(Darmstadt, GAUS-AG Reading Seminar, 26/1/23) ①

E-FUNCTIONS AND GEOMETRY

(joint work with Peter Jossen)

We came to Siegel's question through the study of

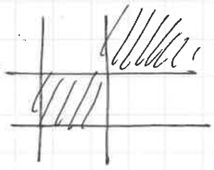
exponential periods $\qquad \int_\sigma e^{-f}\omega \qquad f : X \longrightarrow \mathbb{A}^1$

$\uparrow$ algebraic variety over $\overline{\mathbb{Q}}$

e.g. $\displaystyle\int_0^\infty e^{-x^n}\,dx = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)$ special value of the Γ-function

$$\boxed{\int_0^1\!\!\int_0^1 e^{-xy}\,dx\,dy} - \int_1^\infty\!\!\int_1^\infty e^{-xy}\,dx\,dy = \gamma$$

Euler's constant



$$\oint e^{-(x+\frac{1}{x})}\frac{dx}{x} = 2\pi i \sum_{n=0}^{\infty}\frac{1}{(n!)^2}$$

value at $z=1$ of the E-function $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{(1+n)(n+1)!}$

classical period.

$\hookrightarrow$ value at $z=1$ of E-function $\displaystyle\sum_{n=0}^{\infty}\frac{z^{2n}}{(n!)^2}$ (Bessel)

What can be expected in general?

① PERIODS $\qquad \int_\sigma \omega \qquad \omega \in H^n_{dR}(X,Y)$

$\sigma \in H_n^{sing}(X(\mathbb{C}), Y(\mathbb{C}))$

Conjecture (a form of Bombieri–Dwork)

Periods $\left[\frac{1}{\pi}\right] \overset{?}{=} \mathbb{G}$ ring of special values of G-functions (algebraic points)

We observed recently that it is possible to prove the inclusion $\boxed{\subseteq}$ using Ayoub's ~~theorem~~ work.

Ayoub: period $= \displaystyle\int_{[0,1]^n} f(z_1, \dots, z_n)\, dz_1 \dots dz_n$

power series algebraic over $\overline{\mathbb{Q}}(z_1, \dots, z_n)$ convergent on $|z_i| \leq 1$.

Out of this one fabricates ~~as~~ a G-function by

$$t \longmapsto \int_{[0,t]^n} f(z_1, \dots, z_n)\, dz_1 \dots dz_n$$

(in one variable $\displaystyle\int_0^t f(z)\, dz$ primitive of a G-function)

$f(z)$: G-function

Moreover: $$\frac{1}{\pi} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3} (-1)^n (4n+1)$$

value at $z=1$ of a hypergeometric function

Recent
All these functions are specializations of hypergeometric G-functions in several variables.

$$\{\text{periods}\} \subset \mathbb{G}$$
$$\cap$$
$$\{\text{exponential periods}\} \subset \;??$$

The ring $\mathbb{E}$ of special values of E-functions obviously has to play a role (but it is expected that $\mathbb{E} \cap \mathbb{G} = \overline{\mathbb{Q}}$, etc..).

② EXPONENTIAL PERIOD FUNCTIONS

$X \xrightarrow{f} \mathbb{A}^1$

smooth affine $/ \overline{\mathbb{Q}}$ dim n

$$H^n_{dR}(X, f) = \frac{\Omega^n(X)}{\operatorname{Im}(d - df \wedge \cdot)}$$

"twisted de Rham cohomology"

$$H_n^{rd}(X,f) = \lim_{t\to\infty} H_n(X(\mathbb{C}), \{\mathrm{Re} f \geq t\}; \mathbb{Q})$$

"rapid decay homology"

<u>Theorem</u>: For each $\omega \in H^n_{dR}(X,f)$ and $\sigma \in H_n^{rd}(X,f)$, the function

$$\int_\sigma e^{-zf}\omega \quad (\text{holomorphic on } \{\mathrm{Re}\, z > 0\})$$

extends to a linear combination of

$$z^a (\log z)^b E(z) \qquad \begin{array}{l} a \in \mathbb{Q} \\ b \in \mathbb{Z}_{\geq 0} \end{array}$$

↑ E-function

with coefficients in $\overline{\mathbb{Q}}[\text{periods}, \gamma, \Gamma(\tfrac{p}{q})]$.

<u>Conjecture</u>: All E-functions arise this way.

From this point of view, hypergeometric E-functions admit a very special integral representation, e.g.

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| z\right) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zu} u^{a-1} (1-u)^{b-a-1}\, du$$

$$\sum \frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

$X \longrightarrow \mathbb{A}^1$
Fermat curve

(3) SKETCH OF PROOF

* We consider the $\mathcal{D}_{\mathbb{G}_m}$-module

$$R^n\pi_+ \mathcal{E}^{-zf} \qquad \pi: X \times \mathbb{G}_m \longrightarrow \mathbb{G}_m \text{ projection}$$

$$\mathcal{E}^{-zf} = (\mathcal{O}_{X\times\mathbb{G}_m}, d - d(zf)\wedge\cdot)$$

whose fiber at z is the de Rham cohomology

$$H^n_{dR}(X, zf).$$

A computation shows that this $\mathcal{D}_{\mathbb{G}_m}$-module is isomorphic to $j^* FT(\underbrace{R^n f_+ \mathcal{O}_X}_{\substack{\mathcal{D}_{/\mathbb{A}^1}\text{-module} \\ \text{of type } G \text{ (André)}}})$.

$\therefore$ $\mathcal{D}_{\mathbb{G}_m}$-module of type E,
and hence has a basis of solutions of the form

$$\mathbb{C}\, z^a (\log z)^b E(z)$$

* système local $\mathcal{H}^{rd}_n(X, zf)$ de ~~...~~ rapid decay homology

$$\mathcal{H}^{rd}_n(X, zf) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_{\mathbb{G}_m}\text{-mod}}(\mathcal{H}^n_{dR}(X,zf), \mathrm{Hol}(\,\cdot\,))$$

$$\sigma \longmapsto \left(w \mapsto P_{\sigma,w} = \int_\sigma e^{-zf} w\right)$$

is an isomorphism by Hien–Roucairol.

$$\Rightarrow \quad P_{\sigma,w}(z) = \sum c_{a,b,i}\, z^a (\log z)^b E_i(z)$$

<u>Want</u>: $c_{a,b,i} \in \bar{\mathbb{Q}}[\text{pids}, \Gamma(\tfrac{p}{q}), \gamma]$.

* monodromy around $z=0$ of $\mathcal{H}^{rd}_n(X, zf)$ is <u>quasi-unipotent</u> (André)

<u>Case</u> $X = \mathbb{A}^1$, $f = x^n$ finite monodromy of order n

<u>Case</u> $X = \mathbb{A}^2$, $Y = \#$, $f = xy$ $\qquad M(\gamma)$

quasi-unipotent monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ "basic Jordan block"

<u>Claim</u>: For any quasi-unipotent monodromy, there is a $\otimes$-structure in $H^1(\mathbb{A}^1, x^n)$ and $M(\gamma)$ with the same monodromy.

[<u>Key point</u>: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\otimes N}$ contains $\underbrace{\begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}}_{N+1}$, in fact as Sym^N (same matrix)]

$M = H^n(X, f)$ ⇝ function de périodes $\qquad P = z^A Q(z)$ ← coefficients are entire functions

$M_0 = \otimes$ structure with the same monodromy $\qquad P_0 = z^A Q_0$ ← constant matrix $\overline{\mathbb{Q}}[\gamma, \Gamma(\tfrac{1}{q})]$

Then $M_0^\vee \otimes M$ has period matrix

$$ {}^t P_0^{-1} \otimes P = z^{{}^tA^{-1} \otimes A}\, {}^tQ_0^{-1} \otimes Q $$

with a block of holomorphic functions.

It remains to show:

<u>Lemma</u>: If $\overset{P(z)}{\overset{\parallel}{\int_\sigma}} e^{-zf} w$ (Re $z > 0$) admits a meromorphic continuation around $0 \in \mathbb{C}$, then σ is in the image of the canonical map $H_n(X) \to H_n^{rd}(X, f)$.

In particular, $P(0) = \int_\sigma w$ is a period of $H_n(X)$.

(this was the case for $\oint e^{-z(x+\frac{1}{x})} \frac{dx}{x}$.

The other functions $\int_0^\infty \cdots$ do not extend meromorphically and $0 \in \mathbb{C}$).

Idea: $\begin{array}{c} M \\ f \downarrow \\ S^1 \end{array}$ topological fibration (locally trivial) $\qquad \begin{array}{c} \Gamma_f \subseteq M \times S^1 \\ \downarrow \mathrm{proj} \\ S^1 \end{array}$

$$\mathcal{H}^q(M,f) = R^q \mathrm{proj}_* \mathbb{Q}_{[M\times S^1, \Gamma_f]}$$

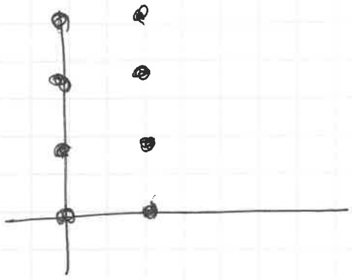
local system on S^1, with fibre $H^q(M, f^{-1}(z))$.

Claim:

$$H_q(M) \twoheadrightarrow H_q(X, f^{-1}(z))^{\mu} \quad \leftarrow \text{monodromy operator}$$

$$E_2^{p,q} = H^p(S^1, \mathcal{H}^q(M,f)) \Rightarrow H^{p+q}(M\times S^1, \Gamma_f)$$

degenerates by cohomological dimension



$$0 \to H^1(S^1, \mathcal{H}^q(M,f)) \to H^{q+1}(M\times S, \Gamma_f) \cong H^q(M\times S, M\times\{1\}) \cong H^q(X)$$

$\rightsquigarrow \mathcal{H}^q(M,f)_\mu \hookrightarrow H^q(X)$ $\square$

We apply this to rapid decay cycles at ∞ of:

$$R^n pr_* \mathbb{Q}_{[X\times \mathbb{A}^1, \Gamma_f]}$$

"perversalization".