ARITHMETIC HOLONOMY BOUNDS AND IRRATIONALITY (after Calegari, Dimitrov, Tang)

→ « The unbounded denominators conjecture » (2021)

→ « The linear independence of 1, $\zeta(2)$, and $L(2, \chi_{-3})$ » (2024) JAMS 2025

<u>Theorem</u> (Calegari - Dimitrov - Tang)

(1) The number

$$L(2, \chi_{-3}) \overset{\text{def}}{=} 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{25} + \frac{1}{49} - \cdots$$

is irrational. Moreover, $1, \zeta(2), L(2, \chi_{-3})$ are $\mathbb{Q}$-linearly independent.

(2) Let $m, n \in \mathbb{Z} \setminus \{0, -1\}$ be integers such that $\left|\frac{m}{n} - 1\right| < 10^{-6}$. Then

$$\log\left(1 + \frac{1}{m}\right) \log\left(1 + \frac{1}{n}\right) \text{ is irrational}$$

Moreover, if $m \neq n$, then

$$1, \log\left(1+\frac{1}{m}\right), \log\left(1+\frac{1}{n}\right), \log\left(1+\frac{1}{m}\right)\log\left(1+\frac{1}{n}\right)$$

are $\mathbb{Q}$-linearly independent.

- <u>Notation</u>: χ_{-3} is the quadratic Dirichlet character modulo 3 given by

$$\chi_{-3}(n) = \begin{cases} 0 & 3 \mid n \\ 1 & n = 3k+1 \\ -1 & n = 3k+2 \end{cases}$$

and $L(s, \chi_{-3}) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}$ the associated Dirichlet series.

- For the values of the Riemann zeta function we know that

$$\zeta(2k) \in \pi^{2k} \cdot \mathbb{Q}^{\times} \qquad (k \geq 1)$$

(and in particular is transcendental)

$\zeta(2k+1)$ is expected to be transcendental and even algebraically independent with π, but this is not known for any value of k.

<u>Best</u>: $\zeta(3) \notin \mathbb{Q}$ (Apéry 1978)

The difference between even and odd values is "explained" by the functional equation for $\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(s)$

$\xi(s) = \xi(1-s)$

At even values, the Γ-factors do not have poles and we may transfer the information from the fact that $\zeta(1-2k) \in \mathbb{Q}$.

- If χ is a primitive quadratic character of conductor D (with sign D = sign $\chi(-1)$)

$$\text{then} \quad L(k,\chi) \in \begin{cases} \pi^k \sqrt{D}\, \mathbb{Q}^\times & k \text{ even } \chi(-1)=1 \quad \Gamma(\tfrac{s}{2}) \\ \pi^k \sqrt{-D}\, \mathbb{Q}^\times & k \text{ odd} \geq 3 \quad \Gamma(\tfrac{s+1}{2}) \\ & \chi(-1) = -1 \end{cases}$$

and in particular is transcendental.

So $L(2,\chi_{-3}) \notin \mathbb{Q}$ covers the 1^{st} case where there is no explicit formula.

Moreover, $L(2,\chi_{-3})$ is not a natural multiple of π^2. Open case: $L(2,\chi_{-4}) = \sum \frac{(-1)^n}{(2n+1)^2}$ Catalan's constant

- For the logarithms, we know thanks to Lindemann's theorem that $\log(\alpha)$ is transcendental for every algebraic number $\alpha \notin \{0,1\}$. It is conjectured that:

$$\log(\alpha_1), \dots \log(\alpha_n) \ \mathbb{Q}\text{-linearly independent} \implies \text{algebraically independent}$$

- How to prove that some number $\beta \in \mathbb{R}$ is irrational?

<u>Lemma</u>: Assume there exists $\delta > 0$ and a sequence $\left(\frac{p_n}{q_n}\right)$ with $\frac{p_n}{q_n} \neq \beta$, $q_n \to \infty$ and $\left|\beta - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}$ for all n. (*)
Then β is irrational.

<u>Proof</u>: If $\beta = \frac{p}{q}$, then $q\, q_n \left|\beta - \frac{p_n}{q_n}\right|$ is a sequence of <u>integers</u> with

$$0 < q\, q_n \left|\beta - \frac{p_n}{q_n}\right| < \frac{q}{q_n^{\delta}} \xrightarrow[n\to\infty]{} 0 ,$$

contradiction! $\square$

- Observe that for this to work we need the exponent of $1/q_n$ in (*) to be $\boxed{> 1}$.

- Given a number of interest, it is <u>extremely hard</u> to produce such a sequence $\frac{p_n}{q_n}$. One of the breakthroughs in the paper by Calegari-Dimitrov-Tang is that they managed to prove $L(2, \chi_{-3}) \notin \mathbb{Q}$ working with a sequence of rational approximations that does **not** converge fast enough:

$$\left| L(2, \chi_{-3}) - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{0{,}52349\ldots}} \qquad = \frac{\log 9}{2+\log 9}$$

- Apéry's proof

Apéry constructs two power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + 5x + 73x^2 + \ldots \in \mathbb{Z}[[x]]$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 6x + \frac{351}{4} x^2 + \ldots \in \mathbb{Q}[[x]]$$

such that $\frac{b_n}{a_n} \xrightarrow[n\to\infty]{} \zeta(3)$ fast enough to deduce the irrationality of $\zeta(3)$.

The coefficients

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$b_n = a_n \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}$$

are solutions of the same recurrence relation

$$(n+1)^3 u_n = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}$$

which translates into

$$L \cdot A(x) = 0$$

$$L \cdot B(x) = 5$$

for the differential operator

$$L = \underbrace{(x^4 - 34x^3 + x^2)}_{\text{roots } 0,\ (\sqrt{2}-1)^4,\ (\sqrt{2}+1)^4} \left(\frac{d}{dx}\right)^3 + (6x^3 - 153x^2 + 3x)\left(\frac{d}{dx}\right)^2 + (7x^2 - 112x + 1)\frac{d}{dx} + (x-5)$$

so $A(x)$ and $B(x)$ have radius of convergence $(\sqrt{2}-1)^4$. But the linear combination

$$P(x) = B(x) - \zeta(3) A(x)$$

converges more, up to exactly $(\sqrt{2}+1)^4$!

[in particular, ∞ many $\neq 0$ coefficients)

Besides, we have control on the denominators of b_n, namely

$$[1, 2, \dots n]^3 b_n \in \mathbb{Z}$$

where $[1, 2 \dots n] \overset{\text{def}}{=} \mathrm{lcm}(1, 2, \dots n)$

Recall that the prime number theorem says $\boxed{[1, 2 \dots n] = e^{n + o(n)}}$ as $n \to \infty$

The key property is that $\boxed{e^3 < (\sqrt{2}+1)^4}$,

so that if $\zeta(3) = \frac{p}{q}$, then

$$\underbrace{q\,[1,\dots n]^3\,|b_n - \zeta(3)\,a_n|}_{\text{sequence of integers}} \xrightarrow[n\to\infty]{} 0$$

- Idea: exploit extra properties of power series A, B such that $B(x) - L(2, \chi_{-3})\,A(x)$ converges more than expected. In fact,

$$H_A(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[[x]] \qquad \text{(Zagier)}$$

$$H_B(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{Q}[[x]]$$

with $[1,\dots n]^2 b_n \in \mathbb{Z}$

$L \cdot H_A = 0$
$L \cdot H_B = 1$ for L of order 2 with singularities in $\{0, \frac{1}{9}, 1, \infty\}$ and

$$H_B(x) - \frac{L(2,\chi_{-3})}{2} H_A(x)$$

has radius of convergence $= 1$.

- Difference: there exist non-polynomial power series with convergence radius $= 1$

and denominators killed by $[1,\dots n]^2$.

For example,

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\log^2(1-x)$$

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

$$\frac{1}{\sqrt{1-x}} \int_0^x \frac{\log(1-t)}{t\sqrt{1-t}}\, dt$$

Then the question becomes how many / how to reach a contradiction if $L(2, \chi_{-3})$ happens to be a rational number...

Algebraicity theorems

Lemma: let $f \in \mathbb{Z}[\![x]\!]$ be a power series with integer coefficients and radius of convergence $R > 1$. Then f is a polynomial.

Weaker hypothesis:

(1) staying in a disk: replace holomorphic with meromorphic

Theorem (Borel, 1894) If $f \in \mathbb{Z}[\![x]\!]$ can be written as a quotient of two power series with convergent radius > 1, then f represents a rational function in $\mathbb{Q}(x)$.

[One checks rationality via vanishing of Hankel determinants $\det(a_{n+i+j})_{0 \le i, j \le N}$]
for all large enough n

(2) changing the domain

$\Omega \subset \mathbb{C}$ simply connected open $0 \in \Omega$

Riemann mapping theorem

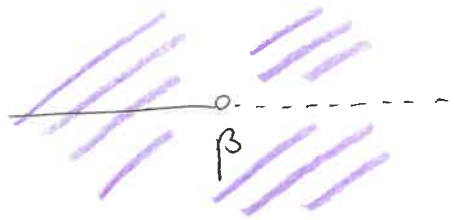
$$\varphi : \mathbb{D} \xrightarrow[\text{biholomorphic}]{} \Omega \qquad \varphi(0) = 0$$

unique up to precomposition with rotations
so $|\varphi'(0)|$ is well defined
$=$ conformal radius of Ω at 0

e.g. $\Omega = D(0, R)$ then $\varphi(z) = Rz$ and conformal radius $= R$

Theorem (Borel - Pólya) Let $f \in \mathbb{Z}[\![x]\!]$ be a power series that extends to a holomorphic function on some simply connected open domain $\Omega \ni 0$ with conformal radius > 1, then $f \in \mathbb{Q}(x)$.

e.g. $\Omega = \mathbb{C} \setminus [\beta, \infty) \xleftarrow{\varphi} \mathbb{D}$



$$\varphi(z) = \frac{4\beta z}{(1+z)^2}$$

$$|\varphi'(0)| = 4\beta$$

Possible application:

$\mathbb{Z}[[x]] \ni A, B$ solutions of ODE with singularities $0, \infty, \alpha, \beta$ $\quad 0 < \alpha < \beta$ real numbers

$P(x) = B(x) - \eta A(x)$ $\boxed{\text{\textbar deg wt}}$

holomorphic at α, analytically continues to a β. If $4\beta > 1$, then η is irrational.

nonmorphic function at (improvement on $\beta > 1$ from the basic lemma)!

Theorem (Calegari - Dimitrov - Tang)

Let $\varphi : \overline{\mathbb{D}} \to \mathbb{C}$ be holomorphic (= extends to some holomorphic function on some open neighbourhood of $\overline{\mathbb{D}}$ in $\mathbb{C}$) with $\varphi(0) = 0$ and $|\varphi'(0)| > 1$. Then the $\mathbb{Q}(x)$-linear span $\mathcal{H}(\varphi)$ of power series $f \in \mathbb{Z}[[x]]$ such that $f(\varphi(z)) \in \mathbb{C}[\![z]\!]$ converges in some neighborhood of $\overline{\mathbb{D}}$ has dimension

$$\dim_{\mathbb{Q}(x)} \mathcal{H}(\varphi) \leq e \cdot \frac{\int_0^1 \max(0, \log|\varphi(e^{2\pi i t})|)\,dt}{\log|\varphi'(0)|}$$

the number e!

$$= \int_{\mathbb{T}} \log^+|\varphi|\mu$$

<u>Remark 1</u> If $f \in \mathbb{Z}[\![x]\!]$ belongs to $\mathcal{H}(\varphi)$, then $f^n \in \mathcal{H}(\varphi)$ for all $n \geq 1$, so finite dimensionality implies that f is algebraic over $\mathbb{Q}(x)$. This was first obtained by André in his book of G-functions.

<u>Remark 2</u>: Calegari–Dimitrov–Tang call this an "arithmetic holonomy bound".

There are never rational approximations with both $A, B \in \mathbb{Z}[X]$, so in general we need to consider power series with denominators of some type. But then powers will have worse denominators.

The right operation to consider is differentiation, so $f, f', f'', \ldots$ live in a finite-dimensional $\mathbb{Q}(x)$-vector space

$\Rightarrow$ f is holonomic

[instead of f is algebraic that we don't want as a conclusion]

e.g. for $$f(x) = \sum_{n=0}^{\infty} \frac{a_n^{\in \mathbb{Z}}}{[1, \ldots b_1 n] \cdots [1, \ldots b_r n]} x^n$$

if there exists $\varphi : \mathbb{D} \to \mathbb{C}$ of conformal size $|\varphi'(0)| > e^{b_1 + \cdots + b_r}$ with $f(\varphi(z))$ germ of meromorphic function in $\mathbb{D}$, then f is holonomic.

(Here for real $b \in \mathbb{R}_{\geq 0}$, $[1, \ldots bn] = [1, \ldots \lfloor bn \rfloor]$

inspired by Perelli-Zannier

$(a_n)_{n \geq 0}$ pseudo polynomial $\limsup_{n \to \infty} |a_n|^{1/n} < e$

$m-n \mid a_m - a_n$ $\sum a_n x^n$ holonomic

Sketch of proof

$\underline{X} = (x_1, \ldots, x_d)$ $\quad \underline{X}^{\underline{j}} = x_1^{j_1} \cdots x_d^{j_d}$ $\quad d \to \infty$

let $f_1, \ldots f_m \in \mathcal{H}(\varphi)$ be $\mathbb{Q}(x)$-linearly independent series.

Goal: bound m $\quad \boxed{d, \alpha \geq 1 \quad \eta \in [0,1)}$

Step 1: use $\boxed{\text{Siegel's lemma}}$ to construct

linear system of equations with $\mathbb{Z}$-coefficients unknowns > equations has non-zero solution of bounded size a non-zero auxiliary function

$$F(\underline{x}) = \sum_{\substack{\underline{i} \in \{1,\dots,n\}^d \\ \underline{k} \in \{0,\dots,D-1\}^d}} a_{\underline{i},\underline{k}}\, \underline{x}^{\underline{k}} \prod_{s=1}^{d} f_{i_s}(\underline{x}) \in \mathbb{Z}[\![\underline{x}]\!]$$

such that • F vanishes to order $\geq \alpha$ at 0

• $|a_{\underline{i},\underline{k}}| \leq \exp(C\eta\alpha + o(\alpha))$ as $\alpha \to \infty$

constant

• $D \leq \dfrac{1}{(d!)^{1/d}} \dfrac{1}{m} \left(1 + \dfrac{1}{\eta}\right)^{1/d} \alpha + o(\alpha)$

Step 2: If $G(\underline{z})$ is a non-zero holomorphic function on the polydisk $|\underline{z}| \leq 1$ and $c\,\underline{z}^{\underline{n}}$ is the smallest monomial for the lexicographic order, then

$$\log|c| \leq \int_{|z_i|=1} \log|G|\,\mu$$

uniform measure on the torus

[In one variable, this follows from Jensen's formula

$$\log|c| = \int_{|z|=1} \log|G|\,d\mu + \sum_{\substack{w_i \in \overline{D} \setminus \{0\} \\ G(w_i)=0}} \log|w_i| \quad]$$

Apply to $G(\underline{z}) = F(\varphi(z_1), \dots, \varphi(z_d))$ to get

$$\log|c| \leq \int_{T^d} \log|F(\varphi(z_1), \dots \varphi(z_d))|\mu$$

$$\leq d D \int_T \log^+|\varphi|\mu + \eta C\alpha + o(\alpha)$$

integrate pointwise
bound from F

$$\log|F(\varphi(z_1), \dots \varphi(z_d))| \leq D \sum_{i=1}^{d} \log^+|\varphi(z_i)| + k C\alpha + o(\alpha)$$

Step 3: let $\beta = \mathrm{ord}_0 F$. The coefficient c from Step 2 is then equal to

$$\varphi'(0)^\beta \times \begin{matrix}\text{lowest} \\ \text{monomial of} \\ F(\underline{x})\end{matrix} \in \mathbb{Z}$$

$\varphi(z) = \varphi'(0)z + \dots$

and hence

$$\log|c| \geq \beta \log|\varphi'(0)| \geq \alpha \log|\varphi'(0)|$$

Step 4:

$$\log|\varphi'(0)| \leq \frac{dD}{\alpha} \int_T \log^+|\varphi|\mu + \eta C + \frac{o(\alpha)}{\alpha}$$

Make $\alpha \to \infty$, then

$$\frac{d\,D}{\alpha} \le \boxed{\frac{d}{(d!)^{1/d}}} \frac{1}{m}\left(1+\frac{1}{\eta}\right)^{1/d} \xrightarrow[\text{then } \eta \to \infty]{\text{first } d \to \infty} \frac{e}{m}$$

Stirling $d! \sim d^d e^{-d} \sqrt{2\pi d}$

so at the end

$$\log|\varphi'(0)| \le \frac{e}{m} \int_T \log^+ |\varphi|\, \mu$$

as we wanted to show $\square$

<u>Application</u>: the unbounded denominators conjecture

$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \} \cup \mathbb{P}^1(\mathbb{Q})$

$f : \mathbb{H} \longrightarrow \mathbb{C}$ holomorphic function

(a) there exists an integer k and a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of finite index such that

$$(*) \qquad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

(b) f locally extends to a meromorphic function on $\mathbb{P}^1(\mathbb{Q})$

Theorem (Calegari - Dimitrov - Tang)

If f admits a Fourier expansion with integer coefficients, $f \in \mathbb{Z}[[q]]$ $q = e^{2\pi i z}$
then $(*)$ holds for all matrices in a congruence subgroup

$$\Gamma(N) = \{ A \in SL_2(\mathbb{Z}) \mid A \equiv I \bmod N \}$$

- Siegel's lemma: $N > M$

$$\sum_{j=1}^{N} a_{ij} X_j = 0 \qquad i = 1, \dots M$$

$$\max |a_{ij}| < B \qquad a_{ij} \in \mathbb{Z} \text{ not all } 0$$

has a non-zero integer solution bounded by $(NB)^{M/N-M}$.

Remark: thanks to the much more sophisticated proof of Bost-Charles, e can be replaced with z in the statement.

Adding denominators

Theorem (Calegari - Dimitrov - Tang 2024)

Let $b \in \mathbb{Q}_{\geq 0}$ and $\sigma \in \mathbb{Z}_{\geq 0}$. Let $\varphi : \overline{\mathbb{D}} \to \mathbb{C}$ be a holomorphic map with $\varphi(0) = 0$ and $|\varphi'(0)| > e^{b\sigma}$. Then the $\mathbb{Q}(x)$-linear span $\mathcal{H}(\varphi, b, \sigma)$ of

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{[1, \dots bn]^{\sigma}} x^n \in \mathbb{Q}[[x]] \qquad a_n \in \mathbb{Z}$$

has dimension

$$\dim_{\mathbb{Q}(x)} \mathcal{H}(\varphi, b, \sigma) \leq 2 \frac{\int_{\mathbb{T}} \log^+ |\varphi| \mu}{\log |\varphi'(0)| - \boxed{b\sigma}}$$

Remark: there is a more general statement in which we fix a matrix of denominators type

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ b_{m1} & b_{m2} & \dots & b_{mr} \end{pmatrix}$$

and we consider functions

$$f_i(x) = \sum \frac{a_n}{[1, \dots b_{i1} n] \dots [1, \dots b_{ir} n]} x^n$$

Why these denominators?

For G-functions coming from geometry (= solution of a Picard-Fuchs differential operator), there exist $A \in \mathbb{Z}_{\geq 1}$

$b \in \mathbb{Q}_{>0}$

$\sigma \in \mathbb{Z}_{\geq 0}$

such that $A^{n+1} [1, \dots, bn + b_0]^{\sigma} a_n \in \mathbb{Z}$

denominator of local exponents around 0

order of diff. equation $- 1$

(Fischler and Rivoal 2017)

In Apéry's example: | order 4, local exponents $= 0$

Let's try to prove the theorem:

assume $a + \frac{b}{2} L(2, \chi_{-3}) + c \frac{\zeta(2)}{4} = 0$

for some integers a, b, c not all 0

Then there exists

$$H(x) = a\, A(x) + b\, B(x) + c\, C(x) \in \mathbb{Q}[\![x]\!]$$

with denominators $[1,\dots n]^2$ solution of inhomogeneous

$$x(1-x)(1-9x)\left(\frac{d}{dx}\right)^2 + \cdots\cdots$$

that converges up to $R=1$.

The series

$$A(x) = 1+3x+15x^2+93x^3+\cdots$$

$$B(x) = x+\frac{23}{4}x^2+\frac{145}{4}x^3+\cdots$$

$$C(x) = x+6x^2+\frac{343}{9}x^3+\cdots$$

are the ones found by Zagier using modular forms

$$x: \quad Y_0(6) = \mathbb{H}/\Gamma_0(6) \longrightarrow \mathbb{P}^1 \setminus \{0, \tfrac{1}{9}, 1, \infty\}$$

$$\Gamma_0(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod 6 \right\}$$

4 cusps $i\infty, 0, \frac{1}{3}, \frac{1}{2}$

$$z \longmapsto q \prod_{n=1}^{\infty} \frac{(1-q^n)^4(1-q^{6n})^8}{(1-q^{2n})^8(1-q^{3n})^4}$$

In fact, $\qquad = q-4q^2+10q^3+\cdots$

$$q = x+4x^2+22x^3+\cdots \in \mathbb{Z}[[x]]$$

so $\mathbb{Z}[[q]] = \mathbb{Z}[[x]]$

The series are obtained through this transformation from modular forms, e.g.

$$\theta_{-3}(z) = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2}$$

$$= 1 + 6\sum_{n=1}^{\infty}\Big(\sum_{d|n}\chi_{-3}(d)\Big)q^n$$

and then forming

$$A = \frac{\theta_{-3}(z)+\theta_{-3}(2z)}{2} \in M_1(\Gamma_0(6), \chi_{-3})$$

$\chi(d)(cz+d)^k f(z)$

let $y = x + \frac{x}{x-1}$ and consider the symmetrization

$$G(y) = H(x) + H\left(\frac{x}{x-1}\right)$$

with denominator type $[1, 2, \ldots 2n]^2$.

We now write 14 functions

$\to$ singularities: $\{0, 4, \infty, -\frac{1}{72}\}$

holomorphic in $\mathbb{C}\setminus[4,\infty)$

symmetrizations of

$f_3(y)$ $\quad$ $f_4(y)$

$1,\ -\log(1-x),\ \log^2(1-x),\ \mathrm{Li}_2(x),$

$$y\ {}_3F_2\left(\begin{matrix}\frac12 & 1 & 1\\ \frac32 & \frac32\end{matrix}\,\middle|\,\frac{y}{4}\right) = \sum_{n=1}^{\infty} \frac{(n-1)!^2}{(2n-1)!\,(2n-1)}\, y^n,$$

$$\int \frac{f_3(y)}{y}\,dy = \sum_{n=1}^{\infty} \frac{(n-1)!^2}{n\,(2n)!}\, y^n$$

$$\int \frac{f_4(y)}{y}\,dy$$

$$\int G(y)\,dy,\quad \int \frac{G(y)-G(0)}{y}\,dy$$

$$\int \frac{G(y)-G(0)-G'(0)\,y}{y^2}\,dy,\quad G(y),\ G'(y),\ G''(y),\ G'''(y),$$

differential equation of order 4

Proposition: These 14 power series are linearly independent over $\mathbb{C}(y)$.

[uses differential equation, monodromy]

Holonomy bound: $\Omega = \mathbb{C} \setminus [4,\infty)$ has conformal radius $|\varphi'(0)| = 16$ which is not bigger than $e^4 = 54{,}59\ldots$

So a better $\varphi : \overline{\mathbb{D}} \to \mathbb{C}$ is needed for the application. Idea:

$$h = \lambda + \frac{\lambda}{\lambda - 1} : \mathbb{D} \longrightarrow \mathbb{C}$$
$$q \longmapsto -256\, q \prod_{n=1}^{\infty} (1+q^n)^{24}$$

and consider $\varphi = h(\Psi(x))$ for some $\Psi : \mathbb{D} \to \mathbb{D}$ such that $\Psi(\mathbb{D})$ only contains one preimage of $-\frac{1}{72}$ under h and making the following bound as small as possible. With this choice we get

$$13{,}9938\ldots < 14$$

linearly independent functions, contradictions!

<u>Bonus</u>: The element $\zeta_2(5)$ of $\mathbb{Q}_2$ is irrational (also obtained by Lai, Sprang, Zudilin by more elementary means)

$$\zeta_p(s) = \lim_{\substack{k \in \mathbb{Z}_{<0} \\ k \equiv s \bmod p-1 \\ k \to s \\ p\text{-adically}}} \zeta(k)$$