An Alternative Proof of the Lindemann-Weierstrass Theorem

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Dedicated to the Memory of Philippe Robba

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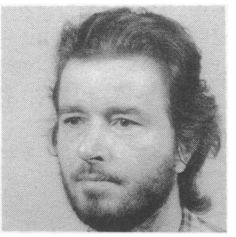

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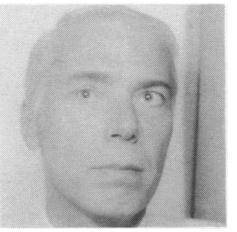


Introduction. In December 1987 J. P. Bézivin and Ph. Robba found a new proof of the Lindemann-Weierstrass theorem as a by-product of their criterion of rationality for solutions of differential equations. Let us recall the Lindemann-Weierstrass theorem, to which we shall refer as LW from now on.

Let $\alpha_1, \ldots, \alpha_t, b_1, \ldots, b_t$ be algebraic numbers such that the b_i are all nonzero and the α_i are mutually distinct. Then

$$b_1 e^{\alpha_1} + b_2 e^{\alpha_2} + \cdots + b_t e^{\alpha_t} \neq 0.$$

It is well known that the transcendence of π follows from LW in the following way. Suppose, on the contrary, that π is algebraic. Then so is $\pi\sqrt{-1}$ and LW now implies that $e^{\pi\sqrt{-1}} + 1 \neq 0$, which is certainly not true. Thus we conclude that π is transcendental.

The usual proof of LW is essentially due to Hilbert and has been polished by a number of authors. One such version can be found in [**2**, Ch. XI] or in [**4**, Ch. I], [**3**]. The new proof of Bézivin and Robba looks totally different. It can be considered as

a direct consequence of their criterion of rationality for solutions of linear differential equations.

The Bézivin-Robba criterion is based on a theorem of Pólya-Bertrandias, which is far from easy and relies heavily on p- adic analysis. We refer the interested reader to [**1**]. In March 1988 F. Beukers found that the use of the Pólya-Bertrandias criterion is much too heavy and can be avoided in a very elementary way. The result is a new proof of LW which is elementary and can compete with the usual one in shortness and simplicity. There is always a possibility that the similarity between the proofs is stronger than one would expect at first sight. In fact, very soon after a first draft of this paper was written (May 1988) Yu. Nesterenko pointed out to us that the numbers $v_n(k)$ which we use are equal to the integrals

$$\sum_{j=1}^{t} b_j e^{\alpha_j} \int_{\alpha_j}^{\infty} e^{-x} x^{n-kt} (x-\alpha_1)^k \cdots (x-\alpha_t)^k \, dx$$

which are used in the Hilbert proof. In spite of such similarities we feel that the arguments of our proof are nice enough to present in front of a wider audience. We would like to thank the referee for several improvements upon our presentation.

THEOREM. *Let* $b_1, \ldots, b_t, \ \alpha_1, \ldots, \alpha_t \in \overline{\mathbb{Q}}$ *such that* $b_i \neq 0 \ \forall i$ *and the* α_i *are mutually distinct. Then*

$$b_1 e^{\alpha_1} + \cdots + b_t e^{\alpha_t} \neq 0.$$

Proof. Consider the Taylor series expansion

$$b_1 e^{\alpha_1 x} + \cdots + b_t e^{\alpha_t x} = \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n,$$

where, clearly,

$$u_n = \sum_{i=1}^{t} b_i \alpha_i^n. \tag{1}$$

Put $(X-\alpha_1)\cdots(X-\alpha_t) = X^t - a_1 X^{t-1} - \cdots - a_t$. Clearly, for any $i = 1, \ldots, t$ and any $n \in \mathbb{Z}_{\geqslant 0}$

$$\alpha_i^{t+n} = a_1 \alpha_i^{t+n-1} + \cdots + a_t \alpha_i^n.$$

By taking suitable linear combinations and using (1) it follows that

$$u_{n+t} = a_1 u_{n+t-1} + \cdots + a_t u_n. \tag{2}$$

Without loss of generality we may assume that $u_n \in \mathbb{Q}$, $\forall n$. If not, then consider the product

$$\prod_{\sigma} \left(\sigma(b_1) e^{\sigma(\alpha_1)x} + \cdots + \sigma(b_n) e^{\sigma(\alpha_t)x} \right)$$

taken over all $\sigma \in \mathrm{Gal}(\mathbb{Q}(b_1, \ldots, b_t, \alpha_1, \ldots, \alpha_t)/\mathbb{Q})$, which, after evaluation, again acquires the form

$$\sum_i b_i' e^{\alpha_i' x},$$

where now the sets $\{b_i'\}$, $\{\alpha_i'\}$ are Galois-stable. This implies that the corresponding numbers a_i' and u_n' are rational. So from now on we assume $u_n \in \mathbb{Q}$, $\forall n$ and

$a_i \in \mathbb{Q}$ $(i = 1, \ldots, t)$. Let D be a common denominator of the a_i. Put $A = \max(1, |\alpha_i|)$. After multiplication with a suitable integer, if necessary, we may assume $u_0, \ldots, u_{t-1} \in \mathbb{Z}$. Hence, using (2) recursively,

$$D^n u_n \in \mathbb{Z}, \tag{3}$$

and by (1),

$$|u_n| \leqslant c_1 A^n$$

for some $c_1 > 0$ and all $n \geqslant 0$. Now suppose that $b_1 e^{\alpha_1} + \cdots + b_t e^{\alpha_t} = 0$, or, equivalently,

$$\sum_{r=0}^{\infty} \frac{u_r}{r!} = 0.$$

Put

$$v_n = n! \sum_{r=0}^{n} \frac{u_r}{r!}$$

and notice

$$|v_n| = n! \left| \sum_{r=0}^{n} \frac{u_r}{r!} \right| = n! \left| \sum_{r=n+1}^{\infty} \frac{u_r}{r!} \right| \leqslant \frac{c_1}{n+1} \sum_{r=n+1}^{\infty} \frac{A^r}{(r=n+1)!} = c_2 \frac{a^{n+1}}{n+1}. \tag{4}$$

If we had $A = D = 1$, like in the high school proof of $e \notin \mathbb{Q}$, inequality (4) gives us a contradiction since we have both $v_n \in \mathbb{Z}$ and $|v_n| \leqslant c_2/(n+1)$, i.e. $v_n = 0$ for all sufficiently large n, in other words $\sum_0^\infty v_n X^n$ is a polynomial. In our general case a similar principle works.

Claim:

$$\sum_{n=0}^{\infty} v_n X^n \in \mathbb{Q}(X).$$

Assuming the claim we proceed with the proof. Define

$$v(X) = \sum_{n=0}^{\infty} v_n X^n.$$

Notice that

$$\frac{v_n}{n!} - \frac{v_{n-1}}{(n-1)!} = \frac{u_n}{n!} \quad \text{or} \quad v_n - n v_{n-1} = u_n.$$

So,

$$\sum_{n=0}^{\infty} (v_n - n v_{n-1}) X^n = \sum_{n=0}^{\infty} u_n X^n. \tag{5}$$

Using (1), the right-hand side of (5) is seen to be

$$\sum_{n=0}^{\infty} u_n X^n = \sum_{i=1}^{t} \frac{b_i}{1 - \alpha_i X},$$

whereas the left-hand side equals

$$v(X) - X \frac{d}{dX}(X v(X)) = (1 - X) v(X) - X^2 \frac{d}{dX} v(X).$$

So (5) becomes

$$\mathscr{L}v(X) = \sum_{i=1}^{t} \frac{b_i}{1-\alpha_i X}, \qquad \mathscr{L} = -X^2\frac{d}{dX} + (1-X). \tag{6}$$

By the claim we know that $v(X) \in \mathbb{Q}(X)$ and so the non-zero poles of $\mathscr{L}v(X)$ have order at least two. However, the right-hand side of (6) has only simple poles. This contradiction proves our theorem, since the assumption $b_1 e^{\alpha_1} + \cdots + b_t e^{\alpha_t} = 0$ has turned out to be untenable. □

It now remains to prove our claim. We first observe that, as v is a solution of the differential equation (6), if v is a rational function then its poles must be at the points $1/\alpha_i$. Therefore we expect that there exists an integer k such that $(1 - a_1X - \cdots - a_tX^t)^k v(X)$ is a polynomial.

DEFINITION. *For any k, $n \in \mathbb{Z}_{\geqslant 0}$ we define $v_n(k)$ as coefficient in the formal power series*

$$\sum_{n=0}^{\infty} v_n(k) X^n = \left(1 - a_1X - \cdots - a_tX^t\right)^k \sum_{n=0}^{\infty} v_n X^n.$$

For later use we also note that

$$v_n(k+1) = v_n(k) - a_1 v_{n-1}(k) - \cdots - a_t v_{n-t}(k) \quad \text{for all } n \geqslant t,\ k \geqslant 0. \tag{7}$$

LEMMA 1. *Let $C = 1 + |a_1| + \cdots + |a_t|$. For all $n \geqslant kt$ we have*

i) $|v_n(k)| \leqslant c_2 A^n C^k$

ii) $D^n v_n(k) \in \mathbb{Z}$

iii) $k!$ *divides* $D^n v_n(k)$.

Proof. The first two assertions follow easily by induction on k from (3), (4) and (7). The third assertion can be shown as follows. Write

$$v_n = u_n + nu_{n-1} + \cdots + n(n-1)\cdots(n-k+2)u_{n-k+1} + w_n,$$

where $w_n = n!\sum_{r=0}^{\infty} u_r/r!$. Notice that $D^{n-k}w_n \in \mathbb{Z}$ and $k!$ divides $D^{n-k}w_n$. Consider the power series

$$v(X) = \sum_{n=0}^{\infty} v_n X^n \qquad w(X) = \sum_{n=0}^{\infty} w_n X^n.$$

Then

$$v(X) - w(X) = \sum_{n=0}^{\infty} \left\{u_n + nu_{n-1} + \cdots + n(n-1)\cdots(n-k+2)u_{n-k+1}\right\} X^n.$$

For any $0 \leqslant r \leqslant k-1$ we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} n(n-1)\cdots(n-r+1)u_{n-r}X^n &= \sum_{i=1}^{t}\sum_{n=0}^{\infty} r!\binom{n}{r} b_i \alpha_i^n X^n \\
&= \sum_{i=1}^{t} r! b_i \alpha_i^r X^r \frac{1}{(1-\alpha_i X)^{r+1}} \\
&= \frac{P_r(X)}{\left(1 - a_1X - \cdots - a_tX^t\right)^{r+1}},
\end{aligned}$$

where $P_r(X)$ is a polynomial of degree $< t(r+1)$. Hence

$$v(X) - w(X) = \frac{P(X)}{\left(1 - a_1X - \cdots - tX^t\right)^k},$$

where $P(X)$ is a polynomial of degree $< tk$. Hence $v_n(k) = w_n(k)\ \forall n \geqslant kt$, where

$$\sum_{n=0}^{\infty} w_n(k)X^n = \left(1 - a_1X - \cdots - a_tX^t\right)^k w(X).$$

Since $k!$ divides $D^{n-k}w_n$, we see that $k!$ divides $D^n w_n(k) = D^n v_n(k)$ as asserted.

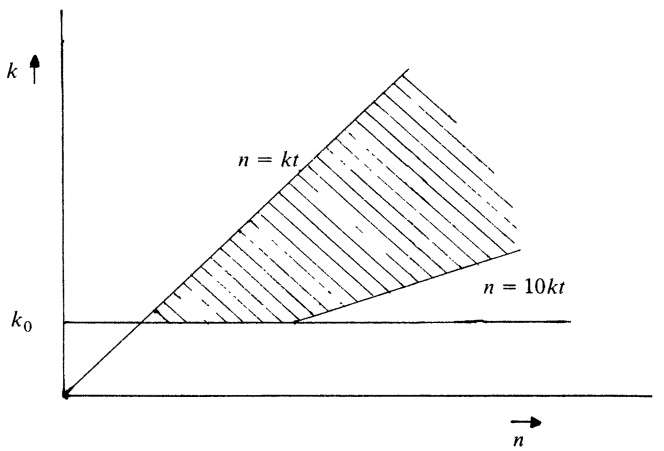
Proof of the Claim. It is sufficient to prove that $\sum_{n=0}^{\infty} v_n(k)X^n \in \mathbb{Q}[X]$ for some $k \in \mathbb{N}$. From the Lemma it follows that if $v_n(k) \neq 0$ and $n \geqslant kt$, then

$$k! \leqslant \left|D^n v_n(k)\right| \leqslant c_2(AD)^n C^k.$$

Hence, if $k! > c_2(AD)^n C^k$ and $n \geqslant kt$ then $v_n(k) = 0$. Choose k_0 so large that $k! > c_2(AD)^{10kt}C^k\ \forall k \geqslant k_0$. Then

$$v_n(k) = 0 \quad \text{for all } k \geqslant k_0,\ kt \leqslant n \leqslant 10kt. \tag{8}$$

This situation can be pictured as follows: If a point (n, k) falls in the shaded region, we have automatically $v_n(k) = 0$ according to (8). So, finally, by (7) and induction on $n - 10kt$, it follows that $v_n(k) = 0$ for all (n, k) in the infinite triangular region $k_0 \leqslant k \leqslant n/10t$. Thus we conclude that $v_n(k_0) = 0\ \forall n \geqslant k_0 t$ and hence $\sum_{n=0}^{\infty} v_n(k)X^n \in \mathbb{Q}[X]$, which proves our claim.



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