REMOLD
Special Semester
Padova 22/4/26

EXPONENTIAL MOTIVES

(joint with P. Jossen)

1) Beginnings

• values of the gamma function

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx \quad \text{at } s \in \mathbb{Q} \setminus \mathbb{Z}$$

They are not expected to be periods but certain products are. Example: E elliptic curve $/\overline{\mathbb{Q}}$

$\int_\sigma \omega$ ← holomorphic differential

in general: new number but if E has CM

$$\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}(\sqrt{-d})$$

then $\int_\sigma \omega$ can be expressed in terms of

$$\Gamma\left(\tfrac{1}{d}\right), \dots, \Gamma\left(\tfrac{d-1}{d}\right) \quad \text{[Chowla–Selberg formula]}$$

Also: $\Gamma\left(\frac{a}{d}\right)^d$ is a product of values of the beta function $B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}\, dx$ which are periods of Fermat curves!

Conjecture (Lang, Rohrlich) For $d \geq 3$,

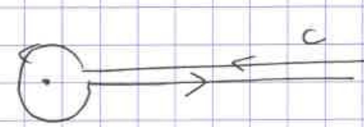
$$\operatorname{trdeg} \overline{\mathbb{Q}}\left(\Gamma\left(\tfrac{1}{d}\right), \dots, \Gamma\left(\tfrac{d-1}{d}\right)\right) = 1 + \frac{\varphi(d)}{2}$$

is NOT in an obvious way an instance of Grothendieck's period conjecture

$$\mathrm{tr\,deg}\ \overline{\mathbb{Q}}\left(\begin{array}{c}\text{periods of a}\\ \text{motive } M\end{array}\right) = \begin{array}{c}\text{dim motivic}\\ \text{Galois group of } M\end{array}$$

I knew for other reasons that $\Gamma(s)$ was a period of a vector bundle with irregular connection (Bloch–Esnault)

$$(e^{2\pi i s}-1)\int_0^\infty e^{-x}x^s\,\frac{dx}{x}$$



$$= \langle c \otimes e^{-x}x^s, \frac{dx}{x}\rangle$$

in $\mathbb{G}_m$, $(\mathcal{O}_X, \nabla(1) = -dx + s\,\frac{dx}{x})$.

Didn't work very well...

$$\Gamma\left(\frac{a}{d}\right) = * \int_0^\infty e^{-x^d}\, x^{a-1}\,dx$$

look at integrals of the shape $\int_\sigma e^{-f}\omega$.

2) <u>Cohomology theories</u>

k subfield of $\mathbb{C}$

X smooth algebraic variety over k (e.g. affine)

$f: X \longrightarrow \mathbb{A}^1_k$ regular function

- $d(e^{-f}\omega) = e^{-f}(d\omega - df \wedge \omega)$

so for a Stokes theorem we introduce

$$H^*_{dR}(X,f) = \mathbb{H}^*_{Zar}(X, \mathcal{O}_X \to \Omega^1_X \to \cdots)$$
$$\omega \mapsto d\omega - df \wedge \omega$$

twisted de Rham cohomology [For f constant: usual de Rham cohomology]

e.g. $X = \mathbb{A}^1$, $x^n = f$
$$\mathbb{Q}[x] \longrightarrow \mathbb{Q}[x]\,dx$$
$$p \longmapsto (p' - nx^{n-1})dx$$

$H^1_{dR}(X,f)$ has basis $dx, x\,dx, \ldots x^{n-2}dx$

different from the analytic de Rham cohomology
$$\mathcal{O}(\mathbb{C}) \longrightarrow \mathcal{O}(\mathbb{C})\,dz$$
$$g \longmapsto (g' - nz^{n-1})\,dz$$
which has trivial H^1 and non-trivial H^0.

$$H^*_{dR,an}(X,f) \cong H^*_{sing}(X(\mathbb{C}), \mathbb{C})$$

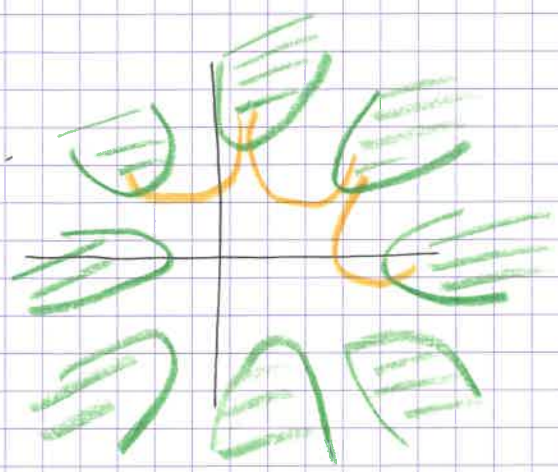
[the connection $\mathcal{E}^f = (\mathcal{O}_X, d - df \wedge \cdot\cdot)$ has irregular singularities at infinity!]

Looking for cycles? Need $\int_\sigma e^{-f}\omega$ to converge on non-compact cycles

$$H_*^{Rd}(X,f) = \lim_{t\to\infty} H_*(X(\mathbb{C}), \{\mathrm{Re} f \geq t\}; \mathbb{Q})$$

rapid decay $\quad \sigma = (\sigma_t)_t \quad \mathrm{Re} f|_{\partial \sigma_t} \geq t$

e.g.



[For f constant: usual Betti cohomology]

Theorem (Hien–Roucairol)

There is a functorial unique isomorphism

$$H^*_{dR}(X,f) \otimes_k \mathbb{C} \xrightarrow{\sim} H^*_{Rd}(X,f) \otimes_{\mathbb{Q}} \mathbb{C}$$

which, for affine X, is dual to the integration pairing

$$H^*_{dR}(X,f) \otimes H_*^{Rd}(X,f) \longrightarrow \mathbb{C}$$

$$([\omega] \quad , \quad [\sigma]) \longmapsto \int_\sigma e^{-f}\omega$$

$$\qquad\qquad = \lim_{t\to\infty} \int_{\sigma_t} e^{-f}\omega$$

Remark: we can also add a subvariety $Y \subset X$, say with simple normal crossings

functorial

1) $\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & X' \arrow[d, "f'"] \\ & \mathbb{A}^1 \end{tikzcd}$ $h(Y) \subseteq Y'$ induces

$$h^* : H_{\substack{dR\\Rd}}(X', Y', f') \to H_{\substack{dR\\Rd}}(X, Y, f)$$

«change of variables»

2) $Z \subseteq Y \subseteq X$ induces

$$\partial : H^{n-1}_{\substack{dR\\Rd}}(Y, Z, f_{|Y}) \longrightarrow H^{n}_{\substack{dR\\Rd}}(X, Y, f)$$

«Stokes»

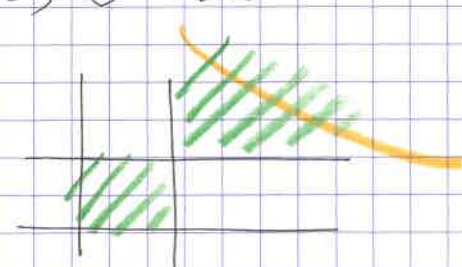
Künneth formula

$$H^*_{\substack{dR\\Rd}}(X \times X', f \boxplus f') = H^*_{\substack{dR\\Rd}}(X, f) \otimes H^*_{\substack{dR\\Rd}}(X', f')$$

We call exponential periods the coefficients of this isomorphism for $K \subseteq \overline{\mathbb{Q}}$. e^{α} for $\alpha \in \overline{\mathbb{Q}}$

e.g. $\sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-x^2} dx$ $\quad H^1(\mathbb{A}^1, x^2)$

$$\gamma = -\Gamma'(1) = -\int_0^{\infty} \log(x)\, e^{-x} dx$$
$$= \iint_{\square - L} e^{-xy} dx\, dy$$



$$I_0(z) = \frac{1}{2\pi i} \oint e^{-\frac{z}{2}\left(x + \frac{1}{x}\right)} \frac{dx}{x}$$

Bessel function $H^1\left(\mathbb{A}^1 \setminus \{0\}, \frac{z}{2}\left(x + \frac{1}{x}\right)\right)$.

3) Exponential motives

Theorem (with Peter Jossen)

There exists a $\mathbb{Q}$-linear neutral tannakian category $M^{exp}(k)$ with realisation functors

$$\begin{tikzcd} & \mathrm{Vect}_k \quad H^n_{dR}(X,Y,f) \\ M^{exp}(k) \arrow[ur, "\omega_{dR}"] \arrow[dr, "\omega_{Rd}"'] & \\ & \mathrm{Vect}_{\mathbb{Q}} \quad H^n_{Rd}(X,Y,f) \end{tikzcd}$$

contains objects $H^n(X,Y,f)$

1) Every object is a subobject of some $H^n(X,Y,f)$

2) $M^{exp}(k)$ contains Nori's category of motives over k as a full subcategory

In particular, for every object M of $M^{exp}(k)$

$$\langle M \rangle^{\otimes} \simeq \mathrm{Rep}\, G_M \qquad G_M \subseteq GL(\omega_{Rd}(M))$$

and there is a conjecture in the style of Grothendieck's period conjecture

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}} (\text{periods of } M) = \dim G_M .$$

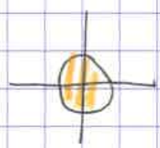
Surprising features

① $H^1(\mathbb{A}^1, x^2)^{\otimes 2} \cong H^1(\{x^2+y^2=1\})$ classical motive!

$\quad\|$

$H^2(\mathbb{A}^2, x^2+y^2)$

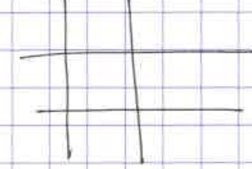
$$(\sqrt{\pi})^2 = \int_{\mathbb{R}^2} e^{-x^2-y^2}\, dx\, dy$$



so a motive with a non zero function can be classical

and there is NO Hodge / ℓ-adic realisation compatible with $\otimes$ product.

② Image of $M(k)$ in $M^{\exp}(k)$ is NOT closed under extension



$$\pi : Bl_{(1,1)} \mathbb{A}^2 \longrightarrow \mathbb{A}^2$$

$$\pi^* : H^2(\mathbb{A}^2, Y, f) \longrightarrow H^2(\tilde{\mathbb{A}}^2, \tilde{Y}, \tilde{f})$$

has rank 2 $\quad \operatorname{Im} \pi^* = M(\gamma)$

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow M(\gamma) \longrightarrow \mathbb{Q}(-1) \longrightarrow 0$$

is a non-classical extension of classical motives: converges to $\begin{pmatrix} 1 & \zeta(1) \\ 0 & 2\pi i \end{pmatrix}$!

The category is constructed using Nori's formalism of quiver representations. It's very useful for transitive aspects and construction of realisation functors but has the big drawback that we have control on morphisms.

4) The perverse realisation

$z \longmapsto \int_\sigma e^{-zf} \omega$ is a solution of a differential equation, essentially the Fourier transform of $f_+ \mathcal{O}_X$ $\quad f: X \to \mathbb{A}^1$

$$\mathrm{PERV}_0 = \begin{pmatrix} \text{constructible} \\ \text{sheaves on } \mathbb{A}^1 \\ \text{with vanishing} \\ \text{cohomology} \end{pmatrix} [1] \subset \mathrm{PERV}$$

e.g. $E(\ ) = j_{S!} j_S^* \mathbb{Q}[1]$ for $j: \ \hookrightarrow \mathbb{A}^1$

Reason to impose the condition:

additive convolution

$$*: D^b_c(\mathbb{A}^1,\mathbb{Q}) \times D^b_c(\mathbb{A}^1,\mathbb{Q}) \longrightarrow D^b_c(\mathbb{A}^1,\mathbb{Q})$$

$$A * B = R\mathrm{sum}_*(A \boxtimes B)$$

in general does not preserve PERV but

$$*: \mathrm{PERV}_0 \times \mathrm{PERV} \longrightarrow \mathrm{PERV}_0$$

so in particular we can define a projector

$$\Pi : \mathrm{PERV} \longrightarrow \mathrm{PERV}_0 \qquad \text{e.g.}$$
$$A \longmapsto A * E(0) \qquad \Pi(\delta_s) = E(s)$$

It's a Tannakian category with fiber functor $\Psi_\infty : \mathrm{PERV}_0 \longrightarrow \mathrm{Vect}_\mathbb{Q}$

$$F[1] \longmapsto \underset{t\to\infty}{\mathrm{colim}}\, F(\mathrm{Re}\, z > t)$$

nearby cycles at infinity

Realisation functor: $\qquad \beta: X \setminus Y \hookrightarrow X$

$$M^{exp}(k) \longrightarrow \mathrm{PERV}_0 \qquad \beta_!\beta^*\mathbb{Q}_X$$

$$H^n(X,Y,f) \longmapsto \Pi\left({}^p\mathcal{H}^n\left(Rf_*\mathbb{Q}_{[X,Y]}\right)\right)$$

$$= R^n pr_* \mathbb{Q}_{[X\times\mathbb{A}^1,\,(Y\times\mathbb{A}^1)\cup\Gamma_f]}[1]$$

recovers the rapid decay realisation!

If $f=0$, then the perverse realization is a sum of copies of $E(0)$. We call it trivial. (trivial representation).

Theorem (with P. Jossen)

An exponential motive is classical if and only if its perverse realization is trivial.

Example: $\Psi : \mathbb{F}_p \to \mathbb{C}^\times$ nontrivial additive character

$$\sum_{\substack{z\in \mathbb{F}_p \\ x \in X(\mathbb{F}_p)}} \Psi(z f(x)) = p \left| \{f=0\}(\mathbb{F}_p) \right|$$

suggests that $H^*(X \times \mathbb{A}^1, zf)$ is always a classical motive. If $\{f=0\}$ is smooth, we can use Gysin

$$H^n(U \times \mathbb{A}^1, zf) \longrightarrow H^{n-1}(Z \times \mathbb{A}^1)(-1) \longrightarrow H^{n+1}(X \times \mathbb{A}^1, zf) \longrightarrow \cdots$$

plus the fact that

$$\begin{tikzcd} U \times \mathbb{A}^1 \arrow[rr] \arrow[dr, "zf"'] & & U \times \mathbb{A}^1 \arrow[dl, "0 \boxplus t"] \\ & \mathbb{A}^1 & \end{tikzcd} \qquad (x,z) \longmapsto (x, zf(x))$$

is an isomorphism.

In general, use the theorem.

Very useful for the applications.

Example: Under the period conjecture, e is transcendental over all classical periods

M motive over k

$M^+ = M \oplus H^0(\operatorname{Spec} k, -1)$ $\qquad \overline{\mathbb{Q}}(\text{periods}, e)$

$$\langle M \rangle^{\otimes} \subset \langle M^+ \rangle^{\otimes} \rightsquigarrow \quad \begin{tikzcd} G_{M^+} \arrow[r, two heads] & G_M \\ G_{R_{per}}(M^+) \arrow[u, hook] \arrow[r] & G_{R_{per}}(M) \arrow[u, hook] \end{tikzcd}$$

$$G_{R_{per}}(M) = \{0\}$$

so $G_{R_{per}}(M^+) = G_{R_{per}}(H^0(\operatorname{Spec} k, -1))$ is in the kernel.

This group is $\mathbb{G}_m$!

(For $H^1(\mathbb{A}^1, x^2)$ it would have been the finite group μ_2).

Remarks: 1) there is a realisation functor with values in $\mathrm{Conn}_0(\mathbb{G}_m)$ the image by Fourier transform of $RS_0(\mathbb{A}^1)$.

We recover $H^*_{dR}(X, f)$ by looking at the fiber at $z = 1$.

Applications to E-functions.

2) suggests how to construct motives in a Voevodsky style

$DM^{exp}(k) \subset DM(k)$ full subcategory

of objects such that $\pi_* M = 0$

[Gallauer - Pepin - Lehalleur
Richard - Scholbach]

In this category we can compute extension groups and $M(\gamma)$ is the only new extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$.

3) Irregular Hodge theory (Deligne, Sabbah, Yu)