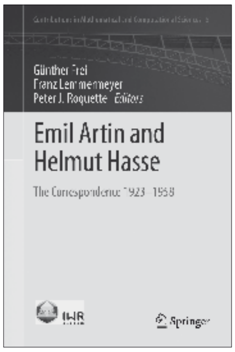

Emil Artin and Helmut Hasse. The Correspondence 1923–1958

Günther Frei, Franz Lemmermeyer and Peter J. Roquette (Eds.)

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If *Parallel Lives* featuring mathematicians were to be written, Emil Artin and Helmut Hasse would certainly deserve one of the chapters. Both born in 1898, they completed their education at around the same time and were soon recognised as rising stars of German number theory. Already, their dissertations contained groundbreaking results: Hasse proved the local-global principle for quadratic forms over the rationals and Artin investigated hyperelliptic curves over finite fields, in particular the analogue of the Riemann hypothesis, which he was the first to consider. They did not stop there: Hasse generalised his results to any number field in his *Habilitation* and Artin developed the theory of L-functions that is nowadays named after him. Within a few years, not only were they the youngest professors in Germany but they also belonged to what Weyl once called the "honours class" of mathematicians who had solved one of Hilbert's problems.

Artin and Hasse probably met for the first time at the annual conference of the German Mathematical Society in 1922. This would be the beginning of a lifelong friendship that was to overcome the dark days of the Third Reich, when Artin was dismissed from university and forced to go into exile, whereas Hasse had – to say the least – ambiguous feelings about Nazism. Although they only co-authored two papers, they maintained an extensive scientific correspondence during the emergence of modern class field theory, sometimes with high exchange frequency. This could come as a surprise, since Hasse was an "ardent letter writer" (p. 10) but Artin was not fond of letter writing: he only wrote in reply and often had to begin his letters "with a long litany of apologies, accusations […] and promises to better myself" (p. 81). He preferred teaching and conversation, at which he excelled. His lectures have been described as "polished diamonds": even when he approached classical topics such as Galois theory or the gamma function, his clarity of presentation was so remarkable that textbooks are still based upon his ideas today.[1]

The correspondence

The volume under review assembles 73 letters and postcards, mainly from Artin to Hasse, written between 1923 and 1958. Most of them had already been published in German, first in a small edition by Günther Frei in 1981, then in collaboration with Peter Roquette and Franz Lemmermeyer in 2008. This second edition was supplemented by voluminous comments, so detailed that it will remain an "introduction to class field theory on a historical basis" written by the leading experts on the subject. In particular, their detective work almost gives the reader the impression that they would be able to reconstruct the contents of Hasse's letters, which seem to be lost. One can only welcome the completion of such an ambitious project, now available to a wider audience thanks to an English translation, which also includes some new letters from 1937 onward, although these are significantly less relevant than the previous ones.

The correspondence opens with an intense exchange of letters, dated July 1923, which led to Artin and Hasse's first joint paper on the so-called second supplementary law for odd prime exponents. Thirty years later, Hasse would melancholically refer to that period as "the old days when we bombed each other rapidly with postcards about reciprocity formulas" (p. 429). Already in these first letters, the influence of Takagi's memoirs on class field theory is manifest. After a decade of "utter scientific solitude", the Japanese mathematician had presented his theorem that all abelian extensions of number fields are class fields during the ICM held in Strasbourg in 1920. Unfortunately, Germans were not allowed to attend and Takagi's contribution went unnoticed. Back in Japan, he had the idea of sending reprints of his work to Siegel, from whom Artin borrowed them.[2] He was deeply impressed by the potential of the techniques and urged Hasse to study them. This would result in Hasse's report on class field theory (*Klassenkörperbericht*), a streamlined presentation of Takagi's work with proofs "reduced to their skeletons", through which a whole new generation of mathematicians encountered class field theory.

Artin *L*-functions

From 1926 to 1930, Artin's L-functions occupy a central place in the correspondence, but the reader will not need to wait that long to witness their birth: as early as in the first letter, Artin announces that he has found "the general L-series attached to Frobenius group characters which accomplish for general fields exactly what the usual L-series accomplish for abelian fields" (p. 50). At the time of writing, the paper *Über eine neue Art von L-Reihen*[3] was already in press. To prove that this new kind of *L*-function was well-behaved, Artin reduced to the abelian case and compared them to Dirichlet *L*-series. For this he needed to show the "general reciprocity law" that there is a canonical isomorphism between the Galois group and the ray class group under which every unramified prime ideal corresponds to its Frobenius substitution.

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[1] To learn more about Artin's teaching, we refer the reader to Zassenhaus's obituary, reprinted on pages 18–24.

[2] One can still find the original reprints at the Sammlung Siegel in the MPIM Library in Bonn. I thank Anke Völzmann for this information.

[3] The title echoes Hecke's *Eine neue Art von Zetafunktion*, where *L*-functions of grössencharacters were defined.

In 1923, Artin was only able to prove the statement for composita of cyclotomic extensions or cyclic extensions of prime degree. However, he formulated it as a theorem instead of a conjecture. From this he could derive what two years later would become Chebotarev's density theorem, a former conjecture of Frobenius (also stated as a theorem). Amusingly enough, Artin could only prove his reciprocity law using Chebotarev's results. All these developments are well documented in the correspondence: on 10 February 1926, Artin asks Hasse if he knows whether Chebotarev's theorem is right,[4] since "if it is correct, we surely will have pocketed the general abelian reciprocity laws" (p. 82). In the following months, Artin and Schreier thoroughly studied the article in a seminar, which led to some simplifications. Once convinced of the validity of the result, it took some time for Artin to work out the details of the proof, but on 17 July 1927 he could inform Hasse that "this semester I gave a two hours course on class field theory and finally proved the "general reciprocity law" in the version that I have given it in my article on L-series" (p. 107).

There followed a second "bombardment of letters" in which the implications of Artin's reciprocity law were discussed. These included Hilbert's conjecture that each ideal of a number field becomes principal in the Hilbert class field (proved by Furtwängler in 1928), the question of whether a tower of successive Hilbert class fields always terminates (finally answered in the negative by Shafarevich and Golod in 1964) and possible generalisations of class field theory to arbitrary Galois extensions. Hasse was particularly interested in how to derive explicit reciprocity formulas for power residues,[5] which was the subject of Part II of his *Klassenkörperbericht*. When Artin achieved his proof, he had almost completed it but decided to rewrite it taking into account the new reciprocity law. "I am sorry that you now have to rewrite the whole report," Artin says, adding right after that: "I believe, however, that it will be worth the trouble" (p. 137). In the years to follow, Hasse's report will come up regularly in the correspondence, until the moment when Artin acknowledges the reception of the galley proofs in August 1930.

The letter dated 18 September 1930 is another remarkable historical document. The presentation of the theory of L-series in Hasse's report "tempted [Artin] to think about […] things that [he had] put aside for such a long time" (p. 268). The 1923 paper still suffered from some defects: first, Artin could only define the local factors of the L-functions at unramified primes, the main difficulty being that Frobenius automorphisms are not uniquely determined in the ramified case. This led to an unnatural detour: to get the whole L-series he needed to reduce to the abelian case, where the complete definition was known, then use the functional equation and a "well known argument due to Hecke" that was not presented in full detail before Hasse's report. Now, prompted by his friend's remarks, Artin could give a uniform definition at all places – essentially the same as that used today – so that "all relations and theorems[6] […] hold exactly right from the start (*von vornherein*)" (p. 273). This also allowed him to define the shape of the functional equation. To do so, he introduces the gamma factor at each infinite prime and gives the definition and the main properties of the conductor, including the "deep" fact that it is an integral ideal. In a subsequent letter he will confess: "I believe that everything depends on guessing. In the case of the conductor as well as that of the functional equation I had to guess everything" (p. 291). So here we have a letter where the local contributions to Artin L-functions, the gamma factors at infinity, the Artin conductor and the Artin root numbers are introduced! If that was not enough, he concludes with the remark that "the new results very much support the conjecture that the $L(\chi, s)$ are entire" (p. 277), a still unproven statement that is considered to be "one of the great challenges of number theory".

An invitation

It should be clear from the above how precious these documents are for the history of number theory in the last century. The Artin-Hasse correspondence contains many other topics that have not been addressed in this short overview. They vary from highly technical pages, where local class field theory emerges, to concrete questions concerning the distribution of the argument of cubic Gauss sums or the existence of unramified icosahedral extensions. Hasse's theory of complex multiplication and his proof of the Riemann hypothesis for elliptic curves are briefly mentioned as well. It only remains for me to invite the reader to accompany the masters on this fascinating journey through class field theory.

Javier Fresán wrote his thesis in arithmetic geometry at the University Paris 13 under the supervision of Christophe Soulé and Jörg Wildeshaus. After a year at the MPIM in Bonn, he is currently an SNF post-doctoral fellow at the ETH in Zürich. His research interests include periods, special values of L-functions and the arithmetic of flat vector bundles. He is also very engaged in science popularisation.

[4] This seems to contradict Emmy Noether's information that Artin was the referee of Chebotarev's article, since it was already published by the Mathematische Annalen when Artin asked Hasse about it.

[5] Artin himself had considered his result as "somewhat strange" since, before him, a reciprocity law was a statement about power residues in number fields containing roots of unity.

[6] As the editors explain, Artin refers here to the functoriality properties of L-series with respect to inflation and induction of representations.