

ON G-FUNCTIONS OF DIFFERENTIAL ORDER 2

(joint with Josh Lurie and Yichen Qin)

(Siegel 1929)

Definition: A G-function is a formal power series with algebraic coefficients

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \bar{\mathbb{Q}}[[z]]$$

such that:

(1) f is solution of a linear differential equation with polynomial coefficients:

$$L \in \bar{\mathbb{Q}}\langle z, \frac{d}{dz} \rangle \quad L \cdot f = 0 \quad \Rightarrow \begin{matrix} \text{coefficients} \\ \text{in a} \\ \text{number} \\ \text{field} \end{matrix}$$

(differential order = minimum degree of such an L)

(2) growth condition: $C > 0$

$$\max_{\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| < C^n$$

$$\text{den}(a_1, \dots, a_n) < C^n$$

"common denominator"

radius of convergence
of $\sum \sigma(a_n) z^n$
 $\hookrightarrow > 0$

Examples: * $\bar{Q}[z]$ otherwise radius of convergence is finite

* all $a_n = 1 \implies$ geometric series explains the name G

* algebraic power series $\bar{Q}[z] \cap \overline{Q(z)}$
(Eisenstein 1852) *stinger: globally bounded*

* hypergeometric series

$${}_pF_{p-1} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{p-1} \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_{p-1})_n} \frac{z^n}{n!}$$

$a_i \in \mathbb{Q}, b_j \in \mathbb{Q} - \mathbb{Z}_{\leq 0}$ $(x)_n = x(x+1)\dots(x+n-1)$

e.g. ${}_2F_1 \left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{z}{16} \right)^n$

solution of $z(z-1) \left(\frac{d}{dz} \right)^2 + (2z-1) \frac{d}{dz} + \frac{1}{4}$

$$= \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-zx)}}$$

realises it as a period function associated with the family of elliptic curves $y^2 = x(1-x)(1-zx)$

${}_2F_1(x)$ transcendental for all non-zero $x \in \bar{\mathbb{Q}}$ in $|x| < 1$.

In general,

$$\begin{array}{c}
 X \\
 \pi \downarrow \boxed{\text{proper}} \\
 \text{smooth} \\
 \text{pt} \in S \subset \mathbb{A}^1
 \end{array}$$

vector bundle
with fiber
 $H_{dR}^p(X_s)$

Gauss-Manin
connection
 \downarrow

$$\rightsquigarrow (\mathcal{H}_{dR}^p(X/S), \nabla_{GM})$$

gives rise to a differential operator $L \in \bar{\mathbb{Q}}\langle z, \frac{d}{dz} \rangle$
 after choosing a coordinate
 «Picard-Fuchs»

Theorem (André): Solutions in $\bar{\mathbb{Q}}\langle \tau \rangle$ of
 Picard-Fuchs differential operators are 6-fuchsians.

\Rightarrow period functions $z \mapsto \int_{\sigma} \omega_z$ are \mathbb{C} -linear
 combinations of 6-fuchsians
 \uparrow
 periods

Conjecture (Bombieri - Dworkin)

Need 6 !
 No 6 !

«All 6-fuchsians come from geometry»

Every 6-fuchsian is annihilated by a
 product of factors of Picard-Fuchs operators.

Today

Theorem (with Josh Lurie & Tucker Austin)

There exist (many...) 6-fuchsians of differential
 \hookrightarrow coming from geometry

order 2 that are not $\bar{\mathbb{Q}}$ -polynomial in

$$\mu(z) \cdot \underset{\substack{p \\ \text{algebraic}}}{F} \underset{p-1}{\left(\begin{array}{c|c} a_1 \dots a_p & \lambda(z) \\ \hline b_1 \dots b_{p-1} & \end{array} \right)} \underset{\substack{\text{algebraic} \\ \text{with } \lambda(0)=0}}{\left. \right)$$

Remark: a) 2 is the minimal differential order because every G -function of order 1 is algebraic (follows from André-Chodura-Katz on minimal differential equations)

b) Explicit examples, e.g. any solution in $\bar{\mathbb{Q}}[z]$ to the differential equation

$$P(z) \left(\frac{d}{dz} \right)^2 + \frac{P'(z)}{z} \frac{d}{dz} + \frac{z-10}{18}$$

where $P(z) = (z-1)(z-2)(z-12)$.

minimal number of singularities

Where does this question come from?

Siegel 1929 was mainly interested by

\mathbb{E} -functions $\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ as he wanted to

approximate

generalize the Lindemann-Weierstrass theorem,

$\alpha_1, \dots, \alpha_r \in \bar{\mathbb{Q}}$
 \mathbb{Q} -linearly independent

$\Rightarrow e^{\alpha_1}, \dots, e^{\alpha_r}$
 algebraically indep.

for example to the Bessel function $J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{z}{2}\right)^{2n}$

He asked the question: is every E-function a $\bar{\mathbb{Q}}$ -polynomial in $\bar{\mathbb{Q}}$

$$\uparrow \uparrow \uparrow \left(\begin{array}{c|c} a_1 \dots a_p & \downarrow \\ b_1 \dots b_q & \lambda \cdot z^{q-p} \end{array} \right) ?$$

True for diff. order ≤ 2 (Gorelov, 2004)

$p < q$

Theorem (with Peter Jones) The answer is no, starting from differential order 3.

Take $f \in \bar{\mathbb{Q}}[z]$ monic of degree 4

expected prod functions $\int_{-\infty}^{+\infty} e^{-zf(x)} dx = \Gamma\left(\frac{1}{4}\right) z^{-1/4} \boxed{E_0(z)} + \Gamma\left(-\frac{1}{4}\right) z^{1/4} \boxed{E_1(z)}$

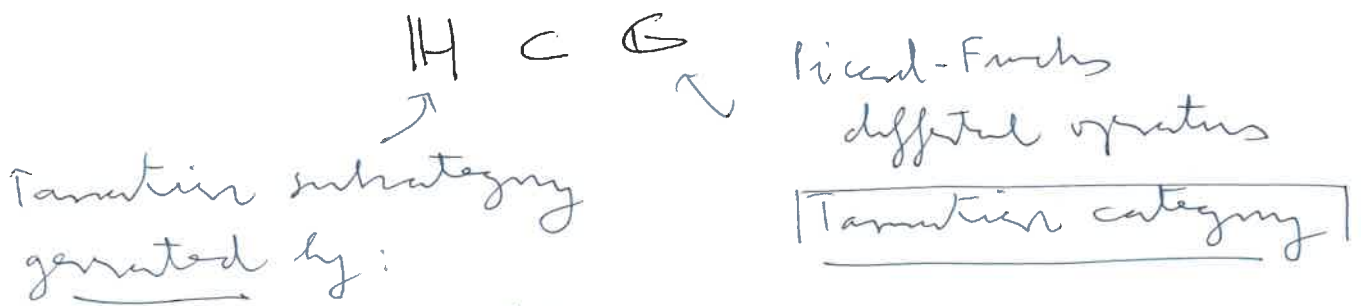
and if the critical values of f do not form an equilateral triangle or do they lie on a line, then E_0 and E_1 are not hypergeometric.

The main theorem answers a variant of Siegel's question for G-functions raised by Fischler and Rivin.

Dreyfus-Rivin:

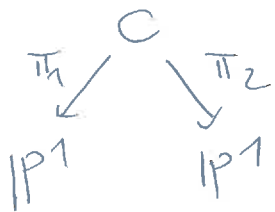
Some ideas of the proof

Step 1: reformulate the problem in terms of categories of $\bar{Q}(z) \langle \frac{d}{dz} \rangle$ -modules (= generic fibres of $\mathcal{D}_{\mathbb{P}^1}$ -modules)



* coverings with finite monodromy

* $\pi_2 + \pi_1^* \mathcal{H} \leftarrow$ hypergeometric differential operator



$$\prod_{i=1}^p (\theta + b_i - 1) - z \prod_{j=1}^p (\theta + a_j)$$

$i=1$ $b_p=1$ $j=1$

Reason: every function $\mu(z) \in \mathbb{F}_{p-1} \left(\begin{matrix} a_1 \dots a_1 \\ b_1 \dots b_{p-1} \end{matrix} \middle| d(z) \right)$

is a solution of an object of \mathbb{H}

Goal: find object of \mathbb{G} of "division" 2 which does not lie in \mathbb{H} .

Step 2: use differential Galois theory to prove that if an object of \mathcal{G} of dimension 2 with differential Galois group SL_2 belongs to \mathcal{H} , then it is generated by a single $\pi_{2*} \pi_1^* \mathcal{H}$ with \mathcal{H} hypergeometric differential operator of order 2.

Step 3: let $\mathcal{H} \begin{pmatrix} a_1 & a_2 \\ b_1 & \end{pmatrix}$ be an irreducible hypergeometric local system of rank 2, seen

as $\rho: \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \rightarrow GL_2(\mathbb{C})$

Consider the adjoint true field, i.e. field $\bar{\mathbb{Q}}(\text{tr ad} \rho(\gamma))$ generated by the traces of $\text{ad} \rho: \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \rightarrow GL(\mathfrak{gl}_2)$

Claim: This adjoint true field is abelian (\mathbb{C} cyclotomic field) uses Katz's rigidity results

+ invariance under passing to finite index subgroups (Mackey-Reid)

Moreover, it cannot be $\mathbb{Q}(\sqrt{D})$ D square free ≥ 7

true field = field of definition local ordinary algebra $\mathbb{C} \left\{ \begin{matrix} 10 \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \\ \sigma \mathcal{H} \simeq \mathcal{H} \end{matrix} \right\}$

Step 4: construct an object of dimension 2
of \mathcal{G} with diff Galois group SL_2 and
non-abelian base field.

Several ways to do that: (Shimura) (CS)

F totally real number field
central simple F -algebra of dim 4

B quaternion algebra over F $B \otimes_F F \cong M_2(F)$
split

\mathcal{O} order in B $Sh_{B,\mathcal{O}}(\mathbb{C}) = \frac{B^\times}{\Gamma}$ $\Gamma =$ image of $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}^{\times}$ in $PSL_2(\mathbb{R})$

non Shimura curve $Sh_{B,\mathcal{O}} \supset \underbrace{Sh_{B,\mathcal{O}}}_{\text{compact of real points}}$

Assume $Sh_{B,\mathcal{O}}^{\circ}$ is normal
and F non-abelian. Then

$$\rho: \pi_1(Sh_{B,\mathcal{O}}^{\circ}) \rightarrow SL_2(\mathbb{C})$$

image is an object of \mathcal{G} with adjoint trace
module space of
abelian varieties of
dimension 2 [$F = \mathbb{R}$]
with
 $\mathcal{O} \hookrightarrow \text{End}(A)$

talks by John Voight
e.g. F cubic of discriminant 148