A construction of the polylogarithm motive

(joint with Clément Dupont)

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$n \geq 1$

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \qquad |z| < 1$$

n-th polylogarithm

$\boxed{n=1}$ $\mathrm{Li}_1(z) = -\log(1-z)$

$\boxed{n \geq 2}$ $\mathrm{Li}_n(z)$ primitive of $\dfrac{\mathrm{Li}_{n-1}(z)}{z}$ that vanishes at 0

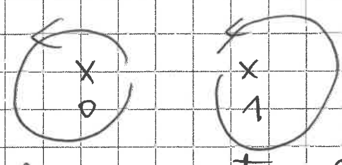
$$\boxed{\mathrm{Li}_n(1) = \zeta(n)}$$

$$\frac{d}{dz}\begin{pmatrix} 1 \\ \mathrm{Li}_1(z) \\ \vdots \\ \mathrm{Li}_n(z) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & & & \\ \frac{dz}{1-z} & 0 & & \\ & \frac{dz}{z} & 0 & \\ & & \ddots & \ddots & \\ & & & \frac{dz}{z} & 0 \end{pmatrix}}_{\Omega_n} \begin{pmatrix} 1 \\ \mathrm{Li}_1(z) \\ \vdots \\ \mathrm{Li}_n(z) \end{pmatrix}$$

* Full basis of fundamental solutions:

$$\Lambda_n(z) = \begin{pmatrix} 1 & \mathrm{Li}_1(z) & \cdots & \cdots & \mathrm{Li}_n(z) \\ & 2\pi i & 2\pi i \log(z) & \cdots & 2\pi i \frac{\log(z)^{n-1}}{(n-1)!} \\ & & (2\pi i)^2 & \cdots & (2\pi i)^2 \frac{\log(z)^{n-2}}{(n-2)!} \\ & & & \ddots & \vdots \\ & & & & (2\pi i)^n \end{pmatrix}$$

multivalued functions on $\underline{\mathbb{P}^1 \setminus \{0, 1, \infty\}}$



normalization so that monodromy matrices are rational

$U_{p_{n+1}}(\mathbb{Q})$

- Deligne (1984): variation of mixed Hodge-Tate structures

$$\left(\mathcal{O}^{n+1}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}, \nabla = d + \Omega_n \right) \qquad W_{2k} = W_{2k+1}$$

$$\langle e_0, \dots, e_k \rangle$$

$\mathcal{L}_n \qquad e_0, \dots, e_n$

$$F^k = \langle e_k, \dots e_n \rangle$$

local system: $\varphi : \mathcal{U} \to \mathbb{C}e_0 \oplus \dots \oplus \mathbb{C}e_n$

$W_{2k} = W_{2k+1}$
sublocal systems
values in $\mathbb{C}e_0 \oplus \dots \oplus \mathbb{C}e_n$

$\Lambda_n(z)\varphi$ locally constant $\mathbb{Q}$-entries

looking at the matrix:

$$\left(\begin{array}{c|c} \boxed{1} & \\ \hline & (2\pi i)\,* \\ & \end{array} \right)$$

$$0 \to \mathbb{Q}(0) \to \mathcal{L}_n \to \mathrm{Sym}^{n-1}(K)(-1) \to 0$$

where K is the Kummer variation $\begin{pmatrix} 1 & \log(z) \\ 0 & 2\pi i \end{pmatrix}$

Indeed: inductive system of such $\quad \mathcal{L} = \operatorname{colim} \mathcal{L}_n$

$$\boxed{0 \to \mathbb{Q}(0) \to \mathcal{L} \to \mathrm{Sym}(K)(-1) \to 0}$$

$\mathcal{L}_n \subset \mathcal{L}_{n+1}$
$\mathbb{Q}(0) \to K$
$\rightsquigarrow \mathrm{Sym}^{n-1}(K)$
$\subset \mathrm{Sym}^n(K)$

Question: does it come from geometry? $\mathbb{Q}(0)$ and K (and hence the symmetric powers do)

$$\log(z) = \int_1^z \frac{dt}{t} \qquad H^1(\mathbb{G}_m, \{1, z\}) \text{ relative cohomology motive} \quad z \neq 1$$

de Rham: $H^1_{dR}(\mathbb{G}_m, \{1, z\}) = \mathbb{Q}\left[\frac{dt}{t}\right] \oplus \mathbb{Q}\left[\frac{dt}{z-1}\right]$

$([\omega], (\eta)) \qquad d\omega = 0 \quad d\eta = \omega|_z$

Betti: $H_1^B(\mathbb{G}_m, \{1,z\}) = \mathbb{Q}[\odot] \oplus \mathbb{Q}[x_1 \rightsquigarrow x_z]$

chains on $\mathbb{C}^\times$ with boundary $\subset \{1,z\}$

$$0 \to H^0(\mathbb{G}_m) \to H^0(\{1,z\}) \to H^1(\mathbb{G}_m, \{1,z\}) \to H^1(\mathbb{G}_m) \to 0$$

$$0 \to \mathbb{Q}(0) \to \boxed{} \to \underset{\mathbb{Q}(-1)}{\parallel} \to 0$$

$$\begin{array}{c|cc} & \frac{dt}{t-1} & \frac{dt}{t} \\ x_1 \to x_z & 1 & \log(z) \\ \odot & 0 & 2\pi i \end{array} \qquad H_0(\{1,z\}) \xleftarrow{\delta} H_1(\mathbb{G}_m, \{1,z\}) \leftarrow H_1(\mathbb{G}_m)$$

$H_0(\mathbb{G}_m)$

More precisely: $S = \mathbb{P}^1_\mathbb{Q} \setminus \{0,1,\infty\}$

$$D(MT(S)) \overset{?}{=} DMT(S) \subset DM(S) =$$ Voevodsky's triangulated category of mixed motives over S

$\mathbb{Q}_S(n)$

abelian category of mixed Tate motives over S

Levine: heart of a t-structure

étale sheaves on Sm/S + $\mathbb{A}^1$-homotopy invariance, 6-functors formalism

Hodge realization functor: $MT(S) \longrightarrow MHT(S)$ variations of mixed Hodge Tate structures $/S$

"Abstract" answer: Beilinson–Deligne, Huber–Wildeshaus, Ayoub

$$\mathrm{Ext}^1_{\mathrm{Ind}(MT(S))}(\mathrm{Sym}(K)(-1), \mathbb{Q}_S(0)) \xrightarrow{\sim} \mathbb{Q}$$

$\mathrm{Ind}(MHT(S))$

So there is an object of $\mathrm{Ind}(MT(S))$ that has Hodge realization the polylogarithm variation.

What is it?

* <u>Motivation / digression</u>: $S = \operatorname{Spec} \mathbb{Q}$

$$\operatorname{Ext}^1_{MT(\mathbb{Q})}(\mathbb{Q}(-n), \mathbb{Q}(0)) \cong K_{2n-1}(\mathbb{Q})_{\mathbb{Q}} \overset{\text{Borel}}{\simeq} \begin{cases} \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} & n = 1 \\ \mathbb{Q} & n \geq 3 \text{ odd} \\ 0 & \text{else} \end{cases}$$

$$\begin{pmatrix} 1 & ? \\ 0 & (2\pi i)^n \end{pmatrix} \qquad \Big\downarrow$$

$$\operatorname{Ext}^1_{MHT}(\mathbb{Q}(-n), \mathbb{Q}(0)) \simeq \mathbb{C}/(2\pi i)^n \mathbb{Q} \qquad \zeta(n).\mathbb{Q}$$

So we know that there exists an extension of $\mathbb{Q}(-n)$ by $\mathbb{Q}(0)$ with "period" $\zeta(n)$, but how to construct it? (<u>open problem</u>)

<u>Candidates</u>: $H^n(\overline{M}_{0,n+3} \setminus A, B \setminus (B \cap A))$

<u>Integral representation</u>

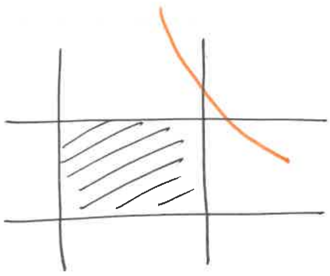
$$Li_n(z) = \int_{[0,1]^n} \frac{z \, dt_1 \ldots dt_n}{1 - z t_1 \ldots t_n} \qquad z \notin [1, \infty)$$

suggests to consider

$$\boxed{\begin{array}{l} S = \mathbb{P}^1 \setminus \{0,1,\infty\} \\ \text{coordinate } z \end{array}} \qquad \underset{t_1, \ldots t_n}{\mathbb{A}^n_S} \setminus \underbrace{\{1 - z t_1 \ldots t_n = 0\}}_{A_n}, \quad \underbrace{\{t_1(1-t_1) \ldots t_n(1-t_n) = 0\}}_{B_n}$$

and the cohomology

$$H^n(\mathbb{A}^n_S \setminus A_n, B_n \setminus (A_n \cap B_n))$$



<u>$n=2$</u>

Theorem (with Clément Dupont) ℓ-adic crystalline in degree n

$\mathcal{L}_n = M(\mathbb{A}^n \setminus A_n, B_n \setminus (A_n \cap B_n))[n]$ is the polylogarithm motive, i.e. fits into an extension

$$0 \to \mathbb{Q}_S(0) \to \mathcal{L}_n \to \mathrm{Sym}^{n-1}(K)(-1) \to 0$$

and has the Hodge realization that we want.

(Here: $M(X,Y) \in DM(S)$ relative cohomology motive $= (p_X)_* (j_{X\setminus Y})_! \mathbb{Q}_{X \setminus Y}(0)$ $\qquad X\setminus Y \overset{j}{\hookrightarrow} X \xrightarrow{p} S$)

Easy: there is a distinguished triangle

$$\begin{array}{c} dt_1 \dots dt_n \quad M(\mathbb{A}^n, B_n)[n] \longrightarrow \mathcal{L}_n \longrightarrow M(A_n, A_n \cap B_n)(-1)[n-1] \\ \| \\ M(\mathbb{A}^1, \{0,1\})^{\otimes n}[n] \\ \| \\ \mathbb{Q}(0) \end{array}$$

Crux of the proof: to show that

$$\boxed{M(A_n, A_n \cap B_n)[n-1] \simeq \mathrm{Sym}^{n-1}(K)}\ !$$

(This was a question that Deligne asked in a letter to Beilinson (2001), when he was looking at Ball and Rivoal's proof that infinitely many of $\zeta(3), \zeta(5), \dots$ are irrational:

$$\int_{[0,1]^n} \frac{t_1^{u_1}(1-t_1)^{v_1} \dots t_n^{u_n}(1-t_n)^{v_n}}{(1-zt_1\dots t_n)^2}\, dt_1 \dots dt_n$$

vanishes at the boundary

$\mathrm{Sym}^{n-1}(K) = M(\quad)^{\mathrm{sgn}}$

$\sigma\begin{pmatrix}1 & \zeta(n)\\ 0 & (2\pi i)^n\end{pmatrix}$ $\quad \int_0^1 f^m w = a_m + b_m \ldots$

linear combination of $1, \mathrm{Li}_1(z), \ldots \mathrm{Li}_n(z)$)
(Hope is to produce linear forms...)

- <u>Funny thing</u>:

(topological space)

$$\mathbb{Q}[\pi_1(M; x, y)] \big/ I^n \xrightarrow{\sim} H_{n-1}\left(M^{n-1}, \{\cancel{x_1} z_1 = x\} \cup \{\cancel{x_1} z_1 = z_2\} \cup \ldots \cup \{\cancel{x_{n-1}} z_{n-1} = y\}\right) \quad \boxed{\text{Beilinson}}$$

$(x \neq y)$

augmentation ideal of the group algebra $\mathbb{Q}[\pi_1(M, x)]$

$$(\gamma : [0,1] \to M) \longmapsto \left(\sigma : \Delta^{n-1} \to M^{n-1},\ (s_1, \ldots s_{n-1}) \mapsto (\gamma(s_1), \ldots \gamma(s_{n-1}))\right)$$

$\Delta^{n-1} = \{0 \le s_1 \le \ldots \le s_{n-1} \le 1\}$

In the case, $M = \mathbb{G}_m$, $x = 1$, $y = \frac{1}{z}$

$$\mathbb{Q}[\pi_1(\mathbb{C}^\times; 1, \tfrac{1}{z})] \big/ I^n \simeq H_{n-1}(\mathbb{G}_m^{n-1}, Z_{n-1})$$

$$\simeq H_{n-1}(A_n, B_n \cap A_n)$$

change of variables

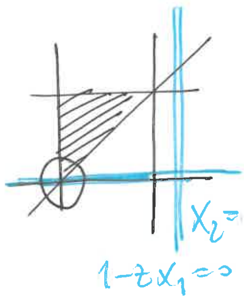
$$(x_1, \ldots, x_n) = (t_1 \ldots t_n, t_2 \ldots t_n, \ldots, t_n)$$

On the other hand, we "know" that (Deligne–Goncharov)

Malcev/unipotent completion $\varprojlim_n \mathbb{Q}[\pi_1(\mathbb{C}^\times; 1, \tfrac{1}{z})] \big/ I^n$ is the Betti realisation of the motive $\mathrm{Sym}(K)$. $\quad n = 2 \quad H_1(\mathbb{C}^\times, \{1, \tfrac{1}{z}\})$ is K

So what we prove is a motivic lift of <u>Beilinson's</u> theorem.

$$\mathrm{Li}_n(z) = \int_{0 \le x_1 \le \ldots \le x_n \le 1} \frac{z\, dx_1}{1 - z x_1} \frac{dx_2}{x_2} \cdots \frac{dx_n}{x_n}$$



Key ingredient: motives of configuration spaces

$j : C_N(X) \hookrightarrow X^N$ stratification (X^π)

configuration space of N points in X

$x_a = x_b$ if a, b in the same block

$\pi \in \Pi(N)$ partitions of N

$i_\pi : X^\pi \hookrightarrow X^N$

There is a filtration system in $\underline{DM(X^N)}$

$$\begin{tikzcd}
0 = F^n \arrow[r] & \cdots \arrow[r] \arrow[dl] & F^1 \arrow[r] \arrow[dl] & F^0 = j_! j^! \underbrace{\mathcal{F}}_{DM(X^n)} \arrow[dl] \\
G^{n-1} \arrow[u] & G^1 \arrow[u] & G^0 \arrow[u, "+1"] &
\end{tikzcd}$$

with
$$\boxed{G^k = \bigoplus_{\substack{\pi \in \Pi(N) \\ |\pi| = n-k}} i_{X^\pi} i^*_{X^\pi} \mathcal{F}[-k] \otimes A(\pi)^\vee}$$

where $A(\pi) = \bigotimes_{B \in \pi} A_B$ ← Arnold module

$H^{|B|-1}(C_N(\mathbb{C}), \mathbb{Q})$

degree $|B|-1$ component of $\langle (e_{i,j})_{i,j \in B} \mid e_{ij} = e_{ji}$

deg 1

$e_{ij} e_{ik} + e_{ik} e_{jk} = e_{ij} e_{jk}$

i, j distinct.

Equivariant for the action of S_N.

$$\boxed{A_B^\vee \simeq \mathrm{sign}_B \otimes \mathrm{Ind}_{C_B}^{S_B}(\zeta)}$$

cyclic subgroup

primitive character

$\Rightarrow (A_B^\vee)^{\mathrm{sign}} = 0$ if $\underline{\underline{|B| \geq 2}}$ □

de Rham realization:

$$\omega_n^{(0)} = dt_1 \ldots dt_n$$

$$\omega_n^{(k)} = \frac{z\, E_{n-k}(z t_1 \ldots t_n)}{(1 - z t_1 \ldots t_n)^{n-k+1}}\, dt_1 \ldots dt_n$$

$$\sum_{j \geq 0} (j+1)^2 x^j = \frac{E_2(x)}{(1-x)^{2+1}}$$

Eulerian polynomials

$E_0(x) = E_1(x) = 1$

$E_2(x) = 1 + x$

$E_3(x) = 1 + 4x + x^2$

$\vdots$

$$\boxed{\int_{[0,1]} \omega_n^{(k)} = \mathrm{Li}_k(z)}$$