# INFINITELY MANY NON-HYPERGEOMETRIC LOCAL SYSTEMS (joint with J. Lam & Y. Qin)

Ullmo 60th Birthday
IHÉS 8/9/25

## ① Motivation

$$\begin{array}{l} X \\ f\big\downarrow \\ S \subset \mathbb{A}^1 \end{array}$$ family of smooth projective varieties defined over $\overline{\mathbb{Q}}$

* $R^n f_* \mathbb{Q}$ local system on $S^{an}$ $\quad \mathbb{V} \quad H^n_B(X_s)$
* $R^n f_* \Omega^\bullet_{X/S}$ vector bundle on $S$ equipped with the Gauss-Manin connection $(\mathcal{E}, \nabla)$ $\quad H^n_{dR}(X_s)$

$$\ker(\mathcal{E}^{an}, \nabla^{an}) \underset{\uparrow}{\simeq} \mathbb{V} \otimes_{\mathbb{Q}} \mathbb{C}$$

Riemann-Hilbert correspondence

Over a dense open subset $U \subset S$ on which $\mathcal{E}$ becomes trivial, the choice of a cyclic vector produces a differential operator $L \in \overline{\mathbb{Q}}[z]\langle \frac{d}{dz} \rangle$.

Definition: A Picard-Fuchs differential operator is a product of irreducible factors of those.

Question: How to characterize them?

1)+2) ⇒ ? $\quad$ 1) regular singularities, quasi-unipotent monodromy

3) ⇒ 1)+? $\quad$ 2) monodromy representation defined over $\overline{\mathbb{Q}}$

… $\quad$ 3) nilpotent $p$-curvature for almost all $p$

4) ⇒ 1)+2). $\quad$ These conditions are expected to characterize

4) basis of solutions given by G-functions

Picard-Fuchs differential operators, e.g.
Simpson conjectures that if $L \in \bar{\mathbb{Q}}[z]\langle \frac{d}{dz} \rangle$ has
regular singularities and the monodromy
defined over $\bar{\mathbb{Q}}$, then $L$ is Picard-Fuchs

~~[illegible]~~

Definition: A G-function is a power series
[Siegel, 1929] $$g(z) = \sum_{n=0}^{\infty} a_n z^n \in \bar{\mathbb{Q}}[[z]]$$

such that ① $L \cdot g = 0$ for some non-zero $L$
($\Rightarrow g$ has coefficients in some number field $K$)

② $\exists C > 0$ $$\max_{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| \leq C^n$$
$$\mathrm{den}(a_0, \dots a_n) \leq C^n$$
($\Rightarrow$ radius of convergence is $\boxed{> 0}$ at every place)

Examples: * $a_n = 1$ $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$
in general all rational functions regular at 0

* algebraic power series (Eisenstein 1852)

* diagonals
$$\sum_{i_1, \dots i_r \geq 0} a_{i_1 \dots i_r} x_1^{i_1} \dots x_r^{i_r} \in \bar{\mathbb{Q}}(x_1, \dots x_n)$$
$$\rightsquigarrow \mathrm{Diag}_r = \sum_{n=0}^{\infty} a_{n \dots n} z^n$$

globally bounded: $A g(Bz) \in \mathcal{O}_K[[z]]$

Not globally bounded: $\displaystyle Li_k(z) = \sum \frac{z^n}{n^k}$

②

* Hypergeometric series:

$\alpha_1, \dots \alpha_p \in \mathbb{Q}$

$\beta_1, \dots \beta_{p-1} \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$

$$ {}_pF_{p-1}\left(\begin{matrix}\alpha_1 \dots \alpha_p \\ \beta_1 \dots \beta_{p-1}\end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_{p-1})_n} \frac{z^n}{n!} $$

where $(x)_n = x(x+1)\dots(x+n-1)$ is the Pochhammer symbol. Solution of

$$ L = \prod_{i=1}^{p} (\theta + \beta_i - 1) - z \prod_{j=1}^{p} (\theta + \alpha_j) \qquad \theta = z\frac{d}{dz} $$

$(\beta_p = 1)$

Sums, products, derivatives, algebraic substitutions of G-functions are again G-functions.

Theorem (1st version) There exist G-functions solutions of a differential operator of order 2 that are not $\overline{\mathbb{Q}}$-polynomials in

(*) $\mu(z)\, {}_pF_{p-1}\left(\begin{matrix}\alpha_1 \dots \alpha_p \\ \beta_1 \dots \beta_{p-1}\end{matrix} \middle| \lambda(z)\right)$ $\quad \mu, \lambda$ algebraic, $\lambda(0) = 0$

Back to the motivation: $L$ Picard-Fuchs has a basis of solutions $\sum (z-\alpha)^{a_i} (\log(z-\alpha))^{b_i} g_i(z-\alpha)$ $\quad g_i \in \overline{\mathbb{Q}}[[z]]$ G-function

at every point $\alpha \in \overline{\mathbb{Q}} \cup \infty$.

Conjecture (Bombieri-Dwork) Every operator with this property (e.g. minimal order diff. equation of a G-function) is a Picard-Fuchs diff. operator.

* Ex: power series solutions of $P(z)\left(\frac{d}{dz}\right)^2 + \frac{P'(z)}{2}\frac{d}{dz} + \frac{z-10}{18}$

$P = (z-1)(z-2)(z-82)$

* OK for diff. order 1 because solutions are algebraic

* Wide open starting from order 1 and 4 singularities

* Suggests that

$$\text{Special values of G-functions} = \text{Periods}\left[\tfrac{1}{\pi}\right] \qquad \frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3}(4n+1)(-1)^n$$

$$\supseteq$$

(with P. Jossen)

A hope by Dwork was that one could prove that every G-operator of order 2 is Picard-Fuchs by obtaining it as an algebraic pullback of the equation for ${}_2F_1\left(\begin{smallmatrix} a & b \\ c & \end{smallmatrix} \middle| z\right)$ ....

Counterexamples by | Krammer (1996)
| Bouw-Möller (2007)
| Dettweiler-Reiter (2010)

↑ Sporadic

$\mathcal{G}$ = category of $\overline{\mathbb{Q}}(z)\langle \frac{d}{dz}\rangle$-modules of Picard-Fuchs type

$\cup$ (= generic fiber of a $\mathcal{D}_{\mathbb{P}^1}$-module) $\qquad \otimes$ diff. Galois groups

$\mathbb{H}$ = $\otimes$-subcategory generated by

$$\begin{array}{ccc} & C & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array} \qquad \pi_{1*}\pi_2^*\mathcal{H}$$

direct image to $\mathbb{P}^1$ of a hypergeometric diff. operator

finite Galois covers

Reason: every $\mu(z)\cdot {}_pF_{p-1}\left( \ldots \middle| \lambda(z)\right)$ is a solution of an object of $\mathbb{H}$.

Theorem ($2^{nd}$ version) There exist infinitely many pairwise non-isomorphic objects of $\mathcal{G}$ of dimension 2 that do not belong to $\mathbb{H}$ (= "non-hypergeometric")

Sketch of proof

(1) Use differential Galois theory (Beukers-Heckman) to prove that if an object of $\mathcal{G}$ with differential Galois group $SL_2$ belongs to $\mathbb{H}$, then it is generated by a single $\pi_{1*}^* \pi_2^* \mathcal{H}$ with $\mathcal{H}$ hypergeometric of order 2. dim 2 middle.

(2) Given a local system

$$\rho : \pi_1(\mathbb{P}^1 \setminus S) \longrightarrow GL_2(\mathbb{C})$$

consider the trace field

$$F = \mathbb{Q}\left(\mathrm{tr}\,\rho(\gamma) \mid \gamma \in \pi_1(\mathbb{P}^1 \setminus S)\right) \subset \text{trace field}$$

or rather the smaller adjoint trace field

$$\mathbb{Q}(\mathrm{tr}\,\mathrm{ad}_\rho(\gamma)) \qquad \mathrm{ad}_\rho : \pi_1(\mathbb{P}^1 \setminus S) \to \mathfrak{gl}_2(\mathbb{C}) \quad GL(\ )$$

which has the advantage of being invariant under passing to finite index subgroups [Maclachlan-Reid]

Key: The local system $\mathbb{H}\begin{pmatrix} a & b \\ & c \end{pmatrix}$ has abelian trace field, different from $\mathbb{Q}(\sqrt{D})$ for $D \geq 7$ squarefree

$\boxed{\text{Why ?}}$ Rigidity [Katz]

$$\{\mathrm{Aut}(\mathbb{C}) \mid \sigma^* \mathbb{H} \simeq \mathbb{H}\}$$

Trace field = field of definition = $\mathbb{C}$

$\subset \mathbb{Q}(\mu_\infty)$

because eigenvalues of local monodromy are roots of unity and fixing them we fix everything.

Upshot: enough to construct objects of $\mathcal{G}$ with non-abelian trace field

(3) Use quaternionic constructions:

$F$ totally real number field

$B$ non-split quaternion algebra over $F$

that splits at exactly one real place

$\mathcal{O} \subset B$ order

$\leadsto$ $\underset{\text{Shimura curve}}{Sh_{B,\mathcal{O}}} \supset Sh^{\circ}_{B,\mathcal{O}}$ = complement of elliptic points

$\boxed{\text{Assume it has genus } 0}$

$\rho: \pi_1(Sh^{\circ}_{B,\mathcal{O}}) \longrightarrow SL_2(\mathbb{C})$ diff. Galois group $SL_2$

adjoint trace field $= F$

tables by John Voight,

real cubic field $F$ of discriminant 148

[other examples coming from Teichmüller curves]

(4) Infinitely many!

Start with a local system $\mathbb{V}$ on $\mathcal{C}_{0,4}$ (universal punctured curve)

$\downarrow$

$\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$

constructed by Lam and Litt

arising from a family of abelian varieties and prove that the restriction to most fibers is not a hypergeometric local system

More precisely, for each $m \geq 1$ we can find $\mathbb{V}$ on

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \longrightarrow & \mathcal{C}_{0,4} \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow[\text{dominant}]{\text{étale}} & M_{0,4} \end{array}$$

with monodromy group $SL_2$ everywhere and trace field $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ $\quad \zeta_m = e^{\frac{2\pi i}{m}}$

$\mathbb{V}$ direct sum of $R^1\pi_*\mathbb{C}$ $\quad \pi : A \to \tilde{\mathcal{C}}$
not pulled back from a curve $\quad$ ppav

**Idea:** $MC_{-1}(f^\circ \mathbb{L})$
Katz's middle convolution

rank 1
torsion order $m$

$$\begin{array}{ccc} E^\circ & \subset & E \quad \text{elliptic curve} \\ f^\circ \downarrow & & \downarrow 2:1 \text{ ramified at } \\ \mathbb{P}^1 \setminus \{0,1,\infty,\lambda\} & \to & \mathbb{P}^1 \quad 0, 1, \infty, \lambda \end{array}$$

$m$ such that $\frac{\varphi(m)}{2}$ odd $\geq 3$ $\quad$ (e.g. $m = 7$)

Then $S_{exc} \subset \tilde{M}(\bar{\mathbb{Q}})$ is finite.

$\Sigma$ set of $p$ such that $\mathbb{V}|_{\mathcal{C}_p}$ has no $\mathbb{H}$.

$\Rightarrow$ adjoint trace field $= \mathbb{Q}(\zeta_m + \zeta_m^{-1}) = K$

May assume all $\mathbb{V}|_{\mathcal{C}_p}$ are isomorphic to the same $\mathbb{H}$

period map: $\check{\mathcal{C}} \longrightarrow$ Shimura variety (Hilbert-modular variety attached to $\mathrm{Res}^K_{\mathbb{Q}} SL_2$)

contradicts the André–Pink–Zannier conjecture [Richard–Yafaev] $\quad$ $\boxed{\dim [K:\mathbb{Q}]}$

So a point, $Z \subset Sh_K(G, X)$ intersects the generalized Hecke orbit of $s_0$ is Zariski dense

$\Longrightarrow Z =$ finite union of weakly special subvarieties

$\infty$ many pts $i(p_1), \ldots i(p_n), \ldots$

$Z =$ Zariski closure $\xrightarrow[\text{Mumford-Tate generic}]{}$ $Z = Sh_K(G, K)$

$\dim Z \leq 2$ contradiction