ON SIMPSON'S STANDARD CONJECTURE FOR UNIPOTENT LOCAL SYSTEMS

(after Tobias Kreutz) (Berlin, 9/1/25)

Starting point: Simpson's 1990 article "Transcendental aspects of the Riemann-Hilbert correspondence"

Riemann-Hilbert:

S smooth complex quasi-projective algebraic variety

$$\begin{tikzcd}
(\mathcal{V}^{an}, \nabla^{an}) & \left\{\begin{array}{c}\text{analytic vector bundles with flat} \\ \text{connection on } S^{an}\end{array}\right\} \arrow[r, leftrightarrow] & \left\{\begin{array}{c}\text{complex local} \\ \text{systems on } S^{an}\end{array}\right\} \\
(\mathcal{V}, \nabla) \arrow[u, mapsto, "\text{Deligne}"] & \left\{\begin{array}{c}\text{algebraic vector bundles} \\ \text{with flat connection} \\ \text{with regular singularities}\end{array}\right\} \arrow[u, "\wr"] \arrow[ur, "RH"'] &
\end{tikzcd}$$

$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes_{\mathcal{O}_{S^{an}}} \Omega^1_{S^{an}}$

$(\mathcal{V}, \nabla) \longmapsto \operatorname{Ker}(\nabla^{an})$

$(\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}}, \mathrm{id} \otimes d) \longleftarrow\!\shortmid \mathbb{V}$

- Now assume $S = X_{\mathbb{C}}^{an}$ for some X defined over the field of algebraic numbers $\overline{\mathbb{Q}} \subseteq \mathbb{C}$.

<u>Question</u> (Simpson) What are the (V, ∇) defined over $\bar{\mathbb{Q}}$ such that $\mathrm{Ker}(\nabla^{an})$ is also defined over $\bar{\mathbb{Q}}$, i.e. corresponds to a representation $\rho : \pi_1(X_{\mathbb{C}}^{an}, x) \longrightarrow GL_n(\bar{\mathbb{Q}}) \subseteq GL_n(\mathbb{C})$?

e.g. $X = \underset{z}{\mathbb{G}_m}$, $V = \mathcal{O}_X$, $\nabla = d + \alpha \frac{dz}{z}$ $\boxed{\alpha \in \bar{\mathbb{Q}}}$

$\mathrm{Ker}(\nabla^{an})$ has monodromy $x \longmapsto e^{2\pi i \alpha}$

If $\left.\begin{array}{l} \alpha \in \bar{\mathbb{Q}} \\ \boxed{e^{2\pi i \alpha} \in \bar{\mathbb{Q}}} \end{array}\right\} \xrightarrow[\text{Gelfond-Schneider}]{} \alpha \in \mathbb{Q}$ finite monodromy

If $d\alpha \in \mathbb{Z}$, then (V, ∇) is a direct factor of $R^0 f_* \underbrace{\mathbb{Q}(\mu_d)}_{\text{need for coeff cat}}$ for $f : \mathbb{G}_m \longrightarrow \mathbb{G}_m$, $z \longmapsto z^d$.

<u>Natural source of examples</u> : $\boxed{\text{Gauß-Manin connections}}$

$$\begin{array}{c} Y \\ f \downarrow \\ X \end{array} \text{ algebraic such that } \begin{array}{c} Y^{an}_{\mathbb{C}} \\ \downarrow \\ X^{an}_{\mathbb{C}} \end{array} \text{ is a locally}$$

trivial fibration, then $(R^i f_* \bar{\mathbb{Q}}, \nabla_{GM})$.
(e.g. f smooth proper)

Also true for "<u>geometric</u>" subquotients (as above)

"Standard" conjecture: any such (V, ∇) is the de Rham realization of a motive over X with coefficient in $\overline{\mathbb{Q}}$

"It is somewhat disingenuous to call this a conjecture ..."

$\boxed{\text{More concrete conjectures:}}$

Abstractness I & II

Existence of a variation of mixed Hodge structure

Galois type conjecture F number field

$X = X_F \otimes_F \overline{\mathbb{Q}}$ Assume (V, ∇) on X_F is such that, for some embedding $\sigma : F \hookrightarrow \mathbb{C}$, $\mathrm{Ker}(\nabla^{an})$ is defined over $\overline{\mathbb{Q}}$. Let ℓ be a prime such that $\rho : \pi_1(X^{an}_{\mathbb{C}}, x) \to GL_n(\overline{\mathbb{Q}})$ is integral at ℓ and $\rho^{et} : \pi_1^{et}(X_{\overline{\mathbb{Q}}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}}_\ell)$ the profinite completion of ρ

Then $\exists\, L/F$ finite extension such that ρ^{et} extends to $\pi_1^{et}(X_L, \bar{x}) \supset \pi_1^{et}(X, \bar{x})$.

$\rightsquigarrow$ arithmetic étale local systems

Remark: This is some kind of "non abelian" analogue of Grothendieck's period conjecture, in the form for X projective every de Rham - Betti class in $H^p_{dRB}(X, \overline{\mathbb{Q}}(q))$ is <u>algebraic</u>

↳ $w \in H^p_{dR}(X)$ such that $\mathrm{comp}_{dRB}(w) \in H^p_{dR}(X, \overline{\mathbb{Q}}(q))$

- It can also be seen as a "<u>bialgebraicity</u>" question for the moduli spaces

$$
\begin{array}{ccc}
\mathcal{M}_{dR}(X)(\mathbb{C}) & \xrightarrow{RH} & \mathcal{M}_B(X)(\mathbb{C}) \\
\cup & & \cup \\
\mathcal{M}_{dR}(X)(\overline{\mathbb{Q}}) & & \mathcal{M}_B(X)(\overline{\mathbb{Q}})
\end{array}
$$

- In his paper, Simpson proves the conjecture for:

(1) rank 1 (V, ∇) [<u>Schneider-Lang</u>]

(2) irreducible ~~local systems~~ (V, ∇) of rank 2 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ [reduce to hypergeometric differential equations]

(3) Absoluteness I & II + mixed Hodge for rank 3 <u>unipotent</u> ~~local systems~~ (V, ∇) on $\mathbb{P}^1 \setminus$ finite set [<u>Baker's theorem</u>: $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}^\times$. If $\log \alpha_1, \dots \log \alpha_n$ are linearly independent over $\mathbb{Q}$, then they are linearly independent over $\overline{\mathbb{Q}}$]

$\boxed{\text{iterated extension of trivial } (\mathcal{O}_X, d)}$

<u>Remark</u>: any unipotent flat connection is a subquotient in the category of connections of the de Rham realisation of a motive regardless of the other side being defined over $\overline{\mathbb{Q}}$.

③

Theorem A (Krentz) $F \subseteq E \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C}$ number fields

$X = \mathbb{P}^1_F \setminus S$ for some $S \subseteq \mathbb{P}^1(F)$ finite, $r \geq 1$

Assume that the de Rham-Betti regulator

$$\mathrm{reg}_j : K_{2j-1}(F) \otimes_{\mathbb{Z}} E \longrightarrow \mathbb{C}/(E + (2\pi i)^j E)$$

is injective for $\boxed{j \leq r-2}$. Then:

every unipotent E-local system $\mathbb{V}$ of $\boxed{\text{index} \leq r}$ on $X^{an}_{\mathbb{C}}$ such that the associated $(\mathcal{V}, \nabla)$ can be defined over E is the de Rham realisation of a mixed Tate motive in

$$MT(X)_{\overline{\mathbb{Q}}}$$

Remark: The assumption holds in the following situations: (1) $r \leq 3$ [Baker's theorem]

(2) F totally real, $r \leq 4$ [Borel: $K_{2j-1}(F) \otimes_{\mathbb{Z}} E = 0$ for even j]

(3) $F = \mathbb{Q} = E$, $r \leq 6$ [irrationality of $\zeta(3)$]

(transcendence would imply $E = \overline{\mathbb{Q}}$)

Theorem B (Krentz) For X as above,

Galois-type conjecture $\Longrightarrow$ standard conjecture

[This is because of a theorem of Soulé on the injectivity of the étale regulator

$$K_{2j-1}(\mathcal{O}_{F,S}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^1(\mathrm{Gal}_{F,S}, \mathbb{Q}_\ell(j))\,]$$

<u>Theorem</u> C (Kings) $F \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C}$ number field

X elliptic curve over F

A $\overline{\mathbb{Q}}$-local system $\mathbb{W}$ with $(\mathcal{V}, \nabla)$ defined over $\overline{\mathbb{Q}}$ is of the form

$$\mathbb{W} = \bigoplus_i \mathbb{W}_i \otimes \chi_i \longleftarrow$$ local system with finite monodromy

where $\mathbb{W}_i$ is the Betti realisation of an object in the tannakian subcategory $\langle \mathcal{H}_{\mathrm{Kum}} \rangle$ of the category of mixed realisations

$$MR(X)_{\overline{\mathbb{Q}}}$$

generated by the Kummer variation

$$\mathcal{H}_{\mathrm{Kum}} = R^1 f_* \overline{\mathbb{Q}} \qquad \begin{tikzcd} Y \arrow[r] \arrow[dr, "f"'] & X^\vee \times X \arrow[d] \\ & X \end{tikzcd}$$

$Y =$ Poincaré line bundle $\setminus \{$zero section$\}$

fiber at $x = H^1\left(\begin{array}{c}\text{semi abelian variety extension of } X^\vee \\ \text{by } \mathbb{G}_m \text{ corresponding to } x\end{array}\right).$

Some ideas of proof (1) $\boxed{\text{Mixed Tate motives}}$

F number field $X = \mathrm{Spec}\, F$ or $\mathbb{P}^1_F \setminus S$

Inside Voevodsky's category $DM(X)$ we can look at the triangulated subcategory $DMT(X)$ generated by $\mathbb{Q}_X(n)$ for $n \in \mathbb{Z}$.

Theorem (Borel, Levine...) $DMT(X) = D(MT(X))$ where $MT(X)$ is a $\mathbb{Q}$-linear Tannakian category

$X = \mathrm{Spec}\, F$

- The Tannakian fundamental group $\mathcal{G}_F^{MT}$ of $MT(F)$ is a semi-direct product $\mathcal{U}_F^{MT} \rtimes \mathbb{G}_m$ where $\mathcal{U}_F^{MT}$ is pro-unipotent and its Lie algebra $\mathfrak{u}_F^{MT}$ is the free pro-nilpotent graded Lie algebra on

$$\bigoplus_{n \geq 1} \mathrm{Ext}^1_{MT(F)}(\mathbb{Q}(0), \mathbb{Q}(n))^\vee .$$

(fiber functor: $\omega(M) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathbb{Q}(n), gr^W_{-2n} M)$)

$MT(X)_E$ $(M, E \to \mathrm{End}_{MT(X)}(M))$ extension of scalars

(2) $\boxed{\text{Motivic fundamental group (Deligne)}}$

- $\pi_1(X^{an}_{\mathbb{C}}, x)$ has prounipotent completion

$$\pi^{un}_{1,B}(X, x) = \mathrm{Spec}\Big(\overset{\text{dual}}{\varinjlim_n}\ \mathbb{Q}[\pi_1(X^{an}_{\mathbb{C}}, x)]/I^n\Big)$$

I: $\boxed{\text{augmentation ideal}}$

$$= \varprojlim_r \pi^{un, r}_{1B}(X, x)$$

quotient by $\pi_1^{un,(r)}$ in the lower central series

$\Gamma \supset [\Gamma, \Gamma] \supset [[\Gamma, \Gamma], \Gamma] \supset \dots$

It's the tannakian fundamental group of the category of unipotent local systems on X^{an}

- $\pi_{1dR}^{un}(X,x) = \varprojlim_{n} \pi_{1dR}^{un,n}(X,x)$

  tannakian fundamental group of the category of unipotent flat connections on X

- Deligne-Goncharov: The Lie algebras $Lie(\pi_{1B}^{un,n})$ and $Lie(\pi_{1dR}^{un,n})$ are the Betti and de Rham realisations of an object in $MT(F)$, i.e. are acted upon by g_F^{MT}.

- Esnault-Levine: There is a split exact sequence

$$1 \to \pi_{1dR}^{un}(X,x) \longrightarrow g_X^{MT} \longrightarrow g_F^{MT} \longrightarrow 1$$

In particular, $MT(X)_E$ is the category of pairs (M,ρ) of $M \in MT(F)_E$

$$\rho: \pi_{1dR}^{un}(X,x)_E \longrightarrow GL(\omega_{dR}(M))$$

such that the associated Lie algebra representation

$$Lie(\pi_{1dR}^{un}(X,x)_E) \to End(\omega_{dR}(M))$$

is equivariant for the action of $g_{F,E}^{MT}$.

(3) de Rham–Betti objects. There is a realisation functor

$$\omega_{dRB} : MT(F)_E \longrightarrow \mathcal{C}_{dRB, E, E}$$

$$M \longmapsto \left(\omega_{dR}(M) \otimes E,\ \omega_B(M) \otimes E,\ \underset{\text{comparison isomorphism}}{} \right)$$

It is fully faithful and its image is stable under subquotients $\Longleftarrow$

$$\mathrm{Ext}^1_{MT(F)_E}(\mathbb{Q}(0), E(j)) \longrightarrow \mathrm{Ext}_{\mathcal{C}_{dR,B,E,E}}(E(0), E(j))$$

$$\mathrm{reg}_j : K_{2j-1}(F) \otimes_{\mathbb{Z}} E \longrightarrow \mathbb{C} / (E + (2\pi i)^j E)$$

is injective. If M has weights $\in [-2(m+1), -2]$ and reg_j is injective for $1 \leq j \leq m$, then

$$\boxed{\mathfrak{g}^{MT}_{F,E}(M) = \mathfrak{g}^{dRB}_{E,E}(M)}$$

the quotients of Tannakian groups associated with $\langle M \rangle^{\otimes}$

- Back to X, $\mathcal{C}^{un}_{dRB, X, E}$ is the category of $(\mathbb{V}, (\mathcal{V}, \nabla), c)$
  - $\mathbb{V}$ unipotent E-local system
  - $(\mathcal{V}, \nabla)$ unipotent flat connection in X_E
  - $c : \mathcal{V}^{an}_{\mathbb{C}} \simeq \mathbb{V} \otimes_E \mathcal{O}^{an}_{X_{\mathbb{C}}}$

In the same spirit as before, $\mathcal{C}^{un}_{dRB, X, E}$ is equivalent to the category of pairs of:

- $V = (V_B, V_{dR}, c)$ in $\mathcal{C}_{dRB, E, E}$
- $\rho_{dR} : \pi^{un}_{1dR}(X, x)_E \longrightarrow GL(V_{dR})$ such that the Lie representation $\mathrm{Lie}(\pi^{un}_{1dR}(X, x)_E) \to \mathrm{End}(V_{dR})$ is equivariant for the action of $\mathfrak{g}^{dRB}_{E, E}$.

We are given a unipotent E-local system $\mathbb{V}$ of index $\leq r$ such that $(\mathcal{V}, \nabla)$ is defined over E, i.e. a representation

$$\rho_{dR} : \mathrm{Lie}(\pi^{un, r}_{1 dR}(X, x)_E) \longrightarrow \mathrm{End}(V_{dR})$$

and we need to construct $\alpha : \mathfrak{g}^{MT}_{F, E} \to GL(V_{dR, \overline{\mathbb{Q}}})$ such that $\rho_{dR, \overline{\mathbb{Q}}}$ is equivariant for α.

Input from transcendence theory (assumption on the equidistr)

$$\mathfrak{g}^{MT}_{F, E}(\mathrm{Lie}\, \pi^{un, r}_1) = \mathfrak{g}^{dRB}_{E, E}(\mathrm{Lie}\, \pi^{un, r}_1)$$

- Construction in 2 steps

$$C \subseteq N \subseteq GL(V_{dR})$$

centralizer normaliser of the image of ρ_{dR} $\underset{\shortparallel}{h_{dR}}$

(i) construct $\bar{\alpha} : \mathfrak{g}^{MT}_{F,E} \longrightarrow N/C$

such that ρ_{dR} is equivariant for the action

(see $N/C \hookrightarrow GL(V_{dR})$). This is simply

$$\begin{tikzcd}
\mathfrak{g}^{MT}_{F,E} \arrow[d] & \mathfrak{g}^{dRB}_{E,E} \arrow[r] \arrow[d] \arrow[rd] & N \subseteq GL(V_{dR}) \arrow[d] \\
\mathfrak{g}^{MT}_{F,E}(\ker) = & \mathfrak{g}^{dRB}_{E,E}(\ker \pi_1^{un,1}) \arrow[r] & N/C
\end{tikzcd}$$

(ii) lift $\bar{\alpha}$ to $\alpha : \mathfrak{g}^{MT}_{F,\bar{\mathbb{Q}}} \longrightarrow N_{\bar{\mathbb{Q}}}$

using the structure of the group $\mathfrak{g}^{MT}_{F}$ and representation theory of $\mathfrak{S}_m$ $\square$

Comments for elliptic curves:

$\pi_1(X^{an}_{\mathbb{C}})$ is abelian, and hence $\overline{\mathrm{Im}\,\rho_B}$ is of the form $\mathbb{G}_a^n \times T$. The semi-simplification is $\pi_1(X^{an}_{\mathbb{C}}) \longrightarrow T$, and satisfies the assumptions of the conjecture, so it is a direct sum of finite characters

$$\mathbb{W} = \bigoplus \mathbb{W}_i \otimes \chi_i$$

↳ unipotent of index 2

In this case, π_1^{un} is the motive $h_1(X)$ and the input from transcendence theory (Wüstholz's analytic subgroup theorem + Chudnovsky's theorem) is that $g^{dRB}(\pi_1^{un}) = g^{MR}(\pi_1^{un})$.