(Columbia Algebraic Geometry Seminar 4/10/24) ①

# ON G-FUNCTIONS OF DIFERENTIAL ORDER 2

(joint with Josh Lam and Yichen Qin)

(Siegel 1929)

Definition: A G-function is a formal power series with algebraic coefficients

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \bar{\mathbb{Q}}[[z]]$$

such that:

(1) $f$ is solution of a linear differential equation with polynomial coefficients: $\Longrightarrow$ coefficients in a number field

$$L \in \bar{\mathbb{Q}}\langle z, \tfrac{d}{dz} \rangle \qquad L \cdot f = 0$$
$$L \neq 0$$

(differential order = minimum degree of such an $L$)

(2) growth condition: $C > 0$

$$\max_{\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| < C^n \qquad den(a_1, \dots, a_n) < C^n$$

"common denominator"

radius of convergence of $\sum \sigma(a_n) z^n$ is $> 0$ $\Leftarrow$

Examples: * $\bar{\mathbb{Q}}[z]$ otherwise radius of convergence is finite

* all $a_n = 1$ ⟶ geometric series explains the name G

* algebraic power series $\bar{\mathbb{Q}}[z] \cap \overline{\mathbb{Q}(z)}$ (Eisenstein 1852)  stronger: globally bounded

* hypergeometric series

$$ {}_pF_{p-1}\left(\begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_{p-1}\end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n} \frac{z^n}{n!} $$

$a_i \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  $(x)_n = x(x+1)\cdots(x+n-1)$

e.g. $$ {}_2F_1\left(\begin{matrix} \frac{1}{2} \; \frac{1}{2} \\ 1 \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{z}{16}\right)^n $$

solution of $z(z-1)\left(\frac{d}{dz}\right)^2 + (2z-1)\frac{d}{dz} + \frac{1}{4}$

$$ = \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-zx)}} $$

realises it as a period function associated with the family of elliptic curves $y^2 = x(1-x)(1-zx)$

${}_2F_1(\alpha)$ transcendental for all non-zero $\alpha \in \bar{\mathbb{Q}}$ in $|\alpha| < 1$.

In general,

vector bundle with fiber $H^p_{dR}(X_s)$

Gauss–Manin connection

$$\begin{array}{c} X \\ \pi \downarrow \ \boxed{\text{proper smooth}} \\ 0 \in S \subset \mathbb{A}^1 \end{array} \rightsquigarrow \left(\mathcal{H}^p_{dR}(X/S), \nabla_{GM}\right)$$

gives rise to a differential operator $L \in \bar{\mathbb{Q}}\langle z, \frac{d}{dz} \rangle$ «Picard-Fuchs»
after choosing a coordinate

<u>Theorem</u> (André): Solutions in $\bar{\mathbb{Q}}[[z]]$ of Picard-Fuchs differential operators are <u>G-functions</u>.

$\Rightarrow$ period functions $z \mapsto \int_\sigma \omega_z$ are $\mathbb{C}$-linear combinations of G-functions

periods

New meaning to G!

<u>Conjecture</u> [Bombieri-Dwork]

« All G functions come from geometry »

Every G-function is annihilated by a product of factors of Picard-Fuchs operators.

$\boxed{\text{Today}}$

<u>Theorem</u> (with Josh Lam & Yichen Qin)

There exist (many...) G-functions of differential

↳ coming from geometry

order 2 that are not $\bar{\mathbb{Q}}$-polynomials in

$$\mu(z) \cdot {}_pF_{p-1}\left(\begin{matrix} a_1 \dots a_p \\ b_1 \dots b_{p-1} \end{matrix} \,\Big|\, \lambda(z)\right)$$

$\mu$ algebraic; $\lambda$ algebraic with $\lambda(0)=0$

<u>Remark</u>: a) 2 is the minimal differential order because every G-function of order 1 is <u>algebraic</u> (follows from André–Chudnovsky–Katz on minimal differential equations)

b) Explicit examples, e.g. any solution in $\bar{\mathbb{Q}}[[z]]$ to the differential equation

$$P(z)\left(\frac{d}{dz}\right)^2 + \frac{P'(z)}{2}\frac{d}{dz} + \frac{z-10}{18}$$

where $P(z) = (z-1)(z-2)(z-82)$. minimal number of singularities

Where does this question come from?

<u>Siegel 1929</u> was mainly interested by E-functions $\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ as he wanted to generalize the Lindemann–Weierstrass theorem,

exponential

$\alpha_1, \dots \alpha_2 \in \bar{\mathbb{Q}}$ $\mathbb{Q}$-linearly independent $\Rightarrow e^{\alpha_1}, \dots, e^{\alpha_2}$ algebraically indep.

for example to the Bessel function $J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{z}{2}\right)^{2n}$

He asked the question: is every E-function a $\overline{\mathbb{Q}}$-polynomial in

$$ {}_pF_q\left(\begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix} \middle| \overset{\overline{\mathbb{Q}}}{\lambda} \cdot z^{q-p}\right) \; ? $$

$\boxed{p<q}$

True for diff. order $\leq 2$ (Gorelov, 2004)

<u>Theorem</u> (with Peter Jossen) The answer is no, starting from differential order 3.

Take $f \in \overline{\mathbb{Q}}[z]$ monic of degree 4

exponential period functions

$$ \int_{-\infty}^{+\infty} e^{-zf(x)}\,dx = \Gamma\left(\tfrac{1}{4}\right) z^{-1/4} \boxed{E_0(z)} + \Gamma\left(-\tfrac{1}{4}\right) z^{1/4} \boxed{E_1(z)} $$

and if the critical values of $f$ do not form a equilateral triangle $\triangle$ nor do they lie on a line, then $E_0$ and $E_1$ are <u>not</u> hypergeometric.

The main theorem answers a variant of Siegel's question for G-functions raised by Fischler and Rivoal.

Dreyfus - Rivoal:

## Some ideas of the proof

Step 1: reformulate the problem in terms of categories of $\bar{\mathbb{Q}}(z)\langle \frac{d}{dz} \rangle$-modules (= generic fibers of $\mathcal{D}_{\mathbb{P}^1}$-modules)

$$\mathbb{H} \subset \mathbb{G}$$

Picard-Fuchs differential operators

$\boxed{\text{Tannakian category}}$

Tannakian subcategory generated by:

* connections with finite monodromy
* $\pi_{2*}\pi_1^*\mathcal{H}$ ← hypergeometric differential operator

$$\begin{tikzcd} & C \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

$$\prod_{i=1}^{p}(\theta + b_i - 1) - z\prod_{j=1}^{p}(\theta + a_j)$$

$b_p = 1$

Reason: every function $\mu(z)\, {}_pF_{p-1}\left(\begin{matrix} a_1 \dots a_p \\ b_1 \dots b_{p-1}\end{matrix} \middle| \lambda(z)\right)$ is a solution of an object of $\mathbb{H}$

Goal: find object of $\mathbb{G}$ of "dimension" 2 which does not lie in $\mathbb{H}$.

(Beukers–Heckman)

Step 2: use differential Galois theory to prove that if an object of $\mathcal{G}$ of dimension 2 with differential Galois group $SL_2$ belongs to $\mathbb{H}$, then it is generated by a single $\pi_{2*}\pi_1^*\mathcal{H}$ with $\mathcal{H}$ hypergeometric differential operator of order 2.

Step 3: let $\mathbb{H}\begin{pmatrix} a_1 & a_2 \\ b_1 & \end{pmatrix}$ be an irreducible hypergeometric local system of rank 2, seen as $\rho : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \longrightarrow GL_2(\mathbb{C})$

Consider the adjoint trace field, i.e. the field $\mathbb{Q}(\mathrm{tr}\,\mathrm{ad}\rho(\gamma))$ generated by the traces of $\mathrm{ad}\rho : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \longrightarrow GL(\mathfrak{sl}_2)$

Claim: This adjoint trace field is $\boxed{\text{abelian}}$ ($\subset$ cyclotomic field) uses Katz's rigidity results

+ invariance under passing to finite index subgroups (Maclachlan–Reid)

Moreover, it cannot be $\mathbb{Q}(\sqrt{D})$ D square free $\geq 7$

trace field = field of definition local whereas cyclotomic $\mathbb{C}^{\{\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \,:\, \sigma\mathbb{H} \simeq \mathbb{H}\}}$

Step 4: construct an object of dimension 2 of $\mathcal{G}$ with diff Galois group $SL_2$ and $\boxed{\text{non-abelian}}$ trace field.

Several ways to do that: (Shimura curves)

$F$ totally real number field

central simple $F$-algebra of dim 4

$B$ quaternion algebra over $F$ $\quad B \otimes_{\mathbb{Q}} F \simeq M_2(F)$ split

$\mathcal{O}$ order in $B$ $\quad Sh_{B,\mathcal{O}}(\mathbb{C}) = \Gamma \backslash \mathfrak{h}$ $\quad \Gamma =$ image of $\mathcal{O}_1^\times$ in $PSL_2(\mathbb{R})$

$\leadsto$ Shimura curve $Sh_{B,\mathcal{O}} \supset \underbrace{Sh^\circ_{B,\mathcal{O}}}_{}$ complement of elliptic points

Assume $Sh^\circ_{B,\mathcal{O}}$ is non-trivial and $F$ non-abelian. Then

$$\rho : \pi_1(Sh^\circ_{B,\mathcal{O}}) \longrightarrow SL_2(\mathbb{C})$$

induces an object of $\mathcal{G}$ with adjoint trace field $F$.

moduli space of abelian varieties of dimension $2[F:\mathbb{Q}]$ with $\mathcal{O} \hookrightarrow \mathrm{End}(A)$

tables by John Voight

e.g. $F$ cubic of discriminant 148