1988 ①

[after André, Dawson, Orr, Papas, Urbanik]

2020 – 2025

# THE METHOD OF G-FUNCTIONS

Cargèse, 1-5/9/25

## ① A formula of Ramanujan

$$\boxed{\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3} (4n+1)(-1)^n}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. → « Modular equations and approximations to $\pi$ » (1914)

- This formula translates the existence of exceptional period relations in the fibers where the elliptic curve

$$E_k : \quad y^2 = (1-x^2)(1-k^2x^2) \qquad k \notin \{0, 1, -1\}$$

acquires complex multiplication (that is to say, $\mathrm{End}(E_k)$ is an order in an imaginary quadratic field). The values of $k$ for which this happens (e.g. $k = \frac{1}{\sqrt{2}}$, $k = \sqrt{2}-1, \ldots$) are classically called singular moduli.

- Indeed, if $\omega, \eta \in H^1_{dR}(E_k)$ is a basis of algebraic de Rham cohomology (with $\omega \in F^0 H^1_{dR}(E_k) = H^0(E_k, \Omega^1_{E_k})$) and $\sigma_1, \sigma_2 \in H_1(E_k(\mathbb{C}), \mathbb{Z})$ is a symplectic basis, then the periods

$$\omega_i = \int_{\sigma_i} \omega \quad \text{and} \quad \eta_i = \int_{\sigma_i} \eta$$

"quasi-period"

satisfy Legendre's relation

$$\underbrace{\eta_1 w_2 - \eta_2 w_1}_{\text{period of } \Lambda^2 H^1(E_K) \simeq H^2(E_K) \simeq H^2(\mathbb{P}^1)} = \boxed{2\pi i}$$

If $E_K$ has complex multiplication, then $\boxed{z = \frac{w_2}{w_1}}$ lies in the imaginary quadratic field and from the equation $\boxed{Az^2 + Bz + C = 0}$ we gets a second relation

$$\boxed{Az\eta_2 - C\eta_1 + \alpha w_1 = 0}$$

for some algebraic $\alpha$ (in fact in the field of definition of $E_K$ adjoint $z$).

- Now, the periods of $E_K$ can be expressed in terms of complete elliptic integrals

1st kind $\quad K(k) = \displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

$$= \frac{\pi}{2} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} \middle| k^2\right) \quad (|k|<1)$$

2nd kind $\quad E(k) = \displaystyle\int_0^1 \sqrt{1-k^2x^2} \frac{dx}{\sqrt{1-x^2}}$

$$= \frac{\pi}{2} \cdot {}_2F_1\left(\begin{matrix} -\frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} \middle| k^2\right)$$

and the corresponding quantities for the complementary modulus $k' = \sqrt{1-k^2}$

$$K(k') \quad \text{and} \quad E(k')$$

For example, $\omega_1 = 4K(k')$

$\eta_1 = 4E(k)$

$\gamma_1 : (0,1) \to (1,0) \to (-1,0) \to (0,1)$

Here we use the hypergeometric function

$${}_2F_1\left(\begin{matrix}\alpha \quad \beta \\ \gamma\end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta_n)}{(\gamma_n)} z^n$$

- The function $E$ is closely related to the derivative of $K$, namely

$$\boxed{\frac{dK}{dk} = \frac{E - k'^2 K}{k(k')^2}}$$

and there is an identity of hypergeometric functions (Clausen, 1828)

$$\boxed{\left(\frac{2K(k)}{\pi}\right)^2 = \frac{1}{k'^2}\, {}_3F_2\left(\begin{matrix}\frac12 \quad \frac12 \quad \frac12 \\ 1 \quad 1\end{matrix} \,\middle|\, -\left(\frac{2k}{k'^2}\right)^2\right)}$$

$(0 \le k \le \sqrt{2}-1)$

Putting everything together we obtain Ramanujan's formula for $k = \sqrt{2}-1$! How to understand the supercongruence

$$\sum_{n=0}^{p-1} \frac{\left(\frac12\right)_n^3}{n!^3}(4n+1) \equiv \left(\frac{-1}{p}\right) \mod p^3$$

## ② A sample of results

The goal of the course is to present a method that in some cases allows us to obtain finiteness of the fibers where exceptional

Elliptic case: $\infty$ many such points but finitely many of bounded degree tical relations among periods occur. Here are two examples due to D. Urbanik (2023).

For $g \geq 2$, consider the universal family of genus $g$ curves

$$\begin{array}{c} \mathscr{C} \\ \downarrow \\ M_g \hookrightarrow \overline{M}_g = \text{Deligne-Mumford compactification} \end{array}$$

(parameterises stable curves $C$ projective connected with at most nodal singularities, finite automorphism group and $\dim H^1(\mathcal{O}_C) = g$)

Let $\mathrm{Exc} \subset M_g(\mathbb{C})$ be the set of points $x$ such that $\mathrm{Jac}(\mathscr{C}_x) \simeq A_1 \times \dots \times A_r$ has a factor with complex multiplication. $\mathrm{End}(A)_{\mathbb{Q}}$ = field of degree $2\dim A$

Theorem 1: Let $S \subset M_g$ be an irreducible Hodge-generic curve such that $\overline{S} \subset \overline{M}_g$ intersects the locus of stable curves

$\mathscr{C}_x = C_1 \cup \dots \cup C_\ell$ such that

$\#$ nodal singularities $\geq \ell + 1$

[Case $g=2$ due to Daw & Orr]

Then $S(\mathbb{C}) \cap \mathrm{Exc}$ is finite.

- "Hodge-generic" means that the Mumford-Tate group of the generic fibre over $S$ is as big

families $y^5 = x(x-1)(x-\lambda)$ $g=4$ $\infty$ many CM fibres.

as possible, namely $GSp_{2g}$. This condition is clearly necessary to obtain finiteness.

- The condition about degeneracy at infinity comes from the method of G-functions. We expect the result to be always true! ⟸ (Zilber-Pink conjecture)
- Another example, outside the realm of abelian varieties (!) is the following:

$$\begin{array}{ccccc} X' & \subset & X & \supset & X_{s_0} \\ \downarrow & & \downarrow & & \downarrow \\ S' & \subset & S & \supset & s_0 \end{array}$$

smooth projective

<u>Theorem 2</u>: Let $X' \to S'$ be a $\boxed{\text{Hodge-generic}}$ family of smooth projective hypersurfaces of degree $d$ in $\mathbb{P}^{n+1}$. Assume that $\boxed{d > n+2}$ and that $X_{s_0}$ is the union of $d$ hyperplanes in general position. Then for every $N$ and every number field $K$ of deg $\leq N$, $\text{Exc} \cap S'(K)$ is finite. ⟸ In fact, $h(\xi) \leq c_1 [\bar{k}(\xi) : k]^{c_2}$

already known by Cadoret & Tamagawa (2013)

Now $\text{Exc} \subset S'(\mathbb{C})$ is the set of points such that $H^n_{prim}(X'_s, \mathbb{Q})$ has complex multiplication in the sense that every simple $\mathbb{Q}$-Hodge structure $W \subset V$ has an algebra of endomorphisms given by algebraic cycles of degree $\dim W$.

$t \in \text{End}_{HS}(H^n(X))$ is a Hodge class in

$H^n(X) \otimes H^n(X)^*$

$= H^n(X) \otimes H^n(X) \otimes \mathbb{Q}(n)$

$\subset H^{2n}(X \times X, \mathbb{Q}(n))$

## ③ G-functions

Definition (Siegel, 1929) A G-function is a formal power series with algebraic coefficients

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \bar{\mathbb{Q}}[[z]]$$

such that:

(1) there exists a non-zero differential operator $L \in \bar{\mathbb{Q}}[z]\langle \frac{d}{dz} \rangle$ $\left(= c_\mu(z)\left(\frac{d}{dz}\right)^\mu + \dots + c_0(z)\right)$

$$\boxed{L \cdot f = 0}$$

(2) there exists a real number $C > 0$

$$\max_{\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| \leq C^n$$

$$\operatorname{den}(a_0, \dots a_n) \leq C^n$$

for all $n \geq 1$. ↑ smallest integer $d_n \geq 1$ such that $d_n a_0, \dots d_n a_n \in \mathbb{Z}$

e.g. for each $k \geq 1$,

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \quad \text{polylogarithm}$$

$$z \frac{d}{dz} Li_k(z) = Li_{k-1}(z) \quad \text{with } L_0(z) = \frac{z}{1-z}$$

so differential equation of ~~[illegible]~~ order $k+1$

$$|a_n| \leq 1$$

$$\operatorname{den}(a_0, \dots a_n) = \operatorname{lcm}(1, \dots n)^k \sim (e^k)^n$$

prime number theorem

<u>Remark</u>: The differential equation implies that $f(z)$ has coefficients in a number field $K$

$$\Sigma_K = \Sigma_f \cup \Sigma_\infty \quad \text{set of places of } K$$

For $v \in \Sigma_K$, we normalize $|\cdot|_v$ so that

* if $v$ is above $p$, $|p|_v = p^{-\frac{[K_v : \mathbb{Q}_p]}{[K:\mathbb{Q}]}}$
* if $v \in \Sigma_\infty$, $|\xi|_v = \begin{cases} |\xi|^{1/[K:\mathbb{Q}]} & v \text{ real} \\ |\xi|^{2/[K:\mathbb{Q}]} & v \text{ complex} \end{cases}$

Then the product formula

$$\sum_{v \in \Sigma_K} \log |\xi|_v = 0$$

holds for every $\xi \in K^\times$, and we define the <u>height</u> as $h : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$

$$h(\xi) = \sum_{v \in \Sigma_K} \log^+ |\xi|_v \in \mathbb{R}_{\geq 0}$$

(independent of choice of $K$)

Northcott: finitely many alg numbers of bounded height & degree

Extended to sequences by $h(\xi_1, \dots, \xi_n) = \sum_{v \in \Sigma_K} \log^+ \max(|\xi_1|_v, \dots)$

$\log^+(x) = \log \max(1, x)$

<u>Remark</u>: Condition (2) in the definition of G-function implies that for every place $v \in \Sigma_K$ the series $i_v(f) = \sum_{n=0}^\infty i_v(a_n) z^n \in \mathbb{C}_v[[z]]$ has radius of convergence

$i_v : K \hookrightarrow \overline{K_v}$

$$R_v(f) \geq \frac{1}{C} \boxed{> 0}$$

$= \limsup_{n \to \infty} |a_n|_v^{1/n}$

Define invariants of $f \in K[\![z]\!]$:

- Size $\quad \sigma(f) = \limsup_{n\to\infty} \frac{1}{n} h(a_0, \dots a_n)$

$$= \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma_K} \log^+ \max(|a_i|_v)$$

$\in [0,\infty]$

- global radius $\quad \rho(f) = \sum_{v\in\Sigma_K} \log^+ \frac{1}{R_v(f)}$

Hadamard's formula $\quad = \sum_{v\in\Sigma_K} \limsup_{n\to\infty} \frac{1}{n} \log^+ \max(a_i)$

Equivalent definition:

A G-function is an $f \in K[\![z]\!]$ solution of a differential equation with $\boxed{\sigma(f) < \infty}$.

Example: diagonals of rational functions in many variables

$f \in K(z)$ such that 0 is not a pole

Write $f = \frac{P}{Q}$ with $P, Q \in \mathcal{O}_K[z]$ and let $N = |N_{K/\mathbb{Q}} Q(0)|$. Then $f \in \mathcal{O}_K[\frac{1}{N}][\![z]\!]$, so for all but finitely many $v$, $|a_i|_v \le 1$ for all $i$, and hence $\sigma(f) < +\infty$

$$\frac{d}{dz} f = \left(\frac{p'}{p} - \frac{q'}{q}\right) f \implies \text{G-function}$$

Now, start with $f \in K(x_1, \dots x_r)$ regular at 0 so that $(r \geq 2)$

$$f = \sum_{i_1, \dots i_r \geq 0} a_{i_1 \dots i_r} x_1^{i_1} \cdots x_r^{i_r}$$

and form the diagonal

$$\mathrm{Diag}_r(f) = \sum_{n=0}^{\infty} a_{n \dots n} z^n \in K[\![z]\!]$$

$$= \frac{1}{(2\pi i)^{r-1}} \int_{\substack{|x_2| = \dots = |x_r| = \varepsilon \\ x_1 \cdots x_r = z}} f \frac{dx_2 \cdots dx_r}{x_2 \cdots x_r}$$

(Deligne) Residue formula

($|z|$, $\varepsilon$ small enough so that $f$ is regular on the polydisc)

is a G-function with coefficients in $\mathcal{O}_K[\frac{1}{N}]$ (called globally bounded!)

e.g. $${}_2F_1\left(\begin{matrix} \frac{1}{2} \; \frac{1}{2} \\ 1 \end{matrix} \middle| 16z\right) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 z^n$$

$$= \underbrace{\sum_{n=0}^{\infty} \binom{2n}{n} z^n}_{\frac{1}{\sqrt{1-4z}}} * \sum_{n=0}^{\infty} \binom{2n}{n} z^n$$

algebraic function $\frac{1}{\sqrt{1-4z}}$

Hadamard product

$$= \mathrm{Diag}\left(\frac{1}{1-x-y} \cdot \frac{1}{1-z-t}\right)$$

Fact: diagonals of rational functions in 2 variables = $K[\![z]\!]$ algebraic over $K(z)$

so ~~G-funct~~ algebraic functions are G-functions (Eisenstein 1852)

Interesting part is that $\mathrm{Diag}_n(f)$ satisfies a differential equation! For this we consider:

$$X = \mathrm{Spec}\left(k(z)[x_1,\dots,x_n] \big/ x_1\cdots x_n - z\right)$$
$$\cup$$
$$Z = \{Q(x_1,\dots x_n) = 0\} \qquad \mathcal{U} = X \setminus Z$$

smooth affine variety over $k(z)$

Every element of $H^{n-1}_{dR}(\mathcal{U}/k(z))$ can be represented by a differential form

$$\frac{R}{Q^s}\,\frac{dx_2\cdots dx_n}{x_2\cdots x_n} \qquad R \in k\left[x_1,\dots x_n, \frac{1}{x_1\cdots x_n}\right]$$

and the $k(z)$-linear map

$$I : H^{n-1}_{dR}(\mathcal{U}/k(z)) \longrightarrow k((z))$$
$$\frac{R}{Q^s}\,\frac{dx_2\cdots dx_n}{x_2\cdots x_n} \longmapsto \mathrm{Diag}_n\left(\frac{R}{Q^s}\right)$$

is horizontal for the Gauss-Manin connection on $H^{n-1}_{dR}(\mathcal{U}/k(z))$ and the usual derivative on $k((z))$

Finite-dimensionality of $H^{n-1}_{dR}(\mathcal{U}/k(z))$ implies differential equation!

# Lecture II:

Recall the definition of a G-function

$$f = \sum_{n=0}^{\infty} a_n z^n \in K[[z]]$$

1) $L \cdot f = 0$

2) $\max_{\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| \leq C^n$

$den(a_0, \dots a_n) \leq C^n$

globally bounded

$\exists A, B \in \mathbb{Z} \quad A f(Bz) \in \mathcal{O}_K[[z]] \Longrightarrow f \in \mathcal{O}_K[\frac{1}{N}][[z]]$

e.g. $Li_k(z)$ is not globally bounded — but stronger

* rational functions $K(z)$
* diagonals $Diag_2\left(\frac{P}{Q}\right)$

e.g. $Diag_2\left(\frac{1}{1-x-y}\right) = \sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}}$

algebraic function

Fact: Diagonals in 2 variables $= K[[z]] \cap \overline{K(z)}$ (Eisenstein 1852)

$${}_2F_1\left(\begin{matrix}\frac{1}{2} \; \frac{1}{2} \\ 1\end{matrix} \,\middle|\, 16z\right) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 z^n$$

transcendental

$$= \sum \binom{2n}{n} z^n * \sum \binom{2n}{n} z^n$$

$$= Diag\left(\frac{1}{1-x_1-x_2} \cdot \frac{1}{1-x_3-x_4}\right)$$

All hypergeometric functions are G-functions, most of them are NOT diagonals.

G-functions are stable under sum, product, derivative, algebraic substitution, ...

Consequences of the definition

$K$ number field of degree $d$

$\Sigma_K = \Sigma_f \cup \Sigma_\infty$ set of places

For $v \in \Sigma_K$, $i_v : K \hookrightarrow K_v \hookrightarrow \widehat{\overline{K_v}} = \mathbb{C}$

Normalize $|\cdot|_v$ so that

- if $v$ is above $p$, $|p|_v = p^{-\frac{[K_v : \mathbb{Q}_p]}{d}}$
- if $v \in \Sigma_\infty$, $|\xi|_v = \begin{cases} |\xi|^{1/d} & v \text{ real} \\ |\xi|^{2/d} & v \text{ complex} \end{cases}$

Then the product formula holds

$$\sum_{v \in \Sigma_K} \log |\xi|_v = 0 \quad \text{for all } \xi \in K^\times$$

and we define the (logarithmic) height as

$$h : \overline{\mathbb{Q}} \longrightarrow \mathbb{R}_{\geq 0}$$

$$\xi \longmapsto h(\xi) = \sum_{v \in \Sigma_K} \log^+ |\xi|_v$$

$\log^+(x) = \log \max(1, x)$

(independent of choice of $K$)

Condition (2) in the definition of G-function implies that for every $v \in \Sigma_K$ the series

$$i_v(f) = \sum_{n=0}^{\infty} i_v(a_n)\, z^n \in \mathbb{C}_v[[z]]$$

has strictly positive radius of convergence

$$\boxed{R_v(f) > 0}$$

(compute using Hadamard's formula)

Define invariants:

size

$$\sigma(f) = \limsup_{n\to\infty} \frac{1}{n} h(a_0, \dots, a_n) = \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma_K} \log^+ \max_{i\le n} |a_i|_v$$

global radius

$$\rho(f) = \sum_{v\in\Sigma_K} \log^+ \frac{1}{R_v(f)} = \sum_{v\in\Sigma_K} \limsup_{n\to\infty} \frac{1}{n} \log^+ \max_{i\le n} |a_i|_v \in [0, \infty]$$

G-function $\iff$ differential equation $+ \boxed{\sigma(f) < 0}$

Questions: 1) structure of differential equation?

2) geometric origin?

3) transcendence properties of special values?

Theorem (André, Bombieri, Chudnovsky$^2$, Katz)

Let $L \in \overline{\mathbb{Q}}[z]\langle \frac{d}{dz} \rangle$ be a differential operator of minimal order annihilating a non-zero G-function. Then around every $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, $L$ has a basis of solutions of the form

$$\sum_i (z-\alpha)^{a_i} \log(z-\alpha)^{b_i} g_i(z-\alpha)$$

where $a_i \in \mathbb{Q}$, $b_i \in \mathbb{Z}_{\geq 0}$, $g_i \in \overline{\mathbb{Q}}[[z]]$ is a G-function. In particular, $1/z$ if $\alpha = \infty$

$L$ has regular singularities everywhere and the associated local system has quasi-unipotent monodromy.

Some ideas in the proof

① Define $\sigma$ and $\rho$ for differential operators

Through companion matrix, $L$ gives rise to a system $\frac{d}{dz} - A(z)$ for some $A \in M_{\mu \times \mu}(K(z))$

Define matrices $A_{[m]} \in M_{\mu\times\mu}(K(z))$ by

$$z^m \left(\frac{d}{dz}\right)^m \cancel{Y} = A_{[m]} \cancel{Y} \quad \text{if} \quad \frac{d}{dz} \cancel{Y} = A \cdot \cancel{Y}$$

$$(A_{[0]} = \mathrm{Id},\ A_{[1]} = A,\ A_{[m+1]} = \frac{d}{dz} A_{[m]} + A_{[m]}(A - mI))$$

$$\sigma(L) = \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma_K} \log^+ \max_{m\leq n} \left\| \frac{A_{[m]}}{m!} \right\|_v$$

Gauss norm on $K(z)$ given on polynomials by $\max |\text{coefficients}|_v$

Definition: $L$ is a G-operator if $\sigma(L) < \infty$ (Galochkin's condition)

Logic: if $Y(z) \in M_{\mu\times\mu}(K[\![z]\!])$ satisfies $\frac{d}{dz} Y = A(z) Y$, then

$$Y(z) = \sum \frac{G_{[m]}(\alpha)}{n!\,\alpha^n} Y(\alpha)\,(z-\alpha)^n$$

at every ordinary point $\alpha$.

$$\rho(L) = \sum_{v\in\Sigma_K} \log^+ \frac{1}{R_v(L)}$$

generic radius of solvability

$$\min\left(\lim_{n\to\infty} \left\| \frac{A_{[m]}}{m!} \right\|_v^{-1/n}, 1\right) \boxed{>0}$$

$\rho(L) < \infty$

Bombieri's condition

G-operator $\Rightarrow$ all power series solutions are G-functions

## 2 Théorème of Chudnovsky

$L$ of minimal order for a non-zero G-function is a G-operator (proof uses approximation diophantine)

③ André: $\sigma(L) < \infty \Leftrightarrow \rho(L) < \infty$

④ The condition $\rho(L) < \infty$ implies that $L$ has nilpotent $p$-curvature for almost all $p$

$\Rightarrow$ $L$ has regular singularities
Katz and rational exponents.

⑤ Finally one shows that the power series in a basis of solutions are $G$-functions by $\sigma(Y) < \infty$ + finding a differential equation $\square$

Picard-Fuchs differential operators

$$\begin{array}{c} X \\ p \downarrow \\ \mathbb{A}^1_z \supset S \ni 0 \end{array}$$ smooth proper

over $\bar{\mathbb{Q}}$

$\mathbb{V} = R^i p_* \mathbb{Q}$ local system on $S(\mathbb{C})$ with fibres $H^i_B(X_s)$

$(\mathcal{E}, \nabla) = R^i p_* \Omega^\bullet_{X/S}$ vector bundle on $S$ with fibres $H^i_{dR}(X_s)$ with connection

$$\nabla_{GM} : R^i p_* \Omega^\bullet_{X/S} \longrightarrow R^i p_* \Omega^\bullet_{X/S} \otimes \Omega^1_S$$

over an open disk where $\mathbb{V}$ is trivial, we can pick a basis $\sigma_i$, sections $\omega_j$ of $\mathcal{E}$ and form the period integrals

$$P(z) = \left( \int_{\sigma_i} \omega_j \right)$$

analytic meaning of the Gauss-Manin connection

$$\frac{d}{dz}\int_{\sigma_i} \omega_j = \int_{\sigma_i} \nabla_{GM,\frac{d}{dz}}(\omega_j)$$

we can write the vector bundle with connection in matrix form and pick a cyclic vector to get a differential operator $L \in \overline{\mathbb{Q}}[z]\langle \frac{d}{dz} \rangle$

Definition: A Picard-Fuchs operator is a product of irreducible factors of those

(geometrically, consider all summands and all extensions of those)

Bombieri-Dwork conjecture

Every G-function is solution of a Picard-Fuchs differential operator.

This suggests that the ring of special values of G-functions is equal to

Small objection C holds

$$\text{Periods}\left[\frac{1}{\pi}\right]$$

because of Ramanujan formulas

* True for operators of order 1 because all solutions are algebraic

Theorem (André) Picard-Fuchs operators are G-operators

$\Longrightarrow$ there exists a fundamental matrix of solutions $G(z) \in M_{\mu \times \mu}$ (G-functions) with $G(0) = \mathrm{Id}$

Since $P(z)$ is another fundamental matrix of solutions, $G(z) = P(z) \underbrace{P(0)^{-1}}$

we need to divide by periods of the fiber at the base point

Already seen in case

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{2} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} \middle| k^2 \right)$$

Key point in the proof

$L$ Picard-Fuchs $\in K[z]\langle \frac{d}{dz} \rangle$

One shows that $\boxed{R_v(L) = 1}$ for almost every place $v$ of $K$. $\Longrightarrow \rho(L) < \infty \Longrightarrow \sigma(L) < \infty$

Enough to do it for $X/K(z)$ smooth proper variety and $L$ associated to $H^i_{dR}(X/K(z))$.

This has an F-crystal structure, i.e. a horizontal isomorphism

completion of $K_v(z)$ for Gauss norm

matrix $pA^\Phi(z^p)$

$$\left(H^i_{dR}(X) \otimes_{K(z)} E_v\right)^\Phi \simeq H^i_{dR}(X) \otimes_{K(z)} E_v$$

and this increases the generic radius from $R$ to $R^{1/p}$. Iterate and take the limit...

# Lecture III

Errata ① $\rho(L) < \infty$ implies that $L$ has nilpotent p-curvature for all $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$ in a set $S$ of Dirichlet density 1   $R_v > |p|^{1/p-1}$

i.e. $$\frac{-1}{\log(s-1)} \sum_{\mathfrak{p} \in S} \frac{1}{N(\mathfrak{p})^s} \longrightarrow 1$$

(not necessarily almost all!)

② Proving that the power series appearing in solutions around a singular point is quite subtle and requires new ideas!

Theorem (André) Picard-Fuchs operators are G-operators

Key point of proof: it suffices to show that $\rho(L) = \sum_{v \in \Sigma_K} \log^+ \frac{1}{R_v(L)} < \infty$, and what we actually show is that $\boxed{R_v(L) = 1}$ ! for almost all $p$. This follows from the existence of a Frobenius structure on de Rham cohomology ☐

$\Rightarrow$ Period functions are period linear combinations of G-functions with monodromy

$$\sum \underbrace{c_i}_{\text{period}} z^{a_i} \log z^{b_i} g_i(z)$$

Bombieri-Dwork conjecture:

G-operators $\boxed{=}$ Picard-Fuchs operators

* True for operators of order 1 because the solutions are algebraic functions

$$c\prod(z-\alpha_i)^{r_i}$$

Open from order 2, as well as any consequence of being geometric!

e.g. $${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n} z^n$$

$$z(1-z)\left(\frac{d}{dz}\right)^2+\left[c-(a+b+1)z\right]\frac{d}{dz}-ab \qquad = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx$$

$\mathrm{Re}(c)>\mathrm{Re}(b)>0$

suggests looking at the family of curves

$$y^N = x^A(x-1)^B(x-z)^C \circlearrowleft \mu_N$$

and realize the differential equation

* Suggests that

$$\text{Periods}\left[\frac{1}{\pi}\right] \overset{?}{=} \text{special values of } G\text{-functions} = \left\{ g(\alpha) \;\middle|\; \begin{matrix}\alpha\in\overline{\mathbb{Q}} \\ \text{inside radius of convergence}\end{matrix} \right\}$$

because of Ramanujan's formula

$\boxed{C}$ is true (observation with Peter)

What happens for the evaluations at finite places?

Bombieri's Hasse principle for special values of G-functions

Let $f_1, \dots, f_\mu \in K[[z]]$ be G-functions satisfying a differential system $\left(\frac{d}{dz}\right)\begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} = A(z) \begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix}$.

A exceptional global relation at $\xi \in \overline{\mathbb{Q}}$ is a polynomial $Q \in K[T_1, \dots, T_\mu]$ homogeneous of degree $\delta$ such that:

(1) $Q$ is not the specialization at $\xi$ of a homogeneous polynomial of the same degree $\tilde{Q} \in K[z][T_1, \dots T_\mu]$ such that

$$\tilde{Q}(f_1(z), \dots f_\mu(z)) = 0$$

(2) For all $v \in \Sigma_{K(\xi)}$ such that

$$|\xi|_v < \min(1, R_v(f_1), \dots R_v(f_\mu))$$

the relation

$$i_v(Q)(i_v(f_1)(i_v(\xi)), \dots i_v(f_\mu)(i_v(\xi))) = 0$$

holds in $\mathbb{C}_v$.

__Example:__ Let $P \in K[X,Y]$ be an irreducible polynomial such that $P(0,Y)$ has a simple root in $K$. By the implicit function theorem, there exists

$$f(z) \in K[[z]] \quad \text{such that } P(z, f(z)) = 0$$

and $f(z), f^2(z), \dots, f^{\mu-1}(z)$ $\quad \mu = \deg_Y P$

are linearly independent G-functions.
Every $\xi \in K$ such that $P(\xi, Y)$ is reducible gives an exceptional global relation at $\xi \in K$.

__Example:__ $$f(z) = \frac{1}{\sqrt{1-z}} = 1 + \frac{z}{2} + \frac{1\cdot 3}{2!}\left(\frac{z}{2}\right)^2 + \frac{1\cdot 3\cdot 5}{3!}\left(\frac{z}{2}\right)^2 + \dots$$

Then $$R_v(f) = \begin{cases} 1 & v = p \neq 2 \text{ or } \infty \\ \frac{1}{4} & v = 2 \end{cases}$$

At $\xi = -15$, the relation $f(\xi) = \frac{1}{4}$ is __not__ global because

$$f(i_3(\xi)) = \frac{1}{4} \in \mathbb{Q}_3$$
$$f(i_5(\xi)) = -\frac{1}{4} \in \mathbb{Q}_5$$

__Theorem__ (Bombieri, André) Let $\Xi(f_1, \dots, f_\mu, \delta)$ be the subset of $\overline{\mathbb{Q}} \ni \xi$ at which there

there exists an exceptional global relation of degree $\delta$ among $f_1(\xi), \dots f_\mu(\xi)$. Then ~~[illegible]~~

$$\sup_{\xi \in \Xi(f_1, \dots f_\mu, \delta)} \boxed{h(\xi) \leq c_1 \delta^{c_2}}$$

with ~~[illegible]~~ constants that only depend on $f_1, \dots f_\mu$ ~~[illegible]~~.

Sketch of proof

Let $f_1, \dots, f_\mu$ be linearly independent G-functions such that $\Lambda \begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} = 0$ $\quad \Lambda = \frac{d}{dz} - A(z)$

Assume that at $\xi \in K^\times$ there are $\mu - r$ linearly independent relations

$$(*) \quad \sum_{i=1}^{\mu} a_{ij} f_i(\xi) = 0 \qquad j = 1, \dots \mu - r$$

Then there exists $Z \in M_{\mu \times r}(K[[z-\xi]])$ such that $\Lambda \cdot Z = 0$ and $\underbrace{\begin{pmatrix} i_v(f_1)(\xi) \\ \vdots \\ i_v(f_\mu)(\xi) \end{pmatrix}}_{\mu \times 1} = \underbrace{i_v(Z)(\xi)}_{\mu \times r} \cdot \underbrace{B_v}_{r \times 1}$

column in $K_v$

at every $v \in \Sigma_K$ at which $(*)$ holds.

Step 1: Find polynomials (not all zero) $P_1, \dots P_\mu \in K[z]$ such that $P \cdot Z$ has vanishing order at $\xi$ $\boxed{\mathrm{ord}_\xi (P \cdot Z) \geq \alpha}$

of controlled degree and height

(Siegel's lemma) and form $R = P_1(z) f_1(z) + \dots + P_\mu(z) f_\mu(z)$

Step 2: $\mathrm{ord}_\xi\, i_v(R) \geq \alpha$

$\beta = \mathrm{ord}_0 R$ is bounded using a zero estimate for polynomial relations of solutions of a differential equation

Step 3: let $\eta = \frac{1}{\beta!}\left(\frac{d}{dz}\right)^\beta R$ the $1^{st}$ non-zero coefficient of $R$. Then

$$i_v(R)\, z^{-\beta}\left(1 - \frac{z}{i_v(\xi)}\right)^{-\alpha}$$ is an

analytic function that takes the value $\boxed{i_v(\eta) \text{ at } z=0}$. We bound $\log|\eta|_v$:

- using the maximum modulus principle at $v \in \Sigma_K$ for which $|\xi|_v < \min(1, R_v(f_i))$
- using the formula $\eta = \frac{1}{\beta!}\left(\frac{d}{dz}\right)^\beta R$ at all the other places

and sum over all $v \in \Sigma_K$, then let $\alpha \to \infty$

Apply this result to symmetric powers of degree $\delta$ of the differential operator ... $\square$

Remarks: 1) effective versions of Hilbert's irreducibility theorem, e.g. $P(p^m, Y) \in \mathbb{Q}[Y]$

st irreducible for primes $p$ such that $p^m$ is big enough (depending only on the polygonal). Spiridzute

2) André uses a refinement of these techniques to show that for all $\xi \in \overline{\mathbb{Q}}^{\times}$ such that

$$|\xi|_v < \min(1, |16|_v)$$

$${}_2F_1\left(\begin{matrix}\frac{1}{2} \ \frac{1}{2}\\ 1\end{matrix}\middle|\, i_v(\xi)\right) \text{ and } {}_2F_1\left(\begin{matrix}-\frac{1}{2} \ \frac{1}{2}\\ 1\end{matrix}\middle|\, i_v(\xi)\right)$$

are algebraically independent over $\mathbb{Q}$.

$$\Longrightarrow \operatorname{trdeg} \overline{\mathbb{Q}}\left(\begin{matrix}\text{periods of}\\ \text{elliptic curves}\end{matrix}\right) \geq 2.$$

An obvious way to produce global relations is to multiply relations at all places. Most of the older applications are restricted to situations where the only $v \in \Sigma_K$ for which $|\xi|_v < \min(1, R_v(f_i))$ holds are the archimedean places.

## Constructions of G-functions

$$\begin{tikzcd} X \arrow[d, "p"'] & \subset & X' \arrow[d, "p'"] & \supset & Y = p'^{-1}(s_0) \arrow[d] \\ S & \subset & S' & \ni & s_0 \end{tikzcd}$$

smooth proper $p$ rel dim $n$

$Y = p'^{-1}(s_0)$ = simple normal crossing divisor = reduced $\bigcup Y_i$ smooth irreducible components

Consider the local system $(R^n p_* \mathbb{Q})^\vee$ over a small punctured disk

$\Delta^*$ around $s_0$. Then $\pi_1(\Delta^*)$ acts with unipotent monodromy $T \circlearrowright R_n f_* \mathbb{Q}|_{\Delta^*}$

Let $2\pi i N$ be the logarithm of $T$ and

$$M_0 R_n f_* \mathbb{Q}|_{\Delta^*} = \mathrm{Im}(2\pi i N)^n \text{ acting on } R_n f_* \mathbb{Q}|_{\Delta^*}$$

André: dimension of this space

$$= h^n(\Sigma_Y)$$

dual complex associated to the simple normal crossing divisor

* vertices $\leftrightarrow$ irreducible components
* attach a $k$-cell for each irreducible component in intersection $Y_{i_1} \cap \dots \cap Y_{i_k}$

Theorem (André) Over a small open disk $\Delta^*$ there exists a basis of sections $w_i$ of $R^n f_* \Omega^{\bullet}_{X/S}$ such that for any $\sigma$ in $M_0 R_n f_* \mathbb{Q}|_{\Delta^*}$,

$$\frac{1}{(2\pi i)^n} \int_\sigma w_i$$ is a G-function in a local parameter $z$ around $s_0$.

From the existence of motivic symmetries, we will get relations between periods, and hence between $\frac{\cancel{\ldots}}{(2\pi i)^n}\cdot$ G-functions. To eliminate the factor $(2\pi i)^n$, we need to place ourselves in a situation where $\boxed{h^n(\Sigma_y) \geq 2\ !}$

<u>e.g.</u> for families of curves $(n=1)$

$$h^1(\Sigma_y) = \#\left(\begin{matrix}\text{double}\\ \text{points}\end{matrix}\right) - \#\text{ components} + 1$$

$$\begin{matrix} X \\ \downarrow \\ S \end{matrix}$$ family of smooth projective hypersurfaces in $\mathbb{P}^{n+1}$ of degree $d > n+2$ such that $Y$ is a union of $d$ hyperplanes in general position

<u>Then:</u> $h^n(\Sigma_y) = W_0 H^n(Y, \mathbb{Q})$ (Deligne)

Let $U = \mathbb{P}^{n+1} \setminus Y$,

$$\cdots \longrightarrow H^n_c(\mathbb{P}^{n+1}) \longrightarrow H^n_c(Y) \longrightarrow H^{n+1}_c(U) \longrightarrow \cdots$$

$$W_0 H^n(Y) = \cancel{\ \ } H^{n+1}_c(U) \quad \text{dimension } \binom{d-1}{n+1}.$$

# Lecture IV: Two theorems of Ulbrich

$X$ smooth projective variety of dimension $n$

An endomorphism of Hodge structures of $H^n(X)$ is a Hodge class in

$$H^n(X) \otimes H^n(X)^\vee \underset{\substack{\text{Poincaré}\\ \text{duality}}}{\cong} H^n(X) \otimes H^n(n) \underset{\substack{\text{Künneth}\\ \text{formula}}}{\subset} H^{2n}(X \times X, \mathbb{Q}(n))$$

So by the Hodge conjecture should correspond to a $\mathbb{Q}$-linear combination of classes of algebraic cycles of codimension $n$ on $X \times X$.

Definition: We say that $H^n(X)$ has motivic complex multiplication if for every simple sub Hodge structure $V \subset H^n(X)$ the algebra of endomorphisms generated by algebraic cycles is a field of degree $\dim_{\mathbb{Q}} V$.

(For $H^1$ (abelian varieties), this coincides with the usual notion)

$\rightarrow$ Why algebraic cycles?

$X/\overline{\mathbb{Q}}$ $\quad Z \in Z^p(X)$ has cycle class

$$cl_{dR}(Z) \in H^{2p}_{dR}(X/\bar{\mathbb{Q}})$$
$$cl_B(Z_{\mathbb{C}}) \in H^{2p}_B(X, \mathbb{Q})$$

and they correspond to each other up to powers of $2\pi i$ through the comparison isomorphism

$$H^{2p}_{dR}(X/\bar{\mathbb{Q}}) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H^{2p}_B(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$
$$cl_{dR}(Z) \longmapsto (2\pi i)^p \, cl_B(Z_{\mathbb{C}})$$

Expressing $cl_{dR}(Z)$ and $cl_B(Z_{\mathbb{C}})$ in a basis of $H^{2p}_{dR}(X/\bar{\mathbb{Q}})$ and a basis of $H^{2p}_B(X, \mathbb{Q})$, this produces linear relations with $\bar{\mathbb{Q}}((2\pi i)^{\pm 1})$ coefficients between the periods of $X$.

Working in $X \times \overset{m}{\cdots} \times X$, we get polynomial relations of degree $m$. $\rightsquigarrow$ $\left|\,\text{All relations expected to arise this way}\,\right|$ Grothendieck period conjecture

<u>Theorem 1</u>: let $S \subset M_g$ be a Hodge-generic curve such that $\bar{S} \subset \bar{M}_g$ intersects the locus of stable curves with $\#\text{nodes} \geq \#\text{smooth compact} + 1$. $(g \geq 2)$

$g = 2$ due to Daw + Orr

Then $S(\mathbb{C})$ contains finitely many points $s$ over which a factor of $\mathrm{Jac}(C_s)$ has CM.

<u>Theorem 2</u>: let $X \longrightarrow S$ be a one-parameter $\boxed{\text{Hodge-generic}}$ family of smooth projective hypersurfaces in $\mathbb{P}^{n+1}$ of degree $d > n+2$ that degenerates

into a union of $d$ hyperplanes in general position. Then for every $s \in S(\bar{\mathbb{Q}})$ such that $X_s$ has motivic complex multiplication

$$\boxed{h(s) \leq c_1 [K(s):K]^{c_2}}$$

- In particular, there are only finitely many in any number field of bounded degree (already known by Orr–Skorobogatov 2014).

## Construction of G-functions

$$\begin{array}{ccccccc} X & \subset & X' & \supset & Y & = p^{-1}(s_0) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow & \\ S & \subset & S' & \ni & s_0 & \end{array}$$

smooth proper relative dimension $n$

reduced divisor with simple normal crossings $\bigcup Y_i$

smooth irreducible components

Consider the local system

$$R_n p_* \mathbb{Q} = (R^n p_* \mathbb{Q})^\vee$$

over a small punctured disk $\Delta^*$ around $s_0$.

Then $\pi_1(\Delta^*)$ acts with unipotent monodromy

$$T \circlearrowright R_n p_* \mathbb{Q}|_{\Delta^*}$$

Let $2\pi i N$ be the logarithm of $T$ and

$$M_0 R_n p_* \mathbb{Q}|_{\Delta^*} = \text{image of } (2\pi i N)^n \text{ acting on } R_n p_* \mathbb{Q}|_{\Delta^*}$$

André computes the dimension of this space to be $h^n(\Sigma_Y)$ where $\Sigma_Y$ = dual complex associated to the simple normal crossing divisor $Y$

* vertices $\leftrightarrow$ irreducible components
* attach a $k$ cell for every irreducible component in intersection $Y_{i_1} \cap \ldots \cap Y_{i_k}$

(notation $W_0$ comes from weight filtration, e.g. $h^n(\Sigma_Y) = \dim W_0 H^n(Y, \mathbb{Q})$)

* Meaning of the degeneration conditions is that $\boxed{h^n(\Sigma_Y) \geq 2}$.

e.g. for families of curves $(n=1)$

$$h^1(\Sigma_Y) = \#\text{nodes} - \#\text{components} + 1$$

e.g. case of $Y \subset \mathbb{P}^{n+1}$ union of $d$ hyperplanes in general position, let $U = \mathbb{P}^{n+1} \setminus Y$

$$\cdots \longrightarrow H^n(\mathbb{P}^{n+1}) \longrightarrow H^n(Y) \longrightarrow H_c^{n+1}(U) \longrightarrow \cdots$$

and $H_c^{n+1}(U)$ is pure of weight 0, so

$$W_0 H^n(Y) \cong H_c^{n+1}(U) \text{ has dimension}$$

$$\binom{d-1}{n+1} \geq 2$$

if $d > n+2$

* Meaning of Hodge-genericity assumption: rule out fractional relations among G-functions!

Theorem (André) Over a dense open subset of $S$, there exists a basis $(\omega_i)$ of sections of $H^n_{dR}(X/S)$ such that for any $\sigma \in M_0 R^n p_* \mathbb{Q}|\Delta^*$

$$\frac{1}{(2\pi i)^n} \int_\sigma \omega_i$$

is a G-function in a local parameter around $s_0$.

Reason: open affine subset $U \subset X$

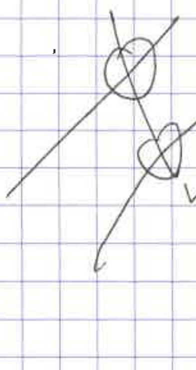
functions $z_1, \dots, z_{n+1}$ on $U$ such that

$dz_1, \dots, dz_{n+1}$ trivialize $\Omega^1_U$

the equation $\boxed{z_1 \cdots z_{n+1} = s}$ defines ~~the~~ ~~[illegible]~~ $p|U$ near $s_0$

Then every class in $H^n_{dR}(X/S)$ restricts to a de Rham class in $U$ that can be represented analytically as

$$h \frac{dz_2 \cdots dz_{n+1}}{z_2 \cdots z_{n+1}}$$



where $h$ is a ~~holomorphic function on power~~ ~~[illegible]~~ holomorphic function on $U$ whose power series representation has coefficients in $\bar{K}$ and the singular part of algebraic $K(z_1, \dots z_{n+1})$

- $\sigma$ vanishing cycle in $X_s \cap U$ product of $n$ simple loops around the divisors $\boxed{z_2 = 0, \dots z_{n+1} = 0}$.

Then $\displaystyle \frac{1}{(2\pi i)^n} \int_\sigma i_S^* \left( h \frac{dz_2 \cdots dz_{n+1}}{dz_2 \cdots dz_{n+1}} \right)$

$i_S : X_S \cap \mathcal{U} \hookrightarrow \mathcal{U}$

is the diagonal of an algebraic power series, and hence of a rational function.

<u>Reinterpretation</u>: there exists a linear functional

$$\sigma^* : H^n_B (X_S, \mathbb{Z}(n)) \longrightarrow \mathbb{Z}(n)$$

such that

$$\frac{1}{(2\pi i)^n} \left( \sigma^*_{\mathbb{C}} \circ \mathrm{comp}^{-1}_{B,dR} (w_i(s)) \right) = g_i(s)$$

G-function

Assumption: at least 2 linearly independent such functionals

* Urbanik observes that the construction of these G-functions can be done $\boxed{\text{purely algebraically}}$ working with $\mathcal{H}$ the canonical extension to $S'$ of $R^n p_* \Omega^\bullet_{X/S}$. Recall that it is given by $R^n p_* \Omega^\bullet_{X'/S'}(\log Y)$. Over $\mathcal{U}$, the sheaf can be computed as the cohomology in degree $n$ of the complex

$$0 \to \mathcal{O}_{\mathcal{U}} \to \Omega^1_{\mathcal{U}/S}(\log(Y\cap\mathcal{U})) \to \cdots \to \Omega^n_{\mathcal{U}/S}(\log(Y\cap\mathcal{U}))$$

We may now restrict to a formal neighborhood of the singular point $q$

$$0 \to \widehat{\mathcal{O}}_{\mathcal{U},(q)} \to \widehat{\Omega}^1_{\mathcal{U}/S}(\log(Y\cap\mathcal{U}))_{(q)} \to \cdots$$

complex of $\widehat{\mathcal{O}}_{S,(s_0)}$-modules $\simeq K[\![s]\!]$

Lemma: cohomology in degree $n \simeq K[\![s]\!]$

So we get a map from

$$\underset{\substack{\text{global}\\ \text{sections}}}{\mathcal{H}(S')} \longrightarrow K[\![s]\!] \quad \text{by restriction}$$

- Advantage: the same construction can be done in rigid analytic de Rham cohomology!

* Urbanik proves that there exists a functional

$$\hat{\sigma}^*: H^n_{\text{ét}}(X_{s,\overline{K}_r}, \mathbb{Z}_p(n)) \longrightarrow \mathbb{Z}_p(n)$$

$$G_{K_r} = \text{Gal}(\overline{K}_r/K_r)$$

equivariant for the action of $G_{K_r}$ on $\mathbb{Z}_p(n)$ through the $n$-th power of the cyclotomic character such that

$$\frac{1}{t^n}\left(\hat{\sigma}^*_{B_{dR}} \circ \text{comp}^{-1}_{\text{ét},dR}(w_i(s))\right) = i_r(g_i)(s)$$

for all $s$ in a rigid analytic disk around $s_0$.

Now:

$$\mathrm{comp}_{\text{ét}, dR} : H^n_{\text{ét}}(X_{s,\bar{K}_r}, \mathbb{Z}_p(n)) \otimes B_{dR} \longrightarrow H^n_{dR}(X_s) \otimes B_{dR}$$

is the p-adic comparison isomorphism and $t \in B_{dR}$ is the analogue of $2\pi i$.

[ Idea: the choice of a compatible system of p-roots of unity that gives rise to $t$ also induces an element of $\pi_1^{\text{ét}}(\mathbb{G}_m)^{(1)}$ and hence a map

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathbb{G}_m, \mathbb{Z}_p(1)) & \longrightarrow & \mathbb{Z}_p(1) \\ \big\downarrow & & \\ H^1_{\text{proét}}(\mathring{\Delta}, \mathbb{Z}_p(1)) & & \end{array}$$

$T \mapsto T^{p^i}$

$T \mapsto \int_1 T$

$\mathring{\Delta} \hookrightarrow \mathbb{G}_m$

where $\mathring{\Delta} = \mathrm{Spa}(K_r\langle T, T^{-1}\rangle, \mathcal{O}_{K_r}\langle T, T^{-1}\rangle)$ and we need to extend this map to define the analogue of a loop around the puncture. In fact, work in higher dimension with $(\mathring{\Delta})^n$ ]

* From this one obtains relations as follows:

if $Z \subset X_s \times X_s$ is an algebraic cycle, then its class $z \in \mathrm{End}(H^n_{\text{ét}}(X_{s,\bar{K}_r}))$ is fixed by ~~a finite index subgroup~~ of $G_{K_r}$. Then

$\hat{\sigma}^*, \hat{\sigma}^* \circ z, \hat{\sigma}^* \circ z^2, \ldots$ all lie in

$$\mathrm{Hom}\left(H^n_{et}(X_{s,\bar{K}_v}, \mathbb{Q}_p(n)), \mathbb{Q}_p(n)\right)(-n)^{G_{K_v}}$$

so there is a linear relation in

$$\mathrm{Hom}\left(H^n_{dR}(X_s) \otimes B_{dR}, B_{dR}\right)$$

and by evaluating on the basis $w_i$ and taking determinants a polynomial relation between $i_v(g_i)(s)$. Bound the dimension in terms of $p$-adic Hodge theory, so that at the end the product of local relations does not get too big!