Milano Winter School 2026 in Algebra e Geometria

A BRAND-NEW IRRATIONALITY THEOREM

18–20/2/26 ①

based on a 2024 paper by F. Calegari, V. Dimitrov and Y. Tang
« The linear independence of 1, $\zeta(2)$ and $L(2,\chi_{-3})$ » 218 pages

- **Theorem** (Calegari - Dimitrov - Tang)

  The number
  $$L(2,\chi_{-3}) = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{25} + \frac{1}{49} - \cdots$$
  $$= \sum_{n=0}^{\infty} \left[\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right]$$
  is irrational. Moreover, the numbers
  $$1, \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad L(2,\chi_{-3})$$
  are $\mathbb{Q}$-linearly independent, that is
  $$a + b\,\zeta(2) + c\,L(2,\chi_{-3}) = 0$$
  with $a,b,c \in \mathbb{Q} \Rightarrow a=b=c=0$.

  [Note that the second statement implies the first one]

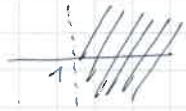
- **Notation**: χ_{-3} is the quadratic Dirichlet character modulo 3

$$\chi_{-3} : \underbrace{(\mathbb{Z}/3\mathbb{Z})^\times}_{\text{modulo } 3} \longrightarrow \underbrace{\{\pm 1\}}_{\text{quadratic}} \qquad \underbrace{\chi_{-3}(-1) = -1}_{\text{nontrivial}}$$

that we can look at as defined on all integers by the formula

$$\chi_{-3}(n) = \begin{cases} 0 & \text{if } 3 \text{ divides } n \\ 1 & \text{if } n = 3k+1 \\ -1 & \text{if } n = 3k+2 \end{cases}$$

and $L(s, \chi_{-3}) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}$ is the



associated Dirichlet series.

[It is a priori only defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ but it extends to a holomorphic function on the whole complex plane]

- For the values of the Riemann zeta function we know that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

Euler 1735

$$\zeta(2k) \in \pi^{2k} \cdot \mathbb{Q}^{\times} \quad \text{for all } k \geq 1$$

(for example, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}, \ldots$)

In particular, since Lindemann proved that π is transcendental in 1882 we deduce that $\zeta(2), \zeta(4), \ldots$ are transcendental.

There is no nonzero $P \in \mathbb{Q}[x]$ such that $P(\pi) = 0$

- The numbers $\zeta(2k+1)$ are also expected to be transcendental, and even algebraically independent with π: we conjecture that

$$P(\pi, \zeta(3), \dots \zeta(2k+1)) = 0 \implies P = 0$$

$$\uparrow$$

$$\mathbb{Q}[x_0, x_1, \dots x_k]$$

but this is not known for a single value of k

"Best": $\zeta(3)$ is irrational (Apéry, 1978)

- The difference between values at even and at odd integers is "explained" by the functional equation for

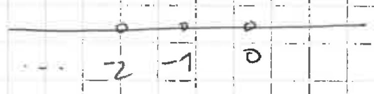
$$\pi^{-s/2}\,\Gamma\!\left(\frac{s}{2}\right)\zeta(s) = \xi(s)$$

completed zeta function

gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \qquad \operatorname{Re} z > 0$$

extends meromorphically with poles at $0, -1, -2, \dots$



Functional equation: $\boxed{\xi(s) = \xi(1-s)}$

If s is an even positive integer, then the gamma factors have no poles and we can transfer the information that $\zeta(1-2k) \in \mathbb{Q}$.

At odd integers there are poles and we gets nothing out of the functional equation.

- If χ is a primitive quadratic Dirichlet character of conductor D [convention: sign of $D = \chi(-1)$]

$$\begin{tikzcd} (\mathbb{Z}/D\mathbb{Z})^\times \arrow[r] \arrow[d] & \{\pm 1\} \\ (\mathbb{Z}/D'\mathbb{Z})^\times \arrow[ur] & \end{tikzcd}$$

$D' | D$

then $$L(k, \chi) \in \begin{cases} \pi^k \sqrt{D}\, \mathbb{Q}^\times & \text{if } k \text{ is even and } \chi(-1) = 1 \\ \pi^k \sqrt{-D}\, \mathbb{Q}^\times & \text{if } k \text{ is odd} \geq 1 \text{ and } \chi(-1) = -1 \end{cases}$$

and in particular it is a transcendental number. So the theorem

$$L(2, \chi_{-3}) \notin \mathbb{Q}$$

covers the first case where there is no explicit formula. Moreover, $L(2, \chi_{-3})$ is not a rational multiple of π^2.

To compare: $$L(3, \chi_{-3}) = \frac{4\pi^3}{243\sqrt{3}}$$

$$L(1, \chi_{-3}) = \frac{\pi}{3\sqrt{3}}$$

- In the case $\chi(-1) = 1$ and $D > 1$, we also know that $L(1, \chi)$ is $\sqrt{D}$

③

thus the logarithm of an algebraic number $\alpha \notin \{0,1\}$ and for those Lindemann's theorem gives $\log(\alpha)$ is transcendental.

- Next open case: Catalan's constant

$$L(2,\chi_{-4}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots$$

- How do we prove that some given real number $\beta \in \mathbb{R}$ is irrational?

<u>Lemma</u>: Assume that there exists a sequence of integers p_n, q_n such that $q_n\beta - p_n \neq 0$ and $|q_n\beta - p_n| \to 0$ as $n \to \infty$. Then β is irrational.

<u>Proof</u>: Since $0 < |q_n\beta - p_n| < 1$ for large n, we have $q_n \neq 0$, and we may assume without loss of generality that $q_n \neq 0$ for all n.

By contradiction, assume $\beta = \frac{p}{q}$. Then

$$\left|\beta - \frac{p_n}{q_n}\right| = \left|\frac{p}{q} - \frac{p_n}{q_n}\right| = \left|\frac{q_n p - p_n q}{q q_n}\right|$$

so $|q_n\beta - p_n| = \left|\frac{q_n p - p_n q}{q}\right|$. Then

$$0 < |q_n p - p_n q| < 1 \text{ for large enough } n,$$

which is a contradiction since there are no integers in the open interval $(0,1)$. $\square$

- Example: Irrationality of e

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

We consider the partial sums

$$\sum_{k=0}^{n} \frac{1}{k!} = \frac{p_n}{q_n} \qquad q_n = n! \qquad p_n = \sum_{k=0}^{n} \frac{n!}{k!}$$

and estimate

$$0 < e - \frac{p_n}{q_n} = \sum_{k=n+1}^{\infty} \frac{1}{k!}$$

$$= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right)$$

$$< \frac{1}{(n+1)!} \frac{1}{1-\frac{1}{n+2}} = \frac{n+2}{(n+1)^2 n!}$$

and hence

$$0 < q_n e - p_n < \frac{n+2}{(n+1)^2} \xrightarrow[n\to\infty]{} 0 \qquad \square$$

Another way to proving irrationality of e is to consider the integrals

$$I_n = \int_0^1 \frac{x^n(1-x)^n}{n!} e^x \, dx$$

On the one hand,

$$0 < I_n \leq \frac{e}{n!} \qquad 0 \leq x(1-x) \leq 1, \quad e^x \leq e$$

On the other hand, using integration by parts we finds

$$I_n = -(4n-2) I_{n-1} + I_{n-2}$$

and from the initial values $I_0 = e-1$ and $I_1 = -e+3$, that

$$I_n = q_n e + p_n \quad \text{for integers } q_n, p_n$$

- Irrationality of $\zeta(2)$ according to Apéry and Beukers

Recall: $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Lemma: (1) For every integer $r \geq 0$,

$$\int_0^1\int_0^1 \frac{x^r y^r}{1-xy} = \sum_{n=1}^{\infty} \frac{1}{(n+r)^2} = \zeta(2) - \left(1 + \frac{1}{4} + \dots + \frac{1}{r^2}\right)$$

(2) For all integers $r > s \geq 0$,

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} = \frac{1}{r-s}\left[\frac{1}{s+1} + \dots + \frac{1}{r}\right]$$

natural number!

[Work with $\int^{1-\varepsilon}$ and $\varepsilon \to 0^+$ if we want to be careful]

<u>Proof</u>: just write $\frac{1}{1-xy} = \sum_{n=0}^{\infty} (xy)^n$ and integrate term by term. The partial fraction decomposition

$$\frac{1}{(n+r+1)(n+s+1)} = \frac{1}{r-s}\left(\frac{1}{n+s+1} - \frac{1}{n+r+1}\right)$$

creates a "telescoping sum" $\square$

Denote $\ell_n = \operatorname{lcm}(1, 2, \dots n)$. Then the formulas in the lemma show that

$$\int_0^1 \int_0^1 \frac{x^r y^r}{1-xy} = \zeta(2) - \frac{a}{\ell_r^2}$$

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} = \frac{b}{\ell_r^2}$$

where a and b are integers. We deduce

$$\int_0^1 \int_0^1 \frac{P(x)\,Q(y)}{1-xy} \in \frac{1}{\ell_n^2}\left(\mathbb{Z}\,\zeta(2) + \mathbb{Z}\right)$$

for all polynomials $P, Q \in \mathbb{Z}[x]$ of degree n.

Choose $\boxed{P_n(x) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n \left(x^n(1-x)^n\right)}$

Claim: $P_n(x) \in \mathbb{Z}[x]$ of degree n

$$I_n = \int_0^1\int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} = \frac{a_n \zeta(2) + b_n}{\ell_n^2}$$

- Integrating by parts n times

$$I_n = (-1)^n \int_0^1\int_0^1 \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^{n+1}}\, dx\, dy$$

[note that all derivatives of $x^n(1-x)^n$ vanish at 0 and 1]

In particular: $\boxed{I_n > 0}$

- The function

$$f(x,y) = \frac{x(1-x)y(1-y)}{1-xy} \qquad \begin{array}{l} 0 \le x < 1 \\ 0 \le y < 1 \end{array}$$

reaches its maximum at

$$x = y = \frac{\sqrt{5}-1}{2}$$

and $f\left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right) = \left(\frac{\sqrt{5}-1}{2}\right)^5$

Therefore, we get:

$$0 < I_n \leq \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \underbrace{\int_0^1\!\!\int_0^1 \frac{1}{1-xy}\,dx\,dy}_{\zeta(2)}$$

- On the other hand, we know that

$$\ell_n = \operatorname{lcm}(1, \dots n) < 3^n \quad \text{(exercise sheet)}$$

for all n, so:

$$0 < \underset{\substack{\| \\ a_n \zeta(2) + b_n}}{\ell_n^2 I_n} < \zeta(2) \cdot \left(3^2 \cdot \left(\frac{\sqrt{5}-1}{2}\right)^5\right)^n$$

and comes the moment of truth! so $\downarrow n \to \infty$ 0

$$3^2 \left(\frac{\sqrt{5}-1}{2}\right)^5 \approx 0{,}812\dots$$

<u>Proof of claim</u>: $\displaystyle\sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n+k}$

so the coefficients of $\frac{1}{n!}\left(\frac{d}{dx}\right)^n$ are

$$\frac{1}{n!}\binom{n}{k}\frac{(n+k)!}{k!}(-1)^k = (-1)^k \binom{n}{k}\binom{n+k}{k} \in \mathbb{Z}$$

$\square$

- The proof of irrationality of $\zeta(3)$ is very similar, except that we work with the integrals

$$\int_0^1\int_0^1\int_0^1 \frac{P_n(x)\cdot P_n(y)}{1-(1-xy)z}\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}}\,dx\,dy\,dz$$

The integrand is this time $\leq (\sqrt{2}-1)^4$
and still $3^3(\sqrt{2}-1)^4 \approx 0,79\ldots$

cuts in the denominators.

Margin of improvement: $\operatorname{lcm}(1,\ldots n) \sim e^n$
by the prime number theorem

For $L(2,\chi_{-3})$ linear forms $a_n + b_n L(2,\chi_{-3})$
were known but at the end of the argument the estimate is not good enough to yield irrationality.

In the work of Calegari–Dimitrov–Tang these linear forms are used in a more clever way.

LECTURE 2

From previous episodes...

- We proved irrationality of $\zeta(2)$ by considering the integrals

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{y^n(1-y)^n x^n (1-x)^n}{(1-xy)^{n+1}} \, dx \, dy$$

$$= \frac{a_n \zeta(2) + b_n}{l_n^2} \qquad l_n = \mathrm{lcm}(1, \dots, n)$$

and exploiting the fact that

$$0 < l_n^2 |I_n| \xrightarrow[n \to \infty]{} 0$$

for which it was crucial that

$$9 \cdot \left(\frac{\sqrt{5}-1}{2}\right)^5 < 1$$

- The proof of irrationality of $\zeta(3)$ follows the same pattern starting with the integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{P_n(x) P_n(y)}{1-(1-xy)z} \, dx \, dy \, dz$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n z^n (1-z)^n}{(1-(1-xy)z)^{n+1}} \, dx \, dy \, dz$$

$$= \frac{a_n \zeta(3) + b_n}{l_n^3}$$

The integrand is this time $\leq (\sqrt{2}-1)^4$ and still

$$3^3 \cdot (\sqrt{2}-1)^4 \approx 0{,}79\ldots < 1$$

[Note the margin of improvement coming from the prime number theorem: $\ell_n \sim e^n$]

Main new point: notice an irrationality proof with linear forms $a_n L(2, \chi_{-3}) + b_n$ that are not good enough to yield irrationality in the straightforward way.

Point of view: form generating series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + 5x + 73x^2 + \cdots \in \mathbb{Z}[[x]]$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 6x + \frac{351}{4} x^2 + \cdots \in \mathbb{Q}[[x]]$$

with $\frac{b_n}{a_n} \xrightarrow[n\to\infty]{} \zeta(3)$. Explicitly, in the situation of Apéry–Beukers

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$b_n = a_n \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}$$

The coefficients are solutions of the same recurrence relation

$$(n+1)^3 u_n = (34n^3 + 51n^2 + 27n + 5)\, u_n - n^3 u_{n-1}$$

with different initial conditions. It translates into differential equations

$$L \cdot A(x) = 0$$
$$L \cdot B(x) = 5$$

for the differential operator

$$L = (x^4 - 34x^3 + x^2)\left(\frac{d}{dx}\right)^3 + (6x^3 - 153x^2 + 3x)\left(\frac{d}{dx}\right)^2$$
$$+ (7x^2 - 112x + 1)\frac{d}{dx} + (x-5)$$

singularities:
$0, (\sqrt{2}-1)^4, (\sqrt{2}+1)^4$

$\Rightarrow$ $A(x)$ and $B(x)$ have radius of convergence equal to $(\sqrt{2}-1)^4$. But the linear combination

$$P(x) = B(x) - \zeta(3) A(x)$$

converges more, exactly up to $(\sqrt{2}+1)^4$!

We also know that $a_n \in \mathbb{Z}$, $\ell_n^3 b_n \in \mathbb{Z}$

The key property is that

$$3^3 < (\sqrt{2}+1)^4 \simeq 33{,}97\ldots$$

so that $\ell_n^3 \, |b_n - \zeta(3) a_n| \xrightarrow[n\to\infty]{} 0$.

- <u>Idea</u>: exploit extra properties of power series $A(x)$, $B(x)$ such that

$$B(x) - L(2, X_3)\, A(x)$$

converges more than expected. In fact,

$$H_A(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[[x]]$$
$$= 1 + 3x + 15x^2 + 93x^3 + \cdots$$

$$H_B(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{Q}[[x]]$$
$$= x + \frac{23}{4}x^2 + \frac{145}{4}x^3 + \frac{3993}{16}x^4 + \cdots$$

with $\ell_n^2 b_n \in \mathbb{Z}$, solution of a differential operator L of order 2 with singularities $\{0, \frac{1}{9}, 1, \infty\}$

$L \cdot H_A = 0$

$L \cdot H_B = 1$

$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$

and such that

$$H_B(x) - \frac{L(2, \chi_{-3})}{2} H_A(x)$$

has radius of convergence $\boxed{= 1.}$

Difference: For $\zeta(3)$, we proved that a power series $\sum_{n=0}^{\infty} c_n x^n$ of radius of convergence $(\sqrt{2}+1)^4$ and $\ell_n^3 c_n \in \mathbb{Z}$ is a polynomial

But there are many non-polynomial power series of radius of convergence 1 with $\ell_n^2 c_n \in \mathbb{Z}$! For example,

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\log^2(1-x)$$

Then $f \in \mathbb{Q}(x)$.

Imagine the <u>following application</u>:

$A(x), B(x) \in \mathbb{Z}[[x]]$ solutions of a differential equation with singularities $0, \alpha < \beta, \infty$
real numbers

$P(x) = B(x) - \eta A(x)$

- holomorphic at α
- analytically does not continue to a meromorphic function at β.

If $4\beta > 1$, then η is irrational.

[This is an improvement on the $\beta > 1$ from the <u>basic lemma</u>]

<u>Proof</u>: Indeed, if $\eta \in \mathbb{Q}$, a multiple of P is a power series with integer coefficients holomorphic on $\mathbb{C} \setminus [\beta, \infty)$ of ~~radius~~ conformal radius > 1.
So $P(x) \in \mathbb{Q}(x)$, contradiction with the assumption that it does not extend meromorphically. $\square$

<u>Theorem</u> (Calegari - Dimitrov - Tang)
Let $\varphi : \overline{\mathbb{D}} \to \mathbb{C}$ be holomorphic (= extends to some holomorphic function on some open neighborhood of $\overline{\mathbb{D}}$ in $\mathbb{C}$) with $\varphi(0) = 0$ and $|\varphi'(0)| > 1$.

Then the $\mathbb{Q}(x)$-linear span $\mathcal{H}(\varphi)$ of power series $f \in \mathbb{Z}[\![x]\!]$ such that

$$f(\varphi(z)) \in \mathbb{C}[\![z]\!]$$

converges in some open neighborhood of $\overline{\mathbb{D}}$ has dimension

$$\dim_{\mathbb{Q}(x)} \mathcal{H}(\varphi) \leq e \; \frac{\int_0^1 \max(0, \log|\varphi(e^{2\pi i z})|)\,dz}{\log|\varphi'(0)|}$$

the number e!

$$\int_{\mathbb{T}} \log^+|\varphi|\,\mu$$

<u>Remark 1</u>: If $f \in \mathbb{Z}[\![x]\!]$ belongs to $\mathcal{H}(\varphi)$, then $f^n \in \mathcal{H}(\varphi)$ for all $n \geq 1$, so finite dimensionality implies that f is <u>algebraic</u> over $\mathbb{Q}(x)$, i.e. there exist polynomials $P_0, \dots P_d \in \mathbb{Q}[x]$ such that

$$P_0(x) + P_1(x) f(x) + \dots + P_d(x) f(x)^d = 0$$

This was first proved by Yves André.

Not all algebraic power series in $\mathbb{Q}[\![x]\!]$ have rational coefficients, e.g.

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{4^n}$$

but a theorem of Eisenstein says this

is true after scaling the variable (only finitely many primes in the denominators).

<u>Remark 2</u>: Calegari - Dimitrov - Tang need to <u>add denominators</u> for the applications to irrationality. In that case, primes will in general have worse denominators but you can consider derivatives $f, f', f'', \ldots$ and finite dimensionality will imply that the power series are solutions of a differential equation.

Example of statement:

$$b \in \mathbb{Q}_{\geq 0}, \quad \sigma \in \mathbb{Z}_{\geq 0} \quad \varphi : \overline{\mathbb{D}} \to \mathbb{C} \quad \varphi(0) = 0$$

$$|\varphi'(0)| > e^{b\sigma}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{\operatorname{lcm}(1, \ldots, \lfloor bn \rfloor)^{\sigma}} x^n \in \mathbb{Q}[[x]] \qquad a_n \in \mathbb{Z}$$

$$\dim_{\mathbb{Q}(x)} \mathcal{H}(\varphi, b, \sigma) \leq 2 \frac{\int_{\mathbb{T}} \log^+ |\varphi| \mu}{\log |\varphi'(0)| - b\sigma}$$

Now, let's try to prove the theorem:

assume $a + \frac{b}{2} L(2, \chi_{-3}) + c \frac{\zeta(2)}{4} = 0$

for some integers a, b, c not all equal to 0. We need to reach a contradiction.

Then there exists a power series

$$H(x) = a A(x) + b B(x) + c C(x) \in \mathbb{Q}[[x]]$$

with denominators $\mathrm{lcm}(1,\dots,n)^2$, solution of a differential equation

$$\underbrace{x(1-x)(1-9x)}_{\substack{\text{singularities} \\ 0,\ \frac{1}{9},\ 1}}\left(\frac{d}{dx}\right)^2 + \cdots$$

that converges up to $R=1$.

The series

$$A(x) = 1 + 3x + 15x^2 + 93x^3 + \cdots$$

$$B(x) = x + \frac{23}{4}x^2 + \frac{145}{4}x^3 + \cdots$$

$$C(x) = x + 6x^2 + \frac{343}{9}x^3 + \cdots$$

were found by Zagier using modular forms. Recall

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}\, z > 0 \}$$

$SL_2(\mathbb{R})$ acts by Möbius transformations

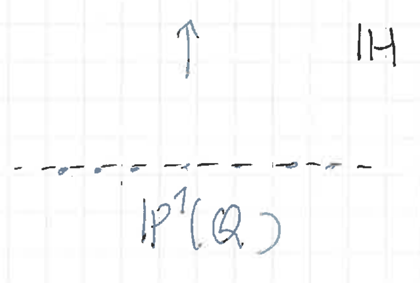
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$\Gamma \subset SL_2(\mathbb{Z})$

A modular form of weight k for Γ is a holomorphic function

$$f : \mathbb{H} \longrightarrow \mathbb{C}$$

satisfying:



$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and that f extends to a meromorphic function on $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$

e.g. $\Gamma_0(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod 6 \right\}$

"congruence subgroup"

Then there are 4 cusps: $0, \frac{1}{3}, \frac{1}{2}, i\infty$

$$X : \mathbb{H}/\Gamma_0(6) \xrightarrow{\sim} \mathbb{P}^1 \setminus \{0, \tfrac{1}{9}, 1, \infty\}$$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$z \longmapsto q \prod_{n=1}^{\infty} \frac{(1-q^n)^4 (1-q^{6n})^8}{(1-q^{2n})^8 (1-q^{3n})^4}$$

so invariant under $z \mapsto z+1$

$$= q - 4q^2 + 10q^3 + \cdots$$

$$\boxed{q = e^{2\pi i z}}$$

In fact,

$$q = X + 4X^2 + 22X^3 + \cdots \in \mathbb{Z}[X]$$

so that we can identify $\mathbb{Z}[q] = \mathbb{Z}[X]$

The series A, B, C are obtained from modular forms through this transformation, e.g.

$$\Theta_{-3}(z) = \sum_{m,n \in \mathbb{Z}} q^{m^2+mn+n^2}$$

$$= 1 + 6 \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_{-3}(d) \right) q^n$$

$$A = \frac{\theta_{-3}(z) + \theta_{-3}(2z)}{2} \in M_1(\Gamma_0(6), \chi_{-3})$$

$\chi_{-3}(d)(cz+d)f(z)$

computed in the variable x.

Modify $H(x)$ as follows: $y = x + \frac{x}{x-1}$

and consider the unique power series

$\mathbb{Q}[\![x]\!] = \mathbb{Q}[\![y]\!]$

$$G(y) = H(x) + H\left(\frac{x}{x-1}\right)$$

$i(x) = \frac{x}{x-1}$

It has denominator $\operatorname{lcm}(1, \dots, 2n)^2$ and singularities at $0, 4, \infty, -\frac{1}{72}$, so it is holomorphic on $\mathbb{C} \setminus [4, \infty)$.

Unfortunately, $\Omega = \mathbb{C} \setminus [4, \infty)$ has conformal radius $|\varphi'(0)| = 16$, which is not bigger than $e^4 \approx 54{,}59\ldots$ so we can't apply the theorem on this domain...

Need a better choice of φ!

<u>Idea</u>: $h : \mathbb{D} \longrightarrow \mathbb{C}$

$$q \longmapsto -256\, q \prod_{n=1}^{\infty} (1+q^n)^{24}$$

and consider $\varphi = h(\Psi(x))$ for some $\Psi : \mathbb{D} \to \mathbb{D}$ such that $\Psi(\mathbb{D})$ only contains one preimage of $-\frac{1}{72}$ under h

and make the following bound as small as possible. Calegari - Dimitrov - Tang find a Φ for which this bound is

$$13{,}9938\ldots < 14$$

So we need 14 linearly independent functions in the space:

$$1, \; -\log(1-x), \; \log^2(1-x), \; \mathrm{Li}_2(x)$$

$$1, \; f_2(y), \; f_3(y), \; f_4(y)$$

$$\sum_{n=1}^{\infty} \frac{(n-1)!^2}{(2n-1)!\,(2n-1)}\, y^n$$

$$\int \frac{f_3(y)}{y}\, dy = \sum_{n=1}^{\infty} \frac{(n-1)!^2}{n\,(2n)!}\, y^n$$

$$\int \frac{f_4(y)}{y}\, dy$$

$$\int G(y)\, dy, \quad \int \frac{G(y) - G(0)}{y}\, dy$$

$$\int \frac{G(y) - G(0) - G'(0)\, y}{y^2}\, dy$$

$$G(y), \; G'(y), \; G''(y), \; G'''(y)$$

<u>Proposition</u> These 14 power series are linearly independent over $\mathbb{C}(y)$.