mathbb{G}_{m}$ of $T$, characters $\chi: T_{\overline{\mathbb{Q}}} \rightarrow \mathbb{G}_{m, \overline{\mathbb{Q}}}$ and vectors $w \in W_{\chi}$. A priori, this representation is only defined over $\overline{\mathbb{Q}}$, and it is the compatibility condition that ensures that it is defined over $\mathbb{Q}$. In these terms, the image of the representation $\rho: T \rightarrow \mathrm{GL}_{V}$ corresponds to the subgroup of $X$ generated by

$$
\left\{\chi \in X \mid W_{\chi} \neq 0\right\} \subseteq X
$$

which is indeed a Galois submodule. In particular $\rho$ is faithful if and only if the characters $\chi$ with $W_{\chi} \neq 0$ generate $X$.
13.1.2. - Let $k$ be a CM field, and write $\Sigma$ for the set of complex embeddings of $k$. The Serre torus $S_{k}$ associated with $k$ is the torus over $\mathbb{Q}$ whose group group of characters is

$$
X_{k}=\{\chi \in \operatorname{Map}(\Sigma, \mathbb{Z}) \mid \sigma \longmapsto \chi(\sigma)+\chi(\bar{\sigma}) \text { is constant. }\}
$$

with its natural $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$-action. It is a torus of dimension $\frac{1}{2}[k: \mathbb{Q}]-1$, except when $k=\mathbb{Q}$, in which case $S_{k}=\mathbb{G}_{m}$. An inclusion of CM fields $k \subseteq k^{\prime}$ induces a map res: $\Sigma^{\prime} \rightarrow \Sigma$ between the respective sets of complex embeddings sending $\sigma^{\prime}: k^{\prime} \rightarrow \mathbb{C}$ to its restriction to $k$. This restriction map in turn induces a Galois equivariant injective group homomorphism $X_{k} \rightarrow X_{k^{\prime}}$ sending $\chi$ to $\chi \circ$ res, and hence a surjective morphism of tori

$$
\left(w, \sum_{\sigma \in \Sigma} n_{\sigma} \sigma\right) \longmapsto\left(w, \sum_{\sigma \in \Sigma^{\prime}} n_{\left.\sigma^{\prime}\right|_{k}} \sigma^{\prime}\right) .
$$

We call Serre torus the protorus

$$
S=\lim _{k \subseteq \mathbb{C}} S_{k}
$$

where the limit runs over all CM fields $k \subseteq \mathbb{C}$ ordered by inclusion. The composite of finitely many CM fields is CM, hence this limit filtered, and to give a finite-dimensional representation of $S$ is to give a finite-dimensional representation of $S_{k}$ for some sufficiently large CM field $k$.
13.1.3. - Let $k \subseteq \mathbb{C}$ be a CM-field, and let $V$ be a finite-dimensional representation of $S_{k}$. We will construct a Hodge structure on the vector space $V$. According to the general discussion of representations of tori, the action of $S_{k}$ on $V$ corresponds to a decomposition

$$
V \otimes \overline{\mathbb{Q}}=\bigoplus_{\chi \in X_{k}} W_{\chi}
$$

which is compatible with Galois actions. We declare the Hodge decomposition of $V$ to be

$$
V^{p, q}=\bigoplus_{\substack{\chi\left(\sigma_{0}\right)=p \\ \chi\left(\bar{\sigma}_{0}\right)=q}} W_{\chi} \otimes_{\overline{\mathbb{Q}}} \mathbb{C},
$$

where $\sigma_{0}: k \rightarrow \mathbb{C}$ is the inclusion.
13.1.4. - Let $V$ be a Hodge structure of weight $w$, and let $k$ be a CM-field. A complex multiplication of $k$ on $V$ is a $\mathbb{Q}$-algebra homomorphism $\iota: k \rightarrow \operatorname{End}(V)$ such that $V$ is a onedimensional $k$-vector space via $\iota$. In other words, $\operatorname{dim}_{\mathbb{Q}} V=[k: \mathbb{Q}]$. Given such a complex multiplication, we obtain

Proposition 13.1.5. - Let $V$ be an irreducible Hodge structure. The following are equivalent:
(1) The Hodge structure $V$ is of CM-type, that is to say the Mumford-Tate group of $V$ is commutative.
(2) The $\mathbb{Q}$-algebra $k=\operatorname{End}(V)$ is a $C M$-field, and $V$ has complex multiplication by $k$.

Theorem 13.1.6. -
13.1.7. - The Fermat curve $C \subseteq \mathbb{P}^{2}$ of degree $n>1$ is the plane curve defined by the equation $x^{n}+y^{n}=z^{n}$. It is smooth of genus $\frac{1}{2}(n-1)(n-2)$. A basis of the de Rham cohomology group $H_{\mathrm{dR}}^{1}(C)$ is given by the forms

$$
\omega_{r, s}=x^{r-1} y^{s-n} d x
$$

for integers $1 \leqslant r, s \leqslant n-1$ satisfying $r+s \neq n$. They are meromorphic forms on $C$. The forms $\left\{\omega_{r, s} \mid r+s<n\right\}$ are holomorphic, and hence form a basis of $H^{0}\left(C, \Omega_{C}^{1}\right)$.

### 13.2. Groups of circulant matrices

Let $n \geqslant 3$ be an integer. We introduce a torus $T_{n} \subseteq \mathrm{GL}_{n}$ of dimension $1+\frac{\varphi(n)}{2}$ and a surjective $\operatorname{map} \pi_{n}: T_{n} \rightarrow S_{\mathbb{Q}\left(\mu_{n}\right)}$ with kernel $\mathbb{Z} / n \mathbb{Z}$. For every quotient $m$ of $n$, we construct a surjective morphism $T_{n} \rightarrow T_{m}$ fitting into a map of short exact sequences

where the vertical map on the left is the canonical projection, and the vertical map on the right is induced by the inclusion $\mathbb{Q}\left(\mu_{m}\right) \rightarrow \mathbb{Q}\left(\mu_{n}\right)$ as explained in 13.1.2.
13.2.1. - Let $n \geqslant 3$ be an integer. An $n \times n$ matrix of the form

$$
C=\left(\begin{array}{ccccc}
c_{0} & c_{1} & c_{2} & \cdots & c_{n-1} \\
c_{n-1} & c_{0} & c_{1} & \cdots & c_{n-2} \\
c_{n-2} & c_{n-1} & c_{0} & \cdots & c_{n-3} \\
\vdots & \vdots & \vdots & & \vdots \\
c_{1} & c_{2} & c_{3} & \cdots & c_{0}
\end{array}\right)
$$

is called a circulant matrix. The polynomial $f_{C}(X)=c_{0}+c_{1} X+\cdots+c_{n-1} X^{n-1}$ is called its associated polynomial. The determinant of $C$ is the product $f(1) f(\zeta) \cdots f\left(\zeta^{n-1}\right)$, where $\zeta$ is a
primitive $n$-th root of unity. Circulant matrices are precisely those matrices which commute with the particular circulant matrix

$$
h=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{array}\right)
$$

and every circulant matrix is a linear combination of $h^{0}, h^{1}, \ldots, h^{n-1}$. In particular, circulant matrices form a subalgebra of the ring of all $n \times n$ matrices that is isomorphic to the group algebra of $\mathbb{Z} / n \mathbb{Z}$. We denote by $\operatorname{Circ}_{n} \subseteq \mathrm{GL}_{n}$ the algebraic group of invertible circulant matrices. As an algebraic group over $\mathbb{Q}$, this group is a torus of dimension $n$, which splits over the cyclotomic field of $n$-th roots of unity. A splitting is given by the isomorphism $\operatorname{Circ}_{n} \rightarrow \mathbb{G}_{m}^{n}$ of algebraic groups over $\mathbb{Q}\left(\mu_{n}\right)$ sending a circulant matrix $C$ to the $n$-tuple $\left(f_{C}\left(\zeta^{j}\right)\right)_{j=0, \ldots, n-1}$, where $\zeta$ is a primitive $n$-th root of unity. The group of characters of $\operatorname{Circ}_{n}$ is the free group generated by the set of $n$-th roots of unity, with its obvious Galois action.

DEfinition 13.2.2. - Let $n \geqslant 3$ be an integer. We denote by $T_{n} \subseteq \operatorname{Circ}_{n}$ the subtorus of circulant matrices $C$ satisfying the following two conditions:
(1) For every divisor $1 \leqslant d<n$ of $n$ and every residue $k$ modulo $d$

$$
\sum_{\substack{j \equiv k(d) \\ 0 \leqslant j<n}} c_{j}= \begin{cases}1 & \text { if } k \equiv 0(d) \\ 0 & \text { otherwise }\end{cases}
$$

(2)

### 13.3. The gamma motive

In this section, we shall compute the motivic Galois group of $M_{n}$ and explain the relation with the Serre torus of the cyclotomic field $\mathbb{Q}\left(\mu_{n}\right)$. From this we will deduce that Lang's conjecture is equivalent to the exponential period conjecture 8.2 .6 for the motive $M_{n}$. This can be seen as a mise au goût du jour of Anderson's theory of ulterior motives [1].
13.3.1 (Motives of Fermat hypersurfaces). - Given two integers $n, m \geqslant 2$, we consider the following variants of the Fermat hypersurface:

$$
\begin{aligned}
& Y=\left\{\left[x_{0}: \cdots: x_{m}\right] \in \mathbb{P}^{m} \mid x_{1}^{n}+\ldots+x_{m}^{n}=x_{0}^{n}\right\} \\
& X=\left\{\left[x_{1}: \cdots: x_{m}\right] \in \mathbb{P}^{m-1} \mid x_{1}^{n}+\ldots+x_{m}^{n}=0\right\} \\
& U=\left\{\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{A}^{m} \mid x_{1}^{n}+\cdots+x_{m}^{n}=1\right\}
\end{aligned}
$$

We shall regard them as varieties over the cyclotomic field $k=\mathbb{Q}\left(\mu_{n}\right)$ and write e.g. $X_{n}^{m-2}$ instead of $X$ when we want to emphasise the degree and the dimension. Observe that the map $\left[x_{1}: \ldots: x_{m}\right] \longmapsto\left[0: x_{1}: \ldots: x_{m}\right]$ induces a closed immersion $\iota: X \hookrightarrow Y$, whose open complement is $U$ under the identification $\mathbb{A}^{m} \simeq \mathbb{P}^{m+1} \backslash\left\{x_{0}=0\right\}$.

Following Anderson [1, 10.2], we make the group $\Lambda=\bigoplus_{i=1}^{m} \mu_{n}$ act on $Y$ by

$$
\left(\xi_{1}, \ldots, \xi_{m}\right) \cdot\left[x_{0}: x_{1}: \cdots: x_{m}\right]=\left[x_{0}: \xi_{1} x_{1}: \cdots: \xi_{m} x_{m}\right] .
$$

This action stabilizes both $X$ and $U$. We identify the character group of $\Lambda(\mathbb{C})$ with $(\mathbb{Z} / n)^{m}$ by associating with an element $\underline{a}=\left(a_{1}, \ldots, a_{m}\right) \in(\mathbb{Z} / n)^{m}$ the character $\left(\xi_{1}, \ldots, \xi_{m}\right) \longmapsto \prod_{i=1}^{m} \xi_{i}^{a_{i}}$. Setting

$$
\Psi=\left\{\underline{a}=\left(a_{1}, \ldots, a_{m}\right) \in(\mathbb{Z} / n)^{m} \mid a_{i} \neq 0, a_{1}+\cdots+a_{m}=0\right\},
$$

there is a direct sum decomposition of classical motives

$$
\begin{equation*}
H^{m-2}(X)=\iota^{*} H^{*}\left(\mathbb{P}^{m-1}\right) \oplus \bigoplus_{a \in \Psi} H_{a}^{m-2}, \tag{13.3.1.1}
\end{equation*}
$$

It follows that the primitive cohomology $H_{\text {prim }}^{m-2}(X)$ is cut out in $H^{m-2}(X)$ by the projector

$$
\theta_{\text {prim }}=\frac{1}{m^{n}} \sum_{\lambda \in \Lambda(\mathbb{C})} \sum_{a \in \Psi}(a, \lambda) \lambda .
$$

The map $\left[x_{1}: \ldots: x_{m}\right] \longmapsto\left[0: x_{1}: \ldots: x_{m}\right]$ induces a closed immersion $\iota: X \hookrightarrow Y$, whose open complement is $U$, under the identification $\mathbb{A}^{m} \simeq \mathbb{P}^{m+1} \backslash\left\{x_{0}=0\right\}$. Noting that $X$ is a smooth divisor on $Y$, the Gysin exact sequence of motives reads:

$$
\begin{equation*}
\cdots \longrightarrow H^{i}(Y) \longrightarrow H^{i}(U) \longrightarrow H^{i-1}(X)(-1) \longrightarrow H^{i+1}(Y) \longrightarrow \cdots \tag{13.3.1.2}
\end{equation*}
$$

Moreover, (13.3.1.2) is $\Lambda$-equivariant, so we can replace each term with its image under the projector $\theta_{\text {prim }}$ and still get an exact sequence. Since $\theta_{\text {prim }}$ annhilates the cohomology of $Y$, it follows that:

$$
H_{\text {prim }}^{m-1}(U) \xrightarrow{\sim} H_{\text {prim }}^{m-2}(X)(-1) .
$$

We introduce the differential form

$$
\Omega=\sum_{\ell=1}^{m}(-1)^{\ell} x_{\ell} d x_{1} \wedge \cdots \wedge \widehat{d x_{\ell}} \wedge \cdots \wedge d x_{m} .
$$

13.3.2 (Tensor powers of the gamma motive). - We now have all the ingredients to prove that, for each integer $m \geqslant 2$, the tensor power $M_{n}^{\otimes m}$ contains a submotive isomorphic to $H_{\text {prim }}^{m-2}(X)(-1)$.

Proposition 13.3.3. - There is an isomorphism of exponential motives

$$
\begin{equation*}
\left(M_{n}^{\otimes m}\right)^{\mu_{n}} \xrightarrow{\sim} H_{\text {prim }}^{m-2}\left(X_{n}^{m-2}\right)(-1) . \tag{13.3.3.1}
\end{equation*}
$$

Proof. The proof is an elaboration on the ideas that were already used in Lemma 12.2.1. We first recall that, by the Künneth formula,

$$
M_{n}^{\otimes m}=H^{m}\left(\mathbb{A}^{m}, x_{1}^{n}+\cdots+x_{m}^{n}\right) .
$$

Consider the morphism $h: U \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{m}$ given by

$$
h\left(\left(x_{1}, \ldots, x_{m}\right), r\right)=\left(r x_{1}, \ldots, r x_{m}\right) .
$$

Since $h$ sends the closed subvariety $U \times\{0\} \subseteq U \times \mathbb{A}^{1}$ to the origin $O \in \mathbb{A}^{1}$ and commutes with the functions $0 \boxplus r^{n}$ on $U \times \mathbb{A}^{1}$ and $x_{1}^{n}+\ldots+x_{m}^{n}$ on $\mathbb{A}^{m}$, it induces a morphism of motives

$$
H^{m}\left(\mathbb{A}^{m}, O, x_{1}^{n}+\ldots+x_{n}^{m}\right) \longrightarrow H^{m}\left(U \times \mathbb{A}^{1}, U \times\{0\}, 0 \boxplus r^{n}\right) .
$$

The source is isomorphic to $H^{m}\left(\mathbb{A}^{m}, x_{1}^{n}+\ldots+x_{n}^{m}\right)$ by the long exact sequence (4.2.4.2) and the target decomposes as a tensor product according to the Künneth formula, so we get a map:

$$
\begin{equation*}
M_{n}^{\otimes m} \longrightarrow H^{m-1}(U) \otimes H^{1}\left(\mathbb{A}^{1},\{0\}, r^{n}\right) \tag{13.3.3.2}
\end{equation*}
$$

We need to show that the morphism

$$
\begin{equation*}
\left(M_{n}^{\otimes m}\right)^{\mu_{n}} \longrightarrow H^{m-1}(U) \otimes H^{1}\left(\mathbb{A}^{1}, r^{n}\right) \tag{13.3.3.3}
\end{equation*}
$$

obtained from (13.3.3.2) by restricting to the submotive $\left(M_{n}^{\otimes m}\right)^{\mu_{n}} \subseteq M_{n}^{\otimes m}$ and composing with the projection $H^{1}\left(\mathbb{A}^{1},\{0\}, r^{n}\right) \rightarrow H^{1}\left(\mathbb{A}^{1}, r^{n}\right)$ is identically zero. This will yield a morphism

$$
\begin{equation*}
\left(M_{n}^{\otimes m}\right)^{\mu_{n}} \longrightarrow H^{m-1}(U) \tag{13.3.3.4}
\end{equation*}
$$

and the proof will show as well that (13.3.3.4) is injective with image $H_{\mathrm{prim}}^{m-1}(U)$.
To carry out this program we look at the de Rham realisation. A basis of $H_{\mathrm{dR}}^{m}\left(\mathbb{A}^{m}, x_{1}^{n}+\ldots+x_{m}^{n}\right)$ is given by the differentials

$$
\omega_{\underline{j}}=x_{1}^{j_{1}-1} \cdots x_{m}^{j_{m}-1} d x_{1} \cdots d x_{m}, \quad \underline{j}=\left(j_{1}, \ldots, j_{m}\right) \in\{1, \ldots, n-1\}^{m}
$$

which are $\mu_{n}$-invariant if and only if $n$ divides $|\underline{j}|=j_{1}+\ldots+j_{m}$. By a straightforward computation, the morphism $h$ sends this basis to

$$
\begin{aligned}
h^{*} \omega_{\underline{j}} & =\sum_{\ell=1}^{m}(-1)^{m-\ell} x_{1}^{j_{1}-1} \cdots x_{\ell}^{j_{\ell}} \cdots x_{m}^{j_{m}-1} d x_{1} \cdots \widehat{d x_{\ell}} \cdots d x_{m} \otimes r^{|\dot{j}|-1} d r \\
& =(-1)^{m} x_{1}^{j_{1}-1} \cdots x_{m}^{j_{m}-1} \Omega \otimes r^{|j|-1} d r .
\end{aligned}
$$

Let us now assume that $n$ divides $|\underline{j}|$. By induction, the relation

$$
r^{a n-1} d r-\frac{1}{(a-1)} r^{(a-1) n-1} d r=d_{r^{n}}\left(-\frac{1}{n} r^{(a-1) n}\right)
$$

implies that the differentials $r^{a n-1} d r$ and $\frac{1}{(a-1)!} r^{n-1} d r$ are cohomologous for all integers $a \geqslant 1$. Taking into account that $r^{n-1} d r$ spans the kernel of the projection $H_{d R}^{1}\left(\mathbb{A}^{1},\{0\}, r^{n}\right) \rightarrow H_{d R}^{1}\left(\mathbb{A}^{1}, r^{n}\right)$, it follows that (13.3.3.3) realises to the zero map in de Rham cohomology, hence it is itself zero. The argument also shows that the resulting morphism

$$
\left(R_{\mathrm{dR}}\left(M_{n}\right)^{\otimes m}\right)^{\mu_{n}} \longrightarrow H_{\mathrm{dR}}^{m-1}(U)
$$

sends the basis $\left[\omega_{\underline{j}}\right]$, where $\underline{j}$ runs through the indices such that $n$ divides $|\underline{j}|$, to

$$
\frac{(-1)^{m}}{(|j|-1)!}\left[x_{1}^{j_{1}-1} \cdots x_{m}^{j_{m}-1} \Omega\right]
$$

To conclude, it suffices to show that these classes form a basis of $H_{\mathrm{dR}, \mathrm{prim}}^{m-1}(U)$.

Remark 13.3.4. - Let us analyse the content of the proposition for $m=2$. Set $\zeta=e^{\frac{\pi i}{n}}$. The variety $X_{n}^{0} \subseteq \mathbb{P}^{1}$ is the finite set of points $P_{r}=\left[1: \zeta^{2 r-1}\right]$ for $r \in \mathbb{Z} / n$. The group $\Lambda=\mu_{n}^{2}$ permutes these points as follows:

$$
\left(e^{\frac{2 \pi i a_{1}}{n}}, e^{\frac{2 \pi i a_{2}}{n}}\right) \cdot P_{r}=P_{a_{2}-a_{1}+r} .
$$

In particular, if $a_{1}+a_{2} \equiv 0$, then $P_{r}$ is sent to $P_{r-2 a_{1}}$. Now recall that the gamma function satisfies

$$
\Gamma\left(\frac{j}{n}\right) \Gamma\left(1-\frac{j}{n}\right)=\frac{\pi}{\sin \left(\frac{\pi j}{n}\right)}=\frac{2 \pi i}{\zeta^{j}+\zeta^{n-j}} .
$$

Remark 13.3.5. - Here is how the fact that $\left(M_{n}^{\otimes m}\right)^{\mu_{n}}$ is isomorphic to a usual motive is reflected at the level of the irregular Hodge filtration. A basis of $R_{\mathrm{dR}}\left(M_{n}^{\otimes m}\right)$ is given by the elements

$$
\begin{equation*}
x_{1}^{j_{1}-1} d x_{1} \otimes \cdots \otimes x_{m}^{j_{m}-1} d x_{m}, \quad 1 \leqslant j_{i} \leqslant n-1, \tag{13.3.5.1}
\end{equation*}
$$

which are pure of Hodge type $\left(\frac{j_{1}+\ldots+j_{m}}{n}, m-\frac{j_{1}+\ldots+j_{m}}{n}\right)$. This type is integral if and only if $j_{1}+\ldots+j_{m}$ is a multiple of $n$. Since $\xi$ acts on (13.3.5.1) by multiplication by $\xi^{j_{1}+\ldots+j_{m}}$, the $\mu_{n}$-invariant differentials are exactly those having integral Hodge type.

## APPENDIX A

## Tannakian formalism

## A.1. Neutral tannakian categories

Recall that a dual of an object $M$ of a tensor category is an object $M^{\vee}$, together with a coevaluation morphism $c: \mathbb{1} \rightarrow M \otimes M^{\vee}$ such that the composition

$$
\operatorname{Hom}(X \otimes M, Y) \longrightarrow \operatorname{Hom}\left(X \otimes M \otimes M^{\vee}, Y \otimes M^{\vee}\right) \longrightarrow \operatorname{Hom}\left(X, Y \otimes M^{\vee}\right)
$$

is bijective. If each object admits a dual, we say that the symmetric monoidal category is closed.

## A.2. Dictionary

A.2.1. - A tannakian category has a finite fundamental group if and only if it is generated as an abelian linear category by finitely many objects.

It suffices to observe that in any tannakian category $\mathbf{T}$ with tannakian fundamental group $G$, the full subcategory $\mathbf{T}_{0}$ consisting of those objects which have finite fundamental groups is a tannakian subcategory, corresponding to representations of $G / G^{0}$.
A.2.2. - Let $\mathbf{T}$ be a tannakian category together with a fibre functor $\omega$, and let $X$ be an object of T. A Jordan-Hölder sequence for $X$ is a filtration by subobjects

$$
\begin{equation*}
0=X_{0} \subsetneq X_{1} \subsetneq \cdots \subsetneq X_{n}=X \tag{A.2.2.1}
\end{equation*}
$$

which has maximal length. Note that such a filtration always exists since the length is bounded by the dimension of the vector space $\omega(X)$. All successive quotients $X_{i} / X_{i-1}$ of a Jordan-Hölder sequence are simple objects of $\mathbf{T}$, and we call semisimplification of $X$ the associated graded object

$$
X^{\mathrm{ss}}=\bigoplus_{i=1}^{n} X_{i} / X_{i-1}
$$

Two distinct Jordan-Hölder filtrations produce the same semisimplification up to reordering of the factors. The object $X$ generates a tannakian subcategory $\langle X\rangle^{\otimes}$ of $\mathbf{T}$ and the semisimplification $X^{\text {ss }}$ generates a tannakian subcategory $\left\langle X^{\mathrm{ss}}\right\rangle^{\otimes}$ of $\langle X\rangle^{\otimes}$. If $G_{X}$ and $G_{X^{\mathrm{ss}}}$ denote the tannakian fundamental groups of $X$ and $X^{\text {ss }}$ respectively, the surjective homomorphism $G_{X} \rightarrow G_{X^{\text {ss }}}$ corresponding to the inclusion $\left\langle X^{\mathrm{ss}}\right\rangle^{\otimes} \subset\langle X\rangle^{\otimes}$ identifies $G_{X^{\text {ss }}}$ with the maximal reductive quotient of $G_{X}$. Let us choose a basis of $\omega(X)$ adapted to the filtration (A.2.2.1). Then we can regard $G_{X}$ as a subgroup of $\mathrm{GL}(\omega(X))$ consisting of upper triangular block matrices with the reductive tannakian fundamental groups of the simple pieces of $X^{\mathrm{ss}}$ on the diagonal.

Remark A.2.3. - The assignment $X \longmapsto X^{\text {ss }}$ is not functorial. For particular tannakian categories such as mixed Hodge structures or motives there is however a functorial filtration (the weight filtration) that induces a non-canonical splitting of the maximal reductive quotient.

## A.3. Exactness criteria

A.3.1. - Let $H \xrightarrow{i} G \xrightarrow{p} F$ be morphisms of affine group schemes over a field $K$. We say that the sequence $1 \rightarrow H \xrightarrow{i} G \xrightarrow{p} F \rightarrow 1$ is exact if the following holds:
(1) The morphism $i$ is a closed immersion.
(2) The morphism $p$ is a faithfully flat.
(3) The composite $p \circ i$ is equal to the trivial morphism $H \rightarrow \operatorname{Spec} K \rightarrow F$
(4) The morphism $H \rightarrow \operatorname{ker}(p)=\operatorname{Spec} K \times{ }_{F} G$ obtained from the universal property of the pull-back

is an isomorphism.

LEMMA A.3.2. - Let $h: G_{1} \rightarrow G_{2}$ be a morphism of affine group schemes over a field and denote by $\omega^{h}: \operatorname{Rep}\left(G_{2}\right) \rightarrow \boldsymbol{\operatorname { R e p }}\left(G_{1}\right)$ the induced functor between their categories of representations.
(1) The morphism $h$ is a closed immersion if and only if every object of $\boldsymbol{\operatorname { R e p }}\left(G_{1}\right)$ is a subquotient of an object in the essential image of $\omega^{h}$.
(2) The morphism $h$ is faithfully flat if and only if $\omega^{h}$ is fully faithful and its essential image is stable under subquotients.

Proof. [26, Proposition 2.21]
A.3.3. - One such criterion is given in [34, Appendix] and another one in section 4 of [29]. The following proposition is a compromise between the two. Notice that condition (1) alone is not sufficient to ensure exactness. It is indeed equivalent to the statement that $\operatorname{ker}(p)$ is equal to the normal subgroup of $G$ generated by $\operatorname{im}(i)$, or also, that the GIT quotient $\operatorname{ker}(p) / \operatorname{im}(i)$ has no non-constant regular functions. The typical example for this situation is the case where $H$ is a parabolic subgroup of $N=G$ and $F=\{1\}$.

Proposition A.3.4. - A sequence of affine group schemes $H \xrightarrow{i} G \xrightarrow{p} F$ over a field of characteristic zero $K$ satisfying $p \circ i=1$ is exact if the following two conditions are satisfied:
(1) For every representation $V$ of $G$, the equality $V^{H}=V^{\mathrm{ker}(p)}$ holds.
(2) Every one-dimensional representation of $H$ which is obtained as a subquotient of some representation of $G$ can be obtained from a one-dimensional representation of $G$. In other words, the restriction map $\operatorname{Hom}\left(G, \mathbb{G}_{m}\right) \rightarrow \operatorname{Hom}\left(i(H), \mathbb{G}_{m}\right)$ is surjective.

Proof. Let us write $N$ for the kernel of $p$, and suppose without loss of generality that $H, G$, and $F$ are linear groups and that $H$ is a subgroup of $N \subseteq G$ via the inclusion $i$. We can deduce from condition (2) that, for every representation $V$ of $G$, the equality

$$
\begin{equation*}
\mathbb{P}(V)^{H}=\mathbb{P}(V)^{N} \tag{A.3.4.1}
\end{equation*}
$$

holds. Indeed, a line $\langle v\rangle$ in $V$ which is stable under $H$ corresponds to a character $\chi: H \rightarrow \mathbb{G}_{m}$. By hypothesis, we can extend this character to $\chi: G \rightarrow \mathbb{G}_{m}$. Let $K\left(\chi^{-1}\right)$ be the one-dimensional representation of $G$ with character $\chi^{-1}$. Then, $v \otimes 1 \in V \otimes K\left(\chi^{-1}\right)$ is fixed by $H$, hence by $N$. It follows that the line $\langle v\rangle$ is also stable under $N$. That the equality $H=N$ follows from (A.3.4.1) is an observation of dos Santos [29, Lemma 4.2 and 4.3]. The argument goes as follows: The quotient $G / H$ is a quasi-projective algebraic variety with $G$-action, $G$ acting by left translation on right cosets. By Chevalley's theorem, there exists a representation $V$ of $G$ and a $G$-equivariant immersion $\alpha: G / H \rightarrow \mathbb{P} V$. The point $\alpha(1) \in \mathbb{P}(V)$ is fixed by $H$, hence by $N$. This means that the equality $N H=H$ holds in $G$, hence $H=N$.

## Bibliography

1. G. W. Anderson, Cyclotomy and an extension of the Taniyama group, Compositio Math. 57 (1986), no. 2, 153-217.
2. Y. André, Une introduction aux motifs (motifs purs, motifs mixtes, périodes), Panoramas et Synthèses, Société Mathématique de France, Paris, 2004.
3. D. Arapura, The Leray spectral sequence is motivic, Invent. Math. 160 (2005), no. 3, 567-589.
4. $\qquad$ , An abelian category of motivic sheaves, Adv. Math. 233 (2013), 135-195.
5. M. Artin, Théorème de finitude pour un morphisme propre; dimension cohomologique des schémas algébriques affines, Théorie des topos et cohomologie étale des schémas. Tome 3, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin-New York, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat, pp. 145-168.
6. J. Ayoub, L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, I, J. Reine Angew. Math. 693 (2014), 1-149.
7. A. A. Beilinson, On the derived category of perverse sheaves, $K$-theory, arithmetic and geometry (Moscow, 19841986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 27-41.
8. A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5-171.
9. P. Belkale and P. Brosnan, Periods and Igusa local zeta functions, Int. Math. Res. Not. (2003), no. 49, 2655-2670.
10. D. Bertrand, Multiplicity and vanishing lemmas for differential and q-difference equations in the Siegel-Shidlovskiu theory, Fundam. Prikl. Mat. 16 (2010), no. 5, 19-30.
11. S. Bloch and H. Esnault, Gauß-Manin determinant connections and periods for irregular connections, Geom. Funct. Anal. (2000), no. Special Volume, Part I, 1-31, GAFA 2000 (Tel Aviv, 1999).
12. $\qquad$ , Homology for irregular connections, J. Théor. Nombres Bordeaux 16 (2004), no. 2, 357-371.
13. S. Bloch, M. Kerr, and P. Vanhove, A Feynman integral via higher normal functions, Compos. Math. 151 (2015), no. 12, 2329-2375.
14. A. Borel, Algebraic D-modules, Perspectives in Mathematics, vol. 2, AcademicPress,Inc., 1987.
15. A. Borel and J.-P. Serre, Corners and arithmetic groups, Comment. Math. Helv. 48 (1973), 436-491, Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.
16. L. Breen, Bitorseurs et cohomologie non abélienne, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 401-476.
17. A. Bruguières, On a tannakian theorem due to Nori, preprint, 2004.
18. K-C. Chen and J-D. Yu, The Künneth formula for the twisted de Rham and Higgs cohomologies, SIGMA Symmetry Integrability Geom. Methods Appl. 14 (2018), Paper No. 055, 14.
19. G. V. Chudnovsky, Contributions to the theory of transcendental numbers, Mathematical Surveys and Monographs, vol. 19, American Mathematical Society, Providence, RI, 1984.
20. M. A. de Cataldo and L. Migliorini, The decomposition theorem, perverse sheaves and the topology of algebraic maps, Bull. Amer. Math. Soc. (N.S.) 46 (2009), no. 4, 535-633.
21. P. Deligne, Équations différentielles à points singuliers réguliers, Lecture Notes in Math., vol. 163, SpringerVerlag, Berlin-New York, 1970.
22. , Théorie de Hodge. III, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5-77.
23. , Catégories tannakiennes, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 111-195.
24. $\qquad$ , Théorie de Hodge irrégulière, Singularités irrégulières, Documents Mathématiques (Paris), vol. 5, Société Mathématique de France, Paris, 2007, Correspondance et documents, pp. xii+188.
25. P. Deligne, B. Malgrange, and J-P. Ramis, Singularités irrégulières, Documents Mathématiques (Paris), vol. 5, Société Mathématique de France, Paris, 2007, Correspondance et documents.
26. P. Deligne, J. S. Milne, A. Ogus, and K-y. Shih, Hodge cycles, motives, and Shimura varieties, Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin-New York, 1982.
27. A. Dimca, Sheaves in topology, Universitext, Springer-Verlag, Berlin, 2004.
28. A. Dimca and M. Saito, On the cohomology of a general fiber of a polynomial map, Compositio Math. 85 (1993), no. 3, 299-309.
29. J. P. dos Santos, The homotopy exact sequence for the fundamental group scheme and infinitesimal equivalence relations, Algebr. Geom. 2 (2015), no. 5, 535-590.
30. A. Douady, Variétés à bord anguleux et voisinages tubulaires, Séminaire Henri Cartan, 1961/62, Exp. 1, Secrétariat mathématique, Paris, 1961/1962, p. 11.
31. Philippe Du Bois, Complexe de de Rham filtré d'une variété singulière, Bull. Soc. Math. France 109 (1981), no. 1, 41-81.
32. S. Eilenberg and S. MacLane, Cohomology theory in abstract groups. I, Ann. of Math. (2) 48 (1947), 51-78.
33. H. Esnault, Variation on Artin's vanishing theorem, Adv. Math. 198 (2005), no. 2, 435-438.
34. H. Esnault, P. H. Hai, and X. Sun, On Nori's fundamental group scheme, Geometry and dynamics of groups and spaces, Progr. Math., vol. 265, Birkhäuser, Basel, 2008, pp. 377-398.
35. H. Esnault, C. Sabbah, and J-D. Yu, E1-degeneration of the irregular Hodge filtration, J. Reine Angew. Math. 729 (2017), 171-227, With an appendix by Morihiko Saito.
36. J. Fresán and P. Jossen, Algebraic cogroups and Nori motives, arXiv:1805.03906v1, 2018.
37. W. Fulton, Intersection theory, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 2, Springer-Verlag, Berlin, 1998.
38. S. I. Gelfand and Y. I. Manin, Methods of homological algebra, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.
39. W. D. Gilliam, Real oriented blowup, Notes available on the author's webpage, 2011.
40. J. Giraud, Cohomologie non abélienne, Springer-Verlag, Berlin-New York, 1971, Die Grundlehren der mathematischen Wissenschaften, Band 179.
41. J. Greenough, Monoidal 2-structure of bimodule categories, J. Algebra 324 (2010), no. 8, 1818-1859.
42. A. Grothendieck, On the de Rham cohomology of algebraic varieties, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 95-103.
43. M. Hien, Periods for irregular singular connections on surfaces, Math. Ann. 337 (2007), no. 3, 631-669.
44. $\qquad$ , Periods for flat algebraic connections, Invent. Math. 178 (2009), no. 1, 1-22.
45. M. Hien and C. Roucairol, Integral representations for solutions of exponential Gauss-Manin systems, Bull. Soc. Math. France 136 (2008), no. 4, 505-532.
46. M. W. Hirsch, Differential topology, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original.
47. A. Huber and S. Müller-Stach, On the relation between Nori motives and Kontsevich periods, arXiv:1105.0865v5, 2014.
48. _ Periods and Nori motives, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 65, Springer, Cham, 2017, With contributions by Benjamin Friedrich and Jonas von Wangenheim.
49. F. Ivorra, Perverse Nori motives, Math. Res. Lett. 24 (2017), no. 4, 1097-1131.
50. K. Joshi, Musings on $\mathbb{Q}(1 / 4)$ : arithmetic spin structures on elliptic curves, Math. Res. Lett. 17 (2010), no. 6, 1013-1028.
51. J. P. Jouanolou, Une suite exacte de Mayer-Vietoris en $K$-théorie algébrique, Algebraic $K$-theory, I: Higher $K$ theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math., vol. 341, Springer, Berlin, 1973, pp. 293-316.
52. M. Kashiwara, d-modules and microlocal calculus, Translations of Mathematical Monographs, vol. 217, Amer. Math. Soc., 2003.
53. M. Kashiwara and P. Schapira, Sheaves on manifolds, Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer-Verlag, Berlin, 1990, With a chapter in French by Christian Houzel.
54. N. M. Katz, Local-to-global extensions of representations of fundamental groups, Ann. Inst. Fourier 36 (1986), no. 4, 69-106.
55. , Exponential sums and differential equations, Annals of Mathematics Studies, vol. 124, Princeton University Press, Princeton, NJ, 1990.
56. $\qquad$ , Convolution and equidistribution, Annals of Mathematics Studies, vol. 180, Princeton University Press, Princeton, NJ, 2012, Sato-Tate theorems for finite-field Mellin transforms.
57. L. Katzarkov, M. Kontsevich, and T. Pantev, Hodge theoretic aspects of mirror symmetry, From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., vol. 78, Amer. Math. Soc., Providence, RI, 2008, pp. 87-174.
58. M. Kontsevich and Y. Soibelman, Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants, Commun. Number Theory Phys. 5 (2011), no. 2, 231-352.
59. M. Kontsevich and D. Zagier, Periods, Mathematics unlimited—2001 and beyond, Springer, Berlin, 2001, pp. 771808.
60. J. C. Lagarias, Euler's constant: Euler's work and modern developments, Bull. Amer. Math. Soc. (N.S.) 50 (2013), no. 4, 527-628.
61. M. Levine, Mixed motives, Handbook of $K$-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 429-521.
62. I. López Franco, Tensor products of finitely cocomplete and abelian categories, J. Algebra 396 (2013), $207-219$.
63. J. S. Milne, Étale cohomology, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.
64. Takuro Mochizuki, Mixed twistor $\mathcal{D}$-modules, Lecture Notes in Mathematics, vol. 2125, Springer, Cham, 2015.
65. M. V. Nori, Lecture at tifr on motives, Unpublished notes.
66. $\qquad$ - Constructible sheaves, Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000), Tata Inst. Fund. Res. Stud. Math., vol. 16, Tata Inst. Fund. Res., Bombay, 2002, pp. 471-491.
67. N. Ramachandran, Values of zeta functions at $s=1 / 2$, Int. Math. Res. Not. (2005), no. 25, 1519-1541.
68. C. Sabbah, On the comparison theorem for elementary irregular $\mathscr{D}$-modules, Nagoya Math. J. 141 (1996), 107124.
69._, Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2, Astérisque (2000), no. 263 , viii +190 .
70._, Hypergeometric periods for a tame polynomial, Port. Math. (N.S.) 63 (2006), no. 2, 173-226.
71._, Fourier-Laplace transform of a variation of polarized complex Hodge structure, II, New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008), Adv. Stud. Pure Math., vol. 59, Math. Soc. Japan, Tokyo, 2010, pp. 289-347.
69. C. Sabbah and C. Schnell, The MHM project, http://www.cmls.polytechnique.fr/perso/sabbah.claude/ MHMProject/mhm.html.
70. Claude Sabbah, Irregular Hodge theory, Mém. Soc. Math. Fr. (N.S.) (2018), no. 156, vi+126, With the collaboration of Jeng-Daw Yu.
71. M. Saito, Thom-Sebastiani for Hodge modules., unpublished.
72. J. Sauloy, Differential Galois theory through Riemann-Hilbert correspondence, Graduate Studies in Mathematics, vol. 177, American Mathematical Society, Providence, RI, 2016, An elementary introduction, With a foreword by Jean-Pierre Ramis.
73. J. Scherk and J. H. M. Steenbrink, On the mixed Hodge structure on the cohomology of the Milnor fibre, Math. Ann. 271 (1985), no. 4, 641-665.
74. C. Schnell, An overview of Morihiko Saito's theory of mixed Hodge modules, https://arxiv.org/abs/1405.3096.
75. O. Schreier, Über die Erweiterung von Gruppen I, Monatsh. Math. Phys. 34 (1926), no. 1, 165-180.
76. J. Schürmann, Topology of singular spaces and constructible sheaves, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series), vol. 63, Birkhäuser Verlag, Basel, 2003.
77. K. Schwede, Gluing schemes and a scheme without closed points, Recent progress in arithmetic and algebraic geometry, Contemp. Math., vol. 386, Amer. Math. Soc., Providence, RI, 2005, pp. 157-172.
78. J-P. Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, Grenoble 6 (1955-1956), 1-42.
79. $\qquad$ , Propriétés conjecturales des groupes de Galois motiviques et des représentations $\ell$-adiques, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 377-400.
80. C. L. Siegel, Über einige Anwendungen diophantischer Approximationen [reprint of Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse 1929, Nr. 1], On some applications of Diophantine approximations, Quad./Monogr., vol. 2, Ed. Norm., Pisa, 2014, pp. 81-138.
81. P. Tauvel and R. W. T. Yu, Lie algebras and algebraic groups, Springer Monographs in Mathematics, SpringerVerlag, Berlin, 2005.
82. T. tom Dieck, Algebraic topology, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008.
83. J-L. Verdier, Stratifications de Whitney et théorème de Bertini-Sard, Invent. Math. 36 (1976), 295-312.
84. C. Voisin, Théorie de Hodge et géométrie algébrique complexe, Cours Spécialisés, vol. 10, Société Mathématique de France, Paris, 2002.
85. G. N. Watson, A treatise on the theory of Bessel functions, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995, Reprint of the second (1944) edition.
86. C. A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
87. J-D. Yu, Irregular Hodge filtration on twisted de Rham cohomology, Manuscripta Math. 144 (2014), no. 1-2, 99-133.

## List of symbols

| $\mathcal{L}_{\psi}$ | he Artin-Schreier sheaf associated with an additive character, page 209 |
| :---: | :---: |
| $E(\alpha)$ | the exponential motive with period $e^{\alpha}$ or its perverse realisation, page 231 |
| EMHS | the category of exponential mixed Hodge structures, page 212 |
| $\mathrm{Q}^{\exp }(k)$ | the quiver of exponential relative varieties over $k$, page 120 |
| $\mathrm{Q}_{\text {aff }}^{\exp }(k)$ | the subquiver of $\mathrm{Q}_{\mathrm{c}}^{\exp }(k)$ where varieties are affine, page 124 |
| $\mathrm{Q}_{\mathrm{c}}^{\exp }(k)$ | the subquiver of $\mathrm{Q}_{\mathrm{c}}^{\exp }(k)$ consisting of cellular objects, page 124 |
| $H_{\text {dR }}^{n}(X, f)$ | de Rham cohomology of the pair ( $X, f$ ), page 181 |
| $H_{n}^{\text {rd }}(X, f)$ | rapid decay homology of the pair ( $X, f$ ), page 7 |
| $H_{\mathrm{rd}}^{n}(X, f)$ | rapid decay cohomology of the pair ( $X, f$ ), page 8 |
| $\langle Q, \rho\rangle$ | the linear hull of a quiver representation $\rho: Q \rightarrow \mathbf{V e c}_{\mathbb{Q}}$, page 113 |
| $\mathrm{Vec}^{\mu}$ | the category of monodromic vector spaces, page 64 |
| $G_{M}$ | the Galois group of an exponential motive $M$, page 146 |
| $\mathrm{G}^{\exp }(k)$ | the exponential motivic Galois group, page 146 |
| $R \Psi_{p}$ | the nearby cycles functor from perverse sheaves on $\mathbb{A}_{\mathbb{Q}}^{1}$ to perverse sheaves on $\mathbb{A}_{\mathbb{F}_{p}}^{1}$, page 210 |
| PS(k) | the category of period structures over $k$, page 199 |
| Perv | the category of $\mathbb{Q}$-perverse sheaves on $\mathbb{A}^{1}(\mathbb{C})$, page 37 |
| Perv ${ }_{0}$ | the category of $\mathbb{Q}$-perverse sheaves on $\mathbb{A}^{1}(\mathbb{C})$ with no global cohomology, page 37 |
| $\Psi_{\infty}$ | the nearby fibre at infinity functor on the category Perv $0_{0}$, page 38 |
| $S_{r}$ | the closed half-plane of complex numbers $z$ with $\operatorname{Re}(z) \geqslant r$, page 7 |


[^0]:    ${ }^{1}$ TODO: introduce vanishing cycles in the discussion about conditions to belong to Perv ${ }_{0}$

[^1]:    ${ }^{1}$ It appears that several, inequivalent definitions of manifolds with corners are in use. Our example fits all of them as far as we know. We use Douady's definition in [30, §I.4], which seems to be the one most adapted to our situation.

[^2]:    ${ }^{1}$ In [47], the proof of Corollary 1.7 is incomplete because of this problem, as is Arapura's [4] proof of Theorem 4.4.2. Levine [61] cites Nori's [65], where Nori draws the right diagram but does not show that it commutes.

[^3]:    ${ }^{2}$ provided a concrete construction of limits in $D^{b}\left(\mathbf{M}_{\mathrm{c}}^{\exp }(k)\right)$ is at disposal.

[^4]:    ${ }^{1}$ We call a continuous map between manifolds with corners piecewise smooth if the domain admits a finite stratification by closed submanifolds with corners such that the restriction of the map to each of them is smooth, in the usual sense that it extends to a smooth function on an open neighbourhood. In our concrete case, the domain is the standard simplex $\Delta^{p}$. One advantage of piecewise smooth maps $\Delta^{p} \rightarrow X$ is that we can pull back and integrate differential $p$-forms, which would not be possible with just continous maps. On the other hand, the restriction of a piecewise smooth map to any face of $\Delta^{p}$ is itself piecewise smooth, and we can construct piecewise smooth maps on $\Delta^{p}$ by specifying compatible piecewise smooth maps on each piece in the barycentric subdivision of $\Delta^{p}$ without worrying about differentiability. This will come in handy when we show compatibility of the comparison isomorphism with cup-products. Finally, we stress that continuous maps $\Delta^{p} \rightarrow X$ can be approximated within the same homotopy class by piecewise smooth maps, hence classical singular homology with respect to continous cycles is the same as singular cohomology using only piecewise smooth cycles.

[^5]:    ${ }^{1}$ This is called join in loc. cit. Note that there is a misprint in the definition of the weight filtration.

[^6]:    ${ }^{1}$ For $\mathbb{Q}(1 / 2)$, see [67] and references given there (Milne). For $\mathbb{Q}(1 / 4)$, see [50].

[^7]:    ${ }^{2}$ To ease the comparison with loc.cit., notice that Bloch and Esnault consider connections given by $\nabla(1)=d f$, so what they call $f$ is our $-f$.

[^8]:    ${ }^{3}$ Since $f$ satisfies a differential equation, which can be thought of as a kind of linear recurrence relation for the coefficients $a_{n}$, Siegel's condition should imply the one we gave in the definition.

