IHP, Rencontre "E-functions, G-functions & periods", 9/1/23

PERIODS AND SPECIAL VALUES OF G-FUNCTIONS

Goal: survey on different presentations of the ring of periods and how one of them can be used to prove the statement

« Every period is the value at $z=1$ of a G-function »

(1) KONTSEVICH–ZAGIER PERIODS

Definition: An effective KZ period is a complex number whose real and imaginary parts can be written as absolute convergent integrals of the form

$$\int_\sigma \frac{P(x_1,\dots x_n)}{Q(x_1,\dots x_n)}\, dx_1\dots dx_n$$

where $P,Q\in\mathbb{Q}[x_1,\dots x_n]$ are polynomials and $\sigma\subset\mathbb{R}^n$ is a $\mathbb{Q}$-semialgebraic subset, i.e. obtained from subsets of the form $\{R(x_1,\dots x_n)\geq 0\}$ by complement and finite $\cup$ and $\cap$. $R\in\mathbb{Q}[x_1,\dots x_n]$

$$\boxed{\mathbb{P}_{KZ}=\mathbb{P}_{KZ}^{eff}\left[\frac{1}{\pi}\right]}$$

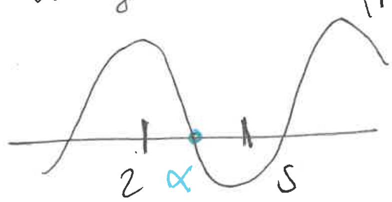
* Periods form a countable subring of $\mathbb{C}$ containing $\overline{\mathbb{Q}}$

$\alpha=\int_0^\alpha dx$ and $\sigma=\{0\le x\le\alpha\}$ is $\mathbb{Q}$-semialgebraic

minimal polynomial $\{P=0\}$

say real positive

$\sigma=\{0\le x\le r\}\cup\{r\le s\le x\}\cap\{P(x)\geq 0\}$



Conjecture: All algebraic relations between periods are obtained as combinations of

(1) additivity of the integral

(2) change of variable

(3) Stokes's formula

Example: $\zeta(n) = \int_{[0,1]^n} \frac{dx_1 \cdots dx_n}{1 - x_1 \cdots x_n} = \sum_{k=1}^{\infty} \frac{1}{k^n}$ $(n \geq 2)$

$$\zeta(n,m) = \sum_{k_1 > k_2 \geq 1} \frac{1}{k_1^n k_2^m} = \int_{1 \geq x_1 \geq \cdots \geq x_{n+m} \geq 0} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_{n-1}(1-x_n) x_{n+1} \cdots x_{n+m}}$$

$(n \geq 2, m \geq 1)$

$\zeta(n)\zeta(m) = \zeta(n,m) + \zeta(m,n) + \zeta(n+m)$ obtained from $\sum$

[change of variables $x_i = t_1 \cdots t_i$]

(2) COHOMOLOGICAL PERIODS

X smooth affine algebraic variety over $\mathbb{Q}$

Betti homology

$$H_n^B(X) = H_n^{sing}(X(\mathbb{C}), \mathbb{Q}) = \frac{\{\text{closed singular chains } \Delta^n \to X(\mathbb{C})\}}{\{\text{boundaries}\}}$$

Algebraic de Rham cohomology

$$H^n_{dR}(X) = H^n(\mathcal{O}(X) \to \Omega^1(X) \to \Omega^2(X) \to \cdots)$$

$$= \frac{\{\text{closed differential forms}\}}{\{\text{exact differential forms}\}}$$

locally of the form $\sum_{|I|=p} f_I(x)\, dx_{i_1} \cdots dx_{i_p}$

f_I: regular function on X

$$H^n_{dR}(X) \times H^B_n(X) \longrightarrow \mathbb{C}$$

$$([\omega], [\sigma]) \longmapsto \boxed{\int_\sigma \omega}$$

well defined by Stokes formula

functorial: $\int_{f_*\sigma} \omega = \int_\sigma f^*\omega$

$f : X \to Y$

<u>Theorem</u> (Grothendieck) This is a perfect pairing

period: $\int_\sigma \frac{dx}{y}$ a period of an elliptic function

$y^2 = x^3 + ax + b$

Conjecturally, not <u>all</u> KZ periods are obtained like this

<u>e.g.</u> $\log(q) = \int_1^q \boxed{\frac{dx}{x}}$ ↳ algebraic differential form on $X = \mathbb{A}^1 \setminus \{0\} = \{(x,y) \mid xy = 1\} \subset \mathbb{A}^2$

but integral is not over a <u>compact cycle</u>

⇝ Relative variant

$Y \subset X$ closed subvariety, also defined over $\mathbb{Q}$

<u>Betti homology</u>: $H^B_n(X, Y)$ = ~~cycles~~ chains on $X(\mathbb{C})$ with $\partial\sigma \subset Y(\mathbb{C})$

<u>de Rham cohomology</u> (Y smooth): $H^n_{dR}(X, Y)$ = closed form ω on X + primitive of $\omega|_Y$, i.e. $d\eta = \omega$

More generally, $Y = \bigcup Y_i$ is a SNCD

$$H^n_{dR}(X, Y) \times H^B_n(X, Y) \longrightarrow \mathbb{C}$$

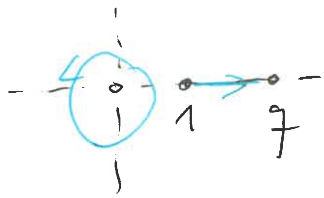
$$[(\omega, \eta)], [\sigma] \longmapsto \int_\sigma \omega + \int_{\partial\sigma} \eta$$

still <u>perfect</u>!

e.g. $H^1_{dR}(\mathbb{A}^1 \setminus \{0\}, \{1, q\}) = \mathbb{Q}\left[\frac{dx}{q-1}\right] \oplus \mathbb{Q}\left[\frac{dx}{x}\right]$

$(x-1)(x-q)$

$H_1^B(\mathbb{A}^1 \setminus \{0\}, \{1, q\}) = \mathbb{Q}[\circlearrowleft] \oplus \mathbb{Q}[\longrightarrow]$



$$
\begin{array}{c|cc}
 & \frac{dx}{q-1} & \frac{dx}{x} \\
\hline
1 \longrightarrow q & 1 & \boxed{\log q} \\
\circlearrowleft & 0 & 2\pi i
\end{array}
$$

Interesting feature: we find other periods ("Galois conjugates")

$$\boxed{\mathbb{P}_{coh} = \mathbb{P}^{eff}_{coh}\left[\frac{1}{\pi}\right]}$$

all coefficients of the integration pairing for relative cohomology. Same if we use any affine variety (we can even suppose $n = \dim X$).

Reformulation de la conjecture de Kontsevich-Zagier

Consider the $\mathbb{Q}$-vector space $\mathbb{P}_{form}$ generated by

$[X, Y, n, \omega, \sigma]$, with X smooth affine, Y SNCD, $\sigma \in H_n^B(X, Y)$, $\omega \in H^n_{dR}(X, Y)$

modulo the relations:

(1) $[X, Y, n, f^*\omega', \sigma] = [X', Y', n, \omega', f_*\sigma]$ $\quad f : X \to X'$

(2) $[X, Y, n, d\omega, \sigma] = [\bigcup Y_i, \bigcup Y_i \cap Y_j, n-1, \omega, \partial\sigma]$ "Stokes"

Conjecture: $\mathbb{P}_{form} \xrightarrow{\int} \mathbb{C}$ is injective.

$\rightsquigarrow$ Galois theory of periods

3 AYOUB'S PERIODS

Definition. An effective Ayoub period is the value of an integral

$$\int_{[0,1]^n} f(z_1,\dots z_n)\, dz_1\dots dz_n$$

where $f = \sum a_{j_1\dots j_n} z_1^{j_1}\dots z_n^{j_n}$ is a power series with $\mathbb{Q}$-coefficients that converges on the closed polydisc $\{|z_i|\le 1\}$ and is algebraic over $\mathbb{Q}(z_1,\dots z_n)$.

$$\boxed{\mathbb{P}_{Ay} = \mathbb{P}_{Ay}^{eff}\left[\frac{1}{\pi}\right]}$$

Conjecture:

$$\int_{[0,1]^\infty} : \mathcal{O}_{\mathbb{Q}\text{-alg}}(\bar{\mathbb{D}}^\infty) \longrightarrow \mathbb{C}$$
$$f \longmapsto \int_{[0,1]^\infty} f$$

has kernel generated by:

$$\frac{\partial g}{\partial z_i} - g|_{z_i=1} + g|_{z_i=0}$$

(only the most basic form of Stokes is needed)

Application

Every Ayoub period is the value of a G-function at $z=1$.

Proof: ① For $\frac{1}{\pi}$, there are formulas by Ramanujan showing that it is the value at $z=-1$ of a hypergeometric function, e.g.

$$\frac{1}{\pi} = \frac{1}{2} \sum_{n=0}^{\infty} \underbrace{\frac{\left(\frac{1}{2}\right)_n^3}{n!^3}}_{\frac{1}{2^{6n}}\binom{2n}{n}^3} (4n+1)(-1)^n$$

② For effective periods, quickest way is to see that $t \longmapsto \int_{[0,t]^n} f(z_1, \dots z_n)\, dz_1 \dots dz_n$ is a G-function. This generalises: $t \longmapsto \int_0^t \boxed{f(z)}\, dz$ algebraic over $\mathbb{Q}(z)$

Eisenstein's theorem $\Rightarrow$ f is a G-function and a primitive of a G-function is again a G-function.

We sometimes prefer to "filter" periods by:

$$\mathbb{P}_n = \left\{ \int_{[0,1]^n} f(z_1, \dots z_n)\, dz_1 \dots dz_n \right\}$$

so that $\mathbb{P}_0 = \overline{\mathbb{Q}} \subset \mathbb{P}_1 = 1\text{-periods} \subseteq \dots$ and prove:

$$G(t) = \int_0^1 \int_{[0,1]^n} f(\underline{z}, t z_n)\, dz_1 \dots dz_{n-1}\, dz_n$$

is holomorphic with Taylor coefficients in $\mathbb{P}_n$. (derivative of an algebraic function is still algebraic)

The function G is solution of a G-operator (they're of Picard-Fuchs type) and here $G(t) = \sum c_i\, g_i(t)$, $c_i \in \mathbb{P}_n$, g_i G-functions □

5 COMPARISON THEOREMS

Theorem: $\boxed{\mathbb{P}_{KZ} = \mathbb{P}_{\mathrm{coh}} = \mathbb{P}_{Ay}}$ as subsets of $\mathbb{C}$

Enough to prove the "effective" versions of the statements.

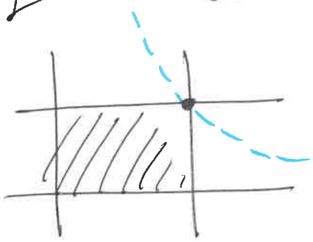
Some ideas

* $\mathbb{P}_{\mathrm{coh}} \subset \mathbb{P}_{KZ}$: $X \subset \mathbb{A}^N_{\mathbb{Q}}$ implies $X(\mathbb{C}) \subset \mathbb{R}^{2N}$ $\mathbb{Q}$-semialgebraic

Łojasiewicz: any class in $H^B_n(X, Y)$ can be represented by a $\mathbb{Q}$-linear combination of $\mathbb{Q}$-semialgebraic singular simplices

* $\mathbb{P}_{KZ} \subset \mathbb{P}_{\mathrm{coh}}$:

Main difficulty: absolute convergent integrals over cycles whose boundary encounters the polar divisor of the differential form, e.g. $\zeta(2) = \int_{[0,1]^2} \frac{dx}{1-xy}$



One would like to consider $X = \mathbb{A}^2 \setminus \{1-xy=0\}$

$Y = \#$

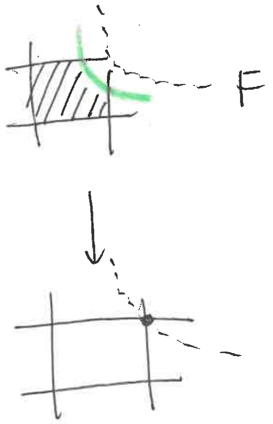
$\frac{dx}{1-xy} \in \Omega^2(X)$ but Y is not a subvariety of X

Solution: blow-up the point $(1,1)$

$$\widetilde{\mathbb{A}}^2 = Bl_{(1,1)}\mathbb{A}^2 \subset \mathbb{A}^2_{(x,y)} \times \mathbb{P}^1_{[s:t]}$$

$$(x-1)s = (y-1)t$$

$$\begin{array}{c} \widetilde{\mathbb{A}}^2 \\ \pi \downarrow \\ \mathbb{A}^2 \end{array}$$



Replace X by $\widetilde{\mathbb{A}}^2 \setminus$ strict transform of $\{1-xy=0\}$

Y by total transform of $\#$

$\pi^*\omega$ does not have poles along E !

so $\zeta(2) = \int_{\#} \pi^*\omega$

This procedure works in general thanks to Hironaka's resolution of singularities.

* $\mathbb{P}_{\mathrm{Ay}} \subseteq \mathbb{P}_{\mathrm{wh}}$:

Popescu's approximation theorem

$\Longrightarrow$ for each $f \in \mathcal{O}_{\mathbb{Q}\text{-alg}}(\overline{\mathbb{D}}^n)$ $\exists$ $U \xrightarrow{u} \mathbb{A}^n_{\mathbb{Q}}$ ~~subset~~ étale map of affine schemes

$$\begin{array}{ccc} \overline{\mathbb{D}}^n & \xrightarrow{j} & U(\mathbb{C}) \\ & \searrow & \downarrow \\ & & \mathbb{C}^n \end{array}$$

such that f is the image of $f' \in \mathcal{O}(U)$ by j^*.

$$Y = u^*\partial\left(z_1\cdots z_n(z_1-1)\cdots(z_n-1)=0\right)$$

$$[U, Y, n, f'dz_1\dots dz_n, \xi]$$

ξ class given by $[0,1]^n \subset \overline{\mathbb{D}}^n \subset U(\mathbb{C})$.