Arithmetic of CM cycles on GSpin-Shimura varieties

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Abstract. These notes are an introduction to the paper "Faltings Heights of Abelian Varieties with Complex Multiplication" [2] by Eyal Goren, Ben Howard, Keerthi Madapusi Pera, and myself. No original results appear.

Contents

1. Introduction

These are the notes for a course given at Monte Verità (Ascona) during the summer school "Motives and Complex Multiplication", organized by ETH in the summer of 2016. The course was intended as an introduction to the paper "Faltings Heights of Abelian Varieties with Complex Multiplication" [2] by Eyal Goren, Ben Howard, Keerthi Madapusi Pera, and myself. No original results appear, but I hope that the simplified presentation will help understanding the main ideas behind the paper.

Let us first specify what the objects mentioned in the title are. The *Shimura varieties* we will consider are those associated to GSpin groups. These will provide the ambient spaces where we will compute the arithmetic intersection between suitable *special cycles*: the Heegner (arithmetic) divisors and certain CM cycles, called *big CM cycles* in [5].

The significance of this computation stems from Kudla's programme that, in this specific case, incarnates in very precise conjectures by Bruinier, Kudla, and Yang [5, Conj. 5.4 & 5.5]. These provide conjectural formulas for the intersection numbers that have been proven in many cases in [2], enough to deduce an averaged form of a conjecture of Colmez [7] leading Tsimerman [18] to provide an unconditional proof of the André–Oort conjecture for Shimura varieties of Hodge type. We refer to [2] for a more systematic introduction to this side of the story.

It is not known that GSpin-Shimura varieties represent a moduli problem, in general. This makes it difficult to construct reasonable integral models over $\mathbb{Z}$ that we need in order to define a meaningful intersection theory. On the other hand, they have a very rich structure, exploited already by Borcherds [3] and Bruinier [4], namely they are endowed with a large supply of Heegner divisors $\{\mathcal{Z}(s)\}_s$ indexed by positive integers $s \in \mathbb{N}$.

Given a big CM cycle $\mathcal{Y}$, the conjecture [5, Conj. 5.4] of Bruinier, Kudla, and Yang predicts, loosely speaking, that the generating series

$$\sum_{s \geq 1} \log(\sharp \mathcal{Y} \cap \mathcal{Z}(s)) q^n,$$

with $q = e^{2\pi\tau}$, is of arithmetic significance, namely it is the non-constant part of the formal Fourier expansion of the diagonal restriction of the derivative of a weight 1 Hilbert modular Eisenstein series. That is to say, the conjecture predicts a way to compute the intersection numbers we are interested in in terms of automorphic forms associated to the reflex field E of $\mathcal{Y}$. We will not discuss this connection further. Here, it suffices to say that the computation of the coefficients on the automorphic side is usually easier and, in our case, has been done essentially by Yang [19] and Kudla–Yang [13] (except at the prime $p = 2$ discussed in [2]). Our goal is to prove that the numbers predicted via automorphic forms indeed match the intersection numbers arising from geometry. In fact, these numbers will decompose on both sides (automorphic and geometric) into factors that geometrically correspond to

- the primes where the intersection is supported;
- for any such prime, the number of points in the intersection;
- for every point of intersection, the length of the Artinian ring defining the intersection.

The key point in [2] that we want to discuss is the computation of the local multiplicities. In order to keep the presentation as simple as possible in these notes we will consider only primes, where the intersection is supported, unramified in E. We refer to the text for a discussion of the issues involved in dealing with more general primes.

The plan of the paper, following closely the plan of the lectures, is the following. We first introduce the group GSpin and the relevant Shimura varieties associated to it both as complex analytic varieties and, especially, as schemes (better, stacks) over $\mathrm{Spec}(\mathbb{Z})$. Here, we rely on recent developments due to Vasiu and Kisin (see [12]) and, in the case of primes of bad reduction, to Madapusi Pera [15].

Second, we introduce the Heegner divisors $\mathcal{Z}(s)$, indexed by positive integers s, and the big CM cycles $\mathcal{Y}$, both complex analytically and arithmetically.

Finally, we will determine the primes of $\mathbb{Z}$ where the intersection $\mathcal{Z}(s)\cap\mathcal{Y}$ is supported and, for almost all such primes, the length of the intersection supported at the given prime, proving that it agrees with the automorphic expectations of [5].

1.1. What is not in this paper

First of all, we will not explain how the automorphic coefficients (of the formal Fourier expansion, constructed from the Hilbert modular Eisenstein series defined above) are computed as the computation involves techniques, typically orbital integrals, that go beyond the scope of these notes.

Another aspect of the theory that will not be considered here is the fact that the divisors $\mathcal{Z}(s)$ are naturally endowed with a metric at infinity, giving rise to arithmetic divisors $\widehat{\mathcal{Z}}(s)$. This is one of the most fascinating parts of the subject and ties up tightly with Borcherds' theory. It is one of the reasons of the richness of the arithmetic of GSpin-Shimura varieties. In conjunction with computations of Bruinier–Kudla–Yang of the contribution at infinity, our computation of the intersection numbers $\mathcal{Z}(s) \cap \mathcal{Y}$ provides the value of the *arithmetic intersection numbers* $\widehat{\mathcal{Z}}(s) \cap \mathcal{Y}$ as predicted by [5, Conj. 5.5]. It is in this context that also the constant term of the formal Fourier expansion mentioned above acquires meaning. This is essentially the special value at 0 of the logarithmic derivative of the complete L-function associated to the quadratic character defining the totally real subfield of the reflex field E of $\mathcal{Y}$. This provides the key tool in proving the averaged version of Colmez's conjecture relating Faltings' height of abelian varieties with CM by $\mathcal{O}_E$ (and varying CM type) with special values of L-functions and their derivatives. We refer to [2] for a more systematic account.

Finally, in the computation of the local multiplicities of the intersections $\mathcal{Z}(s) \cap \mathcal{Y}$ one is naturally led to study the deformation theory of endomorphisms of Lubin–Tate formal groups at a prime p. As mentioned above, we will consider only primes p unramified in E. For the general case, needed for the application to Colmez's conjecture, one has to consider also ramified primes. In this case one uses the theory of Kisin modules, see [2, §2], that is neither discussed nor used in these notes.

1.2. GSpin-Shimura varieties

Let V be a $\mathbb{Q}$-vector space of dimension $n+2$ with $n \geq 0$, and a quadratic form $Q\colon V \to \mathbb{Q}$ which is non degenerate, of signature $(n, 2)$. Consider the associated bilinear form

$$[\cdot,\cdot]\colon V\times V\to\mathbb{Q},\quad [x,y]=Q(x+y)-Q(x)-Q(y).$$

1.2.1. Clifford algebras. To V and Q one associates the Clifford algebra $C(V)=C(V,Q)$. It is a $\mathbb{Q}$-algebra, with an inclusion

$$V\hookrightarrow C(V)$$

as $\mathbb{Q}$-vector spaces satisfying the following universal property: for any $\mathbb{Q}$-algebra R with a $\mathbb{Q}$-linear map $j\colon V\to R$ such that

$$j(v)j(v)=Q(v),$$

there exists a unique homomorphism of $\mathbb{Q}$-algebras

$$C(V)\to R$$

such that the composite with the inclusion $V\subset C(V)$ is j. In particular, for any v and $w\in V$, we have

$$v\cdot w+w\cdot v=[v,w]\in C(V),$$

where $v\cdot w$ (and $w\cdot v$) is the product in $C(V)$.

The construction of the Clifford algebra is quite straightforward. In fact,

$$C(V):=\Big(\bigoplus_{n=0}^{\infty}V^{\otimes m}\Big)/(v\otimes v-Q(v)\,|\,v\in V)$$

is the quotient of the tensor algebra of V by the two-sided ideal generated by the elements $v\otimes v-Q(v)$ for all $v\in V$. As such ideal is generated by elements lying in even degree (in the tensor algebra considered with its natural grading), the $\mathbb{Z}/2\mathbb{Z}$-grading on the tensor algebra (into even and odd tensors) induces a $\mathbb{Z}/2\mathbb{Z}$-grading on the Clifford algebra that correspondingly splits into a direct sum

$$C(V)=C^{+}(V)\oplus C^{-}(V).$$

Note that $C^{+}(V)$ is a $\mathbb{Q}$-subalgebra of $C(V)$ while $C^{-}(V)$ is just a two-sided module for $C^{+}(V)$. Furthermore we have the following formulas:

$$\dim_{\mathbb{Q}}(C(V))=2^{n+2},\quad \dim_{\mathbb{Q}}^{+}(C(V))=\dim_{\mathbb{Q}}(C^{-}(V))=2^{n+1}.$$

If $L\subset V$ is a lattice on which Q is integral-valued, one has variants of the constructions above and one can similarly define the $\mathbb{Z}$-algebras $C(L)$ and $C^{+}(L)$ with the inclusion $L\subset C(L)$.

1.2.2. The group GSpin. Next we construct the algebraic group $\mathrm{GSpin}(V, Q)$ over $\mathbb{Q}$. Given a commutative $\mathbb{Q}$-algebra R, its R-valued points are

$$\mathrm{GSpin}(V)(R) := \{x \in (C^+(V) \otimes_{\mathbb{Q}} R)^* \mid x(V \otimes_{\mathbb{Q}} R)x^{-1} \subset V \otimes_{\mathbb{Q}} R\}.$$

In particular, any $x \in \mathrm{GSpin}(V)(R)$ acts on $V \otimes_{\mathbb{Q}} R$ and, since the equality

$$Q(y) = Q(xyx^{-1})$$

holds for any $y \in V \otimes_{\mathbb{Q}} R$, this action factors through the orthogonal group $\mathrm{O}(V, Q)(R)$. Notice that the units of R form a subgroup of the center of $\mathrm{GSpin}(V)(R)$, in particular they act trivially on $V \otimes_{\mathbb{Q}} R$. Furthermore, one can prove that the action of $\mathrm{GSpin}(V, Q)$ on V factors through the special orthogonal group $\mathrm{SO}(V, Q)$ and one gets an exact sequence of algebraic groups:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V, Q) \longrightarrow 0. \tag{1.1}$$

1.3. Examples of GSpin groups

1.3.1. Example 1: the case $n = 0$. In this case, $V = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$ so that $C^-(V) = V$ and $C^+(V)$ is an algebra of dimension 2. We have

$$C^+(V) = \mathbb{Q} \oplus \mathbb{Q}e_1 \cdot e_2$$

as a $\mathbb{Q}$-vector space. Denote $x = e_1 \cdot e_2$, let $a_1 = Q(e_1)$ and $a_2 = Q(e_2)$. They are negative rational numbers and, if we write $b = [e_1, e_2] \in \mathbb{Q}$, we have

$$x^2 = e_1e_2e_1e_2 = -e_1^2e_2^2 + [e_1, e_2]e_1e_2 = -a_1a_2 + bx,$$

so we have

$$x^2 - bx + a_1a_2 = 0.$$

As an algebra, this gives

$$C^+(V) = \mathbb{Q}[x]/(x^2 - bx + a_1a_2),$$

which is an imaginary quadratic field K, and we see that

$$\mathrm{GSpin}(V) = C^+(V)^\times = \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m.$$

In particular, its base change to $\mathbb{R}$ is the Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{R}}$.

1.3.2. Example 2: the case $n = 1$. Consider the $\mathbb{Q}$-vector space

$$V \subset M_{2\times 2}(\mathbb{Q}) = \{x \in M_{2\times 2} \mid Tr(v) = 0\}.$$

Fix some $N \in \mathbb{N}$ such that $N \geq 1$ and let Q_N be the quadratic form

$$A \mapsto N \cdot \det A.$$

Then

$$\mathrm{GSpin}(V) \cong \mathrm{GL}_2,$$

where GL_2 acts on V by conjugation.

1.3.3. Embedding into symplectic groups. Contrary to $\mathrm{SO}(V,Q)$ the algebraic group $\mathrm{GSpin}(V,Q)$ admits a natural embedding into a symplectic group. Consider the embedding

$$\mathrm{GSpin}(V) \subset C^+(V)^\times \subset \mathrm{GL}(C(V)),$$

where the second map is given by left multiplication of elements of $C^+(V)$ on $C(V)$. One can prove that there exists a symplectic form δ on $C(V)$, not canonical, such that the inclusion factors through the $\mathrm{GSp}(C(V),\delta)$, namely

$$\mathrm{GSpin}(V,Q) \hookrightarrow \mathrm{GSp}(C(V),\delta). \tag{1.2}$$

We refer to [17, Prop. 3, Prop. 4 & Ex. 3] where all possible such δ's are constructed, up to equivalence. It is the existence of these embeddings that makes GSpin-Shimura varieties better behaved than the orthogonal ones. Indeed the former admit an embedding into Siegel modular varieties, i.e., they are Hodge type Shimura varieties, while the latter do not, in general. See the discussion in [8, §3 & 4] for a Hodge theoretic discussion. This is very important for the construction of integral models.

1.3.4. Hermitian symmetric spaces. Fix the algebraic group

$$G := \mathrm{GSpin}(V,Q).$$

As a first step in order to construct a Shimura variety, we provide several descriptions of the Hermitian symmetric space associated to G. We refer to [14, §3.4] for details. Here, we simply provide several realizations that will be used in the sequel:

1. as a complex manifold
$$D_{\mathbb{C}} = \{z \in V_{\mathbb{C}}\backslash\{0\} \mid Q(z)=0, [z,\overline{z}]<0\}/\mathbb{C}^* \subset \mathbb{P}(V_{\mathbb{C}});$$
2. as a Riemannian manifold
$$D_{\mathbb{R}} = \{\text{Negative definite oriented planes } H \subset V_{\mathbb{R}}\};$$
3. using the Deligne torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{R}}$,
$$D = G(\mathbb{R}) \text{ conjugacy class of } h\colon \mathbb{S} \to G_{\mathbb{R}}.$$

Let us explain how we can go back and forth between these incarnations.

Given an $H = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ in $D_{\mathbb{R}}$, we let $z = e_1 + ie_2 \in V_{\mathbb{C}}$ and we take the line $[z]$ to get the realization in $D_{\mathbb{C}}$. Vice versa giving a line $[z] \in D_{\mathbb{C}}$ we get a negative definite oriented plane in $V_{\mathbb{R}}$ by taking the $\mathbb{R}$-span of the real and imaginary parts $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ of z.

To get the realization (3), given an oriented, negative definite plane $H \subset V_{\mathbb{R}}$ we can identify $\mathbb{S} \cong \mathrm{GSpin}\, H$ and simply take $h\colon \mathbb{S} \cong \mathrm{GSpin}\, H \hookrightarrow G_{\mathbb{R}}$ to be the morphism induced by the inclusion $H \subset V_{\mathbb{R}}$.

Example 1. The Hermitian space

$$D_{\mathbb{R}} = \{\text{two possible orientations on } V_{\mathbb{R}} = \mathbb{R}e_1 \oplus \mathbb{R}e_2\}$$

consists of two points.

Example 2. We have $D_{\mathbb{R}} \cong \mathbb{H}^+ \sqcup \mathbb{H}^- \subset \mathbb{C}$ which are the Poincaré upper and lower half planes. The inverse of the map is given as

$$\mathbb{R} \cdot Re \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \oplus \mathbb{R} \cdot \mathrm{im} \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \leftarrow z = x + iy.$$

1.3.5. The dimension of the Hermitian symmetric space. Pick $[z] \in D_{\mathbb{C}}$, then

$$V_{\mathbb{C}} = \mathbb{C}z \oplus (\mathbb{C}z \oplus \mathbb{C}\overline{z})^{\perp} \oplus \mathbb{C}\overline{z}$$

and the tangent space of Q in $\mathbb{P}(V_{\mathbb{C}})$ at $[z]$ can be computed as the Zariski tangent space at $[z]$, namely the set of lines $[z+\delta\epsilon+\gamma\epsilon\overline{z}]$, with $\delta \in (\mathbb{C}z\oplus\mathbb{C}\overline{z})^{\perp}$ and ϵ a formal variable with square $\epsilon^2 = 0$, such that $Q([z + \delta\epsilon + \gamma\epsilon\overline{z}]) = 0$, i.e, if and only if $\gamma = 0$. Thus the tangent space of $D_{\mathbb{C}}$ at $[z]$ is isomorphic to $(\mathbb{C}z \oplus \mathbb{C}\overline{z})^{\perp}$ and $\dim D_{\mathbb{C}} = n$.

1.4. GSpin-Shimura varieties

Given V and Q, the algebraic group $G := \mathrm{GSpin}(V, Q)$ and the Hermitian symmetric space D as in the previous section, define the complex manifold:

$$M_K(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$$

for some compact open K subgroup of the adelic points $G(\mathbb{A}_f)$ of G. For every class $g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$, we let Γ_g be the arithmetic subgroup of $G(\mathbb{Q})$ defined by $\Gamma_g := G(\mathbb{Q}) \cap (gKg^{-1})$. Then, we also have the set theoretic decomposition

$$M_K(\mathbb{C}) = \amalg_{g\in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K}\Gamma_g\backslash D.$$

It is the second description of $M_K(\mathbb{C})$ that endows it with the structure of a complex manifold coming from the one on D; see [14, §2.2].

Given a quadratic lattice $L \subset V$, i.e., a lattice on which Q is integral-valued, one can construct a compact open subgroup K_L by taking

$$K_L = G(\mathbb{A}_f) \cap C^+(\widehat{L})^{\times} \subset C^+(V)^{\times}(\mathbb{A}_f)$$

where $\widehat{L} := L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. We will be especially interested in the case that L is maximal among the integral lattices. This will guarantee the existence of good integral models over $\mathbb{Z}$ for the Shimura variety $M_{K_L}(\mathbb{C})$. If the compact open subgroup is of the type K_L for some lattice L, we simply write $M_L(\mathbb{C})$ for $M_{K_L}(\mathbb{C})$. It follows from [2, Prop. 4.1.1] that, for L maximal, $M_L(\mathbb{C})$ is connected if $n \geq 2$ or the order of the finite group $L^{\vee}/L$ is square free. Looking at the examples again, we have:

Example 1. $M(\mathbb{C})$ consists of finitely many points.

Example 2. We have $V = M_{2\times 2}(\mathbb{Q})^{Tr=0}$, and Q_N makes $\mathrm{GSpin}(V) \cong \mathrm{GL}_2$. Take

$$L := \left\{ \begin{pmatrix} a & \frac{-b}{N} \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

One can check that

$$K_L \cong \Pi_p \widetilde{K}_p$$

where

$$\widetilde{K}_p = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{Z}_p) \mid \gamma \in N\mathbb{Z}_p \right\}.$$

In this case one sees that

$$M_2(\mathbb{C}) \cong Y_0(N)(\mathbb{C})$$

which is the modular curve of level $\Gamma_0(N)$, classifying cyclic isogenies of elliptic curves $\rho\colon E \to E'$ of degree N.

Warning. The case of elliptic curves is misleading as it might appear that $M_L(\mathbb{C})$ could have a natural moduli interpretation as a Shimura variety of PEL type in the sense of [8, §4], i.e., a moduli space of abelian varieties with prescribed polarization, endomorphisms and level structures. If this were the case, one could use the moduli definition to provide integral models. Indeed for small values of n, namely $n \leq 6$, this is the case and one gets instances of well known modular varieties: modular and Shimura curves for $n = 1$, Hilbert modular surfaces and quaternionic variants for $n = 2$, Siegel three-folds and quaternionic variants for $n = 3$, etc. Also for $n = 19$ one has a natural interpretation as the moduli of periods of K3 surfaces, see [9, §6]. On the other hand, if the dimension $n + 2$ of V is large there is no such moduli interpretation of $M_L(\mathbb{C})$ as the moduli space for a PEL type moduli problem. As we will see, this is the source of complications when one attempts to provide integral models for $M_L(\mathbb{C})$.

2. Special Cycles on GSpin-Shimura Varieties

2.1. Extra structures on GSpin-Shimura varieties

Recall the notation. We fixed a vector space V over $\mathbb{Q}$ of dimension $n+2$, and a quadratic form $Q\colon V \to \mathbb{Q}$ of signature $(n, 2)$, with a maximal quadratic lattice $L \subset V$. We let $G = \mathrm{GSpin}(V)$, and then define

$$M_L(\mathbb{C}) = M(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K_L$$

for a particular choice of compact open subgroup K_L associated to L.

We have a natural functor

$$\left\{ \begin{array}{c} \text{Algebraic} \\ \text{Representations of G} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Local Systems of} \\ \mathbb{Q}\text{-vector spaces on } M(\mathbb{C}) \end{array} \right\}$$

given by associating to a representation $G \to GL(W)$ with W a finite dimensional $\mathbb{Q}$-vector space the local system of $\mathbb{Q}$-vector spaces $W_{\mathrm{Betti},\mathbb{Q}}$ over $M(\mathbb{C})$ where

$$W_{\mathrm{Betti},\mathbb{Q}} := G(\mathbb{Q})\backslash(W \times D) \times G(\mathbb{A}_f)/K_L.$$

Note that this gives us a unique locally free $\mathcal{O}_{M_L(\mathbb{C})}$-module with integrable connection $(W_{\mathrm{dR}}, \nabla)$. Namely $W_{\mathrm{dR}} := W_{\mathrm{Betti},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_{M_L(\mathbb{C})}$ with the connection $\nabla = 1 \otimes d$. Then $(W_{\mathrm{dR}}, \nabla)$ is characterized as the vector bundle with integrable connection such that

$$W_{\mathrm{dR}}^{\nabla=0} = W_{\mathrm{Betti},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

One deduces the following two extra properties

a. For any z in the symmetric space D, the map

$$h_z \colon \mathbb{S} \to G_{\mathbb{R}} \to GL(W_{\mathbb{R}})$$

induces a map

$$\mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^* \to \mathrm{GL}(W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{GL}(W_{\mathbb{C}})$$

and the fiber $W_{\mathrm{dR},z}$ at z has a bigrading $\oplus_{p,q} W_{\mathrm{dR},z}^{p,q}$ obtained by the decomposition of $W_{\mathbb{C}}$ according to the decomposition into eigenspaces for the action of $\mathbb{C}^* \times \mathbb{C}^*$.

b. W_{dR} is endowed with a decreasing filtration $\mathrm{Fil}^J(W_{\mathrm{dR}}) \subset W_{\mathrm{dR}}$ by holomorphic sub-bundles of W_{dR}, defined pointwise by

$$\mathrm{Fil}^J(W_{\mathrm{dR},z}) := \oplus_{p \geq J} W_{\mathrm{dR},z}^{p,q}$$

such that the connection ∇ satisfies Griffith's transversality, namely

$$\nabla(\mathrm{Fil}^{J+1}(W_{\mathrm{dR}})) \subset \mathrm{Fil}^J(W_{\mathrm{dR}}) \otimes_{\mathcal{O}_{M_L(\mathbb{C})}} \Omega_{M_L(\mathbb{C})}.$$

A $\mathbb{Q}$-local system on $M_L(\mathbb{C})$ with properties (a) and (b) is called *a variation of $\mathbb{Q}$-Hodge structures*. In particular, we deduce that for any G-representation W the local system $W_{\mathrm{Betti},\mathbb{Q}}$ is a variation of $\mathbb{Q}$-Hodge structures.

2.1.1. The Kuga–Satake construction. Consider the representation W of G given by $W := C(V)$ viewed as a representation of G through the inclusion $G = \mathrm{GSpin}(V, Q) \subset C^+(V)^*$ and the map $C^+(V)^* \subseteq \mathrm{GL}(C(V))$ provided by sending an element $u \in C^+(V)^*$ to the automorphism of $C(V)$ given by *left multiplication* by u.

In this case, for any $z \in D$, we have the following decomposition

$$C(V)_{\mathrm{dR},z} = C(V)_{\mathrm{dR},z}^{(-1,0)} \oplus C(V)_{\mathrm{dR},z}^{(0,-1)}$$

which is equivalent to giving a complex structure on $C(V)_{\mathrm{Betti},\mathbb{Q},z}\otimes\mathbb{R}$. In particular, choosing a maximal quadratic lattice $L\subset V$ so that $M_L(\mathbb{C})=\Gamma\backslash D$ with $\Gamma := G(\mathbb{Q})\cap K_L$ (al least for $n\geq 2$ or $L^\vee/L$ of square free order), this gives a complex structure on the torus $C(V)_{\mathrm{Betti},\mathbb{Q},z}/C(L)$ and via (1.2) a complex abelian variety

$$A_z := C(V)_{\mathrm{Betti},\mathbb{Q},z}\otimes\mathbb{R}/C(L);$$

the choice of polarization in (1.2) will not be relevant for us and is used simply to guarantee that A_z has the structure of an abelian variety, and not simply of a complex torus, called the *Kuga–Satake abelian variety* defined by z. We then get an abelian scheme

$$\pi\colon A\longrightarrow M_L(\mathbb{C})$$

such that

$$C(V)_{\mathrm{Betti},\mathbb{Q}}=\mathrm{R}_1\pi_*\mathbb{Q}$$

as variations of $\mathbb{Q}$-Hodge structures.

Warning. To make sense of this descent we should systematically work with stacks as in [2] as the action of Γ on D is not free in general, i.e., Γ is not a neat subgroup of $G(\mathbb{Q})$ or equivalently K_L is not neat subgroup of $G(\mathbb{A}_f)$ in the terminology of [14, §2.2]; we ignore this issue in the sequel for simplicity.

We also get a variation of $\mathbb{Z}$-Hodge structures

$$C(V)_{\mathrm{Betti}}=\mathrm{R}_1\pi_*\mathbb{Z}.$$

The associated vector bundle with connection $C(V)_{\mathrm{dR}}$ is the relative de Rham homology

$$C(V)_{\mathrm{dR}}=\mathcal{H}_{1,\mathrm{dR}}(A)$$

and the connection discussed above is the so called *Gauss–Manin connection*. The filtration is given by the Hodge filtration

$$0\to\mathrm{R}_1\pi_*(\mathcal{O}_A)^\vee\to\mathcal{H}_{1,\mathrm{dR}}(A)\to\pi_*(\omega_A^1)^\vee\to 0$$

(notice that we are working with $\mathcal{H}_{1,\mathrm{dR}}(A)$ the vector bundle dual to the de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(A)$ that one usually uses). We also have a right action of $C^+(V)$ on $C(V)$, which defines canonically $C^+(V)\subset\mathrm{End}^0(A)$.

Example 1. Recall that $n=0$ and $C^+(L)\subset C^+(V)=K$ is an order in the quadratic imaginary field K. In this case

$$A_z=A_z^+\times A_z^-$$

where A_z^+ is an elliptic curve with complex multiplication by $C^+(L)$ and

$$A_z^-=A_z^+\otimes_{C^+(L)}L.$$

Example 2. In this case $V = M_{2\times 2}(\mathbb{Q})^{Tr=0}$, we have

$$M(\mathbb{C}) \cong Y_0(N)(\mathbb{C}).$$

If $z \in M(\mathbb{C})$ corresponds to the cyclic isogeny $[E_z \to E'_z]$, then $A_z = A_z^+ \times A_z^-$ with

$$A_z^+ = A_z^- = E_z \times E'_z.$$

2.1.2. Another important VHS. Recall that we also have a natural homomorphism

$$G \to \mathrm{SO}(V)$$

given by sending $x \in G$ to the orthogonal transformation

$$V \to V, \quad y \mapsto xyx^{-1}.$$

Consider $\ell\colon W = V \hookrightarrow \mathrm{End}(C(V))$ given by

$$v \mapsto \ell_v := \{\text{left multiplication by } v \text{ on } C(V)\}.$$

Notice that this is a morphism on the category of representations of G, where the action of G on $\mathrm{End}(C(V))$ is defined through its action on $C(V)$ described above.

Now, we get as before a variation of $\mathbb{Q}$-Hodge structures $V_{\mathrm{Betti},\mathbb{Q}}$. Let $\pi\colon A \to M_L(\mathbb{C})$ be the Kuga–Satake abelian scheme. Using the description $M_L(\mathbb{C}) = \Gamma\backslash D$, the lattice L and the inclusion $L \subseteq \mathrm{End}(C(L))$ given by left multiplication, define a variation of $\mathbb{Z}$-Hodge structures $\mathbb{V}_{\mathrm{Betti}}$ and a morphism of $\mathbb{Z}$-Hodge structures

$$\ell_{\mathrm{Betti}}\colon \mathbb{V}_{\mathrm{Betti}} \hookrightarrow \mathrm{End}(C(V)_{\mathrm{Betti}}) = \mathrm{End}(\mathrm{R}_1\pi_*\mathbb{Z}). \tag{2.1}$$

Consider the inclusion $L \subset C(L)$. For every $v \in L$ we have the equality $[v, v] = 2Q(v) = 2v \cdot v$, where $[v, v]$ is the value of the bilinear form $[-,-]$ on L and the multiplication $v \cdot v$ is the multiplication in $C(V)$. This implies that the bilinear form $[\cdot,\cdot]$ induces a pairing $[\cdot,\cdot]$ on $\mathbb{V}_{\mathrm{Betti}}$ as variations of $\mathbb{Z}$-Hodge structures. It has the property that for every local section $\alpha \in \mathbb{V}_{\mathrm{Betti}}$ we have $[\alpha, \alpha] = 2\ell_{\mathrm{Betti}}(\alpha) \circ \ell_{\mathrm{Betti}}(\alpha)$ where $\ell_{\mathrm{Betti}}(\alpha) \circ \ell_{\mathrm{Betti}}(\alpha)$ stands for the composition of elements in $\mathrm{End}(\mathrm{R}_1\pi_*\mathbb{Z})$ and $[\alpha, \alpha]$ is an integer that we view in $\mathrm{End}(C(V)_{\mathrm{Betti}})$ via the natural embedding $\mathbb{Z} \subset \mathrm{End}(C(V)_{\mathrm{Betti}})$.

Also, we have a morphism of vector bundles with connections

$$\ell_{\mathrm{dR}}\colon V_{\mathrm{dR}} \hookrightarrow \mathrm{End}(C(V)_{\mathrm{dR}}) = \mathrm{End}(\mathcal{H}_{1,\mathrm{dR}}(A)). \tag{2.2}$$

We now compute the fibers.

For any $z \in D_{\mathbb{C}}$, where $D_{\mathbb{C}}$ is the incarnation of the symmetric space as the isotropic lines in $V_{\mathbb{C}}$, the morphism h_z defines a decomposition

$$V_{\mathbb{C}} = \mathbb{C}_z \oplus (\mathbb{C}_z \oplus \mathbb{C}_{\overline{z}})^{\perp} \oplus \mathbb{C}_{\overline{z}} \subset \mathrm{End}(H_{1,\mathrm{dR}}(A_z)).$$

The filtration is given by

$$\begin{aligned}\mathrm{Fil}^1(\mathbb{V}_{\mathrm{dR},\mathbb{Z}}) &= \mathbb{C}_z,\\ \mathrm{Fil}^0(\mathbb{V}_{\mathrm{dR},\mathbb{Z}}) &= \mathbb{C}_z \oplus (\mathbb{C}_z \oplus \mathbb{C}_{\overline{z}})^{\perp},\\ \mathrm{Fil}^{-1}(\mathbb{V}_{\mathrm{dR},\mathbb{Z}}) &= \mathbb{C}_z \oplus (\mathbb{C}_z \oplus \mathbb{C}_{\overline{z}})^{\perp} \oplus \mathbb{C}_{\overline{z}}.\end{aligned}$$

Notice that

$$\begin{aligned}&\mathrm{Fil}^j\mathrm{End}(\mathcal{H}_{1,\mathrm{dR}}(A))\\ &\qquad = \left\{f \in \mathrm{End}(\mathcal{H}_{1,\mathrm{dR}}(A)) | f(\mathrm{Fil}^i\mathcal{H}_{1,\mathrm{dR}}(A) \subset \mathrm{Fil}^{i+j}\mathcal{H}_{1,\mathrm{dR}})(A)\right\}\end{aligned}$$

is a three step filtration on $\mathrm{End}(\mathcal{H}_{1,\mathrm{dR}}(A)$ inducing the filtration on $\mathbb{V}_{\mathrm{dR}}$ described above.

By Griffith's transversality the connection ∇ on $\mathbb{V}_{\mathrm{dR}}$ defines a linear map

$$\mathrm{Fil}^1\mathbb{V}_{\mathrm{dR}} \to Gr^0\mathbb{V}_{\mathbb{R}} \otimes \Omega^1_{M(\mathbb{C})}$$

giving rise to the Kodaira–Spencer isomorphism

$$\mathcal{T}_{M(\mathbb{C})} \to \mathrm{Hom}_{M(\mathbb{C})}(\mathrm{Fil}^1 V_{\mathrm{dR}}, Gr^0 V_{\mathrm{dR}})$$

that provides a more conceptual approach to the computation of the tangent space (and bundle) to $M_L(\mathbb{C})$ (compare with 1.3.4).

2.2. Heegner divisors

In this section we will show how, given an element $\lambda \in V$ with $Q(\lambda) > 0$, we can construct a divisor in $M_L(\mathbb{C})$ as a Shimura subvariety. These will give the Heegner divisors mentioned in the introduction. The fact that we have such a large supply of easily constructed divisors, and in general of cycles of higher codimension obtained by intersecting such divisors, makes the theory of GSpin-Shimura varieties extremely rich.

Given λ as above, set $V_\lambda := \lambda^{\perp} \subset V$. This is a dimension $(n-1)+2$ subspace of V and $Q_\lambda := Q|_{V_\lambda}$ is a quadratic form of signature $(n-1, 2)$. Then we get a subgroup

$$G_\lambda = \mathrm{GSpin}(V_\lambda, Q_\lambda) \subset \mathrm{GSpin}(V) = G.$$

The symmetric space D_λ for G_λ is identified with

$$D_\lambda = \left\{[z] \in D_{\mathbb{C}} \subset V_{\mathbb{C}}\backslash\{0\} : z \in V_{\lambda,G} = \lambda^{\perp}\right\}/\mathbb{C}^*.$$

Let $K_\lambda := G_\lambda(\mathbb{A}_f) \cap K$. We get a GSpin-Shimura variety

$$M_\lambda(\mathbb{C}) = G_\lambda(\mathbb{Q})\backslash D_\lambda \times G_\lambda(\mathbb{A}_f)/K_\lambda$$

together with a homomorphism

$$M_\lambda(\mathbb{C}) \to M_K(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K.$$

Notice that such map is in general not an injection but the image of this map consists of divisors of $M_K(\mathbb{C})$. The discrepancy between $M_\lambda(\mathbb{C})$ and its image in $M_K(\mathbb{C})$ makes the intersection theory of Heegner divisors more involved. We will ignore this issue here for the sake of simplicity and pretend that we can identify $M_\lambda(\mathbb{C})$ and its image. We refer to [2] for the correct treatment using stacks.

Consider now $K = K_L$, so that $M_K(\mathbb{C}) = M_L(\mathbb{C})$, for a maximal lattice L and assume that $M_L(\mathbb{C}) = \Gamma\backslash D$ with $\Gamma = G(\mathbb{Q}) \cap K_L$. Here is an intrinsic characterization of the image of $M_\lambda(\mathbb{C})$ in $M_L(\mathbb{C})$. Take $z \in D$ and consider the corresponding element $\ell_{\lambda,\mathrm{Betti}} \in \mathrm{End}(\mathrm{H}_{1,\mathrm{Betti}}(A_z))$ of (2.1) and $\ell_{\lambda,\mathrm{dR}} \in \mathrm{End}(\mathrm{H}_{1,\mathrm{dR}}(A_z))$ of (2.2). Then,

Proposition 2.1. *We have $z \in D_\lambda$ if and only if $\ell_{\lambda,\mathrm{Betti}}$ is the Betti realization of an endomorphism $\ell_\lambda \in \mathrm{End}(A_z)$ if and only if $\ell_{\lambda,\mathrm{dR}}$ is the de Rham realization of an endomorphism $\ell_\lambda \in \mathrm{End}(A_z)$.*

Proof. We have $z \in D_\lambda$ if and only if

$$\lambda \in z^\perp = \mathrm{Fil}^0\mathbb{V}_{\mathrm{dR},z} \subset \mathbb{V}_{\mathrm{dR},z}.$$

This happens if and only if the element $\ell_{\lambda,\mathrm{dR}} \in \mathrm{End}(\mathrm{H}_{1,\mathrm{dR}}(A_z))$ of (2.2) lies in $\mathrm{Fil}^0\mathrm{End}(\mathrm{H}_{1,\mathrm{dR}}(A_z))$, i.e., $\ell_{\lambda,\mathrm{dR}}$ preserves the Hodge filtration of $\mathrm{H}_{1,\mathrm{dR}}(A_z)$. This is equivalent to require that the element $\ell_{\lambda,\mathrm{Betti}}$ defines an endomorphism of A_z. □

Definition 2.2. For any $m \in \mathbb{N}_{>0}$, let $Z(m)(\mathbb{C}) \to M(\mathbb{C})$ be $\Gamma\backslash(\amalg_\lambda M_\lambda(\mathbb{C}))$ where the union is taken over all $\lambda \in L$ such that $Q(\lambda) = m$.

The proposition says that the image of $Z(m)(\mathbb{C})$ singles out points z of $M_L(\mathbb{C})$ where the Kuga–Satake abelian variety A_z acquires an extra endomorphism, f, whose Betti realization (resp. de Rham realization) lies in $\mathbb{V}_{\mathrm{Betti},z}$ (resp. $\mathbb{V}_{\mathrm{dR},z}$) and such that $f \circ f = [m]$ (multiplication by m on A_z). In fact f will be an endomorphism of type ℓ_λ as in loc. cit. for some $\lambda \in L$ such that $Q(\lambda) = m$. The advantage of this reinterpretation of $Z(m)$ is that it is intrinsic on the Kuga–Satake abelian scheme as we will see later, and hence it can be given also over $\mathbb{Z}$ providing integral models of the given Heegner divisor; see §2.5.1. This is why we work with $Z(m)(\mathbb{C})$ instead of the single $M_\lambda(\mathbb{C})$.

2.3. The big CM points

Our next task is to introduce the CM cycles that intersected with the Heegner divisors will give arithmetically significant numbers. Such cycles are associated to the data of a CM field E of degree $2d + 2 = n + 2$ with totally real subfield F, the choice of one field homomorphism

$$\sigma_0 \in \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R}) = \{\sigma_0, \sigma_1, \dots, \sigma_d\}$$

and the choice of $\lambda \in F$ such that $\sigma_0(\lambda) < 0$ and $\sigma_i(\lambda) > 0$ for $i \geq 1$. For each $0 \leq i \leq d$ we also choose extensions σ_i and $\overline{\sigma}_i \colon E \to \mathbb{C}$ of $\sigma_i \colon F \to \mathbb{C}$.

With the data above we take $V := E$ as a $\mathbb{Q}$-vector space, the F-quadratic form $Q_F \colon V \to F$ given by $x \mapsto \lambda \cdot x \cdot \overline{x}$ and the $\mathbb{Q}$-quadratic form $Q \colon V \to \mathbb{Q}$ given by $\mathrm{Tr}_{F/\mathbb{Q}}(Q_F)$. Note that (V, Q) has signature $(n, 2)$ thanks to our choice of λ with $n = 2d$ even.

We also have the decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{i=0}^{d} (V \otimes_F^{\sigma_i} \mathbb{R}) = \bigoplus_{i=0}^{d} V_{\sigma_i}$$

where $V_{\sigma_i} := V \otimes_F^{\sigma_i} \mathbb{R} = \mathbb{C}$. On each vector space V_{σ_i} we have the quadratic form $Q_i(x) := \sigma_i(\lambda) x \overline{x}$, so this gives a quadratic form on $V \otimes_{\mathbb{Q}} \mathbb{R}$ making $V_0 := V_{\sigma_0} \subset V \otimes_{\mathbb{Q}} \mathbb{R}$ a negative definite plane. The choice of $\sigma_0 \colon E \to \mathbb{C}$ provides an identification $V_{\sigma_0} \cong \mathbb{C}$ so that the real plane V_{σ_0} acquires a natural orientation providing a point $z_0 \in D_{\mathbb{R}}$. Fixing a quadratic lattice $L \subset V$ we will get the Shimura variety $M_L(\mathbb{C})$.

The next goal is to define a torus T of dimension $d + 1$ and a homomorphism of algebraic groups $T \to G = \mathrm{GSpin}(V, Q)$ so that, setting $K_T := K_L \cap T(\mathbb{A}_f)$, we can define the Shimura variety

$$Y_L(\mathbb{C}) = T(\mathbb{Q}) \backslash \{z_0\} \times T(\mathbb{A}_f)/K_T$$

which consists of finitely many points, and a natural map

$$Y_L(\mathbb{C}) \to M_L(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K_L,$$

whose image is called the *big CM cycle* associated to (E, σ_0, λ).

Remark 2.3. The word big suggests the existence of a *small CM cycle*. This is the case and it is constructed starting from a quadratic imaginary extension of $\mathbb{Q}$ instead of a CM field extension of degree $2(d + 1) = n + 2$. There are interesting conjectures in this setting as well in the spirit of Kudla's programme, elaborated by Bruinier and Yang in [6]. These conjectures have been proven under some mild assumptions in [1, Thm. A].

The torus T above is defined as the quotient $T_E/T_F^{\mathrm{Nm}=1}$ of the tori $T_E = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_{m,E})$ by the norm 1 elements of $T_F = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_{m,F})$ over $\mathbb{Q}$. The map $T \to G$ arises from a homomorphism of algebraic groups $T_E \to G$ over $\mathbb{Q}$ that for simplicity we will simply describe over $\mathbb{R}$. Note that

$$T_E(\mathbb{R}) = (E \otimes_{\mathbb{Q}} \mathbb{R})^* = \prod_{i=0}^{d} (E \otimes_{\mathbb{Q}}^{\sigma_i} \mathbb{R})^* = \prod_{i=0}^{d} \mathrm{GSpin}(V_{\sigma_i}, Q_i)$$

so that we get a group homomorphism to $\mathrm{GSpin}(V_{\mathbb{R}}) = G(\mathbb{R})$ via the map of multiplicative monoids

$$\prod_{i=0}^{d} \mathrm{GSpin}(V_{\sigma_i}, Q_i) = \prod_{i=0}^{d} C^+_{\mathbb{R}}(V_{\sigma_i}) \xrightarrow{\text{mult}} \bigotimes_{\mathbb{R}, i=1}^{d} C^+_{\mathbb{R}}(V_{\sigma_i}) \longrightarrow C^+(V_{\mathbb{R}}).$$

We refer to [2] for the definition of the morphism $T_E \to G$ over $\mathbb{Q}$: this is defined directly over a Galois closure F' of F as above and one then uses a Galois descent argument.

More concretely, given $x \in T_E(\mathbb{R}) = (E \otimes_{\mathbb{Q}} \mathbb{R})^*$ its image on $\mathrm{SO}(V_{\mathbb{R}})$ is defined by the action of x on $V_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R}$ provided by $x\overline{x}^{-1}$ (using the multiplicative structure of E and complex conjugation on E).

We have a natural map $T_F \subset T_E$ and the elements of $T_F(\mathbb{R})$ act trivially on $V_{\mathbb{R}}$ so that T_F maps trivially onto $\mathrm{SO}(V, Q)$. The induced map to $\mathbb{G}_m$ (the kernel of the morphism $G \to \mathrm{SO}(V, Q)$ given in (1.1)) is the Norm map Nm. In particular, $T_F^{\mathrm{Nm}=1}$ maps trivially to G so that we get the sought-for morphism

$$T = T_E / T_F^{\mathrm{Nm}=1} \longrightarrow G.$$

Notice that V has the extra structure of E-vector space that defines endomorphisms of $V_{\mathrm{Betti}}|_{Y(\mathbb{C})}$ as $\mathbb{Q}$-Hodge structure. In fact, for any $z \in Y(\mathbb{C})$ the E action on $V_{\mathrm{Betti},\mathbb{Q},z}$ induces a decomposition

$$V_{\mathrm{Betti},\mathbb{Q},z} \otimes_{\mathbb{Q}} \mathbb{C} = V_{\mathrm{dR},z} = \bigoplus_{i=0}^{d} V_{\mathrm{dR},z}(\sigma_i) \oplus V_{\mathrm{dR},z}(\overline{\sigma}_i),$$

where $V_{\mathrm{dR},z}(\sigma)$ is the 1-dimensional $\mathbb{C}$-vector space on which E acts via $\sigma \colon E \to \mathbb{C}$. Then

$$\mathrm{Fil}^1 V_{\mathrm{dR},z} = V_{\mathrm{dR},z}(\sigma_0), \quad \mathrm{Gr}^{-1} V_{\mathrm{dR},z} = V_{\mathrm{dR},z}(\overline{\sigma}_0)$$

and

$$\mathrm{Gr}^0 V_{\mathrm{dR},z} = \bigoplus_{i=1}^{d} V_{\mathrm{dR},z}(\sigma_i) \oplus V_{\mathrm{dR},z}(\overline{\sigma}_i).$$

Let now $\lambda \in V$ be an element with $Q(\lambda) > 0$. The next lemma shows that the images of $Y(\mathbb{C})$ and $M_\lambda(\mathbb{C})$ in $M_L(\mathbb{C})$ do not intersect. This will imply that for the associated arithmetic objects, i.e. the associated objects over $\mathbb{Z}$, we have *proper intersection*. This is not at all the case for the small CM points of Remark 2.3 where we have *improper intersection*; see [1] for a discussion.

Lemma 2.4. *The intersection of the images of $Y(\mathbb{C})$ and $M_\lambda(\mathbb{C})$ in $M_L(\mathbb{C})$ is empty.*

Proof. Assume that we have an element $z \in D_\lambda$ whose image on $M_L(\mathbb{C})$ lies in the image of $Y(\mathbb{C})$. This is equivalent to saying that $\lambda \in V_{\mathrm{Betti},\mathbb{Q},z}$ satisfies $\lambda \in (\mathbb{C}z \oplus \mathbb{C}\overline{z})^\perp$. But $V_{\mathrm{Betti},\mathbb{Q},z} = V = E$ as a $\mathbb{Q}$-vector space and we get that

$$V = E \cdot \lambda \subset (\mathbb{C}z \oplus \mathbb{C}\overline{z})^\perp$$

as the pairing on V is E-hermitian. As $(\mathbb{C}z \oplus \mathbb{C}\overline{z})^\perp \oplus (\mathbb{C}z \oplus \mathbb{C}\overline{z}) = V_\mathbb{C}$, this is clearly a contradiction □

2.4. Integral models

In order to proceed with the computation of the intersections numbers we want, we need models for $M_L(\mathbb{C})$, the Heegner divisors $Z(m)$ and the CM cycle $Y(\mathbb{C})$ over $\mathbb{Z}$

Thanks to [8] we know that $M_L(\mathbb{C})$ is the complex analytic space associated to a quasi-projective variety M_L over a number field K called the *reflex field*; see also [14, Thm. 2.4.3]. In the case of GSpin, for $n \geq 1$, the reflex field is $\mathbb{Q}$. For $n = 0$ it is the quadratic imaginary field $K = C^+(V)$.

We assume next that $n \geq 1$. Let $\Delta_L := L^\vee/L$ be the discriminant group, where $L^\vee$ is the $\mathbb{Z}$-dual of L and the inclusion $L \subset L^\vee$ is defined by the bilinear form $[\cdot,\cdot]$ associated to Q. By work of Vasiu and Kisin [12] M_L admits a canonical integral model, smooth over $\mathbb{Z}[2^{-1}|\Delta_L|^{-1}]$. The techniques of loc. cit. do not apply at $p = 2$ or at primes at which L is not self-dual. Under the assumption that L is maximal among the quadratic lattices of V and is self-dual at 2, Madapusi Pera [15] constructed a canonical integral model

$$\mathcal{M} \to \operatorname{Spec} \mathbb{Z}$$

which has singular fibers at the primes dividing $|\Delta_L|$. He also shows that the Kuga–Satake abelian scheme $\pi\colon A \to M(\mathbb{C})$ extends to an abelian scheme

$$A \to \mathcal{M}.$$

Furthermore over $\mathcal{M}[2^{-1}|\Delta_L|^{-1}]$ we have two motives. Here, for the notion of motive we use the one introduced by Deligne [10, §1], i.e., it is a collection of realizations: Betti, de Rham, ℓ-adic étale for every prime ℓ and crystalline at every prime p not dividing $2|\Delta_L|$ with various comparison isomorphisms among them. First of all, we have the motive $\mathbb{H}_1(A)$ associated to A and consisting of

i. a Betti realization over $\mathcal{M}_\mathbb{C}$ given by $C(V)_{\mathrm{Betti}} = \mathrm{R}_1\pi_*\mathbb{Z}$;
ii. a de Rham realization $\mathcal{H}_{1,\mathrm{dR}}(A)$ which is a locally free $\mathcal{O}_\mathcal{M}[2^{-1}|\Delta_L|^{-1}]$-module, endowed with a two step descending filtration by locally free submodules, the Hodge filtration, $\mathrm{Fil}^\bullet\mathcal{H}_{1,\mathrm{dR}}(A)$ and an integrable connection satisfying Griffith's transversality;

iii. for every prime ℓ, an ℓ-adic étale realization $\mathrm{H}_{1,\text{ét}}(A, \mathbb{Z}_\ell)$ which is the ℓ-adic lisse local system over $\mathcal{M}[\ell^{-1}2^{-1}|\Delta_L|^{-1}]$ given simply by the ℓ-adic Tate module of A;

iv. for every prime p not dividing $2|\Delta_L|$, a crystalline realization $\mathrm{H}_{1,\text{cris}}(A_{\mathbb{F}_p})$ over $\mathcal{M}_{\mathbb{F}_p}$, identified with the covariant Dieudonné module of $A_{\mathbb{F}_p}/\mathcal{M}_{\mathbb{F}_p}$ and endowed with a Frobenius action.

Then we have a submotive $\mathbb{V} \subset \mathrm{End}(\mathbb{H}_1(A)) \cong \mathbb{H}_1(A) \otimes \mathbb{H}_1(A)^\vee$ that we define through its realizations:

i. a Betti realization over $\mathcal{M}_\mathbb{C}$ given by $\mathbb{V}_{\text{Betti}} \subset \mathrm{End}(C(V)_{\text{Betti}})$;

ii. a de Rham realization $\mathbb{V}_{\text{dR}}$ which is a locally free $\mathcal{O}_\mathcal{M}[2^{-1}|\Delta_L|^{-1}]$-module, endowed with a descending filtration $\mathrm{Fil}^\bullet\mathbb{V}_{\text{dR}}$ by locally free submodules and an integrable connection satisfying Griffith's transversality. We also have an embedding $\mathbb{V}_{\text{dR}} \hookrightarrow \mathrm{End}(\mathcal{H}_{1,\text{dR}}(A))$ as locally a direct summand, strictly compatible with the filtrations and compatible with the connections;

iii. for every prime ℓ, an étale ℓ-adic realization given by a lisse ℓ-adic subsheaf $\mathbb{V}_{\text{ét},\ell} \hookrightarrow \mathrm{End}(\mathrm{H}_{1,\text{ét}}(A, \mathbb{Z}_\ell))$ over $\mathbb{Z}[\ell^{-1}2^{-1}|\Delta_L|^{-1}]$.

iv. for every prime p not dividing $2|\Delta_L|$, a crystalline realization given by a subcrystal, locally a direct summand, $\mathbb{V}_{\text{crys},p} \hookrightarrow \mathrm{End}(\mathrm{H}_{1,\text{cris}}(A_{\mathbb{F}_p}))$ over $\mathcal{M}_{\mathbb{F}_p}$. Moreover, the Frobenius on $\mathrm{End}(\mathrm{H}_{1,\text{cris}}(A_{\mathbb{F}_p}))$, defined after inverting p, induces a Frobenius on $\mathbb{V}_{\text{crys},p}[p^{-1}]$.

As mentioned above there are various comparison isomorphisms between these realizations namely Betti-étale, Betti-de Rham, de Rham-crystalline, crystalline-étale and the inclusion $\mathbb{V} \subset \mathrm{End}(\mathbb{H}_1(A))$ is compatible with these isomorphisms.

Also, the motive $\mathbb{V}$ is endowed with a quadratic form $\mathbb{V} \to \mathbf{1}$. Here $\mathbf{1}$ is the unit object in each of the categories considered above, the structure sheaf with the usual differentiation in the de Rham case, the constant sheaf $\mathbb{Z}_\ell$ in the ℓ-adic étale case and the structure sheaf in the crystalline case. Furthermore for every local section $v \in \mathbb{V}$ (meaning in one of the realizations) we have $Q(v) = v \circ v$ where the composition $\circ$ is taken in $\mathrm{End}(\mathbb{H}_1(A))$ (where $\mathbb{H}_1(A)$ is the appropriate realization of A).

Remark 2.5. In order to prove the average Colmez conjecture one needs to deal also with primes dividing $2|\Delta_L|$. The trick used in [2] is, for any such prime p, to embed isometrically (L, Q) into a quadratic lattice $(L^\diamond, Q^\diamond)$ that is self-dual at p. We then work consistently with the Shimura variety, its integral model and various motives associated to $L^\diamond$ over $\mathbb{Z}_{(p)}$. We omit these extra complications in these notes.

2.5. Models of Heegner divisors and big CM points

We next define the integral models of the Heegner divisors and of the big CM cycle.

2.5.1. Models of Heegner divisors. For any $n \in \mathbb{Z}_{\geq 1}$, define $\mathcal{Z}(n) \to \mathcal{M}$ over $\operatorname{Spec} \mathbb{Z}[2^{-1}|\Delta_L|^{-1}]$ as the functor representing pairs $(\rho\colon S \to \mathcal{M}, f)$ such that $f \in \operatorname{End}(A \times_{\mathcal{M}} S)$ and we have, for all realizations above:

- $\mathrm{R}_{\mathrm{dR}}(f) \subset \rho^{\times}(\mathbb{V}_{\mathrm{dR}})$;
- $\mathrm{R}_{\text{ét},\ell}(f) \subset \rho^{\times}(\mathbb{V}_{\ell})$;
- $R_{\mathrm{cris},p}(f) \subset \rho^{\times}(\mathbb{V}_{\mathrm{cris},p})$.

Here, $\rho^{\times}(\mathbb{V}_{?})$ is the base change of $\mathbb{V}_{?}$ to S, in the appropriate category. Moreover, the de Rham realization $\mathrm{R}_{\mathrm{dR}}(f)$ of f is the map induced by f on the de Rham homology of $A \times_{\mathcal{M}} S$ relative to S, the étale ℓ-adic realization $\mathrm{R}_{\text{ét},\ell}(f)$ of f is the map induced by f on the ℓ-adic Tate module of $A \times_{\mathcal{M}} S$ and the crystalline realization $R_{\mathrm{cris},p}(f)$ of f is the map induced on the (covariant) Dieudonné module of $A \times_{\mathcal{M}} S \otimes \mathbb{F}_p$.

2.5.2. Integral models of big CM cycles. Consider the integral model of $Y(\mathbb{C})$. It is a scheme over the reflex field E. Define $\mathcal{Y}$ to be the normalization of $\operatorname{Spec} \mathcal{O}_E$ in Y. The morphism $Y(\mathbb{C}) \to M(\mathbb{C})$ gives a map $Y \to M$, which induces a morphism

$$J\colon \mathcal{Y} \longrightarrow \mathcal{M}.$$

Then define $\mathbb{V}_{\mathcal{Y}} = J^{*}(\mathbb{V})$. Let

$$\mathcal{O}'_E = E \cap \operatorname{End}(L) \subset \mathcal{O}_E$$

which is an order in $\mathcal{O}_E$.

The fact that $\mathcal{O}'_E$ acts on $\mathbb{V}_{\mathrm{Betti},\mathbb{Z}}|_{Y(\mathbb{C})}$ (as a $\mathbb{Z}$-Hodge structure) implies that $\mathcal{O}'_E$ acts on $\mathbb{V}_{\mathcal{Y}}$ through endomorphisms (in the category of realizations we are considering).

2.6. The Bruinier–Kudla–Yang conjecture

We can finally compute the intersection number of $\mathcal{Z}(n) \times_{\mathcal{M}} \mathcal{Y}$ via the morphism $\mathcal{Y} \hookrightarrow \mathcal{M}$ and $\mathcal{Z}(n) \hookrightarrow \mathcal{M}[2^{-1}|\Delta_L|^{-1}]$ defined in the previous section. We remark that in [2] we take care of defining these objects and computing the intersection also for (some of the) primes dividing $2|\Delta_L|$.

One can also refine the divisors $\mathcal{Z}(n)$ writing them as a sum $\sum_{\mu \in \Delta_L} \mathcal{Z}_{\mu}(n)$ and one is reduced to compute the intersection numbers of the $\mathcal{Z}_{\mu}(n)$'s. We refer to cit. for a more thorough discussion. For simplicity of the exposition we will assume here that L is unimodular, i.e., $\Delta_L = 0$ or equivalently $L = L^{\vee}$.

The conjecture of Bruinier–Kudla–Yang [5, Conj. 5.4] provides a formula for

$$a(m) := \#(\mathcal{Z}(m) \times_{\mathcal{M}} \mathcal{Y}).$$

We will not discuss how the formula is obtained but here are some consequences that one can draw from it:

i. $a(m)$ can be written as a sum $\sum_{\alpha \in F_+, \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m} a(\alpha)$ over the totally positive elements $\alpha \in F$ of trace m;

ii. for any α as in (i) we have $a(\alpha) \neq 0$ if and only there exists a unique prime $\mathcal{P}$ of $\mathcal{O}_F$ either inert or ramified in $\mathcal{O}_E$ such that α is not a norm of $E \otimes_F F_{\mathcal{P}}$, but for any other prime $\mathcal{Q}$ of $\mathcal{O}_F$ it is.

iii. for any α as in (i) with $a(\alpha) \neq 0$ and with associated prime $\mathcal{P}$ as in (ii), if $\mathcal{P}$ is unramified then we have

$$a(\alpha) = \frac{\mathrm{ord}_{\mathcal{P}}(\alpha) + 1}{2} \cdot \log \mathrm{Norm}_{F/\mathbb{Q}}(\alpha).$$

Recall that these are consequences of the computations, involving automorphic techniques, of the $a(m)$'s using their interpretation as coefficients a formal q-expansion via [5, Conj. 5.4]. The goal of this section is to prove that (i), (ii) and (iii) have a geometric explanation using their definition, as an intersection number $\#(\mathcal{Z}(m) \times_{\mathcal{M}} \mathcal{Y})$.

Remark 2.6. The Bruinier–Kudla–Yang conjecture does not make any assumption on L being unimodular or even being maximal (as a quadratic lattice) and provides an analogue of (iii) also for ramified primes. Indeed its predictions come from computations of the coefficients of a formal q-expansion constructed from a weight 1 Hilbert modular Eisenstein series associated to L. On the other hand, to relate the conjecture to geometry, we need at least the existence of good integral models, and thus we need that (L, Q) is a maximal quadratic lattice. Then assertions (i) and (ii) of the conjecture hold true and do not require any extra assumption. It is for the proof of (iii) that the assumption that L is self-dual is crucial. In [2, Def. 5.3.3] we will further require that the prime p is *good* for L, meaning that:

- For every prime $\mathcal{P}$ of F over p unramified in E, the $\mathbb{Z}_p$-lattice $L_{\mathcal{P}}$ is $\mathcal{O}_{E,\mathcal{P}}$-stable and self-dual.
- For every prime $\mathcal{P}$ of E over p ramified in E, the $\mathbb{Z}_p$-lattice $L_{\mathcal{P}}$ is maximal for the induced $\mathbb{Z}_p$-valued quadratic form, and there exists an $\mathcal{O}_{E,\mathcal{P}}$-stable lattice $\Lambda_{\mathcal{P}} \subset V_{\mathcal{P}}$ such that

$$\Lambda_{\mathcal{P}} \subset L_{\mathcal{P}} \subsetneq \mathrm{Different}_{E_{\mathcal{P}}/F_{\mathcal{P}}}^{-1} \Lambda_{\mathcal{P}}.$$

Notice that, given L, the set of bad primes for L is finite and that we are also allowing cases where $\mathcal{P}$ is ramified. It is this flexibility that allows us to prove in [2, Thm. 6.4.2] sufficiently many cases of the conjecture by Bruinier, Kudla and Yang to get the application to Colmez's conjecture mentioned in the introduction.

In Sections 2.9 and 2.10 we will make the much stronger assumption that p is unramified in E. Assertion (iii) predicts how much we can lift a given special endomorphism of the Kuga–Satake abelian variety at a point of $\mathcal{Y}$ in characteristic p. This can be phrased in terms of the Grothendieck–Messing theory of deformation of endomorphisms of p-divisible groups. If p is unramified one can use directly crystalline theory. If p is ramified we have to resort in [2] to the theory of Kisin modules to make this deformation theory problem computable.

2.7. The decompositions $a(m) = \sum_{\alpha\in F_+, \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m} a(\alpha)$

Let κ be a perfect field of characteristic p and let $y\colon \mathrm{Spec}\kappa \to \mathcal{Y}$ be a κ-valued point. Let A_y be the base change of the Kuga–Satake abelian scheme over $\mathcal{M}$. Let $\mathbf{W}_y \subset \mathrm{End}^0(A_y)$ be the $\mathbb{Q}$-vector space of special endomorphisms of A_y, i.e., of those endomorphisms whose realizations (étale ℓ-adic realizations for ℓ different from p and crystalline) lie in the base change to y of the motive $\mathbb{V}_y \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathrm{End}(\mathbb{H}_1(A_y)) \otimes_{\mathbb{Z}} \mathbb{Q}$. One can prove the following

Lemma 2.7. *There exists a unique structure of E-vector space on $\mathbf{W}_y$ and a unique positive definite hermitian form $Q_y\colon \mathbf{W}_y \to \mathbb{Q}$ such that the realization morphism $\mathbf{W}_y \to \mathbb{V}_y \otimes_{\mathbb{Z}} \mathbb{Q}$ is E-linear and compatible with the given quadratic form on $\mathbb{V}_y \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Moreover, if $\mathbf{W}_y \neq \{0\}$ then $\mathbf{W}_y$ is an E-vector space of dimension 1.

Proof. We refer to [2] for a proof of this result. We just provide some hints. Using the construction of $Y(\mathbb{C})$ one sees that for every $z \in Y(\mathbb{C})$ the torus $T(\mathbb{Q})$ acts via automorphisms on the Hodge structures $V_{\mathrm{Betti},\mathbb{Q},z}$ and $\mathrm{H}_1(A_z, \mathbb{Q})$ of Section 2.3. This is the action that by dévissage one proves to extend to an action on the Kuga–Satake abelian scheme $A|_{\mathcal{Y}}$ and on the motive $\mathbb{V} \otimes \mathbb{Q}|_{\mathcal{Y}}$. We then get an action of $T(\mathbb{Q})$ on $\mathbf{W}_y$ and on its realizations. If $\mathbf{W}_y$ is non zero, using the realizations one checks that the $\mathbb{Q}$-span of $T(\mathbb{Q})$ in $\mathrm{End}(\mathbf{W}_y)$ is E, making $\mathbf{W}_y$ into a 1-dimensional E-vector space. The fact that Q_y is positive definite follows from the existence of a Rosati involution on A_y fixing $\mathbf{W}_y$ (see [1, Prop. 2.6.3]). □

The quadratic form $Q_y\colon \mathbf{W}_y \to \mathbb{Q}$ can be uniquely written as $\mathrm{Tr}_{F/\mathbb{Q}} Q_{y,F}$ for a unique F-quadratic form $Q_{y,F}\colon \mathbf{W}_y \to F$ that is positive definite for every real embedding of F.

As Q_y is positive definite, for every positive integer m the fiber of $\mathcal{Z}(m)$ over y consists of the set $f \in \mathbf{W}_y \cap \mathrm{End}(A_y)$ such that $Q_y(f) = m$. The latter decomposes as the disjoint union over all positive $\alpha \in F$ of trace m of the sets $f \in \mathbf{W}_y \cap \mathrm{End}(A_y)$ such that $Q_{y,F}(f) = \alpha$. This explains the first consequence of the conjecture by Bruinier, Kudla and Yang.

2.8. On the support of $\mathcal{Z}(m) \times_{\mathcal{M}} \mathcal{Y}$

We continue the discussion of the previous section. We fix a κ-valued point y of $\mathcal{Y}$ such that $\mathbf{W}_y \neq \{0\}$. We then have two E-vector spaces, endowed with a hermitian quadratic form, namely (V, Q) and $(\mathbf{W}_y, Q_y)$. We have

- for $\ell \neq p$, the ℓ-adic realization functor defines an isomorphism of $\mathbb{Q}_\ell$-quadratic spaces $\mathbf{W}_y \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \mathbb{V}_{\text{ét},\ell,y} \otimes_{\mathbb{Z}} \mathbb{Q} \cong V \otimes \mathbb{Q}_\ell$;
- let $K_0 := \mathrm{Frac}(\mathbb{W}(\kappa))$. The crystalline realization functor defines an isomorphism of K_0-quadratic spaces $\mathbf{W}_y \otimes_{\mathbb{W}(\kappa)} K_0 \cong \mathbb{V}_{\mathrm{cris},p,y} \otimes_{\mathbb{Z}} \mathbb{Q}$;
- the signature of $(V \otimes_{\mathbb{Q}} \mathbb{R}, Q)$ is $(2, n)$ and $(\mathbf{W}_y \otimes_{\mathbb{Q}} \mathbb{R}, Q_y)$ has signature $(0, n+2)$.

Consider V as F-vector space with quadratic form $Q_F \colon V = E \to F$ given by $x \mapsto \lambda \cdot x \cdot \overline{x}$. For every place v of F consider the invariants $\epsilon_v(V, Q_F)$ and $\epsilon_v(\mathbf{W}_y, Q_{y,F})$ in $F_v^*/(F_v^*)^2$. Let $\mathcal{P}_0$ be the prime determined by the image of $y \in \mathcal{Y}(\kappa)$ via the structural morphism $\mathcal{Y} \to \mathrm{Spec}(\mathcal{O}_E)$. Then it follows from the above, and some extra analysis at primes above p provided in Lemma 2.10, that:

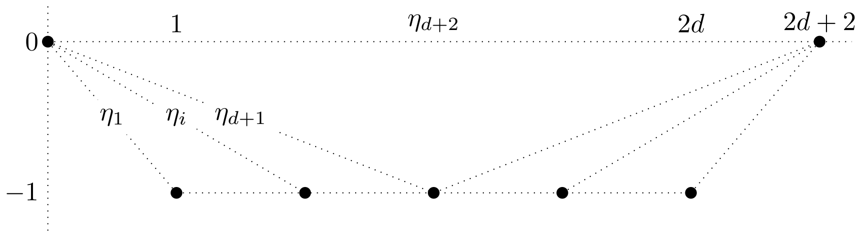
Proposition 2.8. *The invariants $\epsilon_v(V, Q_F)$ and $\epsilon_v(\mathbf{W}_y, Q_{y,F})$ are the same if and only if v is a finite place different from $\mathcal{P}_0$ or if v is a place at infinity different from $\sigma_0 \colon F \to \mathbb{R}$.*

In particular, if we have $f \in \mathbf{W}_y$ of norm $Q_{y,F}(f) = \alpha \in F$, then the difference of invariants characterizes the prime $\mathcal{P}_0$ over which y is supported as the only finite place where α is *not* in the image of $Q_F \colon V_{\mathcal{P}_0} \to F_{\mathcal{P}_0}$. This is in agreement with automorphic expectation (ii).

2.9. The Newton polygon

As in the previous section, let κ be a perfect field of positive characteristic p and let $y \in \mathcal{Y}(\kappa)$. Let K_0 be the fraction field of $\mathbb{W}(\kappa)$. Consider the K_0-vector spaces $\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q} \subset \mathrm{End}(\mathrm{H}_{1,\mathrm{cris}}(A_\kappa)) \otimes K_0$. They are endowed with Frobenius semilinear, bijective maps φ. Here, $\mathrm{H}_{1,\mathrm{cris}}(A_\kappa)$ is the covariant Dieudonné module of A_κ.

The category of finite dimensional K_0-vector spaces endowed with a bijective, Frobenius semilinear map is called the category of Frobenius *isocrystals*. For each of them one can construct a Newton polygon, see [11]. In our case we get a convex polygon in the plane with leftmost endpoint $(0,0)$, integral break points and rightmost endpoint $(2d+2, 0)$ that looks like



We refer to these Newton polygons as $\eta_1, \ldots, \eta_{d+2}$, where the Newton polygon η_i starts at $(0,0)$, ends at $(2d+2,0)$ and has break points $(i,-1)$ and $(2d+2-i,-1)$, except for η_{d+2} which is the straight line from $(0,0)$ to $(2d+2,0)$.

Definition 2.9. We say that $\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q}$ is *supersingular* if its Newton polygon is a straight line from $(0,0)$ to $(2d+2,0)$.

Using the structure of F-vector space we can decompose

$$\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q} = \oplus_v \mathbb{V}(v)$$

according to the places v of F over p. For every such v the $F_v \otimes_{\mathbb{Q}_p} K_0$-module $\mathbb{V}(v)$ is of rank two and is stable under Frobenius. Recall that we can single out one place v_0, namely the one defined by the image of $y \in \mathcal{Y}(\kappa)$ via the structural morphism $\mathcal{Y} \to \mathrm{Spec}(\mathcal{O}_E)$. In [2, §5] one proves the following:

Proposition 2.10. *For every $v \neq v_0$ the slope of $\mathbb{V}(v)$ is 0.*

The slope of $\mathbb{V}(v_0)$ is not 0 if and only if v_0 splits in E.

In particular, $\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q}$ is not supersingular if and only if the place v_0 of F splits in E.

Assuming that p is unramified in $\mathcal{Y}$, that the order $\mathcal{O}'_E$ is maximal at p and that κ contains all the residue fields at primes of $\mathcal{O}_E$ at primes above p we prove the following weaker version of the Proposition:

Claim. The slope of $\mathbb{V}(v_0)$ is 0 if and only if v_0 is inert in E.

First of all, the map $y\colon \mathrm{Spec}(\kappa) \to \mathcal{Y}$ lifts uniquely to a morphism $\widetilde{y}\colon \mathrm{Spec}(\mathbb{W}(\kappa)) \to \mathcal{Y}$ and, due to our hypothesis that p is unramified, the crystal $\mathbb{V}_{\mathrm{cris},y}$ can be identified as $\mathbb{W}(\kappa)$-module with $\mathbb{V}_{\mathrm{dR},\widetilde{y}}$ via the crystalline-de Rham comparison isomorphism of Deligne's formalism [10, §1]. Using these two structures we get a $\mathbb{W}(\kappa)$-module $\mathbb{V}_{\widetilde{y}}$ with:

- a Frobenius semilinear morphism φ;
- an action of $\mathcal{O}'_E \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$ commuting with φ;
- a three step filtration $\mathrm{Fil}^\bullet \mathbb{V}_{\widetilde{y}}$ commuting with the action of $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Using the $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$ action and the fact that $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{W}(\kappa)$ splits completely into a product of copies of $\mathbb{W}(\kappa)$, we get a decomposition

$$\mathbb{V}_{\widetilde{y}} = \bigoplus_{\tau} \mathbb{V}_{\widetilde{y}}(\tau),$$

where the decomposition is taken over all embeddings $\tau\colon \mathcal{O}_E \to \mathbb{W}(\kappa)$ and for each τ the factor $\mathbb{V}_{\widetilde{y}}(\tau)$ is a $\mathbb{W}(\kappa)$-module of rank 1 on which $\mathcal{O}_E$ acts via τ. Frobenius on $\mathbb{V}_{\widetilde{y}}$ sends each factor $\mathbb{V}_{\widetilde{y}}(\tau)$ to the factor $\mathbb{V}_{\widetilde{y}}(\sigma\tau)$ where σ is the Frobenius automorphism on $\mathbb{W}(\kappa)$. Moreover given a place v the $\mathbb{W}(\kappa)$-module $\mathbb{V}(v)$ is the sum $\mathbb{V}_{\widetilde{y}}(\tau)$ of all τ's inducing the place v.

In particular, the rank 1 submodule $\mathrm{Fil}^1\mathbb{V}_{\widetilde{y}}$ will correspond to a factor $\mathbb{V}_{\widetilde{y}}(\tau_0)$ and $\mathrm{Gr}^{-1}\mathbb{V}_{\widetilde{y}}$ will correspond to a factor $\mathbb{V}_{\widetilde{y}}(\overline{\tau}_0)$. A result of Mazur, see [11, Thm. 1.6.1], guarantees that the Newton polygon lies above the Hodge polygon

and that the Newton polygon touches the Hodge polygon at a breaking point of the Newton polygon. This implies that if τ_0 and $\overline{\tau}_0$ lie in different orbits for the action of σ, the Newton polygon can *not* be horizontal and hence $\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q}$ cannot be supersingular.

Assume next that τ_0 and $\overline{\tau}_0$ lie in the same orbit. Let f_0 be the length of the orbit $\{\sigma^i\tau_0\}$ or equivalently the inertia degree of the prime defined by v_0. Write the matrix of Frobenius on $\mathbb{V}(v_0)$ as a subdiagonal matrix

$$[\varphi_0] := \begin{pmatrix} 0 & & & & & \varphi(\tau_0) \\ \varphi(i, \sigma\circ\tau_0) & 0 & & & & \\ & \varphi(i,\sigma^2\circ\tau) & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & \varphi(i,\sigma^{f_0-1}\circ\tau_0) & 0 \end{pmatrix}.$$

The entry $\varphi(\sigma^j\circ\tau_0)$ of this matrix lies in K_0. Then $\varphi_0^{f_0}$ is the diagonal matrix $\mathrm{diag}(\beta_1,\ldots,\beta_{f_0})$ with $\beta_j \in K_0$ having all the same p-adic valuation a_0, which is the sum of the p-adic valuations of $\varphi(\tau_0),\ldots,\varphi(\sigma^{f_i-1}\circ\tau_0)$. Hence, $[\varphi_0]^{f_0}$ is diagonal with entries having all the same p-adic valuation a_0 which implies that the Newton polygon of φ_0 has only one slope, namely a_0.

Together with the cited result of Katz, we conclude that $a_0 = 0$ proving the Claim in this case.

We are now ready for the following Proposition that, together with Proposition 2.10, proves geometrically the second expectation of the Bruinier–Kudla–Yang conjecture.

Proposition 2.11. *Let y be a κ-valued point of $\mathcal{Y}$ as before. Then $\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q}$ is supersingular if and only if $\mathbf{W}_y$ is non zero.*

Proof. If $\mathbf{W}_y$ is non zero the crystalline realization provides an isomorphism $\mathbf{W}_y \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong (\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q})^{\varphi=1}$. This forces the Newton polygon of $\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q}$ to be constant of slope 0.

Vice versa if $\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q}$ is supersingular one can prove that the associated Kuga–Satake abelian scheme is supersingular. In particular, the crystalline realization functor provides an isomorphism

$$\mathrm{End}^0(A_y) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{End}^0(A_y[p^\infty]) \cong \mathrm{End}^0(\mathrm{H}_{1,\mathrm{cris}}(A_y))^{\varphi=1}.$$

In particular, $(\mathbb{V}_{\mathrm{cris},y} \otimes \mathbb{Q})^{\varphi=1}$ is realized within the space $\mathrm{End}^0(A_y) \otimes_{\mathbb{Q}} \mathbb{Q}_p$. The same applies for a prime $\ell \neq p$. An argument of [15] guarantees that all these spaces arise from a $\mathbb{Q}$-vector space of $\mathrm{End}^0(A_y)$ that must be $\mathbf{W}_y$. □

2.10. Deformation theory

To conclude the proof of the conjecture of Bruinier–Kudla–Yang in the setting of the previous section, we are left to compute the length of the complete local ring of $\mathcal{Z}(m) \times_{\mathcal{M}} \mathcal{Y}$ at a point $y \in \mathcal{Y}(\kappa)$ associated to a special endomorphism $f \in \mathbf{W}_y$ of norm $Q_{y,F} = \alpha \in F_+$. Assertion (iii) in Section 2.6 is equivalent to requiring that such length is $\frac{\mathrm{val}_{\mathcal{P}_0}(\alpha)+1}{2}$.

Here, κ is a perfect field of characteristic $p > 0$ with p a prime unramified in $\mathcal{O}_E$ and such that $\mathcal{O}'_E \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Recall that $y\colon \mathrm{Spec}(\kappa) \to \mathcal{Y}$ lifts uniquely to a morphism $\widetilde{y}\colon \mathrm{Spec}(\mathbb{W}(\kappa)) \to \mathcal{Y}$. Let $R := \mathbb{W}(\kappa)$ and let $\mathcal{P}_0$ be the prime of F associated to the place v_0 as in the previous section. We are then left to show the following:

Proposition 2.12. *Let f be a special endomorphism of A_y of norm $\alpha := Q_F(f) \in F^+$. Then f deforms to a special endomorphism of A_{R/p^n} if and only if $n \leq \frac{\mathrm{val}_{\mathcal{P}_0}(\alpha)+1}{2}$.*

Proof. We have $\mathrm{R}_?(f) \in \mathbb{V}_?(A_y)$ for $? = \mathrm{cris}$ or $\mathrm{dR} \pmod{\mathcal{P}}_0$. Since f is an endomorphism, it preserves the Hodge filtration on $\mathrm{H}_{1,\mathrm{dR}}(A_y)$, i.e., $\mathrm{R}_{\mathrm{dR}}(f)$ lies in $\mathrm{Fil}^0\mathrm{End}(\mathrm{H}_{1,\mathrm{dR}}(A_y))$ so that $\mathrm{R}_{\mathrm{dR}}(f) \in \mathrm{Fil}^0(\mathbb{V}_{\mathrm{dR}}(A_\kappa))$.

On the other hand, recalling that $K_0 := \mathrm{Frac}(R)$ and using the decomposition $\mathbb{V}_{\mathrm{dR}}(A_\kappa) = \bigoplus_{\tau\colon E \to K_0} \mathbb{V}_{\mathrm{dR}}(\tau)$ into 1-dimensional κ-vector spaces according to the homomorphisms $\tau\colon E \to K_0$ we have that there exists $\tau_0\colon E \to K_0$ such that $\mathrm{Fil}^1(\mathbb{V}_{\mathrm{dR}}(A_R)) = \mathbb{V}_{\mathrm{dR}}(\tau_0)$ and, denoting by $\bar{\tau}_0$ its complex conjugate, $\mathrm{Fil}^0(\mathbb{V}_{\mathrm{dR}}(A_R)) = \bigoplus_{\tau\colon E\to K_0, \tau\neq\bar{\tau}_0} \mathbb{V}_{\mathrm{dR}}(\tau)$.

Recall from the discussion after Proposition 2.10 that we may canonically identify $\mathbb{V}_{\mathrm{cris},y}$ as $\mathbb{W}(\kappa)$-module with $\mathbb{V}_{\mathrm{dR},\widetilde{y}}$. Using these two structures we get a $\mathbb{W}(\kappa)$-module $\mathbb{V}$ that we can decompose as $\bigoplus_v \mathbb{V}(v)$ according to the places of F over p.

As pR has nilpotent divided powers structure, Grothendieck–Messing theory [16] gives that f lifts to a special endomorphism of R/p^nR (and if so, uniquely) if and only if $\mathrm{R}_{\mathrm{cris}}(f) \in \mathrm{Fil}^0(\mathbb{V}/p^n)$ if and only if the v_0-component $\mathrm{R}_{\mathrm{cris}}(f)_{v_0} \in \mathbb{V}(v_0)$ of $\mathrm{R}_{\mathrm{cris}}(f)$ satisfies

$$\mathrm{R}_{\mathrm{cris}}(f)_{v_0} \in \mathrm{Fil}^0(\mathbb{V}(v_0)/p^n).$$

Using [15, (4.8.2)] one gets a basis for $\mathbb{V}(v_0)$ such that the matrix of Frobenius $\varphi(0)$ is given by

$$[\varphi(0)] := \begin{pmatrix} 0 & & & & & & & u \\ p & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & p^{-1} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & 1 & 0 \end{pmatrix},$$

where u is a unit.

The first remark is that $\mathrm{R}_{\mathrm{cris}}(f)_{v_0}$ is fixed by $\varphi(0)$. We may therefore calculate the $\varphi(0)$ invariant elements:

$$\begin{aligned} (\mathbb{V}(v_0))^{\varphi=1} &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{f_0} \end{pmatrix} \mid \varphi(0) \begin{pmatrix} x_1 \\ \vdots \\ x_{f_0} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{f_0} \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{f_0} \end{pmatrix} \mid x_2 = px_1^\sigma, x_3 = x_2^\sigma, \ldots, x_{r+1} = p^{-1}x_r^\sigma, \ldots, x_1 = u^\sigma x_{f_0}^\sigma \right\} \\ &= \left\{ {}^t(\alpha, p\alpha^\sigma, \ldots, p\alpha^{\sigma^r}, \alpha_{\sigma^{r+1}}, \ldots, \alpha^{\sigma^{f_0-1}}) \mid \alpha = u^\sigma \alpha^{\sigma^{f_0}} \right\}. \end{aligned} \tag{2.3}$$

In particular,

$$\mathrm{R}_{\mathrm{cris}}(f)_{v_0} \in \{(\beta, p\beta^\sigma, \ldots, \beta^{\sigma^{f_0-1}})\},$$

where the τ_0 component is β for some β and the $\bar{\tau}_0$ component is $p\beta^{\sigma^r}$. Let

$$\alpha = Q_F(f), \quad \alpha \in F^+ \subseteq F \subseteq \prod_{\gamma\colon F\to K_0} K_0.$$

The image of α in the component K_0 corresponding to the embedding $\gamma = \tau_0|_F F \to K_0$ is $p\beta\beta^{\sigma^r}$. This implies that

$$\mathrm{val}_{v_0}(\alpha) = 2\mathrm{val}_{v_0}(\beta) + 1.$$

As

$$\mathrm{R}_{\mathrm{cris}}(f)_{v_0} \in \mathrm{Fil}^0(\mathbb{V}(v_0)/p^n) \iff p\beta^{\sigma^r} \equiv 0 \pmod{p^n},$$

we have $\mathrm{val}_{v_0}(\beta) \geq n-1$, which is equivalent to

$$n \leq \frac{\mathrm{val}_{v_0}(\alpha)+1}{2}.$$

□

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