

INFINITELY MANY NON-HYPERGEOMETRIC

①

LOCAL SYSTEMS (joint with J. Lan & Y. Qiu)

Ulm's 60th Birthday

IHES 8/9/25

① Motivation

X family of smooth projective
 $f: X \rightarrow S \subset \mathbb{A}^1$ varieties defined over $\bar{\mathbb{Q}}$

* $R^n f_* \mathbb{Q}$ local system on S^{an} $\forall H_B^n(X_s)$

* $R^n f_* \Omega_{X/S}^\bullet$ vector bundle on S equipped with the Gauss-Manin connection (E, ∇) $H_{dR}^n(X_s)$

$$\ker(E^{\text{an}}, \nabla^{\text{an}}) \cong \bigoplus_{\mathbb{Q}} \mathbb{C} \hat{=} \text{Riemann-Hilbert correspondence}$$

Over a dense open subset $U \subset S$ on which E is trivial, the choice of a cyclic vector produces a differential operator $L \in \mathbb{Q}[z] \langle \frac{d}{dz} \rangle$.

Definition: A Picard-Fuchs differential operator is a product of irreducible factors of those.

Question: How to characterize them?

- 1)+2) $\Rightarrow \text{quasi-unipotent monodromy}$
 - 2) monodromy representations defined over $\bar{\mathbb{Q}}$
 - 3) nilpotent p -curvature for almost all p
 - 4) basis of solutions given by G -functions
- These conditions are expected to characterize

Picard-Fuchs differential operators, e.g.

Simpson conjectures that if $L \in \bar{\mathbb{Q}}[z] \langle \frac{d}{dz} \rangle$ has regular singularities and the monodromy defined over $\bar{\mathbb{Q}}$, then L is Picard-Fuchs

~~XXXXXXXXXXXXXXXXXXXX~~

Definition: A G-function is a power series

[Siegel, 1929]
$$g(z) = \sum_{n=0}^{\infty} a_n z^n \in \bar{\mathbb{Q}}[[z]]$$

such that ① $L \cdot g = 0$ for some non-zero L

($\Rightarrow g$ has coefficients in some number field K)

② $\exists C > 0 \quad \max_{\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| \leq C^n$

$\text{den}(a_0, \dots, a_n) \leq C^n$

(\Rightarrow order of any zero is ≥ 0 at every place)

Examples: * $a_n = 1 \quad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

in general all rational functions regular at 0

* algebraic power series (Eisenstein 1852)

* diagonals

$$\sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \dots x_n^{i_n} \in \bar{\mathbb{Q}}(x_1, \dots, x_n)$$

$$\mapsto \text{Diag}_z = \sum_{n=0}^{\infty} a_{n, \dots, n} z^n$$

globally bounded: $A g(Bz) \in \mathcal{O}_K[[z]]$

not globally bounded!
instead:
$$h_K(z) = \sum_{n \in \mathbb{Z}} \frac{z^n}{|n|^k}$$

* Hypergeometric series:

$\alpha_1, \dots, \alpha_p \in \mathbb{Q}$

$\beta_1, \dots, \beta_{p-1} \in \mathbb{Q}, \beta_i \leq 0$

$$F \left(\begin{matrix} \alpha_1 \dots \alpha_p \\ \beta_1 \dots \beta_{p-1} \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_{p-1})_n} \frac{z^n}{n!}$$

where $(x)_n = x(x+1)\dots(x+n-1)$ is the Pochhammer symbol. Solution of

$$L = \prod_{i=1}^p (\theta + \beta_i - 1) - z \prod_{j=1}^p (\theta + \alpha_j) \quad \theta = z \frac{d}{dz}$$

$(\beta_p = 1)$

Sums, products, derivatives, algebraic substitutions of G-functions are again G-functions.

Theorem (1st main) There exist G-functions solutions of a differential operator of order ≥ 2 that are not $\bar{\mathbb{Q}}$ -polynomials in

(*) $\mu(z) \cdot F \left(\begin{matrix} \alpha_1 \dots \alpha_p \\ \beta_1 \dots \beta_{p-1} \end{matrix} \middle| \lambda(z) \right)$ μ, λ algebraic
 $\lambda(0) = 0$

Back to the motivation: L (Picard-Fuchs) has a basis

of solutions $\sum \binom{a_i}{z^{\alpha_i}} (\log z)^{b_i} g_i(z)^{-\alpha_i}$ $z_i \in \bar{\mathbb{Q}} \setminus \{0\}$
 at every point $z \in \bar{\mathbb{Q}} \cup \infty$. G-function

Conjecture (Bombieri-Dworkin) Every operator with

this property (e.g. mixed order diff. systems of a G-function) is a Picard-Fuchs diff. operator.

* $\bar{\mathbb{Q}}[x]$, piecewise solutions of $P(z) \left(\frac{d}{dz} \right)^2 + \frac{P'(z)}{z} \frac{d}{dz} + \frac{z-10}{18}$
 $P = (z-1)(z-2)(z-82)$

* OK for diff. order 1 because solutions are algebraic

* Wide open starting from order 1 and 4 ingredients

* Suggests that

Special cases of \mathbb{G} -modules = periods $\left[\frac{1}{\pi} \right]$

$$\frac{z}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^3}{n!^3} \frac{(4n+1)}{(-1)^n}$$

\geq
(with P. Tene)

A hope by Dwork was that we could prove that every \mathbb{G} -module of order 2 is rigid-Fuchs by obtaining it as an algebraic pullback of the system for ${}_{\mathbb{Z}/1}F_1 \left(\begin{smallmatrix} a & b \\ c \end{smallmatrix} \middle| z \right) \dots$

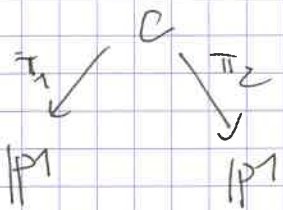
↑
Singular

Counterexamples by Kramer (1990)
Bouw-Müller (2007)
Zethwiler-Ruter (2010)

\mathcal{G} = category of $\mathbb{Q}(z) \left\langle \frac{d}{dz} \right\rangle$ -modules of rigid-Fuchs type

\mathcal{V} (= generic fiber of a $\mathbb{A}_{\mathbb{P}^1}$ -module) \otimes diff Galois group

\mathcal{H} = \otimes -subcategory generated by



$\pi_1 * \pi_2^* \mathcal{H}$

invert map to \mathbb{P}^1
of a hypergeometric
diff. operator

finite Galois cover

Reason: every $\mu(z) \cdot \mathbb{P}_{p-1}^F \left(\dots \middle| \mu(z) \right)$ is a solution of an object of \mathcal{H} .

Theorem (2nd main) There exist infinitely many pairwise non-isomorphic sheets of G of dimension 2 that do not belong to \mathbb{H} (= "non-hyperbolic")

Sketch of proof

(1) Use differential Galois theory (Bertini-Hertzsprung) to prove that if a sheet of G with differential Galois group SL_2 belongs to \mathbb{H} , then it is generated by a single $\pi_{1*} \pi_2^* \mathcal{H}$ with \mathcal{H} hyperbolic of order 2 modifiable.

(2) Given a local system

$$\rho: \pi_1(\mathbb{P}^1 - S) \rightarrow GL_{\mathbb{Z}}(\mathbb{C})$$

consider the trace field

$$F = \mathbb{Q}(\text{tr}_{\rho}(\gamma) \mid \gamma \in \pi_1(\mathbb{P}^1 - S)) \subset \text{trace field}$$

or rather the admit trace field

$$\mathbb{Q}(\text{tr}_{\text{ad} \rho}(\gamma)) \quad \text{ad} \rho: \pi_1(\mathbb{P}^1 - S) \rightarrow GL_{\mathbb{Z}}(\mathbb{C})$$

which has the advantage of being invariant under passing to finite index subgroups [Mackaay-Hurt]

Key: The local system $\mathbb{H} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has abelian trace field, different from $\mathbb{Q}(\sqrt{D})$ for $D \geq 7$ squarefree

Why? Rigidity (Katz)

$$\{ \text{Aut}(\mathbb{C}) \mid \sigma^* \mathbb{H} \cong \mathbb{H} \}$$

Trace field = field of definition = \mathbb{C}

$$\subset \mathbb{Q}(\mu_{\infty})$$

because eigenvalues of local matrices are roots of unity and from there we find everything.

Update: enough to construct objects of G with non-abelian trace field

(3) one prime structure:

F totally real number field

B non-split quaternion algebra over F

that splits at exactly one real place

$\mathcal{O} \subset B$ order

we $\mathcal{H}_{B, \mathcal{O}} \supset \mathcal{H}_{B, \mathcal{O}}^\circ =$ component of unipotent part
Shimura curve [Assume it has genus 0]

$\rho: \pi_1(\mathcal{H}_{B, \mathcal{O}}^\circ) \rightarrow \mathrm{SL}_2(\mathcal{O})$ diff. Galois group SL_2
 adjoint trace field = F

talks by John Voight,

real cubic field F of discriminant 158

[other examples coming from Teichmüller curves]

(4) Infinitely many!

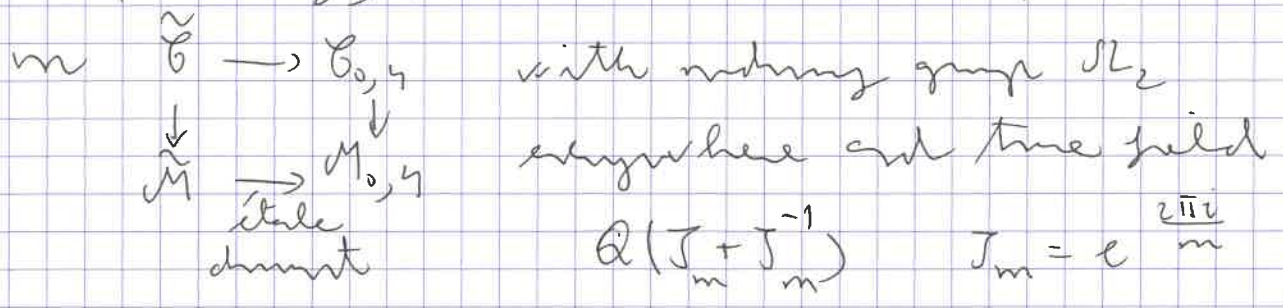
start with a local system \mathbb{V} on
 constructed by Hom and Hilb

arising from a family of abelian varieties

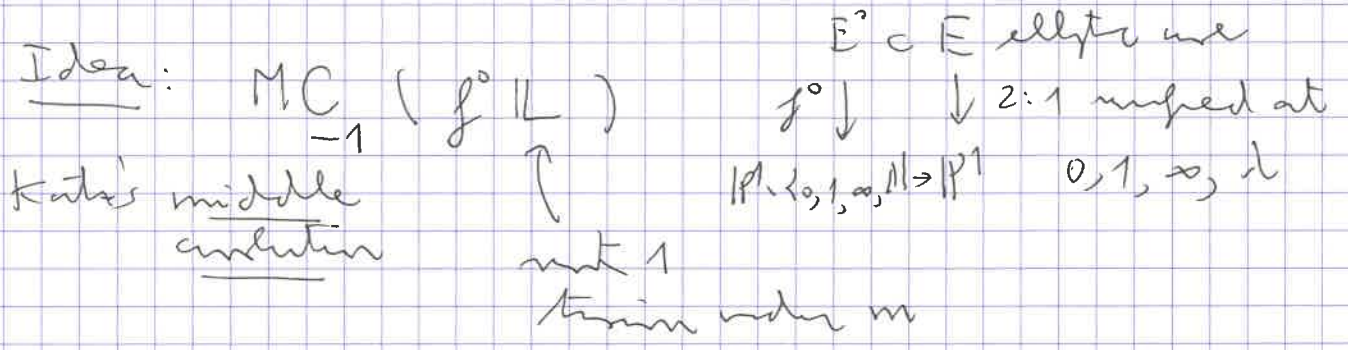
$\mathcal{P}_{0,4}$ universal period mod
 \downarrow
 $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$

and prove that the restriction to most fibres is not a hyperbolic local system

More precisely, for each $m \geq 1$ we can find \mathbb{W}



\mathbb{W} direct sum of $R^1 \pi_* \mathbb{C}$ not pulled from a curve $\pi: A \rightarrow \tilde{\mathcal{G}}$ par



m such that $\frac{\varphi(m)}{2}$ odd ≥ 3 (e.g. $m=7$)

Then $\sum_{x \in \tilde{\mathcal{M}}(\mathbb{Q})} \dots$ is finite.

\mathbb{Z} set of p such that $\mathbb{W}_{1/p}$ has $\sim \mathbb{H}$.

\Rightarrow adjoint trace field $= \mathbb{Q}(J_m + J_m^{-1}) = K$

May assume all $\mathbb{W}_{1/p}$ are isomorphic to the same \mathbb{H}

period map: $\tilde{\mathcal{G}} \rightarrow$ Shimura variety (Hilbert-norm variety attached to $Rg_K^{SL_2}$)

contradicts the André-Pink-Zannier conjecture (Richard-Yafaar) $\dim(K:\mathbb{Q})$

So a part, $Z \subset Sh_K(\mathcal{G}, X)$ intersects the

generalized Hecke orbit of S_0 is Zariski dense

$\Rightarrow Z = \text{finite union of weakly special subsets}$

among pts $i(1), \dots, i(n), \dots$

$Z = \text{Zariski dense} \rightarrow Z = \text{Sh}_K(G, \kappa)$
Mordell-Tate
group

$\text{dim } Z \leq 2$ embedded