Euler's constant and exponential motives

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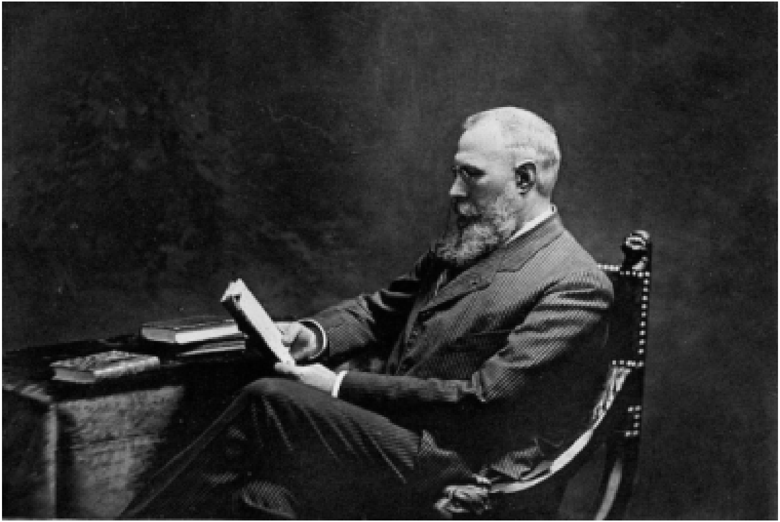
A sealed envelope

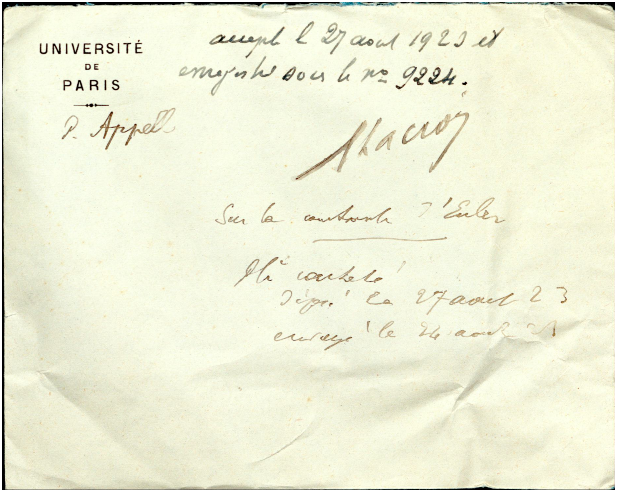
March 21, 2024, 19h31

The attached manuscript is a *pli cacheté* deposited at the French Academy of Sciences by Paul Appell in 1923. In accordance with the rules, it has waited a hundred years and only recently been opened. It contains the proof that Euler's constant is algebraic over the field $\mathbb{Q}(\pi, \log 2)$. You are a better judge than I am, but it seems to me, if the proof is correct, that it's a nice result that doesn't seem to be known. However, the fact that Appell didn't publish it leaves some doubt...

Paul Appell (1855–1930)



Pli cacheté 9224 “Sur la constante C d’Euler”


UNIVERSITÉ DE PARIS

P. Appell

accepté le 27 août 1923 et enregistré sous le n° 9224.

Sur la constante d'Euler

Pli cacheté déposé le 27 août 23 envoyé le 26 août 23

" "	"	9223	MM^rs Edouard et Georges Urbain pas envoyé d'ac. rec Le pli cacheté fut remis à Jules Delaunay par un inconnu qui n'a donné aucune adresse		- ouvert à la demande de Mme S. Urbain-Bost le 31 mai 2001, par la Commission des P.C. Classé 20 août 1923
27 août	"	9224	M. Appell 5 rue de la Sorbonne Paris V^e	27/8/23	

C 1.

Sur la constante d'Euler

par P. Appell

La constante C d'Euler

$$C = \lim \left\{ 1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n} - \log n \right\}$$

joue un rôle important en analyse, (Voir par exemple J. Bertrand, Calcul intégral, intégrales Eulériennes). Je me propose ici de donner quelques propriétés de C, et surtout de montrer que C peut être exprimée algébriquement à l'aide de π et de log…

Euler's constant

$$\underline{\underline{C}} = \lim \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$$

plays an important role in Analysis. (See, for example, J. Bertrand, *Calcul intégral, intégrales Eulériennes*). My goal here is to give some properties of $\underline{\underline{C}}$, **and mainly to prove that** $\underline{\underline{C}}$ **can be expressed algebraically with the help of** π **and** $\log 2$**.**

Appell's computations rely on the fact that $-C$ is equal to the derivative of the gamma function

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$$

at $z = 1$. He uses the notation

$$\Gamma'(1) = \lambda = -C$$
$$\Gamma'(1/2) = \mu$$

and proves the equality

$$\lambda = \frac{\mu}{\sqrt{\pi}} + 2\log 2$$

by taking the logarithmic derivative of

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = \Gamma(2z) 2\sqrt{\pi} 2^{-2z}.$$

Rappel de formules connues

$$\Gamma_z = \int_0^\infty e^{-x} x^{z-1} dx$$

$$\Gamma'_z = \int_0^\infty e^{-x} x^{z-1} \log x \, dx$$ (les log. étant népériens)

$$\Gamma''_z = \int_0^\infty e^{-x} x^{z-1} (\log x)^2 dx$$

Posons $\Gamma'(1) = \lambda = -C = \int_0^\infty e^{-x} \log x \, dx$

$$\Gamma'\left(\frac{1}{2}\right) = \mu = \int_0^\infty \frac{e^{-x} \log x}{\sqrt{x}} dx$$

Remplaçant x par x^2.

$$\Gamma'(1) = \lambda = 4\int_0^\infty e^{-x^2} \log x \cdot x \, dx$$

4) La formule connue

$$\Gamma z\, \Gamma\left(z+\frac{1}{2}\right) = \Gamma(2z)\, 2\sqrt{\pi}\, 2^{-2z}$$

donne en dérivant, après avoir pris les logarithmes

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'\left(z+\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)} = 2\frac{\Gamma'(2z)}{\Gamma(2z)} - 2\log 2$$

$$\frac{\Gamma''(z)}{\Gamma(z)} - \left[\frac{\Gamma'(z)}{\Gamma(z)}\right]^2 + \frac{\Gamma''\left(z+\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)} - \left[\frac{\Gamma'\left(z+\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)}\right]^2$$

$$= 4\left\{\frac{\Gamma''(2z)}{\Gamma(2z)} - \left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right)^2\right\}$$

Et en faisant $z = \frac{1}{2}$, $z+\frac{1}{2} = 1$, $2z = 1$

$$\frac{\mu}{\sqrt{\pi}} + \lambda = 2\lambda - 2\log 2$$

$$\frac{\Gamma''\left(\frac{1}{2}\right)}{\sqrt{\pi}} - \frac{\mu^2}{\pi} + \frac{\Gamma''(1)}{1} - \frac{\lambda^2}{1} = 4\left[\Gamma''(1) - \lambda^2\right],$$

d'où

(2) $\frac{\mu}{\sqrt{\pi}} - \lambda + 2\log 2 = 0$

End of the letter

Then (9) gives

$$\pi\left[4(\log 2)^2 - \frac{\pi^2}{6}\right] = \mu^2(1-\sqrt{\pi}) - \frac{\pi^3\sqrt{\pi}}{2} + 8\pi\log 2\big(\frac{\mu}{\sqrt{\pi}} + 2\log 2\big)$$

equations expressing μ.
We have computed λ or $-C$ by (2).

Alors j'ai donc

$$\pi\left[4(\lg 2)^2 - \frac{\pi^2}{6}\right]\mu_1$$

$$= \mu^2\left(1-\sqrt{\pi_1}\right) - \frac{\pi^3\sqrt{\pi_1}}{2}$$

$$+ 8i\lg 2\left(\frac{\mu}{\sqrt{\pi_1}} + i\lg 2\right)$$

équation donnant μ

On a circuit $\int$ on $- C$

$\mu_2(2)$.

P. Appell

This equality doesn't hold...

$$\pi\left(4(\log 2)^2 - \frac{\pi^2}{6}\right) + \pi(\gamma + \log 4)^2(\sqrt{\pi} - 1) + \frac{\pi^3\sqrt{\pi}}{2} + 8\pi\gamma\log 2 = 0$$

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In [1]: FVSA=pi*(4*log(2)**2-pi**2/6)+pi*(euler_gamma+log(4))**2*(sqrt(pi)-1)+pi**3*sqrt(pi)/2+8*pi*log(2)*euler_gamma

In [3]: gp(FVSA)
Out[3]: 47.759896108845991128993466331306984997

In [4]: pari(FVSA)
Out[4]: 47.7598961088460
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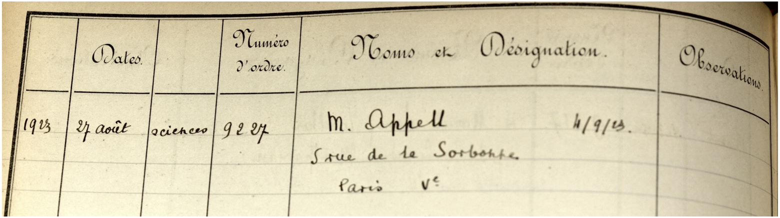
Le pli cacheté 9227

Before leaving, I turned the page of the register book...



A complement to the previous sealed letter, sent a few days later, has not been opened yet!

UNIVERSITÉ
DE
PARIS

A joindre au pli
cacheté sur le
[illegible] d'[illegible]
[illegible]

Accepté le 27 août 1923
Enregistré sous le n° 9227

Em Picard

CABINET

DU

VICE-RECTEUR

Contrexéville 25 août

Mon cher [illegible]

Voici un 2e article à joindre à celui que je vous ai envoyé il y a 3 jours.

Merci et bien à vous

Gréard

Contrexéville (Vosges)
Hotel Cosmopolitain

Cabinet du Recteur

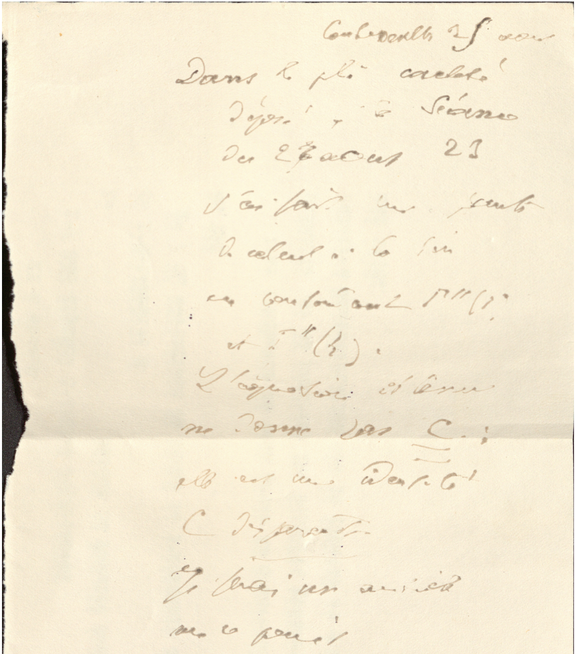
~~Paris~~, le 23 avril 1922

Mon cher Gauja

Voici un pli cacheté (sur la constante d'Euler) que je dépose à l'Académie des Sciences. Vous pourrez m'envoyer le reçu à partir du 29 au Klingenthal par Ottrott (Bas Rhin)

Recevez, cher Mr Gauja, avec mes remerciements [illegible]

After the exceptional opening...

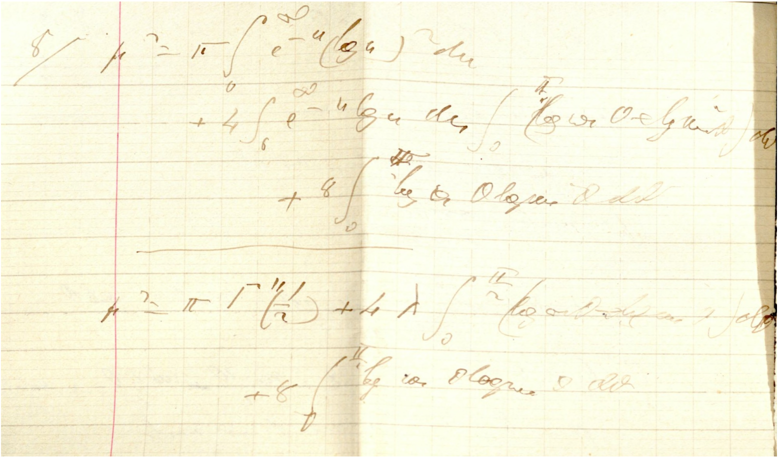


[illegible] 25 août

Dans le pli cacheté
déposé à la Séance
du 2[illegible] août 23
j'écrivais une partie
du calcul de la fin
en confondant [illegible]
et [illegible].
L'expression obtenue
ne donne pas C ;
elle est une identité
C disparait.
[illegible]
[illegible]

In the sealed letter
deposited at the meeting
of August 27,
I made a calculation error
at the end
by confusing $\Gamma''(1)$
with $\Gamma''(1/2)$. The equation
doesn't give C:
it is an identity.
C disappears.
I will write an article
on this point.

The origin of the mistake



The origin of the mistake

In passing from equation

$$\mu^2 = \pi \int_0^\infty e^{-u} (\log u)^2 du + 4 \int_0^\infty e^{-u} \log u du \int_0^{\frac{\pi}{2}} (\log \cos\theta + \log \sin\theta) d\theta + 8 \int_0^{\frac{\pi}{2}} \log\cos\theta \log\sin\theta d\theta$$

to equation

$$\mu^2 = \pi \Gamma''(\frac{1}{2}) + 4\lambda \int_0^{\frac{\pi}{2}} (\log\cos\theta + \log\sin\theta) d\theta + 8 \int_0^{\frac{\pi}{2}} \log\cos\theta \log\sin\theta d\theta$$

The origin of the mistake

When we replace $\Gamma''(1/2)$ with its correct value $\Gamma''(1)$ and compute the integrals, we find

$$\mu^2 = \pi\gamma^2 + 8\pi\gamma \log 2 + 4\pi(\log 2)^2,$$

which is simply the square of the identity $\mu = \sqrt{\pi}(-\gamma - 2\log 2)$ from the beginning.

Quelques intégrales définies se rattachant à la constante d'Euler, Acta Mathematica **45** (1925), 287–302

The n-th derivative of the gamma function at $z = 1$ is equal to

$$\Gamma^{(n)}(1) = \int_0^\infty (\log x)^n e^{-x} dx = P_n(\gamma),$$

where $P_n \in \mathbb{Q}\big(\zeta(2), \dots, \zeta(n)\big)[T]$ is a polynomial of degree n satisfying the recurrence relation $P_n' = -nP_{n-1}$. For example,

$$P_1 = -X, \quad P_2 = X^2 + \zeta(2), \quad P_3 = -X^3 + 3\zeta(2)X - 2\zeta(3),$$
$$P_4 = X^4 + 6\zeta(2)X^2 + 8\zeta(3)X + \frac{135}{2}\zeta(4), \dots$$

We also have

$$\Gamma^{(n)}(1/2) = \sqrt{\pi}Q_n(\gamma + \log 4),$$

where Q_n is obtained by replacing $\zeta(k)$ with $(2^k - 1)\zeta(k)$ in the coefficients of P_n.

Exponential motives

One way to think of the theory of motives is as a

Galois theory for periods $\int_\sigma \omega$

or, more geometrically, for higher dimensional algebraic varieties.

Exponential motives aim to be a Galois theory for integrals

$$\int_\sigma e^{-f}\omega$$

or algebraic varieties along with a regular function.

$k \subset \mathbb{C}$ a subfield (e.g. $k = \mathbb{Q}$),
X a smooth algebraic variety over k (e.g. X affine),
$f \colon X \longrightarrow \mathbb{A}^1_k$ a regular function.

Twisted de Rham cohomology

$$\mathrm{H}^n_{\mathrm{dR}}(X, f) = \mathbb{H}^n_{\mathrm{Zar}}(X, \mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \to \cdots)$$

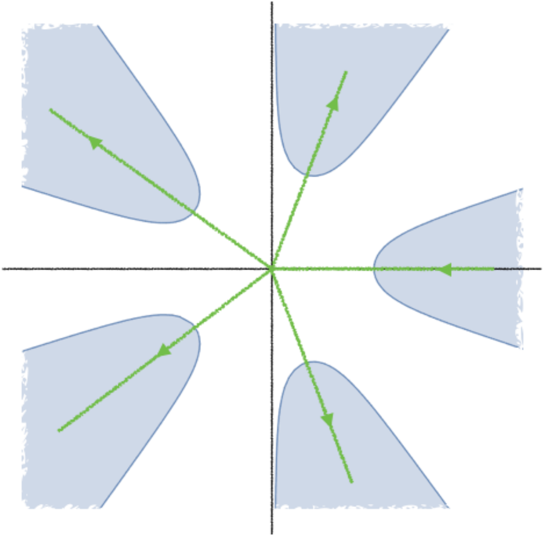
$$\omega \mapsto d\omega - df \wedge \omega$$

Rapid decay homology

$$\mathrm{H}_n^{\mathrm{Rd}}(X, f) = \lim_{t \to +\infty} \mathrm{H}_n^{\mathrm{sing}}(X(\mathbb{C}), \{\mathrm{Re}(f) \geq t\}; \mathbb{Q})$$

$$\sigma = (\sigma_t)_t, \quad \mathrm{Re}(f(x)) \geq t \text{ for } x \in \partial\sigma_t$$

Example: $\mathrm{H}_1^{\mathrm{Rd}}(\mathbb{A}^1, x^5)$



There is an integration pairing

$$\mathrm{H}^n_{\mathrm{dR}}(X, f) \otimes \mathrm{H}_n^{\mathrm{Rd}}(X, f) \longrightarrow \mathbb{C}$$

$$[\omega] \otimes [\sigma] \longmapsto \int_\sigma e^{-f}\omega = \lim_{t \to +\infty} \int_{\sigma_t} e^{-f}\omega$$

which is perfect by results of Bloch–Esnault and Hien–Roucairol.

Variant: for $Y \subset X$ a simple normal crossing divisor,

$$\mathrm{H}^n_{\mathrm{dR}}(X, Y, f) \otimes \mathrm{H}_n^{\mathrm{Rd}}(X, Y, f) \longrightarrow \mathbb{C}.$$

When $k \subset \overline{\mathbb{Q}}$, the coefficients of this pairing are called

exponential periods.

Following ideas of Kontsevich, and Nori, in joint work with Peter Jossen, we built a $\mathbb{Q}$-linear neutral tannakian category $\mathbf{M}^{\exp}(k)$ of exponential motives over k, with fiber functors

$$\mathrm{Vect}_k \xleftarrow{\;\omega_{\mathrm{dR}}\;} \mathbf{M}^{\exp}(k) \xrightarrow{\;\omega_{\mathrm{Rd}}\;} \mathrm{Vect}_{\mathbb{Q}}$$

This category contains objects $\mathrm{H}^n(X, Y, f)$, which are mapped by ω_{dR} and ω_{Rd} to twisted de Rham cohomology and rapid decay cohomology (=linear dual of rapid decay homology).

For each exponential motive M, there exists an algebraic group

$$G_M \subset \mathrm{GL}(\omega_{\mathrm{Rd}}(M))$$

and an equivalence of categories

$$\langle M \rangle^{\otimes} \simeq \mathbf{Rep}_{\mathbb{Q}}(G_M).$$

Conjecture (extension of Grothendieck's period conjecture)

$$\mathrm{trdeg}\, \mathbb{Q}(\textit{exponential periods of } M) = \dim_{\mathbb{Q}} G_M.$$

The category of exponential motives contains Nori's category of classical mixed motives over k as a full subcategory

$$\mathbf{M}(k) \subset \mathbf{M}^{\exp}(k)$$

corresponding to $f = 0$.

However, an object $\mathrm{H}^n(X, f)$ can be classical for non-zero f.

Example 1: The computation of

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy = \pi$$

in polar coordinates leads to an isomorphism

$$\mathrm{H}^2(\mathbb{A}^2, x^2+y^2) \cong \mathrm{H}^1(\{x^2+y^2=1\}).$$

Example 2: From the identity of exponential sums

$$\sum_{(x,z)\in X(\mathbb{F}_p)\times\mathbb{F}_p} e^{\frac{2\pi i}{p} zf(x)} = p|\{f=0\}(\mathbb{F}_p)|$$

we guess that $\mathrm{H}^n(X\times\mathbb{A}^1, zf)$ is always classical.

How to recognize classical motives?

Following Katz and Kontsevich–Soibelman, let

$$\mathbf{Perv}_0 \subset \mathbf{Perv}(\mathbb{A}^1(\mathbb{C}), \mathbb{Q})$$

be the full subcategory of perverse sheaves with rational coefficients A on the complex affine line satisfying $R\pi_* A = 0$

e.g. $E(0) = j_! \mathbb{Q}[1]$ for the inclusion $j \colon \mathbb{G}_m \hookrightarrow \mathbb{A}^1$.

There is a projector (left adjoint to the inclusion)

$$\Pi \colon \mathbf{Perv}(\mathbb{A}^1(\mathbb{C}), \mathbb{Q}) \longrightarrow \mathbf{Perv}_0$$
$$A \longmapsto A * E(0)$$

where $*$ stands for additive convolution.

There is a rather simple criterion in terms of the perverse realisation functor

$$
\begin{aligned}
R_{\mathrm{perv}}\colon \mathbf{M}^{\exp}(k) &\longrightarrow \mathbf{Perv}_0 \\
\mathrm{H}^n(X,f) &\longmapsto \Pi\big({}^{\mathfrak{p}}\mathcal{H}^n(Rf_*\mathbb{Q})\big) \\
&\qquad\qquad = R^{n+1}p_*\mathbb{Q}_{[X\times\mathbb{A}^1,\Gamma_f]}
\end{aligned}
$$

where $p\colon X\times\mathbb{A}^1\to\mathbb{A}^1$ is the projection and Γ_f the graph of f.

Theorem (with Peter Jossen)

An object of $\mathbf{M}^{\exp}(k)$ lies in the essential image of $\mathbf{M}(k)$ if and only if its perverse realisation is trivial (=isomorphic to $E(0)^{\oplus s}$).

Euler's constant

Euler's constant

From the derivative of the gamma function we get the integral representation

$$\begin{aligned}
\gamma &= -\int_0^\infty \log(u) e^{-u} du \\
&= -\int_0^\infty e^{-u} \int_1^u \frac{dv}{v} du \\
&= \int_0^1 e^{-u} \int_u^1 \frac{dv}{v} du - \int_1^\infty e^{-u} \int_1^u \frac{dv}{v} du \\
&\qquad \text{(change of variables } u = xy \text{ and } v = y\text{)} \\
&= \int_0^1 \int_0^1 e^{-xy} dxdy - \int_1^\infty \int_1^\infty e^{-xy} dxdy.
\end{aligned}$$

This leads to consider the exponential motive

$$\mathrm{H}^2(X, Y, f),$$

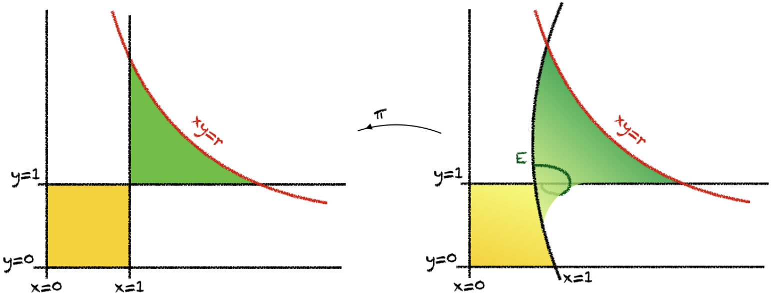
where $X = \mathbb{A}^2_{\mathbb{Q}}$, $Y = \{xy(x-1)(y-1) = 0\}$, and $f = xy$.

It turns out to have dimension 3 and period matrix

$$\begin{pmatrix} \boxed{1} & 0 & \boxed{\gamma} \\ 0 & 1/e & \int_1^\infty e^{-t} dt/t \\ \boxed{0} & 0 & \boxed{2\pi i} \end{pmatrix}$$

The shape of the matrix suggests the existence of a two-dimensional quotient $M(\gamma)$ with period matrix $\left(\begin{smallmatrix} 1 & \gamma \\ 0 & 2\pi i \end{smallmatrix}\right)$: it is the image by the morphism induced by the blow-up map

$$\pi \colon \mathrm{Bl}_{(1,1)} \mathbb{A}^2 \longrightarrow \mathbb{A}^2.$$



The motive $M(\gamma)$ is an extension

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow M(\gamma) \longrightarrow \mathbb{Q}(-1) \longrightarrow 0$$

which is not isomorphic to a classical motive.

Indeed, the perverse realisation of $M(\gamma)$ is a non-trivial extension of $E(0)$ by itself, namely $j_! L[1]$ where L is the rank 2 local system on $\mathbb{G}_m$ with unipotent monodromy $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

(Intuition: the fiber over non-zero $z \in \mathbb{C}^\times$ of $f : (X, Y) \to \mathbb{A}^1$ is the pair $(\mathbb{G}_m, \{1, z\})$ which realises the logarithm).

Using this, one can show that the period conjecture implies

Conjecture

Euler's constant γ is transcendental over the field generated by all classical periods.

Indeed, let M be a classical motive. Then

$$\begin{array}{ccc} \mathbb{G}_a = G_{R_{\mathrm{perv}}(M \oplus M(\gamma))} & \longrightarrow & G_{R_{\mathrm{perv}}(M)} = 0 \\ \downarrow & & \downarrow \\ G_{M \oplus M(\gamma)} & \longrightarrow & G_M \end{array}$$

so $\dim G_{M \oplus M(\gamma)} > \dim G_M$. From the conjecture we deduce

$$\operatorname{trdeg} \mathbb{Q}(\text{periods of } M)(\gamma) > \operatorname{trdeg} \mathbb{Q}(\text{periods of } M).$$

Monodromy factors of E-functions

E-functions

(Siegel, 1929) An *E*-function is a power series

$$E(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[\![z]\!]$$

which solves a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}[z]$ and satisfies the growth condition: there exists $C > 0$ with

$$\max_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| < C^n,$$

$$\mathrm{den}(a_0, \cdots, a_n) < C^n.$$

An example

An example of an E-function is the modified Bessel function

$$I_0(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\Big(\frac{z}{2}\Big)^{2n} = \frac{1}{2\pi i}\int_{|x|=1} e^{-\frac{z}{2}(x+\frac{1}{x})}dx,$$

which is an exponential period function for the family

$$\mathrm{H}^1(\mathbb{G}_m, \tfrac{z}{2}(x+\tfrac{1}{x}))(1).$$

Integrating over the rapid decay cycle $[0, +\infty]$ instead, we get

$$\begin{aligned} K_0(z) &= \frac{1}{2}\int_0^{\infty} e^{-\frac{z}{2}(x+\frac{1}{x})}dx \\ &= -\Big(\log\big(\frac{z}{2}\big) + \gamma\Big) I_0(z) + \underbrace{\sum_{n=1}^{\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n!)^2}\big(\frac{z}{2}\big)^{2n}}_{\text{a new } E\text{-function!}} \end{aligned}$$

Theorem (with Peter Jossen)

Let X be a smooth affine variety of dimension d and $f : X \to \mathbb{A}^1$. For each $[\omega] \in \mathrm{H}^d_{\mathrm{dR}}(X, f)$ and $[\sigma] \in \mathrm{H}_d^{\mathrm{Rd}}(X, f)$, the function

$$\int_\sigma e^{-zf}\omega$$

is a linear combination of

$$z^a(\log z)^b \underbrace{E(z)}_{E\text{-function}} \qquad (a \in \mathbb{Q}, b \in \mathbb{Z}_{\geq 0})$$

with coefficients in the ring

$$\overline{\mathbb{Q}}(\textit{classical periods}, \gamma, \Gamma(r) \text{ for } r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}).$$

A key point of the proof is that the differential equation solved by

$$z \mapsto \int_\sigma e^{-zf}\omega$$

has a regular singularity at 0 with quasi-unipotent monodromy and there is a tensor construction in $M(\gamma)$ and $\mathrm{H}^1(\mathbb{A}^1, x^d)$ whose perverse realisation has the same monodromy.

More on extensions

The computation of extension groups in the category $\mathbf{M}^{\exp}(k)$ is out of reach. We can instead work in

$$\mathrm{DExp}(k) \subset \mathrm{DM}_{\mathrm{gm}}(\mathbb{A}^1),$$

the full triangulated subcategory of objects satisfying $R\pi_* M = 0$ for $\pi \colon \mathbb{A}^1_k \to \mathrm{Spec}(k)$. There are Tate objects

$$\mathbb{Q}(n) = j_! j^* \pi^* \mathbb{1}(n)$$

and extension groups

$$\mathrm{Hom}_{\mathrm{DExp}(k)}(\mathbb{Q}(-n), \mathbb{Q}(0)[j]) \stackrel{?}{=} \mathrm{Ext}^j_{\mathbf{M}^{\exp}(k)}(\mathbb{Q}(-n), \mathbb{Q}(0)).$$

For k a number field, Borel's theorem implies that all extensions groups in $\mathbf{M}^{\exp}(k)$ agree with those in $\mathbf{M}(k)$ except for

$$\mathrm{Ext}^1_{\mathbf{M}^{\exp}(k)}(\mathbb{Q}(-1), \mathbb{Q}(0)) \cong \mathrm{Ext}^1_{\mathbf{M}(k)}(\mathbb{Q}(-1), \mathbb{Q}(0)) \oplus \mathbb{Q},$$

with an extra extension corresponding to $M(\gamma)$, and

$$\mathrm{Ext}^2_{\mathbf{M}^{\exp}(k)}(\mathbb{Q}(-n), \mathbb{Q}(0)) \cong \mathrm{Ext}^1_{\mathbf{M}(k)}(\mathbb{Q}(-n), \mathbb{Q}(-1))$$

with an isomorphism given by cup-product with $M(\gamma)$.

Through Levine's construction, this gives rise to a tannakian category of exponential mixed Tate motives over $\mathbb{Z}$.

Its fundamental group is $U \rtimes \mathbb{G}_m$ with U pro-unipotent, with graded Lie algebra u^{gr} with

generators: $\sigma_1, \sigma_3, \sigma_5, \cdots$

relations: $[\sigma_1, \sigma_3] = [\sigma_1, \sigma_5] = \cdots = 0$

This seems to explain why all exponential iterated integrals that one can write down, such as Appell's

$$\Gamma^{(n)}(1) = \int_0^\infty (\log x)^n e^{-x} dx$$

are always polynomials in Euler's constant γ with coefficients in the algebra generated by multiple zeta value.