TALK B4: FUNCTIONAL BAD APPROXIMABILITY

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① Exponentials

$\alpha_1, \dots, \alpha_m \in \mathbb{C}$ distinct

$$e^{\alpha_i x} = \sum_{n=0}^{\infty} \frac{\alpha_i^n}{n!} x^n \in \mathbb{C}[[x]]$$

Theorem

1) $e^{\alpha_1 x}, \dots, e^{\alpha_m x}$ are $\mathbb{C}[x]$-linearly independent

2) For all $P_1, \dots, P_m \in \mathbb{C}[x]^m \setminus \{0\}$,

$$\mathrm{ord}_{x=0} \left(\sum_{i=1}^{m} P_i(x) e^{\alpha_i x} \right) \leq \underbrace{\sum_{i=1}^{m} (\overset{= d_i}{\deg P_i} + 1) - 1}_{N}$$

non-zero by 1)

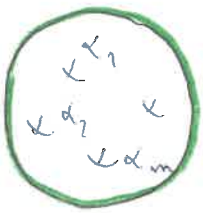
smallest k such that x^k has non-zero coeff

3) There is a unique up to scalar tuple for which equality holds.

4) It is given by

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{e^{zx}}{(z-\alpha_1)^{d_1+1} \cdots (z-\alpha_m)^{d_m+1}} \, dz$$

[Hermite 1893]



Proof: 1) Assume $\sum_{i=1}^{m} P_i(x) e^{\alpha_i x} = 0$.

If all α_i have different real part, pick α_k with maximal $\mathrm{Re}(\alpha_k)$, say >0 and factor $e^{\alpha_k x}$:

$$0 = e^{\alpha_k x}\Big(P_k(x) + \underbrace{\sum_{i\neq k} P_i(x) e^{(\alpha_i-\alpha_k)x}}_{\substack{\downarrow \; x\to+\infty \\ \text{(rapid decay)} \\ 0}}\Big)$$

so $P_k(x) \longrightarrow 0$ as $x \to +\infty$, which means $P_k(x) = 0$. + keep going

[otherwise, choose $\lambda \in \mathbb{C}$ such that all $\alpha_i \lambda$ have different real part λt]

2-3) Consider the differential operator

$$L = \prod_{i=1}^{m} \left(\frac{d}{dx} - \alpha_i\right)^{\deg P_i + 1} \quad \text{of order } N$$

Since $\left(\frac{d}{dx} - \alpha_i\right)\left(P_i(x) e^{\alpha_i x}\right) = P_i'(x) e^{\alpha_i x}$,

this annihilates the linear combination.

But the space of solutions of

$$L = \left(\frac{d}{dx}\right)^N + \cdots$$

and 0 is N-dimensional and

$$f \longmapsto \left(f(0), f'(0), \ldots, f^{(N-1)}(0)\right)$$

is a bijection (Cauchy's theorem).

- If $\mathrm{ord}_{x=0}\left(\sum_{i=1}^{m} P_i(x)\, e^{\alpha_i x}\right) \geq N$, then it has to be zero, impossible by 1). $\longrightarrow \boxed{2)}$
- The space where $\mathrm{ord}_{x=0} = N-1$ is 1-dimensional $\longrightarrow \boxed{3}$

4) By the residue theorem,

$$\frac{1}{2\pi i}\int_{|z|=R} \frac{e^{zx}}{(z-\alpha_1)^{d_1+1}\cdots(z-\alpha_m)^{d_m+1}}\,dz$$

$$= \sum_{i=1}^{m} \underbrace{\mathrm{Res}_{z=\alpha_i}}_{\frac{1}{d_i!}\left(\frac{d}{dz}\right)^{d_i}\left[(z-\alpha_i)^{d_i+1}\times \text{---}\right]} \left(\frac{e^{zx}}{(z-\alpha_1)^{d_1+1}\cdots(z-\alpha_m)^{d_m+1}}\right)$$

$$= \sum_{i=1}^{m} \underbrace{P_i(x)}_{\text{polynomial of degree } d_i} e^{\alpha_i x}$$

On the other hand,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}\, \underbrace{\frac{1}{2\pi i}\int_{|z|=R} \frac{z^n}{(z-\alpha_1)^{d_1+1}\cdots(z-\alpha_m)^{d_m+1}}\,dz}_{\text{residue at } \infty}$$

$$= \frac{x^{N-1}}{(N-1)!} + O(x^N)$$

(explicit formula $\displaystyle\sum_{r=0}^{\infty} \frac{x^{N-1+r}}{(N-1+r)!} \sum_{k_1+\dots+k_m=r} \prod_{j=1}^{m} \binom{d_j+k_j}{k_j} \alpha_j^{k_j}$)

$\square$

Comments: 1) In particular, e^x is a transcendental function [use $(e^x)^k = e^{kx}$!]

2) For a rational function $\frac{P(x)}{Q(x)}$,

$$\mathrm{ord}_{x=0}\left(e^x - \frac{P(x)}{Q(x)}\right) \le \underbrace{\deg P}_{m} + \underbrace{\deg Q}_{n} + 1$$

so the exponential function is badly approximable by rational functions

[analogue of real algebraic numbers of degree ≥ 2 being badly approximable by rational numbers]

Definition. We call Hermite-Padé approximant of order $[m/n]$ the unique such P/Q. Explicit formulas:

$$P_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{(m+n-k)!}{(m+n)!} x^k = {}_1F_1\left(\begin{matrix}-m\\-m-n\end{matrix}\,\middle|\, x\right)$$

$$Q_n(x) = \sum_{k=0}^{n} \cdots\cdots = {}_1F_1\left(\begin{matrix}-n\\-m-n\end{matrix}\,\middle|\, -x\right)$$

3) A way to reformulate part of the results:

$$\mathbb{C}[X]_{\leq d_1} \times \dots \times \mathbb{C}[X]_{\leq d_m} \longrightarrow \mathbb{C}[[x]]/(x^N)$$

$$P_1, \dots P_m \longmapsto \sum_{i=1}^{m} P_i(x)\, e^{\alpha_i x}$$

is a bijection! (injective by 2) + same dimension N)

2 Transcendence of e

Hermite uses these ideas to prove that the number e is transcendental.

$\iff$ $1, e, e^2, \dots e^d$ are $\mathbb{Q}$-linearly independent for all $d \geq 1$

Lemma: Assume for every $d \geq 1$ every $\varepsilon > 0$, there exist $\boxed{d+1}$ linearly independent elements of $\mathbb{Z}^{\boxed{d+1}}$

$$\underline{b_i} = (q_i, p_{1i}, \dots p_{di}) \qquad q_i > 0$$

such that $\max_{\boxed{1 \leq k \leq d}} |q_i \alpha^k - p_{ki}| \leq \varepsilon$ for all i.

Then $\boxed{\alpha \text{ is transcendental.}}$

<u>Proof</u>: If α is algebraic of degree d, write $n_0 + n_1\alpha + \dots + n_d\alpha^d = 0$ with $n_i \in \mathbb{Z}$.

Then for each i

$$0 = \sum_{k=1}^{d} n_k (q_i \alpha^k - p_{ik}) + \underbrace{n_0 q_i + \sum_{k=1}^{d} n_k p_{ik}}_{\in \mathbb{Z} \text{ pick } i \text{ for which nonzero}}$$

so

$$1 \le \left| \sum_{k=1}^{d} n_k (q_i \alpha^k - p_{ik}) \right| \le \varepsilon \sum_{k=1}^{d} |n_k| \quad \lightning \quad \square$$

- Hermite considers the functions

$$e^x, \dots, e^{dx}$$

and

considers simultaneous Padé approximants

$$n_0, \dots, n_d \qquad N = n_0 + \dots + n_d$$

$$\underset{\deg = N - n_0}{Q(x)}\, e^{kx} - \underset{\deg = N - n_k}{P_k(x)} = O(x^{N+1})$$

$P_1/Q, \dots P_d/Q$ are unique and can be chosen so that $\frac{1}{n_0!} Q(x) \in \mathbb{Z}[x]$, $\frac{1}{n_k!} P_k(x) \in \mathbb{Z}[x]$

[If such a Q exists, then the $P_k(x)$ are uniquely determined as truncation, so it suffices to prove it does:

Unknowns: $N - n_0 + 1$ coeff. in degrees

Equations: for each k, $\underbrace{N-n_k+1, \ldots, N}_{n_k \text{ equations}}$ vanish $\to$

$$n_1 + \cdots + n_d = N - n_0$$

independent because $e^x, e^{2x}, \ldots, e^{dx}$ linearly independent

so Q exists and is unique up to scalar]

$$Q(x)e^{kx} - P_k(x)$$
$$= x^{N+1} e^{kx} \int_0^k t^{n_0}(t-1)^{n_1} \cdots (t-d)^{n_d} e^{-tx}\, dt$$

Apply this to parameters:

$$\begin{array}{c|cccc}
 & n_0 & n_1 & \cdots & n_d \\
\hline
 & n-1 & n & \cdots & n \\
d+1 & n & n-1 & \cdots & n \\
 & \cdots & \cdots & \cdots & \cdots \\
 & n & n & \cdots & n-1
\end{array}$$

to get a matrix of Hermite-Padé approximants

$$\begin{pmatrix} Q_0(x) & P_{10}(x) & \cdots & P_{d0}(x) \\ \vdots & \vdots & & \vdots \\ Q_d(x) & P_{1d}(x) & \cdots & P_{dd}(x) \end{pmatrix}$$

Hermite prs

<u>Proposition</u>: $\boxed{\det = \underset{\uparrow \text{ nonzero}}{*} \, X^{d(d+1)n}}$

<u>Proof</u>: compute degrees

$$\begin{pmatrix} dn & dn-1 & \cdots \\ dn-1 & \ddots & \vdots \\ \cdots & & dn \end{pmatrix} = * \, X^{(d+1)dn} + \cdots$$

on the other hand, the ~~discriminant~~ determinant is also equal to

$$\begin{array}{cccc} \left(\begin{matrix} Q_0(x) \\ \vdots \\ Q_d(x) \end{matrix}\right. & \begin{matrix} P_{01}(x) - Q_0(x)e^{x} \\ \vdots \\ P_{1d}(x) - Q_d(x)e^{x} \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} & \left.\begin{matrix} P_{d0}(x) - Q_0(x)e^{dx} \\ \vdots \\ P_{dd}(x) - Q_d(x)e^{dx} \end{matrix}\right) \\ & \uparrow & & \uparrow \\ & O(X^{(d+1)n}) & & O(X^{(d+1)n}) \end{array}$$

$\square$

$$\therefore \quad q_j = \frac{Q_j(1)}{(n-1)!} \qquad p_{jk} = \frac{P_{jk}(1)}{(n-1)!}$$

give linear independent forms and

$$|q_j e^k - p_{jk}| = \frac{1}{(n-1)!}\left| \int_0^k \frac{t^n (t-1)^n \cdots (t-d)^n}{t-j} e^{k-t}\, dt \right|$$

$$\le \frac{e^d d^{dn+1}}{(n-1)!} \xrightarrow[\text{as } n \to \infty]{} 0 \qquad \square$$

③ <u>The Chudnovsky–Osgood theorem</u>

- Kolchin (1959) proved an analogue of Liouville's inequality for holonomic power series and formulated the problem of proving Roth's theorem:

  given $f \in \mathbb{C}[[x]] \setminus \mathbb{C}(x)$ solution of a differential operator L and $\varepsilon > 0$, there exists $C(\varepsilon, f)$ such that

  $$\mathrm{ord}_{x=0}\left(f(x) - \frac{P(x)}{Q(x)} \right) \le (2+\varepsilon) \max(\deg P, \deg Q) + C(\varepsilon, f)$$

[write it as a norm inequality]

More generally, we could look for analogues of Schmidt's subspace theorem about simultaneous approximation of algebraic numbers.

Theorem (Shidlovsky, Chudnovsky^2, Bertrand - Beukers, Osgood...)

Assume $f_1, \dots, f_m \in \mathbb{C}[[x]]$ are $\mathbb{C}(x)$-linear independent power series such that

$$\boxed{\frac{d}{dx}\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}} \quad A \in M_{m\times m}(\mathbb{C}(z)).$$

Then for all $(P_1, \dots, P_m) \in \mathbb{C}[x]^m \setminus \{0\}$

$$\mathrm{ord}_{x=0}\left(\sum_{i=1}^{m} P_i(x) f_i(x)\right)$$

$$\leq \mu \max_i \deg P_i + \mu h + c$$

where
- μ = minimal order of an ODE satisfied by $\sum_{i=1}^{m} P_i(x) f_i(x)$ $(\leq m)$
- h, c real numbers associated to A

* For example, if A has simple poles $S \subset \mathbb{P}^1$ (regular singularities), then around each $s \in S$ there is a fundamental matrix of solutions of the form $\boxed{Y_s (x-s)^{B_s}}$ $\quad Y_s \in GL_m(\mathbb{C}[[x-s]])$

$B_s \in M_m(\mathbb{C})$

exponents = eigenvalues of B_s well defined.

"$x - \infty = \frac{1}{x}$"

$$h = \sum_{s \in S} \left(\begin{matrix} \text{smallest real part} \\ \text{of an exponent at } s \end{matrix} \right)$$

$$c = (|S| - 2) \frac{m(m-1)}{2}$$

Remark: ① If $f_1, \dots, f_m$ are G-functions, we are in this situation and the exponents are rational numbers.

② Tension between this upper bound and Siegel's lemma which implies that we can find polynomials such that order of vanishing is high.

Sketch of proof: I just want to emphasize $\boxed{\text{one}}$ idea coming from diff. Galois theory:

$$F(x) = \sum_{i=1}^{m} l_i(x)\, f_i(x) \qquad L \cdot F(x) = 0$$

L of minimal degree μ can be obtained as follows: the differential system has a differential Galois group G.

Pick a basis $F_1 = F, F_2, \dots F_\mu$ of the orbit of F under G. Then

$$Ly = \frac{1}{W(F_1, \dots, F_\mu)} \begin{array}{cl} \begin{vmatrix} y & y' & \cdots & y^{(\mu)} \\ F_1 & F_1' & \cdots & F_1^{(\mu)} \\ \vdots & & & \vdots \\ F_\mu & F_\mu' & \cdots & F_\mu^{(\mu)} \end{vmatrix} & \begin{array}{l} \\ \sigma_1 \\ \vdots \\ \sigma_\mu \end{array} \end{array}$$

$$= y^{(\mu)} + \dots$$

(Exercise: do this for $e^x + e^{2x}$, $G = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ to find $y'' - 3y' + 2y = 0$)

$$G = \mathrm{Gal}_{diff}\left(\mathbb{C}(x)(\text{solutions}) / \mathbb{C}(x) \right)$$

Consequence: we can write down a basis of solutions of L as $\sum_{i=1}^{m} l_i(x) \cdot g_{ij}(x)$.

solution of the original diff. system: apply $\sigma_1, \dots \sigma_\mu$ to $f_1, \dots f_m$

non-apparent singularities of L $\subset$ singularities of A

Consider the Wronskian at each singular point: as an element of

K_s = diff. extension of $\mathbb{C}[[x-s]]$ generated by $\log(x-s)$ and $(x-s)^{\mathbb{C}}$

Such an element has a generalized order g-ord$_s$ = minimum of real parts of exponents of $(x-s)$.

Fact: $\sum_{s \in \mathrm{Sing}} \text{g-ord}_s\, w = 0$

|| residue of trace of expansion matrix of L

$\frac{d}{dx} w = \text{trace} \cdot w$

Estimate each contribution:

- $\mathrm{ord}_{x=0} F(x)$ comes from 0
- $\max_i \deg P_i$ comes from ∞ $\square$

Theorem (Chudnovsky - Osgood)

$f_1, \dots f_m \in \mathbb{C}[[x]]$ $\mathbb{C}(x)$-linearly independent holonomic power series

For each $\varepsilon > 0$, there exists $C(\varepsilon, f_1, \dots f_m)$ such that setting

$\underline{x} = (x_1, \dots x_d)$ $\quad f_i(\underline{x}) = \prod_{s=1}^{d} f_{i_s}(x_s)$

satisfies: for all

$Q_i(\underline{x}) \in \mathbb{C}[x_1, \dots x_d]$ $\quad \deg_{x_j} Q_i < D$

not all zero

in the power series $\displaystyle\sum_{\underline{i}\in\{1,\dots m\}^d} Q_{\underline{i}}(\underline{x})\, f_{\underline{i}}(\underline{x})$

every lowest-order monomial $\beta\, \underline{x}^{\underline{n}}$

$\underline{n}=(n_1,\dots n_d)$ satisfies

$$n_j \le (m+\varepsilon)D + C(\varepsilon)$$

$\varepsilon=0$ possible if linear span of $f_1,\dots f_m$ stable under derivation.